ON A CLASS OF PLANAR LOGARITHMIC CHOQUARD EQUATIONS WITH STEEP POTENTIAL WELL AND SUPERCRITICAL EXPONENTIAL GROWTH

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Abstract. We establish the existence and asymptotic behavior of nontrivial solutions for the following class of planar logarithmic Choquard equations

$$-\Delta u+\lambda V(x)u=\left[\log\frac{1}{|x|}*G(u)\right]g(u)\text{ in }\mathbb{R}^2,$$

where $V\in\mathcal{C}^0(\mathbb{R}^2,\mathbb{R}^+)$ denotes a potential well with $\lambda>0$ and G is the primitive of g that fulfills the supercritical exponential growth in the Trudinger-Moser sense. Thanks to an asymptotical approximation approach and a powerful truncation argument, we conclude that this equation admits at least a nontrivial solution for all sufficiently large $\lambda>0$ using variational methods, where the decay rate of the obtained solution as $|x|\to+\infty$ and its asymptotic behavior as $\lambda\to+\infty$ are also considered. In particular, we are capable of supposing the "almost optimal" growth condition $\lim\limits_{t\to+\infty}t^\vartheta g(t)G(t)e^{-8\pi t^2}>0$ for $\vartheta\in(0,3]$.

1. Introduction and main results

This article focuses on the existence and asymptotic behavior of nontrivial solutions for the following planar logarithmic Choquard equation

$$-\Delta u+\lambda V(x)u=\left[\log\frac{1}{|x|}*G(u)\right]g(u)\text{ in }\mathbb{R}^2,\tag{1.1}$$

where $V\in\mathcal{C}^0(\mathbb{R}^2,\mathbb{R}^+)$ denotes a potential well with $\lambda>0$ and F is the primitive of f that fulfills the supercritical exponential growth in the Trudinger-Moser sense. The potential V is supposed to satisfy the following set of assumptions:

- (V_1) $V\in\mathcal{C}^0(\mathbb{R}^2,\mathbb{R})$ with $V(x)\geq0$ on $\mathbb{R}^2$;
- (V_2) $\Omega\triangleq\text{int}V^{-1}(0)$ is nonempty and bounded with smooth boundary, and $\overline{\Omega}=V^{-1}(0)$;
- (V_3) there exists a $b>0$ such that the set $\Xi\triangleq\{x\in\mathbb{R}^2:V(x)<b\}$ is nonempty and has a bounded measure.

It is well-known that Bartsch and his collaborators firstly introduced the assumptions like $(V_1)-(V_3)$ in [11,12]. Particularly, the harmonic trapping potential

$$V(x)=\begin{cases}\omega_1|x_1|^2+\omega_2|x_2|^2-\omega, & \text{if } |(\sqrt{\omega_1}x_1,\sqrt{\omega_2}x_2)|^2\geq\omega,\\ 0, & \text{if } |(\sqrt{\omega_1}x_1,\sqrt{\omega_2}x_2)|^2\leq\omega,\end{cases}$$

with $\omega>0$ satisfies $(V_1)-(V_3)$, where $\omega_i>0$ is called by the anisotropy factor of the trap in quantum physics and trapping frequency of the ith-direction in mathematics, see e.g. [13,17,41]. Actually, the potential λV with the above hypotheses is usually denoted by the steep potential well.

Regarding the nonlinearities g and G, throughout the whole paper, we shall always suppose that

$$G(s)=F(s)e^{\sigma s^2},\ \forall s\in\mathbb{R},\tag{1.2}$$

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where $F : \mathbb{R} \to \mathbb{R}$ satisfies the critical exponential growth at infinity and $\sigma > 0$. Owing to the celebrated Trudinger-Moser type inequality, one might say that a function h has the critical exponential growth at infinity if there exists a constant $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|h(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases} \tag{1.3}$$

This definition was introduced by Adimurthi and Yadava [2], see also de Figueiredo, Miyagaki and Ruf [27] for example. Very recently, some results extend it to a class of so-called supercritical exponential growth in [7,8,52], see also [26,30,44] for example. In the present paper, we shall follow the same spirit of [7,8,52], nevertheless, there are some interesting techniques to the essential improvements. Speaking it more clearly, it is supposed that the function f satisfies (1.3) and the following assumptions

- (f_1) $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, $f(s) \equiv 0$ for all $s \le 0$ and $f(s) = o(s)$ as $s \to 0^+$;
- (f_2) there exists a constant $\delta \in (0,1)$ such that
$$\frac{F(s)f'(s)}{f^2(s)} \ge \delta,\ \forall s > 0,\ \text{where } F(s) = \int_0^s f(t)\mathrm{d}t \text{ is given in (1.2)};$$
- (f_3) $\lim\limits_{s\to+\infty} \dfrac{F(s)f'(s)}{f^2(s)} = 1$ or equivalently, $\lim\limits_{s\to+\infty} \dfrac{\mathrm{d}}{\mathrm{d}s}\dfrac{F(s)}{f(s)} = 1$;
- (f_4) there exist some constants $\beta > 0$ and $\vartheta \in (0,3]$ such that
$$\liminf_{s\to+\infty} \frac{s^\vartheta f(s)F(s)}{e^{2\alpha_0 s^2}} \ge \beta > \beta_0,$$
where $\beta_0 = 0$ if $\vartheta < 3$, while if $\vartheta = 3$, then $\beta_0 \triangleq \frac{16}{\pi\alpha_0^2 r_0^4} > 0$ with a sufficiently small $r_0 \in (0,1)$ determined by Lemma 3.8 below.

Over the past several decades, a number of mathematicians have paid considerable attentions to the following class of nonlocal Schrödinger equations

$$-\Delta u + V(x)u = [I_\mu(x) * F(u)]\, f(u),\ x \in \mathbb{R}^N, \tag{1.4}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ denotes the external potential, F is the primitive function of the nonlinearity f satisfying some technical assumptions and $*$ denotes the convolution operator with $0 \le \mu < N$. For all $x \in \mathbb{R}^N \backslash (0)$, the kernel I_μ with $N \ge 2$ is defined by

$$I_\mu(x) \triangleq \begin{cases} \dfrac{\Gamma(\mu/2)}{\Gamma\left((N-\mu)/2\right)\pi^{N/2}2^{N-\mu}|x|^\mu}, & \text{if } 0 < \mu < N, \\ \dfrac{1}{2^{N-1}\pi^{N/2}\Gamma(N/2)} \log\dfrac{1}{|x|}, & \text{if } \mu = 0, \end{cases}$$

where Γ denotes the Euler's Gamma function. Since Γ is positive, the reader will observe that I_μ has totally different properties passing from $\mu > 0$ to the limiting case $\mu = 0$: one is positive definite while the other is indefinite and does not vanish at infinity. As a consequence, there exists a quite different framework to deal with the two cases in Eq. (1.4).

If $0 < \mu < N$ in Eq. (1.4), it is closely associated with the so-called Choquard equation arising in the study of Bose-Einstein condensation. In fact, for $N = 3$, $\mu = 1$ and $f(u) = u$, it becomes the Choquard-Pekar equation proposed by Pekar [49] to describe a polaron at rest in the quantum field theory. In [37], Choquard exploited it to characterization an electron trapped in its own hole as an approximation to the Hartree-Fock theory for a one component plasma. Subsequently, by means of the variational methods, Lieb [35] and Lions [38] established the existence and uniqueness of positive solutions to Eq. (1.4). Let us refer the reader to [42,46] for the regularity, radial symmetry and decay property of its ground state solution. In the meanwhile, Moroz *et al.* [45] regarded Eq. (1.4) as the model for self-gravitating particles in the context because it belongs to the classic Schrödinger-Newton equation, see e.g. [23,50,59]. In recent years, owing to the appearance of the convolution type nonlinearities, a lot of improvements on Eq. (1.4) and its variants, see [1,4,9,33,33,34,46,54] and their references therein, particular by [47], for a very abundant and meaningful review of the Choquard equations.

If $\mu=0$ in Eq. (1.4), when $F(u)=u^2$, then it is of class the following form

$$-\Delta u+V(x)u+\big(\log(|x|)*u^2\big)u=g(u)\quad \text{in } \mathbb{R}^2, \tag{1.5}$$

whose variational functional is defined by

$$\mathcal{I}(u)=\frac{1}{2}\int_{\mathbb{R}^2}\big[|\nabla u|^2+\bar{V}(x)u^2\big]\mathrm{d}x+\frac{m}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}G(u)\mathrm{d}x,$$

where and in the sequel $G(u)\triangleq\int_0^u g(s)\mathrm{d}s$. Alternatively, the functional $\mathcal{I}$ would not be well-defined on $H^1(\mathbb{R}^2)$ in general and it was pointed out by Stubbe in [58]. In order to deal with it, Stubbe introduced a new Hilbert space

$$X=\left\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}\log(1+|x|)u^2\mathrm{d}x<+\infty\right\},$$

endowed with the inner product and norm

$$(u,v)_X=\int_{\mathbb{R}^2}\big[\nabla u\nabla v+uv+\log(1+|x|)uv\big]\mathrm{d}x \text{ and } \|u\|_X=\sqrt{(u,u)_X}.$$

As we know, Stubbe's argument relies strongly on the vital identity

$$\log r=\log(1+r)-\log\left(1+\frac{1}{r}\right),\ \forall r>0,$$

because it permits us to define the variational functionals $V_1,V_2:X\to\mathbb{R}$ by

$$V_1(u)\triangleq\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(1+|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,\ \forall u\in X,$$

and

$$V_2(u)\triangleq\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,\ \forall u\in X.$$

In [58], it has been deduced that $V_1,V_2\in\mathcal{C}^1(X,\mathbb{R})$ and found the equality

$$V_0(u)\triangleq\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y=V_1(u)-V_2(u),\ \forall u\in X,$$

which implies that $\mathcal{I}$ given in (1.5) is of class $\mathcal{C}^1(X)$. Afterwards, taking full advantage of this powerful argument introduced in [58], there are some other interesting results in [10,15,21,24,25,29,56] and the references therein.

We would like to mention here that the space dimension of Eq. (1.5) is two and so it causes some interesting point. As a matter of fact, for every bounded or unbounded domain $\Omega\subset\mathbb{R}^2$, the imbedding $H_0^1(\Omega)\hookrightarrow L^p(\Omega)$ is continuous for all $p\in[2,+\infty)$. However, the nonlinearity g in (1.5) might behave like $e^{\alpha t^2}$ for sufficiently large $t\in\mathbb{R}$. In reality, one cannot conclude the fact that $H_0^1(\Omega)\hookrightarrow L^\infty(\Omega)$ from the above imbedding. Thereby, to overcome this difficulty, the celebrated Trudinger-Moser inequality introduced in [48,51,60] could act as an ideal candidate to be the suitable substitute of the Sobolev inequality. First of all, let us exhibit the case on bounded domain Ω instead of the whole space $\mathbb{R}^2$. In [48,51,60], the authors developed following sharp maximal exponential integrability for functions in $H_0^1(\Omega)$:

$$\sup_{u\in H_0^1(\Omega):\|\nabla u\|_{L^2(\Omega)}\le 1}\int_\Omega e^{\alpha u^2}\mathrm{d}x\le C|\Omega| \text{ if } \alpha<4\pi, \tag{1.6}$$

where $C=C(\alpha)>0$ is a constant and $|\Omega|$ denotes the Lebesgue measure of Ω. Afterwards, the so called concentration-compactness principle in the Trudinger-Moser inequality sense was established by Lions [39]: Let (u_n) be a sequence of functions in $H_0^1(\Omega)$ with $\|\nabla u_n\|_{L^2(\Omega)} = 1$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, it holds that

$$\sup_{n\in\mathbb{N}} \int_\Omega e^{4\pi p u_n^2} \mathrm{d}x < +\infty, \ \forall 0 < p < \bar{p}_{\alpha_0}(u) \triangleq \frac{1}{1 - \|\nabla u_0\|_{L^2(\Omega)}^2}. \tag{1.7}$$

It would be obvious to observe that the supremum in (1.6) becomes infinite if the domain $\Omega \subset \mathbb{R}^2$ satisfies $|\Omega| = \infty$. The inequality above is therefore unavailable for the unbounded domains. To handle it, the authors in [14, 16] established the following version of the Trudinger-Moser inequality: For all $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{L^2(\mathbb{R}^2)} \leq M < +\infty$, there is a positive constant $C = C(M, \alpha)$ such that

$$\sup_{u\in H^1(\mathbb{R}^2):\|\nabla u\|_{L^2(\mathbb{R}^2)}\leq 1} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq C \text{ if } \alpha < 4\pi. \tag{1.8}$$

In spirit of [39], de Souza and do Ó [28] generalized the Lions's concentration-compactness principle to $\mathbb{R}^2$: Let (u_n) be in $W_0^{1,2}(\mathbb{R}^2)$ with $\|u_n\|_{W_0^{1,2}(\mathbb{R}^2)} = 1$ and suppose that $u_n \rightharpoonup u_0$ in $W_0^{1,2}(\mathbb{R}^2)$, there holds

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} \left(e^{4\pi p u_n^2} - 1\right) \mathrm{d}x < \infty, \ \forall 0 < p < \tilde{p}_{\alpha_0}(u) \triangleq \frac{1}{1 - \|u_0\|_{W_0^{1,2}(\mathbb{R}^2)}^2}. \tag{1.9}$$

We would like to cite the results in [3,27] and the references therein concerning some other generalizations, extensions and applications of the Trudinger-Moser inequalities for bounded and unbounded domain.

Up to our best knowledge, Alves and Figueiredo [5] firstly applied (1.3) to the Schrödinger-Poisson equation (1.5) and studied the existence of ground state solutions via using Nehari manifold method. Along this direction, there are more and more research works concerning this topic including the two dimensional Choquard problem with logarithmic kernel, see [18, 20, 40] for example.

Very recently, by establishing a Pohožaev-Trudinger log-weighted inequality, Cassani and Tarsi [20] concluded that the following equation

$$-\Delta u + V(x)u = \left[\log \frac{1}{|x|} * F(u)\right] f(u) \text{ in } \mathbb{R}^2, \tag{1.10}$$

admits a nontrivial finite energy solution in the space $H_V^1 L_{w_0}^q(\mathbb{R}^2)$ which is equipped with the norm

$$\|u\|^2 \triangleq \|u\|_{H^1(\mathbb{R}^2)}^2 + \left(\int_{\mathbb{R}^2} |u|^q \log(1 + |x|)\mathrm{d}x\right)^{\frac{2}{q}}, \ q > 2,$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ is positive and periodic and the nonlinearity f satisfies (1.3) and $(f_1) - (f_3)$ as well as (f_4) with $\vartheta = 3$. Taking into account an asymptotical approximation approach introduced in [40], Cassani, Du and Liu [19] investigated the existence of a positive solution in $H^1(\mathbb{R}^2)$ for Eq. (1.10) with $V \equiv 1$ under the assumptions $(f_1) - (f_3)$ and (f_4) with $\vartheta = 1$. Explaining it more precisely, given a $\tau \in (0, 1)$, with the help of

$$\lim_{\tau\to 0^+} \mathbb{G}_\tau(x) \triangleq \lim_{\tau\to 0^+} \frac{|x|^{-\tau} - 1}{\tau} = -\log|x|,$$

the authors showed that

$$-\Delta u + u = [\mathbb{G}_\tau(x) * F(u)]\, f(u) \text{ in } \mathbb{R}^2,$$

admits a positive solution u_τ for all $\tau \in (0, 1)$. Then, by tending $\tau \to 0^+$, they demonstrated that the weak limit of u_τ is in fact a positive solution of Eq. (1.10) with $V \equiv 1$. Another similar application to deal with Eq. (1.10) can be also found in [22].

Whereas, as far as we are concerned, there seems no related existence results for Eq. (1.10) with steep potential well. Moreover, we further perceive that no attempts on the nonlinearity f involving supercritical exponential growth in Eq. (1.10) have been made yet up to now. We shall fill these blanks in this paper by introducing some subtle techniques. First of all, in order to exhibit the main results legibly, let us first introduce the work space. Following as [55], for all fixed $\lambda > 0$, by (V_1), we define the space

$$E_\lambda \triangleq \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} \lambda V(x)|u|^2 \mathrm{d}x < +\infty \right\}$$

which is indeed an Hilbert space equipped with the inner product and norm

$$(u,v)_{E_\lambda} = \int_{\mathbb{R}^2} \big[\nabla u \nabla v + \lambda V(x) uv\big] \mathrm{d}x \text{ and } \|u\|_{E_\lambda} = \sqrt{(u,u)_{E_\lambda}},\ \forall u, v \in E_\lambda.$$

From here onwards, we shall denote E and $\|\cdot\|_E$ by E_λ and $\|\cdot\|_{E_\lambda}$ for $\lambda = 1$, respectively. It is simple to observe that $\|\cdot\|_E \le \|\cdot\|_{E_\lambda}$ for all $\lambda \ge 1$. Moreover, thanks to [55], there exists a $\lambda_0 > 0$ such that E_λ could be continuously imbedded into $H^1(\mathbb{R}^2)$ for all $\lambda > \lambda_0$.

Our first main result is concerned with the existence of nontrivial solutions for Eq. (1.1).

Theorem 1.1. *Suppose that $(V_1) - (V_3)$ hold and the nonlinearity G defined in (1.2) satisfies (1.3) with $(f_1) - (f_4)$. Then, there exist some constants $\sigma^* > 0$ and $\lambda^* > 0$ such that for every $\sigma \in (0, \sigma^*)$ and $\lambda > \lambda^*$, Eq. (1.1) has at least a nontrivial solution in $E_\lambda \cap L^\infty(\mathbb{R}^2)$.*

Remark 1.2. It should be pointed out here that the supercritical exponential growth used in Theorem 1.1 was proposed in [53,57] for a suitable function $\bar{g}(s) = e^{\alpha|s|^\tau}$ and it is one of the following cases

$$\text{(I) } \tau > 2 \text{ is arbitrary and } \alpha > 0 \text{ is fixed; (II) } \alpha > 0 \text{ is arbitrary and } \tau \ge 2 \text{ is fixed,} \tag{1.11}$$

see [7,8] in detail. As a matter of fact, we call (I) and (II) the ***subcritical-supercritical exponential growth*** and ***critical-supercritical exponential growth***, respectively. The reader is invited to find that the nonlinearity G satisfying (1.2) and (1.3) belongs to (II). Because of the indefinite logarithmic kernel, we have to make some suitable adjustments to the truncation technique in [53,57] to adapt to our settings in Theorem 1.1.

Remark 1.3. In contrast to [19,20,22], we are going to take some delicate analyses to depend on the assumption (f_4) with $\vartheta = 1$ and $\vartheta = 3$ in a unified way. It is worth highlighting that our arguments indicate that the constant $\beta > 0$ in (f_4) can be accurately characterized to some extent. To demonstrate the proof of Theorem 1.1, we can never repeat the methods exploited in these quoted papers simply since a new maximal growth condition on the nonlinearity G has been imposed.

We are now in a position to exhibit the main ideas of the proof of Theorem 1.1. To look for nontrivial solutions associated with Eq. (1.1), due to a variational method point of view, it will be found critical points for the corresponding variational functional. Alternatively, since the arguments adopted in [20] strongly relies on the periodicity of the potential V, we mainly borrow the amazing approach from [40] to address this issue. On the other hand, the nonlinearity G possesses the supercritical exponential growth at infinity, it seems unable to investigate the existence of nontrivial solutions for the following equation directly

$$-\Delta u + \lambda V(x) u = [\mathbb{G}_\tau(x) * G(u)]\, g(u) \text{ in } \mathbb{R}^2, \tag{1.12}$$

since the variational functional is not well-defined in $H^1(\mathbb{R}^2)$ or even in E_λ, where and in the sequel for all $\tau \in (0,1)$, the function $\mathbb{G}_\tau : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\mathbb{G}_\tau(x) \triangleq \frac{|x|^{-\tau} - 1}{\tau},\ \forall x \in \mathbb{R}^2 \backslash (0). \tag{1.13}$$

Motivated by [53,57], we are going to take advantage of a so-called truncation argument to explore the existence of nontrivial solutions for Eq. (1.1). Speaking it more clearly, for each $n \in \mathbb{N}^+$, let us first define

$$\eta_n(s) = \eta\left(\frac{s}{n}\right),\ \mathfrak{F}_n(s) = s^2 \eta_n(s) \text{ and } \mathfrak{F}_n'(s) = \mathfrak{f}_n(s),\ \forall s \in \mathbb{R}, \tag{1.14}$$

where $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ denotes an even function with $0 \le \eta \le 1$ and satisfies

$$\eta(s) = \begin{cases} 1, & |s| \le 1, \\ 0, & |s| \ge 2, \end{cases} \text{ with } |\eta'(s)| \le 2 \text{ and } |\eta''(s)| \le 4, \ \forall s \in \mathbb{R}.$$

We remark here that the cut-off function η above is more restrictive than that of in [53, 57]. With η_n in hands, it enables us to replace G and g in Eq. (1.12) with

$$G_n(s) \triangleq F(s)e^{\sigma \mathfrak{F}_n(s)}, \ \forall s \in \mathbb{R} \tag{1.15}$$

and $G_n(s) = \int_0^s g_n(t)\mathrm{d}t$, respectively. In other words, we shall consider the existence results for

$$-\Delta u + \lambda V(x)u = [\mathbb{G}_\tau(x) * G_n(u)]\, g_n(u) \text{ in } \mathbb{R}^2. \tag{1.16}$$

As one knows, Eq. (1.16) admits a variational structure and, for all $n \in \mathbb{N}^+$, its variational functional $J_n : E_\lambda \to \mathbb{R}$ is defined by

$$J_n(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + \lambda V(x)|u|^2\right]\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2} [\mathbb{G}_\tau(x) * G_n(u)]\, G_n(u)\mathrm{d}x.$$

In Section 2 below, we shall verify that J_n is not only well-defined, but also belongs to $\mathcal{C}^1(E_\lambda, \mathbb{R})$. As a consequence, each critical point of J_n is in fact a (weak) solution of Eq. (1.16). Moreover, according to the definition of η_n in (1.14), any nontrivial critical point, saying it u, of J_n satisfying $\|u\|_{L^\infty(\mathbb{R}^2)} < n$ is a nontrivial solution of Eq. (1.12).

Next, we would like to explore the decay of the obtained nontrivial solution at infinity. The following result reveals that the nontrivial solutions of Eq. (1.1) decay exponentially as $|x| \to \infty$.

Theorem 1.4. *Suppose that $(V_1) - (V_3)$ hold and the nonlinearity G defined in (1.2) satisfies (1.3) with $(f_1) - (f_4)$. Let $u_\lambda \in E_\lambda \cap H^1(\mathbb{R}^2)$ be a nontivial solution of Eq. (1.1) for every $\sigma \in (0, \sigma^*)$ and $\lambda > \lambda^*$. Then, we have*

$$|u_\lambda(x)| \le \mathbb{A}\lambda^{-\frac{1}{2}} \exp\left[-\mathbb{B}\lambda^{\frac{1}{2}}(|x| - R)\right], \ \forall |x| > R,$$

and the positive constants $\mathbb{A}$, $\mathbb{B}$, R are independent of σ and λ.

Remark 1.5. Although a similar result regarding the exponential decay of nontrivial solutions for Eq. (1.10) with $V \equiv 1$ has been studied in [19, Lemma 4.2], as far as we know, it seems that Theorem 1.4 is a new result for planar logarithmic Choquard equation with steep potential well and supercritical exponential growth. The adaptation procedure to our problem is therefore nontrivial at all due to the presences of them.

Finally, let us focus on the asymptotic behaviors of the nontrivial solutions as $\sigma \to 0^+$ and $\lambda \to +\infty$, respectively. With aid of L^∞-estimate of the obtained solution, we have the following results.

Theorem 1.6. *Under the assumptions of Theorem 1.1, let $u_{\lambda,\sigma} \in E_\lambda \cap H^1(\mathbb{R}^2)$ be a nontrivial solution of Eq. (1.1) for every $\sigma \in (0, \sigma^*)$ and $\lambda > \lambda^*$. If $\sigma \in (0, \sigma^*)$ is fixed, then, going to a subsequence if necessary, we have $u_{\lambda,\sigma} \to u_{0,\sigma}$ in $H^1(\mathbb{R}^2)$ as $\lambda \to +\infty$, where $u_{0,\sigma} \in H_0^1(\Omega)$ is a nontrivial solution of*

$$\begin{cases} -\Delta u = \left[\int_\Omega \log\left(\frac{1}{|x-y|}\right) G(u(y))\mathrm{d}y\right] g(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

Theorem 1.7. *Under the assumptions of Theorem 1.1, let $u_{\lambda,\sigma} \in E_\lambda \cap H^1(\mathbb{R}^2)$ be a nontrivial solution of Eq. (1.1) for every $\sigma \in (0, \sigma^*)$ and $\lambda > \lambda^*$. If $\lambda > \lambda^*$ is fixed, then we have $u_{\lambda,\sigma} \to u_{\lambda,0}$ in E_λ as $\sigma \to 0^+$ along a subsequence, where $u_{\lambda,0} \in E_\lambda$ is a nontrivial solution of*

$$-\Delta u + \lambda V(x)u = \left[\log\frac{1}{|x|} * F(u)\right] f(u) \text{ in } \mathbb{R}^2. \tag{1.18}$$

Remark 1.8. The similar results for a class of local Schrödinger equations with subcritical growths exhibited in Theorems 1.4 and 1.6 have been explored in [11,12], whereas as what have mentioned in Remark 1.5, there exist some additional efforts to overcome the difficulties in our problems. Last but not the least, we believe that the result in Theorem 1.7 seems the first attempt to make a thorough inquiry about the asymptotic behavior of the nontrivial solutions from supercritical exponential problems to the critical ones in the literature.

Although the approaches to deal with a class of elliptic equations involving supercritical exponential growth have already appeared in the previous papers [53,57], we have to try best to clean the unpleasant obstacles existing in planar logarithmic Choquard equation with steep potential well and thereby it is believed that the results in this article are new up to now.

In our opinion, one of the most significant contributions is that we succeed in dealing with the biggest challenge that how to balance the mutual interactions between the too loose sign-changing logarithm kernel and the supercritical exponential growth rate of a fairly general nonlinearity. Consequently, we are certainly confident that our results would prompt some further explorations on related topics.

The outline of the paper is organized as follows. In Section 2, we mainly exhibit some preliminary results including the truncation argument that will be exploited frequently in the whole article. Section 3 is mainly devoted to the existence results for the auxiliary problems (1.12) and (1.16). In Section 4, we will focus on the existence and decaying property of nontrivial solutions for Eq. (1.1), so the detailed proofs of Theorems 1.1 and 1.4 conclude. Finally, the asymptotic behaviors of the nontrivial solutions are studied in Section 5.

Notations: From now on in this paper, otherwise mentioned, we utilize the following notations:

- $C, C_1, C_2, \cdots$ denote any positive constant, whose value is not relevant and $\mathbb{R}^+ \triangleq (0, +\infty)$.
- Let $(Z, \|\cdot\|_Z)$ be a Banach space with dual space $(Z^{-1}, \|\cdot\|_{Z^{-1}})$, and Ψ be functional on Z.
- The Cerami sequence at a level $c \in \mathbb{R}$ ($(C)_c$ sequence in short) corresponding to Φ means that $\Phi(x_n) \to c$ and $(1 + \|x_n\|_Z)\|\Phi'(x_n)\|_{Z^{-1}} \to 0$ as $n \to \infty$, where $(x_n) \subset Z$.
- For any $\varrho > 0$ and every $x \in \mathbb{R}^2$, $B_\varrho(x) \triangleq \{y \in \mathbb{R}^2 : |y - x| < \varrho\}$.
- $o_k(1)$ denotes the real sequences by $o_k(1) \to 0$ as $k \to +\infty$.
- " $\to$ " and " $\rightharpoonup$ " stand for the strong and weak convergence in the related function spaces, respectively.

2. Variational Framework and Preliminaries

In this section, we shall formulate the variational structure for our problems and then present some preliminary results that will play crucial roles in the next sections.

First of all, we recall the well-known Hardy-Littlewood-Sobolev inequality and it appears repeatedly throughout the whole paper.

Proposition 2.1. *(Hardy-Littlewood-Sobolev inequality [36, Theorem 4.3]). Suppose that $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$, $g \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then, there exists a sharp constant $C_{HLS} = C_{HLS}(s, N, \mu, r) > 0$, independent of g and h, such that*

$$\int_{\mathbb{R}^N} [|x|^{-\mu} * g(x)]h(x)\mathrm{d}x \le C_{HLS}|g|_s|h|_r. \tag{2.1}$$

In the sequel we collect some basic estimates whose proofs are omitted.

Lemma 2.2. *Let $\tau \in (0, 1]$, then*

$$\frac{t^{-\tau} - 1}{\tau} \ge \log t^{-1}, \ \forall t \in (0, 1].$$

Moreover, for all $\tau' > \tau$, there is a constant $C_{\tau'} > 0$ such that

$$\frac{t^{-\tau}-1}{\tau} \leq C_{\tau'} t^{-\tau'}, \ \forall t \in (0,+\infty).$$

Let us point out some immediate consequences of the nonlinearity f satisfying (1.3) and $(f_1)-(f_4)$. Due to (1.3) and (f_1), for all $\epsilon > 0$ and $\alpha > \alpha_0$, there is a constant $C_\epsilon > 0$ such that

$$|f(s)| \leq \epsilon|s| + C_\epsilon |s|^{\bar{q}-1}\left(e^{\alpha s^2}-1\right), \ \forall s \in \mathbb{R}, \tag{2.2}$$

where $\bar{q} \geq 2$ can be arbitrarily chosen later. By means of (f_2), there holds

$$|F(s)| \leq \epsilon|s|^2 + C_\epsilon |s|^{\bar{q}}\left(e^{\alpha s^2}-1\right), \ \forall s \in \mathbb{R}. \tag{2.3}$$

Moreover, without mentioning any longer, let us exploit directly the following inequality (see e.g. [62, Lemma 2.1]):

$$\left(e^{\alpha s^2}-1\right)^m \leq \left(e^{\alpha m s^2}-1\right), \ \forall s \in \mathbb{R}, \ \alpha > 0 \text{ and } m > 1.$$

In view of (f_2), one sees $f'(s) \geq 0$ for all $s > 0$ and then f is nondecreasing on $s \in (0,+\infty)$. Therefore, it holds that

$$0 < F(s) = \int_0^s f(t)\mathrm{d}t \leq f(s)s, \ \forall s > 0. \tag{2.4}$$

In addition, we can use (f_2) and (2.4) to show that

$$0 < F(s) \leq (1-\delta)f(s)s, \ \forall s > 0. \tag{2.5}$$

Actually, by (f_2), we observe that $(F(s)/f(s))' \leq 1-\delta$ for any $s > 0$, then for all $\varepsilon \in (0,s)$, one has

$$\frac{F(s)}{f(s)} - \frac{F(\varepsilon)}{f(\varepsilon)} = \int_\varepsilon^s \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(t)}{f(t)}\right)\mathrm{d}t \leq (1-\delta)\int_\varepsilon^s \mathrm{d}t = (1-\delta)(s-\varepsilon)$$

which together with $\lim\limits_{\varepsilon\to 0^+} F(\varepsilon)/f(\varepsilon) = 0$ by (2.4) yields that (2.5) holds true. We repeat the calculations in [19, (1.3)] to find two constants $M_0 > 0$ and $s_0 > 0$ such that

$$F(s) \leq M_0 f(s), \ \forall s \geq s_0. \tag{2.6}$$

With the assumptions $(f_1)-(f_4)$ and the above properties of f in hands, we next conclude the ones for G_n defined in (1.15). Beginning with them, inspired by [52, 53], there are the following facts

$$0 \leq \mathfrak{F}_n(s) \leq 4n^2 \text{ and } |\mathfrak{f}_n(s)| \leq 12n, \ \forall n \in \mathbb{N}^+ \text{ and } s \in \mathbb{R}, \tag{2.7}$$

and

$$0 \leq \mathfrak{F}_n(s) \leq s^2, \ |\mathfrak{f}_n(s)| \leq 6|s| \text{ and } |\mathfrak{f}_n'(s)| \leq 34, \ \forall n \in \mathbb{N}^+ \text{ and } s \in \mathbb{R}. \tag{2.8}$$

Lemma 2.3. *Suppose (1.3) and $(f_1)-(f_4)$, then for all $n \in \mathbb{N}^+$, we have the following conclusions:*

- (g_1) *$g_n \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, $g_n(s) \equiv 0$ for all $s \leq 0$ and $g_n(s) = o(s)$ as $s \to 0^+$ uniformly in $\sigma \in (0, \frac{1}{4n^2})$;*
- (g_2) *for the constant $\delta \in (0,1)$ given in (f_2), there is a $\sigma_1 > 0$ such that for all $\sigma \in (0,\sigma_1)$,*
$$\frac{G_n(s)g_n'(s)}{g_n^2(s)} \geq \frac{\delta}{4}, \ \forall s > 0, \ \text{where } g_n(s) = \int_0^s g_n'(t)\mathrm{d}t;$$
- (g_3) $\lim\limits_{s\to+\infty} \dfrac{G_n(s)g_n'(s)}{g_n^2(s)} = 1$ *or equivalently,* $\lim\limits_{s\to+\infty} \dfrac{\mathrm{d}}{\mathrm{d}s}\dfrac{G_n(s)}{g_n(s)} = 1$ *uniformly in $n \in \mathbb{N}^+$;*
- (g_4) *there is a $\sigma_2 > 0$ such that for all $\sigma \in (0,\sigma_2)$, it holds that $g_n(s)G_n(s) \geq \frac{1}{2}f(s)F(s)$ for $s > 0$;*
- (g_5) *for all $\sigma \in (0,\sigma_2)$, we have $G_n(s) \leq 2M_0 g_n(s)$ for all $s \geq s_0$, where $s_0 > 0$ comes from (2.6).*

Proof. According to the definition of G_n defined in (1.15), one simply has that

$$g_n(s) = [f(s) + \sigma F(s)\mathfrak{f}_n(s)]\, e^{\sigma \mathfrak{F}_n(s)}, \ \forall s \in \mathbb{R}.$$

Obviously, using (f_1), we have that $g_n \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $g_n(s) \equiv 0$ for all $s \leq 0$. For all $\sigma \in (0, \frac{1}{4n^2})$, we are able to apply (2.8) to derive $\lim\limits_{s\to 0} \dfrac{g_n(s)}{s} = 0$ easily and so point (g_1) holds true. To prove point (g_2), we claim that

$$F(s)f(s)|\mathfrak{f}_n(s)| \leq 24n^2 f^2(s) \text{ and } F^2(s)|\mathfrak{f}_n'(s)| \leq 136n^2 f^2(s), \ \forall s > 0. \tag{2.9}$$

Indeed, we recall the definition of η_n defined in (1.14) to see that $\mathfrak{f}_n(s) \equiv 0$ if $s \geq 2n$. Then, with the aid of (2.8), we have that $|\mathfrak{f}_n(s)s| \leq 24n^2$ for all $s > 0$ which together with (2.4) indicates the first part of this claim. Similarly, using the fact that $\mathfrak{f}_n'(s) \equiv 0$ if $s \geq 2n$, we exploit (2.8) again to obtain $|\mathfrak{f}_n'(s)s^2| \leq 136n^2$ for all $s > 0$ and so (2.4) gives us the second part of this claim. Combining (2.9) and (f_2) with $\delta \in (0, 1)$, there is a $\sigma_1 \triangleq \frac{\delta}{736n^2} > 0$ such that we have that

$$\begin{aligned}
&F(s)\left[f'(s) + 2\sigma f(s)\mathfrak{f}_n(s) + \sigma F(s)\mathfrak{f}_n'(s) + \sigma^2 F(s)\mathfrak{f}_n^2(s)\right] - \frac{\delta}{2}\left[f^2(s) + \delta^2 F^2(s)\mathfrak{f}_n^2(s)\right] \\
&\geq \frac{\delta}{2} f^2(s) + F(s)\left[2\sigma f(s)\mathfrak{f}_n(s) + \sigma F(s)\mathfrak{f}_n'(s)\right] + \frac{1}{2}\sigma^2 F(s)\mathfrak{f}_n^2(s) \\
&\geq \frac{\delta}{2} f^2(s) - 2\sigma F(s)f(s)|\mathfrak{f}_n(s)| - \sigma F^2(s)|\mathfrak{f}_n'(s)| \\
&\geq \left(\frac{\delta}{2} - 184\sigma n^2\right) f^2(s) \geq \frac{\delta}{4} f^2(s).
\end{aligned}$$

It is simple to calculate that

$$g_n'(s) = \left[f'(s) + 2\sigma f(s)\mathfrak{f}_n(s) + \sigma F(s)\mathfrak{f}_n'(s) + \sigma^2 F(s)\mathfrak{f}_n^2(t)\right] e^{\sigma \mathfrak{F}_n(s)}, \ \forall s \in \mathbb{R}.$$

As a consequence, the above two formulas shows us that

$$\begin{aligned}
\frac{G_n(s)g_n'(s)}{g_n^2(s)} &= \frac{F(s)\left[f'(s) + 2\sigma f(s)\mathfrak{f}_n(s) + \sigma F(s)\mathfrak{f}_n'(s) + \sigma^2 F(s)\mathfrak{f}_n^2(s)\right]}{[f(s) + \sigma F(s)\mathfrak{f}_n(s)]^2} \\
&\geq \frac{F(s)\left[f'(s) + 2\sigma f(s)\mathfrak{f}_n(s) + \sigma F(s)\mathfrak{f}_n'(s) + \sigma^2 F(s)\mathfrak{f}_n^2(s)\right]}{2f^2(s) + 2\sigma^2 F^2(s)\mathfrak{f}_n^2(s)} \geq \frac{\delta}{4}
\end{aligned}$$

finishing the proof of point (g_2) if $\sigma \in (0, \sigma_1)$. Recalling the verification of point (g_2), one will find that point (g_3) is a direct corollary of (f_3). In order to verify point (g_4), we continue depending on the fact that $\mathfrak{f}_n(s) \equiv 0$ if $s \geq 2n$. Then, choosing $\sigma_2 = \frac{1}{48n^2}$, it follows from (2.4) and (2.8) that

$$\begin{aligned}
g_n(s) &= [f(s) + \sigma F(s)\mathfrak{f}_n(s)]\, e^{\sigma \mathfrak{F}_n(s)} \geq [f(s) - \sigma F(s)|\mathfrak{f}_n(s)|]\, e^{\sigma \mathfrak{F}_n(s)} \\
&\geq \left(1 - 24\sigma n^2\right) f(s) e^{\sigma \mathfrak{F}_n(s)} \geq \frac{1}{2} f(s) e^{\sigma \mathfrak{F}_n(s)} \geq \frac{1}{2} f(s), \ \forall s > 0,
\end{aligned} \tag{2.10}$$

provided $\sigma \in (0, \sigma_2)$. Thereby, point (g_4) concludes. If the last second inequality in (2.10) is considered, then point (g_5) follows (2.6) immediately. The proof is completed. □

Let $\tau \in (0, 1)$ be fixed, for all $n \in \mathbb{N}^+$, we define the functionals $\Psi_n^1 : E_\lambda \to \mathbb{R}$ and $\Psi_n^2 : E_\lambda \to \mathbb{R}$ by

$$\Psi_n^1(u) = \frac{1}{2}\left(\int_{\mathbb{R}^2} G_n(u)\mathrm{d}x\right)^2$$

and

$$\Psi_n^2(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u)\right] G_n(u)\mathrm{d}x.$$

We have the following results.

Lemma 2.4. *Suppose that f satisfies (1.3) and (f_1), for all $n \in \mathbb{N}^+$, then Ψ_n^j is well-defined on E_λ for $\lambda \geq 1$, where $j = 1, 2$. Actually, $\Psi_n^j \in \mathcal{C}^1$ with $j = 1, 2$ and the derivatives given by*

$$(\Psi_n^1)'(u)(\psi) = \int_{\mathbb{R}^2} G_n(u)\mathrm{d}x \int_{\mathbb{R}^2} g_n(u)\psi \mathrm{d}x$$

for all $\psi \in E_\lambda$ and

$$(\Psi_n^2)'(u)(\psi) = \int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u)\right] g_n(u)\psi \mathrm{d}x.$$

Proof. The verifications of Ψ_n^1 can follow [53, Lemma 2.6] lines by lines, and so we omit the details. In order to derive that Ψ_n^2 is well-defined on E_λ, we find that $G_n(s) \leq F(s)e^{4\sigma n^2}$ for all $n \in \mathbb{N}^+$ by (1.15) and (2.7). Combining (2.1) and (2.3), for all $u \in E_\lambda$, one has that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left[|x|^{-\tau} * F(u)\right] F(u)\mathrm{d}x \leq C_{HLS} \left(\int_{\mathbb{R}^2} |F(u)|^{\frac{4}{4-\tau}} \mathrm{d}x\right)^{\frac{4-\tau}{2}} \\
&\quad \leq C_0\epsilon \left(\int_{\mathbb{R}^2} |u|^{\frac{8}{4-\tau}} \mathrm{d}x\right)^{\frac{4-\tau}{2}} + C_1 C_\epsilon \left(\int_{\mathbb{R}^2} |u|^{\frac{4\bar{q}\nu'}{4-\tau}} \mathrm{d}x\right)^{\frac{4-\tau}{2\nu'}} \left(\int_{\mathbb{R}^2} \left(e^{\frac{4\alpha\nu}{4-\tau}u^2} - 1\right) \mathrm{d}x\right)^{\frac{4-\tau}{2\nu}}.
\end{aligned}$$

From which, we conclude that Ψ_n^2 is well-defined on E_λ for all $n \in \mathbb{N}^+$ thanks to the classic Trudinger-Moser inequality. We shall show that Ψ_n^2 has a continuous Gateaux derivative on E_λ, then $\Psi_n^2 \in \mathcal{C}^1$ by [61, Proposition 1.3].

Existence of the Gateaux derivative. Let $u, \psi \in E_\lambda$. Given $x \in \mathbb{R}^2$ and $0 < |t| < 1$, taking into account the mean value theorem, there exists a $\chi \in (0, 1)$ such that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u + t\psi)\right] G_n(u + t\psi)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u)\right] G_n(u)\mathrm{d}x \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u + t\psi)\right] G_n(u + t\psi)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u + t\psi)\right] G_n(u)\mathrm{d}x \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u + t\psi)\right] G_n(u)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u)\right] G_n(u)\mathrm{d}x \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u + t\psi)\right] g_n(u + \chi t\psi)t\psi\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} \left[|x|^{-\tau} * G_n(u)\right] g_n(u + \chi t\psi)t\psi\mathrm{d}x
\end{aligned}$$

The standard arguments enable us to derive that $|x|^{-\tau} * G_n(u + t\psi) \to |x|^{-\tau} * G_n(u)$ a.e. in $\mathbb{R}^2$ as $t \to 0$, then the Lebesgue's Dominated Convergence theorem indicates that

$$(\Psi_n^2)'(u)(\psi) = \lim_{t \to 0} \frac{\Psi_n^2(u + t\psi) - \Psi_n^2(u)}{t},$$

from where we know that the Gateaux derivative exists and can be computed as above.

Continuity of the Gateaux derivative. We suppose $u_k \to u$ in E_λ, then $u_k \to u$ a.e. in $\mathbb{R}^2$ along a subsequence. Some simple calculations that can be found above provide us that

$$\left|(\Psi_n^2)'(u_k)(\psi) - (\Psi_n^2)'(u)(\psi)\right| = o_k(1)\|\psi\|_{E_\lambda}, \ \forall \psi \in E_\lambda,$$

which gives the desired result. The proof of this lemma is completed. $\square$

We conclude this section by the following convergent results related to the nonlinearity G_n and g_n.

Lemma 2.5. *Let g be given by (1.2) and satisfy (1.3) with $(f_1) - (f_4)$. Then, for all $\sigma \in (0, \sigma_2)$ with $\sigma_2 > 0$ in Lemma 2.3-(g_5), if $(u_k) \subset E_\lambda$ satisfies $u_k \to u$ in $L^p(\mathbb{R}^2)$ with $2 < p < +\infty$ and $u_k \to u$ a.e. in $\mathbb{R}^2$ with a constant $C > 0$, which is independent of $k \in \mathbb{N}^+$, such that*

$$\sup_{k \in \mathbb{N}^+} \int_{\mathbb{R}^2} g_n(u_k)u_k \mathrm{d}x \leq C \text{ and } \sup_{k \in \mathbb{N}^+} \int_{\mathbb{R}^2} \left[G_n(u_k)\right]^\kappa \mathrm{d}x \leq C,$$

where $1<\kappa<(1-\varepsilon)^{-\frac{1}{2}}$ *with* $\varepsilon\in(0,1)$ *given in* (3.28) *below, we have the following results*

$$\left\{\begin{array}{l}\lim\limits_{k\to\infty}\displaystyle\int_{\mathbb{R}^2}G_n(u_k)\mathrm{d}x=\int_{\mathbb{R}^2}G_n(u)\mathrm{d}x,\\ \lim\limits_{k\to\infty}\displaystyle\int_{\mathbb{R}^2}g_n(u_k)\psi\mathrm{d}x=\int_{\mathbb{R}^2}g_n(u)\psi\mathrm{d}x,\ \forall\psi\in C_0^\infty(\mathbb{R}^2),\end{array}\right.\tag{2.11}$$

in the sense of subsequences if necessary. Moreover, if in addition $0<\tau<\tau_\kappa\triangleq\dfrac{2(\kappa-1)}{\kappa}$, *it holds that, along some subsequences,*

$$\left\{\begin{array}{l}\lim\limits_{k\to\infty}\displaystyle\int_{\mathbb{R}^2}\left[|x|^{-\tau}*G_n(u_k)\right]G_n(u_k)\mathrm{d}x=\int_{\mathbb{R}^2}\left[|x|^{-\tau}*G_n(u)\right]G_n(u)\mathrm{d}x,\\ \lim\limits_{k\to\infty}\displaystyle\int_{\mathbb{R}^2}\left[|x|^{-\tau}*G_n(u_k)\right]g_n(u_k)\psi\mathrm{d}x=\int_{\mathbb{R}^2}\left[|x|^{-\tau}*G_n(u)\right]g_n(u)\psi\mathrm{d}x,\ \forall\psi\in C_0^\infty(\mathbb{R}^2).\end{array}\right.\tag{2.12}$$

Proof. Recalling Lemma 2.3-(g_5), the verification of (2.11) is very similar to that of [53, Lemma 2.9], and so we omit the details. To derive (2.12), we firstly find that

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{G_n(u_k(y))}{|x-y|^\tau}\mathrm{d}y&=\int_{|x-y|<1}\frac{G_n(u_k(y))}{|x-y|^\tau}\mathrm{d}y+\int_{|x-y|\ge1}\frac{G_n(u_k(y))}{|x-y|^\tau}\mathrm{d}y\\&\le\left(\int_{|x-y|<1}|x-y|^{-\frac{\tau\kappa}{\kappa-1}}\mathrm{d}y\right)^{\frac{\kappa-1}{\kappa}}\left(\int_{\mathbb{R}^2}[G_n(u_k)]^\kappa\mathrm{d}y\right)^{\frac{1}{\kappa}}+\int_{\mathbb{R}^2}G_n(u_k)\mathrm{d}x.\end{aligned}$$

On the one hand, due to Lemma 2.3-(g_1), we argue as the same way in (2.4) to see that $G_n(s)\le g_n(s)s$ for all $s>0$. On the other hand, since $\tau<\tau_\kappa$ reveals that $\frac{\tau\kappa}{\kappa-1}<2$, we are capable of adopting Lemma 2.3-(g_1) again to obtain

$$\int_{\mathbb{R}^2}\frac{G_n(u_k(y))}{|x-y|^\tau}\mathrm{d}y\le C\tag{2.13}$$

for some $C>0$ independent of $k\in\mathbb{N}^+$. The proof of (2.12) would be done immediately if we can show that $|x|^{-\tau}*G_n(u_k)\to|x|^{-\tau}*G_n(u)$ a.e. in $\mathbb{R}^2$ as $k\to\infty$. Actually, it is correct via applying (2.11) and (2.13) to the generalized Dominated Lebesgue's Convergence theorem. Alternatively, we shall not demonstrate the detailed proof since it is essentially similar to that of [6, Lemma 4.6]. So, the proof is completed. $\square$

3. Existence results for the auxiliary problems

In this section, we are going to investigate the existence results for the auxiliary problems (1.12) and (1.16) under the assumptions $(V_1)-(V_3)$ and (1.3) as well as $(f_1)-(f_4)$ for all $\tau\in(0,1)$ and $n\in\mathbb{N}^+$.

On the one hand, we shall mainly focus on the following planar logarithmic Choquard equation

$$-\Delta u+\lambda V(x)u=\left[\mathbb{G}_\tau(x)*G_n(u)\right]g_n(u)\ \text{in}\ \mathbb{R}^2,\tag{3.1}$$

where $\mathbb{G}_\tau$ and G_n are defined by (1.13) and (1.15), respectively. We recall that a solution $u\in E_\lambda$ to the Problem (3.1) corresponds to a critical point of the variational functional $\mathcal{J}_{\lambda,\tau,n}:E_\lambda\to\mathbb{R}$ below

$$\mathcal{J}_{\lambda,\tau,n}(u)=\frac12\int_{\mathbb{R}^2}[|\nabla u|^2+\lambda V(x)|u|^2]\mathrm{d}x-\frac12\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u)\right]G_n(u)\mathrm{d}x.\tag{3.2}$$

For all fixed $\lambda\ge\lambda_0$, we can derive from the discussions in Lemma 2.4 that the variational functional $\mathcal{J}_{\lambda,\tau,n}$ is well-defined and belongs to $\mathcal{C}^1(E_\lambda,\mathbb{R})$ with its derivative given by

$$\mathcal{J}'_{\lambda,\tau,n}(u)(v)=\int_{\mathbb{R}^2}[\nabla u\nabla v+\lambda V(x)uv]\mathrm{d}x-\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u)\right]g_n(u)v\mathrm{d}x,\ \forall u,v\in E_\lambda.$$

As a matter of fact, due to (1.13), we can rewrite $\mathcal{J}_{\lambda,\tau,n}$ as the form below

$$\mathcal{J}_{\lambda,\tau,n}(u)=\frac{1}{2}\int_{\mathbb{R}^2}[|\nabla u|^2+\lambda V(x)|u|^2]\mathrm{d}x+\frac{1}{2\tau}\left(\int_{\mathbb{R}^2}G_n(u)\mathrm{d}x\right)^2-\frac{1}{2\tau}\int_{\mathbb{R}^2}\left[|x|^{-\tau}*G_n(u)\right]G_n(u)\mathrm{d}x.$$

Then, the conclusion follows by Lemma 2.4.

The main result concerning Eq. (3.1) is the following:

Theorem 3.1. *Let V satisfy $(V_1)-(V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1)-(f_3)$. Then there exist some constants $\sigma_*>0$, $\tau_*>0$ and $\lambda_*>0$ such that for all $\sigma\in(0,\sigma_*)$, $\tau\in(0,\tau_*)$ and $\lambda>\lambda_*$, Eq. (3.1) admits at least a nontrivial solution for all $n\in\mathbb{N}^+$.*

The proof of the above theorem will be divided into several lemmas.

As a start, we shall verify that the variational functional $\mathcal{J}_{\lambda,\tau,n}$ satisfies the mountain-pass geometry structure for all $n\in\mathbb{N}^+$.

Lemma 3.2. *Let V satisfy $(V_1)-(V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1)-(f_3)$. Then, for all $\sigma\in(0,\frac{1}{4n^2})$, $\tau\in(0,\frac{1}{2})$ and $\lambda>\lambda_0$, there exists a constant $\varrho>0$, independent of σ,τ,λ and $n\in\mathbb{N}^+$, such that*

$$m_\rho\triangleq\inf\left\{\mathcal{J}_{\lambda,\tau,n}(u):u\in E_\lambda,\|u\|_{E_\lambda}=\rho\right\}>0,\ \forall\rho\in(0,\varrho],\tag{3.3}$$

and

$$\bar{m}_\rho\triangleq\inf\left\{\mathcal{J}'_{\lambda,\tau,n}(u)(u):u\in E_\lambda,\|u\|_{E_\lambda}=\rho\right\}>0,\ \forall\rho\in(0,\varrho].\tag{3.4}$$

Proof. In view of (2.7), we apply $\tau'=\frac{1}{2}$ in Lemma 2.2 to have that

$$\begin{aligned}\mathcal{J}_{\lambda,\tau,n}(u)&\geq\frac{1}{2}\|u\|_{E_\lambda}^2-\frac{1}{2}\int_{\mathbb{R}^2}\left(\int_{|x-y|\leq1}\frac{|x-y|^{-\tau}-1}{\tau}G_n(u(y))\mathrm{d}y\right)G_n(u(x))\mathrm{d}x\\&\geq\frac{1}{2}\|u\|_{E_\lambda}^2-\frac{C}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(u(x))F(u(y))}{|x-y|^{\frac{1}{2}}}\mathrm{d}x\mathrm{d}y,\end{aligned}\tag{3.5}$$

for some $C>0$ independent of σ,τ,λ and $n\in\mathbb{N}^+$. Let $\epsilon=1$ in (2.3), we choose $\varrho=\sqrt{\frac{7\pi}{2\alpha\nu}}$ with $\frac{1}{\nu}+\frac{1}{\nu'}$ and $\nu,\nu'>1$, then it depends on (2.1) and (1.8) to arrive at

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(u(x))F(u(y))}{|x-y|^{\frac{1}{2}}}\mathrm{d}x\mathrm{d}y&\leq C_1\left(\int_{\mathbb{R}^2}|u|^{\frac{16}{7}}\mathrm{d}x\right)^{\frac{7}{4}}+C_2\left[\int_{\mathbb{R}^2}|u|^{\frac{8}{7}\bar{q}}\left(e^{\frac{8}{7}\alpha u^2}-1\right)\mathrm{d}x\right]^{\frac{7}{4}}\\&\leq C_1\left(\int_{\mathbb{R}^2}|u|^{\frac{16}{7}}\mathrm{d}x\right)^{\frac{7}{4}}+C_2\left[\int_{\mathbb{R}^2}|u|^{\frac{8}{7}\bar{q}\nu'}\mathrm{d}x\right]^{\frac{7}{4\nu'}}\left[\int_{\mathbb{R}^2}\left(e^{\frac{8}{7}\alpha\nu u^2}-1\right)\mathrm{d}x\right]^{\frac{7}{4\nu}}\\&\leq C_3\|u\|_{E_\lambda}^4+C_4\|u\|_{E_\lambda}^{2\bar{q}},\end{aligned}\tag{3.6}$$

where $C_j>0$ is independent of σ,τ,λ and $n\in\mathbb{N}^+$ with $j\in\{1,2,3,4\}$. With the aid of (3.5) and (3.6), we can conclude (3.3) and (3.4) immediately. The proof is completed. □

Lemma 3.3. *Let V satisfy $(V_1)-(V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1)-(f_3)$. Then, for all $\sigma\in(0,1)$, $\tau\in(0,1)$ and $\lambda>\lambda_0$, there exists a function $e\in E_\lambda$ with $\|e\|_{E_\lambda}>\varrho$ such that $\mathcal{J}_{\lambda,\tau,n}(e)<0$, where e is independent of σ,τ,λ and $n\in\mathbb{N}^+$.*

Proof. Choosing a nonnegative function $\psi\in C_0^\infty(\mathbb{R}^2)$ with $\psi(x)\equiv1$ for all $|x|\leq\frac{1}{8}$, $\psi(x)\equiv0$ for all $|x|\geq\frac{1}{4}$ and $|\nabla\psi|_\infty\leq16$, then it simply has that

$$\mathcal{J}_{\lambda,\tau,n}(t\psi)=\frac{t^2}{2}\|\psi\|_{E_\lambda}^2-\frac{1}{2}\int_{|x-y|\leq\frac{1}{2}}\left(\int_{|x-y|\leq\frac{1}{2}}\frac{|x-y|^{-\tau}-1}{\tau}G_n(t\psi(y))\mathrm{d}y\right)G_n(t\psi(x))\mathrm{d}x$$

$$\leq \frac{t^2}{2}\|\psi\|_{E_\lambda}^2 - \frac{1}{2}\int_{|x-y|\leq\frac{1}{2}}\left(\int_{|x-y|\leq\frac{1}{2}}\frac{|x-y|^{-\tau}-1}{\tau}F(t\psi(y))\mathrm{d}y\right)F(t\psi(x))\mathrm{d}x$$

$$\leq \frac{t^2}{2}\|\psi\|_{E_\lambda}^2 - \frac{\log 2}{2}\left(\int_{\mathbb{R}^2}F(t\psi(x))\mathrm{d}x\right)^2, \tag{3.7}$$

where we have used the facts $\psi(x)\psi(y)\equiv 0$ if $|x-y|\geq\frac{1}{2}$, (2.3) and Lemma 2.2, respectively. Define

$$\xi(t) \triangleq \frac{1}{2}\left(\int_{\mathbb{R}^2}F(t\psi(x))\mathrm{d}x\right)^2, \ \forall t>0,$$

and so

$$\xi'(t) = \frac{1}{t}\left(\int_{\mathbb{R}^2}F(t\psi(x))\mathrm{d}x\right)\left(\int_{\mathbb{R}^2}f(t\psi(x))t\psi(x)\mathrm{d}x\right), \ \forall t>0.$$

In view of (2.5), one has that $\dfrac{\xi'(t)}{\xi(t)} \geq \dfrac{2}{(1-\delta)t}$ for all $t>0$. Integrating it on $[1,s]$, it holds that

$$\frac{1}{2}\left(\int_{\mathbb{R}^2}F(s\psi(x))\mathrm{d}x\right)^2 = \xi(s) \geq \xi(1)s^{\frac{2}{1-\delta}} = \frac{1}{2}\left(\int_{\mathbb{R}^2}F(\psi(x))\mathrm{d}x\right)^2 s^{\frac{2}{1-\delta}}, \ \forall s>1. \tag{3.8}$$

Combining (3.7) and (3.8), we reach

$$\mathcal{J}_{\lambda,\tau,n}(t\psi) \leq \frac{t^2}{2}\|\psi\|_{E_\lambda}^2 - \frac{\log 2}{2}\left(\int_{\mathbb{R}^2}F(\psi(x))\mathrm{d}x\right)^2 t^{\frac{2}{1-\delta}}, \ \forall t>1.$$

Since $\delta\in(0,1)$ in (f_2), then $\mathcal{J}_{\lambda,\tau,n}(t\psi)\to-\infty$ as $t\to+\infty$. As a consequence, we can find a sufficiently $t_0>0$ such that $e=t_0\psi$ will be the desired function. The proof is completed. □

Relying on Lemmas 3.2 and 3.3, we shall exploit the following critical point theorem without the (C) condition introduced in [43] to construct a (C) sequence for $\mathcal{J}_{\lambda,\tau,n}$.

Proposition 3.4. *Let Z be a Banach space and $\varphi\in\mathcal{C}^1(Z,\mathbb{R})$ Gateaux differentiable for all $v\in Z$, with G-derivative $\varphi'(v)\in Z^{-1}$ continuous from the norm topology of Z to the weak $*$ topology of Z^{-1} and $\varphi(0)=0$. Let S be a closed subset of Z which disconnects (archwise) Z. Let $v_0=0$ and $v_1\in Z$ be points belonging to distinct connected components of $\bar{X}\backslash Z$. Suppose that*

$$\inf_S \varphi \geq \varrho > 0 \ \textit{and} \ \varphi(v_1)\leq 0$$

and let $\Gamma=\{\gamma\in C([0,1],Z):\gamma(0) \text{ and } \gamma(1)=v_1\}$. Then

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t)) \geq \varrho > 0$$

and there is a $(\mathrm{C})_c$ sequence for φ.

Combining Lemmas 3.2 and 3.3 as well as Proposition 3.4, for all $\sigma\in(0,\frac{1}{4n^2})$, $\tau\in(0,\frac{1}{2})$ and $\lambda>\lambda_0$, there exists a sequence $(u_k)\subset E_\lambda$ such that

$$\mathcal{J}_{\lambda,\tau,n}(u_k)\to c_{\lambda,\tau,n} \text{ and } (1+\|u_k\|_{E_\lambda})\|\mathcal{J}'_{\lambda,\tau,n}(u_k)\|_{E_\lambda^{-1}}\to 0, \tag{3.9}$$

for all $n\in\mathbb{N}^+$, where

$$c_{\lambda,\tau,n} \triangleq \inf_{\gamma\in\Gamma_{\lambda,\tau,n}}\max_{t\in[0,1]}\mathcal{J}_{\lambda,\tau,n}(\gamma(t))>0 \tag{3.10}$$

with $\Gamma_{\lambda,\tau,n}=\{\gamma\in\mathcal{C}([0,1],E_\lambda):\gamma(0)=0 \text{ and } \mathcal{J}_{\lambda,\tau,n}(\gamma(1))<0\}$.

Remark 3.5. The reader is invited to observe from the proofs of Lemmas 3.2 and 3.3 that there exist some constants $\bar{c},\hat{c}>0$, independent of σ,τ,λ and $n\in\mathbb{N}^+$, such that $\bar{c}\leq c_{\lambda,\tau,n}\leq\hat{c}$.

Lemma 3.6. *Let V satisfy $(V_1)-(V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1)-(f_3)$. Then, for all $\sigma\in(0,\sigma_3)$ with $\sigma_3\triangleq\min\{\sigma_1,\sigma_2,\frac{1}{4n^2}\}>0$, any sequence $(u_k)\subset E_\lambda$ satisfying (3.9) and (3.10) is uniformly bounded in $k\in\mathbb{N}^+$. Moreover, there is a constant $\mathcal{C}_0>0$ independent of $k\in\mathbb{N}^+$ such that*

$$\left|\sup_{k\in\mathbb{N}^+}\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]G_n(u_k)\mathrm{d}x\right|\le\mathcal{C}_0 \text{ and } \left|\sup_{k\in\mathbb{N}^+}\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]g_n(u_k)u_k\mathrm{d}x\right|\le\mathcal{C}_0. \tag{3.11}$$

Proof. To verify that $(u_k)\subset E_\lambda$ is uniformly bounded in $k\in\mathbb{N}^+$, we introduce a suitable test function below

$$v_k\triangleq\begin{cases}\dfrac{G_n(u_k)}{g_n(u_k)}, & \text{if } u_k>0,\\ \left(1-\dfrac{\delta}{4}\right)u_k, & \text{if } u_k\le 0,\end{cases}$$

where $\delta\in(0,1)$ is given by (f_2). For all $\sigma\in(0,\sigma_1)$ with $\sigma_1>0$ in Lemma 2.3-(g_2), we claim that

$$G_n(s)\le\left(1-\frac{\delta}{4}\right)g_n(s)s,\ \forall s>0. \tag{3.12}$$

In fact, Lemma 2.3-(g_2) infers that $\dfrac{\mathrm{d}}{\mathrm{d}s}\dfrac{G_n(s)}{g_n(s)}\le 1-\dfrac{\delta}{4}$ for any $s>0$, then for all $\varepsilon\in(0,s)$, one has

$$\frac{G_n(s)}{g_n(s)}-\frac{G_n(\varepsilon)}{g_n(\varepsilon)}=\int_\varepsilon^s\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{G_n(t)}{g_n(t)}\right)\mathrm{d}t\le\left(1-\frac{\delta}{4}\right)\int_\varepsilon^s\mathrm{d}t=\left(1-\frac{\delta}{4}\right)(s-\varepsilon)$$

which together with $\lim\limits_{\varepsilon\to0^+}\dfrac{G_n(\varepsilon)}{g_n(\varepsilon)}=0$ uniformly in $n\in\mathbb{N}^+$ yields the claim by tending $\varepsilon\to0^+$. Thereby, we are derived from (3.12) that $(v_k)\subset E_\lambda$. To further show that $\|v_k\|_{E_\lambda}\le C\|u_k\|_{E_\lambda}$ for some constant $C>0$ independent of $k\in\mathbb{N}^+$, due to Lemma 2.3-(g_1), it suffices to show that

$$\frac{G_n(s)g_n'(s)}{g_n^2(s)}\le C,\ \forall s>0. \tag{3.13}$$

Indeed, it is a direct consequence of Lemma 2.3-(g_3) and $g_n\in\mathcal{C}^1$ in (g_1). Adopting $\mathcal{J}_{\lambda,\tau,n}(u_k)\to c_{\lambda,\tau,n}$ in (3.9), it has that

$$c_{\lambda,\tau,n}+o_k(1)=\frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u_k|^2+\lambda V(x)|u_k|^2\right]\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]G_n(u_k)\mathrm{d}x. \tag{3.14}$$

Combining Lemma 2.3-(g_1), (g_2) and (3.12) as well as (3.14), we reach

$$\begin{aligned}o_k(1)\|u_k\|_{E_\lambda}&=\left(1-\frac{\delta}{4}\right)\int_{u_k\le0}\left[|\nabla u_k|^2+\lambda V(x)|u_k|^2\right]\mathrm{d}x+\int_{u_k>0}|\nabla u_k|^2\left[1-\frac{G_n(u_k)g_n'(u_k)}{g_n^2(u_k)}\right]\mathrm{d}x\\&\quad+\int_{\mathbb{R}^2}\lambda V(x)u_k\frac{G_n(u_k)}{g_n(u_k)}\mathrm{d}x-\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]G_n(u_k)\mathrm{d}x\\&\le\left(1-\frac{\delta}{4}\right)\int_{\mathbb{R}^2}\left[|\nabla u_k|^2+\lambda V(x)|u_k|^2\right]\mathrm{d}x-\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]G_n(u_k)\mathrm{d}x\\&=2c_{\lambda,\tau,n}-\frac{\delta}{4}\|u_k\|_{E_\lambda}^2+o_k(1)\end{aligned}$$

which reveals that $\|u_k\|_{E_\lambda}\le C$ for some $C>0$ independent of $k\in\mathbb{N}^+$. Since $\|u_k\|_{E_\lambda}\|\mathcal{J}'_{\lambda,\tau,n}(u_k)\|_{E_\lambda^{-1}}\to 0$ in (3.9), then

$$o_k(1)=\frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u_k|^2+\lambda V(x)|u_k|^2\right]\mathrm{d}x-\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]g_n(u_k)u_k\mathrm{d}x. \tag{3.15}$$

The remaining parts follow directly by (3.14) and (3.15). The proof is completed. □

Remark 3.7. As a direct corollary of Lemma 3.6, without loss of generality, we shall always assume that the Cerami sequence $(u_k)\subset E_\lambda$ in (3.9) is nonnegative. In fact, we define $u_k^-=\min\{u_k,0\}\in E_\lambda$ and so $u_k^-\leq 0$ for all $k\in\mathbb{N}^+$. Recalling $g_n(s)\equiv 0$ for all $s\leq 0$ by Lemma 2.3-(g_1), then

$$\begin{aligned}o_k(1)=\|u_k^-\|_{E_\lambda}\|\mathcal{J}'_{\lambda,\tau,n}(u_k)\|_{E_\lambda^{-1}}&\geq \mathcal{J}'_{\lambda,\tau,n}(u_k)(u_k^-)\\&=\|u_k^-\|_{E_\lambda}^2-\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]g_n(u_k)u_k^-\mathrm{d}x\\&=\|u_k^-\|_{E_\lambda}^2\geq 0.\end{aligned}$$

Hence, we must have that $u_k^-\to 0$ in E_λ and it permits us to conclude that (u_k^+) is a Cerami sequence, where $u_k^+=\max\{u_k,0\}\geq 0$.

Consider that the nonlinearity G_n possesses the supercritical exponential growth at infinity and it causes the lack of compactness. To restore it, we proceed as the Brézis-Lieb method to pull the mountain-pass level $c_{\lambda,\tau,n}$ down below a critical value. Have this aim in mind, motivated by [2,3,16,27,28,32,62], for a fixed constant $r_0\in(0,1]$, we shall consider the Moser sequence functions defined by

$$\bar{w}_k(x)\triangleq\frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log k}, & \text{if } 0\leq|x|\leq\frac{r_0}{k},\\ \frac{\log(\frac{1}{|x|})}{\sqrt{\log k}}, & \text{if } \frac{r_0}{k}<|x|\leq r_0,\\ 0, & \text{if } |x|>r_0,\end{cases}$$

where $r_0>0$ is sufficiently small to satisfy $B_{r_0}(0)\subset\Omega$. In fact, since $\Omega=\mathrm{int}V^{-1}(0)$ is open, without loss of generality, we can suppose that $0\in\Omega$ and so such an r_0 is available.

Lemma 3.8. *Let V satisfy $(V_1)-(V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1)-(f_4)$. Then, for all $\sigma\in(0,\sigma_3)$ with $\sigma_3\triangleq\min\{\sigma_1,\sigma_2,\frac{1}{4n^2}\}>0$, we have that*

$$0<\inf_{n\in\mathbb{N}^+}c_{\lambda,\tau,n}\leq\sup_{n\in\mathbb{N}^+}c_{\lambda,\tau,n}<\frac{2\pi}{\alpha_0},\ \forall\tau\in\left(0,\frac{1}{2}\right)\text{ and }\lambda>\lambda_0.$$

Proof. The first inequality is a corollary of Remark 3.5 and so we shall just exhibit the detailed proof of the last inequality. First of all, some elementary calculations provide us that $(\bar{w}_k)\subset E_\lambda$ and $|\nabla w_k|_2^2=1$ and $|\sqrt{\lambda V}\bar{w}_k|_2^2=0$ for all $k\in\mathbb{N}^+$. We define $w_k=\frac{\bar{w}_k}{\|\bar{w}_k\|_{E_\lambda}}$, then $\|w_k\|_{E_\lambda}\equiv 1$ for all $k\in\mathbb{N}^+$.

To conclude the proof, it suffices to determine a suitable $\mathcal{B}>0$, independent of σ,τ,λ and $k,n\in\mathbb{N}^+$, such that there is a $k_0\in\mathbb{N}^+$ satisfying

$$\max_{t\geq 0}\mathcal{J}_{\lambda,\tau,n}(tw_{k_0})<\mathcal{B}.\tag{3.16}$$

Arguing it indirectly, we could suppose that there is a $t_k>0$ such that

$$\mathcal{J}_{\lambda,\tau,n}(t_kw_k)=\max_{t\geq 0}\mathcal{J}_{\lambda,\tau,n}(tw_k)\geq\mathcal{B}.$$

Consequently, one can infer that

$$\frac{t_k^2}{2}\geq\frac{1}{2}\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(t_kw_k)\right]G_n(t_kw_k)\mathrm{d}x+\mathcal{B},\tag{3.17}$$

and

$$t_k^2=\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(t_ku_k)\right]g_n(t_kw_k)t_kw_k\mathrm{d}x.\tag{3.18}$$

In view of the support of $\bar{w}_k$, we choose $r_0<\frac{1}{2}$ and so $G_\tau(x)\geq 0$ for all $x\in\operatorname{supp}w_k$. Using Lemma 2.3-$(g_1)$ and (3.17), there holds

$$t_k^2\geq 2\mathcal{B}.\tag{3.19}$$

Our next aim is to show that $\limsup\limits_{k\to+\infty} t_k^2 \le 2\mathcal{B}$. Otherwise, by (3.19), there would be a $\delta_0 > 0$ such that

$$t_k^2 \ge 2\mathcal{B} + \delta_0. \tag{3.20}$$

for some sufficiently large $k \in \mathbb{N}^+$. Hence, we are able to make full use of Lemma 2.2, Lemma 2.3-(g_1), (g_4), and Hölder's inequality to reach

$$\begin{aligned}
\hat{\mathbb{B}} &\triangleq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|x-y|^{-\tau}-1}{\tau} G_n(t_kw_k(y))g_n(t_kw_k(x))t_kw_k(x)\mathrm{d}x\mathrm{d}y \\
&\ge \int_{B_{\frac{r_0}{k}}(0)}\int_{B_{\frac{r_0}{k}}(0)} \log\left(\frac{1}{|x-y|}\right) G_n(t_kw_k(y))g_n(t_kw_k(x))t_kw_k(x)\mathrm{d}x\mathrm{d}y \\
&\ge \log\left(\frac{k}{2r_0}\right)\left(\int_{B_{\frac{r_0}{k}}(0)} G_n(t_kw_k(x))\mathrm{d}x\right)\left(\int_{B_{\frac{r_0}{k}}(0)} g_n(t_kw_k(x))t_kw_k(x)\mathrm{d}x\right) \\
&\ge \log\left(\frac{k}{2r_0}\right)\left(\int_{B_{\frac{r_0}{k}}(0)} \sqrt{G_n(t_kw_k(x))g_n(t_kw_k(x))t_kw_k(x)}\mathrm{d}x\right)^2 \\
&\ge \frac{1}{2}\log\left(\frac{k}{2r_0}\right)\left(\int_{B_{\frac{r_0}{k}}(0)} \sqrt{F(t_kw_k(x))f(t_kw_k(x))t_kw_k(x)}\mathrm{d}x\right)^2.
\end{aligned}$$

According to (f_4) and the definition of w_k, for some sufficiently large $k \in \mathbb{N}^+$, it holds that

$$\hat{\mathbb{B}} \ge \frac{\beta}{4}\pi^2 r_0^4 \log\left(\frac{k}{2r_0}\right) t_k^{1-\vartheta}\|\bar{w}_k\|_{E_\lambda}^{\vartheta-1}\left(\frac{\log k}{2\pi}\right)^{\frac{1-\vartheta}{2}} \exp\left[\left(\frac{\alpha_0}{\pi}t_k^2\|\bar{w}_k\|_{E_\lambda}^{-2} - 4\right)\log k\right]$$

which together with (3.18) indicates that

$$t_k^{1+\vartheta} \ge \frac{\beta}{4}\pi^2 r_0^4 \log\left(\frac{k}{2r_0}\right)\|\bar{w}_k\|_{E_\lambda}^{\vartheta-1}\left(\frac{\log k}{2\pi}\right)^{\frac{1-\vartheta}{2}} \exp\left[\left(\frac{\alpha_0}{\pi}t_k^2\|\bar{w}_k\|_{E_\lambda}^{-2} - 4\right)\log k\right]. \tag{3.21}$$

Obviously, we must have that $t_k^2 \le \frac{4\pi}{\alpha_0}$ for large $k \in \mathbb{N}^+$ and so we choose

$$\mathcal{B} = \frac{2\pi}{\alpha_0}, \tag{3.22}$$

which enables us to arrive at a contradiction, namely it implies that $\limsup\limits_{k\to+\infty} t_k^2 \le \dfrac{4\pi}{\alpha_0}$. Owing to (3.20), without loss of generality, we shall suppose that, up to a subsequence if necessary,

$$\lim_{k\to+\infty} t_k^2 = \frac{4\pi}{\alpha_0}, \tag{3.23}$$

Finally, we are capable of finishing the proof. If $\vartheta < 3$ in (f_4), i.e. $1 + \frac{1-\vartheta}{2} > 0$, then $(\log k)^{1+\frac{1-\vartheta}{2}} \to +\infty$ as $k \to +\infty$ and so there would be a contraction by (3.21) and (3.23). In this situation, the proof is done immediately because of $\beta > \beta_0 = 0$ due to (3.16) and (3.22). If $\vartheta = 3$ in (f_4), then we apply this and (3.23) in (3.21) to get

$$\left(\frac{4\pi}{\alpha_0}\right)^2 \ge \frac{\beta}{4}\pi^2 r_0^4(2\pi),$$

which contradicts with the choice of β_0 in (f_4). Consequently, we always conclude a contradiction when (f_4) holds true and so (3.16) is true. With (3.22) in hands, the proof of this lemma is completed. □

Because of the sign-changing logarithmic kernel, the following lemma is a key ingredient.

Lemma 3.9. *Let V satisfy $(V_1)-(V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1)-(f_3)$. Then, for all $\sigma\in(0,\sigma_3)$ with $\sigma_3\triangleq\min\{\sigma_1,\sigma_2,\frac{1}{4n^2}\}>0$, any sequence $(u_k)\subset E_\lambda$ satisfying (3.9) and (3.10) has the following conclusions*

$$\sup_{k\in\mathbb{N}^+}\int_{\mathbb{R}^2}g_n(u_k)u_k\mathrm{d}x\le\mathcal{C}_1 \text{ and } \sup_{k\in\mathbb{N}^+}\int_{\mathbb{R}^2}[G_n(u_k)]^\kappa\,\mathrm{d}x\le\mathcal{C}_1, \tag{3.24}$$

for some $\mathcal{C}_1>0$ independent of $k\in\mathbb{N}^+$, where $1<\kappa<(1-\varepsilon)^{-\frac12}$ with $\varepsilon\in(0,1)$ given in (3.28) below.

Proof. First of all, inspired by [19], we introduce the auxiliary function

$$H_n(s)\triangleq\int_0^s\sqrt{\frac{G_n(t)g_n'(t)}{g_n^2(t)}}\mathrm{d}t,\ \forall s>0,$$

and define $w_k\triangleq H_n(u_k)$. Let us take some key estimate for $\|w_k\|_{E_\lambda}$. To end it, we choose v_k determined by Lemma 3.6 and then Remark 3.7 jointly with $\|v_k\|_{E_\lambda}\|\mathcal{J}'_{\lambda,\tau,n}(u_k)\|_{E_\lambda^{-1}}\to0$ in (3.9) implies that

$$\begin{aligned}o_k(1)&=\int_{\mathbb{R}^2}|\nabla u_k|^2\left[1-\frac{G_n(u_k)g_n'(u_k)}{g_n^2(u_k)}\right]\mathrm{d}x+\int_{\mathbb{R}^2}\lambda V(x)u_k\frac{G_n(u_k)}{g_n(u_k)}\mathrm{d}x\\&\quad-\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]G_n(u_k)\mathrm{d}x.\end{aligned}$$

Due to the above formula, we apply $\mathcal{J}_{\lambda,\tau,n}(u_k)\to c_{\lambda,\tau,n}$ in (3.9) to get

$$\begin{aligned}2c_{\lambda,\tau,n}+o_k(1)&=\int_{\mathbb{R}^2}\left[|\nabla u_k|^2+\lambda V(x)|u_k|^2\right]\mathrm{d}x-\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]G_n(u_k)\mathrm{d}x\\&=\|u_k\|_{E_\lambda}^2-\int_{\mathbb{R}^2}|\nabla u_k|^2\left[1-\frac{G_n(u_k)g_n'(u_k)}{g_n^2(u_k)}\right]\mathrm{d}x-\int_{\mathbb{R}^2}\lambda V(x)u_k\frac{G_n(u_k)}{g_n(u_k)}\mathrm{d}x\\&=\int_{\mathbb{R}^2}|\nabla u_k|^2\frac{G_n(u_k)g_n'(u_k)}{g_n^2(u_k)}\mathrm{d}x+\int_{\mathbb{R}^2}\lambda V(x)\left[u_k^2-u_k\frac{G_n(u_k)}{g_n(u_k)}\right]\mathrm{d}x.\end{aligned}\tag{3.25}$$

As a consequence, combining (3.12) and (3.25), it holds that

$$\begin{aligned}|\nabla w_k|_2^2=\int_{\mathbb{R}^2}|\nabla H_n(u_k)|^2\mathrm{d}x&=\int_{\mathbb{R}^2}|\nabla u_k|^2\frac{G_n(u_k)g_n'(u_k)}{g_n^2(u_k)}\mathrm{d}x\\&=2c_{\lambda,\tau,n}+\int_{\mathbb{R}^2}\lambda V(x)\left[u_k\frac{G_n(u_k)}{g_n(u_k)}-u_k^2\right]\mathrm{d}x+o_k(1)\\&\le2c_{\lambda,\tau,n}+o_k(1).\end{aligned}\tag{3.26}$$

Moreover, it follows from Lemma 3.6 and (3.13) that

$$\int_{\mathbb{R}^2}\lambda V(x)w_k^2\mathrm{d}x=\int_{\mathbb{R}^2}\lambda V(x)H_n^2(u_k)\mathrm{d}x\le C\int_{\mathbb{R}^2}\lambda V(x)u_k^2\mathrm{d}x\le C\|u_k\|_{E_\lambda}^2\le C.\tag{3.27}$$

Secondly, we recall Lemma 2.3-(g_3), for all $\varepsilon\in(0,1)$, there is a $s_\varepsilon>0$ such that

$$\frac{G_n(s)g_n'(s)}{g_n^2(s)}\ge1-\varepsilon,\ \forall s\ge s_\varepsilon.$$

With the help of it, we exploit Lemma 2.3-(g_2) to have that

$$w_k=\int_0^{s_\varepsilon}\sqrt{\frac{G_n(t)g_n'(t)}{g_n^2(t)}}\mathrm{d}t+\int_{s_\varepsilon}^{u_k}\sqrt{\frac{G_n(t)g_n'(t)}{g_n^2(t)}}\mathrm{d}t\ge\frac{\sqrt{\delta}}{2}s_\varepsilon+\sqrt{1-\varepsilon}(u_k-s_\varepsilon)\ge\sqrt{1-\varepsilon}(u_k-s_\varepsilon)$$

from where it follows that

$$u_k\le s_\varepsilon+\frac{w_k}{\sqrt{1-\varepsilon}},\ \forall x\in\mathbb{R}^2.\tag{3.28}$$

Finally, we are ready to exhibit the verifications of (3.24) in detail. In view of Lemma 3.8 and (3.26), we choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu > 1$ sufficiently close to 1 in such a way that $\frac{1}{\nu} + \frac{1}{\nu'} = 1$ with $\nu > 1$ and

$$\alpha\nu|\nabla w_k|_2^2 < \frac{4\pi(1-\varepsilon)^2}{(1+\varepsilon)^2},$$

where $\varepsilon \in (0,1)$ comes from (3.28). We define

$$\hat{w}_k = \sqrt{\frac{\alpha\nu(1+\varepsilon)^2}{4\pi(1-\varepsilon)^2}}w_k, \ \forall k \in \mathbb{N}^+.$$

So, we conclude that $|\nabla \hat{w}_k|_2^2 \le 1$ and $|\sqrt{\lambda V}\hat{w}_k|_2^2 \le C$ by (3.27) for some $C > 0$ independent of $k \in \mathbb{N}^+$. Thanks to (1.8), these facts show us that

$$\sup_{k\in\mathbb{N}^+}\int_{\mathbb{R}^2}\left[e^{\alpha\nu(1+\varepsilon)^2(1-\varepsilon)^{-1}w_k^2} - 1\right]\mathrm{d}x = \sup_{k\in\mathbb{N}^+}\int_{\mathbb{R}^2}\left[e^{4\pi(1-\varepsilon)\hat{w}_k^2} - 1\right]\mathrm{d}x \le C < +\infty. \tag{3.29}$$

To continue to proof, for the above fixed $\varepsilon \in (0,1)$ given in (3.28), we need the following two types of Young's inequality

$$|a+b|^2 \le (1+\varepsilon)|a|^2 + (1+\varepsilon^{-1})|b|^2, \ \forall a, b \in \mathbb{R}$$

and

$$e^{a+b} - d \le \frac{1}{1+\varepsilon}\left[e^{(1+\varepsilon)a} - d\right] + \frac{\varepsilon}{1+\varepsilon}\left[e^{(1+\varepsilon^{-1})b} - d\right], \ \forall a, b, d \in \mathbb{R}.$$

Letting $\bar{q} \ge 2$ be given as (2.2), we are derived from Lemma 3.6 and (3.28)-(3.29) that

$$\begin{aligned}
&\int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}}\left(e^{\alpha u_k^2} - 1\right)\mathrm{d}x \le \int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}}\left[e^{\alpha\left(s_\varepsilon + \frac{w_k}{\sqrt{1-\varepsilon}}\right)^2} - 1\right]\mathrm{d}x \\
\le\ & \int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}}\left[e^{\alpha(1+\varepsilon^{-1})s_\varepsilon^2 + \alpha(1+\varepsilon)(1-\varepsilon)^{-1}w_k^2} - 1\right]\mathrm{d}x \\
\le\ & \int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}}\left[\left(\frac{\varepsilon}{1+\varepsilon}e^{\alpha(1+\varepsilon^{-1})^2 s_\varepsilon^2} - 1\right) + \left(\frac{1}{1+\varepsilon}e^{\alpha(1+\varepsilon)^2(1-\varepsilon)^{-1}w_k^2} - 1\right)\right]\mathrm{d}x \\
\le\ & C_\varepsilon\int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}}\mathrm{d}x + \frac{1}{1+\varepsilon}\int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}}\left(e^{\alpha(1+\varepsilon)^2(1-\varepsilon)^{-1}w_k^2} - 1\right)\mathrm{d}x \\
\le\ & C_\varepsilon\int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}}\mathrm{d}x + \frac{1}{1+\varepsilon}\left(\int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}\nu'}\mathrm{d}x\right)^{\frac{1}{\nu'}}\left[\int_{u_k\ge s_\varepsilon}\left(e^{\alpha\nu(1+\varepsilon)^2(1-\varepsilon)^{-1}w_k^2} - 1\right)\mathrm{d}x\right]^{\frac{1}{\nu}} \\
\le\ & C_\varepsilon.
\end{aligned} \tag{3.30}$$

Since $1 < \kappa < (1-\varepsilon)^{-\frac{1}{2}}$, some simple calculations reveal that

$$\sup_{k\in\mathbb{N}^+}\int_{\mathbb{R}^2}\left[e^{\alpha\kappa\nu(1+\varepsilon)^2(1-\varepsilon)^{-1}w_k^2} - 1\right]\mathrm{d}x \le \sup_{k\in\mathbb{N}^+}\int_{\mathbb{R}^2}\left[e^{4\pi\sqrt{1-\varepsilon}\hat{w}_k^2} - 1\right]\mathrm{d}x \le C < +\infty.$$

From which, we proceed as the same way in (3.30) to arrive at

$$\int_{u_k\ge s_\varepsilon}|u_k|^{\bar{q}\kappa}\left(e^{\alpha\kappa u_k^2} - 1\right)\mathrm{d}x \le C_\varepsilon < +\infty. \tag{3.31}$$

In view of (2.2) and (2.3) as well as (2.7), we conclude the proofs of this lemma immediately by taking advantage of (3.30) and (3.31). The proof is completed. □

Lemma 3.10. *Let V satisfy $(V_1) - (V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1) - (f_3)$. Then there exist some constants $\sigma_* > 0$, $\tau_* > 0$ and $\lambda_* > 0$ such that for all $\sigma \in (0, \sigma_*)$, $\tau \in (0, \tau_*)$ and $\lambda > \lambda_*$, any sequence $(u_k) \subset E_\lambda$ satisfying (3.9) and (3.10) contains a strongly convergent subsequence.*

Proof. First of all, we choose $\sigma \in (0,\sigma_3)$ with $\sigma_3 \triangleq \min\{\sigma_1,\sigma_2,\frac{1}{4n^2}\} > 0$, $\sigma \in (0,\frac{1}{2})$ and $\lambda > \lambda_0$, then all the conclusions above in this Section remain true. By Lemma 3.6, (u_k) is bounded in E_λ and so, there exists a $u \in E_\lambda$ such that $u_k \rightharpoonup u$ in E_λ, $u_k \to u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ with $s \in [1,+\infty)$ and $u_k \to u$ a.e. in $\mathbb{R}^2$. To conclude the proof clearly, we shall split it into several steps:

Step 1: Define $v_k \triangleq u_k - u$, then there exists a $\hat{\Lambda} > 0$ such that $v_k \to 0$ in $L^q(\mathbb{R}^2)$ for all $q \in (2,+\infty)$ along a subsequence as $k \to \infty$ when $\lambda > \hat{\Lambda}$.

Actually, since (v_k) is uniformly bounded in $k \in \mathbb{N}$ for all $\lambda > \Lambda_0$, then we have one of the following two possibilities for some $r > 0$:

$$\text{(i): } \lim_{k\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_r(y)} |v_k|^2 \mathrm{d}x > 0 \text{ and (ii): } \lim_{k\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_r(y)} |v_k|^2 \mathrm{d}x = 0.$$

If (i) was true, there is a constant $\tilde{\delta} > 0$ independent of $\lambda > \Lambda_0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_r(y)} |v_n|^2 \mathrm{d}x \geq \tilde{\delta}$$

for some $r > 0$. Since (u_k) is uniformly bounded in E_λ, without loss of generality, we are assuming that $\lim\limits_{n\to\infty} \|u_k\|^2_{E_\lambda} \leq \Theta$ for some $\Theta \in (0,+\infty)$. Clearly, there holds $\lim\limits_{n\to\infty} \|v_k\|^2_{E_\lambda} \leq 4\Theta$. Recalling $v_k \to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^2)$ with $q \in (2,+\infty)$ and $|\mathcal{A}_R| \to 0$ as $R \to +\infty$ by (V_2), where $\mathcal{A}_R \triangleq \{x \in \mathbb{R}^2 \backslash B_R(0) : V(x) < b\}$, we can determine a sufficiently large but fixed $R > 0$ to satisfy

$$\limsup_{k\to\infty} \int_{B_R(0)} |v_k|^2 \mathrm{d}x < \frac{\tilde{\delta}}{4} \tag{3.32}$$

and

$$|\mathcal{A}_R| < \left(\frac{\tilde{\delta}}{16\Theta}\right)^{\frac{q}{q-2}} (|\Xi|\kappa_{\mathrm{GN}})^{-\frac{2}{q-2}}, \tag{3.33}$$

where $\kappa_{\mathrm{GN}} > 0$ symbols as the optimal constant related to Gagliardo-Nirenberg inequality. Proceeding as [55, (3.21)], one sees that

$$\limsup_{k\to\infty} \int_{\mathcal{A}_R} |v_k|^2 \mathrm{d}x \leq \limsup_{k\to\infty} \left(\int_{\mathcal{A}_R} |v_k|^q \mathrm{d}x\right)^{\frac{2}{q}} |\mathcal{A}_R|^{\frac{q-2}{q}} \leq 4\Theta(|\Xi|\kappa_{\mathrm{GN}})^{\frac{2}{q}} |\mathcal{A}_R|^{\frac{q-2}{q}} < \frac{\hat{\delta}}{4}. \tag{3.34}$$

Let us choose $\hat{\Lambda} = \max\left\{1, \Lambda_0, \frac{16\Theta}{\tilde{\delta}b}\right\}$, then for all $\lambda > \hat{\Lambda}$, we reach

$$\limsup_{k\to\infty} \int_{\mathcal{B}_R} |v_k|^2 \mathrm{d}x \leq \limsup_{n\to\infty} \frac{1}{\lambda b} \int_{\mathcal{B}_R} \lambda V(x) |v_k|^2 \mathrm{d}x \leq \frac{4\Theta}{\lambda b} < \frac{\tilde{\delta}}{4}, \tag{3.35}$$

where $\mathcal{B}_R \triangleq \{x \in \mathbb{R}^2 \backslash B_R(0) : V(x) \geq b\}$. We gather (3.36), (3.37) and (3.35) to derive

$$\begin{aligned}
\tilde{\delta} &\leq \lim_{k\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_r(y)} |v_k|^2 \mathrm{d}x \leq \limsup_{k\to\infty} \int_{\mathbb{R}^2} |v_k|^2 \mathrm{d}x \\
&= \limsup_{k\to\infty} \left(\int_{\mathbb{R}^2 \backslash B_R(0)} |v_k|^2 \mathrm{d}x + \int_{B_R(0)} |v_k|^2 \mathrm{d}x\right) \leq \frac{3\tilde{\delta}}{4}
\end{aligned}$$

which is impossible. The proof of this step is done.

Step 2: For all $\tau \in (0,\tau_\kappa)$ with τ_κ in Lemma 2.5, we have that $u \neq 0$, $\mathcal{J}'_{\lambda,\tau,n}(u) = 0$ and $\mathcal{J}_{\lambda,\tau,n}(u) \geq 0$.

We suppose, by contradiction, that $u \equiv 0$ and thus the Step 1 gives us that $u_n \to 0$ in $L^q(\mathbb{R}^2)$ for all $q \in (2,+\infty)$ when $\Lambda > \hat{\Lambda}$. According to Lemma 3.9, we realize that Lemma (2.5) is available and thus

(2.11)-(2.12) can give us that

$$\lim_{k\to\infty}\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]G_n(u_k)\mathrm{d}x=0. \tag{3.36}$$

As a consequence, combining $\mathcal{J}_{\lambda,\tau,n}(u_k)\to c_{\lambda,\tau,n}$ in (3.9), (3.36) and Lemma 3.8, it holds that

$$\limsup_{k\to\infty}\int_{\mathbb{R}^2}[|\nabla u_k|^2+\lambda V(x)u_k^2]\mathrm{d}x=2c_{\lambda,\tau,n}<\frac{4\pi}{\alpha_0}.$$

Thereby, we shall choose $\alpha>\alpha_0$ sufficiently close to α_0 and $\nu'>1$ sufficiently close to 1 in such a way that $\frac{1}{\nu}+\frac{1}{\nu'}=1$ and

$$\alpha\nu\|u_k\|_{E_\lambda}^2<4\pi(1-\varepsilon),\ \forall k\in\mathbb{N}^+,$$

where $\varepsilon\in(0,1)$ comes from (3.28). It follows from (2.2), (2.7) with $\sigma<\frac{1}{4n^2}$ and the Holder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^2}g_n(u_k)u_k\mathrm{d}x&\le\epsilon\int_{\mathbb{R}^2}|u_k|^2\mathrm{d}x+C_\epsilon\int_{\mathbb{R}^2}|u_k|^{\bar q}(e^{\alpha u_k^2}-1)\mathrm{d}x\\&\le\epsilon\int_{\mathbb{R}^2}|u_k|^2\mathrm{d}x+C_\epsilon\left(\int_{\mathbb{R}^2}|u_k|^{q\nu'}\mathrm{d}x\right)^{\frac{1}{\nu'}}\left(\int_{\mathbb{R}^2}(e^{4\pi(1-\varepsilon)^2(u_k/\|u_k\|_{E_\lambda})^2}-1)\mathrm{d}x\right)^{\frac{1}{\nu}}.\end{aligned}$$

Recalling (1.8), we shall deduce that $\int_{\mathbb{R}^2}g_n(u_k)u_k\mathrm{d}x\to0$ by letting $k\to\infty$ and then tending $\epsilon\to0^+$. Thanks to this and (2.13), it holds that $\int_{\mathbb{R}^2}[|x|^{-\tau}*G_n(u_k)]g_n(u_k)u_k\mathrm{d}x\to0$. It is, therefore, to reach

$$\lim_{k\to\infty}\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]g_n(u_k)u_k\mathrm{d}x=0. \tag{3.37}$$

Adopting $\mathcal{J}'_{\lambda,\tau,n}(u_k)(u_k)\to0$ in (3.9) and (3.37), it derives that $\|u_k\|_{E_\lambda}\to0$ which together with (3.36) and (3.9) reveals that $c_{\lambda,\tau,n}=0$. It is absurd because of Lemma 3.8. So, $u\neq0$ concludes. Moreover, $\mathcal{J}'_{\lambda,\tau,n}(u)=0$ is a direct consequence of $\mathcal{J}'_{\lambda,\tau,n}(u_k)\to0$ in E_λ^{-1} and (2.11)-(2.12). In view of Remark 3.7, we are capable of supposing that $u\ge0$. Choosing $v=\frac{G_n(u)}{g_n(u)}$, it belongs to E_λ in the same spirit of Lemma 3.6 and so $\mathcal{J}'_{\lambda,\tau,n}(u)(v)=0$ which is equivalent to

$$\int_{\mathbb{R}^2}|\nabla u|^2\left[1-\frac{G_n(u)g_n'(u)}{g_n^2(u)}\right]\mathrm{d}x+\int_{\mathbb{R}^2}\lambda V(x)u\frac{G_n(u)}{g_n(u)}\mathrm{d}x-\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u)\right]G_n(u)\mathrm{d}x=0.$$

Using it, one has that

$$\begin{aligned}\mathcal{J}_{\lambda,\tau,n}(u)&=\frac12\|u\|_{E_\lambda}^2-\frac12\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u)\right]G_n(u)\mathrm{d}x\\&=\frac12\|u\|_{E_\lambda}^2-\frac12\int_{\mathbb{R}^2}|\nabla u|^2\left[1-\frac{G_n(u)g_n'(u)}{g_n^2(u)}\right]\mathrm{d}x-\frac12\int_{\mathbb{R}^2}\lambda V(x)u\frac{G_n(u)}{g_n(u)}\mathrm{d}x\\&=\frac12\int_{\mathbb{R}^2}|\nabla u|^2\frac{G_n(u)g_n'(u)}{g_n^2(u)}\mathrm{d}x+\frac12\int_{\mathbb{R}^2}\lambda V(x)\left[u^2-u\frac{G_n(u)}{g_n(u)}\right]\mathrm{d}x\end{aligned}\tag{3.38}$$

finishing the proof of this step.

Step 3: Choosing $\sigma_*=\sigma_3$, $\tau_*=\min\{\frac12,\tau_\kappa\}$ and $\lambda_*=\max\{\lambda_0,\hat\Lambda\}$, then we have that $u_k\to u$ in E_λ along a subsequence as $k\to\infty$ for all $\sigma\in(0,\sigma_*)$, $\tau\in(0,\tau_*)$ and $\lambda>\lambda_*$.

Combining $\mathcal{J}_{\lambda,\tau,n}(u_k)\to c_{\lambda,\tau,n}$ in (3.9), the Fatou's lemma and (2.11)-(2.12), there holds

$$\begin{aligned}c_{\lambda,\tau,n}&=\frac12\|u_k\|_{E_\lambda}^2-\frac12\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x)*G_n(u_k)\right]G_n(u_k)\mathrm{d}x+o_k(1)\\&=\frac12\|u_k-u\|_{E_\lambda}^2+\mathcal{J}_{\lambda,\tau,n}(u)+o_k(1)\ge\frac12\|u_k-u\|_{E_\lambda}^2+o_k(1).\end{aligned}$$

Due to Lemma 3.8, it permits us to choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu > 1$ sufficiently close to 1 in such a way that $\frac{1}{\nu} + \frac{1}{\nu'} = 1$ with $\nu, \nu' > 1$ and

$$\alpha\nu\|u_k - u\|_{E_\lambda}^2 < \frac{4\pi(1-\varepsilon)}{(1+\varepsilon)^2}, \ \forall k \in \mathbb{N}^+,$$

where $\varepsilon \in (0,1)$ comes from (3.28). Therefore, we apply (1.8) to arrive at

$$\sup_{k\in\mathbb{N}^+} \int_{\mathbb{R}^2} \left[e^{\alpha\nu(1+\varepsilon)^2|u_k-u|^2} - 1\right] \mathrm{d}x \le \sup_{k\in\mathbb{N}^+} \int_{\mathbb{R}^2} \left[e^{4\pi(1-\varepsilon)(|u_k-u|/\|u_k-u\|_{E_\lambda})^2} - 1\right] \mathrm{d}x \le C,$$

where $C > 0$ is independent of $k \in \mathbb{N}^+$. Exploiting again the two types of Young's inequality introduced in Lemma 3.9, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \left(e^{\alpha\nu u_k^2} - 1\right) \mathrm{d}x &\le \int_{\mathbb{R}^2} \left[e^{\alpha\nu(1+\varepsilon^{-1})u^2 + \alpha\nu(1+\varepsilon)|u_k-u|^2} - 1\right] \mathrm{d}x \\
&\le \frac{\varepsilon}{1+\varepsilon} \int_{\mathbb{R}^2} \left[e^{\alpha\nu(1+\varepsilon^{-1})^2u^2} - 1\right] \mathrm{d}x + \frac{1}{1+\varepsilon} \int_{\mathbb{R}^2} \left[e^{\alpha\nu(1+\varepsilon)^2|u_k-u|^2} - 1\right] \mathrm{d}x \\
&\le C
\end{aligned}$$

for some $C > 0$ independent of $k \in \mathbb{N}^+$. So, letting $k \to \infty$ and then tending $\epsilon \to 0^+$ we derive

$$\begin{aligned}
\int_{\mathbb{R}^2} |g_n(u_k)(u_k - u)| \,\mathrm{d}x &\le \epsilon \int_{\mathbb{R}^2} |u_k||u_k - u|\mathrm{d}x + C_\epsilon \int_{\mathbb{R}^2} |u_k - u||u_k|^{\bar{q}-1}(e^{\alpha u_k^2} - 1)\mathrm{d}x \\
&\le \epsilon|u_k|_2|u_k - u|_2 + C_\epsilon|u_k|_{\bar{q}\nu'}^{\bar{q}-1}|u_k - u|_{\bar{q}\nu'} \left[\int_{\mathbb{R}^2} \left(e^{\alpha\nu u_k^2} - 1\right) \mathrm{d}x\right]^{\frac{1}{\nu}} \\
&\to 0.
\end{aligned}$$

As a by-product of it and (2.13), there holds $\int_{\mathbb{R}^2}[|x|^{-\tau} * G_n(u_k)]g_n(u_k)(u_k - u)\mathrm{d}x \to 0$ and so

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} \left[\mathbb{G}_\tau(x) * G_n(u_k)\right] g_n(u_k)(u_k - u)\mathrm{d}x = 0,$$

which together with (3.9) yields that

$$\begin{aligned}
o_k(1) &= \mathcal{J}'_{\lambda,\tau,n}(u_k)(u_k - u) - \mathcal{J}'_{\lambda,\tau,n}(u)(u_k - u) \\
&= \|u_k - u\|_{E_\lambda}^2 - \int_{\mathbb{R}^2} \left[\mathbb{G}_\tau(x) * G_n(u_k)\right] g_n(u_k)(u_k - u)\mathrm{d}x \\
&= \|u_k - u\|_{E_\lambda}^2 + o_k(1).
\end{aligned}$$

The proof is completed. □

At this stage, we are capable of showing the proof of Theorem 3.1.

Proof of Theorem 3.1. Given $\sigma \in (0, \sigma_*)$, $\tau \in (0, \tau_*)$ and $\lambda > \lambda_*$, hence the existence of a sequence $(u_k) \subset E_\lambda$ is available. In view of Lemma 3.10, there is a nontrivial $u \in E_\lambda$ such that $\mathcal{J}_{\lambda,\tau,n}(u) = c_{\lambda,\tau,n}$ and $\mathcal{J}'_{\lambda,\tau,n}(u) = 0$ in E_λ^{-1}. The proof is completed. □

Now, we are in a position to contemplate the existence of a nontrivial solution for Eq. (1.12). More precisely, let us consider the following planar logarithmic Choquard equation

$$-\Delta u + \lambda V(x)u = \left[\mathbb{G}_\tau(x) * G(u)\right] g(u) \text{ in } \mathbb{R}^2. \tag{3.39}$$

In this direction, we have the result below.

Theorem 3.11. *Let V satisfy $(V_1)-(V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1)-(f_3)$. Then there exist some constants $\hat{\sigma}_* > 0$, $\hat{\tau}_* > 0$ and $\hat{\lambda}_* > 0$ such that for all $\sigma \in (0, \hat{\sigma}_*)$, $\tau \in (0, \hat{\tau}_*)$ and $\lambda > \hat{\lambda}_*$, Eq. (3.39) has at least a nontrivial solution.*

Let us recall the definition of η_n defined in (1.14), one may observe that if a solution $v \in E_\lambda$ of Eq. (3.1) satisfies $|v|_\infty < n$ and thereby it is in fact a solution of Eq. (3.39). In order to conclude Theorem 3.11, we are going to verify that the solution $u \in E_\lambda$ obtained in Theorem 3.1 belongs to $L^\infty(\mathbb{R}^2)$ and its L^∞-norm can be controlled by a positive constant which is independent of $n \in \mathbb{N}^+$.

To proceed with the proof, we claim that the constants, C, $\mathcal{C}_0$, $\mathcal{C}_1$, etc., are independent of $n \in \mathbb{N}^+$. As a matter of fact, the reader is invited to observe that this would be correct if we are able to verify that the constant $C > 0$ in (3.13) does not depend on $n \in \mathbb{N}^+$. On the one hand, combining (f_1) and (f_3), there exists a constant $C > 0$ independent of $n \in \mathbb{N}^+$ such that $F(s)f'(s) \le Cf^2(s)$ for all $s > 0$. Using it and (2.9) as well as (2.8), one has that

$$F(s)\left[f'(s) + 2\sigma f(s)\mathfrak{f}_n(s) + \sigma F(s)\mathfrak{f}_n'(s) + \sigma^2 F(s)\mathfrak{f}_n^2(s)\right] \le \left[C + (184 + 36\sigma)\sigma n^2\right] f^2(s),\ \forall s > 0.$$

On the other hand, in view of (2.10), we are choosing $\sigma_4 \triangleq \frac{1}{2}\sigma_* > 0$ and the claim therefore concludes for all $\sigma \in (0, \sigma_4)$. From now on until the end of this Section, we would like to highlight here that the constants adopted are independent of $n \in \mathbb{N}^+$.

Before showing that $u \in E_\lambda$ belongs to $L^\infty(\mathbb{R}^2)$, we recall from Lemma 3.10 jointly with Remark 3.7 that there is a function $\varpi \in E_\lambda$ which is independent of $k, n \in \mathbb{N}^+$ such that

$$0 \le u(x) \le \varpi(x),\ \forall x \in \mathbb{R}^2. \tag{3.40}$$

Lemma 3.12. *Under the assumptions of Theorem 3.1 and let $u \in E_\lambda$ be a nontrivial solution of Eq. (3.1). Then, for all $\sigma \in (0, \sigma_4)$, we have that $u \in L^\infty(\mathbb{R}^2)$. In particular, it holds that*

$$|u|_\infty \le \left(\frac{2}{\tilde{q}}\right)^{\frac{2\tilde{q}}{(2-\tilde{q})^2}} \left(\bar{\mathcal{C}}_0^{\tilde{q}'} \int_{\mathbb{R}^2} (e^{\alpha \tilde{q}' \varpi^2} - 1)\mathrm{d}x\right)^{\frac{\tilde{q}}{2\tilde{q}'(2-\tilde{q})}} |u|_4^4,$$

where $\bar{\mathcal{C}}_0 > 0$ is a constant independent of $n \in \mathbb{N}^+$, ϖ comes from (3.40) and $\tilde{q} \in (1,2)$ with $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$.

Proof. Let $\gamma > 1$ and $z \in \mathbb{N}^+$ and we introduce the sets $\mathfrak{A}_z \triangleq \{x \in \mathbb{R}^2 : u^{\gamma - 1} \le z\}$ and $\mathfrak{B}_z \triangleq \mathbb{R}^2 \backslash A_z$. Consider the sequences

$$u_z = \begin{cases} u^{2\gamma-1}, & \text{in } \mathfrak{A}_z, \\ z^2 u, & \text{in } \mathfrak{B}_z, \end{cases} \quad \text{and } v_z = \begin{cases} u^\gamma, & \text{in } \mathfrak{A}_z, \\ zu, & \text{in } \mathfrak{B}_z. \end{cases}$$

It is simple to observe that $u_z, v_z \in E_\lambda$, $|u_z| \le |u|^{2\gamma-1}$ and $|v_z|^2 = uu_z \le |u|^{2\gamma}$ in $\mathbb{R}^2$. Moreover,

$$\nabla u_z = \begin{cases} (2\gamma - 1)u^{2(\gamma-1)}\nabla u, & \text{in } \mathfrak{A}_z, \\ z^2 \nabla u, & \text{in } \mathfrak{B}_z, \end{cases} \quad \text{and } \nabla v_z = \begin{cases} \gamma u^{\gamma-1}\nabla u, & \text{in } \mathfrak{A}_z, \\ z\nabla u, & \text{in } \mathfrak{B}_z, \end{cases}$$

which imply that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^2} \nabla u \nabla u_z \mathrm{d}x = (2\gamma - 1)\int_{\mathfrak{A}_z} u^{2(\gamma-1)}|\nabla u|^2 \mathrm{d}x + z^2 \int_{\mathfrak{B}_z} |\nabla u|^2 \mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla v_z|^2 \mathrm{d}x = \gamma^2 \int_{\mathfrak{A}_z} u^{2(\gamma-1)}|\nabla u|^2 \mathrm{d}x + z^2 \int_{\mathfrak{B}_z} |\nabla u|^2 \mathrm{d}x. \end{cases} \tag{3.41}$$

Combining (3.41) and the fact that $\gamma > 1$, one obtains

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla v_z|^2 \mathrm{d}x &= \int_{\mathbb{R}^2} \nabla u \nabla u_z \mathrm{d}x + (\gamma - 1)^2 \int_{\mathfrak{A}_z} u^{2(\gamma-1)}|\nabla u|^2 \mathrm{d}x \\ &\le \left[1 + \frac{(\gamma-1)^2}{2\gamma - 1}\right] \int_{\mathbb{R}^2} \nabla u \nabla u_z \mathrm{d}x \le \gamma^2 \int_{\mathbb{R}^2} \nabla u \nabla u_z \mathrm{d}x. \end{aligned} \tag{3.42}$$

Since $u \in E_\lambda$ is a nontrivial critical point of $\mathcal{J}_{\lambda,\tau,n}$, then $\mathcal{J}'_{\lambda,\tau,n}(u)(u_z) = 0$ which gives that

$$\int_{\mathbb{R}^2} \left[\nabla u \nabla u_z + \lambda V(x) u u_z\right] \mathrm{d}x = \int_{\mathbb{R}^2} \left[\mathbb{G}_\tau(u) * G_n(u)\right] g_n(u) u_z \mathrm{d}x. \tag{3.43}$$

Next, we need to take some careful analyses for the two items in (3.43). Firstly, one may note that $uu_z = v_z^2$, it then follows from (3.42) and $\gamma > 1$ that

$$\int_{\mathbb{R}^2} \left[|\nabla v_z|^2 + \lambda V(x)|v_z|^2\right] \mathrm{d}x \le \gamma^2 \int_{\mathbb{R}^2} \left[\nabla u \nabla u_z + \lambda V(x) u u_z\right] \mathrm{d}x. \tag{3.44}$$

Using (2.13) and Lemma 3.9, there is a constant $C > 0$ independent of $n \in \mathbb{N}^+$ such that $|\mathbb{G}_\tau(u) * G_n(u)| \le C$. Moreover, for all $\sigma \in (0, \sigma_4)$, we apply (2.4) and (2.7) to get

$$g_n(u)u_z = \left[f(u) + \sigma F(u)\mathfrak{f}_n(u)\right] u_z e^{\sigma \mathfrak{F}_n(u)} \le \left(1 + 24\sigma n^2\right) f(u) u_z e^{\sigma \mathfrak{F}_n(u)} \le 2e f(u) u_z. \tag{3.45}$$

As a consequence of the above two facts, it infers from (2.2) with $\bar{q} = 2$ and (3.40) that

$$\begin{aligned}\int_{\mathbb{R}^2} \left[\mathbb{G}_\tau(u) * G_n(u)\right] g_n(u) u_z \mathrm{d}x &\le \epsilon C \int_{\mathbb{R}^2} |v_z|^2 \mathrm{d}x + C_\epsilon \int_{\mathbb{R}^2} |v_z|^2 \left(e^{\alpha |u|^2} - 1\right) \mathrm{d}x \\ &\le \frac{1}{8}\|v_z\|_{E_\lambda}^2 + C I_{\alpha,\tilde{q}'} \left(\int_{\mathbb{R}^2} |v_z|^{2\tilde{q}} \mathrm{d}x\right)^{\frac{1}{\tilde{q}}},\end{aligned} \tag{3.46}$$

where and in the sequel

$$I_{\alpha,\tilde{q}'} \triangleq \left[\int_{\mathbb{R}^2} \left(e^{\alpha \tilde{q}' |\varpi|^2} - 1\right) \mathrm{d}x\right]^{\frac{1}{\tilde{q}'}}.$$

According to (3.43), (3.44) and (3.46), we have that

$$\int_{\mathbb{R}^2} \left[|\nabla v_z|^2 + \lambda V(x)|v_z|^2\right] \mathrm{d}x \le C I_{\alpha,\tilde{q}'} \gamma^2 \left(\int_{\mathbb{R}^2} |v_z|^{2\tilde{q}} \mathrm{d}x\right)^{\frac{1}{q}}.$$

We fix $\tilde{q} \in (1,2)$ with $\tilde{q}' = \tilde{q}/(\tilde{q}-1)$ and $E_\lambda \hookrightarrow L^4(\mathbb{R}^2)$, then there is a constant $\bar{\mathcal{C}}_0 > 0$ independent γ and $n \in \mathbb{N}^+$ such that

$$\left(\int_{\mathbb{R}^2} |v_z|^4 \mathrm{d}x\right)^{\frac{1}{2}} \le \bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'} \gamma^2 \left(\int_{\mathbb{R}^2} |v_z|^{2\tilde{q}} \mathrm{d}x\right)^{\frac{1}{q}}.$$

Once $v_z = u^\gamma$ in $\mathfrak{A}_z$ and $v_z \le u^\gamma$ in $\mathbb{R}^2$, there holds

$$\left(\int_{\mathfrak{A}_z} |u|^{4\gamma} \mathrm{d}x\right)^{\frac{1}{2}} \le \bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'} \gamma^2 \left(\int_{\mathbb{R}^2} |u|^{2\tilde{q}\gamma} \mathrm{d}x\right)^{\frac{1}{\tilde{q}}}, \ \forall z \in \mathbb{N}^+.$$

Applying the Lebesgue's Dominated Convergence theorem with $z \to \infty$ to the above formula, one has

$$|u|_{4\gamma}^{2\gamma} \le \bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'} \gamma^2 |u|_{2\tilde{q}\gamma}^{2\gamma}. \tag{3.47}$$

We choose the constant $\mu = 2/\tilde{q}$, then $\mu > 1$ because $\tilde{q} \in (1,2)$. For every $j \in \mathbb{N}^+$, define $\gamma_j = \mu^j$ and thus $2\tilde{q}\gamma_{j+1} = 2\tilde{q}\mu\gamma_j = 4\gamma_j$. For $j = 1$, $\gamma_1 = \mu > 1$ which can be applied in (3.47) to derive

$$|u|_{4\mu} \le \mu^{\frac{1}{\mu}} (\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2\mu}} |u|_4. \tag{3.48}$$

For $j = 2$, $\gamma_2 = \mu^2 > 1$ and $2\tilde{q}\gamma_2 = 4\gamma_1 = 4\mu$ and by (3.47),

$$|u|_{4\mu^2} \le (\mu^2)^{\frac{1}{\mu^2}} (\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2\mu^2}} |u|_{4\mu}. \tag{3.49}$$

For $j = 3$, $\gamma_3 = \mu^3 > 1$ and $2\tilde{q}\gamma_3 = 4\gamma_2 = 4\mu^2$ and by (3.47),

$$|u|_{4\mu^3} \le (\mu^3)^{\frac{1}{\mu^3}} (\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2\mu^3}} |u|_{4\mu^2}. \tag{3.50}$$

Similar to (3.48), (3.49) and (3.50), proceeding this iteration procedure j times, we can infer that

$$|u|_{4\mu^j} \le \mu^{\sum_{i=1}^j \frac{i}{\mu^i}} (\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2}\sum_{i=1}^j \frac{1}{\mu^i}} |u|_4 \tag{3.51}$$

invoking that $u \in L^{4\mu^j}(\mathbb{R}^2)$ for every $j \in \mathbb{N}^+$. Clearly, $\sum_{i=1}^{\infty} \frac{i}{\mu^i} = \frac{\mu}{(\mu-1)^2}$ and $\sum_{i=1}^{\infty} \frac{1}{\mu^i} = \frac{1}{\mu-1}$, thereby we can take the limit in (3.51) as $j \to \infty$ to obtain

$$|u|_\infty \leq \mu^{\frac{\mu}{(\mu-1)^2}} (\bar{\mathcal{C}}_0 I_{\alpha,\tilde{q}'})^{\frac{1}{2(\mu-1)}} |u|_4^4$$

finishing the proof of this lemma. □

Proof of Theorem 3.11. Choosing $\hat{\sigma}_* = \sigma_*$, $\hat{\tau}_* = \frac{1}{2}\tau_*$ and $\hat{\lambda}_* = \lambda_*$, then for all $\sigma \in (0, \hat{\sigma}_*)$, $\tau \in (0, \hat{\tau}_*)$ and $\lambda > \hat{\lambda}_*$, we know that the $u \in E_\lambda$ obtained in Theorem 3.1 is still a nontrivial solution of Eq. (3.1). Due to (3.38) and Lemma 3.8, we are derived from Lemma 2.3-(g_2) and (3.12) that

$$\|u\|_{E_\lambda}^2 \leq \frac{4\pi}{\alpha_0 \min\{1 - \frac{\delta}{4}, \frac{\delta}{4}\}} \tag{3.52}$$

which together with Lemma 3.12 yields that

$$|u|_\infty \leq \left(\frac{2}{\tilde{q}}\right)^{\frac{2\tilde{q}}{(2-\tilde{q})^2}} \left[\bar{\mathcal{C}}_0^{\tilde{q}'} \int_{\mathbb{R}^2} (e^{\alpha \tilde{q}' \varpi^2} - 1)\mathrm{d}x\right]^{\frac{\tilde{q}}{2\tilde{q}'(2-\tilde{q})}} \mathbb{T}_4^4 \left(\frac{4\pi}{\alpha_0 \min\{1 - \frac{\delta}{4}, \frac{\delta}{4}\}}\right)^2 \triangleq \hat{\mathcal{C}}_0,$$

where $T_4 > 0$ denotes an imbedding constant of $E_\lambda \hookrightarrow L^4(\mathbb{R}^2)$. Now, we arrive at the proof of Theorem 3.11 by fixing $n > \hat{\mathcal{C}}_0$, because in this scenario u is a nontrivial solution of (3.39) due to the definition of η_n in (1.14). The proof is completed. □

4. Existence and decaying property of solutions for Eq. (1.1)

In this section, we are concerned with the existence and decaying property of nontrivial solutions for Eq. (1.1). To reach the proof of Theorem 1.1, we are based on the results obtained in Theorem 3.11.

In what follows, let us denote $u_{\lambda,\tau}$ by the nontrivial solution obtained in Theorem 3.11 to emphasize the dependence of the parameters λ and τ. As a matter of fact, we have that $\mathcal{J}_{\lambda,\tau,n}(u_{\lambda,\tau}) = c_{\lambda,\tau,n}$ and $\mathcal{J}'_{\lambda,\tau,n}(u_{\lambda,\tau}) = 0$ in E_λ^{-1}. In view of Remark 3.5 or Lemma 3.8, we are able to take the same arguments to deduce that $(u_{\lambda,\tau})$ is uniformly bounded in $\tau \in (0, \hat{\tau}_*)$. Consequently, as $\tau \to 0^+$, up to subsequences if necessary, there is a function $u_{\lambda,0} \in E_\lambda$ such that

$$u_{\lambda,\tau} \rightharpoonup u_{\lambda,0} \text{ in } E_\lambda,\ u_{\lambda,\tau} \to u_{\lambda,0} \text{ in } L^s_{\mathrm{loc}}(\mathbb{R}^2) \text{ with } s \geq 2,\ u_{\lambda,\tau} \to u_{\lambda,0} \text{ a.e. in } \mathbb{R}^2. \tag{4.1}$$

To restore the compactness, regarding as a counterpart of Lemma 3.9, we shall establish the following result.

Lemma 4.1. *Let V satisfy $(V_1) - (V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1) - (f_4)$. Then, for all $\tau \in (0, \hat{\tau}_*)$, any sequence $(u_{\lambda,\tau}) \subset E_\lambda$ satisfying (4.1) has the following conclusions*

$$\sup_{\tau \in (0,\hat{\tau}_*)} \int_{\mathbb{R}^2} g(u_{\lambda,\tau}) u_{\lambda,\tau} \mathrm{d}x \leq \mathcal{C}_2 \text{ and } \sup_{\tau \in (0,\hat{\tau}_*)} \int_{\mathbb{R}^2} [G(u_{\lambda,\tau})]^\kappa \mathrm{d}x \leq \mathcal{C}_2, \tag{4.2}$$

for some $\mathcal{C}_2 > 0$ independent of $\tau \in (0, \hat{\tau}_*)$, where $1 < \kappa < (1-\varepsilon)^{-\frac{1}{2}}$ with $\varepsilon \in (0,1)$ given in (3.28).

Proof. Let us go back to the proof of Lemma 3.9, one might observe that the constant C and C_ε are independent of $\tau \in (0, \hat{\tau}_*)$. Setting $w_{\lambda,\tau} \triangleq H_n(u_{\lambda,\tau})$, then we are able to proceed as (3.26) and (3.27) to derive

$$|\nabla w_{\lambda,\tau}|_2^2 \leq 2c_{\lambda,\tau,n} + o_k(1). \tag{4.3}$$

and there is a constant $C > 0$ independent of $\tau \in (0, \hat{\tau}_*)$ such that

$$\int_{\mathbb{R}^2} \lambda V(x) w_{\lambda,\tau}^2 \mathrm{d}x \leq C. \tag{4.4}$$

Doe the $\varepsilon > 0$ in (3.28), we also can find an $s_\varepsilon > 0$ to satisfy

$$u_{\lambda,\tau} \leq s_\varepsilon + \frac{w_{\lambda,\tau}}{\sqrt{1-\varepsilon}}, \ \forall x \in \mathbb{R}^2. \tag{4.5}$$

With (4.3), (4.3) and (4.3) in hands, repeating the calculations exhibited in Lemma 3.9, we are capable of applying Lemma 2.5 to reach the proof of this lemma, where a similar fact in (3.45) is used.. □

Our next aim is to investigate the decaying property of $u_{\lambda,\tau}$ which yields the proof of Theorem 1.4.

Lemma 4.2. *Let V satisfy $(V_1)-(V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1)-(f_4)$. Then, there is a $\tilde{\lambda}_* > 0$ such that for all $\lambda > \tilde{\lambda}_*$, there are $\mathbb{A}, \mathbb{B} > 0$, independent of σ, τ, λ and $n \in \mathbb{N}^+$ such that*

$$|u_{\lambda,\tau}(x)| \leq \mathbb{A}\lambda^{-\frac{1}{2}} \exp\left[-\mathbb{B}\lambda^{\frac{1}{2}}\left(|x|-R\right)\right], \ \forall |x| > R, \tag{4.6}$$

where the consatnt R is independent of σ, τ, λ and $n \in \mathbb{N}^+$.

Proof. First of all, we shall conclude that $u_{\lambda,\tau} \in L^\infty(\mathbb{R}^2)$. To demonstrate it, we realize that $u_{\lambda,\tau}(x) \geq 0$ for all $x \in \mathbb{R}^2$ by Remark 3.7. Since $\mathcal{J}'_{\lambda,\tau,n}(u_{\lambda,\tau}) = 0$ in E_λ^{-1}, it allows us to obtain

$$-\Delta u_{\lambda,\tau} + \lambda V(x) u_{\lambda,\tau} \leq \left(\int_{|x-y|<1} \frac{|x-y|^{-\tau}-1}{\tau} G_n(u_{\lambda,\tau})(y)\mathrm{d}y\right) g_n(u_{\lambda,\tau}) \text{ in } \mathbb{R}^2. \tag{4.7}$$

Recalling $\hat{\tau}_* = \frac{1}{2}\tau_* < \frac{1}{2}\tau_\kappa = \frac{\kappa-1}{\kappa}$, because $\sigma \in (0, \hat{\sigma}_*)$ and $\tau \in (0, \hat{\tau}_*)$, we apply (4.2) to have

$$\begin{aligned} \Pi &\triangleq \int_{|x-y|<1} \frac{|x-y|^{-\tau}-1}{\tau} G_n(u_{\lambda,\tau}(y))\mathrm{d}y \leq C_\kappa e \int_{|x-y|<1} \frac{G(u_{\lambda,\tau}(y))}{|x-y|^{\frac{\kappa-1}{\kappa}}}\mathrm{d}y \\ &\leq C_\kappa e \left(\int_{|x-y|<1} \frac{1}{|x-y|}\mathrm{d}y\right)^{\frac{\kappa-1}{\kappa}} \left(\int_{\mathbb{R}^2} [G(u_{\lambda,\tau}(y))]^\kappa \mathrm{d}y\right)^{\frac{1}{\kappa}} \leq \mathcal{C}_3 \end{aligned} \tag{4.8}$$

for some $\mathcal{C}_3 > 0$ independent of τ, where we have adopted Lemma 2.2 with $\tau' = \dfrac{\kappa-1}{\kappa} > \tau$. Combining (4.7) and (4.8), it has that

$$-\Delta u_{\lambda,\tau} \leq -\Delta u_{\lambda,\tau} + \lambda V(x) u_{\lambda,\tau} \leq \mathcal{C}_3 g_n(u_{\lambda,\tau}) \text{ in } \mathbb{R}^2.$$

Proceeding as the same calculations exhibited in Lemma 3.12, we can determine a suitable constant $\mathcal{C}_4 > 0$ independent of τ such that $|u_{\lambda,\tau}|_\infty \leq \mathcal{C}_4$. By exploiting $\sigma \in (0, \hat{\sigma}_*)$ again, $g_n(u_{\lambda,\tau}) \leq \mathcal{C}_5 u_{\lambda,\tau}^{\bar{q}-1}$ for some $\bar{q} > 2$ given in (2.2), where $\mathcal{C}_5 > 0$ is independent of τ. Let $c(x) = -u_{\lambda,\tau}^{\bar{q}-2}(x)$ for all $x \in \mathbb{R}^3$, then one sees $|c(x)|_\infty \leq \mathcal{C}_4^{\bar{q}-2}$ and

$$-\Delta u_{\lambda,\tau} + c(x) u_{\lambda,\tau} \leq 0 \text{ in } \mathbb{R}^2.$$

Consequently, we follow [31, Theorem 8.17] to find a $\mathcal{C}_6 > 0$ independent of τ such that

$$\sup_{x \in B_1(y)} u_{\lambda,\tau}(x) \leq \mathcal{C}_6 |u_{\lambda,\tau}|_{L^2(B_2(y))}, \ \forall y \in \mathbb{R}^2. \tag{4.9}$$

By (V_3), there exists an $R_1 > 0$ such that $\Xi \subset B_{R_1}(0)$, and so $V(x) \geq b$ for all $|x| \geq R_1$ which together with (3.52) implies that

$$\int_{L^2(B_2(y))} |u_{\lambda,\tau}|^2 \mathrm{d}x \leq \frac{4\pi}{\alpha_0 \min\{1-\frac{\delta}{4}, \frac{\delta}{4}\}} (\lambda b)^{-1}, \ \forall |y| \geq R_1 + 2. \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$u_{\lambda,\tau}(x) \leq \mathcal{C}_6 \sqrt{\frac{4\pi}{\alpha_0 \min\{1-\frac{\delta}{4}, \frac{\delta}{4}\}}} (\lambda b)^{-\frac{1}{2}} \triangleq \mathcal{C}_7 \lambda^{-\frac{1}{2}}, \ \forall |x| \geq R_1 + 1. \tag{4.11}$$

Finally, to continue the proof, we define

$$W_{\lambda,\tau}(x) \triangleq \lambda V(x) - \mathcal{C}_5 u_{\lambda,\tau}^{\bar{q}-2},\ \forall x \in \mathbb{R}^2.$$

Since $|u_{\lambda,\tau}|_\infty \le \mathcal{C}_4$ and $V(x) \ge b$ for all $|x| \ge R_1$, there is a $\tilde{\lambda}_* > 0$ such that for all $\lambda > \tilde{\lambda}_*$, it has that

$$W_{\lambda,\tau}(x) \ge \frac{1}{2}\lambda b \triangleq \lambda \mathbb{B}^2,\ \forall |x| \ge R_1 + 1.$$

As a consequence, we reach

$$-\Delta u_{\lambda,\tau} + \lambda \mathbb{B}^2 u_{\lambda,\tau} \le 0,\ \forall |x| \ge R_1 + 1. \tag{4.12}$$

Choosing $\psi_{\lambda,\tau}(x) = \mathbb{A}\lambda^{-\frac{1}{2}} \exp\left[-\mathbb{B}\lambda^{\frac{1}{2}}(|x| - R)\right]$ with $\mathbb{A} = \mathcal{C}_7$ and $R = R_1 + 1$, it therefore follows from (4.11) that

$$u_{\lambda,\tau}(x) \le \psi_{\lambda,\tau}(x),\ \forall |x| = R. \tag{4.13}$$

Moreover, it simply derives that

$$-\Delta \psi_{\lambda,\tau} + \lambda \mathbb{B}^2 \psi_{\lambda,\tau} \ge 0,\ \forall |x| \ne 0. \tag{4.14}$$

Setting $\varphi_{\lambda,\tau} = \psi_{\lambda,\tau} - u_{\lambda,\tau}$, then we make use of (4.12), (4.13) and (4.14) to have that

$$\begin{cases} -\Delta \varphi_{\lambda,\tau} + \lambda \mathbb{B}^2 \varphi_{\lambda,\tau} \ge 0, & \text{in } |x| > R, \\ \varphi_{\lambda,\tau} \ge 0, & \text{on } |x| = R. \end{cases}$$

According to the maximum principle (see e.g. [31, Theorem 8.1]), we shall demonstrate that $\varphi_{\lambda,\tau}(x) \ge 0$ for all $|x| \ge R$. The proof is completed. $\square$

With Lemma 4.1 and Lemma 4.2 in hands, we are capable of arriving at the compact result below.

Lemma 4.3. *Let V satisfy $(V_1) - (V_3)$ and suppose g given by (1.2) to require (1.3) with $(f_1) - (f_4)$. Then, for all $\sigma \in (0, \hat{\sigma}_*)$, $\tau \in (0, \hat{\tau}_*)$ and $\lambda > \max\{\hat{\lambda}_*, \tilde{\lambda}_*\}$, passing to a subsequence if necessary,*

$$\lim_{\tau \to 0^+} \int_{\mathbb{R}^2} \left[\mathbb{G}_\tau(x) * G(u_{\lambda,\tau})\right] G(u_{\lambda,\tau}) \mathrm{d}x = \int_{\mathbb{R}^2} \left[\log\left(\frac{1}{|x|}\right) * G(u_{\lambda,0})\right] G(u_{\lambda,0}) \mathrm{d}x.$$

Proof. Since $\tilde{\lambda} > \hat{\lambda}$, we can repeat the same way in the Step 1 of the proof of Lemma 3.10 to conclude that $u_{\lambda,\tau} \to u_{\lambda,0}$ in $L^s(\mathbb{R}^2)$ for all $s > 2$ along a subsequence as $\tau \to 0^+$.

On the one hand, due to (4.2) and Lemma 2.3-(g_5) with $\tau \in (0, \hat{\tau}_*)$, it simply has that

$$\lim_{\tau \to 0^+} \int_{\mathbb{R}^2} G(u_{\lambda,\tau}) \mathrm{d}x = \int_{\mathbb{R}^2} G(u_{\lambda,0}) \mathrm{d}x. \tag{4.15}$$

Recalling Π which is defined in (4.8) is uniformly bounded in $\mathbb{R}^2$, jointly with (4.15), we can show that $\dfrac{|x-y|^{-\tau} - 1}{\tau} 1_{|x-y|<1} G(u_{\lambda,\tau}(y)) \to \log\left(\dfrac{1}{|x-y|}\right) 1_{|x-y|<1} G(u_{\lambda,0}(y))$ a.e. in $\mathbb{R}^2$ as $\tau \to 0^+$. Therefore, we it follows from the generalized Lebesgue's Dominated Convergence theorem that

$$\begin{aligned} &\lim_{\tau \to 0^+} \int_{\mathbb{R}^2} \left(\int_{|x-y|<1} \frac{|x-y|^{-\tau} - 1}{\tau} G(u_{\lambda,\tau}(y)) \mathrm{d}y \right) G(u_{\lambda,\tau}) \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \left[\int_{|x-y|<1} \log\left(\frac{1}{|x-y|}\right) G(u_{\lambda,0}(y)) \mathrm{d}y \right] G(u_{\lambda,0}) \mathrm{d}x. \end{aligned}$$

On the other hand, we begin with the term $\dfrac{|x-y|^{-\tau} - 1}{\tau} 1_{|x-y|\ge 1}$. As a matter of fact, according to the intermediate mean value theorem, there is a such that

$$\frac{|x-y|^{-\tau} - 1}{\tau} 1_{|x-y|\ge 1} = -|x-y|^{-\tau\zeta} \log(|x-y|) 1_{|x-y|\ge 1}$$

where $\zeta = \zeta(|x-y|) \in (0,1)$. Some elementary calculations provide us that

$$\begin{aligned}\left|\frac{|x-y|^{-\tau}-1}{\tau}1_{|x-y|\geq 1}G(u_{\lambda,\tau}(y))G(u_{\lambda,\tau}(x))\right| &\leq \left|\log(|x-y|)1_{|x-y|\geq 1}G(u_{\lambda,\tau}(y))G(u_{\lambda,\tau}(x))\right| \\ &\leq |x|G(u_{\lambda,\tau}(y))G(u_{\lambda,\tau}(x)) + |y|G(u_{\lambda,\tau}(y))G(u_{\lambda,\tau}(x)).\end{aligned}$$

As a consequence, we are able to observe observe that

$$\int_{\mathbb{R}^2}\left(\int_{|x-y|\geq 1}\left|\frac{|x-y|^{-\tau}-1}{\tau}\right|G(u_{\lambda,\tau}(y))\mathrm{d}y\right)G(u_{\lambda,\tau})\mathrm{d}x \leq 2\left(\int_{\mathbb{R}^2}G(u_{\lambda,\tau})\mathrm{d}x\right)\left(\int_{\mathbb{R}^2}|x|G(u_{\lambda,\tau})\mathrm{d}x\right).$$

We claim that there is a constant $\mathcal{C}_8 > 0$ independent of τ such that $|xG(u_{\lambda,\tau})|_1 \leq \mathcal{C}_8$. To see it, with aid of (4.2), it suffices to deduce that

$$\begin{aligned}\int_{|x|\geq R}|x|G(u_{\lambda,\tau})\mathrm{d}x &\leq e\int_{|x|\geq R}|x|\left[|u_{\lambda,\tau}|^2 + |u_{\lambda,\tau}|^2\left(e^{\alpha|u_{\lambda,\tau}|_\infty^2}-1\right)\right]\mathrm{d}x \\ &\leq e^{1+\alpha\mathcal{C}_4}\int_{|x|\geq R}|x||u_{\lambda,\tau}|^2\mathrm{d}x \triangleq \mathcal{C}_8 < +\infty,\end{aligned}$$

where we have used $|u_{\lambda,\tau}|_\infty \leq \mathcal{C}_4$ and (4.6) in Lemma 4.2. Owing to (4.15), we obtain

$$\begin{aligned}&\lim_{\tau\to 0^+}\int_{\mathbb{R}^2}\left(\int_{|x-y|\geq 1}\frac{|x-y|^{-\tau}-1}{\tau}G(u_{\lambda,\tau}(y))\mathrm{d}y\right)G(u_{\lambda,\tau})\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left[\int_{|x-y|\geq 1}\log\left(\frac{1}{|x-y|}\right)G(u_{\lambda,0}(y))\mathrm{d}y\right]G(u_{\lambda,0})\mathrm{d}x.\end{aligned}$$

The proof is completed. □

Now,we can exhibit the proof of Theorem 1.1 in detail as follows.

Proof of Theorem 1.1. Choosing $\sigma^* = \hat{\sigma}_*$, $\tau^* = \hat{\tau}_*$ and $\lambda^* = \max\{\tilde{\lambda}_*, \hat{\lambda}_*\}$, then for all $\sigma \in (0,\sigma^*)$, $\tau \in (0,\tau^*)$ and $\lambda > \lambda^*$, we take the similar arguments adopted in the proof of Lemma 3.10 to conclude the proof. Since $\lambda^* > \hat{\lambda}$, as explained in Lemma 4.3, passing to some sequences if necessary, $u_{\lambda,\tau} \rightharpoonup u_{\lambda,0}$ in E_λ, $u_{\lambda,\tau} \to u_{\lambda,0}$ in $L^s(\mathbb{R}^2)$ for all $s > 2$ and $u_{\lambda,\tau} \to u_{\lambda,0}$ a.e. in $\mathbb{R}^2$ as $\tau \to 0^+$. We next claim that $u_{\lambda,0} \neq 0$. Otherwise, combining Lemma 4.3 and $\mathcal{J}_{\lambda,\tau,n}(u_{\lambda,\tau}) = c_{\lambda,\tau,n}$ with Lemma 3.8, it indicates that $\limsup\limits_{\tau\to 0^+}\|u_{\lambda,\tau}\|_{E_\lambda}^2 < \dfrac{4\pi}{\alpha_0}$. Then, some standard calculations give that $\lim\limits_{\tau\to 0^+}\displaystyle\int_{\mathbb{R}^2}g(u_{\lambda,\tau})u_{\lambda,\tau}\mathrm{d}x = 0$. We adopt $\mathcal{J}'_{\lambda,\tau,n}(u_{\lambda,\tau}) = 0$ in E_λ^{-1} and (4.8) to find that

$$\begin{aligned}0 &= \|u_{\lambda,\tau}\|_{E_\lambda}^2 - \int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x) * G(u_{\lambda,\tau})\right]g(u_{\lambda,\tau})u_{\lambda,\tau}\mathrm{d}x \\ &\geq \|u_{\lambda,\tau}\|_{E_\lambda}^2 - \int_{\mathbb{R}^2}\left(\int_{|x-y|\leq 1}\mathbb{G}_\tau(x-y)G(u_{\lambda,\tau}(y))\mathrm{d}y\right)g(u_{\lambda,\tau})u_{\lambda,\tau}\mathrm{d}x \\ &\geq \|u_{\lambda,\tau}\|_{E_\lambda}^2 - C_\kappa\int_{\mathbb{R}^2}\left(\int_{|x-y|\leq 1}\frac{G(u_{\lambda,\tau}(y))}{|x-y|}\mathrm{d}y\right)g(u_{\lambda,\tau})u_{\lambda,\tau}\mathrm{d}x \\ &\geq \|u_{\lambda,\tau}\|_{E_\lambda}^2 - C_\kappa\Pi\int_{\mathbb{R}^2}g(u_{\lambda,\tau})u_{\lambda,\tau}\mathrm{d}x \\ &= \|u_{\lambda,\tau}\|_{E_\lambda}^2 + o_\tau(1).\end{aligned}$$

Hence, we are derived from it and Lemma 4.3 that $\lim\limits_{\tau\to 0^+}c_{\lambda,\tau,n} = \lim\limits_{\tau\to 0^+}\mathcal{J}_{\lambda,\tau,n}(u_{\lambda,\tau}) = 0$ which contradicts with Remark 3.5. So, the claim holds true.

Using a very similar argument in Lemma 4.3, for all $v \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have

$$\lim_{\tau\to 0^+}\int_{\mathbb{R}^2}\left[\mathbb{G}_\tau(x) * G(u_{\lambda,\tau})\right] g(u_{\lambda,\tau})v\mathrm{d}x = \int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x|}\right) * G(u_{\lambda,0})\right] g(u_{\lambda,0})v\mathrm{d}x.$$

which together with $\mathcal{J}'_{\lambda,\tau,n}(u_{\lambda,\tau})(v) = 0$ implies that $\mathcal{J}'_{\lambda,\tau,n}(u_{\lambda,0})(v) = 0$. Thereby, $u_{\lambda,0}$ is a nontrivial solution of Eq. (1.1). A similar idea in (3.38) shows us that $\mathcal{J}_{\lambda,\tau,n}(u_{\lambda,0}) \geq 0$. Thereby, proceeding as the same way in the proof of Lemma 3.10, we are capable of demonstrating that $u_{\lambda,\tau} \to u_{\lambda,0}$ in E_λ as $\tau \to 0^+$. The proof is completed. $\square$

Remark 4.4. As some by-products of $u_{\lambda,\tau} \to u_{\lambda,0}$ in E_λ as $\tau \to 0^+$, we have the results below

(1) Since $\|u_{\lambda,\tau}\| \leq \mathcal{C}_9$ for some $\mathcal{C}_9 > 0$ independent of σ, τ and λ, then

$$\left|\int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x|}\right) * G(u_{\lambda,0})\right] G(u_{\lambda,0})\mathrm{d}x\right| \leq \mathcal{C}_9^2 + 2\bar{c}, \tag{4.16}$$

where $\bar{c} > 0$ comes from Remark 3.5. Analogously, for all $v \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, it holds that

$$\left|\int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x|}\right) * G(u_{\lambda,0})\right] g(u_{\lambda,0})v\mathrm{d}x\right| \leq \mathcal{C}_9\|v\|_{E_\lambda}, \tag{4.17}$$

(2) The proof of Theorem 1.4 follows immediately by (4.6) and so we omit it.

5. Asymptotic behaviors

In this section, we shall mainly study the asymptotic behaviors of the obtained solutions in Theorem 1.1 as $\lambda \to +\infty$ and $\sigma \to 0^+$, respectively.

First of all, according to Theorem 1.1, we shall denoted the obtained solution by $u_{\lambda,\sigma} \in E_\lambda$ for all $\sigma \in (0, \sigma^*)$ and $\lambda > \lambda^*$.

5.1. Case 1: $\sigma \in (0, \sigma^*)$ is fixed and $\lambda \to +\infty$.

For any $u \in H_0^1(\Omega)$, we denote by $\tilde{u} \in H^1(\mathbb{R}^2)$ its trivial extension, namely

$$\tilde{u} \triangleq \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \Omega^c = \{x : x \in \mathbb{R}^2\backslash\Omega\}. \end{cases}$$

We now define $\mathcal{J}_{\Omega,\tau,n} : H_0^1(\Omega) \to \mathbb{R}$ as

$$\mathcal{J}_{\Omega,\tau,n}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\mathrm{d}x - \int_\Omega\left[\int_\Omega \log\left(\frac{1}{|x-y|}\right) G_n(u(y))\mathrm{d}y\right] G_n(u)\mathrm{d}x.$$

Note that, as what we have done in Section 4, we always exploited the following fact directly without mentioning if there is no misunderstanding

$$F(u_{\lambda,\sigma}) \leq G_n(u_{\lambda,\sigma}) = F(u_{\lambda,\sigma})e^{\sigma\mathfrak{F}_n(u_{\lambda,\sigma})} = F(u_{\lambda,\sigma})e^{\sigma|u_{\lambda,\sigma}|^2} \leq eF(u_{\lambda,\sigma}) \tag{5.1}$$

for all $\sigma \in (0, \sigma^*)$.

Proof of Theorem 1.6. Let $\lambda_k \to +\infty$ as $k \to +\infty$ and $(u_{\lambda_k,\sigma}) \subset E_{\lambda_k}$ be a sequence of nontrivial solutions of Eq. (1.1), that is, $\mathcal{J}'_{\lambda_k,\sigma,n}(u_{\lambda_k,\sigma}) = 0$ in $E_{\lambda_k}^{-1}$ and $\mathcal{J}_{\lambda_k,\sigma,n}(u_{\lambda_k,\sigma}) = c_{\lambda_k,\sigma,n}$. According to Remark 3.5, we can argue as Lemma 3.6 show that $(u_{\lambda_k,\sigma})$ is uniformly bounded in $k \in \mathbb{N}^+$. Passing to some subsequence if necessary, $u_{\lambda_k,\sigma} \rightharpoonup u_{0,\sigma}$ in $H^1(\mathbb{R}^2)$, $u_{\lambda_k,\sigma} \to u_{0,\sigma}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for all $2 < s < +\infty$ and $u_{\lambda_k,\sigma} \to u_{0,\sigma}$ a.e. in $\mathbb{R}^2$ as $k \to +\infty$. We claim that $u_{0,\sigma} \equiv 0$ in Ω^c. Otherwise, there is a compact subset $\Theta_{u_{0,\sigma}} \subset \Omega^c$ with $\mathrm{dist}(\Theta_{u_{0,\sigma}}, \partial\Omega^c) > 0$ such that $u_{0,\sigma} \not\equiv 0$ on $\Theta_{u_{0,\sigma}}$ and by Fatou's lemma

$$\liminf_{k\to\infty}\int_{\mathbb{R}^2} u_{\lambda_k,\sigma}^2\mathrm{d}x \geq \int_{\Theta_{u_{0,\sigma}}} {u_{0,\sigma}}^2\mathrm{d}x > 0. \tag{5.2}$$

Moreover, there exists $\varepsilon_0 > 0$ such that $V(x) \geq \varepsilon_0$ for any $x \in \Theta_{u_{0,\sigma}}$ by the assumptions (V_1) and (V_2). Since $\|u_{\lambda_k,\sigma}\|_{E_{\lambda_k}}$ is uniformly bounded, then the Fatou's lemma gives us that

$$+\infty > \liminf_{k\to\infty} \int_{\mathbb{R}^2} \lambda_k V(x) u_{\lambda_k,\sigma}^2 \mathrm{d}x \geq \varepsilon_0 \left(\int_{\Theta_{u_{0,\sigma}}} u_{0,\sigma}^2 \mathrm{d}x \right) \liminf_{k\to\infty} \lambda_k = +\infty,$$

a contradiction. Therefore, $u_{0,\sigma} \in H_0^1(\Omega)$ by the fact that $\partial\Omega$ is smooth. In order to finish the proof, we are ready to verify that $u_{\lambda_k,\sigma} \to u_{0,\sigma}$ in $L^s(\mathbb{R}^2)$ for all $2 < s < +\infty$ in the sense of a subsequence as $k \to +\infty$. Arguing it indirectly, we follow the arguments in [55] to make full use of the Lions' vanishing lemma to find some $\check{\delta}, \check{r} > 0$ and $\check{x}_k \in \mathbb{R}^2$ such that

$$\int_{B_{\check{r}}(\check{x}_k)} |u_{\lambda_k,\sigma} - u_{0,\sigma}|^2 \mathrm{d}x \geq \check{\delta},$$

which implies that $|\check{x}_n| \to \infty$ and hence $|B_{\check{r}}(\check{x}_k) \cap \Xi| \to 0$. Recalling $\|u_{\lambda_k,\sigma}\|_{E_{\lambda_k}}$ is uniformly bounded, then the Hölder's inequality yields that

$$\lim_{k\to+\infty} \int_{B_{\check{r}}(\check{x}_k)\cap\Xi} |u_{\lambda_k,\sigma} - u_{0,\sigma}|^2 \mathrm{d}x = 0.$$

As a consequence, we are derived from the above two facts that

$$\begin{aligned}
+\infty > \liminf_{k\to\infty} \lambda_k b \int_{B_{\check{r}}(\check{x}_k)\cap\Xi^c} u_{\lambda_k,\sigma}^2 \mathrm{d}x &= \liminf_{k\to\infty} \lambda_k b \int_{B_{\check{r}}(\check{x}_k)\cap\Xi^c} |u_{\lambda_k,\sigma} - u_{0,\sigma}|^2 \mathrm{d}x \\
&= \liminf_{k\to\infty} \lambda_k b \left(\int_{B_{\check{r}}(\check{x}_k)} |u_{\lambda_k,\sigma} - u_{0,\sigma}|^2 \mathrm{d}x - \int_{B_{\check{r}}(\check{x}_k)\cap\Xi} |u_{\lambda_k,\sigma} - u_{0,\sigma}|^2 \mathrm{d}x \right) \\
&\geq \frac{\check{\delta} b}{2} \liminf_{k\to\infty} \lambda_k = +\infty,
\end{aligned}$$

which is impossible. So, $u_{\lambda_k,\sigma} \to u_{0,\sigma}$ in $L^s(\mathbb{R}^2)$ for all $2 < s < +\infty$. In view of (4.17), it is simple to observe that, for all $v \in \mathcal{C}_0^\infty(\mathbb{R}^2)$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} \left[\log\left(\frac{1}{|x|}\right) * G(u_{\lambda_k,\sigma}) \right] g(u_{\lambda_k,\sigma}) v \mathrm{d}x = \int_{\mathbb{R}^2} \left[\int_\Omega \log\left(\frac{1}{|x-y|}\right) G(u_{0,\sigma}(y)) \mathrm{d}y \right] g(u_{0,\sigma}) v \mathrm{d}x$$

jointly with $\mathcal{J}'_{\lambda_k,\sigma,n}(u_{\lambda_k,\sigma}) = 0$ in $E_{\lambda_k}^{-1}$ yields that $\mathcal{J}'_{\Omega,\sigma,n}(u_{0,\sigma}) = 0$ in $(H_0^1(\Omega))^{-1}$. Adopting the same idea above, we are able to rule out the case $u_{0,\sigma} = 0$ standardly. Therefore, $u_{0,\sigma} \in H_0^1(\Omega)$ is a nontrivial solution of Eq. (1.17). The proof is completed. □

5.2. **Case 2: $\lambda > \lambda^*$ is fixed and $\sigma \to 0^+$.**

The reader is invited to retrace the contents exhibited above and we are very sure that the constants C and $\mathcal{C}_j$ are independent of σ.

We define the variational variational functional $\mathcal{J}_{\lambda,0} : E_\lambda \to \mathbb{R}$ by

$$\mathcal{J}_{\lambda,0}(u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) F(u(y)) \mathrm{d}y \right] F(u) \mathrm{d}x$$

Let us show the proof of Theorem 1.7 as follows.

Proof of Theorem 1.7. Let $\sigma_k \to 0^+$ as $k \to +\infty$ and $(u_{\lambda,\sigma_k}) \subset E_\lambda$ be a sequence of nontrivial solutions of Eq. (1.1), that is, $\mathcal{J}'_{\lambda,\sigma_k,n}(u_{\lambda,\sigma_k}) = 0$ in E_λ^{-1} and $\mathcal{J}_{\lambda,\sigma_k,n}(u_{\lambda,\sigma_k}) = c_{\lambda,\sigma_k,n}$. According to Remark 3.5, we can follow as Lemma 3.6 show that (u_{λ,σ_k}) is uniformly bounded in $k \in \mathbb{N}^+$. Passing to some subsequence if necessary, $u_{\lambda,\sigma_k} \rightharpoonup u_{\lambda,0}$ in E_λ, $u_{\lambda,\sigma_k} \to u_{\lambda,0}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for all $2 < s < +\infty$ and $u_{\lambda,\sigma_k} \to u_{\lambda,0}$ a.e. in $\mathbb{R}^2$ as $k \to +\infty$. In view of the Step 2 of the proof of Lemma 3.10, we still have that $u_{\lambda,\sigma_k} \to u_{\lambda,0}$ in $L^s(\mathbb{R}^2)$ for all $2 < s < +\infty$ as $k \to +\infty$.

In view of Lemma 4.2, we have showed that $|u_{\lambda,\sigma_k}|_\infty \leq \mathcal{C}_4$ for some $\mathcal{C}_4 > 0$ independent of $k \in \mathbb{N}^+$. Then, taking into account (2.2) and (2.3), it simply sees that

$$\begin{cases} \lim\limits_{k\to\infty} \displaystyle\int_{\mathbb{R}^2} F(u_{\lambda,\sigma_k})\mathrm{d}x = \int_{\mathbb{R}^2} F(u_{\lambda,0})\mathrm{d}x, \\ \lim\limits_{k\to\infty} \displaystyle\int_{\mathbb{R}^2} f(u_{\lambda,\sigma_k})u_{\lambda,\sigma_k}\mathrm{d}x = \int_{\mathbb{R}^2} f(u_{\lambda,\sigma_0})u_{\lambda,0}\mathrm{d}x, \\ \lim\limits_{k\to\infty} \displaystyle\int_{\mathbb{R}^2} f(u_{\lambda,\sigma_k})v\mathrm{d}x = \int_{\mathbb{R}^2} f(u_{\lambda,0})v\mathrm{d}x,\ \forall v \in \mathcal{C}_0^\infty(\mathbb{R}^2). \end{cases} \tag{5.3}$$

With (4.16), (5.1) and the first one in (5.3) in hands, the generalized Lebesgue's Dominated Convergence indicates that

$$\begin{aligned} &\lim_{k\to\infty} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) G_n(u_{\lambda,\sigma_k}(y))\mathrm{d}y\right] G_n(u_{\lambda,\sigma_k})\mathrm{d}x \\ &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) F(u_{\lambda,0}(y))\mathrm{d}y\right] F(u_{\lambda,0})\mathrm{d}x. \end{aligned}$$

In the same spirit of it, we also conclude that

$$\begin{aligned} &\lim_{k\to\infty} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) G_n(u_{\lambda,\sigma_k}(y))\mathrm{d}y\right] g_n(u_{\lambda,\sigma_k})u_{\lambda,\sigma_k}\mathrm{d}x \\ &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) F(u_{\lambda,0}(y))\mathrm{d}y\right] f(u_{\lambda,0})u_{\lambda,0}\mathrm{d}x \end{aligned}$$

and for all $v \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned} &\lim_{k\to\infty} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) G_n(u_{\lambda,\sigma_k}(y))\mathrm{d}y\right] g_n(u_{\lambda,\sigma_k})v\mathrm{d}x \\ &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) F(u_{\lambda,0}(y))\mathrm{d}y\right] f(u_{\lambda,0})v\mathrm{d}x. \end{aligned}$$

Consequently, we are capable of taking advantage of the above three formulas to deduce that $u_{\lambda,\sigma_k} \to u_{\lambda,0}$ in E_λ along a subsequence as $k \to +\infty$ and $u_{\lambda,0}$ is in fact a nontrivial solution of Eq. (1.18). The proof is completed. □

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