ON EXISTENCE AND MULTIPLICITY RESULTS FOR A CLASS OF GAUGED SCHRÖDINGER EQUATIONS WITH INDEFINITE POTENTIALS

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Abstract. We study a class of indefinite Schrödinger equations coupled with the Chern-Simons theory

$$\begin{cases} -\Delta u + V_\omega(|x|)u + \lambda \left(\displaystyle\int_{|x|}^{\infty} \frac{h_u(s)}{s} u^2(s)\mathrm{d}s + \frac{h_u^2(|x|)}{|x|^2} \right) u = g(x,u) \text{ in } \mathbb{R}^2, \\ u(x) = u(|x|), \end{cases}$$

where $V_\omega(|x|) = |x|^2 - \omega$ with $\omega \in \mathbb{R}$ such that the operator $-\Delta + V_\omega$ is invertible and $h_u(s) = \int_0^s \frac{r}{2} u^2(r)\mathrm{d}r$. If g fulfills the supercritical exponential growth at infinity in the Trudinger-Moser sense, due to a subtle truncation argument, we take advantage of some analytic techniques and the elliptic regularity theory to deduce that this equation admits a nontrivial solution for all sufficiently small $\lambda > 0$ by using variational method. If $g(x,u) = \xi(x)|u|^{p-2}u$ with $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$ and $1 < p < 2$, as applications of the arguments above, we conclude the existence of infinitely many nontrivial solutions whose energies converge to 0 for $\lambda > 0$ small enough. As far as we know, the results above have not been considered yet in the literature.

1. Introduction and main results

In this article, we mainly focus on the existence and multiplicity results for a class of gauged nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + V_\omega(|x|)u + \lambda \left(\displaystyle\int_{|x|}^{\infty} \frac{h_u(s)}{s} u^2(s)\mathrm{d}s + \frac{h_u^2(|x|)}{|x|^2} \right) u = g(x,u) \text{ in } \mathbb{R}^2, \\ u(x) = u(|x|), \end{cases} \tag{1.1}$$

where the indefinite potential $V_\omega(|x|) = |x|^2 - \omega$ satisfies that the constant $\omega > 0$ is sufficiently large to make the operator $-\Delta + V_\omega$ non-degenerate and $h_u(s) = \int_0^s \frac{r}{2} u^2(r)\mathrm{d}r$.

The study of equation (1.1) is mainly motivated by the Chern-Simons-Schrödinger system introduced in [24, 25]

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi + \varrho(\phi) = 0, \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}(\overline{\phi} D_2 \phi), \\ \partial_0 A_2 - \partial_2 A_0 = \mathrm{Im}(\overline{\phi} D_1 \phi), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2. \end{cases} \tag{1.2}$$

This system consists of the nonlinear Schrödinger equation augmented by the gauge field $A_j : \mathbb{R}^{1+2} \to \mathbb{R}$, where i denotes the imaginary unit, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ represents the complex scalar field and $\partial_0 = \partial/\partial t$, $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$ as well as $D_j = \partial_j i A_j$ stands for the covariant derivative for $j = 0, 1, 2$. Given a $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{1+2})$, system (1.2) is invariant under the gauge transformation

$$\phi \to \phi e^{i\chi}, \ A_j \to A_j - \partial_j \chi,$$

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due to the celebrated Chern-Simons theory [16].

As a matter of fact, there are a lot of interesting applications of this model in the high-temperature superconductor, Aharovnov-Bohm scattering and quantum Hall effect, see e.g. [24–26] in detail. For some other further physical motivations on (1.2), we refer the reader to [16,21,35,36] and the references therein. To explore the existence of standing waves of system (1.2), the ansatz $\phi(t,x) = u(x)\exp(-i\omega t)$ with $\omega \in \mathbb{R}$ and $x \in \mathbb{R}^2$ is usually considered, where u is radially symmetric or not.

If u is radially symmetric, Byeon, Huh and Seok [8] investigated the existence of solutions of type

$$A_0(t,x) = k(|x|),\ A_1(t,x) = \frac{x_2}{|x|^2}h(|x|),\ A_2(t,x) = \frac{x_1}{|x|^2}h(|x|), \tag{1.3}$$

where $\omega > 0$ symbols as the frequency and u, k, h are real value functions depending only on $|x|$. Note that (1.3) satisfies the Coulomb gauge condition with $\chi = ct + n\pi$, where n is an integer and c is a real constant. Indeed, inserting (1.3) into (1.2), it can be reduced to the following semilinear elliptic equation

$$-\Delta u + (\omega+\zeta)u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2}\right) u = \varrho(u) \text{ in } \mathbb{R}^2, \tag{1.4}$$

where $\varrho(u) = \bar{\lambda}|u|^{p-2}u$ with $\bar{\lambda} > 0$, $h(s) = \int_0^s \frac{r}{2}u^2(r)\mathrm{d}r$, and $\zeta \in \mathbb{R}$ stands for an integration constant of A_0 which takes the form

$$A_0(r) = \zeta + \int_r^{\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s.$$

Since the constant $\omega + \zeta$ is a gauge invariant of the stationary solutions, one might take $\zeta = 0$ in (1.4) for simplicity in what follows and thereby $\lim\limits_{|x|\to\infty} A_0(x) = 0$ which was assumed in [6,24,41]. If $\bar{u}$ solves (1.4), inspired by [14], $u = \lambda^{\frac{1}{p-2}}\bar{u}$ satisfies

$$-\Delta u + \omega u + \lambda\left(\int_{|x|}^{\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2}\right) u = |u|^{p-2}u \text{ in } \mathbb{R}^2, \tag{1.5}$$

where $\lambda = \bar{\lambda}^{-\frac{4}{p-2}}$. Over the past several decades, Eq. (1.5) and its variants have attracted considerable attentions due to the appearance of the nonlocal Chern-Simons term

$$\int_{|x|}^{\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2}, \tag{1.6}$$

which indicates that it is not a pointwise identity any longer. In [8], Byeon *et al.* obtained the existence of ground state solutions for all $p > 4$ by means of a suitable constraint minimization method, existence and nonexistence of nontrivial solutions depending on $\lambda > 0$ for $p = 4$, and the existence of minimizers under L^2-constraint for all $p \in (2,4)$. In the meanwhile, Pomponio and Ruiz [41] concluded that there is a sharp constant $\omega_0 > 0$ such that the corresponding variational functional to Eq. (1.5) is bounded from below if $\omega \geq \omega_0$ and not bounded from below for every $\omega \in (0,\omega_0)$ with $p \in (2,4)$. Concerning a more general nonlinearity in (1.5), namely replacing $|u|^{p-2}u$ with $f(u)$, the authors in [11] investigated the multiplicity results when f satisfies the planar version of Berestycki-Lions type assumptions. As a matter of fact, they especially supposed that

- $(\bar{f}_1)$ $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ is an odd function;
- $(\bar{f}_2)$ $\limsup\limits_{s\to\infty} \dfrac{f(s)}{e^{\alpha s^2}} \leq 0$ for all $\alpha > 0$;
- $(\bar{f}_3)$ $-\infty < \liminf\limits_{s\to\infty} \dfrac{f(s)}{s} \leq \limsup\limits_{s\to\infty} \dfrac{f(s)}{s} < 0$;
- $(\bar{f}_4)$ There exists a $\zeta_0 > 0$ such that $F(\zeta_0) > 0$, where and in the sequel $F(s) = \displaystyle\int_0^s f(t)\mathrm{d}t$.

With aid of a particular truncation argument, the authors in [11] obtained the multiplicity of nontrivial solutions by using the Symmetric mountain-pass theorem. We remark that the assumption $(\bar{f}_2)$ reveals that the nonlinearity f possesses the subcritical exponential growth at infinity in the Trudinger-Moser sense. There are some other results on (1.5), we refer the reader to [4,14,20,23,27,29,32,37,44–47,51,52,59–62] and the references therein for example even if these references are far to be exhaustive.

If u is non-radially symmetric, instead, it seems more complex to contemplate (1.2) to some extent. Generally speaking, mathematicians usually consider the case $A_j(t,x) = A_j(x)$ for all $(t,x_1,x_2) \in \mathbb{R}^{1+2}$ and $j = 0,1,2$ for simplicity. Owing to this, Eq. (1.2) is reduced to be the following form of type

$$\begin{cases} -\Delta u + \omega u + A_0 u + \displaystyle\sum_{j=1}^{2} A_j^2 u = f(x,u), \\ \partial_1 A_2 - \partial_2 A_1 = -\dfrac{1}{2}|u|^2, \\ \partial_1 A_0 = A_2|u|^2, \ \partial_2 A_0 = -A_1|u|^2. \end{cases} \tag{1.7}$$

Let A_j satisfy the Coulomb gauge condition $\sum_{j=0}^{2} \partial_j A_j = 0$, then (1.7) becomes

$$\begin{cases} -\Delta u + \omega u + A_0 u + \displaystyle\sum_{j=1}^{2} A_j^2 u = f(x,u), \\ \partial_1 A_0 = A_2|u|^2, \ \partial_2 A_0 = -A_1|u|^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\dfrac{1}{2}|u|^2, \ \partial_1 A_1 + \partial_2 A_2 = 0. \end{cases} \tag{1.8}$$

Combining $\partial_1 A_0 = A_2|u|^2$ and $\partial_2 A_0 = -A_1|u|^2$ in (1.8), one has that

$$\Delta A_0 = \partial_1\big(A_2|u|^2\big) - \partial_2\big(A_1|u|^2\big),$$

leading to

$$A_0[u](x) = \frac{x_1}{2\pi|x|^2} * \big(A_2|u|^2\big) - \frac{x_2}{2\pi|x|^2} * \big(A_1|u|^2\big).$$

It follows from $\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2$ and $\partial_1 A_1 + \partial_2 A_2 = 0$ in (1.8) to derive

$$\Delta A_1 = \partial_2\left(\frac{|u|^2}{2}\right) \text{ and } \Delta A_2 = -\partial_1\left(\frac{|u|^2}{2}\right).$$

As a consequence, one might observe that the components A_j for $j = 1,2$ in (1.8) can be represented as

$$\begin{cases} A_1[u](x) = \dfrac{x_2}{2\pi|x|^2} * \left(\dfrac{|u|^2}{2}\right) = -\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^2} \frac{(x_2-y_2)u^2(y)}{|x-y|^2}\mathrm{d}y, \\ A_2[u](x) = -\dfrac{x_1}{2\pi|x|^2} * \left(\dfrac{|u|^2}{2}\right) = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^2} \frac{(x_1-y_1)u^2(y)}{|x-y|^2}\mathrm{d}y. \end{cases} \tag{1.9}$$

With (1.9) in hands, then we are able to deal with the system (1.8) using variational methods and so it has received many attentions which can be found in [10,19,21,22,30,49,50,58] the references therein.

According to our best knowledge, it seems that the first attempt on system (1.8) in $\mathbb{R}^2$ is due to [22], where the existence of infinitely many solutions was established for $f(x,u) = |u|^{p-2}u$ with $p > 6$. After that, with the help of the constraint minimization method, Wan and Tan [54] concluded a ground state solution of Nehari-Pohožaev type for $f(x,u) = |u|^{p-2}u$ with $4 < p \le 6$. What's more, Gou and Zhang in [19] considered the case $f(x,u) = |u|^{p-2}u$ with $p > 2$ for the existence of L^2-normalized solutions to system (1.8). In reality, there exist also a lot of other interesting generalizations and improvements related to the variants of system (1.8) including a non-constant potential. Explaining it more clearly, replacing ω with a potential $V : \mathbb{R}^2 \to \mathbb{R}$. For example, if V is positive and satisfies the Rabinowitz type condition, the authors in [55] established the existence of semi-classical solutions to this system when $f(x,u) = |u|^{p-2}u$ with $p > 6$, see e.g. [10] and the references therein.

Very recently, Pomponio, Shen, Zeng and Zhang [42] especially obtained the existence and multiplicity of nontrivial solution for system (1.8) with an indefinite potential. As a matter of fact, they studied particularly the following system

$$\begin{cases} -\Delta u + [V(x) - \omega]\, u + A_0 u + \sum_{j=1}^{2} A_j^2 u = f(x,u), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \ \partial_1 A_1 + \partial_2 A_2 = 0, \ A_1 \partial_1 u + A_2 \partial_2 u = 0, \\ \partial_1 A_0 = A_2 |u|^2, \ \partial_2 A_0 = -A_1 |u|^2, \end{cases} \tag{1.10}$$

where the assumptions on the potential V and the nonlinearity f are supposed to satisfy

(V) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ with $V(x) \geq 0$ on $\mathbb{R}^2$ and $\liminf_{|x| \to \infty} V(x) = +\infty$.

$(\tilde{f}_1)$ $f(x,t) \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R})$ and there exist two constants $C_1 > 0$ and $p > 6$ such that

$$|f(x,t)| \leq C_1(|t| + |t|^{p-1}), \ \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R};$$

$(\tilde{f}_2)$ $f(x,t)t \geq 6F(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ and

$$\lim_{|t| \to +\infty} \frac{f(x,t)t - 6F(x,t)}{t^6} = +\infty \text{ uniformly in } x \in \mathbb{R}^2. \tag{1.11}$$

They proved the following existence results.

Theorem 1.1. *Suppose* (V) *and* $(\tilde{f}_1) - (\tilde{f}_2)$,

(i) *if* $f(x,t) = o(t)$ *as* $t \to 0^+$ *uniformly in* $x \in \mathbb{R}^2$ *in addition, then system* (1.10) *has a nontrivial solution;*

(ii) *if* $f(x,-t) = -f(x,t)$ *for all* $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ *in addition, then system* (1.10) *has infinitely many nontrivial solutions whose energies converge to* $+\infty$.

We would like to mention here that the existence result exhibited in Theorem 1.1-(i) has also been explored in [28, 30], where the nonlinearity f satisfies several slightly different conditions. Nevertheless, it should be emphasized that these articles including [42] strongly depend on the local linking argument because of the Chern-Simons term. Unfortunately, as pointed out in [28], the corresponding variational functional is required to satisfy the global compactness condition whenever the critical point theorem involving local linking structure is applied up to now. As a consequence, there seems no related results for the case that f admits the *critical*, or even *supercritical exponential growth* which shall be explained later. Last but not the least, the assumption $(\tilde{f}_2)$ and its mild modification play crucial role in verifying that every (C) sequence is uniformly bounded. In a word, we shall try our best to introduce some new techniques to consider the mentioned issues above in this paper.

The reader is invited to observe that the spatial dimension of (1.5) and (1.8), is two, therefore the case is special and quite delicate. Since the Sobolev embedding theorem ensures $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ with $q \in [1, \infty)$ for every bounded domain $\Omega \subset \mathbb{R}^2$, but $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$, to get rid of the obstacle in the limiting case, the Trudinger-Moser inequality [39, 40, 53] seems to be an ideal candidate as it exhibits the sharp maximal exponential integrability for functions in $H_0^1(\Omega)$:

$$\sup_{u \in H_0^1(\Omega): \|\nabla u\|_{L^2(\Omega)} \leq 1} \int_\Omega e^{\alpha u^2} \mathrm{d}x \leq C|\Omega|, \quad \text{if } \alpha \leq 4\pi, \tag{1.12}$$

where the constant $C > 0$ relies only on α, and $|\Omega|$ denotes the Lebesgue measure of Ω. Subsequently, this inequality was improved by P. L. Lions in [34]: Let (u_n) be a sequence of functions in $H_0^1(\Omega)$ with $\|\nabla u_n\|_{L^2(\Omega)} = 1$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, then for all $p < \frac{1}{(1 - \|\nabla u_0\|_{L^2(\Omega)}^2)}$, it holds that

$$\limsup_{n \to \infty} \int_\Omega e^{4\pi p u_n^2} \mathrm{d}x < +\infty.$$

Motivated by the Trudinger-Moser type inequality, we say that a function $f(\cdot,s)$ has *critical exponential growth* at infinity in the Trudinger-Moser sense if there is a constant $\alpha_0>0$ such that

$$\lim_{|s|\to+\infty}\frac{|f(x,s)|}{e^{\alpha s^2}}=\begin{cases}0, & \forall\alpha>\alpha_0,\\ +\infty, & \forall\alpha<\alpha_0,\end{cases}\ \text{uniformly in } x\in\mathbb{R}^2. \tag{1.13}$$

This definition was introduced by Adimurthi and Yadava [1], see also de Figueiredo, Miyagaki and Ruf [17] for example.

Whereas, the supremum in (1.12) would become infinite for the domain Ω with $|\Omega|=\infty$ and thereby the Trudinger-Moser inequality seems unavailable for the unbounded domains. As to the whole space $\mathbb{R}^2$, authors in [7,9] developed the following Trudinger-Moser inequality:

$$\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)\mathrm{d}x<+\infty,\ \forall\alpha>0 \text{ and } u\in H^1(\mathbb{R}^2).$$

Moreover, for all $u\in H^1(\mathbb{R}^2)$ with $\|u\|_{L^2(\mathbb{R}^2)}\le M<+\infty$, there is a $C=C(M,\alpha)>0$ such that

$$\sup_{u\in H^1(\mathbb{R}^2):\|\nabla u\|_{L^2(\mathbb{R}^2)}\le1}\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)\mathrm{d}x\le C \text{ if } \alpha<4\pi. \tag{1.14}$$

Concerning some other generalizations, extensions and applications of the Trudinger-Moser inequalities for bounded and unbounded domains, we refer to [17] and the references therein. Let us note here that this inequality due to Cao [9] keeps effective for $\alpha<4\pi$, i.e. with subcritical growth. For the sharp case, based on symmetrization and blow-up analysis, Ruf [43], Li and Ruf [33] proved that

$$\sup_{u\in W_0^{1,N}(\mathbb{R}^N):\|u\|_{L^N}^N+\|\nabla u\|_{L^N}^N\le1}\int_{\mathbb{R}^N}\left(e^{\alpha|u|^{\frac{N}{N-1}}}-\sum_{k=0}^{N-2}\frac{\alpha^k|u|^{kN/(N-1)}}{k!}\right)\mathrm{d}x<\infty,\ \text{if } \alpha\le\alpha_N,$$

by replacing the L^N-norm of ∇u in the supremum with the standard Sobolev norm. This inequality was also generalized by de Souza and do Ó [13] for $N=2$.

Associated with the so-called *supercritical exponential growth* at infinity in the Trudinger-Moser sense on a nonlinearity, there exist some diverse understandings in several directions, see [2,3,12,18,38,46] and their references therein for example. In this article, we shall retrace the previous papers [2,3,46]. Explaining it more explicitly, denoted $G(x,\cdot)$ by the primitive of $g(x,\cdot)$ for all $x\in\mathbb{R}^2$ throughout the whole paper, we are supposing that

$$G(x,t)=F(x,t)e^{\sigma t^2},\ \forall(x,t)\in\mathbb{R}^2\times\mathbb{R}, \tag{1.15}$$

where and in the sequel $F(x,t)=\int_0^t f(x,s)\mathrm{d}s$ with $f:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ satisfying the critical exponential growth at infinity in (1.13) and $\sigma>0$. Obviously, we realize that the nonlinearity g has the supercritical growth at infinity. As a matter of fact, this seems introduced firstly by Alves and Shen in [2,3]. According to them, taking into account a suitable function $\bar g(x,t)=e^{\alpha|t|^\tau}$ for all $(x,t)\times\mathbb{R}^2\times\mathbb{R}$, our nonlinearity g satisfying the supercritical exponential growth belongs to a special case of the following two alternatives

$$\begin{cases}\text{(I) } \tau>2 \text{ is arbitrary and } \alpha>0 \text{ is fixed;}\\ \text{(II) } \alpha>0 \text{ is arbitrary and } \tau\ge2 \text{ is fixed.}\end{cases} \tag{1.16}$$

What's more, as pointed out in [3, Remark 1.11], one can call (I) and (II) the ***subcritical-supercritical exponential growth*** and ***critical-supercritical exponential growth***, respectively. As we can derive from [2,3], the Case (I) keeps simpler to handle and so, throughout this paper, we shall always consider the Case (II) if it refers to the supercritical exponential growth. As a consequence, one of main purposes in this paper is to investigate the existence of nontrivial solutions for the gauged Schrödinger equations with supercritical exponential growth. To this end, we are going to suppose that $f:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying $f(x,t)\equiv0$ for all $t\le0$ and the following assumptions

(f_1) $f(x,t)=o(t)$ as $t\to 0^+$ uniformly in $x\in\mathbb{R}^2$;

(f_2) There is a $\theta>2$ such that $f(x,t)t\geq \theta F(x,t)\geq 0$ for all $(x,t)\in\mathbb{R}^2\times\mathbb{R}$ and

$$\lim_{|t|\to+\infty}\frac{F(x,t)}{t^2}=+\infty \text{ uniformly in } x\in\mathbb{R}^2. \tag{1.17}$$

(f_3) There exist some constants $\hat{t}_0>0$ and $M_0>0$ such that

$$0<F(x,t)\leq M_0 f(x,t),\ \forall (x,t)\in\mathbb{R}^2\times[\hat{t}_0,+\infty);$$

(f_4) There exist some constants $\beta_0>0$ and $\vartheta\in[0,1]$ such that

$$\liminf_{t\to+\infty}\frac{t^{\vartheta}f(x,t)}{e^{\alpha_0 t^2}}\geq \beta_0\begin{cases}>0, & \text{if } \vartheta\in[0,1),\\ =+\infty, & \text{if } \vartheta=1,\end{cases}\quad \text{uniformly in } x\in\mathbb{R}^2.$$

Our first main result is concerned with the existence of a nontrivial solution for Eq. (1.1).

Theorem 1.2. *Let g satisfy (1.15) with (1.13) and suppose $(f_1)-(f_4)$. Then, given some large $\omega>0$ such that the operator $-\Delta+V_\omega$ is non-degenerate, there exist some $\sigma_*>0$ and $\lambda_*>0$ such that Eq. (1.1) admits at least a nontrivial solution for all $\sigma\in(0,\sigma_*)$ and $\lambda\in(0,\lambda_*)$.*

Remark 1.3. As far as we know, there seems no related results for gauged Schrödinger equations with indefinite potential and supercritical exponential growth. As a matter of fact, apart from [46,48,51], A very few attempts have been made currently to adapt the supercritical exponential growth to the Chern-Simons-Schrödinger equations. On the one hand, the truncation argument exploited in [2,3,46,51] is unapplicable any longer because it would fail to demonstrate the linking structures of the corresponding variational functional which also leads to the failure of the aforementioned approach utilized in [11]. On the other hand, because of the indefinite potential, we are unable to repeat the methods introduced in [48] to arrive at the proof of Theorem 1.2, either. Regardless of taking the indefinite potential into account, we make use of a weak version of Ambrosetti-Rabinowitz type condition in (f_2) above, instead of $\vartheta=6$ in $(\tilde{f}_2)$ in [42]. Consequently, some subtle ideas have been proposed in this article to certificate that every (C) sequence is uniformly bounded and contains a strongly convergent subsequence. What's more, in order to make the L^∞-estimate for the obtained solution, we shall have to carry on some more delicate and careful analyses caused by the indefinite feature in our problem.

Remark 1.4. In contrast to [28,30,42], to the best knowledge of us, the consideration of Theorem 1.2 provides some conspicuously interesting contributions below

(1) We succeed in deriving the existence of a nontrivial solution of the indefinite gauged Schrödinger equation with (super)critical exponential growth which has not been studied yet in the literature.

(2) Let us date back to the conditions $(\tilde{f}_2)$ and (f_2), hence it has been relaxed the restriction on f to a large extent. Nevertheless, one can never improve $(\tilde{f}_2)$ completely by (f_2) since it seems impossible to suppose in our problem that $\vartheta=2$ and

$$\lim_{|t|\to+\infty}\frac{f(x,t)t-2F(x,t)}{t^2}=+\infty \text{ uniformly in } x\in\mathbb{R}^2.$$

(3) Despite disposing of the indefinite case, we are capable of imposing the "almost optimal" growth condition (f_4) on the nonlinearity to restore the compactness.

The next goal in this article is to investigate the multiplicity results for Eq. (1.1) and it should be regarded as a supplement of Theorem 1.1-(ii) to some extent.

Theorem 1.5. *Let $g(x,t)=\xi(x)|t|^{p-2}t$ with $0\leq\xi\in L^{\frac{2}{2-p}}(\mathbb{R}^2)$ and $1<p<2$ for all $(x,t)\in\mathbb{R}^2\times\mathbb{R}$. Then, given some large $\omega>0$ such that the operator $-\Delta+V_\omega$ is non-degenerate, there exist a constants $\lambda^*>0$ such that for all $\lambda\in(0,\lambda^*)$, Eq. (1.1) admits infinitely many nontrivial solutions (u_m) whose energies converge to 0 as $m\to+\infty$.*

As a matter of fact, by making some straightforward adjustments, we are capable of taking advantage of the methods adopted in Theorem 1.5 to conclude the following result without detailed proof.

Corollary 1.6. *Let $\sigma \equiv 0$ in (1.15). Suppose $|f(x,t)| \le \xi(x)|t|^{p-1}$ with $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$ and $1 < p < 2$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ as well as*

(f_0) *there is a constant $1 < \nu < 2$ such that*

$$0 \le f(x,t)t \le \nu F(x,t) \text{ for all } (x,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

If $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ in addition, then, for all large $\omega > 0$ such that the operator $-\Delta + V_\omega$ is non-degenerate, there exist a constant $\hat{\lambda}^* > 0$ such that for all $\lambda \in (0, \hat{\lambda}^*)$, Eq. (1.1) has infinitely many nontrivial solutions (u_m) whose energies converge to 0 as $m \to +\infty$.

Remark 1.7. We know that the condition (f_0) indicates that the nonlinearity f possesses the *subcritical exponential growth* at infinity in the Trudinger-Moser sense and it has been supposed in [28, 30]. Since we depend on a new generalized fountain theorem developed by Ding and Dong [15] to derive the proof of Theorem 1.5, it is extremely required to verify the corresponding variational functional satisfies the global compactness condition and thus we are unable to establish the existence of infinitely many nontrivial solutions for the indefinite gauged Schrödinger equations with (super)critical exponential growth at present. Alternatively, what we would like to point out here is that this is solvable when $\omega \le 0$, or $\omega > 0$ is sufficiently small, see [48, Theorem 1.3] in detail. In consideration of the significant differences, there are some additional efforts to arrive at the proof of Theorem 1.5.

Remark 1.8. As the reader might observe that the results in Theorem 1.2 and Theorem 1.5 are true provided that the parameter $\lambda > 0$ is small enough. It seems very natural to wonder that whether these results would remain valid for all arbitrary $\lambda > 0$ or not, and we are working hard in this direction. In fact, there are also some other interesting questions worth further explorations which shall be collected as follows

- Can we replace the potential V_ω with a more general one? In other words, if V_ω is continuous and periodic as well as $-\Delta + V_\omega$ is inevitable, does Eq. (1.1) admit a nontrivial solution?
- Can we demonstrate some similar existence and multiplicity results concluded in Theorem 1.2 and Theorem 1.5 for the Chern-Simons-Schrödinger system like (1.10)?
- Can we establish the multiplicity results in Theorem 1.5 when $\xi(x) \equiv 1$ and $2 < p < +\infty$?
- What happens when the sufficiently large $\omega > 0$ satisfies the operator $-\Delta + V_\omega$ to be generate in Theorem 1.2 and Theorem 1.5?

We note that, up to our best knowledge, it is the first time to deal with the existence and multiplicity results for a class of gauged nonlinear Schrödinger equations with indefinite potential and less restrictive nonlinearities. It seems standard to consider the indefinite problem via using linking argument by now, but we would like to highlight here that there are two fundamental difficulties arising in Theorems 1.2 and 1.5. On the one hand, because of the appearances of the Chern-Simons term and the supercritical exponential case in Theorem 1.2, it seems technical to introduce a suitable truncation argument to ensure that the corresponding variational functional is well-defined and of class $\mathcal{C}^1$ and it possesses the global linking structures at the same time. Motivated by [48], there is a such one but we are confronted with the barrier of how to conclude that every (C) sequence is uniformly bounded. On the other hand, in light of the operator $-\Delta + V_\omega$ is non-degenerate, we cannot adapt directly the L^∞-estimate explored in [48] to our problems. Therefore, we prefer to regard it as one of the most striking novelties in this paper.

Again the results established in Theorem 1.2 and Theorem 1.5 are new in some sense that we discuss the existence and multiplicity results for indefinite gauged nonlinear Schrödinger equation with a wider class of nonlinearities. We anticipate that our results would prompt some extensive applications on related topics.

The outline of this article is organized as follows. In Section 2, some preliminary results including the truncation argument are provided and will be used frequently in the whole paper. Sections 3 and 4 are focused on the proofs of Theorem 1.2 and Theorem 1.5, respectively.

Notations: From now on in this paper, otherwise mentioned, we ultilize the following notations:

- $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \cdots$ denote any positive constant, whose value is not relevant and $\mathbb{R}^+ \triangleq (0, +\infty)$.
- Let $(Z, \|\cdot\|_Z)$ be a Banach space with dual space $(Z^{-1}, \|\cdot\|_{Z^{-1}})$, and Ψ be functional on Z.
- The Cerami sequence at a level $c \in \mathbb{R}$ ($(C)_c$ sequence in short) corresponding to Φ means that $\Phi(x_n) \to c$ and $(1 + \|x_n\|_Z)\|\Phi'(x_n)\|_{Z^{-1}} \to 0$ as $n \to \infty$, where $\{x_n\} \subset Z$.
- For any $\varrho > 0$ and every $x \in \mathbb{R}^2$, $B_\varrho(x) \triangleq \{y \in \mathbb{R}^2 : |y - x| < \varrho\}$.
- $|\cdot|_p$ denotes the usual norm of the Lebesgue space Ω, for every $p \in [1, \infty]$, where $\Omega \subset \mathbb{R}^2$.
- $H^1(\mathbb{R}^2)$ represents the usual Sobolev space equipped with the standard norm so that it is the completion of $C_0^\infty(\mathbb{R}^2)$.
- Let $H_r^1(\mathbb{R}^2) = \{u \in H^1(\mathbb{R}^2) : u(x) = u(|x|)\}$ and its norm is labeled by $\|\cdot\| = \sqrt{|\nabla \cdot|_2^2 + |\cdot|_2^2}$.
- $o_k(1)$ denotes the real sequences by $o_k(1) \to 0$ as $k \to +\infty$.
- " $\to$ " and " $\rightharpoonup$ " stand for the strong and weak convergence in the related function spaces, respectively.

2. Variational Framework and Preliminaries

In this section, on the one hand, we shall formulate the variational structures for our problems and on the other hand, there rare some preliminary results which shall be crucial in the next sections.

To look for nontrivial solutions associated with Eq. (1.1), due to a variational method point of view, it will be found critical points for the corresponding variational functional. So, as a start, we need to introduce the work space adopted in the whole paper. Let us define

$$X \triangleq \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x < +\infty \text{ and } \int_{\mathbb{R}^2} |x|^2 |u|^2 \mathrm{d}x < +\infty \right\}$$

and it is an Hilbert space equipped with the norm

$$\|\cdot\|_X = \left[\int_{\mathbb{R}^2} \left(|\nabla \cdot|^2 + |x|^2 |\cdot|^2 \right) \mathrm{d}x \right]^{\frac{1}{2}}.$$

It is widely known that X_r can be continuously imbedded into $H_r^1(\mathbb{R}^2)$ and compactly imbedded into $L^q(\mathbb{R}^2)$ for all $2 \le q < +\infty$, where and in the sequel $X_r = \{u \in X : u(x) = u(|x|)\}$.

According to the compactness of the imbedding $X_r \hookrightarrow L^2(\mathbb{R}^2)$, by virtute of the spectral theory of self-adjoint compact operators, one sees that the eigenvalue problem

$$-\Delta u + |x|^2 u = \hat{\mu} u, \ u \in X_r, \tag{2.1}$$

admits a complete sequence of eigenvalues

$$0 < \hat{\mu}_1 \le \hat{\mu}_2 \le \cdots, \ \hat{\mu}_j \to +\infty \text{ as } j \to +\infty,$$

where each $\hat{\mu}_j$ has been repeated in the sequence due to its finite multiplicity. Moreover, we denote by ϕ_j the eigenfunction of $\hat{\mu}_j$. Taking $\hat{\mu}_j \to +\infty$ as $j \to +\infty$ into account, for all $\omega > 0$, there is a $j_0 \in \mathbb{N}^+$ such that $0 < \hat{\mu}_1 \le \hat{\mu}_2 \le \cdots \le \hat{\mu}_{j_0} < \omega < \hat{\mu}_{j_0+1} \le \cdots$ since $\omega > 0$ satisfies the operator $-\Delta + V_\omega(|x|)$ is non-degenerate. Setting

$$X_r^- = \mathrm{span}\{\phi_1, \phi_2, \cdots \phi_{j_0}\} \text{ and } X_r^+ \triangleq \left(X_r^-\right)^\perp$$

and so $X_r = X_r^- \bigoplus X_r^+$. Accordingly, there are some constants $\mu^\pm > 0$ such that

$$\pm \int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\omega(|x|) u^2 \right] \mathrm{d}x \ge \mu_\pm \|u\|_X^2, \ \forall u \in X_r^\pm. \tag{2.2}$$

Now, we recall the related Euler-Lagrange functional $J : X_r \to \mathbb{R}$ of Eq. (1.1) which is defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\omega(|x|)u^2\right] \mathrm{d}x + \frac{\lambda}{2}N(u) - \int_{\mathbb{R}^2} G(x,u)\mathrm{d}x,$$

where the functional $N : H_r^1(\mathbb{R}^2) \to \mathbb{R}$ is given by

$$N(u) = \int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}u^2(s)\mathrm{d}s\right)^2 \mathrm{d}x. \tag{2.3}$$

In view of [8, Proposition 2.2], we can deduce that N is of class $\mathcal{C}^1(H_r^1(\mathbb{R}^2),\mathbb{R})$. Unfortunately, it seems impossible to verify that J belongs to $\mathcal{C}^1(X_r,\mathbb{R})$ because of the nonlinearity G involving supercritical exponential growth. In spirit of [48], we are going to take advantage of the subtle truncation argument to overcome this difficulty. Explaining it more clearly, for each $n \in \mathbb{N}^+$, let us first define

$$\eta_n(t) = \eta\left(\frac{t}{n}\right),\ \mathfrak{F}_n(t) = t^2\eta_n(t) \text{ and } \mathfrak{F}_n'(t) = \mathfrak{f}_n(t),\ \forall t \in \mathbb{R}, \tag{2.4}$$

where $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ denotes an even function with $0 \le \eta \le 1$ and satisfies

$$\eta(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & |t| \ge 2, \end{cases} \text{ with } |\eta'(t)| \le 2,\ \forall t \in \mathbb{R}.$$

With η_n in hands, we are able to replace G and g in Eq. (1.1) with

$$G_n(x,t) \triangleq F(x,t)e^{\sigma\mathfrak{F}_n(t)},\ \forall t \in \mathbb{R} \tag{2.5}$$

and $G_n(x,t) = \int_0^t g_n(x,s)\mathrm{d}s$, respectively. In the meanwhile, in light of the linking structures, we set

$$h_{u,n}(s) = \int_0^s \frac{r}{2}\mathfrak{F}_n(u(r))\mathrm{d}r,\ \forall u \in H_r^1(\mathbb{R}^2).$$

At this stage, we shall contemplate the following auxiliary semilinear elliptic equation

$$-\Delta u + V_\omega(|x|)u + \lambda\left(\int_{|x|}^\infty \frac{h_{u,n}(s)}{s}\mathfrak{F}_n(u(s))\mathrm{d}s + \frac{h_{u,n}^2(|x|)}{|x|^2}\right)\frac{\mathfrak{f}_n(u)}{2} = g_n(x,u) \text{ in } \mathbb{R}^2, \tag{2.6}$$

As we shall conclude later, Eq. (2.6) possesses a variational structure and, for all $n \in \mathbb{N}^+$, its variational functional $J_n : X_r \to \mathbb{R}$ is defined by

$$J_n(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\omega(|x|)u^2\right]\mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^2} \frac{\mathfrak{F}_n(u)}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}\mathfrak{F}_n\left(u(s)\right)\mathrm{d}s\right)^2 \mathrm{d}x - \int_{\mathbb{R}^2} G_n(x,u)\mathrm{d}x.$$

In what follows, we will also certify that J_n is not only well-defined, but also belongs to $\mathcal{C}^1(X_r,\mathbb{R})$. As a consequence, each critical point of J_n is in fact a (weak) solution of Eq. (2.6). Moreover, according to the definition of η_n in (2.4), every nontrivial critical point, saying it u, of J_n satisfying $|u|_\infty < n$, is a nontrivial solution of Eq. (1.1).

Next, we mainly focus on the two modified terms above that permit us to treat Eq. (2.6) variationally. To begin with, motivated by [48], it is simple to derive that there are the following facts

$$0 \le \mathfrak{F}_n(t) \le 4n^2 \text{ and } |\mathfrak{f}_n(t)| \le 12n,\ \forall n \in \mathbb{N}^+ \text{ and } t \in \mathbb{R}, \tag{2.7}$$

and

$$0 \le \mathfrak{F}_n(t) \le t^2 \text{ and } |\mathfrak{f}_n(t)| \le 6|t|,\ \forall n \in \mathbb{N}^+ \text{ and } t \in \mathbb{R}. \tag{2.8}$$

With aid of (2.7) and (2.8), there is a constant $\mathcal{C}_\theta > 0$ which is only dependent of θ in (f_2) that

$$|\theta\mathfrak{F}_n(t) - 2\mathfrak{f}_n(t)t| \le \mathcal{C}_\theta t^2,\ \forall n \in \mathbb{N}^+ \text{ and } t \in \mathbb{R}. \tag{2.9}$$

On the one hand, we are concerned with the modified Chern-Simons term $N_n : H_r^1(\mathbb{R}^2) \to \mathbb{R}$ for all $n \in \mathbb{N}^+$ below

$$N_n(u) \triangleq \int_{\mathbb{R}^2} \frac{\mathfrak{F}_n(u)}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} \mathfrak{F}_n(u(s))\,\mathrm{d}s \right)^2 \mathrm{d}x.$$

Lemma 2.1. *For all $n \in \mathbb{N}^+$, we have the following conclusions:*

(i) *$N_n(u) \le N(u)$ for all $u \in H_r^1(\mathbb{R}^2)$ and thus N_n is well-defined. Besides, N_n is of class $\mathcal{C}^1$ and, for all $\psi \in H_r^1(\mathbb{R}^2)$, its derivative is characterized as*

$$\begin{aligned} N_n'(u)(\psi) = 2 \int_{\mathbb{R}^2} \frac{\mathfrak{F}_n(u)}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} \mathfrak{F}_n(u(s))\,\mathrm{d}s \right) \left(\int_0^{|x|} \frac{s}{2} \mathfrak{f}_n(u(s))\,\psi(s)\mathrm{d}s \right) \mathrm{d}x \\ + \int_{\mathbb{R}^2} \frac{\mathfrak{f}_n(u)\psi}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} \mathfrak{F}_n(u(s))\,\mathrm{d}s \right)^2 \mathrm{d}x. \end{aligned}$$

(ii) *Suppose $u_k \rightharpoonup u$ in $H_r^1(\mathbb{R}^2)$ as $k \to \infty$, then, going to some subsequences if necessary, for all $n \in \mathbb{N}^+$ and $\psi \in H_r^1(\mathbb{R}^2)$,*

$$N_n(u_k) \to N_n(u),\ N_n'(u_k)(u_k) \to N_n'(u)(u) \text{ and } N_n'(u_k)(\psi) \to N_n'(u)(\psi).$$

(iii) *There exists a constant $\mathbb{T}_2 > 0$, which is only dependent of the imbedding constant of $X \hookrightarrow L^2(\mathbb{R}^2)$, such that $N_n(u) \le \mathbb{T}_2 n^4 \|u\|_X^2$ and $|N_n'(u)(u)| \le \mathbb{T}_2 n^4 \|u\|_X^2$.*

(iv) *Let $\mathbb{T}_2 > 0$ above be sufficiently large if necessary, then $|\theta N_n(u) - 2N_n'(u)(u)| \le \mathbb{T}_2 n^4 \|u\|_X^2$.*

Proof. We shall omit the details and the reader can refer to [48, Lemmas 2.2, 2.4 and 2.5]. □

On the other hand, let us focus on the modified nonlinearities g_n and G_n for all $n \in \mathbb{N}^+$.

Lemma 2.2. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1) - (f_4)$. Then, for all $n \in \mathbb{N}^+$, we have the following conclusions:*

(g_1) *$g_n(x,t) = o(t)$ as $t \to 0^+$ uniformly in $x \in \mathbb{R}^2$ and $n \in \mathbb{N}^+$;*

(g_2) *There is a $\sigma_1 > 0$ such that for all $\sigma \in (0, \sigma_1)$, then it holds that $g_n(x,t)t - \theta G_n(x,t) \ge -\frac{\theta-2}{8\mathbb{T}_2^2} t^2$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$. Moreover, for all $\sigma > 0$,*

$$\lim_{|t| \to +\infty} \frac{G_n(x,t)}{t^2} = +\infty \text{ uniformly in } x \in \mathbb{R}^2 \text{ and } n \in \mathbb{N}^+. \tag{2.10}$$

(g_3) *There is a $\sigma_2 > 0$ such that for all $\sigma \in (0, \sigma_2)$, we have for $\hat{t}_0 > 0$ and $M_0 > 0$ in (f_3),*

$$0 < G_n(x,t) \le 2M_0 g_n(x,t),\ \forall (x,t) \in \mathbb{R}^2 \times [\hat{t}_0, +\infty) \text{ and } n \in \mathbb{N}^+;$$

(g_4) *Given a $\sigma \in (0, \sigma_2)$, then for all $\beta_0 > 0$ and $\vartheta \in [0,1]$ in (f_4), it holds that*

$$\liminf_{t \to +\infty} \frac{t^\vartheta g_n(x,t)}{e^{\alpha_0 t^2}} \ge \frac{\beta_0}{2} \begin{cases} > 0, & \text{if } \vartheta \in [0,1), \\ = +\infty, & \text{if } \vartheta = 1, \end{cases} \quad \text{uniformly in } x \in \mathbb{R}^2 \text{ and } n \in \mathbb{N}^+.$$

Proof. It follows from the definition of G_n defined in (2.5) that

$$g_n(x,t) = [f(x,t) + \sigma F(x,t)\mathfrak{f}_n(t)]\, e^{\sigma \mathfrak{F}_n(t)},\ \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

Obviously, taking (f_1) in (2.8) into account, we have the point (g_1). In order to deduce point (g_2), due to (1.13) and (f_1), for all $\epsilon > 0$ and $\alpha > \alpha_0$, there is a constant $C_\epsilon > 0$ such that

$$|f(x,t)| \le \epsilon |t| + C_\epsilon |t|^{\bar{q}-1} \left(e^{\alpha t^2} - 1 \right),\ \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \tag{2.11}$$

where $\bar{q} \ge 2$ can be arbitrarily chosen later. Using (f_2), there holds

$$|F(x,t)| \le \epsilon |t|^2 + C_\epsilon |t|^{\bar{q}} \left(e^{\alpha t^2} - 1 \right),\ \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.12}$$

Moreover, without mentioning any longer, let us exploit directly the following inequality (see e.g. [57, Lemma 2.1]):

$$\left(e^{\alpha t^2}-1\right)^m \leq \left(e^{\alpha m t^2}-1\right),\ \forall t\in\mathbb{R},\ \alpha>0 \text{ and } m>1.$$

Choosing $\epsilon=1$ and $\bar{q}=2$ in (2.11), we apply (f_2) and (2.7) to get

$$\begin{aligned} g_n(x,t)t-\theta G_n(x,t) &= \left[f(x,t)t-\theta F(x,t)+\sigma F(x,t)\mathfrak{f}_n(t)t\right]e^{\sigma\mathfrak{F}_n(t)} \\ &\geq \sigma F(x,t)\mathfrak{f}_n(t)te^{\sigma\mathfrak{F}_n(t)} \geq -\sigma F(x,t)|\mathfrak{f}_n(t)t|e^{4\sigma n^2} \\ &\geq -24\mathcal{C}_1\sigma n^2\left(e^{4\alpha n^2}-1\right)e^{4\sigma n^2}t^2, \end{aligned}$$

where $\mathcal{C}_1>0$ is independent of $n\in\mathbb{N}^+$. As a consequence, we define

$$\sigma_1 \triangleq \min\left\{\frac{1}{4n^2},\frac{\theta-2}{192\mathcal{C}_1\mathbb{T}_2^2en^2\left(e^{4\alpha n^2}-1\right)}\right\}$$

and then it yields the first part of point (g_2) for all $\sigma\in(0,\sigma_1)$. Recalling (1.15) and (1.17), we see the remaining part in point (g_2) immediately. Combining $(f_2)-(f_3)$ and (2.7), we set

$$\sigma_2=\frac{1}{24n}$$

and so for all $\sigma\in(0,\sigma_2)$, it has that

$$\begin{aligned} g_n(x,t) &= f(x,t)+\sigma F(x,t)\mathfrak{f}_n(t) \geq f(x,t)-12\sigma nF(x,t) \\ &\geq (1-12\sigma n)f(x,t) \geq \frac{1}{2}f(x,t) \end{aligned} \tag{2.13}$$

from where we adopt again (f_3) to find that

$$\frac{G_n(x,t)}{g_n(x,t)}=\frac{2F(x,t)}{f(x,t)}\leq 2M_0,\ \forall(x,t)\in\mathbb{R}^2\times[\hat{t}_0,+\infty),$$

which is the desired point (g_3). The point (g_4) is an immediate consequence of (2.13) and (f_4) for all $\sigma\in(0,\sigma_2)$. The proof is completed. □

As some by-products of Lemma 2.2, we collect some growth conditions for the nonlinearities g_n and G_n for all $n\in\mathbb{N}^+$ as follows. For all $\sigma\in(0,\sigma_1)$, then

$$|g_n(x,t)|\leq 7e\epsilon|t|+7eC_\epsilon|t|^{\bar{q}-1}\left(e^{\alpha t^2}-1\right),\ \forall(x,t)\in\mathbb{R}^2\times\mathbb{R}, \tag{2.14}$$

and

$$|G_n(x,t)|\leq e\epsilon|t|^2+eC_\epsilon|t|^{\bar{q}}\left(e^{\alpha t^2}-1\right),\ \forall(x,t)\in\mathbb{R}^2\times\mathbb{R}, \tag{2.15}$$

where the constants ϵ, C_ϵ and $\bar{q}$ are appearing in (2.11) and (2.12). To see them, we are derived from (f_2) and (2.7) that

$$\begin{aligned} g_n(x,t) &\leq \left[1+\sigma|\mathfrak{f}_n(t)t|\right]f(x,t)e^{\sigma\mathfrak{F}_n(t)} \\ &\leq \left(1+24\sigma n^2\right)f(x,t)e^{4\sigma n^2} \\ &\leq 7ef(x,t),\ \forall(x,t)\in\mathbb{R}^2\times\mathbb{R}, \end{aligned}$$

provided $\sigma\in(0,\sigma_1)$. So, (2.14) and (2.15) conclude. As a consequence, we are able to make sure that the functional $\Psi_n: H_r^1(\mathbb{R}^2)\to\mathbb{R}$ defined by

$$\Psi_n(u)=\int_{\mathbb{R}^2}G_n(x,u)\mathrm{d}x$$

is well-defined and of class of $\mathcal{C}^1$, see e.g. [48, Lemma 2.6] in detail.

As this stage, gathering the discussions above, one shall demonstrate that the variational functional J_n is well-defined and belongs to $\mathcal{C}^1(X_r, \mathbb{R})$ which permit us to make full use of variational methods to find nontrivial solution for Eq. (2.6). What's more, it is elementary to take some calculations that the derivative of J_n is given by

$$J_n'(u)(\psi) = \int_{\mathbb{R}^2} [\nabla u \nabla \psi + V_\omega(|x|)u\psi]\, \mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^2} \frac{\mathfrak{f}_n(u)\psi}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}\mathfrak{F}_n\left(u(s)\right)\mathrm{d}s\right)^2 \mathrm{d}x$$
$$+ \lambda \int_{\mathbb{R}^2} \frac{\mathfrak{F}_n(u)}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}\mathfrak{F}_n\left(u(s)\right)\mathrm{d}s\right)\left(\int_0^{|x|} \frac{s}{2}\mathfrak{f}_n\left(u(s)\right)\psi(s)\mathrm{d}s\right)\mathrm{d}x - \int_{\mathbb{R}^2} g_n(x,u)\psi \mathrm{d}x.$$

We conclude this section by the following convergent results related to the nonlinearity G_n and g_n.

Lemma 2.3. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1)-(f_4)$. Then, for all $n \in \mathbb{N}^+$ and $0 < \sigma < \min\{\sigma_1, \sigma_2\}$, if $(u_k) \subset H_r^1(\mathbb{R}^2)$ and $u_k \rightharpoonup u$ in $H_r^1(\mathbb{R}^2)$ with*

$$\sup_{k\in\mathbb{N}^+} \int_{\mathbb{R}^2} g_n(x,u_k)u_k \mathrm{d}x < +\infty,$$

then for all $n \in \mathbb{N}^+$, along a subsequence, it holds that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} G_n(x,u_k)\mathrm{d}x = \int_{\mathbb{R}^2} G_n(x,u)\mathrm{d}x \tag{2.16}$$

Moreover, passing to a subsequence if necessary, there holds

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} g_n(x,u_k)\psi \mathrm{d}x = \int_{\mathbb{R}^2} g_n(x,u)\psi \mathrm{d}x \text{ for all } \psi \in C_0^\infty(\mathbb{R}^2). \tag{2.17}$$

Proof. With Lemma 2.2-(g_3) and (2.14)-(2.15) in hands, the proof is standard and we onit it. $\square$

3. Existence results for (super)critical problem

In this section, we shall investigate the existence results for the auxiliary semilinear elliptic equation (2.6) under the assumptions (1.15) and (1.13) as well as $(f_1)-(f_4)$ for all $n \in \mathbb{N}^+$.

The main result concerning Eq. (2.6) is the following:

Theorem 3.1. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1)-(f_4)$. Then, there exist some constants $\sigma_0 > 0$ and $\lambda_0 > 0$ such that for all $\sigma \in (0, \sigma_0)$ and $\lambda \in (0, \lambda_0)$, Eq. (2.6) admits at least a nontrivial solution for all $n \in \mathbb{N}^+$.*

The proof of the above theorem will be divided into several lemmas.

First of all, due to the indefinite settings in this paper, we shall make full use of the following type of linking theorem.

Proposition 3.2. *(see [5]) Let $\mathbb{X} = \mathbb{Y} \bigoplus \mathbb{Z}$ be an Hilbert space with $\dim \mathbb{Y} < \infty$ and let $z \in \mathbb{Z}$ with $\|z\|_{\mathbb{X}} = 1$ be fixed. Assume that $\Phi \in \mathcal{C}^1(\mathbb{X}, \mathbb{R})$ satisfies there exist $\hat{\rho} > \rho > 0$ such that*

$$\inf_{\mathbb{S}_\rho} \Phi > \sup_{\partial\mathbb{Q}} \Phi,$$

where

$$\mathbb{S}_\rho \triangleq \{u \in \mathbb{Z} : \|u\|_{\mathbb{X}} = \rho\} \text{ and } \mathbb{Q} \triangleq \{v + sz : v \in \mathbb{Y}, 0 \le s \le \hat{\rho}, \|v\|_{\mathbb{X}} \le \hat{\rho}\}.$$

Then, there exists a (C) sequence of Φ at the level

$$c = \inf_{\gamma\in\Gamma} \sup_{u\in\mathbb{Q}} \Phi(\gamma(u)), \tag{3.1}$$

where

$$\Gamma = \{\gamma \in \mathcal{C}(\mathbb{Q}, \mathbb{X}) : \gamma|_{\partial\mathbb{Q}} = \mathrm{id}\}.$$

In order to apply Proposition 3.2, we set $\mathbb{X} = X_r$ with $\mathbb{Y} = X_r^-$ and $\mathbb{Z} = X_r^+$ since $\dim X_r^- = j_0 < +\infty$ in Section 2. Moreover, setting $\Phi = J_n \in \mathcal{C}^1(\mathbb{X}, \mathbb{R})$ for all $n \in \mathbb{N}^+$, then we prove the following results regarding the linking geometry structure.

Lemma 3.3. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1)-(f_4)$. Then, for all $\sigma \in (0, \sigma_1)$, there is a $\rho > 0$ independent of σ and $n \in \mathbb{N}^+$ such that*

$$\inf_{S_\rho} J_n > 0,$$

for all $n \in \mathbb{N}^+$, where $S_\rho \triangleq \{u \in X_r^+ : \|u\|_X = \rho\}$.

Proof. Recalling (2.15), if $\|u\|_X^2 < \dfrac{2\pi}{\alpha}$, then for all $u \in X$, (1.14) shows that

$$\begin{aligned}\int_{\mathbb{R}^2} G_n(x,u)\mathrm{d}x &\le e\epsilon \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x + eC_\epsilon \int_{\mathbb{R}^2} |u|^4 \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \\ &\le e\epsilon \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x + eC_\epsilon \left(\int_{\mathbb{R}^2} |u|^8 \mathrm{d}x\right)^{\frac{1}{2}} \left[\int_{\mathbb{R}^2} \left(e^{2\alpha u^2} - 1\right) \mathrm{d}x\right]^{\frac{1}{2}} \\ &\le e\mathbb{T}_2^2 \epsilon \|u\|_X^2 - eC_\epsilon \mathcal{C}_2 \mathbb{T}_8^4 \|u\|_X^4,\end{aligned}$$

where $\mathbb{T}_s > 0$ is associated with the imbedding constant on $X \hookrightarrow L^s(\mathbb{R}^2)$ with $s = 2, 8$, and $\mathcal{C}_2 > 0$ is independent of $n \in \mathbb{N}^+$. Choosing $\epsilon = \dfrac{\mu_+}{4e\mathbb{T}_2^2} > 0$ with $\mu_+ > 0$ given in (2.2), we obtain

$$J_n(u) \ge \frac{\mu_+}{4} \|u\|_X^2 - \mathcal{C}_3 \|u\|_X^4, \ \forall u \in X_r^+.$$

for some $\mathcal{C}_3 > 0$ independent of $n \in \mathbb{N}^+$. Now, we set $\rho \triangleq \left\{\sqrt{\dfrac{2\pi}{\alpha}}, \sqrt{\dfrac{\mu_+}{8\mathcal{C}_3}}\right\} > 0$ and so

$$\inf_{S_\rho} J_n \ge \frac{\mu_+}{8} \rho^2,$$

where $S_\rho \triangleq \{u \in X_r^+ : \|u\|_{X_r} = \rho\}$. The proof is completed. □

Lemma 3.4. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1)-(f_4)$. Then, there is a $\lambda_1 > 0$ such that for all $\lambda \in (0, \lambda_1)$, there exists a $\hat{\rho} > \rho$ independent of σ and $n \in \mathbb{N}^+$ such that*

$$\sup_{\partial Q} J_n \le 0$$

for all $n \in \mathbb{N}^+$, where $Q = \{v + sz : v \in X_r^-, 0 \le s \le \hat{\rho}, \|v\|_X \le \hat{\rho}\}$ with $z = \dfrac{\phi_{j_0+1}}{\|\phi_{j_0+1}\|_X} \in X_r^+$ and ϕ_{j_0+1} denoting the eigenfunction of μ_{j_0+1} associated with (2.1).

Proof. Choosing $\lambda_1 = \dfrac{\mu_-}{\mathbb{T}_2 n^4}$ with $\mu_- > 0$ given in (2.2), then for all $\lambda \in (0, \lambda_1)$, Lemma 2.1-(iii) shows that

$$\lambda N_n(w) \le \mu_- \|w\|_X^2, \ \forall w \in X_r. \tag{3.2}$$

From which, according to (f_2), (2.2) and (2.5), it holds that

$$J_n(w) = -\frac{\mu_-}{2} \|w\|_X^2 + \frac{\lambda}{2} N_n(w) - \int_{\mathbb{R}^2} G_n(x,w)\mathrm{d}x \le 0, \ \forall w \in X_r^-, \tag{3.3}$$

provided $\lambda \in (0, \lambda_1)$. On the one hand, for all $u = v + s_u z \in Q$, we define $\hat{u} = \dfrac{u}{\|u\|_X}$. If $\hat{u} \equiv 0$, one sees that $s_u = 0$ and so $u = v \in X_r^-$. In this scenario, it follows from (3.3) that

$$J_n(u) \le 0. \tag{3.4}$$

On the other hand, if $\hat{u} \neq 0$, then there is a set $\Upsilon_{\hat{u}} \subset \mathbb{R}^2$ with Lebesgue measure $|\Upsilon_{\hat{u}}| > 0$ such that $|\hat{u}(x)| > 0$ for every $x \in \Upsilon_{\hat{u}}$. As a consequence, we shall demonstrate that $|u(x)| = |\hat{u}(x)| \|u\|_X \to +\infty$ on $\Upsilon_{\hat{u}}$ as $\|u\|_X \to +\infty$. Hence, due to Lemma 2.2-(g_2),

$$\liminf_{\|u\|_X \to +\infty} \int_{\mathbb{R}^2} \frac{G_n(x,u)}{u^2} \hat{u}^2 \mathrm{d}x \geq \liminf_{\|u\|_X \to +\infty} \int_{\Upsilon_{\hat{u}}} \frac{G_n(x,u)}{u^2} \hat{u}^2 \mathrm{d}x = +\infty \text{ uniformly in } n \in \mathbb{N}^+. \tag{3.5}$$

Taking again (3.2) into account, for all $\lambda \in (0, \lambda_1)$, we apply (3.5) to obtain

$$\frac{J_n(u)}{\|u\|_X^2} \leq 1 - \frac{1}{\|u\|_X^2} \int_{\mathbb{R}^2} G_n(x,u) \mathrm{d}x \to -\infty \text{ as } \|u\|_X \to +\infty.$$

Thus, with aid of (3.4), there exists $\hat{\rho} > \rho$ independent of σ and $n \in \mathbb{N}^+$ such that

$$\max_{\partial Q} \Phi \leq 0$$

finishing the proof of this lemma. □

Combining Lemmas 3.3 and 3.4 as well as Proposition 3.2, for all $\sigma \in (0, \sigma_1)$ and $\lambda \in (0, \lambda_1)$, there exists a sequence $(u_k) \subset X_r$ such that

$$J_n(u_k) \to c_n \text{ and } (1 + \|u_k\|_X) \|\mathcal{J}_n'(u_k)\|_{X^{-1}} \to 0, \tag{3.6}$$

for all $n \in \mathbb{N}^+$, where

$$c_n \triangleq \inf_{\gamma \in \Gamma_n} \max_{u \in Q} J_n(\gamma(u)) > 0 \tag{3.7}$$

with

$$\Gamma_n = \{\gamma \in \mathcal{C}(Q, X_r) : \gamma|_{\partial \mathbb{Q}} = \mathrm{id}\}.$$

Remark 3.5. The reader is invited to observe from the proofs of Lemmas 3.3 and 3.4 that there exist some constants $\bar{c}, \hat{c} > 0$, independent of σ, λ and $n \in \mathbb{N}^+$, such that $\bar{c} \leq c_n \leq \hat{c}$.

Lemma 3.6. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1)-(f_4)$. Then, for all $\sigma \in (0, \sigma_1)$ and $\lambda \in (0, \lambda_2)$ with $\lambda_2 = \frac{\theta - 2}{8\mathbb{T}_2 n^4} > 0$, if $(u_k) \subset X_r$ satisfies (3.6) and (3.7), we deduce that the sequence $(\|u_k\|_X)$ is uniformly bounded in $k, n \in \mathbb{N}^+$ along a subsequence. In particular, it holds that*

$$\sup_{k \in \mathbb{N}^+} \int_{\mathbb{R}^2} g_n(x, u_k) u_k \mathrm{d}x \leq \mathcal{C}_4 \tag{3.8}$$

for some $\mathcal{C}_4 > 0$ independent of $n, k \in \mathbb{N}^+$.

Proof. Suppose, by contradiction, that $\|u_k\|_X \to +\infty$ as $k \to +\infty$. We define $v_k = \frac{u_k}{\|u_k\|_X}$, then, passing to a subsequence if necessary, there exists a function $v \in X$ such that $v_k \rightharpoonup v$ in X, $v_k \to v$ in $L^p(\mathbb{R}^2)$ for all $2 \leq p < +\infty$ and $v_k \to v$ a.e. in $\mathbb{R}^2$. On the one hand, we shall conclude that $v \equiv 0$. Otherwise, there is a set $\Upsilon_v \triangleq \{x \in \mathbb{R}^2 : |v(x)| > 0\} \subset \mathbb{R}^2$ with positive Lebesgue measure, that is $|\Upsilon_v| > 0$ such that $|u_k(x)| \to \infty$ as $k \to \infty$ for all $x \in \Upsilon_v$. For all $\sigma \in (0, \sigma_1)$, we are derived from Lemma 2.2-(g_2) that

$$\liminf_{k \to +\infty} \int_{\mathbb{R}^2} \frac{G_n(x,u_k)}{u_k^2} v_k^2 \mathrm{d}x \geq \liminf_{k \to +\infty} \int_{\Upsilon_v} \frac{G_n(x,u_k)}{u_k^2} v_k^2 \mathrm{d}x = +\infty \text{ uniformly in } n \in \mathbb{N}^+. \tag{3.9}$$

Recalling (3.7) and Remark 3.5, we are capable of taking advantage of (3.2) and (3.9) to reach

$$0 = \limsup_{k \to +\infty} \frac{J_n(u_k)}{\|u_k\|_X^2} \leq 1 - \liminf_{k \to +\infty} \int_{\mathbb{R}^2} \frac{G_n(x,u_k)}{u_k^2} v_k^2 \mathrm{d}x = -\infty \text{ uniformly in } n \in \mathbb{N}^+,$$

which is impossible. So, we conclude that $v \equiv 0$. In this situation, we must have that $v_k \to v$ in $L^p(\mathbb{R}^2)$ for all $2 \leq p < +\infty$.

With the above discussions in hands, by applying $\lambda < \lambda_2 = \frac{\theta-2}{2\mathbb{T}_2 n^4}$ to Lemma 2.1-(iv), we then exploit Lemma 2.2-(g_2) to see that

$$\begin{aligned}\frac{\theta c_n + o_k(1)}{\|u_k\|_X^2} &= \frac{\theta J_n(u_k) - J_n'(u_k)(u_k)}{\|u_k\|_X^2}\\ &= \frac{\theta-2}{2} - \frac{\theta-2}{2}\int_{\mathbb{R}^2}\omega|v_k|^2 \mathrm{d}x + \frac{\lambda}{2\|u_k\|_X^2}\left[\theta N_n(u_k) - 2N_n'(u_k)(u_k)\right]\\ &\quad + \frac{1}{\|u_k\|_X^2}\int_{\mathbb{R}^2}\left[\theta g_n(x,u_k)u_k - G_n(x,u_k)\right]\mathrm{d}x\\ &\geq \frac{\theta-2}{8} + o_k(1)\end{aligned}$$

which contradicts with Remark 3.5 if we tend $k \to +\infty$. The verification of (3.8) follows immediately with the help of $J_n'(u_k)(u_k) = o_k(1)$ and Lemma 2.1-(iii) with $\lambda < \lambda_2$. The proof is completed. □

In light of the nonlinearity G_n possesses the supercritical exponential growth at infinity and it causes the lack of compactness. To restore it, we proceed as the Brézis-Lieb method to pull the linking level c_n down below a critical value. Have this aim in mind, motivated by [1,9,13,17,31,57], for a sufficiently small but fixed constant $r_0 \in (0,1]$ which shall be determined later, we make use of the Moser sequence functions defined by

$$w_k(x) \triangleq \frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log k}, & \text{if } 0 \leq |x| \leq \frac{r_0}{k},\\ \frac{\log(\frac{1}{|x|})}{\sqrt{\log k}}, & \text{if } \frac{r_0}{k} < |x| \leq r_0,\\ 0, & \text{if } |x| > r_0.\end{cases} \tag{3.10}$$

Whereas, due to the indefinite settings in our problem, it would be much more complicated concerning the estimate of minimax level than those of [42,47–49,52]. Roughly speaking, we are going to rely on a new norm associated with the indefinite operator $-\Delta + V_\omega(|x|)$. As a consequence, let us included the detailed proof of the following result in the Appendix.

Lemma 3.7. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1)-(f_4)$. Then, for all $\sigma \in (0,\sigma_2)$ and $\lambda \in (0, \frac{1}{\mathbb{T}_2 n^4})$, we have that for some $k_0 \in \mathbb{N}^+$*

$$\max_{t\geq 0, v \in X_r^-} J_n(v + tw_{k_0}) < c_* \triangleq \frac{2\pi}{\alpha_0}.$$

Now, we show the proof of Theorem 3.1 and it is regarded as a direct corollary of the result below.

Lemma 3.8. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1)-(f_4)$. Then, for all $n \in \mathbb{N}^+$, Eq. (2.6) possesses a nontrivial solution for all $\sigma \in (0,\sigma_0)$ and $\lambda \in (0,\lambda_0)$, where $\sigma_0 = \min\{\sigma_1,\sigma_2\} > 0$ and $\lambda_0 = \min\{\lambda_1,\lambda_2\} > 0$.*

Proof. For all $\sigma \in (0,\sigma_1)$ and $\lambda \in (0,\lambda_1)$, we are derived from Lemmas 3.3 and 3.4 as well as Proposition 3.2 that there exists a sequence $(u_k) \subset X_r$ satisfying (3.6) and (3.7). Since $\lambda < \lambda_2$, then Lemma 3.6 reveals that $(u_k) \subset X_r$ is uniformly bounded in X_r, that is, there exists a constant $\mathcal{C}_5 > 0$, independent of $n, k \in \mathbb{N}^+$ such that $\|u_k\|_X \leq \mathcal{C}_5$. Passing to a subsequence if necessary, there is a function $u \in X_r$ such that $u_k \rightharpoonup u$ in X_r, $u_k \to u$ in $L^p(\mathbb{R}^2)$ for all $2 \leq p < +\infty$ and $u_k \to u$ a.e. in $\mathbb{R}^2$. Using Lemma 3.6 again, we shall obtain (2.17) from (3.8) and thus Lemma 2.1-(ii) indicates that $J_n'(u) = 0$. In other words, we deduce that u is a solution of Eq. (2.6) for all $n \in \mathbb{N}^+$. The proof would be done if we verify that $u \neq 0$. Arguing in indirectly, we suppose that $u \equiv 0$. Let us take into account (2.17) and Lemma 2.1-(ii) once more, then

$$\lim_{k\to\infty}\int_{\mathbb{R}^2} G_n(x,u_k)\mathrm{d}x = 0 \text{ and } \lim_{k\to\infty} N_n(u_k) = 0 \text{ uniformly in } n \in \mathbb{N}^+.$$

Combining $u_k \to 0$ in $L^2(\mathbb{R}^2)$ and $J_n(u_k) \to c_n$ in (3.6) as well as Lemma 3.7, one finds that

$$\limsup_{k\to\infty} \|u_k\|_X^2 < \frac{4\pi}{\alpha_0}.$$

Thereby, we shall choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu > 1$ sufficiently close to 1 in such a way that $\frac{1}{\nu} + \frac{1}{\nu'} = 1$ with $\nu' > 1$ and

$$\alpha\nu' \|u_k\|_X^2 < 4\pi(1-\epsilon) \text{ for some suitable } \epsilon \in (0,1).$$

We define

$$\hat{u}_k = \sqrt{\frac{\alpha\nu'}{4\pi(1-\epsilon)}}\, u_k, \ \forall k \in \mathbb{N}^+,$$

and so $\|\hat{u}_k\|_X \le 1$. In view of (2.14) with $\bar{q} \ge 2$, for all $\sigma \in (0, \sigma_1)$, it holds that

$$\begin{aligned}\int_{\mathbb{R}^2} g_n(x,u_k)u_k \mathrm{d}x &\le \int_{\mathbb{R}^2} |u_k|^2 \mathrm{d}x + \mathcal{C}_6 \int_{\mathbb{R}^2} |u_k|^{\bar{q}} \left(e^{\alpha u_k^2} - 1\right) \mathrm{d}x \\ &\le \int_{\mathbb{R}^2} |u_k|^2 \mathrm{d}x + \mathcal{C}_6 \left(\int_{\mathbb{R}^2} |u_k|^{\bar{q}\nu} \mathrm{d}x\right)^{\frac{1}{\nu}} \left[\int_{\mathbb{R}^2} \left(e^{4\pi(1-\epsilon)\hat{u}_k^2} - 1\right) \mathrm{d}x\right]^{\frac{1}{\nu'}}.\end{aligned}$$

Recalling (1.14), we can deduce that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} g_n(x,u_k)u_k \mathrm{d}x = 0 \text{ uniformly in } n \in \mathbb{N}^+.$$

From which, combining $u_k \to 0$ in $L^2(\mathbb{R}^2)$ and $J_n'(u_k)(u_k) \to 0$ in (3.6) and Lemma 2.1-(ii), we conclude that $\|u_k\|_X \to 0$ uniformly in $n \in \mathbb{N}^+$. Therefore, we must have $c_n = \lim_{k\to\infty} J_n(u_k) = 0$ which contradicts with Remark 3.5 and so the claim $u \neq 0$ concludes. The proof is completed. □

At this stage, according to the observations in the Introduction, we are going to take the L^∞-estimate for the obtained solution u explored in Theorem 3.1. Unfortunately, in light of the "almost optimal" growth condition (f_4) was supposed, then the arguments exploited in [2,3,46] would become unavailable any longer. As a consequence, we need the following result which is crucial in this paper.

Lemma 3.9. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1) - (f_4)$. Then, for all $n \in \mathbb{N}^+$, there are $\bar{\sigma}_0 < \sigma_0$ and $\bar{\lambda}_0 < \lambda_0$ such that the sequence $(u_n) \subset X_r$ in Lemma 3.8 contains a strongly convergent subsequence uniformly in $n, k \in \mathbb{N}^+$ for all $\sigma \in (0, \bar{\sigma}_0)$ and $\lambda \in (0, \bar{\lambda}_0)$.*

Proof. Recalling the proof of Lemma 3.8, we know that $\|u_k\|_X \le \mathcal{C}_5$. Moreover, $u_k \rightharpoonup u$ in X_r, $u_k \to u$ in $L^p(\mathbb{R}^2)$ for all $2 \le p < +\infty$ and $u_k \to u$ a.e. in $\mathbb{R}^2$. In order to conclude that $u_k \to u$ in X_r along a subsequence, it is sufficient to demonstrate that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} g_n(x,u_k)(u_k - u)\mathrm{d}x = 0 \text{ uniformly in } n \in \mathbb{N}^+. \tag{3.11}$$

As a matter of fact, it is very similar to the verification of (3.11) to have that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} g_n(x,u)(u_k - u)\mathrm{d}x = 0 \text{ uniformly in } n \in \mathbb{N}^+. \tag{3.12}$$

With (3.11) and (3.12) in hands, we take advantage of $J_n'(u_k)(u_k - u) = o_k(1)$ by (3.6) and $J'(u) = 0$ in Lemma 3.8 to reach

$$\begin{aligned}o_k(1) &= J_n'(u_k)(u_k - u) - J_n'(u)(u_k - u) \\ &= \|u_k - u\|_X^2 - \omega \int_{\mathbb{R}^2} |u_k - u|^2 \mathrm{d}x + \frac{\lambda}{2}\left[N_n'(u_k)(u_k - u) - N_n'(u)(u_k - u)\right]\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^2} g_n(x,u_k)(u_k-u)\mathrm{d}x+\int_{\mathbb{R}^2} g_n(x,u)(u_k-u)\mathrm{d}x\\
&=\|u_k-u\|_X^2+o_k(1)
\end{aligned}
$$

which yields the claim. Thereby, our next goal is to exhibit the verification of (3.11). First of all, using (2.10), there is a constant $\mathcal{K}>0$, which is independent of $n,k\in\mathbb{N}^+$, such that

$$
G_n(x,t)\ge t^2,\ \forall (x,|t|)\in\mathbb{R}^2\times[\mathcal{K},+\infty). \tag{3.13}
$$

To continue showing (3.11), we shall split the proof into the following two cases.

Case 1. $|u|\le 2\mathcal{K}$.

In this case, we shall prove (3.11) for all $\sigma\in(0,\sigma_0)$ and $\lambda\in(0,\lambda_0)$. Taking into account $u_k\to u$ a.e. in $\mathbb{R}^2$, without loss of generality, one would suppose that $|u_k|\le 3\mathcal{K}$ for all $k\in\mathbb{N}^+$. With the help of (2.14) with $\bar q=2$, for all $\sigma\in(0,\sigma_1)$, we see

$$
\begin{aligned}
\int_{\mathbb{R}^2}|g_n(x,u_k)(u_k-u)|\,\mathrm{d}x&\le\int_{\mathbb{R}^2}|u_k(u_k-u)|\,\mathrm{d}x+\mathcal{C}_6\int_{\mathbb{R}^2}|u_k-u|\,|u_k|\left(e^{\alpha u_k^2}-1\right)\mathrm{d}x\\
&\le|u_k|_2|u_k-u|_2+\mathcal{C}_6\left(e^{9\alpha\mathcal{K}^2}-1\right)|u_k|_2|u_k-u|_2
\end{aligned}
$$

which gives us (3.11) immediately, where $\mathcal{C}_6>0$ is independent of $n,k\in\mathbb{N}^+$.

Case 2. $|u|>2\mathcal{K}$.

In this case, we are going to look for some small $\bar\sigma_0<\sigma_0$ and $\bar\lambda_0<\lambda_0$ such that (3.11) remains valid for all $\sigma\in(0,\bar\sigma_0)$ and $\lambda\in(0,\bar\lambda_0)$. What's more, since we recall that $u_k\to u$ a.e. in $\mathbb{R}^2$, without loss of generality, one is allowed to suppose that $|u_k|\ge\mathcal{K}$ for all $k\in\mathbb{N}^+$. So, (3.13) reveals that

$$
G_n(x,u_k)\ge u_k^2,\ \forall x\in\mathbb{R}^2. \tag{3.14}
$$

Proceeding as [8, Proposition 2.2], we easily conclude $N_n(u)\le\frac{1}{16\pi}|u|_2^2|u|_4^4\le\frac{1}{16\pi}\mathbb{T}_4^4\mathcal{C}_5^4|u|_2^2$, where $\mathbb{T}_4>0$ is related to the imbedding constant of $X_r\hookrightarrow L^4(\mathbb{R}^2)$. Analogously, there exists a $\mathcal{C}_7>0$, independent of $n,k\in\mathbb{N}^+$, such that

$$
\left|2N_n(u)-N_n'(u)(u)\right|\le\mathcal{C}_7|u|_2^2. \tag{3.15}
$$

Denoting $\hat\theta=\frac{\theta+2}{2}\in(2,\theta)$, we claim that there is a $\bar\sigma_0<\sigma_0$ such that for all $\sigma\in(0,\bar\sigma_0)$, it holds that

$$
g(x,t)t-\hat\theta G(x,t)\ge 0,\ \forall (x,t)\in\mathbb{R}^2\times\mathbb{R}\text{ and }n\in\mathbb{N}^+. \tag{3.16}
$$

Indeed, according to the definition of G_n in (2.5), we apply (f_2) and (2.7) to deduce that

$$
\begin{aligned}
g_n(x,t)t-\hat\theta G_n(x,t)&=\left[f(x,t)t-\hat\theta F(x,t)+\sigma F(x,t)\mathfrak{f}_n(t)t\right]e^{\sigma\mathfrak{F}_n(t)}\\
&\ge\left[\frac14(\theta-2)-24\sigma n^2\right]F(x,t)e^{\sigma\mathfrak{F}_n(t)},\ \forall(x,t)\in\mathbb{R}^2\times\mathbb{R}.
\end{aligned}
$$

Choosing $\bar\sigma_0=\min\left\{\sigma_0,\dfrac{\theta-2}{96n^2}\right\}>0$, we then deduce that (3.16) holds true for all $\sigma\in(0,\bar\sigma_0)$. As a consequence, we are capable of taking advantaging of (3.14), (3.15) and (3.16) to reach

$$
\begin{aligned}
J_n(u)&=J_n(u)-\frac12J_n'(u)(u)\\
&=\frac{\lambda}{4}\left[2N_n(u)-N_n'(u)(u)\right]+\frac12\int_{\mathbb{R}^2}\left[g_n(x,u_k)u_k-2G_n(x,u_k)\right]\mathrm{d}x\\
&\ge\frac{\hat\theta-2}{2}\int_{\mathbb{R}^2}G_n(x,u_k)\mathrm{d}x-\frac{\lambda}{4}\mathcal{C}_7|u|_2^2\ge\left[\frac14(\theta-2)-\frac{\lambda}{4}\mathcal{C}_7\right]|u|_2^2.
\end{aligned}
$$

Let us determine $\bar{\lambda}_0 = \min\left\{\lambda_0, \frac{\theta-2}{\mathcal{C}_7}\right\} > 0$, then for all $\lambda \in (0, \bar{\lambda}_0)$, one finds that $J_n(u) \geq 0$. Thanks to this crucial conclusion, we are ready to verify (3.11) in detail. We gather $J_n(u_k) \to c_n$ in (3.6), the Fatou's lemma, $u_k \to u$ in $L^2(\mathbb{R}^2)$ and Lemma 2.1-(ii) as well as (2.16) to derive

$$\begin{aligned} c_n &= \frac{1}{2}\|u_k\|_X^2 - \frac{\omega}{2}\int_{\mathbb{R}^2} |u_k|^2 \mathrm{d}x + \frac{\lambda}{2}N_n(u_k) - \int_{\mathbb{R}^2} G_n(x, u_k)\mathrm{d}x + o_k(1) \\ &= \frac{1}{2}\|u_k - u\|_X^2 + \frac{1}{2}\|u\|_X^2 - \frac{\omega}{2}\int_{\mathbb{R}^2} |u|^2 \mathrm{d}x + \frac{\lambda}{2}N_n(u) - \int_{\mathbb{R}^2} G_n(x, u)\mathrm{d}x + o_k(1) \\ &= \frac{1}{2}\|u_k - u\|_X^2 + J_n(u) + o_k(1) \\ &\geq \frac{1}{2}\|u_k - u\|_X^2 + o_k(1). \end{aligned}$$

In view of Lemma 3.7, it holds that $\limsup\limits_{k\to\infty} \|u_k - u\|_X^2 < \frac{4\pi}{\alpha_0}$. Consequently, we shall choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu > 1$ sufficiently close to 1 in such a way that $\frac{1}{\nu} + \frac{1}{\nu'} = 1$ with $\nu' > 1$ and

$$\alpha\nu'\|u_k - u\|_X^2 < \frac{4\pi}{(1+\epsilon)^3} \text{ for some suitable } \epsilon \in (0, 1).$$

Setting

$$\breve{u}_k = \sqrt{\frac{\alpha\nu'(1+\epsilon)^3}{4\pi}}(u_k - u), \ \forall k \in \mathbb{N}^+.$$

Obviously, one finds that $\|\breve{u}_k\|_X^2 \leq 1$ for all $k \in \mathbb{N}^+$. Moreover, for the above fixed $\epsilon \in (0, 1)$, we need the following two types of Young's inequality

$$|a + b|^2 \leq (1 + \epsilon)|a|^2 + (1 + \epsilon^{-1})|b|^2, \ \forall a, b \in \mathbb{R}$$

and

$$e^{a+b} - d \leq \frac{1}{1+\epsilon}\left[e^{(1+\epsilon)a} - d\right] + \frac{\epsilon}{1+\epsilon}\left[e^{(1+\epsilon^{-1})b} - d\right], \ \forall a, b, d \in \mathbb{R}.$$

By means of the above facts with (1.14) and $\|\breve{u}_k\|_X^2 \leq 1$, one has

$$\begin{aligned} \int_{\mathbb{R}^2} \left(e^{\alpha\nu'|u_k|^2} - 1\right)\mathrm{d}x &\leq \int_{\mathbb{R}^2} \left[e^{\alpha\nu'(1+\epsilon)|u_k-u|^2 + \alpha\nu'(1+\epsilon^{-1})|u|^2} - 1\right]\mathrm{d}x \\ &\leq \frac{1}{1+\epsilon}\int_{\mathbb{R}^2} \left[e^{4\pi(1+\epsilon)^{-1}|\breve{u}_k|^2} - 1\right]\mathrm{d}x + \frac{\epsilon}{1+\epsilon}\int_{\mathbb{R}^2} \left[e^{\alpha(1+\epsilon^{-1})^2|u|^2} - 1\right]\mathrm{d}x \\ &\leq \mathcal{C}_8 + \frac{\epsilon}{1+\epsilon}\int_{\mathbb{R}^2} \left[e^{\alpha(1+\epsilon^{-1})^2|u|^2} - 1\right]\mathrm{d}x \\ &< +\infty, \end{aligned}$$

from where it infers from (2.14) that

$$\int_{\mathbb{R}^2} |g_n(x, u_k)(u_k - u)|\,\mathrm{d}x \leq |u_k|_2|u_k - u|_2 + \mathcal{C}_9|u_k|_{2(\bar{q}-1)\nu}^{(\bar{q}-1)}|u_k - u|_{2\nu}\left[\int_{\mathbb{R}^2}\left(e^{\alpha\nu'|u_k|^2} - 1\right)\mathrm{d}x\right]^{\frac{1}{\nu'}}.$$

So, we can prove (3.11) in this Case. The proof is completed. □

As a byproduct of Lemma 3.9, we conclude that $u_k \to u$ in X_r along a subsequence. Moreover, one knows that $\|u_k\|_X \leq \mathcal{C}_5$ by Lemma 3.8 for some $\mathcal{C}_5 > 0$, independent of $n, k \in \mathbb{N}^+$. Therefore, for all $\sigma \in (0, \bar{\sigma}_0)$ and $\lambda \in (0, \bar{\lambda}_0)$, there exists a function $\varpi \in X_r$ that is independent of $n, k \in \mathbb{N}^+$ such that

$$\max\left\{\sup_{k\in\mathbb{N}^+} |u_k(x)|, |u(x)|\right\} \leq \varpi(x), \ \forall x \in \mathbb{R}^2. \tag{3.17}$$

Lemma 3.10. *Let G be given by (1.15) and satisfy (1.13) as well as $(f_1)-(f_4)$. Then, for all $n \in \mathbb{N}^+$, there exist some $\sigma_* = \bar{\sigma}_0$ and $\lambda_* < \bar{\lambda}_0$ such that for all $\sigma \in (0, \sigma_*)$ and $\lambda \in (0, \lambda_*)$, the obtained solution $u \in X_r$ established in Theorem 3.1 belongs to $L^\infty(\mathbb{R}^2)$. In particular, it holds that*

$$|u|_\infty \le \left(\frac{2}{\tilde{q}}\right)^{\frac{2\tilde{q}}{(2-\tilde{q})^2}} \left[\bar{\mathcal{C}}_0^{\tilde{q}'} \int_{\mathbb{R}^2} (e^{\alpha \tilde{q}' \varpi^2} - 1)\mathrm{d}x\right]^{\frac{\tilde{q}}{2\tilde{q}'(2-\tilde{q})}} |u|_4^4,$$

where $\bar{\mathcal{C}}_0 > 0$ is a constant independent of $n \in \mathbb{N}^+$, ϖ comes from (3.17) and $\tilde{q} \in (1, 2)$ with $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$.

Proof. For all $\omega > 0$, according to (2.10), there is a $\mathfrak{T}_\omega > 0$ independent of $n, k \in \mathbb{N}^+$ such that

$$G_n(x,t) \ge \omega t^2, \ \forall (x, |t|) \in \mathbb{R}^2 \times [\mathfrak{T}_\omega, +\infty). \tag{3.18}$$

Without loss of generality, we could suppose that $|u|_\infty > \mathfrak{T}_\omega$. Otherwise, the proof is done immediately. Because we have showed that $u_k \to u$ in X_r by Lemma 3.9, there holds that $J_n(u) = c_n + o_k(1)$ which together with Remark 3.5 reveals that $J_n(u) \ge \bar{c} > 0$ for all $n \in \mathbb{N}^+$. Proceeding as the very similar calculations in Case 2 of the proof of Lemma 3.9, we use $J_n(u) > 0$ and (3.19) to arrive at

$$\begin{aligned}\frac{1}{2}\|u\|_X^2 &\ge \frac{\omega}{2}\int_{\mathbb{R}^2} u^2\mathrm{d}x - \frac{\lambda}{2}N_n(u) + \int_{\mathbb{R}^2} G_n(x,u)\mathrm{d}x \\ &\ge \left(\frac{3\omega}{2} - \frac{\lambda}{32\pi}\mathbb{T}_4^4\mathcal{C}_5^4\right)\int_{\mathbb{R}^2} u^2\mathrm{d}x.\end{aligned}$$

If we choose $\lambda_3 = \dfrac{16\omega\pi}{\mathbb{T}_4^4\mathcal{C}_5^4} > 0$, then for all $\lambda \in (0, \lambda_3)$, one has

$$\|u\|_X^2 \ge 2\omega \int_{\mathbb{R}^2} u^2\mathrm{d}x. \tag{3.19}$$

Next, we shall begin with the verification of $u \in L^\infty(\mathbb{R}^2)$. As a start, we suppose that $u \ge 0$. Given $\gamma > 1$ and $z \in \mathbb{N}^+$, we will introduce the measurable sets $\mathfrak{A}_z \triangleq \{x \in \mathbb{R}^2 : u^{\gamma-1} \le z\}$ and $\mathfrak{B}_z \triangleq \mathbb{R}^2 \backslash A_z$. Consider the sequences

$$u_z = \begin{cases} u^{2\gamma-1}, & \text{in } \mathfrak{A}_z, \\ z^2 u, & \text{in } \mathfrak{B}_z, \end{cases} \text{ and } v_z = \begin{cases} u^\gamma, & \text{in } \mathfrak{A}_z, \\ zu, & \text{in } \mathfrak{B}_z. \end{cases}$$

It is simple to observe that $u_z, v_z \in X_r$, $|u_z| \le |u|^{2\gamma-1}$ and $|v_z|^2 = uu_z \le |u|^{2\gamma}$ in $\mathbb{R}^2$. Moreover,

$$\nabla u_z = \begin{cases} (2\gamma-1)u^{2(\gamma-1)}\nabla u, & \text{in } \mathfrak{A}_z, \\ z^2\nabla u, & \text{in } \mathfrak{B}_z, \end{cases} \text{ and } \nabla v_z = \begin{cases} \gamma u^{\gamma-1}\nabla u, & \text{in } \mathfrak{A}_z, \\ z\nabla u, & \text{in } \mathfrak{B}_z, \end{cases}$$

which imply that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^2} \nabla u \nabla u_z \mathrm{d}x = (2\gamma-1)\int_{\mathfrak{A}_z} u^{2(\gamma-1)}|\nabla u|^2\mathrm{d}x + z^2\int_{\mathfrak{B}_z}|\nabla u|^2\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla v_z|^2\mathrm{d}x = \gamma^2\int_{\mathfrak{A}_z} u^{2(\gamma-1)}|\nabla u|^2\mathrm{d}x + z^2\int_{\mathfrak{B}_z}|\nabla u|^2\mathrm{d}x. \end{cases} \tag{3.20}$$

Combining (3.20) and the fact that $\gamma > 1$, one obtains

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla v_z|^2\mathrm{d}x &= \int_{\mathbb{R}^2}\nabla u\nabla u_z\mathrm{d}x + (\gamma-1)^2\int_{\mathfrak{A}_z} u^{2(\gamma-1)}|\nabla u|^2\mathrm{d}x \\ &\le \left[1 + \frac{(\gamma-1)^2}{2\gamma-1}\right]\int_{\mathbb{R}^2}\nabla u\nabla u_z\mathrm{d}x \le \gamma^2\int_{\mathbb{R}^2}\nabla u\nabla u_z\mathrm{d}x.\end{aligned} \tag{3.21}$$

Because $u \in X_r$ is a nontrivial critical point of J_n, that is, $J_n'(u)(u_z) = 0$ which gives that

$$\int_{\mathbb{R}^2}\left[\nabla u\nabla u_z + \left(|x|^2 - \omega\right)uu_z\right]\mathrm{d}x = -\frac{\lambda}{2}N_n'(u)(u_z) + \int_{\mathbb{R}^2} g_n(x,u)u_z\mathrm{d}x. \tag{3.22}$$

Then, we shall take some careful analyses for each item in (3.22). First of all, let us note that $uu_z = v_z^2$, it follows from (3.19), (3.21) and $\gamma > 1$ that, for all $\lambda \in (0, \lambda_3)$

$$\int_{\mathbb{R}^2} \left(|\nabla v_z|^2 + |x|^2 |v_z|^2\right) \mathrm{d}x \le 2\gamma^2 \int_{\mathbb{R}^2} \left[\nabla u \nabla u_z + \left(1 + |x|^2\right) uu_z\right] \mathrm{d}x. \tag{3.23}$$

Due to (2.8), some simple calculations provide us that

$$\frac{h_{u,n}^2(|x|)}{|x|^2} \le \mathcal{C}_\pi |u|_4^4 \le \mathcal{C}_\pi \mathbb{T}_4^4 \|u\|_X^4$$

and

$$\begin{aligned}\left|\int_{|x|}^{\infty} \frac{h_{u,n}(s)}{s} \mathfrak{F}_n(u(s)) \mathrm{d}s\right| &\le \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{h_{u,n}(|x|)}{|x|^2} \mathfrak{F}_n(u) \mathrm{d}x \le \mathcal{C}_\pi |u|_4^2 \int_{\mathbb{R}^2} \frac{\mathfrak{F}_n(u)}{|x|} \mathrm{d}x \\ &\le \mathcal{C}_\pi |u|_4^2 \left(\int_{|x|<1} \frac{1}{|x|^{\frac{3}{2}}} \mathrm{d}x\right)^{\frac{2}{3}} \left(\int_{|x|<1} \mathfrak{F}_n^3(u) \mathrm{d}x\right)^{\frac{1}{3}} + \mathcal{C}_\pi |u|_4^2 |u|_2^2 \\ &\le \mathcal{C}_\pi |u|_4^2 \left(|u|_6^2 + |u|_2^2\right) \le \mathcal{C}_\pi \mathbb{T}_4^2 \left(\mathbb{T}_2^2 + \mathbb{T}_6^2\right) \|u\|_X^4,\end{aligned}$$

where $\mathcal{C}_\pi > 0$ is a constant which is only dependent of π, and $\mathbb{T}_i$ denotes the best imbedding constant of the imbedding $X \hookrightarrow L^s(\mathbb{R}^2)$ with $s \in \{2, 4, 6\}$. Decreasing λ if necessary, it holds that

$$\left|-\frac{\lambda}{2} N_n'(u)(u_z)\right| \le \lambda \mathcal{C}_\pi \mathbb{T}_4^2 \left(\mathbb{T}_2^2 + \mathbb{T}_4^2 + \mathbb{T}_6^2\right) \|u\|_X^4 \int_{\mathbb{R}^2} uu_z \mathrm{d}x \le \frac{1}{4} \int_{\mathbb{R}^2} uu_z \mathrm{d}x. \tag{3.24}$$

In view of $J_n(u) < c^*$ and $J_n'(u) = 0$, for all $\sigma \in (0, \sigma_1)$ and $\lambda \in (0, \lambda_2)$ we take advantage of Lemma 2.1-(iv) and Lemma 2.2-(g_2) as well as (3.19) to have that

$$\begin{aligned}\theta c^* &= \theta J_n(u) - J_n'(u)(u) \\ &= \frac{\theta - 2}{2} \|u\|_X^2 - \frac{\theta - 2}{2} \int_{\mathbb{R}^2} \omega |u|^2 \mathrm{d}x + \frac{\lambda}{2} \left[\theta N_n(u) - 2N_n'(u)(u)\right] \\ &\quad + \int_{\mathbb{R}^2} \left[\theta g_n(x, u)u - G_n(x, u)\right] \mathrm{d}x \\ &\ge \frac{\theta - 2}{8} \|u\|_X^2. \end{aligned} \tag{3.25}$$

Consequently, (3.24) and (3.25) guarantees a $\lambda_* < \min\{\bar{\lambda}_0, \lambda_3\}$ which is independent of $n \in \mathbb{N}^+$ and $\gamma > 1$ such that for all $\lambda \in (0, \lambda_*)$

$$\left|-\frac{\lambda}{2} N_n'(u)(u_z)\right| \le \frac{1}{4} \int_{\mathbb{R}^2} uu_z \mathrm{d}x. \tag{3.26}$$

It follows from (2.14) that

$$\begin{aligned}\int_{\mathbb{R}^2} g_n(x, u) u \mathrm{d}x &\le \frac{1}{4} \int_{\mathbb{R}^2} uu_z \mathrm{d}x + \mathcal{C}_9 \int_{\mathbb{R}^2} uu_z \left(e^{\alpha |u|^2} - 1\right) \mathrm{d}x \\ &\le \frac{1}{4} \int_{\mathbb{R}^2} uu_z \mathrm{d}x + \mathcal{C}_9 I_{\alpha, \tilde{q}'} \left(\int_{\mathbb{R}^2} |v_z|^{2\tilde{q}} \mathrm{d}x\right)^{\frac{1}{\tilde{q}}},\end{aligned} \tag{3.27}$$

where $\mathcal{C}_9 > 0$ is independent of $n \in \mathbb{N}^+$ and

$$I_{\alpha, \tilde{q}'} \triangleq \left(\int_{\mathbb{R}^2} \left(e^{\alpha \tilde{q}' |\varpi|^2} - 1\right) \mathrm{d}x\right)^{\frac{1}{\tilde{q}'}}.$$

Here ϖ comes from (3.17) and $\tilde{q} \in (1, 2)$ with $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$.

As a consequence of (3.22), (3.23), (3.26) and (3.27), we have

$$\int_{\mathbb{R}^2}\left[|\nabla v_z|^2+\left(1+|x|^2\right)|v_z|^2\right]\mathrm{d}x \le \mathcal{C}_{10} I_{\alpha,\tilde{q}'}\gamma^2\left(\int_{\mathbb{R}^2}|v_z|^{2\tilde{q}}\mathrm{d}x\right)^{\frac{1}{\tilde{q}}}.$$

We fix $\tilde{q}\in(1,2)$ with $\tilde{q}'=\tilde{q}/(\tilde{q}-1)$ and $X\hookrightarrow L^4(\mathbb{R}^2)$, there is a constant $\bar{\mathcal{C}}_0>0$ independent γ such that

$$\left(\int_{\mathbb{R}^2}|v_z|^4\mathrm{d}x\right)^{\frac{1}{2}} \le \bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'}\gamma^2\left(\int_{\mathbb{R}^2}|v_z|^{2\tilde{q}}\mathrm{d}x\right)^{\frac{1}{\tilde{q}}}.$$

Once $v_z=u^\gamma$ in $\mathfrak{A}_z$ and $v_z\le u^\gamma$ in $\mathbb{R}^2$, there holds

$$\left(\int_{\mathfrak{A}_z}|u|^{4\gamma}\mathrm{d}x\right)^{\frac{1}{2}} \le \bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'}\gamma^2\left(\int_{\mathbb{R}^2}|u|^{2\tilde{q}\gamma}\mathrm{d}x\right)^{\frac{1}{\tilde{q}}},\ \forall z\in\mathbb{N}.$$

Applying the Lebesgue's Dominated Convergence theorem with $z\to\infty$ to the above formula, one has

$$|u|_{4\gamma}^{2\gamma}\le \bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'}\gamma^2|u|_{2\tilde{q}\gamma}^{2\gamma}. \tag{3.28}$$

We choose the constant $\mu=2/\tilde{q}$, then $\mu>1$ because $\tilde{q}\in(1,2)$. For every $j\in\mathbb{N}^+$, define $\gamma_j=\mu^j$ and thus $2\tilde{q}\gamma_{j+1}=2\tilde{q}\mu\gamma_j=4\gamma_j$. For $j=1$, $\gamma_1=\mu>1$ which can be applied in (3.28) to derive

$$|u|_{4\mu}\le \mu^{\frac{1}{\mu}}(\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2\mu}}|u|_4. \tag{3.29}$$

For $j=2$, $\gamma_2=\mu^2>1$ and $2\tilde{q}\gamma_2=4\gamma_1=4\mu$ and by (3.28),

$$|u|_{4\mu^2}\le (\mu^2)^{\frac{1}{\mu^2}}(\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2\mu^2}}|u|_{4\mu}. \tag{3.30}$$

For $j=3$, $\gamma_3=\mu^3>1$ and $2\tilde{q}\gamma_3=4\gamma_2=4\mu^2$ and by (3.28),

$$|u|_{4\mu^3}\le (\mu^3)^{\frac{1}{\mu^3}}(\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2\mu^3}}|u|_{4\mu^2}. \tag{3.31}$$

Similar to (3.29), (3.30) and (3.31), proceeding this iteration procedure j times, we can infer that

$$|u|_{4\mu^j}\le \mu^{\sum_{i=1}^{j}\frac{i}{\mu^i}}(\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2}\sum_{i=1}^{j}\frac{1}{\mu^i}}|u|_4 \tag{3.32}$$

invoking that $u\in L^{4\mu^j}(\mathbb{R}^2)$ for each $j\in\mathbb{N}^+$. Clearly, $\displaystyle\sum_{i=1}^{\infty}\frac{i}{\mu^i}=\frac{\mu}{(\mu-1)^2}$ and $\displaystyle\sum_{i=1}^{\infty}\frac{1}{\mu^i}=\frac{1}{\mu-1}$, thereby we can take the limit in (3.32) as $j\to\infty$ to obtain

$$|u|_\infty\le \mu^{\frac{\mu}{(\mu-1)^2}}(\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2(\mu-1)}}|u|_4^4.$$

Finally, if u changes sign, then it is enough to argue as before by contemplating once the positive part $u^+\triangleq\max\{u,0\}$ and once the negative part $u^-\triangleq\max\{-u,0\}$ in place of u in the definition of u_z. As a result, we shall conclude the verification of $u\in L^\infty(\mathbb{R}^2)$ for the nontrivial solution u finishing the proof of this lemma. □

With the help of Theorem 3.1 and Lemma 3.10, we are capable of exhibiting the detailed proof of Theorem 1.2.

Proof of Theorem 1.5. Recalling Theorem 3.1, we have showed that the function u is a nontrivial solution of Eq. (2.6) for every $\lambda\in(0,\lambda_*)$ and $\sigma\in(0,\sigma_*)$. Moreover, it satisfies $\|u\|_X^2<\dfrac{16\pi\theta}{\alpha_0(\theta-2)}$ by (3.25) and so Lemma 3.10 indicates that

$$|u|_\infty\le\left(\frac{2}{\tilde{q}}\right)^{\frac{2\tilde{q}}{(2-\tilde{q})^2}}\left[\bar{\mathcal{C}}_0^{\tilde{q}'}\int_{\mathbb{R}^2}(e^{\alpha\tilde{q}'\varpi^2}-1)\mathrm{d}x\right]^{\frac{\tilde{q}}{2\tilde{q}'(2-\tilde{q})}}\mathbb{T}_2^2\left[\frac{16\pi\theta}{\alpha_0(\theta-2)}\right]^2\triangleq\hat{\mathcal{C}}_0,$$

where $\mathbb{T}_2 > 0$ denotes an imbedding constant of $X \hookrightarrow L^2(\mathbb{R}^2)$. Now, we are able to reach the proof of Theorem 1.2 by fixing $n > \hat{\mathcal{C}}_0$, because in this case u is a nontrivial solution of Eq. (1.1) according to the definition of η_n in (2.4). The proof is completed. $\square$

4. Multiplicity results for subcritical problem

In this section, we are concerned with the existence of infinitely many solutions for the auxiliary semilinear elliptic equation (2.6) with a sublinear type nonlinearity for all $n \in \mathbb{N}^+$. Speaking it more clearly, we study the following class of gauged nonlinear Schrodinger equation with indefinite potential

$$-\Delta u + V_\omega(|x|)u + \lambda \left(\int_{|x|}^{\infty} \frac{h_{u,n}(s)}{s} \mathfrak{F}_n(u(s)) \mathrm{d}s + \frac{h_{u,n}^2(|x|)}{|x|^2} \right) \frac{\mathfrak{f}_n(u)}{2} = \xi(x)|u|^{p-2}u \text{ in } \mathbb{R}^2, \tag{4.1}$$

where $\omega > 0$ is sufficiently large such that the operator $-\Delta + V_\omega$ is non-degenerate, $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$ with $\xi \geq 0$ and $1 < p < 2$.

The main result concerning Eq. (4.1) can be stated as follows.

Theorem 4.1. *Let $\omega > 0$ be large and satisfy the operator $-\Delta + V_\omega$ is non-degenerate. If $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$ with $\xi \geq 0$ and $1 < p < 2$, then for all $n \in \mathbb{N}^+$, there is $\lambda^0 > 0$ such that for all $\lambda \in (0, \lambda^0)$ possesses a sequence of solutions $(u_m) \subset X_r$ satisfying*

$$\mathcal{J}_n(u_m) < 0 \text{ and } \mathcal{J}_n(u_m) \to 0 \text{ as } m \to +\infty \text{ uniformly in } n \in \mathbb{N}^+, \tag{4.2}$$

where the variational functional $\mathcal{J}_n : X_r \to \mathbb{R}$ corresponding to Eq. (4.1) is of class $\mathcal{C}^1$ and defined by

$$\mathcal{J}_n(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\omega(|x|)u^2 \right] \mathrm{d}x + \frac{\lambda}{2} N_n(u) - \frac{1}{p} \int_{\mathbb{R}^2} \xi(x)|u|^p \mathrm{d}x.$$

As before, we also divide the proof of the above theorem into several lemmas. Taking into account the existence of infinitely many solutions for Eq. (4.1), we introduce the following minimax argument which is known as the generalized fountain theorem due to Ding and Dong [15].

In what follows, we denote $\mathbb{X}$ by an Hilbert space equipped with the norm $\|\cdot\|_{\mathbb{X}}$. Let $\mathbb{X} = \mathbb{X}^- \bigoplus \mathbb{X}^+$ satisfy $\mathbb{X}^- = \bigoplus_{i=-\infty}^{-1} \mathbb{X}_i$ and $\mathbb{X}^+ = \bigoplus_{i=1}^{\infty} \mathbb{X}_i$, then there is the following result.

Proposition 4.2. *(see [15, Theorem 3.2]) Let $\Phi \in \mathcal{C}^1(\mathbb{X}, \mathbb{R})$ be an even functional which is $\mathcal{P}$-lower semicontinuous, that is, for all $\mathcal{C} \in \mathbb{R}$ the set $\{u \in \mathbb{X} : \Phi(u) \leq \mathcal{C}\}$ is $\mathcal{P}$-closed, and such that $\nabla\Phi$ is weakly sequentially continuous. If there is a constant $m_0 > 0$, for all $m > m_0$, such that Φ satisfies the following conditions:*

- (i) *there exists $\tau_m > 0$ such that $a^m \triangleq \inf \Phi(\partial\mathbb{B}^m) \geq 0$, where $\partial\mathbb{B}^m \triangleq \{u \in \mathbb{Y}_m : \|u\|_{\mathbb{X}} = \tau_m\}$ and $\mathbb{Y}_m = \bigoplus_{j=m}^{\infty} \mathbb{X}_j$;*
- (ii) *there exists a finite dimensional G-invariant subsequence $\hat{\mathbb{X}}_m \subset \mathbb{X}_m$ and there exists $0 < \nu_m < \tau_m$ such that $b^m \triangleq \sup \Phi(\mathcal{N}^m) < 0$, where $\mathcal{N}^m \triangleq \{u \in \mathbb{X}^- \bigoplus \hat{\mathbb{X}}_m : \|u\|_{\mathbb{X}} = \nu_m\}$;*
- (iiii) *$d^m \triangleq \inf \Phi(\mathbb{B}^m) \to 0$ as $m \to +\infty$, where $\mathbb{B}^m \triangleq \{u \in \mathbb{Y}_m : \|u\|_{\mathbb{X}} \leq \tau_m\}$.*

If Φ satisfies the (PS) *condition, then Φ possesses a sequence (u_m) of nontrivial critical points such that $\Phi(u_m) < 0$ and $\Phi(u_m) \to 0$ as $m \to +\infty$.*

Here, the Palais-Smale sequence at level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ sequence in short) corresponding to Φ assumes that $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ as $n \to \infty$, where $(x_n) \subset \mathbb{X}$. If for any $(\mathrm{PS})_c$ sequence (x_n) in $\mathbb{X}$, there exists a subsequence (x_{n_k}) such that $x_{n_k} \to x_0$ in $\mathbb{X}$ for some $x_0 \in \mathbb{X}$, then we say that the variational functional Φ satisfies the so-called $(\mathrm{PS})_c$ condition.

Proceeding as what we have done in Section 3, to apply Proposition 4.2, we continue setting $\mathbb{X} = X_r$. In view of the eigenvalue problem (2.1), let $\hat{\mu}_j - \omega$ be the eigenvalue of the non-degenerate operator $L \triangleq -\Delta + V_\omega(|x|)$ and $\mathbb{V}_j$ stands for the eigenfunction spaces related to $\hat{\mu}_j - \omega$, where $j \in \{1, 2, \cdots\}$. At this stage, we choose $\mathbb{X}^- = X_r^- \triangleq \bigoplus_{j=1}^{j_0} \mathbb{V}_j$ and $\mathbb{X}^+ = X_r^+ \triangleq \bigoplus_{j=j_0+1}^{+\infty} \mathbb{V}_j$, then $\mathbb{X} = \mathbb{X}^- \bigoplus \mathbb{X}^+$ and hence $X_r = X_r^- \bigoplus X_r^+$, where L is positive definite on the infinite dimensional space X_r^+ and negative definite on the finite dimensional space X_r^-. Hereafter, we are going to denote $P^\pm$ by the orthogonal projections from X_r to $X_r^\pm$ the decomposition above, respectively. Moreover, the spaces P^- and P^+ are also orthogonal with respect to L^2-inner product. For any $u \in X_r$, we define

$$\|u\|_L^2 = \|P^+u\|_L^2 + \|P^-u\|_L^2,$$

where

$$\|P^+u\|_L^2 = \int_{\mathbb{R}^2} \left[|\nabla P^+u|^2 + V_\omega(|x|)|P^+u|^2\right] \mathrm{d}x$$

and

$$\|P^-u\|_L^2 = -\int_{\mathbb{R}^2} \left[|\nabla P^-u|^2 + V_\omega(|x|)|P^-u|^2\right] \mathrm{d}x.$$

Obviously, one also obtains that

$$\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\omega(|x|)|u|^2\right] \mathrm{d}x = \|P^+u\|_L^2 - \|P^-u\|_L^2, \ \forall u \in X_r.$$

Consequently, we define $\|\cdot\|_{\mathbb{X}} = \|\cdot\|_L$. In other words, it is the case that $(\mathbb{X}, \|\cdot\|_{\mathbb{X}}) = (X_r, \|\cdot\|_L)$. As a matter of fact, we shall conclude that X_r endowed with the norm $\|\cdot\|_L$ can be compactly imbedded into $L^s(\mathbb{R}^2)$ for all $2 \le s < +\infty$, see Lemma A.1 below in detail. In this scenario, we define $\Phi = \mathcal{J}_n$ for all $n \in \mathbb{N}^+$ and it is simple to observe that $\mathcal{J}_n \in \mathcal{C}^1(X_r, \mathbb{R})$ by Lemma 2.1. What's more, let us recall the definition of η in (2.4), then one might realize that the functional N_n is even for all $n \in \mathbb{N}^+$ and so is $\mathcal{J}_n$. In the meanwhile, it would be very standard to certify that $\mathcal{J}_n$ is $\mathcal{P}$-lower semicontinuous and $\nabla \mathcal{J}_n$ is weakly sequentially continuous for all $n \in \mathbb{N}^+$. As a consequence, the application of Proposition 4.2 with $m_0 = j_0$ becomes available.

We next focus on the verifications of the geometry conditions on $\mathcal{J}_n$.

Lemma 4.3. *Let $1 < p < 2$ and $0 \le \xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$. Then, there exists a $\tau_m > 0$ independent of $\lambda > 0$ and $n \in \mathbb{N}^+$ such that*

$$a^m \triangleq \inf \mathcal{J}_n(\partial \mathbb{B}^m) \ge 0$$

for all $n \in \mathbb{N}^+$, where $\partial \mathbb{B}^m \triangleq \{u \in \mathbb{Y}_m : \|u\|_L = \tau_m\}$ and $\mathbb{Y}_m = \bigoplus_{j=m}^{\infty} \mathbb{V}_j$. In addition,

$$d^m \triangleq \inf \mathcal{J}_n(\mathbb{B}^m) \to 0 \text{ as } m \to +\infty \text{ uniformly in } n \in \mathbb{N}^+,$$

where $\mathbb{B}^m \triangleq \{u \in \mathbb{Y}_m : \|u\|_L \le \tau_m\}$.

Proof. For all $m > m_0$, we define

$$\beta_m \triangleq \sup_{u \in \mathbb{Y}_m \setminus (0)} \frac{|u|_2}{\|u\|_L} > 0. \tag{4.3}$$

Due to the compact imbedding $(X_r, \|\cdot\|_L) \hookrightarrow L^2(\mathbb{R}^2)$, we can proceed as the proof of [56, Lemma 3.8] to conclude that $\beta_m \to 0$ as $m \to \infty$. So, given $u \in \mathbb{Y}_m$, we are derived from (4.3) that

$$\begin{aligned} \mathcal{J}_n(u) &= \frac{1}{2}\|u\|_L^2 + \frac{\lambda}{2} N_n(u) - \frac{1}{p} \int_{\mathbb{R}^2} \xi(x)|u|^p \mathrm{d}x \\ &\ge \frac{1}{2}\|u\|_L^2 - \frac{1}{p}|\xi|_{\frac{2}{2-p}} \beta_m^p \|u\|_L^p = \frac{1}{4}\|u\|_L^2 \end{aligned}$$

provided $\|u\|_L = \left(\dfrac{4|\xi|_{\frac{2}{2-p}}\beta_m^p}{p}\right)^{\frac{1}{2-p}} \triangleq \tau_m$. So, it holds that

$$\mathcal{J}_n(u) \geq \frac{1}{4}\tau_m^2 \geq 0,\ \forall u \in \partial\mathbb{B}^m,$$

which indicates that $a^m \geq 0$. For all $u \in \mathbb{B}^m = \{u \in \mathbb{Y}_m : \|u\|_L \leq \tau_m\}$, it has that

$$0 \leftarrow -\frac{1}{p}|\xi|_{\frac{2}{2-p}}\beta_m^p\tau_m^p \leq -\frac{1}{p}|\xi|_{\frac{2}{2-p}}\beta_m^p\|u\|_L^p \leq \mathcal{J}_n(u) \leq \frac{1}{2}\|u\|_L^2 + C\|u\|_L^6 \leq \frac{1}{2}\tau_m^2 + C\tau_m^6 \to 0$$

as $m \to \infty$, where $C > 0$ is independent of $n \in \mathbb{N}^+$. Thus, $d^m \to 0$. The proof is completed. $\square$

Lemma 4.4. *Let $1 < p < 2$ and $0 \leq \xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$. Then, there exists a constant $\lambda^1 > 0$ such that for all $\lambda \in (0, \lambda^1)$, there exist a finite dimensional G-invariant subsequence $\hat{\mathbb{X}}_m \subset \mathbb{X}_m$ and $0 < \nu_m < \tau_m$ independent of $n \in \mathbb{N}^+$ such that*

$$b^m \triangleq \sup \Phi(\mathcal{N}^m) < 0,$$

where $\mathcal{N}^m \triangleq \{u \in \mathbb{X}^- \bigoplus \hat{\mathbb{X}}_m : \|u\|_L = \nu_m\}$

Proof. Given a $u = u^- + u^+ \in \mathbb{X}^- \bigoplus \hat{\mathbb{X}}_m$, where $\hat{\mathbb{X}}_m$ is a finite dimensional subsequence of $\mathbb{X}_m$, it is sufficient to demonstrate that $\mathcal{J}_n(u) < 0$ as $\|u\|_L \to 0$. Arguing it by a contradiction, we would suppose that there is a sequence $(u_k) \subset \mathbb{X}^- \bigoplus \hat{\mathbb{X}}_m$ with $\|u_k\|_L \to 0$ such that $\mathcal{J}_n(u_k) \geq 0$ for all $k \in \mathbb{N}^+$. Then, we set $v_k \triangleq \dfrac{u_k}{\|u_k\|_L}$ and so $\|v_k\|_L \equiv 1$. Passing to a subsequence if necessary, there exists a $v \in X_r$ such that $v_k \rightharpoonup v$, $v_k^- \rightharpoonup v^-$ and $v_k^+ \to v^+$ in X_r, $v_k \to v$ in $L^s(\mathbb{R}^2)$ for all $2 \leq s < +\infty$ and $v_k \to v$ a.e. in $\mathbb{R}^2$. Because of Remark A.2, without loss of generality, we choose $\lambda^1 = \frac{1}{2\mathbb{T}_2 n^4}$, then for all $\lambda \in (0, \lambda^1)$, Lemma 2.3-(iii) shows us that

$$\begin{aligned}
0 \leq \frac{\mathcal{J}_n(u_k)}{\|u_k\|_L^2} &= \frac{1}{2}\left(\|v_k^+\|_L^2 - \|v_k^-\|_L^2\right) + \frac{\lambda}{2\|u_k\|_L^2}N_n(u_k) - \frac{1}{p}\int_{\mathbb{R}^2}\xi(x)\frac{|u_k|^p}{\|u_k\|_L^2}\mathrm{d}x \\
&\leq \frac{1}{2}\left(\|v_k^+\|_L^2 - \|v_k^-\|_L^2\right) + \frac{1}{4}\left(\|v_k^+\|_L^2 + \|v_k^-\|_L^2\right) - \frac{1}{p}\int_{\mathbb{R}^2}\xi(x)\frac{|u_k|^p}{\|u_k\|_L^2}\mathrm{d}x \\
&= \frac{3}{4}\|v_k^+\|_L^2 - \frac{1}{4}\|v_k^-\|_L^2 - \frac{1}{p}\int_{\mathbb{R}^2}\xi(x)\frac{|u_k|^p}{\|u_k\|_L^2}\mathrm{d}x.
\end{aligned}$$

We claim that $v^+ \neq 0$. Otherwise, with aid of $v_k^+ \to 0$ in X_r, we would obtain

$$0 \leq \lim_{k\to+\infty}\frac{\mathcal{J}_n(u_k)}{\|u_k\|_L^2} \leq \frac{3}{4}\lim_{k\to+\infty}\|v_k^+\|_L^2 = 0$$

which, in turn, yields that $\|v_k^-\|_L \to 0$. Therefore, one deduces that $1 = \|v_k\|_L^2 = \|v_k^+\|_L^2 + \|v_k^-\|_L^2 \to 0$, a contradiction. This claim gives us that $v \neq 0$ and so $|\Theta_v| > 0$, where $\Theta_v \triangleq \{x \in \mathbb{R}^2 : |v(x)| > 0\}$. It follows from the Fatou's lemma and $p < 2$ that

$$\begin{aligned}
0 \leq \limsup_{k\to+\infty}\frac{\mathcal{J}_n(u_k)}{\|u_k\|_L^2} &\leq \frac{3}{4}\|v^+\|_L^2 - \frac{1}{4}\liminf_{k\to+\infty}\|v_k^-\|_L^2 - \frac{1}{p}\liminf_{k\to+\infty}\int_{\mathbb{R}^2}\xi(x)\frac{|u_k|^p}{\|u_k\|_L^2}\mathrm{d}x \\
&\leq \frac{3}{4}\|v^+\|_L^2 + \frac{1}{4} - \frac{1}{p}\liminf_{k\to+\infty}\frac{1}{\|u_k\|_L^{2-p}}\int_{\Theta_v}\xi(x)|v_k|^p\mathrm{d}x \\
&\to -\infty
\end{aligned}$$

which is an absurd. The proof is completed. $\square$

Lemma 4.5. *Let $1 < p < 2$ and $0 \leq \xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$ for all $x \in \mathbb{R}^2$. Then, there is a $\lambda^2 > 0$ such that for all $\lambda \in (0, \lambda^2)$ and for all $n \in \mathbb{N}^+$, $\mathcal{J}_n$ satisfies the (PS) condition.*

Proof. Given a $c \in \mathbb{R}$ which is independent of $n \in \mathbb{N}^+$, let $(u_k) \subset X_r$ be a (PS) sequence of $\mathcal{J}_n$ at the level c such that

$$J_n(u_k) \to c \text{ and } J_n'(u_n) \to 0. \tag{4.4}$$

On the one hand, we shall certify that (u_k) is uniformly bounded in $k \in \mathbb{N}^+$. According to Remark A.2, we take advantage of $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$ to have that

$$\begin{aligned}
o_k(1)\|u_k\|_L &\geq J_n'(u_k)(u_k^+ - u_k^-) \\
&= \|u_k\|_L^2 + \frac{\lambda}{2} N_n'(u_k)(u_k^+ - u_k^-) - \int_{\mathbb{R}^2} \xi(x)|u_k|^{p-2}u_k(u_k^+ - u_k^-)\mathrm{d}x \\
&\geq \|u_k\|_L^2 - \frac{\lambda}{2}\left|N_n'(u_k)(u_k^+ - u_k^-)\right| - |\xi|_{\frac{2}{2-p}}|u_k|_2^{p-1}|u_k^+ - u_k^-|_2 \\
&\geq \|u_k\|_L^2 - \frac{\lambda}{2}\left(\left|N_n'(u_k)(u_k)\right| + 2\left|N_n'(u_k)(u_k^-)\right|\right) - \mathbb{T}_2^{\frac{p}{2}}|\xi|_{\frac{2}{2-p}}\|u_k\|_L^p
\end{aligned} \tag{4.5}$$

For all $u \in X_r$, it follows from (2.7) that

$$\begin{aligned}
\frac{h_{u,n}(|x|)}{|x|} &= \frac{1}{4\pi|x|}\int_{B_{|x|}(0)} \mathfrak{F}_n(u(y))\mathrm{d}y \leq \frac{1}{4\pi|x|}\left(\int_{B_{|x|}(0)}\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{B_{|x|}(0)}\mathfrak{F}_n^2(u(y))\mathrm{d}y\right)^{\frac{1}{2}} \\
&\leq \frac{1}{4\sqrt{\pi}}\left(\int_{B_{|x|}(0)}\mathfrak{F}_n^2(u(y))\mathrm{d}y\right)^{\frac{1}{2}} \leq n^2|x|.
\end{aligned}$$

As a consequence, by means of (2.8) and Remark A.2, we obtain

$$\begin{aligned}
\left|N_n'(u_k)(u_k^-)\right| &= \Bigg|2\int_{\mathbb{R}^2}\frac{\mathfrak{F}_n(u_k)}{|x|^2}\left(\int_0^{|x|}\frac{s}{2}\mathfrak{F}_n(u_k(s))\,\mathrm{d}s\right)\left(\int_0^{|x|}\frac{s}{2}\mathfrak{f}_n(u_k(s))\,u_k^-(s)\mathrm{d}s\right)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2}\frac{\mathfrak{f}_n(u_k)u_k^-}{|x|^2}\left(\int_0^{|x|}\frac{s}{2}\mathfrak{F}_n(u_k(s))\,\mathrm{d}s\right)^2\mathrm{d}x\Bigg| \\
&\leq \frac{n^2}{\sqrt{2\pi}}\int_{\mathbb{R}^2}\frac{\mathfrak{F}_n(u_k)}{|x|}\left(\int_0^{|x|}s\mathfrak{f}_n^2(u_k(s))\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_{B_{|x|}(0)}|u_k^-(y)|^2\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}x \\
&\quad + n^4\left(\int_{\mathbb{R}^2}|x|^2|\mathfrak{f}_n(u_k)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|x|^2|u_k^-(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \frac{6n^3}{\sqrt{\pi}}\int_{\mathbb{R}^2}\frac{\mathfrak{F}_n(u_k)}{|x|^{\frac{1}{2}}}\left(\int_{B_{|x|}(0)}|u_k(y)|^2\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}x + 6n^4\mathbb{T}_2^2\|u_k\|_L\|u_k^-\|_L \\
&\leq 12n^4|u_k|_2^2 + 6n^4\mathbb{T}_2^2\|u_k\|_L\|u_k^-\|_L \leq 18n^4\mathbb{T}_2^2\|u_k\|_L^2,
\end{aligned}$$

form where and Lemma 2.1-(iii), we can determine a sufficiently small $\lambda^2 > 0$ such that for all $\lambda \in (0, \lambda^2)$

$$\lambda\left(\left|N_n'(u_k)(u_k)\right| + 2\left|N_n'(u_k)(u_k^-)\right|\right) \leq \|u_k\|_L^2.$$

Recalling (4.5) and $p < 2$, we can deduce that (u_k) is uniformly bounded in $k \in \mathbb{N}^+$ for all $\lambda \in (0, \lambda^2)$.

On the other hand, let us conclude that (u_n) contains a strongly convergent subsequence. Since (u_n) is uniformly bounded in X_r, going to a subsequence if necessary, there is a $u \in X_r$ such that $u_n \rightharpoonup u$ in X_r, $u_n \to u$ in $L^s(\mathbb{R}^2)$ for all $2 \leq s < +\infty$ and $u_n \to u$ a.e. in $\mathbb{R}^2$. Taking $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^2)$ into account, one finds that

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\xi(x)\left(|u_k|^{p-2}u_k-|u|^{p-2}u\right)(u_k^+-u^+)\mathrm{d}x\right| &\le |\xi|_{\frac{2}{2-p}}\left||u_k|^{p-2}u_k-|u|^{p-2}u\right|_{\frac{\frac{2}{p-1}}{\frac{2}{p-1}}}|u_k^+-u^+|_2\\ &\le |\xi|_{\frac{2}{2-p}}\left(|u_k|_2^{p-1}+|u|_2^{p-1}\right)|u_k-u|_2\\ &= o_k(1)\end{aligned}$$

which together with Lemma 2.1-(ii) and (4.4) indicates that

$$\begin{aligned}o_k(1) &= J_n'(u_k)(u_k^+-u^+)-J_n'(u)(u_k^+-u^+)\\ &= \|u_n^+-u^+\|_L^2+\lambda N_n'(u_k)(u_k^+-u^+)-\lambda N_n'(u)(u_k^+-u^+)-\int_{\mathbb{R}^2}\xi(x)|u_k|^{p-2}u_k(u_k^+-u^+)\mathrm{d}x\\ &= \|u_n^+-u^+\|_L^2+o_k(1).\end{aligned}$$

So, we have that $u_k^+\to u^+$ in X_r^+. Analogously, one can deduce that $u_k^-\to u^-$ in X_r^-. As a consequence, (u_n) contains a strongly convergent subsequence. The proof is completed. □

With the help of Lemma 4.3, Lemma 4.4 and Lemma 4.5, for $\lambda^0=\min\{\lambda^1,\lambda^2\}$, we shall demonstrate that Eq. (4.1) has a sequence of solutions $(u_m)\subset X_r$ satisfying (4.2) for all $\lambda\in(0,\lambda^0)$ and $n\in\mathbb{N}^+$. As a matter of fact, it is the desired conclusion exhibited in Theorem 4.1 and so we would not write its proof any more.

Based on the discussions above, we are able to give the proof of Theorem 1.5.

Proof of Theorem 1.5. According to Theorem 4.1, one knows that, for all $\lambda\in(0,\lambda^0)$ and $n\in\mathbb{N}^+$, $(u_m)\subset X_r$ is a sequence of solutions of (4.1) satisfying (4.2). Proceeding as the very similar arguments showed in the proof of Lemma 4.5, we have that $(\|u_m\|_L)$ is uniformly bounded in $n,m\in\mathbb{N}^+$ for all $\lambda\in(0,\lambda^0)$. Next, we clam that there is a $\lambda^*<\lambda^0$ such that for all $\lambda\in(0,\lambda^*)$, there exists a constant $\tilde{\mathcal{C}}>0$ independent of $n,m\in\mathbb{N}^+$ such that $\sup\limits_{m\in\mathbb{N}^+}|u_m|_\infty\le\tilde{\mathcal{C}}$. Without loss of generality, we would like to suppose that $\sup\limits_{m\in\mathbb{N}^+}|u_m|_\infty\ge 1$. Otherwise, the proof would be done immediately. In this scenario,

$$\xi(x)|u_m|^{p-1}\le\xi(x)|u_m|^p$$

and

$$\int_{\mathbb{R}^2}\xi^2(x)|u_m|^{2p}\mathrm{d}x\le\left(\int_{\mathbb{R}^2}|\xi|^{\frac{2}{2-p}}\mathrm{d}x\right)^{2-p}\left(\int_{\mathbb{R}^2}|u_m|^{\frac{2p}{p-1}}\mathrm{d}x\right)^{p-1}$$

form where it follows that $\xi(x)|u_m|^{p-1}$ is uniformly bounded in $n,m\in\mathbb{N}^+$ in $L^2(\mathbb{R}^2)$ for all $\lambda\in(0,\lambda^0)$. As a consequence, it is standard to take advantage of the arguments in the proof of Lemma 3.10 to find a small $\lambda^*<\lambda^0$ such that for all $\lambda\in(0,\lambda^*)$,

$$\left|\lambda\left(\int_{|x|}^{\infty}\frac{h_{u_m,n}(s)}{s}\mathfrak{F}_n(u_m(s))\mathrm{d}s+\frac{h_{u_m,n}^2(|x|)}{|x|^2}\right)\right|\le 1.$$

In view of (2.8) and (4.1), it holds that

$$-\Delta u_m+|x|^2u_m\le(\omega+3)|u_m|+\xi(x)|u_m|^{p-1}\text{ in }\mathbb{R}^2.$$

Because $(\omega+3)|u_m|+\xi(x)|u_m|^{p-1}$ is uniformly bounded in $n,m\in\mathbb{N}^+$ in $L^2(\mathbb{R}^2)$ for each $\lambda\in(0,\lambda^*)$, the classic elliptic regularity results reveal that $\sup\limits_{m\in\mathbb{N}^+}|u_m|_\infty\le\tilde{\mathcal{C}}$ holds true. Owing to the definition of η_n, we are able to derive that $(u_m)\subset X_r$ is a sequence of solutions of (1.1) if we choose $n>\tilde{\mathcal{C}}$. The proof is completed. □

Appendix A. The proof of Lemma 3.7

In this section, we exhibit the detailed proof of Lemma 3.7. In order to begin with it, let us introduce some preliminary results.

First of all, we define $V_\omega^{\pm}(|x|) \triangleq \max\{\pm V_\omega(|x|), 0\}$ for all $x \in \mathbb{R}^2$. Given a $u \in X_r$, since $V_\omega^+(|x|) \leq |x|^2$ for all $x \in \mathbb{R}^2$, then

$$\|\cdot\|_+ = \left(\int_{\mathbb{R}^2} \left[|\nabla \cdot|^2 + V_\omega^+(|x|)|\cdot|^2\right] \mathrm{d}x\right)^{\frac{1}{2}}$$

can be regarded as a norm on X_r.

Lemma A.1. *For any $\omega > 0$, the norms $\|\cdot\|_+$, $\|\cdot\|_L$ and $\|\cdot\|_X$ are equivalent to each other, where $\|\cdot\|_X$ comes from Section 2.*

Proof. On the one hand, we shall certify that $\|\cdot\|_+$ is equivalent to $\|\cdot\|_X$. The reader might find that $V_\omega^{\pm}(|x|) \leq |x|^2$ for all $x \in \mathbb{R}^2$, then it holds that $\|u\|_+ \leq \|u\|_X$ for all $u \in X_r$. Moreover, it is simple to observe that $\lim\limits_{|s|\to+\infty} \dfrac{V_\omega^+(|x|)}{|x|^2} = 1$. So, there is an $R_\omega > 1$ such that

$$|x| > R_\omega \Longrightarrow V_\omega^+(|x|) > \frac{1}{2}|x|^2.$$

With aid of the above fact, for all $u \in X_r$, we obtain

$$\begin{aligned}
\|u\|_X^2 &= \int_{|x|>R_\omega} \left(|\nabla u|^2 + |x|^2|u|^2\right) \mathrm{d}x + \int_{|x|\leq R_\omega} \left(|\nabla u|^2 + |x|^2|u|^2\right) \mathrm{d}x \\
&\leq 2\int_{|x|>R_\omega} \left[|\nabla u|^2 + V_\omega^+(|x|)|u|^2\right] \mathrm{d}x + R_\omega^2 \int_{|x|\leq R_\omega} \left(|\nabla u|^2 + |u|^2\right) \mathrm{d}x \\
&\leq 2\int_{|x|>R_\omega} \left[|\nabla u|^2 + V_\omega^+(|x|)|u|^2\right] \mathrm{d}x + R_\omega^2(1+\mathcal{C}_{R_\omega}^2) \int_{|x|\leq R_\omega} |\nabla u|^2 \mathrm{d}x \\
&\leq \left[2 + R_\omega^2(1+\mathcal{C}_{R_\omega}^2)\right] \int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\omega^+(|x|)|u|^2\right] \mathrm{d}x,
\end{aligned}$$

where $\mathcal{C}_{R_\omega} > 0$ is a constant independent of u. Therefore, we have that $\|u\|_X \leq \mathcal{C}\|u\|_+$ for some $\mathcal{C} > 0$ independent of u. Thus, $\|\cdot\|_+$ is equivalent to $\|\cdot\|_X$.

On the other hand, let us show that $\|\cdot\|_+$ is equivalent to $\|\cdot\|_L$. Since $V_\omega^-(|x|) = \max\{\omega - |x|^2, 0\} \leq \omega$ for all $x \in \mathbb{R}^2$, then for all $u = P^+u + P^-u \in X_r$ with $P^{\pm}u \in X_r^{\pm}$, there holds

$$\begin{aligned}
\|u\|_+^2 &= \int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\omega(|x|)|u|^2\right] \mathrm{d}x + \int_{\mathbb{R}^2} V_\omega^-(|x|)|u|^2 \mathrm{d}x \\
&\leq \|P^+u\|_L^2 - \|P^-u\|_L^2 + \omega \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x \\
&\leq \|P^+u\|_L^2 + \|P^-u\|_L^2 + 2\omega\left(|P^+u|_2^2 + |P^-u|_2^2\right) \\
&\leq \|u\|_L^2 + 2\omega\left(\frac{\|P^+u\|_L^2}{\hat{\mu}_{j_0+1} - \omega} + \frac{\|P^-u\|_L^2}{\omega - \hat{\mu}_{j_0}}\right) \leq \mathcal{C}_{\omega,\hat{\mu}_{j_0+1},\hat{\mu}_{j_0}}\|u\|_L^2
\end{aligned} \tag{A.1}$$

for some $\mathcal{C}_{\omega,\hat{\mu}_{j_0+1},\hat{\mu}_{j_0}} > 0$ independent of u. Due to the definition of $\|\cdot\|_L$, one has that

$$\|P^+u\|_L^2 = \int_{\mathbb{R}^2} \left[|\nabla P^+u|^2 + V_\omega(|x|)|P^+u|^2\right] \mathrm{d}x \leq \|P^+u\|_+^2 \tag{A.2}$$

and

$$\|P^-u\|_L^2 = -\int_{\mathbb{R}^2} \left[|\nabla P^-u|^2 + V_\omega(|x|)|P^-u|^2\right] \mathrm{d}x \leq \omega \int_{\mathbb{R}^2} |P^-u|^2 \mathrm{d}x \leq \omega \hat{\mathbb{T}}_2 \|P^-u\|_+^2, \tag{A.3}$$

where $\hat{\mathbb{T}}_2 > 0$ denotes the best imbedding constant of $(X_r, \|\cdot\|_+)$ into $L^2(\mathbb{R}^2)$. Since P^+ and P^- are orthogonal and $0 \le V_\omega^-(|x|) \le \omega$ for all $x \in \mathbb{R}^2$, we conclude that

$$\begin{aligned}
\|u\|_+^2 &= \int_{\mathbb{R}^2} \left[|\nabla P^+u + \nabla P^-u|^2 + V_\omega^+(|x|)|P^+u + P^-u|^2\right] \mathrm{d}x \\
&= \|P^+u\|_+^2 + \|P^-u\|_+^2 + 2\int_{\mathbb{R}^2} \left[\nabla P^+u \nabla P^-u + V_\omega^+(|x|)P^+uP^-u\right] \mathrm{d}x \\
&= \|P^+u\|_+^2 + \|P^-u\|_+^2 + 2\int_{\mathbb{R}^2} V_\omega^-(|x|)P^+uP^-u\mathrm{d}x \\
&\ge \|P^+u\|_+^2 + \|P^-u\|_+^2 - 2\omega\hat{\mathbb{T}}_2\|u\|_+^2,
\end{aligned}$$

which together with (A.2) and (A.3) indicates that

$$\begin{aligned}
\|u\|_L^2 &= \|P^+u\|_L^2 + \|P^-u\|_L^2 \le \|P^+u\|_+^2 + \omega\hat{\mathbb{T}}_2\|P^-u\|_+^2 \\
&\le \left(1 + \omega\hat{\mathbb{T}}_2\right)\left(\|P^+u\|_+^2 + \|P^-u\|_+^2\right) \le \left(1 + 2\omega\hat{\mathbb{T}}_2\right)^2 \|u\|_+^2. \quad \text{(A.4)}
\end{aligned}$$

Combining (A.1) and (A.4), we can derive that $\|\cdot\|_+$ is equivalent to $\|\cdot\|_L$. The proof is completed. □

Remark A.2. According to Lemma A.1 above, without loss of generality, we would not distinguish the norm $\|\cdot\|_X$ from $\|\cdot\|_L$ on X_r when there is no misunderstanding in the whole paper.

Before making an estimate for the mountain-pass value c_n in (3.7), in some similar spirit of [3, Lemma 4.5], we need to introduce some significant observations that play fundamental roles in the proof of Lemma 3.7. Regarding the Moser sequence of functions defined in (3.10), some elementary computations give us that

$$\int_{\mathbb{R}^2} |\nabla w_k|^2 \mathrm{d}x = 1 \text{ and } \int_{\mathbb{R}^2} |w_k|^2 \mathrm{d}x = \frac{1}{4\log k}\left(1 - \frac{1 + 2\log k}{k^2}\right) \triangleq \delta_k^1, \quad \text{(A.5)}$$

$$\begin{aligned}
\int_{\mathbb{R}^2} |x|^2|w_k|^2 \mathrm{d}x &= \int_{0\le|x|\le\frac{r_0}{k}} |x|^2|w_k|^2 \mathrm{d}x + \int_{\frac{r_0}{k}<|x|\le 1} |x|^2|w_k|^2 \mathrm{d}x \\
&= \frac{\log k}{4k^4} + \frac{1}{32\log k}\left(1 - \frac{1 + 4\log k + 8\log^2 k}{k^4}\right) \triangleq \delta_k^2, \quad \text{(A.6)}
\end{aligned}$$

$$\int_{\mathbb{R}^2} |\nabla w_k| \mathrm{d}x = \frac{\sqrt{2\pi}r_0}{\sqrt{\log k}}\left(1 - \frac{1}{k}\right) \triangleq \delta_k^3, \quad \text{(A.7)}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^2} |w_k| dx &= \int_{B_{\frac{r_0}{k}}(0)} |w_k| \mathrm{d}x + \int_{B_{r_0}(0)\setminus B_{\frac{r_0}{k}}(0)} |w_k| \mathrm{d}x \\
&= \frac{\pi r_0^2\sqrt{\log k}}{k^2\sqrt{2\pi}} + \frac{\pi r_0^2 \log r_0}{\sqrt{\log k}\sqrt{2\pi}}\left(1 - \frac{1}{k^2}\right) - \frac{\sqrt{2\pi}}{\sqrt{\log k}}\int_{\frac{\rho}{k}}^{\rho} r\log r \mathrm{d}r \\
&= \frac{\pi r_0^2\sqrt{\log k}}{k^2\sqrt{2\pi}} + \frac{\sqrt{2\pi}r_0^2}{4\sqrt{\log k}}\left(1 - \frac{1}{k^2}\right) - \frac{\sqrt{2\pi}r_0^2\sqrt{\log k}}{2k^2} \\
&\triangleq \delta_k^4 \le \frac{\pi r_0^2\sqrt{\log k}}{k^2\sqrt{2\pi}} + \frac{\sqrt{2\pi}r_0^2}{4\sqrt{\log k}} \le \frac{\sqrt{2\pi}r_0^2}{2\sqrt{\log k}}. \quad \text{(A.8)}
\end{aligned}$$

Moreover, recalling $w_k = w_k^+ + w_k^-$ as well as P^+ and P^- are orthogonal, we derive

$$0 \le \int_{\mathbb{R}^2} |\nabla w_k^-|^2 \mathrm{d}x = -\|w_k^-\|_L^2 - \int_{\mathbb{R}^2} V_\omega(x)|w_k^-|^2 \mathrm{d}x \le \omega\int_{\mathbb{R}^2} |w_k^-|^2 \mathrm{d}x \le \omega\int_{\mathbb{R}^2} |w_k|^2 \mathrm{d}x \to 0$$

yielding that $\|w_k^-\|_L^2 \to 0$, and so, (A.5) and (A.6) reveal that

$$\|w_k^+\|_L^2 = \|w_k^-\|_L^2 + \int_{\mathbb{R}^2} \left[|\nabla w_k|^2 + V_\omega(x)|w_k|^2\right] \mathrm{d}x = \int_{\mathbb{R}^2} |\nabla w_k|^2 dx + o_k(1) = 1 + o_k(1). \tag{A.9}$$

By the inner product $(\cdot,\cdot)_L$ in Section 4, for all $v \in X_r^-$, it follows from (A.7) and (A.8) that

$$\begin{aligned}|(w_k^-, v)_L| &= \left|\int_{\mathbb{R}^2} [\nabla w_k \nabla v + V_\omega(|x|) w_k v]\, \mathrm{d}x\right| \le |\nabla v|_\infty \int_{\mathbb{R}^2} |\nabla w_k| \mathrm{d}x + (1+\omega)|v|_\infty \int_{\mathbb{R}^2} |w_k| \mathrm{d}x \\ &\le \frac{\sqrt{2\pi} r_0}{\sqrt{\log k}} (c_1 |\nabla v|_2 + c_2(1+\omega)|v|_2) \le \frac{\sqrt{2\pi} r_0 c_3}{\sqrt{\log k}} \|v\|_{H^1(\mathbb{R}^2)} \le \frac{\sqrt{2\pi}}{\sqrt{\log k}} \|v\|_L \triangleq A_k^{-1} \|v\|_L, \end{aligned} \tag{A.10}$$

where we used the fact that $\dim X_r^- < +\infty$ and $r_0 \in (0,1)$ can be chosen arbitrarily small.

Lemma A.3. *Given (w_k) defined above, if (t_k) and $(\|v_k\|_L)$ with $(v_k) \subset X_r^-$ are uniformly bounded in $n, k \in \mathbb{N}^+$, then for all $\lambda \in (0, \frac{1}{\mathbb{T}_2 n^4})$, it holds that*

$$\lambda N_n(v_k + t_k w_k) \le \|v_k + t_k w_k^-\|_L^2 + o_k(1).$$

Proof. Because we have verified that $\|w_k^-\|_L \to 0$, we apply the facts that (t_k) and $(\|v_k\|_L)$ are uniformly bounded in (A.10) to reach

$$\|v_k + t_k w_k^-\|_L^2 = \|v_k\|_L^2 + t_k^2 \|w_k^-\|_L^2 + 2t_k (v_k, w_k)_L = \|v_k\|_L^2 + o_k(1).$$

On the other hand, for all $u \in X_r$, it follows from (2.7) that

$$\begin{aligned}\frac{h_{u,n}(|x|)}{|x|} &= \frac{1}{4\pi|x|} \int_{B_{|x|}(0)} \mathfrak{F}_n(u(y)) \mathrm{d}y \le \frac{1}{4\pi|x|} \left(\int_{B_{|x|}(0)} \mathrm{d}y\right)^{\frac12} \left(\int_{B_{|x|}(0)} \mathfrak{F}_n^2(u(y)) \mathrm{d}y\right)^{\frac12} \\ &\le \frac{1}{4\sqrt{\pi}} \left(\int_{B_{|x|}(0)} \mathfrak{F}_n^2(u(y)) \mathrm{d}y\right)^{\frac12} \le n^2 |x|\end{aligned}$$

which implies that

$$\begin{aligned}\lambda N_n(v_k + t_k w_k) &= \lambda \int_{\mathbb{R}^2} \frac{h_{v_k + t_k w_k, n}(|x|)}{|x|^2} \mathfrak{F}_n(v_k + t_k w_k) \mathrm{d}x \le \lambda n^4 \int_{\mathbb{R}^2} |x|^2 |v_k + t_k w_k|^2 \mathrm{d}x \\ &\le \frac{1}{\mathbb{T}_2} \left\{ \int_{\mathbb{R}^2} |x|^2 |v_k|^2 \mathrm{d}x + t_k^2 \int_{\mathbb{R}^2} |x|^2 |w_k|^2 \mathrm{d}x + 2t_k \left(\int_{\mathbb{R}^2} |x|^2 |v_k|^2 \mathrm{d}x\right)^{\frac12} \left(\int_{\mathbb{R}^2} |x|^2 |w_k|^2 \mathrm{d}x\right)^{\frac12} \right\} \\ &= \frac{1}{\mathbb{T}_2} \int_{\mathbb{R}^2} |x|^2 |v_k|^2 \mathrm{d}x + o_k(1) \le \|v_k\|_L^2 + o_k(1),\end{aligned}$$

where we have used Remark A.2 and (A.6). The proof is completed. □

We are now in a position to show the proof of Lemma 3.7 in detail.

Proof of Lemma 3.7. Arguing it indirectly, for all $n \in \mathbb{N}^+$, we could suppose that

$$\max_{t \ge 0, v \in X_r^-} J_n(v + t w_k) \ge c^*, \ \forall k \in \mathbb{N}^+. \tag{A.11}$$

In view of the proofs of Lemmas 3.3 and 3.4, there exist $t_k > 0$ and $\tilde{v}_k \in X_r^-$ such that

$$\begin{aligned} c^* \le \max_{t \ge 0, v \in X_r^-} J_n(v + t w_k) &= \max_{t \ge 0, v \in X_r^-} J_n(v + t w_k^+) = J_n(\tilde{v}_k + t_k w_k^+) \\ &= J_n(\tilde{v}_k - t_k w_k^- + t_k w_k) \triangleq J_n(v_k + t_k w_k),\end{aligned}$$

from where it follows that

$$\frac{1}{2}\left(t_k^2\|w_k^+\|_L^2 - \|v_k + t_k w_k^-\|_L^2\right) + \frac{\lambda}{2} N_n(v_k + t_k w_k) \geq c^* + \int_{\mathbb{R}^2} G_n(x, v_k + t_k w_k)\mathrm{d}x \tag{A.12}$$

and

$$\left(t_k^2\|w_k^+\|_L^2 - \|v_k + t_k w_k^-\|_L^2\right) + \frac{\lambda}{2} N_n'(v_k+t_kw_k)(v_k+t_kw_k) = \int_{\mathbb{R}^2} g_n(x, v_k+t_kw_k)(v_k+t_kw_k)\mathrm{d}x. \tag{A.13}$$

Combining (A.5), (A.6), (A.9) and (A.10), we have

$$\begin{aligned} t_k^2\|w_k^+\|_L^2 - \|v_k + t_k w_k^-\|_L^2 &= t_k^2(\|w_k^+\|_L^2 - \|w_k^-\|_L^2) - \|v_k\|_L^2 - 2t_k(w_k^-, v_k)_L \\ &= t_k^2 \int_{\mathbb{R}^2} [|\nabla w_k|^2 + V_\omega(|x|) w_k^2]\mathrm{d}x - \|v_k\|_L^2 - 2t_k(w_k^-, v_k)_L \\ &\leq t_k^2(1 + \omega\delta_k^1 + \delta_k^2) - \|v_k\|_L^2 + 2t_k A_k^{-1}\|v_k\|_L, \end{aligned}$$

and, according to Remark A.2, we are capable of taking advantage of Lemma 2.1-(iii) to see that

$$\begin{aligned} \left|N_n'(v_k + t_k w_k)(v_k + t_k w_k)\right| &\leq \mathbb{T}_2 n^4 \|v_k + t_k w_k\|_L^2 = \mathbb{T}_2 n^4 \left(\|t_k w_k^+\|_L^2 + \|v_k + t_k w_k^-\|_L^2\right) \\ &\leq \mathbb{T}_2 n^4 \left[t_k^2(1 + o_k(1)) + \|v_k\|_L^2 + 2t_k A_k^{-1}\|v_k\|_L\right]. \end{aligned}$$

Since $\lambda \in (0, \frac{1}{\mathbb{T}_2 n^4})$ and $g_n(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ by (2.13), then (A.13) gives us that

$$\frac{3}{2} t_k^2(1 + o_k(1)) - \frac{1}{2}\|v_k\|_L^2 + 3t_k A_k^{-1}\|v_k\|_L \geq 0 \tag{A.14}$$

which indicates that

$$\frac{\|v_k\|_L}{t_k} \leq 3A_k^{-1} + \sqrt{9A_k^{-2} + 3(1 + o_k(1))} = \frac{A_k(1 + o_k(1))}{\sqrt{1 + \frac{1}{3}A_k(1 + o_k(1))} - 1}, \ \forall k \in \mathbb{N}^+. \tag{A.15}$$

Using $\dim X_r^- < +\infty$ again and (A.15), for all $x \in B_{\frac{r_0}{k}}(0)$, we can conclude that

$$\begin{aligned} t_k w_k(x) + v_k(x) &\geq \frac{t_k\sqrt{\log k}}{\sqrt{2\pi}} - |v_k|_\infty \geq \frac{t_k\sqrt{\log k}}{\sqrt{2\pi}} - c_4\|v_k\|_L \\ &\geq t_k A_k \left(1 - \frac{c_4(1 + o_k(1))}{\sqrt{1 + \frac{1}{3}A_k(1 + o_k(1))} - 1}\right) \triangleq t_k A_k(1 - \tilde{A}_k). \end{aligned} \tag{A.16}$$

We claim that there is a $t_0 > 0$ independent of $n, k \in \mathbb{N}^+$ such that $\liminf\limits_{k\to+\infty} t_k \geq t_0$ along a subsequence. Otherwise, we would assume that $t_k \to 0$ and so $\|v_k\|_L \to 0$ by (A.15) as $k \to +\infty$. In view of (A.12), one could derive a contraction. With aid of this claim and (A.16), since $A_k \to +\infty$ and $\tilde{A}_k \to 0$, we are to make full use of Lemma 2.2-(g_3) and (g_4). So, for all $\epsilon \in (0, \beta_0/2)$, there is a constant $R_\epsilon = R(\epsilon) > \hat{t}_0$ such that

$$g_n(x,t) \geq (\beta_0/2 - \epsilon)\, t^{-\vartheta} e^{\alpha_0 |t|^2}, \ \forall x \in \mathbb{R}^2 \text{ and } t \geq R_\epsilon.$$

For some sufficiently large $k \in \mathbb{N}^+$, one knows that $t_k w_k(x) + v_k(x) \geq R_\epsilon$ on $B_{r_0/k}(0)$. Hence,

$$\begin{aligned} &\left(t_k^2\|w_k^+\|_L^2 - \|v_k + t_k w_k^-\|_L^2\right) + \frac{\lambda}{2} N_n'(v_k + t_k w_k)(v_k + t_k w_k) \\ &\geq \int_{\mathbb{R}^2} g_n(x, v_k + t_k w_k)(v_k + t_k w_k)\mathrm{d}x \geq \int_{B_{r_0/k}(0)} g_n(x, v_k + t_k w_k)(v_k + t_k w_k)\mathrm{d}x \\ &\geq \pi\,(\beta_0/2 - \epsilon)\left[t_k A_k(1 - \tilde{A}_k)\right]^{1-\vartheta} e^{\alpha_0|t_kA_k(1-\tilde{A}_k)|^2}\left(\frac{r_0}{k}\right)^2 \\ &= \pi {r_0}^2\,(\beta_0/2 - \epsilon)\left[(2\pi)^{-1} t_k \log k(1 - \tilde{A}_k)\right]^{1-\vartheta} e^{\left[\alpha_0(2\pi)^{-1}t_k^2(1-\tilde{A}_k)^2 - 2\right]\log k}. \end{aligned} \tag{A.17}$$

It is similar to (A.14) that

$$\begin{aligned}\log\left[\frac{3}{2}t_k^2(1+o_k(1))-\frac{1}{2}\|v_k\|_L^2+3t_kA_k^{-1}\|v_k\|_L\right]\\ \geq \mathcal{C}\log(1-\tilde{A}_k)+(1-\vartheta)\left[\log t_k+\log(\log k)\right]+\left[\alpha_0(2\pi)^{-1}t_k^2(1-\tilde{A}_k)^2-2\right]\log k\end{aligned} \tag{A.18}$$

for some $\mathcal{C}>0$ independent of $n,k\in\mathbb{N}^+$. Thereby, with the help of (A.15), we must conclude that (t_k) is uniformly bounded in $n,k\in\mathbb{N}^+$ and without loss of generality, we are supposing that $t_k\to t_0>0$ as $k\to+\infty$ along a subsequence. Owing to Lemma A.3, we are derived from (A.12) that $t_0^2\geq 2c^*=\frac{4\pi}{\alpha_0}$.

If $\vartheta\in[0,1)$ in (f_4), since $\|v_k\|_L$ is uniformly bounded in $n,k\in\mathbb{N}^+$ by (A.15), it follows from (A.18) that

$$\tilde{\mathcal{C}}+\log t_0^2\geq(1-\vartheta)\log(\log k)+o_k(1)$$

which is impossible if one tends $k\to+\infty$.

If $\vartheta=1$ in (f_4), since $\beta_0=+\infty$ in this situation, we are derived from (A.17) that

$$\tilde{\mathcal{C}}+\log t_0^2\geq\frac{\beta_0}{4}+o_k(1)$$

which also yields a contradiction. In summary, we would always demonstrate a contradiction if (A.11) is false. The proof is completed. □

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