**Topological Methods in Nonlinear Analysis**

Vol. 48, No. 2 December 2016

# GROUND STATES OF NONLOCAL SCALAR FIELD EQUATIONS WITH TRUDINGER–MOSER CRITICAL NONLINEARITY

João Marcos do Ó — Olímpio H. Miyagaki — Marco Squassina

Topol. Methods Nonlinear Anal. **48** (2016), 477–492
DOI: 10.12775/TMNA.2016.045

Published by the
**Juliusz Schauder Center**
**TORUŃ, 2016**

ISSN 1230-3429

Topological Methods in Nonlinear Analysis
Volume 48, No. 2, 2016, 477–492
DOI: 10.12775/TMNA.2016.045



# GROUND STATES OF NONLOCAL SCALAR FIELD EQUATIONS WITH TRUDINGER–MOSER CRITICAL NONLINEARITY

João Marcos do Ó — Olímpio H. Miyagaki — Marco Squassina

(*Submitted by Mónica Clapp*)

Abstract. We investigate the existence of ground state solutions for a class of nonlinear scalar field equations defined on the whole real line, involving a fractional Laplacian and nonlinearities with Trudinger–Moser critical growth. We handle the lack of compactness of the associated energy functional due to the unboundedness of the domain and the presence of a limiting case embedding.

## 1. Introduction and main result

The goal of this paper is to investigate the existence of ground state solutions $u \in H^{1/2}(\mathbb{R})$ for the following class of nonlinear scalar field equations:

$$(-\Delta)^{1/2}u + u = f(u) \quad \text{in } \mathbb{R}, \tag{1.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity in the critical growth range. Precisely, we focus here on the case when $f$ has *maximal growth* which allows to study problem (1.1) variationally in the Sobolev space $u \in H^{1/2}(\mathbb{R})$, see Section 2. We are motivated by the following Trudinger–Moser type inequality due to Ozawa [27].

2010 *Mathematics Subject Classification.* Primary: 35J60, 35B09; Secondary: 35B33.
*Key words and phrases.* Trudinger–Moser inequality; fractional Laplacian; ground states.
The research was partially supported by INCTmat/MCT/Brazil, CNPq, CAPES/Brazil. The paper was completed while the second author was visiting the Department of Mathematics of Rutgers University, whose hospitality he gratefully acknowledges.

Theorem 1.1. *There exists* $0 < \omega \leq \pi$ *such that, for all* $\alpha \in (0, \omega)$, *there exists* $H_\alpha > 0$ *with*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, dx \leq H_\alpha \|u\|_{L^2}^2, \tag{1.2}$$

*for all* $u \in H^{1/2}(\mathbb{R})$ *with* $\|(-\Delta)^{1/4} u\|_{L^2}^2 \leq 1$.

From inequality (1.2) we have naturally associated notions of *subcriticality* and *criticality* for this class of problems. Precisely, we say that $f\colon \mathbb{R} \to \mathbb{R}$ has subcritical growth at $\pm\infty$ if

$$\limsup_{s \to \pm\infty} \frac{f(s)}{e^{\alpha s^2} - 1} = 0, \quad \text{for all } \alpha > 0,$$

and has $\alpha_0$-critical growth at $\pm\infty$ if there exist $\omega \in (0, \pi]$ and $\alpha_0 \in (0, \omega)$ such that

$$\begin{aligned} \limsup_{s \to \pm\infty} \frac{f(s)}{e^{\alpha s^2} - 1} &= 0, &&\text{for all } \alpha > \alpha_0, \\ \limsup_{s \to \pm\infty} \frac{f(s)}{e^{\alpha s^2} - 1} &= \pm\infty, &&\text{for all } \alpha < \alpha_0. \end{aligned}$$

For instance, let $f$ be given by

$$f(s) = s^3 e^{\alpha_0 |s|^\nu} \quad \text{for all } s \in \mathbb{R}.$$

If $\nu < 2$, $f$ has subcritical growth, while if $\nu = 2$ and $\alpha_0 \in (0, \omega]$, $f$ has critical growth. By a *ground state* solution to problem (1.1) we mean a nontrivial weak solution of (1.1) with the least possible energy.

The following assumptions on $f$ will be needed throughout the paper:

(f1) $f\colon \mathbb{R} \to \mathbb{R}$ is a $C^1$, odd, convex function on $\mathbb{R}^+$, and

$$\lim_{s \to 0} \frac{f(s)}{s} = 0.$$

(f2) $s \mapsto s^{-1} f(s)$ is an increasing function for $s > 0$.

(f3) There are $q > 2$ and $C_q > 0$ with

$$F(s) \geq C_q |s|^q, \quad \text{for all } s \in \mathbb{R}.$$

(AR) There exists $\vartheta > 2$ such that

$$\vartheta F(s) \leq s f(s), \quad \text{for all } s \in \mathbb{R}, \qquad F(s) = \int_0^s f(\sigma)\, d\sigma.$$

The main result of the paper is the following:

Theorem 1.2. *Let* $f(s)$ *and* $f'(s)s$ *have* $\alpha_0$*-critical growth and satisfy* (f1)–(f3) *and* (AR). *Then problem* (1.1) *admits a ground state solution* $u \in H^{1/2}(\mathbb{R})$ *provided* $C_q$ *in* (f3) *is large enough.*

The nonlinearity

$$f(s) = \lambda s|s|^{q-2} + |s|^{q-2}se^{\alpha_0 s^2}, \quad q > 2 \text{ and } s \in \mathbb{R},$$

satisfies all hypotheses of Theorem 1.2 provided that $\lambda$ is sufficiently large. More examples of nonlinearities which satisfy the above assumptions can be found in [18]. In $\mathbb{R}^2$ one can use radial estimates, then apply, for instance, the Strauss lemma [33] to recover some compactness results. In $\mathbb{R}$ analogous compactness results fail, but in [20], the authors used the concentration compactness principle due to Lions [35] for problems with polynomial nonlinearities. In this paper, we use the minimization technique over the Nehari manifold in order to get ground state solutions. We adopt some arguments from [4] combined with those used in [10] and [21].

**1.1. Quick overview of the literature.** In [28], P. Rabinowitz studied the semi-linear problem

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \quad u > 0, \tag{1.3}$$

when $V$ is a positive potential and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ has subcritical growth, that is it behaves at infinity like $s^p$ with $2 < p < 2^* - 1$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent, $N \geq 3$. This was extended or complemented in several ways, see e.g. [35].

For $N = 2$ formally $2^* \rightsquigarrow +\infty$, but $H^1(\mathbb{R}^N) \not\hookrightarrow L^\infty(\mathbb{R}^N)$. Instead, the Trudinger–Moser inequality [26], [34] states that $H^1$ is continuously embedded into an Orlicz space defined by the Young function $\phi(t) = e^{\alpha t^2} - 1$. In [1], [14], [13], [24], with the help of Trudinger–Moser embedding, problems in a bounded domain were investigated, when the nonlinear term $f$ behaves at infinity like $e^{\alpha s^2}$ for some $\alpha > 0$. We refer the reader to [12] for a recent survey on this subject. In [11] the Trudinger–Moser inequality was extended to the whole $\mathbb{R}^2$ and the authors gave some applications to study equations like (1.3) when the nonlinear term has critical growth of Trudinger–Moser type. For further results and applications, we would like to mention also [2], [3], [16], [29] and references therein. When the potential $V$ is a positive constant and $f(x,s) = f(s)$ for $(x,s) \in \mathbb{R}^N \times \mathbb{R}$, that is the autonomous case, the existence of ground states for subcritical nonlinearities was established in [6] for $N \geq 3$ and [7] for $N = 2$ respectively, while in [3] the critical case for $N \geq 3$ and $N = 2$ was treated. For fractional problem of the form

$$(-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

with $N > 2s$ and $s \in (0,1)$, we refer to [10, 19] where positive ground states were obtained in subcritical situations. For instance, [10] extends the results in [6] to the fractional Laplacian. In [19] regularity and qualitative properties of the ground state solution are obtained, while in [31] a ground state solution is obtained for coercive potential. For fractional problems in bounded domains of $\mathbb{R}^N$ with $N > 2s$ involving critical nonlinearities we cite [5], [9], [22], [30] and [17] for the whole space with vanishing potentials. In [20] the authors investigated properties of the ground state solutions of $(-\Delta)^s u + u = u^p$ in $\mathbb{R}$. Recently, in [21], nonlocal problems defined in bounded intervals of the real line involving the square root of the Laplacian and exponential nonlinearities were investigated, using a version of the Trudinger–Moser inequality due to Ozawa [27]. As it was remarked in [21], the nonlinear problem involving exponential growth with fractional diffusion $(-\Delta)^s$ requires $s = 1/2$ and $N = 1$. In [18] some nonlocal problems in $\mathbb{R}$ with vanishing potential, thus providing compactifying effects, are considered. See also [32] for related results on the existence of solutions for fractional Schrödinger equations involving exponential critical growth.

## 2. Preliminary stuff

We recall that

$$H^{1/2}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\, dx\, dy < \infty \right\},$$

endowed with the norm

$$\|u\| = \left( \|u\|_{L^2}^2 + \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\, dx\, dy \right)^{1/2}.$$

The square root of the Laplacian, $(-\Delta)^{1/2}$, of a smooth function $u\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^{1/2} u)(\xi) = |\xi| \mathcal{F}(u)(\xi),$$

where $\mathcal{F}$ denotes the Fourier transform, that is,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi\cdot x} \phi(x)\, dx,$$

for functions $\phi$ in the Schwartz class. Also $(-\Delta)^{1/2} u$ can be equivalently represented [15] as

$$(-\Delta)^{1/2} u = -\frac{1}{2\pi} \int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2}\, dy.$$

Also, in light of [15, Propostion 3.6], we have

$$\|(-\Delta)^{1/4} u\|_{L^2}^2 := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\, dx\, dy, \quad \text{for all } u \in H^{1/2}(\mathbb{R}), \tag{2.1}$$

and, sometimes, we identify these two quantities by omitting the normalization constant $1/2\pi$. From [25, Theorem 8.5 (iii)] we also know that, for any $m \geq 2$, there exists $C_m > 0$ such that

$$\|u\|_{L^m} \leq C_m \|u\|, \quad \text{for all } u \in H^{1/2}(\mathbb{R}). \tag{2.2}$$

PROPOSITION 2.1. *The integral*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, dx$$

*is finite for any positive* $\alpha$ *and* $u \in H^{1/2}(\mathbb{R})$.

PROOF. Let $\alpha_0 \in (0, \omega)$ and consider the convex function defined by

$$\phi(t) = \frac{e^{\alpha_0 t^2} - 1}{H_{\alpha_0}}, \quad t \in \mathbb{R},$$

where $H_{\alpha_0} > 0$ is defined as in Theorem 1.1. We introduce the Orlicz norm induced by $\phi$ by setting

$$\|u\|_\phi := \inf\left\{\gamma > 0 : \int_{\mathbb{R}} \phi\left(\frac{u}{\gamma}\right) dx \leq 1\right\},$$

and the corresponding Orlicz space $L_{\phi^*}(0,1)$, see the monograph by Krasnosel'skiĭ and Rutickiĭ [23, Chapter II, in particular pp. 78–81] for properties of this space. We claim that $\|v\|_\phi \leq \|v\|$, for all $v \in H^{1/2}(\mathbb{R})$. Let $v \in H^{1/2}(\mathbb{R}) \setminus \{0\}$ and set $w = \|v\|^{-1} v$, so that by formula (2.1) we conclude

(2.3)
$$\|(-\Delta)^{1/4} w\|_{L^2} = \frac{1}{(2\pi)^{1/2}\|v\|}\left(\int_{\mathbb{R}^2} \frac{(v(x) - v(y))^2}{|x-y|^2}\, dxdy\right)^{1/2} \leq (2\pi)^{-1/2} < 1.$$

Therefore, in light of Theorem 1.1, we have

$$\int_{\mathbb{R}} \phi\left(\frac{v}{\|v\|}\right) dx = \int_{\mathbb{R}} \frac{e^{\alpha_0 w^2} - 1}{H_{\alpha_0}}\, dx \leq \|w\|_{L^2}^2 \leq 1,$$

which proves the claim by the very definition of $\|\cdot\|_\phi$. Fix now an arbitrary function $u \in H^{1/2}(\mathbb{R})$. Hence, there exists a sequence $(\psi_n)$ in $C_c^\infty(\mathbb{R})$ such that $\psi_n \to u$ in $H^{1/2}(\mathbb{R})$, as $n \to \infty$. By the claim this yields $\|\psi_n - u\|_\phi \to 0$, as $n \to \infty$.

Fix now $n = n_0$ sufficiently large that $\|\psi_{n_0} - u\|_\phi < 1/2$. Then we have, in light of [23, Theorem 9.15, p. 79], that

$$\int_{\mathbb{R}} \phi(2u - 2\psi_{n_0})\, dx \leq \|2u - 2\psi_{n_0}\|_\phi < 1.$$

Finally, writing $u = (2u - 2\psi_{n_0})/2 + (2\psi_{n_0})/2$, and since

$$\int_{\mathbb{R}} \phi(2\psi_{n_0})\, dx = \frac{1}{H_{\alpha_0}} \int_{\mathbb{R}} (e^{4\alpha_0 \psi_{n_0}^2} - 1)\, dx = \frac{1}{H_{\alpha_0}} \int_{\mathrm{supt}(\psi_{n_0})} (e^{4\alpha_0 \psi_{n_0}^2} - 1)\, dx < \infty,$$

the convexity of $\phi$ yields $\int_{\mathbb{R}} \phi(u)\, dx < \infty$. Hence, the assertion follows by the arbitrariness of $u$. A different proof can be given writing (in the above notations)

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, dx = \int_{\mathbb{R}} (e^{\alpha \psi_n^2} - 1)\, dx + \int_{\mathbb{R}} (e^{\alpha u^2} - e^{\alpha \psi_n^2})\, dx,$$

estimating the right-hand side by

$$|e^{\alpha u^2} - e^{\alpha \psi_n^2}| \le 2\alpha(|\psi_n - u| + |\psi_n|)e^{2\alpha|\psi_n - u|^2} e^{2\alpha|\psi_n|^2}|\psi_n - u|,$$

using the Hölder inequality, the smallness of $\|\psi_n - u\|$ and Theorem 1.1 to conclude, for $n$ large enough. $\square$

Define the functional $J\colon H^{1/2}(\mathbb{R}) \to \mathbb{R}$ associated with problem (1.1), given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}} (|(-\Delta)^{1/4}u|^2 + u^2)\,dx - \int_{\mathbb{R}} F(u)\,dx.$$

Under our assumptions on $f$, by Proposition 2.1, we can easily see that $J$ is well defined. Also, it is standard to prove that $J$ is a $C^1$-functional and

$$J'(u)v = \int_{\mathbb{R}} (-\Delta)^{1/4}u(-\Delta)^{1/4}v\,dx + \int_{\mathbb{R}} uv\,dx - \int_{\mathbb{R}} f(u)v\,dx,$$

for all $u, v \in H^{1/2}(\mathbb{R})$. Thus, the critical points of $J$ are precisely the solutions of (1.1), namely $u \in H^{1/2}(\mathbb{R})$ with

$$\int_{\mathbb{R}} (-\Delta)^{1/4}u(-\Delta)^{1/4}v\,dx + \int_{\mathbb{R}} uv\,dx = \int_{\mathbb{R}} f(u)v\,dx, \quad \text{for all } v \in H^{1/2}(\mathbb{R}),$$

is a (weak) solution to (1.1).

Lemma 2.2. *Let $u \in H^{1/2}(\mathbb{R})$ and $\rho_0 > 0$ be such that $\|u\| \le \rho_0$. Then*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,dx \le \Lambda(\alpha, \rho_0), \quad \textit{for every } 0 < \alpha\rho_0^2 < \omega.$$

Proof. Let $0 < \alpha\rho_0^2 < \omega$. Then, by Theorem 1.1, we have

$$\int_{\mathbb{R}} \big(e^{\alpha u^2} - 1\big)\,dx \le \int_{\mathbb{R}} \big(e^{\alpha\rho_0^2(u/\|u\|)^2} - 1\big)\,dx \le H_{\alpha\rho_0^2}\frac{\|u\|_{L^2}^2}{\|u\|^2} \le H_{\alpha\rho_0^2} := \Lambda(\alpha, \rho_0),$$

since $\|(-\Delta)^{1/4}u\|u\|^{-1}\|_{L^2}^2 < 1$, see inequality (2.3). $\square$

Remark 2.3. From (f1)–(f2) and (AR) we see that, for $s \in \mathbb{R} \setminus \{0\}$,

$$s^2 f'(s) - sf(s) > 0, \tag{2.4}$$

$$f'(s) > 0,$$

$$\mathcal{H}(s) := sf(s) - 2F(s) > 0, \tag{2.5}$$

$$\mathcal{H} \text{ is even, and increasing on } \mathbb{R}^+,$$

$$\mathcal{H}(s) > \mathcal{H}(\lambda s), \quad \text{for all } \lambda \in (0,1). \tag{2.6}$$

Suppose that $u \neq 0$ is a critical point of $J$, that is, $J'(u) = 0$, then necessarily $u$ belongs to $\mathcal{N} := \{u \in H^{1/2}(\mathbb{R})\setminus\{0\} : J'(u)u = 0\}$. So $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of $J$.

Lemma 2.4. *Under assumptions* (f1)–(f3) *and* (AR), *$\mathcal{N}$ satisfies the following properties*:

(a) *$\mathcal{N}$ is a manifold and $\mathcal{N} \neq \emptyset$.*
(b) *For $u \in H^{1/2}(\mathbb{R}) \setminus \{0\}$ with $J'(u)u < 0$, there is a unique $\lambda(u) \in (0,1)$ with $\lambda u \in \mathcal{N}$.*
(c) *There exists $\rho > 0$ such that $\|u\| \geq \rho$ for any $u \in \mathcal{N}$.*
(d) *If $u \in \mathcal{N}$ is a constrained critical point of $J|_{\mathcal{N}}$, then $J'(u) = 0$ and $u$ solves* (1.1).
(e) $m = \inf\limits_{u \in \mathcal{N}} J(u) > 0$.

PROOF. Consider the $C^1$-functional $\Phi\colon H^{1/2}(\mathbb{R}) \setminus \{0\} \to \mathbb{R}$ defined by

$$\Phi(u) = J'(u)u = \|u\|^2 - \int_{\mathbb{R}} f(u)u\,dx.$$

Note that $\mathcal{N} = \Phi^{-1}(0)$ and $\Phi'(u)u < 0$, if $u \in \mathcal{N}$. Indeed, if $u \in \mathcal{N}$, then

$$\Phi'(u)u = \int_{\mathbb{R}} (f(u)u - f'(u)u^2)\,dx < 0,$$

where we have used (2.4). Then $c = 0$ is a regular value of $\Phi$ and consequently $\mathcal{N} = \Phi^{-1}(0)$ is a $C^1$-manifold, proving (a).

Now we prove $\mathcal{N} \neq \emptyset$ and that (b) holds. Fix $u \in H^{1/2}(\mathbb{R}) \setminus \{0\}$ and consider the function $\Psi\colon \mathbb{R}^+ \to \mathbb{R}$,

$$\Psi(t) = \frac{t^2}{2}\|u\|^2 - \int_{\mathbb{R}} F(tu)\,dx.$$

Then $\Psi'(t) = 0$ if and only if $tu \in \mathcal{N}$, in which case it holds

$$\|u\|^2 = \int_{\mathbb{R}} \frac{f(tu)}{t}\,u\,dx. \tag{2.7}$$

In light of (2.4), the function on the right-hand side of (2.7) is increasing. Whence, it follows that a critical point of $\Psi$, if exists, is *unique*. Now, there exist $\delta > 0$ and $R > 0$ such that

$$\Psi(t) > 0 \quad \text{if } t \in (0,\delta) \quad \text{and} \quad \Psi(t) < 0 \quad \text{if } t \in (R,\infty).$$

In fact, by virtue of (f3), there exist $C, C' > 0$ such that

$$\Psi(t) = \frac{t^2}{2}\|u\|^2 - \int_{\mathbb{R}} F(tu)\,dx \leq Ct^2 - C't^q < 0,$$

provided that $t > 0$ is chosen large enough. Using (f1) and the fact that $f$ has $\alpha_0$-Trudinger–Moser critical growth at $+\infty$, for some $\alpha \in (\alpha_0, \omega)$ and for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$F(s) \leq \varepsilon[s^2 + s^4(e^{\alpha s^2} - 1)] + C_\varepsilon s^4, \quad s \in \mathbb{R}.$$

Then, for any $u \in H^{1/2}(\mathbb{R}) \setminus \{0\}$,

$$\Psi(t) \geq \frac{t^2}{2}\|u\|^2 - \varepsilon t^2\|u\|_{L^2}^2 - C_\varepsilon t^4\|u\|_{L^4}^4 - \varepsilon t^4 \int_{\mathbb{R}} u^4(e^{\alpha(tu)^2} - 1)\,dx.$$

For $0 < t < \tau < (\omega/(2\alpha\|u\|^2))^{1/2}$, by Lemma 2.2 and (2.2), there is $C = C(\|u\|,\alpha) > 0$ such that

$$\int_{\mathbb{R}} u^4(e^{\alpha(tu)^2}-1) \le \|u\|_{L^8}^4\bigg(\int_{\mathbb{R}} e^{2\alpha\tau^2u^2}-1\bigg)^{1/2} \le C.$$

Then, for some $B, B' > 0$, we have

$$\Psi(t) \ge Bt^2 - B't^4 > 0, \quad \text{for } t > 0 \text{ small enough.}$$

Thus, we conclude that there exists a unique maximum $t_0 = t_0(u) > 0$ such that $t_0u \in \mathcal{N}$, and consequently $\mathcal{N}$ is a nonempty set. Given $u \in H^{1/2}(\mathbb{R}) \setminus \{0\}$ with $J'(u)u < 0$, we have

$$\Psi'(1) = \|u\|^2 - \int_{\mathbb{R}} f(u)u\,dx = J'(u)u < 0,$$

which implies $t_0 < 1$.

Let us prove (c). Let $\alpha \in (\alpha_0,\omega)$ and $\rho_0 > 0$ with $\alpha\rho_0^2 < \omega$. By the growth conditions on $f$, there exists $r > 1$ so close to 1 that $r\alpha\rho_0^2 < \omega$, $\ell > 2$ and $C > 0$ with

$$f(s)s \le \frac{1}{4}s^2 + C(e^{r\alpha s^2}-1)^{1/r}|s|^\ell, \quad \text{for all } s \in \mathbb{R}.$$

Let now $u \in \mathcal{N}$ with $\|u\| \le \rho \le \rho_0$. Then, by Lemma 2.2 and (2.2), we have for $u \in \mathcal{N}$

$$\begin{aligned}0 = \Phi(u) &\ge \|u\|^2 - \frac{1}{4}\|u\|_{L^2}^2 - C\int_{\mathbb{R}} \big(e^{r\alpha u^2}-1\big)^{1/r}|u|^\ell\,dx \\ &\ge \frac{3}{4}\|u\|^2 - C\bigg(\int_{\mathbb{R}}\big(e^{r\alpha u^2}-1\big)\,dx\bigg)^{1/r}\bigg(\int_{\mathbb{R}}|u|^{r'\ell}\,dx\bigg)^{1/r'} \\ &\ge \frac{3}{4}\|u\|^2 - C\|u\|^\ell,\end{aligned} \tag{2.8}$$

which yields $0 < \widehat{\rho} := (3/(4C))^{1/(\ell-2)} \le \|u\| \le \rho$, a contradiction if $\rho < \min\{\widehat{\rho},\rho_0\}$. Then $u \in \mathcal{N}$ implies $\|u\| \ge \min\{\widehat{\rho},\rho_0\}$, proving (c).

Concerning (d), if $u \in \mathcal{N}$ is a minimizer, then $J'(u) = \lambda\Phi'(u)$ for some $\lambda \in \mathbb{R}$. Testing with $u$ and recalling the previous conclusions yields $\lambda = 0$, hence the assertion.

Finally, assertion (e) follows by condition (AR) and (c), since $u \in \mathcal{N}$ implies $J(u) \ge (1/2 - 1/\vartheta)\|u\|^2 \ge (1/2-1/\vartheta)\rho^2 > 0$. □

Lemma 2.5. *Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence for $J$ on $\mathcal{N}$, that is,*

$$J'(u_n)u_n = 0 \quad \textit{and} \quad J(u_n) \to m := \inf_{u\in\mathcal{N}} J(u) \quad \textit{as } n \to \infty, \tag{2.9}$$

*then the following facts hold*:

(a) *$(u_n)$ is bounded in $H^{1/2}(\mathbb{R})$. Thus, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^{1/2}(\mathbb{R})$.*
(b) *$\limsup_n \|u_n\| < \rho_0$, for some $\rho_0 > 0$ sufficiently small.*

(c) $(u_n)$ *does not converge strongly to zero in* $L^\sigma(\mathbb{R})$, *for some* $\sigma > 2$.

Proof. Let $(u_n) \subset H^{1/2}(\mathbb{R})$ satisfy (2.9). Using (AR) condition, we have for $\vartheta > 2$,

$$m + o(1) = J(u_n) \geq \frac{\|u_n\|^2}{2} - \frac{1}{\vartheta}\int_{\mathbb{R}} f(u_n)u_n\,dx = \left(\frac{1}{2} - \frac{1}{\vartheta}\right)\|u_n\|^2, \tag{2.10}$$

which implies (a).

To prove (b) we use assumption (f3) and the fact that, by (2.2),

$$\mathcal{S}_q := \inf_{v \in H^{1/2}(\mathbb{R})\setminus\{0\}} \mathcal{S}_q(v) > 0, \qquad \mathcal{S}_q(v) = \frac{\|v\|}{\|v\|_{L^q}}. \tag{2.11}$$

Let $(u_n) \subset \mathcal{N}$ and $u \in \mathcal{N}$ satisfy (2.9). Then inequality (2.10) yields

$$\limsup_n \|u_n\|^2 \leq \frac{2\vartheta}{\vartheta - 2}\, m. \tag{2.12}$$

Notice that, for every $v \in H^{1/2}(\mathbb{R}) \setminus \{0\}$, arguing as for the proof of (b) of Lemma 2.4, one finds $t_0 > 0$ such that $t_0 v \in \mathcal{N}$. Hence $m \leq J(t_0 v) \leq \max_{t \geq 0} J(tv)$.

Now, using assumption (f3) and formula (2.11), for every $\psi \in H^{1/2}(\mathbb{R}) \setminus \{0\}$, we can estimate

$$\begin{aligned} m &\leq \max_{t\geq 0} J(t\psi) \leq \max_{t\geq 0}\left(\frac{t^2}{2}\|\psi\|^2 - C_q t^q \|\psi\|^q_{L^q}\right) \\ &\leq \max_{t\geq 0}\left(\frac{\mathcal{S}_q(\psi)^2}{2} t^2 \|\psi\|^2_{L^q} - C_q t^q \|\psi\|^q_{L^q}\right) = \left(\frac{1}{2} - \frac{1}{q}\right)\frac{\mathcal{S}_q(\psi)^{2q/(q-2)}}{(qC_q)^{2/(q-2)}}, \end{aligned}$$

which together with (2.12) implies that

$$\limsup_n \|u_n\|^2 \leq \frac{2\vartheta}{\vartheta - 2}\left(\frac{1}{2} - \frac{1}{q}\right)\frac{\mathcal{S}_q(\psi)^{2q/(q-2)}}{(qC_q)^{2/(q-2)}}.$$

Taking the infimum over $\psi \in H^{1/2}(\mathbb{R}) \setminus \{0\}$, we get

$$\limsup_n \|u_n\|^2 \leq \frac{\vartheta}{\vartheta - 2}\,\frac{q-2}{q}\,\frac{\mathcal{S}_q^{2q/(q-2)}}{(qC_q)^{2/(q-2)}} < \rho_0^2,$$

provided $C_q$ is large enough, proving (b).

Let us prove (c). By Lemma 2.4 (c), we have

$$\|u_n\|^2 = \int_{\mathbb{R}} f(u_n)u_n\,dx \geq \rho^2 > 0.$$

In view of assertion (b) the norm $\|u_n\|$ is small (precisely, we can assume that $r\alpha\|u_n\|^2 < r\alpha\rho_0^2 < \omega$ for $r$ very close to 1). Arguing as in the proof of (2.8), we can find $\varepsilon \in (0,1)$ and $C > 0$ such that

$$\begin{aligned} \|u_n\|^2 &= \int_{\mathbb{R}} f(u_n)u_n\,dx \leq \varepsilon\|u_n\|^2_{L^2} + C\int_{\mathbb{R}} \left(e^{r\alpha u_n^2} - 1\right)^{1/r}|u_n|^\ell\,dx \\ &\leq \varepsilon\|u_n\|^2 + C\left(\int_{\mathbb{R}}\left(e^{r\alpha u_n^2} - 1\right)dx\right)^{1/r}\|u_n\|^\ell_{L^{r'\ell}} \leq \varepsilon\|u_n\|^2 + C\|u_n\|^\ell_{L^{r'\ell}}, \end{aligned}$$

which implies $0 < (1-\varepsilon)\rho^2 \leq (1-\varepsilon)\|u_n\|^2 \leq C\|u_n\|^{\ell}_{L^{r'\ell}}$, and, consequently, $(u_n)$ cannot vanish in $L^{r'\ell}(\mathbb{R})$, as $n \to \infty$. □

Next, we formulate a Brezis–Lieb type lemma in our framework.

Lemma 2.6. *Let $(u_n) \subset H^{1/2}(\mathbb{R})$ be a sequence such that $u_n \rightharpoonup u$ weakly in $H^{1/2}(\mathbb{R})$ and $\|u_n\| < \rho_0$ with $\rho_0 > 0$ small. Then, as $n \to \infty$, we have*

$$\int_{\mathbb{R}} f(u_n)u_n\,dx = \int_{\mathbb{R}} f(u_n-u)(u_n-u)\,dx + \int_{\mathbb{R}} f(u)u\,dx + o(1),$$
$$\int_{\mathbb{R}} F(u_n)\,dx = \int_{\mathbb{R}} F(u_n-u)\,dx + \int_{\mathbb{R}} F(u)\,dx + o(1).$$

Proof. We shall apply [8, Lemma 3 and Theorem 2]. Since $f$ is convex on $\mathbb{R}^+$ and by the properties collected in Remark 2.3, we have that the functions $F(s)$ and $G(s) := f(s)s$ are convex on $\mathbb{R}$ with $F(0) = G(0) = 0$. We let $\alpha \in (\alpha_0, \omega)$ and $\rho_0 \in (0, 1/2)$ with $\alpha\rho_0^2 < \omega$. Thus, by Lemma 2.2, we have

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} \big(e^{\alpha u_n^2} - 1\big)dx < \infty. \tag{2.13}$$

Choose $k \in (1, (1-\rho_0)/\rho_0)$ and let $\varepsilon > 0$ with $\varepsilon < 1/k$. Then, in light of [8, Lemma 3], the functions

$$\phi_\varepsilon(s) := j(ks) - kj(s) \geq 0 \quad \psi_\varepsilon(s) := |j(C_\varepsilon s)| + |j(-C_\varepsilon s)|, \quad C_\varepsilon = \frac{1}{\varepsilon(k-1)},$$

satisfy the inequality $|j(a+b) - j(a)| \leq \varepsilon\phi_\varepsilon(a) + \psi_\varepsilon(b)$, for all $a, b \in \mathbb{R}$, and, if $v_n := u_n - u$ and $u_n$ satisfies (2.13), we claim that

(i) $v_n \to 0$ almost everywhere;
(ii) $j(u) \in L^1(\mathbb{R})$;
(iii) $\int_{\mathbb{R}} \phi_\varepsilon(v_n)\,dx \leq C$ for some constant $C$ independent of $n \geq 1$;
(iv) $\int_{\mathbb{R}} \psi_\varepsilon(u)\,dx < \infty$, for all $\varepsilon > 0$ small.

Under this claim, then, by [8, Theorem 2], it holds

$$\lim_n \int_{\mathbb{R}} |j(u_n) - j(v_n) - j(u)|\,dx = 0, \tag{2.14}$$

with $j = F$ and with $j = G$. Next we are going to prove the claim. Item (i) follows by the weak convergence of $(u_n)$. To prove (ii) it is enough to use Proposition 2.1 (see the growth conditions below). To check (iii) for $j = F$ and $j = G$, we find $\alpha \in (\alpha_0, \omega)$, $D > 0$ and $q > 2$ such that

$$F(s) \leq \big(s^2 + e^{\alpha s^2} - 1\big) + D|s|^q, \quad \text{for all } s \in \mathbb{R}, \tag{2.15}$$

$$G(s) \leq \big(s^2 + e^{\alpha s^2} - 1\big) + D|s|^q, \quad \text{for all } s \in \mathbb{R}, \tag{2.16}$$

$$|f(s)| \leq \big(s + e^{\alpha s^2} - 1\big) + D|s|^{q-1}, \quad \text{for all } s \in \mathbb{R}, \tag{2.17}$$

$$|f'(s)s| \leq \big(s + e^{\alpha s^2} - 1\big) + D|s|^{q-1}, \quad \text{for all } s \in \mathbb{R}. \tag{2.18}$$

We claim that $\phi_\varepsilon(v_n)$ verifies (iii). First let us consider the case $j = F$, that is, $\phi_\varepsilon(v_n) = F(kv_n) - kF(v_n)$. In fact, by the Mean Value Theorem, there exists $\vartheta \in (0,1)$ with $w_n = v_n(k(1-\vartheta)+\vartheta)$ such that

$$\begin{aligned}\phi_\varepsilon(v_n) &= F(kv_n) - F(v_n) + F(v_n) - kF(v_n)\\ &= f(w_n)v_n(k-1) + (1-k)F(v_n) \leq f(w_n)v_n(k-1),\end{aligned}$$

since $k > 1$ and $F \geq 0$. Analogously, for $j = G$, we have

$$\begin{aligned}\phi_\varepsilon(v_n) &= G(kv_n) - G(v_n) + G(v_n) - kG(v_n)\\ &= f'(w_n)w_nv_n(k-1) + f(w_n)v_n(k-1) + (1-k)f(v_n)v_n\\ &\leq f'(w_n)w_nv_n(k-1) + f(w_n)v_n(k-1),\end{aligned}$$

since $k > 1$ and $f(s)s \geq 0$ for all $s \in \mathbb{R}$. Thus, to prove (iii) for $F$ and $G$, it is sufficient to see that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} f(w_n)v_n\,dx < \infty, \qquad \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} f'(w_n)w_nv_n\,dx < \infty. \tag{2.19}$$

We know that $\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1)$, as $n \to \infty$, so that $\limsup_n \|v_n\| \leq \rho_0$. In turn, by the choice of $k$, we also have

$$\limsup_n \|w_n\| = \|v_n\|(k(1-\vartheta)+\vartheta) \leq \rho_0(k(1-\vartheta)+\vartheta) \leq \rho_0(k+1) < 1.$$

Since $\alpha_0 < \alpha < \omega$, we can find $m > 1$ very close to 1 such that $m\alpha < \omega$. Then, by (2.17), we get

$$\begin{aligned}&\int_{\mathbb{R}} f(w_n)v_n\,dx\\ &\leq \int_{\mathbb{R}} |w_n||v_n|\,dx + \int_{\mathbb{R}} (e^{\alpha w_n^2} - 1)|v_n|\,dx + D\int_{\mathbb{R}} |w_n|^{q-1}|v_n|\,dx\\ &\leq \|w_n\|_{L^2}\|v_n\|_{L^2} + D\|w_n\|_{L^q}^{q-1}\|v_n\|_{L^q} + \left(\int_{\mathbb{R}} (e^{m\alpha w_n^2} - 1)\,dx\right)^{1/m} \|v_n\|_{L^{m'}}\\ &\leq C\|w_n\|\|v_n\| + C\|w_n\|^{q-1}\|v_n\| + C\left(\int_{\mathbb{R}} (e^{m\alpha w_n^2} - 1)\,dx\right)^{1/m} \|v_n\|\\ &\leq C + C\left(\int_{\mathbb{R}} (e^{m\alpha w_n^2} - 1)\,dx\right)^{1/m} \leq C.\end{aligned}$$

The last integral is bounded via Lemma 2.2, since $\|w_n\| \leq 1$ and $m\alpha < \omega$. The second term in (2.19) can be treated in a similar fashion, using the growth condition (2.18) in place of (2.17). We claim that $\psi_\varepsilon$ verifies (iv) for both $F$ and $G$. It suffices to prove

$$\int_{\mathbb{R}} F(C_\varepsilon u)\,dx < \infty, \quad \text{for all } \varepsilon > 0.$$

By (2.15) this occurs since by Proposition 2.1, we have

$$\int_{\mathbb{R}} \big(e^{\alpha C_\varepsilon^2 u^2} - 1\big)\, dx < \infty.$$

Analogous proof holds for $G$ via (2.16). We can finally apply [8, Theorem 2] yielding (2.14). Thus

$$\int_{\mathbb{R}} j(u_n)\, dx = \int_{\mathbb{R}} j(v_n)\, dx + \int_{\mathbb{R}} j(u)\, dx + o(1),$$

for $j = F$ and $j = G$. □

The previous Lemma 2.6 yields the following useful technical results.

Lemma 2.7. *Let $(u_n) \subset H^{1/2}(\mathbb{R})$ be as in Lemma 2.5 then, for $v_n = u_n - u$, we have*

$$J'(u)\, u + \liminf_n J'(v_n)\, v_n = 0,$$

*so that either $J'(u)u \le 0$ or $\liminf_n J'(v_n)v_n < 0$.*

Proof. Recalling that $v_n = u_n - u$, we get $\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1)$. Then, by Lemma 2.6,

$$\int_{\mathbb{R}} f(u_n)u_n\, dx = \int_{\mathbb{R}} f(v_n)v_n\, dx + \int_{\mathbb{R}} f(u)u\, dx + o(1).$$

Since $u_n \in \mathcal{N}$, by using the above equality, the assertion follows. □

Lemma 2.8. *Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence for $J$ on $\mathcal{N}$, such that $u_n \rightharpoonup u$ weakly in $H^{1/2}(\mathbb{R})$ as $n \to \infty$. If $u \in \mathcal{N}$, then $J(u) = m$.*

Proof. Let $(u_n) \subset \mathcal{N}$ and $u \in \mathcal{N}$ be as above, thus

$$m + o(1) = J(u_n) - \frac{1}{2}J'(u_n)u_n = \frac{1}{2}\int_{\mathbb{R}} \mathcal{H}(u_n)\, dx$$

which together with Fatou's lemma (recall that (2.5) holds) implies

$$m = \frac{1}{2}\liminf_n \int_{\mathbb{R}} \mathcal{H}(u_n)\, dx \ge \frac{1}{2}\liminf_n \int_{\mathbb{R}} \mathcal{H}(u)\, dx = J(u) - \frac{1}{2}J'(u)u = J(u),$$

which yields the conclusion. □

## 3. Proof of Theorem 1.2 concluded

Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence for $J$ on $\mathcal{N}$. From Lemma 2.5 (a), $(u_n)$ is bounded in $H^{1/2}(\mathbb{R})$. Thus, up to a subsequence, we have $u_n \rightharpoonup u$ weakly in $H^{1/2}(\mathbb{R})$.

Assertion 3.1. There exist a sequence $(y_n) \subset \mathbb{R}$ and constants $\gamma, R > 0$ such that

$$\liminf_n \int_{y_n - R}^{y_n + R} |u_n|^2\, dx \ge \gamma > 0.$$

If not, for any $R > 0$,

$$\liminf_n \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |u_n|^2 \, dx = 0.$$

Using a standard concentration-compactness principle due to P.L. Lions (it is easy to see that the argument remains valid for the case studied here) we can conclude that $u_n \to 0$ in $L^q(\mathbb{R})$ for *any* $q > 2$, which is a contradiction with Lemma 2.5 (c).

Define $\overline{u}_n(x) = u_n(x + y_n)$. Then $J(u_n) = J(\bar{u}_n)$ and without of loss generality we can assume $y_n = 0$ for any $n$. Notice that $(\overline{u}_n)$ is also a minimizing sequence for $J$ on $\mathcal{N}$, which it is bounded and satisfies

$$\liminf_n \int_{-R}^{R} |\bar{u}_n|^2 \, dx \geq \gamma, \quad \text{for some } \gamma > 0,$$

and $\overline{u}_n \rightharpoonup \overline{u}$ weakly in $H^{1/2}(\mathbb{R})$, then $\overline{u} \neq 0$ ($u \neq 0$).

ASSERTION 3.2. $J'(u)u = 0$.

If Assertion 3.2 holds, then combining (d) of Lemmas 2.4 and 2.8 we have the result.

We shall now prove Assertion 3.2. Suppose by contradiction that $J'(u)u \neq 0$.

If $J'(u)u < 0$, by Lemma 2.4 (b), there exists $0 < \lambda < 1$ such that $\lambda u \in \mathcal{N}$. Thus

$$\lambda \|u\|^2 = \int_{\mathbb{R}} f(\lambda u) u \, dx.$$

Using (2.5) in combination with Fatou's lemma, we obtain

$$\begin{aligned} m = \liminf_n \frac{1}{2} \int_{\mathbb{R}} \mathcal{H}(u_n) \, dx &\geq \frac{1}{2} \int_{\mathbb{R}} \mathcal{H}(u) \, dx \\ &> \frac{1}{2} \int_{\mathbb{R}} \mathcal{H}(\lambda u) \, dx = J(\lambda u) - \frac{1}{2} J'(\lambda u) \lambda u = J(\lambda u), \end{aligned}$$

which implies $J(\lambda u) < m$ and, hence, a contradiction. Here we have used (2.6).

If $J'(u)u > 0$, by Lemma 2.7, we get $\liminf_n J'(v_n)v_n < 0$. Taking a subsequence, we have $J'(v_n)v_n < 0$, for $n$ large enough. By Lemma 2.4 (b), there exists $\lambda_n \in (0, 1)$ such that $\lambda_n v_n \in \mathcal{N}$.

ASSERTION 3.3. $\limsup_n \lambda_n < 1$.

If $\limsup_n \lambda_n = 1$, up to a sub-sequence, we can assume that $\lambda_n \to 1$, then

$$J'(v_n)v_n = J'(\lambda_n v_n)\lambda_n v_n + o(1).$$

This follows provided that

$$\int_{\mathbb{R}} f(v_n)v_n \, dx = \int_{\mathbb{R}} f(\lambda_n v_n)\lambda_n v_n \, dx + o(1). \tag{3.1}$$

In fact, notice that if $\eta_n := v_n + \tau v_n(\lambda_n - 1)$ for some $\tau \in (0,1)$, it follows

$$f(v_n)v_n - f(\lambda_n v_n)\lambda_n v_n = (f'(\eta_n)\eta_n + f(\eta_n))v_n(1-\lambda_n).$$

Since $\|\eta_n\| = \|v_n + \tau v_n(\lambda_n - 1)\| \le \lambda_n\|v_n\| \le \rho_0$, it follows by arguing as for the justification of formula (2.19), that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} |f'(\eta_n)\eta_n + f(\eta_n)||v_n|\,dx < \infty,$$

so that (3.1) follows, since $\lambda_n \to 1$. Since $\lambda_n v_n \in \mathcal{N}$ we have $J'(\lambda_n v_n)\lambda_n v_n = 0$ what implies that $J'(v_n)v_n = o(1)$, which is a contradiction with $\lim_n J'(v_n)v_n < 0$. Thus, up to subsequence, we may assume that $\lambda_n \to \lambda_0 \in (0,1)$. Arguing as before, from (2.6) we infer

$$m + o(1) = \frac{1}{2}\int_{\mathbb{R}} \mathcal{H}(u_n)\,dx \ge \frac{1}{2}\int_{\mathbb{R}} \mathcal{H}(\lambda_n u_n)\,dx,$$

since $\mathcal{H}(u_n) \ge \mathcal{H}(\lambda_n u_n)$. By means of Lemma 2.6 applied to $w_n = \lambda_n u_n$ (whose norm is small, being smaller than the norm of $u_n$) and $w = \lambda_0 u$, we have in turn

$$\int_{\mathbb{R}} \mathcal{H}(\lambda_n u_n)\,dx = \int_{\mathbb{R}} \mathcal{H}(\lambda_n u_n - \lambda_0 u)\,dx + \int_{\mathbb{R}} \mathcal{H}(\lambda_0 u)\,dx + o(1).$$

Furthermore, we have

$$\int_{\mathbb{R}} \mathcal{H}(\lambda_n u_n - \lambda_0 u)\,dx = \int_{\mathbb{R}} \mathcal{H}(\lambda_n v_n)\,dx + o(1). \tag{3.2}$$

In fact, notice that $\lambda_n u_n - \lambda_0 u = \lambda_n v_n + \gamma_n u$, where $\gamma_n := \lambda_n - \lambda_0 \to 0$ as $n \to \infty$. We have

$$\mathcal{H}(\lambda_n u_n - \lambda_0 u) - \mathcal{H}(\lambda_n v_n) = \mathcal{H}'(\widehat{\eta}_n)u\gamma_n, \qquad \widehat{\eta}_n := \tau u\gamma_n + \lambda_n v_n$$

for $\tau \in (0,1)$ and $\|\widehat{\eta}_n\| = \|\tau u\gamma_n + \lambda_n v_n\| \le \gamma_n\|u\| + \lambda_n\|v_n\| \le \rho_0$ for $n$ large enough. Then, arguing as for the justification of (2.19), we get

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} |\mathcal{H}'(\widehat{\eta}_n)||u|\,dx \le \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} |f'(\widehat{\eta}_n)\widehat{\eta}_n + f(\widehat{\eta}_n)||v_n|\,dx < \infty,$$

which yields (3.2) since $\gamma_n \to 0$ as $n \to \infty$. Therefore, we obtain

$$\begin{aligned} m + o(1) &\ge \frac{1}{2}\int_{\mathbb{R}} \mathcal{H}(\lambda_n v_n)\,dx + \frac{1}{2}\int_{\mathbb{R}} \mathcal{H}(\lambda_0 u)\,dx \\ &= J(\lambda_n v_n) - \frac{1}{2}J'(\lambda_n v_n)\lambda v_n + \frac{1}{2}\int_{\mathbb{R}} \mathcal{H}(\lambda_0 u)\,dx \\ &= J(\lambda_n v_n) + \frac{1}{2}\int_{\mathbb{R}} \mathcal{H}(\lambda_0 u)\,dx. \end{aligned}$$

Since $u \ne 0$, we have $\int_{\mathbb{R}} \mathcal{H}(\lambda_0 u)\,dx > 0$. Then $J(\lambda_n v_n) < m$ for large $n$ enough, namely a contradiction. $\square$

## References


[1] Adimurthi and S.L. Yadava, *Multiplicity results for semilinear elliptic equations in a bounded domain of* $\mathbb{R}^2$ *involving critical exponent*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (1990), 481–504.

[2] C. Alves, J. M. do Ó and O. Miyagaki, *On nonlinear perturbations of a periodic elliptic problem in* $\mathbb{R}^2$ *involving critical growth*, Nonlinear Anal. **56** (2004), 781–791.

[3] C. Alves, M. Souto and M. Montenegro, *Existence of a ground state solution for a nonlinear scalar field equation with critical growth*, Calc. Var. Partial Differential Equations **43** (2012), 537–554.

[4] A. Ambrosetti and Z.-Q. Wang, *Positive solutions to a class of quasilinear elliptic equations on* $\mathbb{R}$, Discrete Contin. Dyn. Syst. **9** (2003), 55–68.

[5] B. Barrios, E. Colorado, A. de Pablo and U. Sánchez, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations **252** (2012), 613–6162.

[6] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations, I-existence of a ground state*, Arch. Rat. Mech. Anal. **82** (1983), 313–346.

[7] H. Berestycki, T. Gallouet and O. Kavian, *Equations de Champs scalaires euclidiens non lineaires dans le plan*, C.R. Acad. Sci. Paris Sér. I Math. **297** (1984), 307–310.

[8] H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[9] X. Cabré and J.G. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224** (2010), 2052–2093.

[10] X. Chang and Z.-Q. Wang, *Ground state of scalar field equations involving a fractional laplacian with general nonlinearity*, Nonlinearity **26** (2013), 479–494.

[11] D. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations **17** (1992), 407–435.

[12] D.G. de Figueiredo, J.M. do Ó and B. Ruf, *Elliptic equations and systems with critical Trudinge-r-Moser nonlinearities*, Discrete Contin. Dyn. Syst. **30** (2011), 455–476.

[13] ———, *On an inequality by N. Trudinger and J. Moser and related elliptic equations*, Comm. Pure Appl. Math. **55** (2002), 135–152.

[14] D.G. de Figueiredo, O.H. Miyagaki and B. Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (1995), 139–153.

[15] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[16] J.M. do Ó, E. Medeiros and U. Severo, *A nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. **345** (2008), 286–304.

[17] J.M. do Ó, O.H. Miyagaki and M. Squassina, *Critical and subcritical fractional problems with vanishing potentials*, Comm. Contemporary Math., DOI: 10.1142/S0219199715500637.

[18] ———, *Nonautonomous fractional problems with exponential growth*, Nonlinear Differential Equations Applications, DOI: 10.1007/s00030-015-0327-0.

[19] P. Felmer, A. Quaas and J. Tan, *Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh A **142** (2012), 1237-1262.

[20] R. Frank and E. Lenzmann, *Uniqueness of non-linear ground states for fractional laplacians in* $\mathbb{R}$, Acta Math. **210** (2013), 261–318.

[21] A. Iannizzotto and M. Squassina, 1/2*-Laplacian problems with exponential nonlinearity*, J. Math. Anal. Appl. **414** (2014), 372–385.

[22] T. Jin, Y. Li and J. Xiong, *On a fractional Nirenberg problem*, part I: *blow up analysis and compactness of solutions*, J. Eur. Math. Soc. **16** (2014), 1111–1171.

[23] M.A. Krasnosel'skiĭ and J.B. Rutickiĭ, *ConvexFfunctions and Orlicz Spaces*, Noordhoff, Groningen, Holland, 1961.

[24] N. Lam and G. Lu, *Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti–Rabinowitz condition*, J. Geom. Anal. **24** (2014), 118–143.

[25] E. H. Lieb and M. Loss, *Analysis*, AMS, 2001.

[26] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970), 1077–1092.

[27] T. Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal. **127** (1995), 259–269.

[28] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.

[29] B. Ruf and F. Sani, *Ground states for elliptic equations in $\mathbb{R}^2$ with exponential critical growth*, Geometric properties for parabolic and elliptic PDE's, Springer INdAM 2, Springer, Milan, 2013, 251–267.

[30] X. Shang, J. Zhang and Y. Yang, *On fractional Schödinger equation in $\mathbb{R}^N$ with critical growth,* J. Math. Phys. **54** (2013), 121502-19 pages.

[31] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$*, J. Math. Phys. **54** (2013), 031501-17 pages.

[32] M. Souza and Y. Araúujo, *On nonlinear perturbations of a periodic fractional Schrödinger equation with critical exponential growth*, Math. Nachr., to appear.

[33] W. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.

[34] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[35] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, **24**, Birkhäuser 1996.

*Manuscript received May 1, 2015*

*accepted September 2, 2015*

João Marcos do Ó
Department of Mathematics
Federal University of Paraíba
58051-900, João Pessoa-PB, BRAZIL

*E-mail address*: jmbo@pq.cnpq.br

Olímpio H. Miyagaki
Department of Mathematics
Federal University of Juiz de Fora
36036-330 Juiz de Fora, Minas Gerais, BRAZIL

*E-mail address*: olimpio@ufv.br

Marco Squassina
Dipartimento di Informatican
Università degli Studi di Verona
Cá Vignal 2, Strada Le Grazie 15
I-37134 Verona, ITALY

*E-mail address*: marco.squassina@univr.it