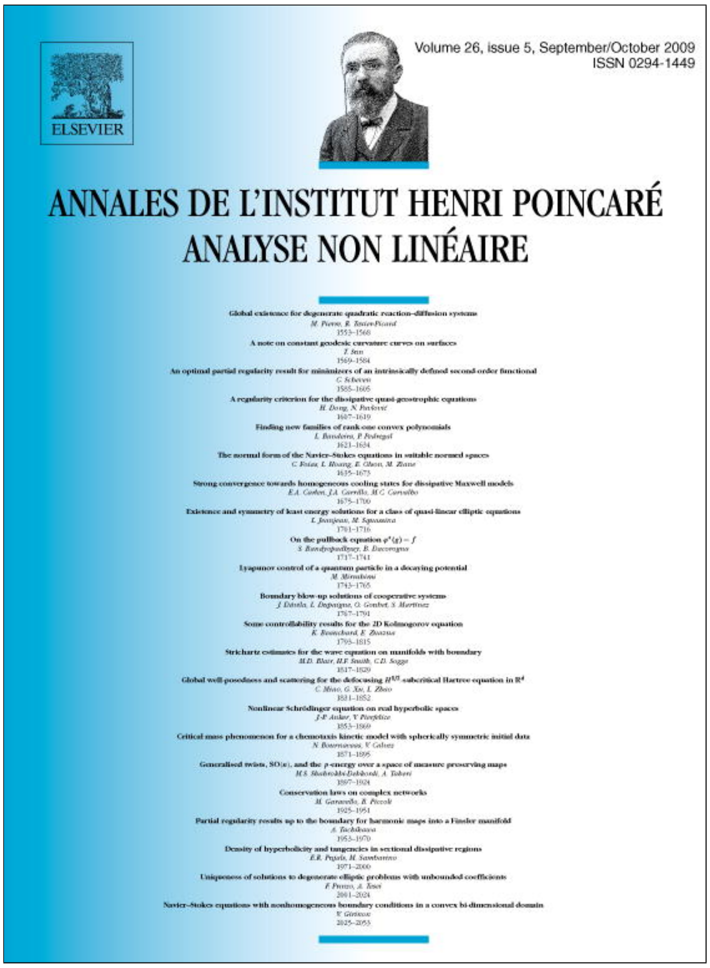
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Existence and symmetry of least energy solutions for a class of quasi-linear elliptic equations

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Abstract

For a general class of autonomous quasi-linear elliptic equations on $\mathbb{R}^n$ we prove the existence of a least energy solution and show that all least energy solutions do not change sign and are radially symmetric up to a translation in $\mathbb{R}^n$.

Résumé

Pour une large classe d'équations quasilinéaires elliptiques autonomes sur $\mathbb{R}^N$, on montre l'existence d'une solution de moindre énergie. On montre aussi que toutes les solutions de moindres énergies ont un signe constant et sont, à une translation près, radiales.

MSC: 35J40; 58E05

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1. Introduction

In this paper we show the existence, radial symmetry and sign of the least energy solutions for a class of quasi-linear elliptic equations,

$$-\operatorname{div}\bigl(j_\xi(u, Du)\bigr) + j_s(u, Du) = f(u) \quad \text{in } \mathcal{D}'\bigl(\mathbb{R}^n\bigr), \tag{1.1}$$

where $\{\xi \mapsto j(s,\xi)\}$ is p-homogeneous. We look for solutions of (1.1) in $D^{1,p}(\mathbb{R}^n)$ where $1 < p \leqslant n$. If we set $F(s) = \int_0^s f(t)$, Eq. (1.1) is formally associated with the functional

$$I(u) = \int_{\mathbb{R}^n} j(u, Du) - \int_{\mathbb{R}^n} F(u),$$

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which is non-smooth on $D^{1,p}(\mathbb{R}^n)$, under natural growth assumptions on the integrand $j(s,\xi)$ (see conditions (1.8), (1.9) below), although it admits directional derivatives along the smooth directions. By least energy solution of (1.1) we mean a non-trivial function $u \in D^{1,p}(\mathbb{R}^n)$ such that

$$I(u) = \inf\bigl\{I(v):\ v \in D^{1,p}(\mathbb{R}^n),\ v \neq 0 \text{ is a solution of (1.1)}\bigr\}.$$

Our work is motivated by [8] where *abstract* conditions are given under which problems of type (1.1) admit a least energy solution and all least energy solutions do not change sign and are radially symmetric, up to a translation in $\mathbb{R}^n$. In this paper, under quite general assumptions on $j(s,\xi)$ and $f(s)$, we prove that, in fact, these *abstract* conditions hold. In the special case of the p-Laplace equation $(1 < p \leqslant n)$

$$-\Delta_p u = f(u) \quad \text{in } \mathcal{D}'\bigl(\mathbb{R}^n\bigr), \tag{1.2}$$

there are various achievements regarding the existence of solutions. For $p = 2$, namely for

$$-\Delta u = f(u) \quad \text{in } \mathcal{D}'\bigl(\mathbb{R}^n\bigr),$$

we refer to the classical paper by Berestycki and Lions [3] for the scalar case and to the paper by Brezis and Lieb [7] for both scalar and systems cases. In [3,7] the existence of a least energy solution is obtained. When $p \neq 2$ we refer to the papers [18,20] and the references therein for the existence of solutions. The issue of least energy solutions is not considered in these papers. We also mention [16] where under assumptions on f, allowing to work with regular functionals in $W^{1,p}(\mathbb{R}^n)$, the existence of a least energy solution is derived. For a more general $j(s,\xi)$ the only previous result about existence of least energy solutions is [21, Theorem 3.2] which actually played the rôle of a technical lemma therein. However it requires significant restrictions on f that we completely removed in this paper.

In [3,7] the existence of a least energy solution is obtained by solving a constrained minimization problem under suitable assumptions on F and f. In [7] the authors assume that F is a C^1 function on $\mathbb{R} \setminus \{0\}$, locally Lipschitz around the origin and having suitable sub-criticality controls at the origin and at infinity. In Theorem 1 of [18], the authors extend the existence results of [3] to the p-Laplacian case and need more regularity on the function F (for instance f is taken in $\mathrm{Lip}_{\mathrm{loc}}$). In our general setting, we consider a set of assumptions on F which is close to that of [7] and some natural assumptions on j which are often considered in the current literature of this kind of problems.

1.1. Main result in the case $1 < p < n$

Let $F : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $F(0) = 0$. Denoting by p^* the critical Sobolev exponent we assume that:

$$\limsup_{s \to 0} \frac{F(s)}{|s|^{p^*}} \leqslant 0; \tag{1.3}$$

$$\text{there exists } s_0 \in \mathbb{R} \text{ such that } F(s_0) > 0. \tag{1.4}$$

Moreover, if $f(s) = F'(s)$ for any $s \in \mathbb{R}$,

$$\lim_{s \to \infty} \frac{f(s)}{|s|^{p^*-1}} = 0. \tag{1.5}$$

Finally,

$$\text{if } u \in D^{1,p}\bigl(\mathbb{R}^n\bigr) \text{ and } u \not\equiv 0 \text{ then } f(u) \not\equiv 0. \tag{1.6}$$

Condition (1.6) is satisfied for instance if $f(s) \neq 0$ for $s \neq 0$ and small because if $u \in D^{1,p}(\mathbb{R}^n)$ and $u \not\equiv 0$ the measure of the set $\{x \in \mathbb{R}^n:\ \eta \leqslant |f(u(x))| \leqslant 2\eta\}$ is positive for $\eta > 0$ and small.

Let $j(s,\xi) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 in s and ξ and denote by j_s and j_ξ the derivatives of j with respect of s and ξ respectively. We assume that:

$$\text{for all } s \in \mathbb{R} \text{ the map } \bigl\{\xi \mapsto j(s,\xi)\bigr\} \text{ is strictly convex and } p\text{-homogeneous;} \tag{1.7}$$

there exist positive constants c_1, c_2, c_3, c_4 and R such that

$$c_1|\xi|^p \leqslant j(s,\xi) \leqslant c_2|\xi|^p, \quad \text{for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^n; \tag{1.8}$$

$$\left|j_s(s,\xi)\right| \leqslant c_3|\xi|^p, \quad \left|j_\xi(s,\xi)\right| \leqslant c_4|\xi|^{p-1}, \quad \text{for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^n; \tag{1.9}$$

$$j_s(s,\xi)s \geqslant 0, \quad \text{for all } s \in \mathbb{R} \text{ with } |s| \geqslant R \text{ and } \xi \in \mathbb{R}^n. \tag{1.10}$$

Conditions (1.7)–(1.10) on j are quite natural assumptions and were already used, e.g., in [1,10,11,30,31]. See also Remark 2.18 for further comments regarding the rôle played by condition (1.10).

Theorem 1.1. *Assume that conditions* (1.3)–(1.10) *hold. Then Eq.* (1.1) *admits a least energy solution* $u \in D^{1,p}(\mathbb{R}^n)$*. Furthermore any least energy solution of* (1.1) *has a constant sign and is radially symmetric, up to a translation in* $\mathbb{R}^n$*.*

1.2. Main result in the case $p = n$

Let $F : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $F(0) = 0$. We assume that:

$$\text{there exists } \delta > 0 \text{ such that } F(s) < 0 \text{ for all } 0 < |s| \leqslant \delta; \tag{1.11}$$

$$\text{there exists } s_0 \in \mathbb{R} \text{ such that } F(s_0) > 0; \tag{1.12}$$

$$\text{there exist } q > 1 \text{ and } c > 0 \text{ such that } \left|f(s)\right| \leqslant c + c|s|^{q-1} \text{ for all } s \in \mathbb{R}. \tag{1.13}$$

$$\text{if } u \in D^{1,n}\big(\mathbb{R}^n\big) \text{ and } u \not\equiv 0 \text{ then } f(u) \not\equiv 0. \tag{1.14}$$

Concerning the Lagrangian j we still assume conditions (1.7)–(1.10) (with $p = n$).

Theorem 1.2. *Assume that conditions* (1.7)–(1.14) *hold. Then Eq.* (1.1) *admits a least energy solution* $u \in D^{1,n}(\mathbb{R}^n)$*. Furthermore any least energy solution of* (1.1) *has a constant sign and if a least energy solution* $u \in D^{1,n}(\mathbb{R}^n)$ *satisfies* $u(x) \to 0$ *as* $|x| \to \infty$ *then it is radially symmetric, up to a translation in* $\mathbb{R}^n$*.*

Our approach to prove Theorems 1.1 and 1.2 is based in an essential way on the work [8]. There *abstract* conditions (see (C1)–(C3) and (D1)–(D3) below) are given which, if they are satisfied, guarantee the conclusions of our Theorems 1.1 and 1.2.

We point out that the way we prove the existence of least energy solutions, by solving a constrained minimization problem, is crucial in order to get the symmetry and sign results of all least energy solutions. First we show that the problem

$$\min\left\{\int_{\mathbb{R}^n} j(u,Du)\colon\ u \in D^{1,p}\big(\mathbb{R}^n\big),\ F(u) \in L^1\big(\mathbb{R}^n\big),\ \int_{\mathbb{R}^n} F(u) = 1\right\} \tag{1.15}$$

admits a solution, which is the hardest step. To do this, we exploit some tools from non-smooth critical point theory, such as the weak slope, developed in [12,13,24,25] (see Section 2.2). Then we prove that any minimizer is of class C^1 and satisfy the Euler–Lagrange equation as well as the Pucci–Serrin identity. This allow us to check the *abstract* conditions of [8] which provide a link between least action solutions of (1.1) and solutions of problem (1.15). Roughly speaking if the *abstract* conditions hold then there exist a least energy solution and to any least energy solution of (1.1) correspond, up to a rescaling, a minimizer of (1.15). It is proved in [27] that any such minimizer are radially symmetric. In addition it is shown in [8] that any minimizer has a constant sign.

Let us point out that, in our setting, the existence results that we obtain have no equivalent in the literature. Also, even assuming the existence of least energy solutions, our results of symmetry and sign are new. In particular we observe that, under our assumptions, to try to show that they are radial using moving plane methods or rearrangements arguments is hopeless. We definitely need to use the approach of [8] which, in turn, is based on the remarkable paper [27]. In [27] results of symmetry for C^1 mimimizers are obtained for general functionals under one or several constraints. Let us finally mention that in [8], and thus in our paper, the results of radial symmetry are obtained without using the fact that our solutions have a constant sign.

2. The case $1 < p < n$

2.1. Conditions (C1)–(C3)

In the following $D^{1,p}(\mathbb{R}^n)$ will denote the closure of the space $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\| = (\int_{\mathbb{R}^n} |Du|^p)^{1/p}$ and D^* is the dual space of $D^{1,p}(\mathbb{R}^n)$. Let us consider the problem

$$-\operatorname{div}\bigl(j_\xi(u, Du)\bigr) + j_s(u, Du) = f(u) \quad \text{in } \mathcal{D}'\bigl(\mathbb{R}^n\bigr), \tag{2.1}$$

associated with the functional

$$I(u) = \int_{\mathbb{R}^n} j(u, Du) - \int_{\mathbb{R}^n} F(u),$$

where $F(s) = \int_0^s f(t)\,dt$. Moreover, introducing the functionals,

$$J(u) = \int_{\mathbb{R}^n} j(u, Du), \qquad V(u) = \int_{\mathbb{R}^n} F(u), \qquad u \in D^{1,p}\bigl(\mathbb{R}^n\bigr),$$

we consider the following constrained problem

$$\text{minimize } J(u) \text{ subject to the constraint } V(u) = 1. \tag{P$_1$}$$

More precisely, let us set

$$X = \bigl\{u \in D^{1,p}\bigl(\mathbb{R}^n\bigr):\ F(u) \in L^1\bigl(\mathbb{R}^n\bigr)\bigr\},$$

and

$$T = \inf_{\mathcal{C}} J, \qquad \mathcal{C} = \bigl\{u \in X:\ V(u) = 1\bigr\}.$$

Consider the following conditions:

(C1) $T > 0$ and problem (P$_1$) has a minimizer $u \in X$;
(C2) any minimizer $u \in X$ of (P$_1$) is a C^1 solution and satisfies the equation

$$-\operatorname{div}\bigl(j_\xi(u, Du)\bigr) + j_s(u, Du) = \mu f(u) \quad \text{in } \mathcal{D}'\bigl(\mathbb{R}^n\bigr), \tag{2.2}$$

for some $\mu \in \mathbb{R}$;
(C3) any solution $u \in X$ of Eq. (2.2) satisfies the identity

$$(n - p)J(u) = \mu n V(u).$$

From [8] we have the following

Proposition 2.1. *Assume that* $1 < p < n$ *and that conditions* (C1)–(C3) *hold. Then* (1.1) *admits a least energy solution and each least energy solution has a constant sign and is radially symmetric, up to a translation in* $\mathbb{R}^n$.

Indeed X is an admissible function space in the sense introduced in [8]. Then Proposition 3 of [8] gives the existence of a least energy solution and that any least energy solution is radially symmetric. Finally, the sign result follows directly from Proposition 5 of [8].

In view of Proposition 2.1, our aim is now to prove that conditions (C1)–(C3) are fulfilled under assumptions (1.3)–(1.10).

Remark 2.2. In [8], in the scalar case, the equation considered is precisely (1.2) and the corresponding functional

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p - \int_{\mathbb{R}^n} F(u).$$

However, in order to show that conditions (C1)–(C3) imply the conclusion of Proposition 2.1, the only property of $|\nabla u|^p$ that it is used is p-homogeneity, namely that (1.7) hold.

2.2. Some recalls of non-smooth critical point theory

In this section we recall some abstract notions that will be used in the sequel. We refer the reader to [12,13,24,25], where this theory is fully developed.

Let X be a metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We set

$$\mathrm{dom}(f) = \{u \in X:\ f(u) < +\infty\} \quad \text{and} \quad \mathrm{epi}(f) = \{(u,\eta) \in X \times \mathbb{R}:\ f(u) \leqslant \eta\}.$$

The set $\mathrm{epi}(f)$ is endowed with the metric

$$d\big((u,\eta),(v,\mu)\big) = \big(d(u,v)^2 + (\eta-\mu)^2\big)^{1/2}.$$

Let us define the function $\mathcal{G}_f : \mathrm{epi}(f) \to \mathbb{R}$ by setting

$$\mathcal{G}_f(u,\eta) = \eta.$$

Note that $\mathcal{G}_f$ is Lipschitz continuous of constant 1. In the following $B(u,\delta)$ denotes the open ball of center u and of radius δ. We recall the definition of the weak slope for a continuous function.

Definition 2.3. Let X be a complete metric space, $g : X \to \mathbb{R}$ a continuous function, and $u \in X$. We denote by $|dg|(u)$ the supremum of the real numbers σ in $[0,\infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B(u,\delta) \times [0,\delta] \to X,$$

such that, for every v in $B(u,\delta)$, and for every t in $[0,\delta]$ it results

$$d\big(\mathcal{H}(v,t),v\big) \leqslant t,$$
$$g\big(\mathcal{H}(v,t)\big) \leqslant g(v) - \sigma t.$$

The extended real number $|dg|(u)$ is called the weak slope of g at u.

According to the previous definition, for every lower semicontinuous function f we can consider the metric space $\mathrm{epi}(f)$ so that the weak slope of $\mathcal{G}_f$ is well defined. Therefore, we can define the weak slope of a lower semicontinuous function f by using $|d\mathcal{G}_f|(u, f(u))$.

Definition 2.4. For every $u \in \mathrm{dom}(f)$ let

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u,f(u))}{\sqrt{1 - |d\mathcal{G}_f|\big(u,f(u)\big)^2}}, & \text{if } |d\mathcal{G}_f|\big(u,f(u)\big) < 1, \\ +\infty, & \text{if } |d\mathcal{G}_f|(u,f(u)) = 1. \end{cases}$$

The previous notion allows to give, in this framework, the definition of critical point of f (namely a point $u \in \mathrm{dom}(f)$ with $|df|(u) = 0$) as well as the following

Definition 2.5. Let X be a complete metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function and let $c \in \mathbb{R}$. We say that f satisfies the Palais–Smale condition at level c ($(PS)_c$ in short), if every sequence (u_n) in $\mathrm{dom}(f)$ such that $|df|(u_n) \to 0$ and $f(u_n) \to c$ ($(PS)_c$ sequence, in short) admits a subsequence (u_{n_k}) converging in X.

We now recall a consequence of Ekeland's variational principle [17] in the framework of the weak slope (just apply [13, Theorem 3.3] with $r = r_h = \sigma = \sigma_h = \varepsilon_h$ for a sequence $\varepsilon_h \to 0$; see also [13, Corollary 3.4]).

Proposition 2.6. *Let X be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function which is bounded from below. Assume that $(u_h) \subset \mathrm{dom}(f)$ is a minimizing sequence for f, that is $f(u_h) \to c = \inf_X f$. Then there exists a sequence $(\varepsilon_h) \subset \mathbb{R}^+$ with $\varepsilon_h \to 0$ as $h \to \infty$ and a sequence $(v_h) \subset X$ such that*

$$|df|(v_h) \leqslant \varepsilon_h, \qquad d(v_h,u_h) \leqslant \varepsilon_h, \qquad f(v_h) \leqslant f(u_h).$$

In particular (v_h) is a minimizing sequence and a $(PS)_c$ sequence for f.

Finally, we mention the notion of subdifferential as introduced in [9].

Definition 2.7. For a function $f : X \to \mathbb{R}$, we set

$$\partial f(x) = \big\{\alpha \in X':\ (\alpha, -1) \in N_{\mathrm{epi}(f)}\big(x, f(x)\big)\big\},$$

where X' is the dual space of X, $N_C(x) = \{\nu \in X':\ \langle \nu, v\rangle \leqslant 0$, for all $v \in T_C(x)\}$ is the normal cone (and $T_C(x)$ the tangent cone) to the set C at the point x.

More precisely, see [9, Definitions 3.1, 3.3 and 4.1].

2.3. Verification of conditions (C1)–(C3)

In this section we assume that (1.3)–(1.10) hold and then show that (C1)–(C3) are fulfilled.

First we extend $J|_{\mathcal{C}}$ to the functional $J^* : D^{1,p}(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$,

$$J^*(u) = \begin{cases} J(u) & \text{if } u \in \mathcal{C}, \\ +\infty & \text{if } u \notin \mathcal{C}, \end{cases} \tag{2.3}$$

which turns out to be lower semicontinuous. Since $\mathcal{C}$ can be regarded as a metric space endowed with the metric of $D^{1,p}(\mathbb{R}^n)$, the weak slope $|dJ|_{\mathcal{C}}|(u)$ and the Palais–Smale condition for $J|_{\mathcal{C}}$ may be defined.

Lemma 2.8. *For all $u \in \mathcal{C}$ there exists $\mu \in \mathbb{R}$ such that*

$$|dJ|_{\mathcal{C}}|(u) \geqslant \sup\big\{J'(u)(v) - \mu V'(u)(v):\ v \in C_c^\infty\big(\mathbb{R}^n\big),\ \|Dv\|_p \leqslant 1\big\}.$$

In particular, for each $(PS)_c$-sequence (u_h) for $J|_{\mathcal{C}}$ there exists $(\mu_h) \subset \mathbb{R}$ such that

$$\lim_h \sup\big\{J'(u_h)(v) - \mu_h V'(u_h)(v):\ v \in C_c^\infty\big(\mathbb{R}^n\big),\ \|Dv\|_p \leqslant 1\big\} = 0.$$

Proof. By condition (1.9), for all $u \in \mathcal{C}$ and any $v \in C_c^\infty(\mathbb{R}^n)$ the directional derivative of J at u along v exists and it is given by

$$J'(u)(v) = \int_{\mathbb{R}^n} j_\xi(u, Du) \cdot Dv + \int_{\mathbb{R}^n} j_s(u, Du)v. \tag{2.4}$$

Moreover the function $\{u \mapsto J'(u)(v)\}$ is continuous from $\mathcal{C}$ into $\mathbb{R}$. Of course, we may assume that $|dJ|_{\mathcal{C}}|(u) < +\infty$. If J^* is defined as in (2.3), we have $|dJ^*|(u) = |dJ|_{\mathcal{C}}|(u)$, so that by virtue of [9, Theorem 4.13] there exists $\omega \in \partial J^*(u)$ with $|dJ^*|(u) \geqslant \|\omega\|_{D^*}$. Moreover, by [9, Corollary 5.9(ii)], we have $\partial J^*(u) \subseteq \partial J(u) + \mathbb{R}V'(u)$. Finally, by [9, Theorem 6.1(ii)], we get $\partial J(u) = \{\eta\}$ where, for any function $v \in C_c^\infty(\mathbb{R}^n)$, $\langle \eta, v\rangle = J'(u)(v)$. This concludes the proof. $\square$

Remark 2.9. Assuming only that F is C^1 on $\mathbb{R} \setminus \{0\}$ and it is locally Lipschitz around the origin (as in [7] as it follows by [7, Assumption 2.8] which is used in the proof of Theorem 2.2 therein) Lemma 2.8 cannot hold in the form it is stated. In this more general case, there would exist $\mu \in \mathbb{R}$ and some function $\varphi \in L^\infty(\mathbb{R}^n)$ such that the solutions of the minimum problem satisfy

$$-\operatorname{div}\big(j_\xi(u, Du)\big) + j_s(u, Du) = \mu f(u)\chi_{\{u \neq 0\}} + \varphi\chi_{\{u=0\}} \quad \text{in } \mathcal{D}'\big(\mathbb{R}^n\big). \tag{2.5}$$

In fact, notice that this is exactly what is obtained at the bottom of p. 103 in [7]. Then, in light of the strong regularity of their solutions (that is $W^{2,\sigma}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\sigma > 1$), the equation is satisfied pointwise and as $\Delta u = 0$ a.e. in $\{u = 0\}$ (by a result of Stampacchia, see [32]) they infer $\varphi = 0$. On the other hand, in our degenerate framework we cannot reach this regularity level and concluding that $\varphi = 0$ (and hence that u solves (2.2)) seems, so far, out of reach.

Now we recall (see [30, Theorem 2]) the following

Lemma 2.10. *Let (u_h) be a bounded sequence in $D^{1,p}(\mathbb{R}^n)$ and, for each $v \in C_c^\infty(\mathbb{R}^n)$, set*

$$\langle w_h, v\rangle = \int_{\mathbb{R}^n} j_\xi(u_h, Du_h)\cdot Dv + \int_{\mathbb{R}^n} j_s(u_h, Du_h)v = J'(u_h)(v).$$

If the sequence (w_h) is strongly convergent to some w in $D^*(\Omega)$ for each open and bounded subset $\Omega \subset \mathbb{R}^n$, then (u_h) admits a strongly convergent subsequence in $D^{1,p}(\Omega)$ for each open and bounded subset $\Omega \subset \mathbb{R}^n$.

Lemma 2.11. *Assume* (1.3)–(1.10). *Then condition* (C1) *holds.*

Proof. In view of assumption (1.4) the constraint $\mathcal{C}$ is not empty (see Step 1 at page 324 in [3]). Let then $(u_h) \subset \mathcal{C}$ be a minimizing sequence for $J|_{\mathcal{C}}$. Therefore, we have

$$\lim_h \int_{\mathbb{R}^n} j(u_h, Du_h) = T, \qquad F(u_h) \in L^1(\mathbb{R}^n), \qquad \int_{\mathbb{R}^n} F(u_h) = 1, \quad \text{for all } h \in \mathbb{N}.$$

After extracting a subsequence, still denoted by (u_h), we get using (1.8),

$$u_h \rightharpoonup u \quad \text{in } L^{p^*}(\mathbb{R}^n), \qquad Du_h \rightharpoonup Du \quad \text{in } L^p(\mathbb{R}^n), \quad u_h(x) \to u(x) \text{ a.e.} \tag{2.6}$$

As $j(s,\xi)$ is positive, convex in the ξ argument, $u_h \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $Du_h \rightharpoonup Du$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, by well-known lower semicontinuity results (cf. [22,23]), it follows

$$\int_{\mathbb{R}^n} j(u, Du) \leqslant \liminf_h \int_{\mathbb{R}^n} j(u_h, Du_h) = T. \tag{2.7}$$

Moreover, setting $F = F^+ - F^-$ with $F^+ = \max\{F, 0\}$ and $F^- = \max\{-F, 0\}$, in view of assumptions (1.3) and (1.5) (which implies that $|F(s)|/|s|^{p^*}$ goes to zero as $s \to \infty$), fixing some $c > 0$ one can find $r_2 > r_1 > 0$ such that $F^+(s) \leqslant c|s|^{p^*}$ for all $|s| \leqslant r_1$ and $|s| \geqslant r_2$, so that

$$1 + \int_{\mathbb{R}^n} F^-(u_h) = \int_{\mathbb{R}^n} F^+(u_h) \leqslant c\int_{\mathbb{R}^n} |u_h|^{p^*}\chi_{\{|u_h|\leqslant r_1\}\cup\{|u_h|\geqslant r_2\}} + \beta\mathcal{L}^n\big(\{|u_h| > r_1\}\big)$$

where $\beta = \max\{F^+(s):\ r_1 \leqslant |s| \leqslant r_2\}$ and $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$. Clearly $\mathcal{L}^n(\{|u_h| > r_1\})$ remains uniformly bounded, as (u_h) is bounded in $L^{p^*}(\mathbb{R}^n)$. Hence, by Fatou's lemma, this yields $F^+(u), F^-(u) \in L^1(\mathbb{R}^n)$ and thus, finally, $F(u) \in L^1(\mathbb{R}^n)$. We have proved that $u \in X$. Notice that, still by assumptions (1.3) and (1.5), in light of [7, Lemma 2.1], we find two positive constants $\varepsilon_1, \varepsilon_2$ such that

$$\mathcal{L}^n\big(\big\{x \in \mathbb{R}^n:\ |u_h(x)| > \varepsilon_1\big\}\big) \geqslant \varepsilon_2, \quad \text{for all } h \in \mathbb{N}. \tag{2.8}$$

Hence, in view of [26, Lemma 6] (cf. the proof due to H. Brezis at the end of p. 447 in [26]) there exists a shifting sequence $(\xi_h) \subset \mathbb{R}^n$ such that $(u_h(x + \xi_h))$ converges weakly to a non-trivial limit. Thus, in (2.6), we may assume that $u \not\equiv 0$. Applying Proposition 2.6 to the lower semicontinuous functional J^* defined in (2.3), we can replace the minimizing sequence $(u_h) \subset \mathcal{C}$ by a minimizing sequence $(v_h) \subset \mathcal{C}$ with $\|v_h - u_h\|_{D^{1,p}} = o(1)$ as $h \to \infty$ (we shall rename v_h again as u_h) such that the weak slope vanishes, namely $|dJ|_{\mathcal{C}}|(u_h) \leqslant \varepsilon_h$, with $\varepsilon_h \to 0$ as $h \to \infty$. It follows by Lemma 2.8 that there exists a sequence $(\mu_h) \subset \mathbb{R}$ of Lagrange multipliers such that

$$J'(u_h)(v) = \mu_h V'(u_h)(v) + \langle \eta_h, v\rangle, \quad \text{for all } h \in \mathbb{N} \text{ and } v \in C_c^\infty(\mathbb{R}^n), \tag{2.9}$$

where η_h strongly converges to 0 in D^*. Also for any bounded domain $\Omega \subset \mathbb{R}^n$, by (1.5) (which implies that, for each $\varepsilon > 0$, there exists $a_\varepsilon \in \mathbb{R}$ such that $|f(s)| \leqslant a_\varepsilon + \varepsilon|s|^{p^*-1}$ for all $s \in \mathbb{R}$) it follows that the map $D^{1,p}(\Omega) \ni v \mapsto f(v) \in D^*(\Omega)$ is completely continuous. Thus by condition (1.6), since $u \neq 0$, there exists a function $\psi_0 \in C_c^\infty(\mathbb{R}^n)$ such that, setting $K_0 = \mathrm{supp}(\psi_0)$, it holds

$$V'(u_h)(\psi_0) = \int_{K_0} f(u_h)\psi_0 \not\to 0, \quad \text{as } h \to \infty.$$

Also the sequence $(J'(u_h)(\psi_0))$ is bounded. In fact, denoting by C a generic positive constant, we have by (1.9) that

$$\int_{K_0} \big|j_\xi(u_h, Du_h)\big||D\psi_0| \leqslant C \int_{K_0} |Du_h|^{p-1} \leqslant C\bigg(\int_{K_0} |Du_h|^p\bigg)^{\frac{p-1}{p}} \leqslant C,$$

$$\int_{K_0} \big|j_s(u_h, Du_h)\big||\psi_0| \leqslant C \int_{K_0} |Du_h|^p \leqslant C,$$

and we conclude using (2.4). Now, formula (2.9) yields

$$\mu_h V'(u_h)(\psi_0) + \langle \eta_h, \psi_0\rangle = J'(u_h)(\psi_0)$$

with $\eta_h \to 0$ in D^*. We deduce that the sequence (μ_h) is bounded in $\mathbb{R}$ and thus we can assume that it converges to some $\mu \in \mathbb{R}$. It follows that $w_h = \mu_h V'(u_h) + \eta_h = \mu_h f(u_h) + \eta_h$ converges strongly to some w in $D^*(\Omega)$. Therefore, by Lemma 2.10, we infer that (u_h) admits a subsequence which strongly converges in $D^{1,p}(\Omega)$. Thus, we have proved that the sequence (u_h) is locally compact in $D^{1,p}(\mathbb{R}^n)$ (fact that is useful in the forthcoming steps). From this we can easily deduce that $J'(u)(v) = \mu V'(u)(v)$ for all $v \in C_c^\infty(\mathbb{R}^n)$, namely (2.2). However this is not enough (nor necessary) to show that u is a minimizer of (P_1) since we do not know if $V(u) = 1$. In this aim we set $u_\sigma(x) = u(x/\sigma)$. Then it holds $J(u_\sigma) = \sigma^{n-p} J(u)$ and $V(u_\sigma) = \sigma^n V(u)$ and hence, by a simple scaling argument, we get that

$$\int_{\mathbb{R}^n} j(w, Dw) \geqslant T\bigg(\int_{\mathbb{R}^n} F(w)\bigg)^{\frac{n-p}{n}}, \quad \text{for all } w \in D^{1,p}\big(\mathbb{R}^n\big) \text{ with } V(w) > 0. \tag{2.10}$$

We follow now an argument in the spirit of the perturbation method developed in the proof of [7, Lemma 2.3]. Taking any function $\phi \in L^{p^*}(\mathbb{R}^n)$ with compact support, we claim that

$$\int_{\mathbb{R}^n} F(u_h + \phi) = 1 + \int_{\mathbb{R}^n} F(u + \phi) - \int_{\mathbb{R}^n} F(u) + \mathrm{o}(1), \quad \text{as } h \to \infty.$$

Indeed, if we set $K = \mathrm{supp}(\phi)$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} F(u_h + \phi) &= \int_{\mathbb{R}^n} F(u_h) + \int_K F(u_h + \phi) - F(u_h)\\ &= 1 + \int_K F(u + \phi) - F(u) + \mathrm{o}(1)\\ &= 1 + \int_{\mathbb{R}^n} F(u + \phi) - F(u) + \mathrm{o}(1), \quad \text{as } h \to \infty,\end{aligned}$$

where the second equality follows by the dominated convergence theorem in light of (1.5) and the strong convergence of u_h to u in $L^{p^*-1}(K)$. Moreover, we have

$$\begin{aligned}\int_{\mathbb{R}^n} j(u_h + \phi, Du_h + D\phi) &= T + \int_{\mathbb{R}^n} j(u_h + \phi, Du_h + D\phi) - j(u_h, Du_h) + \mathrm{o}(1)\\ &= T + \int_K j(u_h + \phi, Du_h + D\phi) - j(u_h, Du_h) + \mathrm{o}(1)\\ &= T + \int_K j(u + \phi, Du + D\phi) - j(u, Du) + \mathrm{o}(1)\\ &= T + \int_{\mathbb{R}^n} j(u + \phi, Du + D\phi) - j(u, Du) + \mathrm{o}(1),\end{aligned}$$

as $h \to \infty$, where the third equality is justified again by the dominated convergence theorem, since as $Du_h \to Du$ in $L^p(K)$ for $h \to \infty$ we have

$$\big|j(u_h+\phi, Du_h+D\phi)-j(u_h,Du_h)\big|\leqslant c_2|Du_h+D\phi|^p\chi_K+c_2|Du_h|^p\chi_K,$$
$$j(u_h+\phi, Du_h+D\phi)-j(u_h,Du_h)\to j(u+\phi,Du+D\phi)-j(u,Du)\quad \text{a.e. in } \mathbb{R}^n.$$

Therefore, choosing $w=u_h+\phi$ inside inequality (2.10), where $\phi\in D^{1,p}(\mathbb{R}^n)$ has compact support and $1+\int_{\mathbb{R}^n}F(u+\phi)-F(u)>0$, and taking the limit as $h\to\infty$, it follows that

$$T+\int_{\mathbb{R}^n} j(u+\phi,Du+D\phi)-\int_{\mathbb{R}^n} j(u,Du)\geqslant T\biggl(1+\int_{\mathbb{R}^n}F(u+\phi)-\int_{\mathbb{R}^n}F(u)\biggr)^{\frac{n-p}{n}}.$$

Fixed λ close to 1, we consider for some $r>1$ a C^∞ function $\Lambda:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\Lambda(t)=\lambda\quad\text{if } t\leqslant 1,\qquad \Lambda(t)=1\quad \text{if } t\geqslant r,\qquad \rho=\inf_{t\in\mathbb{R}^+}\Lambda(t)>\frac{1}{2},\qquad \sup_{t\in\mathbb{R}}\big|\Lambda'(t)\big|<\frac{\rho}{r},\tag{2.11}$$

and we introduce the smooth and bijective map $\Pi:\mathbb{R}^n\to\mathbb{R}^n$ by setting $\Pi(x)=\Lambda(|x|)x$, for all $x\in\mathbb{R}^n$. Finally, we set

$$\Pi_h(x)=h\Pi\biggl(\frac{x}{h}\biggr)=\begin{cases}\lambda x & \text{if } |x|\leqslant h,\\ \Lambda\Bigl(\dfrac{|x|}{h}\Bigr)x & \text{if } h\leqslant|x|\leqslant rh,\\ x & \text{if } |x|\geqslant rh,\end{cases}\qquad \phi_h(x)=u\bigl(\Pi_h(x)\bigr)-u(x).$$

In particular, it follows that $\phi_h\in D^{1,p}(\mathbb{R}^n)$ is a compact support function which satisfies

$$1+\int_{\mathbb{R}^n}F(u+\phi_h)-F(u)>0,$$

at least for all values of λ sufficiently close to 1 (see Eq. (2.12) below). Hence, for any $h\in\mathbb{N}$, we conclude

$$T+\int_{\mathbb{R}^n} j(u+\phi_h,Du+D\phi_h)-\int_{\mathbb{R}^n} j(u,Du)\geqslant T\biggl(1+\int_{\mathbb{R}^n}F(u+\phi_h)-\int_{\mathbb{R}^n}F(u)\biggr)^{\frac{n-p}{n}}.$$

Notice that, we have

$$\int_{\mathbb{R}^n} j(u+\phi_h,Du+D\phi_h)=\int_{\mathbb{R}^n} j\bigl(u\bigl(\Pi_h(x)\bigr),Du\bigl(\Pi_h(x)\bigr)\bigr)=I_1+I_2+I_2.$$

In view of assumptions (1.7) and (1.8), by dominated convergence we have

$$\begin{aligned}I_1&=\int_{\mathbb{R}^n} j\bigl(u(\lambda x),\lambda(Du)(\lambda x)\bigr)\chi_{\{|x|\leqslant h\}}\\ &=\lambda^p\int_{\mathbb{R}^n} j\bigl(u(\lambda x),(Du)(\lambda x)\bigr)\chi_{\{|x|\leqslant h\}}\\ &=\lambda^p\int_{\mathbb{R}^n} j\bigl(u(\lambda x),(Du)(\lambda x)\bigr)+\mathrm{o}(1)\\ &=\lambda^{p-n}\int_{\mathbb{R}^n} j(u,Du)+o(1),\quad \text{as } h\to\infty.\end{aligned}$$

If $L_h=(\partial_j\Pi_h^i)$ and $M_h=(\partial_j(\Pi_h^{-1})^i)$ denote the $n\times n$ Jacobian matrices of the maps Π_h,Π_h^{-1} respectively, taking into account that $\gamma=\sup\{\|L_h\|_{n\times n}\colon h\in\mathbb{N}\}<\infty$ and $\gamma'=\sup\{|\det M_h|\colon h\in\mathbb{N}\}<\infty$ and using the growth condition (1.8) on j, we get

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^n} j\big(u\big(\Pi_h(x)\big), Du\big(\Pi_h(x)\big)\big)\chi_{\{h\leqslant|x|\leqslant rh\}} \\
&\leqslant c_2 \int_{\mathbb{R}^n} \big|Du\big(\Pi_h(x)\big)\big|^p \chi_{\{h\leqslant|x|\leqslant rh\}} \\
&= c_2 \int_{\mathbb{R}^n} \big|L_h(Du)\big(\Pi_h(x)\big)\big|^p \chi_{\{h\leqslant|x|\leqslant rh\}} \\
&\leqslant c_2\gamma^p \int_{\mathbb{R}^n} \big|(Du)\big(\Pi_h(x)\big)\big|^p \chi_{\{h\leqslant|x|\leqslant rh\}} \\
&= c_2\gamma^p\gamma' \int_{\mathbb{R}^n} \big|Du(y)\big|^p \chi_{\{h\leqslant|\Pi_h^{-1}(y)|\leqslant rh\}} = \mathrm{o}(1), \quad \text{as } h\to\infty.
\end{aligned}$$

Concerning the last equality notice that, as $|\Pi_h(x)| \geqslant \rho|x| \geqslant \rho h$, where ρ is the constant appearing in definition (2.11), the condition $|\Pi_h^{-1}(y)| \geqslant h$ implies $|y| \geqslant \rho h$ and hence the integrand goes to zero pointwise. Finally, of course, we have

$$I_3 = \int_{\mathbb{R}^n} j(u, Du)\chi_{\{|x|>rh\}} = \mathrm{o}(1), \quad \text{as } h\to\infty.$$

In conclusion, we get

$$\int_{\mathbb{R}^n} j(u+\phi_h, Du+D\phi_h) = \lambda^{p-n}\int_{\mathbb{R}^n} j(u, Du) + \mathrm{o}(1), \quad \text{as } h\to\infty.$$

In the same way, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} F(u+\phi_h) = \int_{\mathbb{R}^n} F\big(u\big(\Pi_h(x)\big)\big) &= \int_{\mathbb{R}^n} F\big(u(y)\big)\big|\det(M_h)\big| \\
&= \lambda^{-n}\int_{\mathbb{R}^n} F(u) + \mathrm{o}(1), \quad \text{as } h\to\infty.
\end{aligned} \tag{2.12}$$

Finally, collecting the previous formulas, we reach the inequality

$$T + \big(\lambda^{p-n}-1\big)\int_{\mathbb{R}^n} j(u, Du) \geqslant T\bigg(1+\big(\lambda^{-n}-1\big)\int_{\mathbb{R}^n} F(u)\bigg)^{\frac{n-p}{n}}$$

which holds for every λ sufficiently close to 1. Choosing $\lambda = 1+\omega$ and $\lambda = 1-\omega$ with $\omega > 0$ small and then letting $\omega \to 0^+$, we conclude that

$$\int_{\mathbb{R}^n} j(u, Du) = T\int_{\mathbb{R}^n} F(u).$$

Since $u \not\equiv 0$, it follows that $\int_{\mathbb{R}^n} F(u) > 0$, so that plugging $w = u$ into (2.10) one entails $\int_{\mathbb{R}^n} F(u) \geqslant 1$. On the other hand, inequality (2.7) yields $\int_{\mathbb{R}^n} j(u, Du) \leqslant T$. This, of course, forces

$$T = \int_{\mathbb{R}^n} j(u, Du), \qquad \int_{\mathbb{R}^n} F(u) = 1, \tag{2.13}$$

which concludes the proof. $\square$

Remark 2.12. In the proof of Lemma 2.11, in order to show that the (minimizing) sequence (u_h) is strongly convergent in $D^{1,p}(\Omega)$ for any bounded domain Ω of $\mathbb{R}^n$, we have exploited the sub-criticality assumption (1.5) on f, which is stronger than the corresponding assumption (2.6) in [7] on F, that is

$$\lim_{s\to\infty}\frac{F(s)}{|s|^{p^*}}=0.$$

In [7], due to the particular structure of j, namely the model case $j(s,\xi)=\frac{1}{2}|\xi|^2$, to conclude the proof the weak convergence of (u_h) to u in $D^{1,2}(\mathbb{R}^n)$ turns out to be sufficient, while to cover the general case $j(s,\xi)$ the local convergence seems to be necessary to handle the perturbation argument devised at the end of Lemma 2.11. We point out that, also in [18], the authors assume condition (1.5) on f, although they are allowed to take a spherically symmetric minimizing sequence which provides compactness.

Lemma 2.13. *Assume* (1.3)–(1.10). *Then condition* (C2) *holds.*

Proof. Let $u\in D^{1,p}(\mathbb{R}^n)$ be a minimizer for problem (P_1). Then the sequence $u_h=u$ is minimizing for (P_1). By Proposition 2.6 we can find a sequence $(v_h)\subset D^{1,p}(\mathbb{R}^n)$ such that $\|v_h-u\|_{D^{1,p}}=o(1)$ and $|dJ|_{\mathcal{C}}|(v_h)\to 0$ as $h\to\infty$. Hence, by Lemma 2.8 there exists a sequence $(\mu_h)\subset\mathbb{R}$ such that

$$J'(v_h)(\varphi)=\mu_h V'(v_h)(\varphi)+\langle\eta_h,\varphi\rangle,\quad \text{for all } \varphi\in C_c^\infty(\mathbb{R}^n) \tag{2.14}$$

where η_h converges strongly to 0 in D^*. As in the proof of Lemma 2.11 we can assume that $\mu_h\to\mu$ and since $v_h\to u$ in $D^{1,p}(\mathbb{R}^n)$, obviously

$$J'(u)(\varphi)=\mu V'(u)(\varphi),\quad \text{for all } \varphi\in C_c^\infty(\mathbb{R}^n).$$

Namely Eq. (2.2) is satisfied in the sense of distributions for some $\mu\in\mathbb{R}$. By means of assumptions (1.8), (1.9) and (1.10), a standard argument yields $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ (see, e.g., [29, Theorem 1 and Remark at p. 261]). By the regularity results contained in [15,33], it follows that $u\in C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^n)$, for some $0<\beta<1$. □

Let $\varphi\in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and let $L(s,\xi):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ be a function of class C^1 in s and ξ such that, for any $s\in\mathbb{R}$, the map $\{\xi\mapsto L(s,\xi)\}$ is strictly convex. We recall, in the autonomous setting, a Pucci–Serrin variational identity for locally Lipschitz continuous solutions of a general class of equations, recently obtained in [14].

Lemma 2.14. *Let $u:\mathbb{R}^n\to\mathbb{R}$ be a locally Lipschitz solution of*

$$-\operatorname{div}\big(L_\xi(u,Du)\big)+L_s(u,Du)=\varphi\quad \text{in } \mathcal{D}'\big(\mathbb{R}^n\big).$$

Then

$$\sum_{i,j=1}^n\int_{\mathbb{R}^n} D_i h^j D_{\xi_i}L(u,Du)D_j u-\int_{\mathbb{R}^n}(\operatorname{div} h)L(u,Du)=\int_{\mathbb{R}^n}(h\cdot Du)\varphi \tag{2.15}$$

for every $h\in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$.

Remark 2.15. The classical Pucci–Serrin identity [28] is not applicable here, since it requires the C^2 regularity of the solutions while in our degenerate setting (for $p\neq 2$) the maximal allowed regularity is $C^{1,\beta}_{\mathrm{loc}}$ (see [15,33]).

Lemma 2.16. *Assume* (1.3)–(1.10). *Then condition* (C3) *holds.*

Proof. Let $u\in D^{1,p}(\mathbb{R}^n)$ be any solution of Eq. (2.2). In light of conditions (1.8), (1.9) and (1.10), as we observed in the proof of Lemma 2.13, it follows that $u\in C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $0<\beta<1$. Then, since $\{\xi\mapsto j(s,\xi)\}$ is strictly convex, we can use Lemma 2.14 by choosing in (2.15) $\varphi=0$ and

$$L(s,\xi)=j(s,\xi)-\mu F(s),\quad \text{for all } s\in\mathbb{R}^+ \text{ and } \xi\in\mathbb{R}^n,$$

$$h(x)=h_k(x)=T\left(\frac{x}{k}\right)x,\quad \text{for all } x\in\mathbb{R}^n \text{ and } k\geqslant 1, \tag{2.16}$$

being $T \in C_c^1(\mathbb{R}^n)$ such that $T(x) = 1$ if $|x| \leqslant 1$ and $T(x) = 0$ if $|x| \geqslant 2$. In particular, for every k we have that $h_k \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ and

$$D_i h_k^j(x) = D_i T\left(\frac{x}{k}\right)\frac{x_j}{k} + T\left(\frac{x}{k}\right)\delta_{ij}, \quad \text{for all } x \in \mathbb{R}^n,\ i, j = 1, \dots, n,$$

$$(\operatorname{div} h_k)(x) = DT\left(\frac{x}{k}\right) \cdot \frac{x}{k} + nT\left(\frac{x}{k}\right), \quad \text{for all } x \in \mathbb{R}^n.$$

Then it follows by identity (2.15) that

$$\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} D_i T\left(\frac{x}{k}\right)\frac{x_j}{k} D_j u D_{\xi_i} L(u, Du) + \int_{\mathbb{R}^n} T\left(\frac{x}{k}\right) D_\xi L(u, Du) \cdot Du$$
$$- \int_{\mathbb{R}^n} DT\left(\frac{x}{k}\right) \cdot \frac{x}{k} L(u, Du) - \int_{\mathbb{R}^n} nT\left(\frac{x}{k}\right) L(u, Du) = 0,$$

for every $k \geqslant 1$. Since there exists $C > 0$ with

$$\left| D_i T\left(\frac{x}{k}\right)\frac{x_j}{k}\right| \leqslant C \quad \text{for every } x \in \mathbb{R}^n,\ k \geqslant 1 \textit{ and } i, j = 1, \dots, n,$$

by the Dominated Convergence Theorem (recall that by (1.8) and the p-homogeneity of $\{\xi \mapsto j(s, \xi)\}$, of course one has $L(u, Du)$, $D_\xi L(u, Du) \cdot Du \in L^1(\mathbb{R}^n)$), letting $k \to \infty$, we conclude that

$$\int_{\mathbb{R}^n} \left[nL(u, Du) - D_\xi L(u, Du) \cdot Du\right] = 0,$$

namely, by (2.16) and, again, the p-homogeneity of $\{\xi \mapsto j(s, \xi)\}$,

$$(n - p) \int_{\mathbb{R}^n} j(u, Du) = \mu n \int_{\mathbb{R}^n} F(u), \tag{2.17}$$

namely $(n - p)J(u) = \mu n V(u)$, proving that condition (C3) is fulfilled. □

Proof of Theorem 1.1. From Lemmas 2.11, 2.13 and 2.16 we see that the conditions (C1)–(C3) hold. The conclusion follows directly from Proposition 2.1. □

Remark 2.17. In light of formula (2.17) and the positivity of j, it holds $\int_{\mathbb{R}^n} F(u) > 0$ as soon as u is a non-trivial solution of (2.2).

Remark 2.18. Assumption (1.10) was already considered e.g. in [2,5,10,11,21,30,31]. We exploited it in order to get existence (in (C1)), regularity (in (C2)) and hence also for the Pucci–Serrin identity (in (C3)), and it seems hard to drop, mainly concerning the boundedness (and hence C^1 regularity) issue of solutions. In fact, in lack of (1.10) some problems may occur, already in the case of bounded domains and $p = 2$. For instance, as shown by J. Frehse in [19], if $B(0, 1)$ is the unit ball in $\mathbb{R}^n$ centered at zero with $n \geqslant 3$,

$$j(x, s, \xi) = \left(1 + \frac{1}{|x|^{12(n-2)} e^s + 1}\right)|\xi|^2$$

and $f(s) = 0$, then $u(x) = -12(n - 2)\log|x|$ is a weak solution to the corresponding Euler equation with $u = 0$ on $\partial B(0, 1)$. In particular $u \notin L^\infty(B(0, 1))$ although j is very regular. It is immediate to check that $j_s(x, s, \xi)s \leqslant 0$ for any $s \geqslant 0$, so (1.10) fails. Although this counterexample involves an x-dependent Lagrangian (while we deal with autonomous problems) these pathologies in regularity are related to the s-dependence in the Lagrangian j.

3. The case $p = n$

We consider now the following constrained minimization problem

$$\text{minimize } J(u) \text{ for } u \neq 0 \text{ subject to the constraint } V(u) = 0. \tag{P$_0$}$$

More precisely, let us set

$$X_0 = \{u \in D^{1,n}(\mathbb{R}^n):\ F(u) \in L^1(\mathbb{R}^n)\}, \tag{3.1}$$

and

$$T_0 = \inf_{\mathcal{C}_0} J, \quad \mathcal{C}_0 = \{u \in X_0:\ u \neq 0,\ V(u) = 0\}.$$

Consider the following conditions:

(D1) $T > 0$ and problem (P$_0$) has a minimizer $u \in X_0$;
(D2) any minimizer $u \in X_0$ of (P$_0$) is a C^1 solution and satisfies the equation

$$-\operatorname{div}(j_\xi(u, Du)) + j_s(u, Du) = \mu f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{3.2}$$

for some $\mu \in \mathbb{R}$;
(D3) any solution $u \in X_0$ of Eq. (3.2) with $\mu > 0$ satisfies $V(u) = 0$.

From [8] we have the following

Proposition 3.1. *Assume that $p = n$ and that* (D1)–(D3) *hold. Then* (1.1) *admits a least energy solution and each least energy solution has a constant sign. Moreover if $u \in D^{1,n}(\mathbb{R}^n)$ is a least energy solution such that $u(x) \to 0$ as $|x| \to \infty$ it is radially symmetric, up to a translation in $\mathbb{R}^n$.*

Proposition 3.1 follows directly from Propositions 4 and 6 in [8]. See also Remark 2.2.
Let us now show that (D1)–(D3) hold. First we recall a regularity result (see [6]).

Lemma 3.2. *Let $u, v \in D^{1,n}(\mathbb{R}^n)$, $\eta \in L^1(\mathbb{R}^n)$ and $w \in D^*(\mathbb{R}^n)$ with*

$$j_s(u, \nabla u)v \geqslant \eta,$$

and for all $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\langle w, \varphi\rangle = \int_{\mathbb{R}^n} j_\xi(u, \nabla u) \cdot D\varphi + \int_{\mathbb{R}^n} j_s(u, Du)\varphi.$$

Then $j_s(u, \nabla u)v \in L^1(\mathbb{R}^n)$ and

$$\langle w, v\rangle = \int_{\mathbb{R}^n} j_\xi(u, \nabla u) \cdot Dv + \int_{\mathbb{R}^n} j_s(u, \nabla u)v.$$

Now we have

Proposition 3.3. *Assume* (1.7)–(1.14). *Then conditions* (D1)–(D3) *hold.*

Proof. In view of (1.12) the constraint $\mathcal{C}_0$ is not empty (see again Step 1 at p. 324 in [3]). Let then $(u_h) \subset \mathcal{C}_0$ be a minimizing sequence for $J|_{\mathcal{C}_0}$. Therefore, we have

$$\lim_h \int_{\mathbb{R}^n} j(u_h, Du_h) = T_0, \quad u_h \neq 0, \qquad F(u_h) \in L^1(\mathbb{R}^n), \qquad \int_{\mathbb{R}^n} F(u_h) = 0,$$

for all $h \in \mathbb{N}$. Since $u_h \neq 0$ and by (1.11), it holds

$$\int\limits_{\{|u_h|>\delta\}} F(u_h) = \int\limits_{\{0\leqslant|u_h|\leqslant\delta\}} \left|F(u_h)\right| > 0,$$

of course $\mathcal{L}^n(\{|u_h| > \delta\}) > 0$ for every $h \in \mathbb{N}$. Then, since the map $\{u \mapsto J(u)\}$ is invariant under scaling on $D^{1,n}(\mathbb{R}^n)$, it is readily seen that there exists $\varrho > 0$ such that

$$\mathcal{L}^n\big(\big\{|u_h| > \delta\big\}\big) \geqslant \varrho, \quad \text{for all } h \in \mathbb{N}. \tag{3.3}$$

Arguing as in [7, Lemma 3.1], we have that

$$\sup_{h\in\mathbb{N}} \int\limits_{\{|u_h|>\delta\}} |u_h|^r < \infty, \quad \text{for all } r > 1. \tag{3.4}$$

Then the sequence (u_h) is bounded in $L^q(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}^n$ and, after extracting a subsequence still denoted by (u_h), we have $u_h \to u$ in $L^q(\Omega)$ for all $q \geqslant 1$, $Du_h \rightharpoonup Du$ in $L^n(\mathbb{R}^n)$, and $u_h(x) \to u(x)$ a.e. $x \in \mathbb{R}^n$. As $j(s,\xi)$ is positive, convex in the second argument, $u_h \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $Du_h \rightharpoonup Du$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by lower semi-continuity it follows

$$\int\limits_{\mathbb{R}^n} j(u, Du) \leqslant \liminf_h \int\limits_{\mathbb{R}^n} j(u_h, Du_h) = T_0. \tag{3.5}$$

Now from (3.3), as in the proof of Lemma 2.11 we get that, after a shift, the weak limit of (u_h) is non-trivial, that is $u \neq 0$. Notice also that, in view of (1.11), (1.13) and the bound furnished by (3.4) we get, for any $h \in \mathbb{N}$,

$$\int\limits_{\mathbb{R}^n} F_-(u_h) = \int\limits_{\mathbb{R}^n} F_+(u_h) = \int\limits_{\{|u_h|>\delta\}} F_+(u_h) \leqslant C \int\limits_{\{|u_h|>\delta\}} |u_h|^q \leqslant C,$$

where C is a generic positive constant. In particular, by Fatou's lemma, it follows $F \in L^1(\mathbb{R}^n)$. We have proved thus $u \in X_0$. Arguing as in Lemma 2.11, up to substituting $(u_h) \subset \mathcal{C}_0$ with a new minimizing sequence $(v_h) \subset \mathcal{C}_0$, we may assume that $|dJ|_{\mathcal{C}_0}|(u_h) \leqslant \varepsilon_h$, with $\varepsilon_h \to 0$ as $h \to \infty$. By Lemma 2.8 there exists a sequence $(\mu_h) \subset \mathbb{R}$ such that

$$J'(u_h)(v) = \mu_h V'(u_h)(v) + \langle \eta_h, v\rangle, \quad \text{for all } h \in \mathbb{N} \text{ and } v \in C_c^\infty\big(\mathbb{R}^n\big), \tag{3.6}$$

where η_h strongly converges to 0 in D^* as $h \to \infty$. As in Lemma 2.11, it can be proved that (μ_h) is bounded (and hence it converges to some value $\mu \in \mathbb{R}$). Now since (3.6) hold and $u_h \rightharpoonup u$ in $D^{1,n}(\mathbb{R}^n)$ using the classical convergence result of Murat (see Theorem 2.1 of [4]) we get that $Du_h(x) \to Du(x)$ a.e. $x \in \mathbb{R}^n$. At this point it follows easily that (3.2) is satisfied (see e.g. [31, Theorem 3.4] for details).

Let us now prove that, actually, $\mu \neq 0$. If, by contradiction, it was $\mu = 0$, then we would have

$$\int\limits_{\mathbb{R}^n} j_\xi(u, Du)\cdot Dv + \int\limits_{\mathbb{R}^n} j_s(u, Du)v = 0, \quad \text{for all } v \in C_c^\infty\big(\mathbb{R}^n\big).$$

Let now $\zeta : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$\zeta(s) = \begin{cases} M|s| & \text{if } |s| \leqslant R, \\ MR & \text{if } |s| \geqslant R, \end{cases} \tag{3.7}$$

being $R > 0$ the constant defined in (1.10) and M a positive number (which exists by combining the growths conditions (1.8), (1.9)) such that

$$\big|j_s(s,\xi)\big| \leqslant nMj(s,\xi) \tag{3.8}$$

for $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Notice that, by combining (1.10) and (3.8), we obtain

$$\big[j_s(s,\xi) + n\zeta'(s)j(s,\xi)\big]s \geqslant 0, \quad \text{for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^n. \tag{3.9}$$

Taking into account (1.10), by Lemma 3.2 we are allowed to choose $v = ue^{\zeta(u)}$ and hence

$$\int_{\mathbb{R}^n} ne^{\zeta(u)} j(u, Du) + \int_{\mathbb{R}^n} e^{\zeta(u)}\big[j_s(u, Du) + n\zeta'(u) j(u, Du)\big]u = 0.$$

Then, by (3.9) and (1.8) we get

$$nc_1 \int_{\mathbb{R}^n} |Du|^n \leqslant 0,$$

so that $u = 0$, which is not possible. Hence $\mu \neq 0$. Arguing as in Lemma 2.16 the Pucci–Serrin identity follows, namely, as $p = n$

$$\int_{\mathbb{R}^n} F(u) = \frac{n-p}{\mu n} \int_{\mathbb{R}^n} j(u, Du) = 0.$$

The same conclusion obviously hold for any solution $u \in X_0$ for (3.2) with $\mu > 0$, this shows that (D3) hold. Now since $u \in X_0$ and $\int_{\mathbb{R}^n} F(u) = 0$, we have $u \in \mathcal{C}_0$, so that by (3.5)

$$\int_{\mathbb{R}^n} j(u, Du) = T_0.$$

As in Lemma 2.13, one can prove that any minimizer is C^1 and satisfies the Euler–Lagrange equation (3.2), which concludes the proof. □

Remark 3.4. The check that (D1) holds is actually simpler than in the case of (C1). In particular, to check (D1) we do not need to use any kind of strong local convergence, as for the case $1 < p < n$, using classical convergence results due to Murat suffices. Observe also that, in the case $p = n$, if we have a non-trivial function $v \in X_0$ which is a solution to the problem

$$-\operatorname{div}\big(j_\xi(v, Dv)\big) + j_s(v, Dv) = f(v) \quad \text{in } \mathcal{D}'\big(\mathbb{R}^n\big), \tag{3.10}$$

then, by the Pucci–Serrin identity it follows that $\int_{\mathbb{R}^n} F(v) = 0$, so that $v \in \mathcal{C}_0$ and hence

$$I(v) = \int_{\mathbb{R}^n} j(v, Dv) \geqslant T_0 = \int_{\mathbb{R}^n} j(u, Du) = I(u),$$

proving that u is, automatically, a least energy solution of (3.10).

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