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## PLANAR SCHRÖDINGER-POISSON SYSTEM WITH STEEP POTENTIAL WELL: SUPERCRITICAL EXPONENTIAL CASE

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ABSTRACT. We study a class of planar Schrödinger-Poisson systems

$$\begin{cases} -\Delta u + \lambda V(x)u + \phi u = f(u), & x \in \mathbb{R}^2, \\ \Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases}$$

where  $\lambda > 0$  is a parameter,  $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R}^+)$  has a potential well  $\Omega :=$ int  $V^{-1}(0)$ , and the nonlinearity f fulfills the supercritical exponential growth at infinity in the Trudinger-Moser sense. By exploiting the mountain-pass theorem and elliptic regularity theory, we establish the existence and concentrating behavior of ground state solutions for sufficiently large  $\lambda$ .

1. Introduction and main results. In this paper, we focus on the existence and concentrating behavior of ground state solutions for a class of planar Schrödinger-Poisson systems

$$\begin{cases} -\Delta u + \lambda V(x)u + \phi u = f(u), & x \in \mathbb{R}^2, \\ \Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases}$$
(1)

where  $\lambda > 0$  is a parameter,  $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R}^+)$  has a potential well  $\Omega := \operatorname{int} V^{-1}(0)$ , and the nonlinearity f possesses the *supercritical exponential growth* at infinity in the Trudinger-Moser sense. More precisely, throughout the whole paper, we shall suppose f has the form

$$f(t) = h(t)e^{\alpha |t|^{\tau}}, \quad \forall t \in \mathbb{R},$$
(2)

for some  $\alpha > 0$  and  $\tau \ge 2$ . In what follows, we assume that  $h : \mathbb{R} \to \mathbb{R}$  is a function satisfying:

- $(h_1)$   $h \in \mathcal{C}(\mathbb{R})$  vanishes for every  $t \in (-\infty, 0]$  and h(t) = o(t) as  $t \to 0$ ;
- $(h_2)$  The function  $h(t)/t^3$  is increasing for all t > 0;
- (h<sub>3</sub>) There exist  $\delta \in (0,2)$  and  $\gamma, M > 0$  such that  $|h(t)| \leq M(e^{\gamma|t|^{\delta}} 1)$  for each  $t \in \mathbb{R}^+ = (0, +\infty)$ .

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Over the past several decades, a lot of attentions has been paid to the standing, or solitary, wave solutions of Schrödinger-Poisson systems of the type

$$\begin{cases} i\frac{\partial\psi}{\partial t} = -\Delta\psi + W(x)\psi + m\phi\psi - \tilde{f}(|\psi|)\psi, & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\ \Delta\phi = |\psi|^2, & \text{in } \mathbb{R}^d, \end{cases}$$
(3)

where  $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$  denotes the time-dependent wave function,  $W : \mathbb{R}^d \to \mathbb{R}$  is a real external potential,  $m \in \mathbb{R}$  is a parameter,  $\phi$  represents an internal potential for a nonlocal self-interaction of the wave function, and the nonlinear term  $f(\psi) := \tilde{f}(|\psi|)\psi$  describes the interaction effect among particles. Let  $\psi(x,t) = \exp(-i\omega t)u(x)$  with  $\omega \in \mathbb{R}$ . Then, the standing wave solutions of (3) also lead to the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + m\phi u = f(u), & \text{in } \mathbb{R}^d, \\ \Delta \phi = u^2, & \text{in } \mathbb{R}^d, \end{cases}$$
(4)

where here and in what follows  $\overline{V}(x) = W(x) + \omega$ . In view of [27], the second equation in (4) determines  $\phi : \mathbb{R}^d \to \mathbb{R}$  only up to harmonic functions. Conversely, it is natural to choose  $\phi$  as the negative Newton potential of  $u^2$ , that is, the convolution of  $u^2$  with the fundamental solution  $\Phi_d$  of the Laplacian, which is denoted by  $\Phi_d(x) = -1/(d(d-2)\omega_d)|x|^{2-d}$  if  $d \ge 3$ , and  $\Phi_2(x) = -\frac{1}{2\pi}\log(|x|)$  if d = 2. Here,  $\omega_d$  denotes the volume of the unit ball in  $\mathbb{R}^d$ . With this inversion of the second equation in (4), we arrive at the integro-differential equation

$$-\Delta u + \bar{V}(x)u + m(\Phi_d * u^2)u = f(u), \text{ in } \mathbb{R}^d.$$
(5)

When  $m \neq 0$ , the Poisson term  $(\Phi_d * u^2)u$  results in that (5) is not a pointwise identity any longer such that there are some mathematical difficulties which make the study of it more interesting. In the three-dimensional case, because of the relevances in physics, we have a rich literature associated with (5) and its generalizations under the variant assumptions on  $\bar{V}$  and f via variational methods. The interested reader can refer to [1, 8, 20, 33, 42, 48] and the references therein.

For d = 2, the Schrödinger-Poisson equation (5) can be rewritten as the form

$$-\Delta u + \bar{V}(x)u + \frac{m}{2\pi} \left(\log(|x|) * u^2\right)u = f(u) \quad \text{in } \mathbb{R}^2, \tag{6}$$

whose variational functional is defined by

$$\begin{split} I(u) = &\frac{1}{2} \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + \bar{V}(x)u^2 \right] dx \\ &+ \frac{m}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)u^2(x)u^2(y) dx dy - \int_{\mathbb{R}^2} F(u) dx, \end{split}$$

where here and in what follows  $F(u) := \int_0^u f(t) dt$ . As mentioned by Stubbe in [51], the functional I would not be well-defined on  $H^1(\mathbb{R}^2)$ . To overcome this difficulty, he introduced a new Hilbert space

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(1+|x|) u^2 dx < +\infty \right\},\$$

endowed with the inner product and norm

$$(u,v)_X = \int_{\mathbb{R}^2} \left[ \nabla u \nabla v + uv + \log(1+|x|)uv \right] dx \text{ and } \|u\|_X = \sqrt{(u,u)_X}.$$

In Stubbe's argument, it depends strongly on the crucial identity

$$\log r = \log(1+r) - \log\left(1+\frac{1}{r}\right), \ \forall r > 0,$$

because it permits us to define the variational functionals  $V_1, V_2: X \to \mathbb{R}$  by

$$V_1(u) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|) u^2(x) u^2(y) dx dy, \ \forall u \in X,$$

and

$$V_2(u) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right) u^2(x) u^2(y) dx dy, \ \forall u \in X.$$

Stubbe [51] proved that  $V_1, V_2 \in \mathcal{C}^1(X, \mathbb{R})$ , and found the equality

$$V_0(u) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|) u^2(x) u^2(y) dx dy = V_1(u) - V_2(u), \ \forall u \in X,$$

which implies that I given in (6) is of class  $\mathcal{C}^1(X)$ .

In [27], Cingolani and Weth made full use of the above variational framework to study the existence and multiplicity of nontrivial solutions for the equation

$$-\Delta u + u + \frac{m}{2\pi} (\log(|x|) * u^2) u = b|u|^{p-2} u \quad \text{in } \mathbb{R}^2,$$
(7)

where  $b \ge 0$  and  $p \ge 4$ . With the help of variational methods and the action of groups, they investigated the existence of multiple solutions and obtained a ground state solution by minimizing the energy functional over the Nehari manifold for Eq. (7). Subsequently, considering (7) with b = 1 and  $p \in (2, 4]$ , Du-Weth [32] proved that it possesses a nontrivial solution if  $p \in (2,3)$ , and a ground state solution if  $p \in (3, 4]$  by constructing a asymptotic (PS) sequence, where the Pohožaev identity was established. In [14], Bonheure *et al.* concluded the asymptotic decay of the unique positive, radially symmetric solution to (7) with b = 0. Afterwards, Chen, Shi, and Tang [22] extended the main results in [32] to a general nonlinearity. It is important to note that the above cited papers rely on the fact that the potential is constant or  $\mathbb{Z}^2$ -periodic. To handle the obstacle, the authors in [24,25], restricting in an axially symmetric space which is weaker than the classic radially symmetric one. succeeded in finding nontrivial solutions for (7) with a non-constant or non-periodic potential. Regarding some other meaningful research works associated with (7), we suggest the reader to refer to [10, 26, 31] and the references therein, even if these references are far from exhaustive.

Another interesting point for considering Schrödinger-Poisson equations, like Eq. (6), is the space dimension d = 2. Moreover, to our best knowledge, the existence result for the Schrödinger-Poisson equation with steep potential well has not been studied yet, especially with the nonlinearity involving the *supercritical exponential growth*. Therefore, one of the main purposes of the present paper is to fill these blanks. Generally speaking, we aim at establishing the existence and concentrating behavior of ground state solutions for Eq. (1).

As is known, for every bounded domain  $\Omega \subset \mathbb{R}^2$ , the imbedding  $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ with  $1 \leq p < +\infty$  does not imply that  $H_0^1(\Omega) \hookrightarrow L^{\infty}(\Omega)$ . As a consequence, one naturally wonders if there exists another kind of *maximal growth* in this situation. Indeed, the authors in [44, 45, 52] got the following sharp maximal exponential integrability for functions in  $H_0^1(\Omega)$ :

$$\sup_{u \in H_0^1(\Omega), \, \|\nabla u\|_{L^2(\Omega)} \le 1} \int_{\Omega} e^{\alpha u^2} dx \le C \mathrm{meas}(\Omega) \quad \text{if } \alpha \le 4\pi, \tag{8}$$

where the constant  $C = C(\alpha) > 0$ , and meas( $\Omega$ ) stands for the Lebesgue measure of  $\Omega$ . As is known, (8) is the celebrated Trudinger-Moser type inequality, related to elliptic problems with *critical exponential growth*, meaning that a function fsatisfies the following: There exists  $\alpha_0 > 0$  such that

$$\lim_{|t| \to +\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$
(9)

This definition was introduced by Adimurthi and Yadava [2]; see also de Figueiredo, Miyagaki, and Ruf [28] for example. Motivated by the previous works in [6,7], there exist two ways to understand that the function h, defined in (2) together with  $(h_3)$ , satisfies the so-called *supercritical exponential growth* in the following sense:

$$\begin{cases} (I) \ \tau > 2 \text{ is arbitrary and } \alpha > 0 \text{ is fixed;} \\ (II) \ \alpha > 0 \text{ is arbitrary and } \tau \ge 2 \text{ is fixed.} \end{cases}$$
(10)

Moreover, one could call Cases (I) and (II) in (10) the subcritical-supercritical exponential growth and critical-supercritical exponential growth, respectively.

As to the whole space  $\mathbb{R}^2$ , the author in [30] established the following version of the Trudinger-Moser inequality (see also [17] for example):

$$e^{\alpha u^2} - 1 \in L^2(\mathbb{R}^2), \ \forall \alpha > 0 \text{ and } u \in H^1(\mathbb{R}^2).$$

$$(11)$$

Moreover, for all  $u \in H^1(\mathbb{R}^2)$  with  $||u||_{L^2(\mathbb{R}^2)} \leq M < +\infty$ , there is a  $C = C(M, \alpha) > 0$  such that

$$\sup_{u \in H^1(\mathbb{R}^2), \|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1} \int_{\mathbb{R}^2} \left( e^{\alpha u^2} - 1 \right) dx \le C \text{ if } \alpha < 4\pi.$$
(12)

Concerning some other generalizations, extensions, and applications of the Trudinger-Moser inequalities for bounded and unbounded domains, we refer to [28, 29, 36, 47] and the references therein.

As far as we are concerned, Alves and Figueiredo [4] first applied (9) to the Schrödinger-Poisson equation (6), and investigated the existence of ground state solutions by using the Nehari manifold method. In this direction, there are more and more research works concerning this topic, including the two-dimensional Choquard problem with logarithmic kernel, see [3, 19, 21, 23, 25, 38-40, 49] for example.

Before stating the main results in this paper, on V we shall impose the following assumptions:

- $(V_1)$   $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$  with  $V(x) \ge 0$  on  $\mathbb{R}^2$ .
- (V<sub>2</sub>)  $\Omega := \operatorname{int} V^{-1}(0)$  is nonempty and bounded with smooth boundary, and  $\overline{\Omega} = V^{-1}(0)$ .
- (V<sub>3</sub>) There exists b > 0 such that  $\Xi := \{x \in \mathbb{R}^2 : V(x) < b\}$  is nonempty and has finite measure.

Assumptions like  $(V_1) - (V_3)$  were first proposed by Bartsch *et al.* in [11, 12]. Particularly, the harmonic trapping potential

$$V(x) = \begin{cases} \omega_1 |x_1|^2 + \omega_2 |x_2|^2 - \omega, & \text{if } |(\sqrt{\omega_1} x_1, \sqrt{\omega_2} x_2)|^2 \ge \omega, \\ 0, & \text{if } |(\sqrt{\omega_1} x_1, \sqrt{\omega_2} x_2)|^2 \le \omega, \end{cases}$$

with  $\omega > 0$  satisfies  $(V_1) - (V_3)$ , where  $\omega_i > 0$  is called by the anisotropy factor of the trap in quantum physics and trapping frequency of the *i*th-direction in mathematics, see e.g. [13,18,41]. Indeed, the potential  $\lambda V(x)$  with the above hypotheses is usually known as the steep potential well.

 $\mathbf{5}$ 

For all fixed  $\lambda > 0$ , by  $(V_1)$ , define the Hilbert space

$$E_{\lambda} := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} \lambda V(x) |u|^2 dx < +\infty \right\}$$

equipped with the inner product and norm

$$(u,v)_{E_{\lambda}} = \int_{\mathbb{R}^2} \left[ \nabla u \nabla v + \lambda V(x) uv \right] dx \text{ and } \|u\|_{E_{\lambda}} = \sqrt{(u,u)_{E_{\lambda}}}, \ \forall u,v \in E_{\lambda}.$$

Hereafter, let us denote E and  $\|\cdot\|_E$  by  $E_{\lambda}$  and  $\|\cdot\|_{E_{\lambda}}$  for  $\lambda = 1$ , respectively. It is simple to observe that  $\|\cdot\|_E \leq \|\cdot\|_{E_{\lambda}}$  for every  $\lambda \geq 1$ . Therefore, due to Lemma 2.4 below,  $E_{\lambda}$  could be continuously imbedded into  $H^1(\mathbb{R}^2)$ , and then into X. With these discussions, we could introduce the work space

$$X_{\lambda} := \left\{ u \in X : \int_{\mathbb{R}^2} \lambda V(x) |u|^2 dx < +\infty \right\}$$

which is a Hilbert space equipped with the inner product and norm

$$(u,v)_{X_{\lambda}} = \int_{\mathbb{R}^2} \left[ \nabla u \cdot \nabla v + (\lambda V(x) + \log(1+|x|))uv \right] dx \text{ and } \|u\|_{X_{\lambda}} = \sqrt{(u,u)_{X_{\lambda}}},$$

for all  $u, v \in X_{\lambda}$ . Obviously,  $\|\cdot\|_{X_{\lambda}} = \sqrt{\|\cdot\|_{E_{\lambda}}^2 + \|\cdot\|_*^2}$ , where  $\|u\|_* = (\int_{\mathbb{R}^2} \log(1 + |x|)|u|^2 dx)^{\frac{1}{2}}$  for all  $u \in X$ .

Now, we shall exhibit the first main result in this paper as follows.

**Theorem 1.1.** Let V satisfy  $(V_1) - (V_3)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1) - (h_3)$ . Then, for each  $\tau > 2$ , there are  $\alpha^* = \alpha^*(\tau) > 0$  and  $\lambda_0 > 0$  such that Eq. (1) has a nonnegative ground state solution in  $X_{\lambda}$  for all  $\alpha \in (0, \alpha^*)$  and  $\lambda > \lambda_0$ . Moreover, if we suppose that

(h<sub>4</sub>) there are constants  $\xi > 0$  and p > 4 such that  $H(t) = \int_0^t h(s) ds \ge \xi t^p$  for all  $t \in [0, 1]$ ,

then for every  $\alpha > 0$ , there exist  $\tau_* = \tau_*(\alpha) > 2$ ,  $\lambda'_0 > 0$ , and  $\xi_0 > 0$  such that Eq. (1) possesses a nonnegative ground state solution in  $X_{\lambda}$  for every  $\tau \in [2, \tau_*)$ ,  $\lambda > \lambda'_0$ , and  $\xi > \xi_0$ .

From a variational method point of view, in order to find nontrivial solutions corresponding to Eq. (1), it would be equivalent to finding critical points for the variational functional  $J_{\lambda}: X_{\lambda} \to \mathbb{R}$  defined by

$$J_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] dx + \frac{1}{4} V_0(u) - \int_{\mathbb{R}^2} F(u) dx, \ \forall u \in X_{\lambda}.$$

Alternatively, although the property of  $X_{\lambda}$  is good enough, adopting (11), one could never show that  $\mathcal{J}_{\lambda}$  is well-defined in  $X_{\lambda}$ , which is caused by the appearance the nonlinearity f involving the supercritical exponential growth (10) in the Trudinger-Moser sense. As one will see later, this is the biggest difference from [4,25], which prevents us from reusing the approaches in the cited papers to conclude Theorem 1.1.

Inspired by [6, 7], given a fixed constant R > 0, we need to introduce a cutoff function  $f^{R,\bar{\delta}}$  and consider an auxiliary equation which involves a (sub)critical exponential growth. Roughly speaking, we define  $f^{R,\bar{\delta}}: \mathbb{R} \to \mathbb{R}$  as follows:

$$f^{R,\bar{\delta}}(t) = \begin{cases} 0, & t \le 0, \\ h(t)e^{\alpha t^{\tau}}, & 0 \le t \le R, \\ h(t)e^{\alpha R^{\tau-\bar{\delta}}t^{\bar{\delta}}}, & t \ge R, \end{cases}$$
(13)

where

$$\bar{\delta} := \begin{cases} \delta, & \text{if Case I in (10) is considered,} \\ 2, & \text{if Case II in (10) is considered.} \end{cases}$$

In light of such a  $f^{R,\bar{\delta}}$ , we consider the auxiliary equation below

$$-\Delta u + \lambda V(x)u + \left(\log(|x|) * u^2\right)u = f^{R,\bar{\delta}}(u) \text{ in } \mathbb{R}^2, \tag{14}$$

where here and in what follows  $F^{R,\bar{\delta}}(t) = \int_0^t f^{R,\bar{\delta}}(s)ds$ . Obviously, for each fixed R > 0, by  $(h_3)$ , it is simple to find that  $f^{R,\bar{\delta}}$  admits a subcritical or critical exponential growth at infinity. Hence, the variational functional  $J^{R,\bar{\delta}}_{\lambda} : X_{\lambda} \to \mathbb{R}$  given by

$$J_{\lambda}^{R,\bar{\delta}}(u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] dx + \frac{1}{4} V_0(u) - \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx$$

associated with Eq. (14) is well-defined and of class  $C^1(X_{\lambda})$ . Moreover, we could certify that every critical point of it is a (weak) solution of Eq. (14). Recalling [53, Subsection 4.1], let us call the solution  $u_R \in E$  the ground state solution of Eq. (14) if it satisfies

$$J_{\lambda}^{R,\bar{\delta}}(u) = \inf_{u \in \mathcal{N}_{\lambda}^{R,\bar{\delta}}} J_{\lambda}^{R,\bar{\delta}}(v) := m_{\lambda}^{R,\bar{\delta}},\tag{15}$$

where the corresponding Nehari manifold is given as

$$\mathcal{N}_{\lambda}^{R,\bar{\delta}} := \left\{ u \in X_{\lambda} \setminus \{0\} : (J_{\lambda}^{R,\bar{\delta}})'(u)[u] = 0 \right\}.$$

It can easily be concluded that if every ground state solution  $u_R \in E$  of Eq. (14) satisfies  $|u_R|_{\infty} \leq R$ , then  $u_R$  is a ground state solution of Eq. (1), where  $|\cdot|_q$  stands for the standard  $L^q$ -norm with  $1 \leq q \leq \infty$ . With this in mind, we should establish such a solution  $u_R$  to derive the proof of Theorem 1.1. So, it is necessary to show the following result.

**Theorem 1.2.** Let V satisfy  $(V_1) - (V_3)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1) - (h_3)$ . Then, for every fixed R > 0, there exists a  $\lambda_0(R) > 0$ dependent of R such that Eq. (14) with  $\overline{\delta} = \delta$  has a nonnegative ground state solution in  $X_{\lambda}$  for all  $\lambda > \lambda_0(R)$ . Moreover, if, in addition, we suppose that  $(h_4)$ , then there exist  $\lambda'_0(R) > 0$  and  $\xi_0(R) > 0$  such that Eq. (14) with  $\overline{\delta} = 2$  possesses a nonnegative ground state solution in  $X_{\lambda}$  for all  $\lambda > \lambda'_0(R)$  and  $\xi > \xi_0(R)$ .

**Remark 1.3.** It is worth pointing out that, to the best of our knowledge, Theorem 1.2 seems to be the first result concerning the planar Schrödinger-Poisson system with steep potential well involving subcritical or critical exponential growth. Although there may be existence results for such problem with non-constant or non-periodic potential, the results in Theorem 1.2 are new in our setting. We mention the paper [46] in which the authors obtained the existence and concentrating behavior of ground state solutions for the generalized Chern-Simons-Schrödinger equation with steep potential well in the critical exponential case. Nevertheless, compared to it, we have to present some new tricks and analytic skills because of the indefinite Poisson term  $\log(|x|) * u^2$ , bringing additionally distinct challenges.

Next, we shall contemplate the asymptotical behavior of the ground state solutions obtained in Theorem 1.1 as  $\lambda \to +\infty$ . Let  $u \in X_{\lambda}$  be a ground state solution for Eq. (1). There is no doubt that it depends on the parameter  $\lambda > \lambda_0 > 0$  (or,  $\lambda > \lambda'_0 > 0$ ), so we relabeled it by  $u_{\lambda}$  to emphasize this dependence, where  $\lambda_0 > 0$  (or,  $\lambda > \lambda'_0 > 0$ ) is a constant appearing in the proof of Theorem 1.1 below. Therefore, we can prove the following result.

**Theorem 1.4.** Under the assumptions in Theorem 1.1, going to a subsequence,  $u_{\lambda} \rightarrow u_0$  in X as  $\lambda \rightarrow +\infty$ , where  $u_0$  is a ground state solution for the Schrödinger-Poisson equation

$$\begin{cases} -\Delta u + \left(\int_{\Omega} \log |x - y| u^2(y) dy\right) u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$
(16)

**Remark 1.5.** Let us recall the main results in Theorems 1.1 and 1.2, dealing with the planar Schrödinger-Poisson system with non-constant or non-periodic potential involving critical exponential growth. The well-known Ambrosetti-Rabinowitz condition, namely  $f(t)t - \mu F(t) \ge 0$  with  $\mu > 4$  for all  $t \in \mathbb{R}$ , plays a significant role in concluding the boundness of (C) sequence corresponding to the variational functional. In view of  $(h_2)$  (or (28) below), nevertheless, we can remove it successfully by the steep potential well  $\lambda V$  under the assumptions  $(V_1) - (V_3)$ .

Although we have briefly introduced the essential ideas of the main results in this article, it is still far from enough to get the detailed proofs of Theorems 1.1, 1.2, and 1.4. Alternatively, the key to considering such supercritical problem is to transform it into a subcritical or critical one. As to the  $L^{\infty}$ -estimate on nontrivial solutions for a class of supercritical problems, the elliptic regular result or Nash-Moser iteration procedure are effective tools. Moreover, we truly anticipate that the results in the present paper shall be conducive to further studies on the planar Schrödinger-Poisson system with (sub)critical or supercritical exponential growth.

The outline of the paper is organized as follows. In Section 2, we mainly present some preliminary results. In Section 3, we consider the (sub)critical problem (14) and give the proof of Theorem 1.2. Section 4 is devoted to the proofs of Theorems 1.1 and 1.4. Some remarks are given in Section 5.

**Notations.** From now on in this paper, unless otherwise mentioned, we ultilize the following notations:

- $C, C_1, C_2, \cdots$  denote any positive constant, whose value is not relevant, and  $\mathbb{R}^+ := (0, +\infty).$
- Let  $(Z, \|\cdot\|_Z)$  be a Banach space with dual space  $(Z^{-1}, \|\cdot\|_{Z^{-1}})$ , and  $\Phi$  be functional on Z.
- The (C) sequence at a level  $c \in \mathbb{R}$  ((C)<sub>c</sub> sequence in short) corresponding to  $\Phi$ means that  $\Phi(x_n) \to c$  and  $(1 + ||x_n||_Z) ||\Phi'(x_n)||_{Z^{-1}} \to 0$  in  $X^{-1}$  as  $n \to \infty$ , where  $\{x_n\} \subset Z$ .
- $| |_p$  stands for the usual norm of the Lebesgue space  $L^p(\mathbb{R}^2)$  for all  $p \in [1, +\infty]$ , and  $|| ||_{H^i}$  denotes the usual norm of the Sobolev space  $H^i(\mathbb{R}^2)$  for i = 1, 2.
- For any  $\rho > 0$  and every  $x \in \mathbb{R}^2$ ,  $B_{\rho}(x) := \{y \in \mathbb{R}^2 : |y x| < \rho\}.$
- $o_n(1)$  denotes real sequences by  $o_n(1) \to 0$  as  $n \to +\infty$ .
- " → " and " → " stand for the strong and weak convergence in the related function spaces, respectively.

2. **Preliminaries.** In this section, we exhibit some preliminary results which shall be exploited frequently in this paper. Let us begin defining the following three auxiliary symmetric bilinear forms  $X \times X \to \mathbb{R}$ 

$$(u,v) \mapsto B_1(u,v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|)u(x)v(y)dxdy,$$
  

$$(u,v) \mapsto B_2(u,v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)u(x)v(y)dxdy,$$
  

$$(u,v) \mapsto B(u,v) = B_1(u,v) - B_2(u,v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)u(x)v(y)dxdy.$$

The above definitions are understood as being over measurable functions  $u, v : \mathbb{R}^2 \to \mathbb{R}$ , such that the integrals are defined in the Lebesgue sense. Then,  $V_i(u) = B_i(u^2, u^2)$ , where  $V_i$  is given as in the introduction for  $i \in \{0, 1, 2\}$ .

**Lemma 2.1.** ([27, Lemma 2.2]) (i) The space X is compactly embedded in  $L^s(\mathbb{R}^2)$ , for all  $s \in [2, \infty)$ .

(ii)  $0 \le V_1(u) \le 2|u|_2^2|u|_*^2 \le 2||u||_X^4$  and  $V_1$  is weakly semicontinuous in  $H^1(\mathbb{R}^2)$ . (iii)  $V'_i(u)[v] = 4B_i(u^2, uv)$ , and so  $V'_i(u)[u] = 4V_i(u)$  for  $u, v \in X$  and  $i \in \{0, 1, 2\}$ .

(iv) There is  $K_0 > 0$  such that  $|B_2(u,v)| \le K_0 |u|_{\frac{4}{3}} |v|_{\frac{4}{3}}, \forall u, v \in L^{\frac{4}{3}}(\mathbb{R}^2)$ . Hence,

$$|V_2(u)| \le K_0 |u|_{\frac{8}{3}}^4, \ \forall u \in L^{\frac{8}{3}}(\mathbb{R}^2).$$
(17)

(v)  $V_2$  is completely continuous in X, that is,

$$u_n \rightharpoonup u \text{ in } X \Longrightarrow V_2(u_n) \rightarrow V_2(u).$$

**Lemma 2.2.** ([27, Lemma 2.6]) Let  $\{u_n\}$ ,  $\{v_n\}$ , and  $\{w_n\}$  be bounded sequences in X such that  $u_n \rightharpoonup u$  in X. Then, for every  $z \in X$ , we have  $B_1(v_nw_n, z(u_n-u)) \rightarrow 0$ , as  $n \rightarrow +\infty$ .

Taking into account the uniform  $L^{\infty}$ -estimate for the nontrivial solutions established in Theorem 1.2, we obtain the result below.

**Lemma 2.3.** It holds that  $g_u(x) := \log(1+|\cdot|^{-1})*|u|^2 \in L^{\infty}(\mathbb{R}^2)$  for all  $u \in H^1(\mathbb{R}^2)$ . Moreover,

$$|g_u(x)|_{\infty} \le 4\pi (|u|_2^2 + |u|_6^2).$$
(18)

*Proof.* The original idea comes from [5,50], and we exhibit it in detail for the sake of the reader's continence. For all  $x \in \mathbb{R}^2$ , there holds

$$|g_{u}(x)| = \int_{B_{1}(x)} \log\left(1 + \frac{1}{|x-y|}\right) |u(y)|^{2} dy + \int_{\mathbb{R}^{2} \setminus B_{1}(x)} \log\left(1 + \frac{1}{|x-y|}\right) |u(y)|^{2} dy$$
$$\leq \int_{B_{1}(x)} \frac{|u(y)|^{2}}{|x-y|} dy + \log 2 \int_{\mathbb{R}^{2} \setminus B_{1}(x)} |u(y)|^{2} dy.$$
(19)

It follows from Hölder's inequality that

$$\int_{B_1(x)} \frac{|u_R(y)|^2}{|x-y|} dy \le \left( \int_{B_1(x)} \frac{1}{|x-y|^{\frac{3}{2}}} dy \right)^{\frac{4}{3}} \left( \int_{\mathbb{R}^2} |u(y)|^6 dy \right)^{\frac{1}{3}} = (4\pi)^{\frac{2}{3}} \left( \int_{\mathbb{R}^2} |u(y)|^6 dy \right)^{\frac{1}{3}}.$$
(20)

Combining (19) and (20), we get the desired result (18). The proof is complete.  $\Box$ 

9

Next, we have to prove the following imbedding result, which guarantees the reasonableness of the work space  $X_{\lambda}$ , whence  $\lambda \geq 1$ .

**Lemma 2.4.** Suppose that  $(V_1) - (V_3)$ . Then, the imbedding  $E \hookrightarrow H^1(\mathbb{R}^2)$  is continuous. In particular,  $E_{\lambda}$  can be continuously imbedded into  $H^1(\mathbb{R}^2)$  for all  $\lambda \geq 1$ .

*Proof.* For every  $u \in H^1(\mathbb{R}^2)$ , by adopting  $(V_1)$  and  $(V_3)$  as well as the Gagliardo-Nirenberg inequality, one has

$$\begin{split} \int_{\mathbb{R}^2} |u|^2 dx &= \int_{\Xi} |u|^2 dx + \int_{\mathbb{R}^2 \setminus \Xi} |u|^2 dx \\ &\leq \sqrt{\mathrm{meas}(\Xi)} \bigg( \int_{\mathbb{R}^2} |u|^4 dx \bigg)^{\frac{1}{2}} + \frac{1}{b} \int_{\mathbb{R}^2 \setminus \Xi} V(x) |u|^2 dx \\ &\leq \kappa_{\mathrm{GN}} \sqrt{\mathrm{meas}(\Xi)} \bigg( \int_{\mathbb{R}^2} |u|^2 dx \bigg)^{\frac{1}{2}} \bigg( \int_{\mathbb{R}^2} |\nabla u|^2 dx \bigg)^{\frac{1}{2}} + \frac{1}{b} \int_{\mathbb{R}^2} V(x) |u|^2 dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^2} |u|^2 dx + \frac{1}{2} \kappa_{\mathrm{GN}}^2 \mathrm{meas}(\Xi) \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{b} \int_{\mathbb{R}^2} V(x) |u|^2 dx, \end{split}$$

where  $\kappa_{GN} > 0$  denotes the best constant associated with the Gagliardo-Nirenberg inequality. From this inequality, we obtain

$$\begin{split} \int_{\mathbb{R}^2} |u|^2 dx &\leq \kappa_{\rm GN}^2 \operatorname{meas}(\Xi) \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{2}{b} \int_{\mathbb{R}^2} V(x) |u|^2 dx \\ &\leq \max\left\{\kappa_{\rm GN}^2 \operatorname{meas}(\Xi), \frac{2}{b}\right\} \|u\|_E^2 \end{split}$$

indicating that  $||u||_{H^1(\mathbb{R}^2)}^2 \leq C_{\Xi,b} ||u||_E^2$ , where  $C_{\Xi,b} := \max\{1 + \kappa_{GN}^2 \max(\Xi), \frac{b+2}{b}\} > 0$ . The proof of this lemma is finished.  $\Box$ 

**Remark 2.5.** With the help of Lemma 2.4, for all  $\lambda \geq 1$  we can redefine the space  $X_{\lambda}$  by

$$X_{\lambda} := \bigg\{ u \in E_{\lambda} : \int_{\mathbb{R}^2} \log(1+|x|)|u|^2 dx < +\infty \bigg\}.$$

Finally, we consider the compact results for the nonlinearity  $f^{R,\bar{\delta}}$  (defined in (13)) which would be very important in receiving the boundedness of the (C) sequence for  $J_{\lambda}$ .

**Lemma 2.6.** Suppose  $(h_1)$  and  $(h_3)$ . Let  $\{u_n\} \subset H^1(\mathbb{R}^2)$  be a sequence such that  $u_n \to u_0$  in  $L^p(\mathbb{R}^2)$  and  $u_n \to u_0$  a.e. in  $\mathbb{R}^2$ . Then, for all fixed R > 0, passing to a subsequence if necessary,

$$\begin{cases} \lim_{n \to \infty} \int_{\mathbb{R}^2} f^{R,\delta}(u_n) u_n dx = \int_{\mathbb{R}^2} f^{R,\delta}(u_0) u_0 dx, \\ \lim_{n \to \infty} \int_{\mathbb{R}^2} F^{R,\delta}(u_n) dx = \int_{\mathbb{R}^2} F^{R,\delta}(u_0) u dx. \end{cases}$$
(21)

If in addition we suppose that  $\limsup_{n\to\infty} \|u_n\|_{H^1(\mathbb{R}^2)}^2 < \frac{\pi}{\gamma + \alpha R^{\tau-2}}$ , then for every fixed R > 0, passing to a subsequence if necessary,

$$\begin{cases} \lim_{n \to \infty} \int_{\mathbb{R}^2} f^{R,2}(u_n) u_n dx = \int_{\mathbb{R}^2} f^{R,2}(u_0) u_0 dx, \\ \lim_{n \to \infty} \int_{\mathbb{R}^2} F^{R,2}(u_n) dx = \int_{\mathbb{R}^2} F^{R,2}(u_0) u dx. \end{cases}$$
(22)

*Proof.* It suffices to verify the first formulas in (21) and (22), respectively. First, we study (21). In this situation, without loss of generality, we can suppose that  $\sup_{n \to \infty} \|u_n\|_{H^1(\mathbb{R}^2)}^2 < \Sigma(R)$ , where the constant  $\Sigma(R) > 0$  is dependent on R > 0, and  $n \in \mathbb{N}$ independent of  $n \in \mathbb{N}$ . Then, we claim that for every  $\varepsilon > 0$  independent of R, there is a constant  $C_{\varepsilon}(R) > 0$ , which may depend on R, such that

$$|f^{R,\delta}(t)| \le \varepsilon |t| + C_{\varepsilon}(R)|t|^{q-1} [e^{(\Sigma(R)+1)^{-1}|t|^2} - 1], \ \forall t \in \mathbb{R},$$
(23)

where q > 1 is arbitrary. To see it, via  $(h_1)$ ,

$$\lim_{t \to 0} \frac{f^{R,\delta}(t)}{t} = \lim_{t \to 0} e^{\alpha t^{\tau}} \lim_{t \to 0} \frac{h(t)}{t} = 0 \text{ uniformly in } R > 0.$$

On the other hand, we need the following two facts: (1) For  $t_0 > 0$ , there are b > a > 0 such that

$$ae^{t^{\delta}} \leq be^{t^2}, \ \forall t \geq t_0;$$

(2) For  $t_0 > 0$ , since  $\lim_{t \to +\infty} \frac{e^{t^2}}{e^{t^2} - 1} = 1$ , there is a constant C > 0 such that  $e^{t^2} \le C(e^{t^2} - 1), \ \forall t \ge t_0.$ 

$$e^{t^2} \le C(e^{t^2} - 1), \ \forall t \ge t_0.$$

Thereby, for all  $|t| \ge 1$ , by using  $(h_3)$ , there are  $M_2(R) > M_1(R) > M$  such that

$$\begin{split} |f^{R,\delta}(t)| &\leq M(e^{\gamma|t|^{\delta}} - 1)e^{\alpha R^{\tau-\delta}|t|^{\delta}} \\ &\leq Me^{(\gamma+\alpha R^{\tau-\delta})|t|^{\delta}} \\ &\leq M_1(R)e^{(\Sigma(R)+1)^{-1}|t|^2} \\ &\leq M_2(R)[e^{(\Sigma(R)+1)^{-1}|t|^2} - 1] \\ &\leq M_2(R)|t|^{q-1}[e^{(\Sigma(R)+1)^{-1}|t|^2} - 1]. \end{split}$$

So, we derive (23) by the above two formulas. Combining (23) and (12), we obtain that

$$\begin{split} \int_{\mathbb{R}^2} |f^{R,\delta}(u_n)u_n| dx &\leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 dx + C_{\varepsilon}(R) \int_{\mathbb{R}^2} |u_n|^q [e^{(\Sigma(R)+1)^{-1}|u_n|^2} - 1] dx \\ &\leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 dx + C_{\varepsilon}(R) \bigg( \int_{\mathbb{R}^2} |u_n|^{2q} dx \bigg)^{\frac{1}{2}} \\ &\qquad \times \bigg( \int_{\mathbb{R}^2} [e^{\frac{2||u_n|^2_{H^1}(\mathbb{R}^2)}{\Sigma(R)+1} \big(|u_n|/||u_n||^2_{H^1}(\mathbb{R}^2)\big)^2} - 1] dx \bigg)^{\frac{1}{2}} \\ &\leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 dx + \bar{C}_{\varepsilon}(R) \bigg( \int_{\mathbb{R}^2} |u_n|^{2q} dx \bigg)^{\frac{1}{2}}. \end{split}$$

11

Thanks to  $u_n \to u_0$  in  $L^{2q}(\mathbb{R}^2)$ , since q > 1, we could exploit the generalized Lebesgue's dominated convergence theorem to get the desired result by  $n \to \infty$ , and then  $\varepsilon \to 0^+$ .

Next, we consider (22). Similar to (23), we claim that there is a  $C'_{\varepsilon}(R) > 0$  such that

$$|f^{R,2}(t)| \le \varepsilon |t| + C'_{\varepsilon}(R)|t|^{q-1} [e^{(\gamma + \alpha R^{\tau-2})|t|^2} - 1], \ \forall t \in \mathbb{R}.$$
 (24)

Indeed, for all  $|t| \ge 1$ , by using  $(h_3)$ , there are  $\tilde{M}(R) > \bar{M} > M$  such that

$$\begin{split} |f^{R,2}(t)| &\leq M(e^{\gamma|t|^{\delta}} - 1)e^{\alpha R^{\tau-2}|t|^2} \\ &\leq M e^{\gamma|t|^{\delta}} e^{\alpha R^{\tau-2}|t|^2} \\ &\leq \bar{M} e^{\gamma|t|^2} e^{\alpha R^{\tau-2}|t|^2} \\ &\leq \tilde{M}(R)[e^{\gamma|t|^2} - 1][e^{\alpha R^{\tau-2}|t|^2} - 1] \\ &\leq \tilde{M}(R)[e^{(\gamma+\alpha R^{\tau-2})|t|^2} - 1] \\ &\leq \tilde{M}(R)|t|^{q-1}[e^{(\gamma+\alpha R^{\tau-2})|t|^2} - 1] \end{split}$$

jointly with  $f^{R,2}(t) = o(t)$  uniformly in R > 0 as  $t \to 0$ . As a consequence of (24) and (12), there holds

$$\begin{split} \int_{\mathbb{R}^2} |f^{R,2}(u_n)u_n| dx &\leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 dx + C'_{\varepsilon}(R) \int_{\mathbb{R}^2} |u_n|^q [e^{(\gamma + \alpha R^{\tau - 2})|u_n|^2} - 1] dx \\ &\leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 dx + C'_{\varepsilon}(R) \bigg( \int_{\mathbb{R}^2} |u_n|^{2q} dx \bigg)^{\frac{1}{2}} \\ &\qquad \times \bigg( \int_{\mathbb{R}^2} [e^{2(\gamma + \alpha R^{\tau - 2})||u_n||^2_{H^1(\mathbb{R}^2)}} \bigg( \frac{|u_n|}{||u_n||^2_{H^1(\mathbb{R}^2)}} \bigg)^2 - 1] dx \bigg)^{\frac{1}{2}} \\ &\leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 dx + \bar{C}'_{\varepsilon}(R) \bigg( \int_{\mathbb{R}^2} |u_n|^{2q} dx \bigg)^{\frac{1}{2}}. \end{split}$$

Repeating the above arguments, we can verify the validity of (22). So, we can accomplish the proof of this lemma.  $\hfill\square$ 

**Remark 2.7.** Thanks to the calculations in Lemma 2.6, for all  $u \in H^1(\mathbb{R}^2)$ , we immediately derive that

$$\int_{\mathbb{R}^2} |f^{R,\bar{\delta}}(u)u| dx \le \varepsilon \int_{\mathbb{R}^2} |u|^2 dx + C_{\varepsilon}(R) \left(\int_{\mathbb{R}^2} |u|^{2q} dx\right)^{\frac{1}{2}}$$
(25)

provided  $||u||^2_{H^1(\mathbb{R}^2)}$  is sufficiently small, where q > 1 could be selected as required.

3. On problem (14): The (sub)critical case. In this section, we mainly make full use of the mountain-pass theorem introduced in [16,43] to establish the existence of ground state solutions for Eq. (14). Now, we state the theorem, which is a consequence of the Ekeland variational principle developed in [9] as follows.

**Proposition 3.1.** Let Z be a Banach space and  $\Psi \in C^1(Z, \mathbb{R})$  Gateaux differentiable for all  $v \in Z$ , with G-derivative  $\Phi'(v) \in Z^{-1}$  continuous from the norm topology of Z to the weak \* topology of  $Z^{-1}$  and  $\Phi(0) = 0$ . Let S be a closed subset of Z which disconnects (archwise) Z. Let  $v_0 = 0$  and  $v_1 \in Z$  be points belonging to distinct connected components of  $Z \setminus S$ . Suppose that

$$\inf \Phi \ge A > 0 \text{ and } \Phi(v_1) \le 0$$

and let  $\Gamma = \{\gamma \in \mathcal{C}([0,1], Z) : \gamma(0) \text{ and } \gamma(1) = v_1\}$ . Then,  $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) \ge A > 0$ 

and there is a  $(C)_c$  sequence for  $\Phi$ .

In what follows, if not otherwise specified, we shall always assume  $(V_1) - (V_3)$ and  $(h_1) - (h_3)$  for simplicity.

**Lemma 3.2.** Let R > 0 be fixed, then for all  $\lambda \ge 1$ , we have the following conclusions:

- There exist  $A, \rho > 0$  independent of  $\lambda \ge 1$  such that  $J_{\lambda}^{R,\overline{\delta}}(u) \ge A$  for all  $\|u\|_{E_{\lambda}} = \rho;$
- There exists  $\bar{u} \in X_{\lambda}$  such that  $J_{\lambda}^{R,\bar{\delta}}(\bar{u}) \leq 0$  with  $\|\bar{u}\|_{E_{\lambda}} > \rho$ .

*Proof.* For all  $u \in X_{\lambda}$ , it follows from (17) and (25) with  $||u||_{\lambda}$  small enough that

$$J_{\lambda}^{R,\bar{\delta}}(u) = \frac{1}{2} \int_{\mathbb{R}^{2}} [|\nabla u|^{2} + \lambda V(x)|u|^{2}] dx + \frac{1}{4} V_{0}(u) - \int_{\mathbb{R}^{2}} F^{R,\bar{\delta}}(u) dx$$
  

$$\geq \frac{1}{2} ||u||_{E_{\lambda}}^{2} - \frac{K_{0}}{4} |u|_{\frac{8}{3}}^{3} - \varepsilon |u|_{2}^{2} - C_{\varepsilon}(R)|u|_{2q}^{q}$$
  

$$\geq \left(\frac{1}{2} - \varepsilon C_{0}\right) ||u||_{E_{\lambda}}^{2} - C_{1} ||u||_{E_{\lambda}}^{4} - C_{2} ||u||_{E_{\lambda}}^{q}, \qquad (26)$$

where  $C_i > 0$  with  $i \in \{1, 2\}$  is dependent on R and the imbedding constant for  $E_{\lambda} \hookrightarrow L^p(\mathbb{R}^2)$  for each  $2 \leq p < +\infty$  by Lemma 2.4. Choosing  $\varepsilon > 0$  sufficiently small and q > 4 in (25), there exist A > 0 and  $\rho > 0$  which are independent of  $\lambda \geq 1$  such that  $J_{\lambda}^{R,\overline{\delta}}|_{S_{\lambda}} \geq A$ , where  $S_{\lambda} := \{u \in X_{\lambda} : ||u||_{E_{\lambda}} = \rho\}$ . The closed set  $S_{\lambda}$  disconnects  $X_{\lambda}$  in the two arcwise connected components

$$X_{\lambda}^{1} := \{ u \in X_{\lambda} : \|u\|_{E_{\lambda}} < \rho \} \text{ and } X_{\lambda}^{2} := \{ u \in X_{\lambda} : \|u\|_{E_{\lambda}} > \rho \}.$$

Furthermore, we can conclude that  $0 \in X_{\lambda}^1$  and there exists a  $\bar{u} \in X_{\lambda}^2$  such that  $J_{\lambda}^{R,\bar{\delta}}(\bar{u}) \leq 0$  because  $\lim_{t \to \infty} J_{\lambda}^{R,\bar{\delta}}(tu) = -\infty$  for all  $u \in X_{\lambda} \setminus \{0\}$ . Then, the value

$$c_{\lambda}^{R,\bar{\delta}} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{\lambda}^{R,\bar{\delta}}(\gamma(t)) \ge A > 0$$
(27)

determined by Propositions 3.1 is well-defined, where  $\Gamma_{\lambda}^{R,\bar{\delta}} = \{\gamma \in \mathcal{C}([0,1], X_{\lambda}) : \gamma(0) = 0, \gamma(1) \in X_{\lambda}^2 \text{ and } J_{\lambda}^{R,\bar{\delta}}(\gamma(1)) \leq 0\}.$  The proof is complete.  $\Box$ 

To characterize the mountain-pass level  $c_{\lambda}^{R,\bar{\delta}}$ , we have the following lemma.

**Lemma 3.3.** Let R > 0 be fixed, and  $\lambda \ge 1$ . Then, for every  $u \in X_{\lambda} \setminus \{0\}$ , there exists a unique  $t_u > 0$  such that  $t_u u \in \mathcal{N}_{\lambda}^{R,\overline{\delta}}$ . Moreover, the maximum of  $J_{\lambda}^{R,\overline{\delta}}(tu)$  for  $t \ge 0$  is achieved at  $t = t_u$ .

*Proof.* According to the definition of  $f^{R,\bar{\delta}}$  defined by (13), due to  $(f_2)$ , it simply concludes that

the function 
$$f^{R,\delta}(t)/t^3$$
 is increasing on  $t \in \mathbb{R}^+$ . (28)

Let  $u \in X_{\lambda} \setminus \{0\}$  be fixed, and define the function  $\zeta(t) := J_{\lambda}^{R,\bar{\delta}}(tu)$  for  $t \ge 0$ . It can be easily noticed that  $\zeta'(t) := (J_{\lambda}^{R,\bar{\delta}})'(tu)[u]$  if and only if  $t_u u \in \mathcal{N}_{\lambda}^{R,\bar{\delta}}$ . Moreover,  $\zeta'(t) = 0$  is equivalent to

$$\frac{1}{t^2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] dx + V_0(u) = \int_{\mathbb{R}^2} \frac{f^{R,\delta}(tu)}{(tu)^3} u^4 dx$$

By using (28), the functions on the left and right-hand sides are decreasing and increasing, respectively. Moreover, we compute that

$$\zeta(t) = \frac{1}{t^4} \left( \frac{t^2}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] dx + \frac{1}{4} V_0(u) - \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(tu) dx \right).$$

Using some very similar calculations in Lemma 3.2,  $\zeta(t) > 0$  for t > 0 small. Moreover,  $\zeta(0) = 0$  and  $\zeta(t) = J_{\lambda}^{R,\bar{\delta}}(tu) < 0$  for t > 0 large. Therefore, from the previous conclusions, there exists a unique  $t_u > 0$  such that  $\zeta'(t_u) = 0$ , that is,  $t_u u \in \mathcal{N}_{\lambda}^{R,\bar{\delta}}$ . Furthermore,  $\zeta(t_u) = \max_{t \ge 0} \zeta(t)$ .

Next, we consider the number

$$d_{\lambda}^{R,\bar{\delta}} := \inf_{u \in X_{\lambda} \setminus \{0\}} \max_{t \ge 0} J_{\lambda}^{R,\bar{\delta}}(tu).$$
<sup>(29)</sup>

13

We derive the characterization with respect to  $c_{\lambda}^{R,\bar{\delta}}$ .

**Lemma 3.4.** Let R > 0 be fixed, Then, for every  $\lambda \ge 1$ , there holds  $m_{\lambda}^{R,\bar{\delta}} = c_{\lambda}^{R,\bar{\delta}} = d_{\lambda}^{R,\bar{\delta}}$ , where  $m_{\lambda}^{R,\bar{\delta}}$  and  $c_{\lambda}^{R,\bar{\delta}}$  are defined by (15) and (27), respectively.

*Proof.* The preceding lemma implies that  $m_{\lambda}^{R,\bar{\delta}} = d_{\lambda}^{R,\bar{\delta}}$ . Since  $\zeta(t) = J_{\lambda}^{R,\bar{\delta}}(t_0u) < 0$  for  $u \in X_{\lambda} \setminus \{0\}$  and  $t_0$  large, define  $\gamma_{\lambda}^{R,\bar{\delta}}[0,1] \to X_{\lambda}$  by  $\gamma_{\lambda}^{R,\bar{\delta}}(t) = tt_0u$ . It follows that  $\gamma_{\lambda}^{R,\bar{\delta}} \in \Gamma_{\lambda}^{R,\bar{\delta}}$ , and, consequently,  $c_{\lambda}^{R,\bar{\delta}} \leq d_{\lambda}^{R,\bar{\delta}}$ . Next, we show that  $m_{\lambda}^{R,\bar{\delta}} \leq c_{\lambda}^{R,\bar{\delta}}$ . To end it, it suffices to prove that the manifold  $\mathcal{N}_{\lambda}^{R,\bar{\delta}}$  separates  $X_{\lambda}$  into two components. Proceeding as in Lemma 3.2, we have

$$(J_{\lambda}^{R,\bar{\delta}})'(u)[u] \ge \left(\frac{1}{2} - \varepsilon \bar{C}_0\right) \|u\|_{E_{\lambda}}^2 - \bar{C}_1 \|u\|_{E_{\lambda}}^4 - \bar{C}_2 \|u\|_{E_{\lambda}}^q$$

with  $||u||_{E_{\lambda}}$  small and q > 4. So, there exists  $\bar{\rho} > 0$  independent of  $\lambda \ge 1$ , such that  $(J_{\lambda}^{R,\bar{\delta}})'(u)[u] > 0$  when  $0 < ||u||_{E_{\lambda}} < \bar{\rho}$ . This proves that the component containing the origin also contains a small ball around the origin. Moreover,  $J_{\lambda}^{R,\bar{\delta}}(u) \ge 0$  for all u in this component, because  $(J_{\lambda}^{R,\bar{\delta}})'(tu)[u] \ge 0$  for all  $0 \le t \le t_u$ . Thus,  $\gamma_{\lambda}^{R,\bar{\delta}}(0) = 0$  and  $\gamma_{\lambda}^{R,\bar{\delta}}(1)$  are in different components, which indicates that every path  $\gamma_{\lambda}^{R,\bar{\delta}} \in \Gamma_{\lambda}^{R,\bar{\delta}}$  has to cross  $\mathcal{N}_{\lambda}^{R,\bar{\delta}}$ . Therefore, we must have  $m_{\lambda}^{R,\bar{\delta}} \le c_{\lambda}^{R,\bar{\delta}}$  and  $d_{\lambda}^{R,\bar{\delta}} \le c_{\lambda}^{R,\bar{\delta}}$ , and the lemma is proved.

Now, let us turn to concentrate ourself on the properties for the (C) sequence at the mountain-pass level  $c_{\lambda}^{R,\bar{\delta}}$ .

**Lemma 3.5.** Let R > 0 be fixed, and for all  $\lambda \ge 1$ , there is a constant  $c^{R,\delta} > 0$ independent of  $\lambda \ge 1$  such that  $c_{\lambda}^{R,\delta} \le c^{R,\delta}$  for every  $\lambda \ge 1$ . Moreover, if we additionally suppose  $(h_4)$ , then there exists a  $\xi_0 = \xi_0(R) > 0$  such that, for all  $\xi > \xi_0$ ,

$$c_{\lambda}^{R,2} < \frac{\pi}{4C_{\Xi,b}(1+\gamma+\alpha R^{\tau-2})}, \ \forall \lambda \geq 1,$$

where  $\gamma > 2$  and  $C_{\Xi,b} > 0$  come from  $(h_3)$  and Lemma 2.4, respectively.

*Proof.* Let  $\psi \in C_0^{\infty}(\Omega)$  satisfy  $0 \le \psi \le 1$ . Since  $V(x) \equiv 0$  for all  $x \in \Omega$  by  $(V_2)$ , then

$$J_{\lambda}^{R,\delta}(t\psi) = \frac{t^2}{2} \int_{\Omega} |\nabla\psi|^2 dx + \frac{t^4}{4} V_0(\psi) - \int_{\Omega} F^{R,\delta}(t\psi) dx$$

In view of the proof of Lemma 5.16,  $J_{\lambda}^{R,\delta}(t\psi) > 0$  for t > 0 small and  $J_{\lambda}^{R,\delta}(t\psi) \leq 0$  for t > 0 large, and so the maximum of  $J_{\lambda}^{R,\delta}(t\psi)$  for  $t \geq 0$  is achieved at  $t = t_{\psi} > 0$ , namely  $\max_{t\geq 0} J_{\lambda}^{R,\delta}(t\psi) = J_{\lambda}^{R,\delta}(t_{\psi}\psi)$ . We choose  $c^{R,\delta} = J_{\lambda}^{R,\delta}(t_{\psi}\psi)$  independent of  $\lambda \geq 1$ , and get the desired result by Lemma 29.

Without loss of generality, we could suppose that  $0 \in \Omega$ . Since  $\Omega$  is an open set, there is a constant  $\rho > 0$  such that  $B_{\rho}(0) \subset \Omega$ . Let us assume that  $\rho = 1$  just for the convenience of calculations. Now, we shall choose a  $\varphi_0 \in C_0^{\infty}(B_1(0))$  satisfying  $0 \leq \varphi_0 \leq 1, \varphi_0(x) \equiv 1$  if  $|x| \leq 1/2, \varphi_0(x) \equiv 0$  if  $|x| \geq 1$ , and  $|\nabla \varphi_0| \leq 1$  for all  $x \in \mathbb{R}^2$ . Recalling the definition of  $f^{R,2}$  and  $(h_4), F^{R,2}(t) \geq \xi t^p$  with p > 4 for all  $t \in [0, 1]$ . Thus, using Lemma 2.1-(ii),

$$\begin{aligned} J_{\lambda}^{R,2}(\varphi_{0}) &\leq \frac{1}{2} \int_{B_{1}(0)} |\nabla \varphi_{0}|^{2} dx + \frac{1}{2} \int_{B_{1}(0)} |\varphi_{0}|^{2} dx \int_{B_{1}(0)} \log(1+|x|) |\varphi_{0}|^{2} dx \\ &- \int_{B_{1}(0)} F^{R,2}(\varphi_{0}) dx \\ &\leq \frac{1}{2} (1+\pi \log 2)\pi - \xi \int_{B_{1/2}(0)} |\varphi_{0}|^{p} dx \\ &\leq \frac{1}{2} (1+\pi \log 2)\pi - \frac{\xi_{1}}{4}\pi \\ &= 0. \end{aligned}$$
(30)

where  $\xi_1 = 2(1 + \pi \log 2)$ . In particular, invoking (30), we have that

$$\frac{1}{2} \int_{B_1(0)} [|\nabla \varphi_0|^2 + \lambda V(x)|\varphi_0|^2] dx + \frac{1}{4} V_0(\varphi_0) < \xi_1 \int_{B_{1/2}(0)} |\varphi_0|^p dx.$$
(31)

Defining  $\gamma_0^{R,2}(t) = t\varphi_0$ , one deduces that  $\gamma_0^{R,2} \in \Gamma_{\lambda}^{R,2} = \{\gamma \in \mathcal{C}([0,1], X_{\lambda}) : \gamma(0) = 0, J_{\lambda}^{R,2}(\gamma(1)) < 0\}$  by (31). Therefore, we have that

$$\max_{t \in [0,1]} J_{\lambda}^{R,2}(t\varphi_0) \le \max_{t \in [0,1]} \left\{ \frac{t^2}{2} (1 + \pi \log 2)\pi - \xi t^p \int_{B_{1/2}(0)} |\varphi_0|^p dx \right\}$$
$$\le \max_{t \ge 0} \left\{ \frac{t^2}{2} (1 + \pi \log 2) - \frac{\pi \xi}{4} t^p \right\}$$
$$= \frac{\pi (p-1)(1 + \pi \log 2)}{2p} \left[ \frac{2(1 + \pi \log 2)}{p\xi} \right]^{\frac{2}{p-2}}.$$

As a consequence, we can let the constant  $\xi_0 = \xi_0(R)$  be as follows:

$$\xi_0 = \max\left\{\xi_1, \frac{2(1+\pi\log 2)}{p} \left[\frac{4C_{\Xi,b}(p-1)(1+\pi\log 2)(1+\gamma+\alpha R^{\tau-2})}{2p}\right]^{\frac{p-2}{2}}\right\}.$$

According to the definition of  $c_{\lambda}^{R,2}$ , we conclude it. So, we can accomplish the proof of this lemma.

With Lemma 3.5 in hand, we start verifying the boundedness of the (C) sequence at the level  $c_{\lambda}^{R,\bar{\delta}}$  of  $J_{\lambda}^{R,\bar{\delta}}$ . Specifically, if  $\{u_n\} \subset X_{\lambda}$  is a  $(C)_{c_{\lambda}^{R,\bar{\delta}}}$  sequence of  $J_{\lambda}^{R,\bar{\delta}}$ , we aim at showing  $||u_n||_{X_{\lambda}}$  is uniformly bounded in  $n \in \mathbb{N}$  for all  $\lambda \geq 1$  if R > 0 is fixed. Before proceeding, we introduce the following two lemmas developed by P. L. Lions [37].

**Lemma 3.6.** Let  $\{\rho_n\} \subset L^1(\mathbb{R}^2)$  be a bounded sequence and  $\rho_n \geq 0$ . Then, there exists a subsequence, still denoted by  $\rho_n$ , such that one of the following two possibilities occurs:

- (i) (Vanishing)  $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} \rho_n dx = 0$  for all  $\varrho > 0$ ;
- (ii) (Non-Vanishing) there are  $\beta > 0$  and  $\rho < +\infty$  such that

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_{\varrho}(y)} \rho_n dx = \beta$$

**Lemma 3.7.** Suppose that  $\{u_n\}$  is bounded in  $L^2(\mathbb{R}^2)$ , and  $\{|\nabla u_n|\}$  is bounded in  $L^2(\mathbb{R}^2)$  as well as

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_{\varrho}(y)} |u_n|^2 dx = 0.$$

Then,  $u_n \to 0$  in  $L^s(\mathbb{R}^2)$  for  $s \in (2, +\infty)$ .

**Lemma 3.8.** Let R > 0 be fixed, and for all  $\lambda \ge 1$ , suppose that  $\{u_n\} \subset X_{\lambda}$  is a (C) sequence at the level  $c_{\lambda}^{R,\bar{\delta}}$  of  $J_{\lambda}^{R,\bar{\delta}}$ . If  $(h_4)$  is additionally satisfied for  $\bar{\delta} = 2$ , then  $\{\|u_n\|_{E_{\lambda}}\}$  is uniformly bounded in  $n \in \mathbb{N}$ . Moreover, for all  $\varrho > 0$ ,

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_{\varrho}(y)} |u_n|^2 dx = 0$$
(32)

could never occur.

*Proof.* It follows from (28) that

$$f^{R,\bar{\delta}}(t)t - 4F^{R,\bar{\delta}}(t) \ge 0, \ \forall t \in \mathbb{R}^+$$

From this inequality, we obtain that

$$c_{\lambda}^{R,\bar{\delta}} + o_n(1) \ge J_{\lambda}^{R,\bar{\delta}}(u_n) - \frac{1}{4} (J_{\lambda}^{R,\bar{\delta}})'(u_n)[u_n] \ge \frac{1}{4} \|u_n\|_{E_{\lambda}}^2$$
(33)

showing the first part of this lemma. In view of Lemma 2.4, both  $\{|u_n|_r\}$  and  $\{|\nabla u_n|_2\}$  are uniformly bounded in  $n \in \mathbb{N}$  for some r > 2. Suppose by contradiction, we suppose that (32) holds true. Thus,  $u_n \to 0$  in  $L^s(\mathbb{R}^2)$  for all  $s \in (2, +\infty)$  by Lemma 3.7, which, together with (33) as well as Lemmas 2.4 and 3.5, we have obtained that  $\{u_n\} \subset H^1(\mathbb{R}^2)$  satisfies the assumptions in Lemma 2.6. Hence,

$$||u_n||_{E_{\lambda}}^2 + V_1(u_n) = V_2(u_n) + \int_{\mathbb{R}^2} f^{R,\bar{\delta}}(u_n)u_n dx + o_n(1) = o_n(1)$$

where we have used Lemma 2.1-(v) and (22). By the positivity of  $V_1$ , there holds

$$||u_n||_{E_{\lambda}}^2 = o_n(1) \text{ and } V_1(u_n) = o_n(1)$$

jointly with Lemma 2.1-(v), and (21) again indicates that

$$c_{\lambda}^{R,\bar{\delta}} = \frac{1}{2} \|u_n\|_{E_{\lambda}}^2 + \frac{1}{4} V_1(u_n) - \frac{1}{4} V_2(u_n) - \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx + o_n(1) = o_n(1).$$

Thereby, we arrive at a contradiction by (27). The proof is complete.

Thanks to Lemma 3.6, with the help of Lemma 3.8, we derive the following result which is crucial to prove that the sequence  $\{u_n\} \subset X_{\lambda}$  is uniformly bounded in  $X_{\lambda}$ .

**Lemma 3.9.** Under the assumptions in Lemma 3.8, there exists a constant  $\beta_0 > 0$ , independent of  $\lambda \ge 1$ , such that

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_{\varrho}(y)} |u_n|^2 dx = \beta_0.$$

*Proof.* Letting  $\rho_n = |u_n|^2 \in L^1(\mathbb{R}^2)$ , we know that only the **Non-Vanishing** in Lemma 3.6 occurs because of Lemma 3.8. Then, we divide the proof into intermediate steps.

**Step 1.** There exists a constant  $\beta_{\lambda} = \beta(\lambda) > 0$  such that

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_{\varrho}(y)} |u_n|^2 dx = \beta_{\lambda}$$

Suppose, by contradiction, that  $u_n \to 0$  in  $L^s(\mathbb{R}^2)$  for  $s \in (2, +\infty)$ . It is very similar to the proof of Lemma 3.8, and one derives a contradiction.

Step 2. Conclusion.

Suppose by contradiction that the uniform control from below of the  $L^2(\mathbb{R}^2)$ norm is false. So, for any  $k \in \mathbb{N}, k \neq 0$ , there exist  $\lambda_k > 1$  and a  $(C)_{c_{\lambda_i}^{R,\bar{\delta}}}$  sequence

 $\{u_{k,n}\}$  of  $J_{\lambda}^{R,\overline{\delta}}$  such that

$$|u_{k,n}|_2 < \frac{1}{k}$$
, definitely.

Then, by a diagonalization argument, for any  $k \ge 1$ , we can find an increasing sequence  $\{n_k\}$  in  $\mathbb{N}$  and  $u_{n_k} \in X_{\lambda_{n_k}}$  such that

$$J_{\lambda_{n_k}}^{R,\bar{\delta}}(u_{n_k}) = c_{\lambda_{n_k}}^{R,\bar{\delta}} + o_k(1), \ (1 + \|u_{n_k}\|_{X_{\lambda_{n_k}}}) \|(J_{\lambda_{n_k}}^{R,\bar{\delta}})'(u_{n_k})\|_{X_{\lambda_{n_k}}^{-1}} = o_k(1), \ |u_{n_k}|_2 = o_k(1),$$

where  $o_k(1)$  is a positive quantity which goes to zero as  $k \to +\infty$ . In this situation, we can repeat the proof of Lemma 3.8 to reach a contradiction, again. The proof of this lemma is finished.

Now, we can prove that the sequence  $\{u_n\} \subset X_{\lambda}$  in Lemma 3.8 is uniformly bounded in  $X_{\lambda}$  for some sufficiently large  $\lambda > 0$ .

**Lemma 3.10.** Let R > 0 be fixed, and suppose that  $\{u_n\} \subset X_{\lambda}$  is a (C) sequence at the level  $c_{\lambda}^{R,\bar{\delta}}$  of  $J_{\lambda}^{R,\bar{\delta}}$ . Moreover, we shall additionally suppose  $(h_4)$ , whence  $\bar{\delta} = 2$ . Then, there is a  $\lambda_0 = \lambda_0(R) > 1$  ( $\lambda'_0 = \lambda'_0(R) > 1$  for  $\bar{\delta} = 2$ ) such that the sequence  $\{||u_n||_{X_{\lambda}}\}$  is uniformly bounded in  $n \in \mathbb{N}$ , and  $\lambda > \lambda_0$  (or  $\lambda > \lambda'_0$  for  $\bar{\delta} = 2$ ).

*Proof.* Combining Lemmas 3.8 and 3.9, there exists a constant  $\beta_0 > 0$ , independent of  $\lambda \ge 1$ , such that

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} |u_n|^2 dx = \beta_0,$$

where we have supposed that  $\rho = 1$  in Lemma 3.9. Up to a subsequence if necessary, there exists a sequence  $\{y_n\} \subset \mathbb{R}^2$  such that

$$\int_{B_1(y_n)} |u_n|^2 dx = \frac{1}{2}\beta_0.$$
(34)

17

We claim that  $\{y_n\}$  is uniformly bounded in  $n \in \mathbb{N}$ . Otherwise, we could suppose that  $|y_n| \to \infty$  in the sense of a subsequence. Define

$$\Xi_n^1 := \{ x \in B_1(y_n) : V(x) < b \} \text{ and } \Xi_n^2 := \{ x \in B_1(y_n) : V(x) \ge b \}.$$

Since the set  $\Xi := \{x \in \mathbb{R}^2 : V(x) < b\}$  is nonempty and has finite measure, one concludes that

$$\max(\Xi_n^1) \le \max(\{x \in \mathbb{R}^2 : |x| \ge |y_n| - 2, V(x) < b\}) \to 0 \text{ as } n \to \infty.$$
(35)

In view of Lemma 3.8,  $|u_n|_r$  with r > 2 is uniformly bounded in  $n \in \mathbb{N}$ . Then, using (35),

$$\int_{\Xi_n^1} |u_n|^2 dx \le [\max(\Xi_n^1)]^{\frac{r-2}{r}} |u_n|_r^2 = o_n(1)$$

which reveals that

$$\int_{\Xi_n^2} |u_n|^2 dx = \int_{B_1(y_n)} |u_n|^2 dx - \int_{\Xi_n^1} |u_n|^2 dx = \frac{1}{2}\beta_0 + o_n(1).$$

Thanks to  $V(x) \geq 0$  for all  $x \in \mathbb{R}^2$  by  $(V_1)$ , using the definition of  $\Xi_n^2$ ,

$$\int_{\mathbb{R}^2} V(x) |u_n|^2 dx \ge \int_{\Xi_n^2} V(x) |u_n|^2 dx \ge b \int_{\Xi_n^2} |u_n|^2 dx = \frac{1}{2} b\beta_0 + o_n(1).$$
(36)

Besides, in view of the proof of Lemma 3.8 again, we have that

$$\{V_2(u_n)\}\$$
 and  $\left\{\int_{\mathbb{R}^2} F^{R,\overline{\delta}}(u_n)dx\right\}$  are uniformly bounded in  $n \in \mathbb{N}$  and  $\lambda \ge 1$ . (37)  
So, combining (36) and (37), we derive

So, ombining (36) and (37),

$$c_{\lambda}^{R,\bar{\delta}} \geq \frac{1}{2} \int_{\mathbb{R}^2} \lambda V(x) |u_n|^2 dx - \frac{1}{4} V_2(u_n) - \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx + o_n(1)$$
  
$$\geq \frac{\lambda b \beta_0}{4} - C + o_n(1), \tag{38}$$

where the positive constants  $b, \beta_0$ , and C are independent of  $\lambda \ge 1$ . Using Lemma 3.5, there exists a sufficiently large  $\lambda_0 = \lambda_0(R) > 1$  ( $\lambda'_0 = \lambda'_0(R) > 1$  for  $\bar{\delta} = 2$ ) such that (38) is false provided  $\lambda > \lambda_0$ . Hence, the sequence  $\{y_n\} \subset \mathbb{R}^2$  appearing in (34) is uniformly bounded in  $n \in \mathbb{N}$ .

Consequently, passing to a subsequently if necessary, we suppose that  $y_n \to y_0$ in  $\mathbb{R}^2$ . Taking (34) into account, there holds

$$\int_{B_2(y_0)} |u_n|^2 dx \ge \frac{1}{4}\beta_0 > 0.$$
(39)

Next, we shall investigate that  $|u_n|_* = (\int_{\mathbb{R}^2} \log(1+|x|) u_n^2 dx)^{\frac{1}{2}}$  is uniformly bounded in  $n \in \mathbb{N}$ . Let us choose a constant  $\delta > 0$  large enough to satisfy  $\delta > |y_0| + 2$ . Moreover, one has

$$1+|x-y| \ge 1+\frac{|y|}{2} \ge \sqrt{1+|y|}, \ \forall x \in B_{\delta}(0), \ \forall y \in \mathbb{R}^2 \setminus B_{2\delta}(0).$$

Due to this choice for  $\delta$  implying that  $B_2(y_0) \subset B_{\delta}(0)$ , by means of (39),

$$V_1(u_n) = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \log(1+|x-y|) u_n^2(x) dx \right) u_n^2(y) dy$$
$$\geq \int_{\mathbb{R}^2 \setminus B_{2\delta}(0)} \left( \int_{B_{\delta}(0)} \log(1+|x-y|) u_n^2(x) dx \right) u_n^2(y) dy$$

LIEJUN SHEN AND MARCO SQUASSINA

$$\geq \left( \int_{B_{\delta}(0)} u_{n}^{2}(x) dx \right) \left[ \int_{\mathbb{R}^{2} \setminus B_{2\delta}(0)} \log\left(1 + \frac{|y|}{2}\right) u_{n}^{2}(y) dy \right]$$

$$\geq \frac{\beta_{0}}{8} \int_{\mathbb{R}^{2} \setminus B_{2\delta}(0)} \log(1 + |y|) u_{n}^{2}(y) dy$$

$$= \frac{\beta_{0}}{8} \left( \|u_{n}\|_{*}^{2} - \int_{B_{2\delta}(0)} \log(1 + |y|) u_{n}^{2}(y) dy \right)$$

$$\geq \frac{\beta_{0}}{8} (\|u_{n}\|_{*}^{2} - \log(1 + 2\delta) |u_{n}|_{2}^{2}).$$

$$(40)$$

Since we have proved that  $|u_n|_2$  is uniformly bounded in  $n \in \mathbb{N}$  for all  $\lambda > \lambda_0$ , with (40), it suffices to show that  $\{V_1(u_n)\}$  is uniformly bounded in  $n \in \mathbb{N}$  for all  $\lambda > \lambda_0$ . In fact, adopting (37),

$$0 \le V_1(u_n) \le 4c_{\lambda}^{R,\bar{\delta}} + V_2(u_n) + 4\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n)dx + o_n(1)$$

finishing the proof of this lemma.

We are in a position to present the proof of Theorem 1.2.

Proof of Theorem 1.2. Combining Proposition 3.1 and Lemma 3.2, for every fixed R > 0, the variational functional  $J_{\lambda}^{R,\bar{\delta}}$  admits a (C) sequence  $\{u_n\} \subset X_{\lambda}$  at the level  $c_{\lambda}^{R,\bar{\delta}}$  for all  $\lambda \geq 1$ . With the help of Lemma 3.10,  $\{u_n\}$  is bounded in  $X_{\lambda}$  whenever  $\lambda \geq \lambda_0$ , where  $\lambda_0 > 0$  depending on R > 0 is determined by Lemma 3.10. Passing to a subsequence if necessary, there is a  $u \in X_{\lambda}$  such that  $u_n \to u$  in  $X_{\lambda}$ ,  $u_n \to u$  in  $L^s(\mathbb{R}^2)$  for every  $2 \leq s < \infty$  by Lemma 2.1-(i), and  $u_n \to u$  a.e. in  $\mathbb{R}^2$  as  $n \to \infty$ . So, the remaining part is to verify that  $u_n \to u$  in  $X_{\lambda}$  by (27) and Lemma 3.4. Moreover,  $u \geq 0$  by  $(h_1)$ , and we omit the details.

It follows from Lemma 2.1-(ii) and (iii) as well as Lemma 2.2 that

$$V_{1}'(u_{n})[u_{n} - u] = 4B_{1}(u_{n}^{2}, u_{n}(u_{n} - u))$$

$$= 4B_{1}(u_{n}^{2}, (u_{n} - u)^{2}) + 4B_{1}(u_{n}^{2}, u(u_{n} - u))$$

$$= 4B_{1}(u_{n}^{2}, (u_{n} - u)^{2}) + o_{n}(1)$$

$$\leq 8|u_{n} - u|_{2}||u_{n}||_{*}^{2} + o_{n}(1)$$

$$= o_{n}(1).$$
(41)

Using Lemma 2.1-(i) and (iv),

$$V_{2}'(u_{n})[u_{n}-u]| = 4|B_{2}(u_{n}^{2}, u_{n}(u_{n}-u))||$$

$$\leq 4|B_{2}(u_{n}^{2}, (u_{n}-u)^{2})| + 4|B_{2}(u_{n}^{2}, u(u_{n}-u))||$$

$$= 4K_{0}|u_{n}^{2}|_{\frac{4}{3}}|(u_{n}-u)^{2}|_{\frac{4}{3}} + 4K_{0}|u_{n}^{2}|_{\frac{4}{3}}|u(u_{n}-u)|_{\frac{4}{3}}$$

$$\leq 4K_{0}|u_{n}|_{\frac{8}{3}}^{2}|u_{n}-u|_{\frac{8}{3}}^{2} + 4K_{0}|u_{n}|_{\frac{8}{3}}^{2}|u|_{\frac{8}{3}}|u_{n}-u|_{\frac{8}{3}}$$

$$= o_{n}(1). \qquad (42)$$

Using the same arguments in Lemma 3.8, one could verify  $\{u_n\}$  satisfies all of the assumptions in Lemmas 2.6, and so

$$\int_{\mathbb{R}^2} f^{R,\overline{\delta}}(u_n)(u_n-u)dx = o_n(1).$$
(43)

As a consequence of (41), (42), and (43), we obtain

$$o_n(1) = (J_{\lambda}^{R,\delta})'(u_n)[u_n - u]$$
  
=  $\int_{\mathbb{R}^2} \left[ \nabla u_n \nabla (u_n - u) + \lambda V(x) u_n (u_n - u) \right] dx + o_n(1)$   
=  $\|u_n\|_{E_{\lambda}}^2 - \|u\|_{E_{\lambda}}^2 + o_n(1)$   
=  $\|u_n - u\|_{E_{\lambda}}^2 + o_n(1).$ 

Next, we deduce that  $||u_n - u||_* = o_n(1)$  and then we are done. Indeed, proceeding exactly as (40), one has

$$o_n(1) = B_1(u_n^2, (u_n - u)^2) \ge \frac{\beta_0}{8} (||u_n - u||_*^2 - \log(1 + 2\delta)|u_n - u|_2^2)$$

yielding the desired result. The proof is complete.

4. Proofs of Theorems 1.1 and 1.4. In this section, following our discussions in the introduction, we must take the uniform  $L^{\infty}$ -estimate for the nontrivial solution obtained in Theorem 1.2. Let  $u_R \in X_{\lambda}$  be a ground state solution associated with Eq. (14), according to the definition of  $f^{R,\bar{\delta}}$ , which is defined as in (10). Then, it is a ground state solution for Eq. (1) provided  $|u_R|_{\infty} \leq R$ . So, the key idea is to find a constant  $C_0 > 0$  which is independent of R > 0 satisfying  $|u_R|_{\infty} \leq C_0$ . Therefore, we define R to equal to such a constant  $C_0$ . With this in mind, we derive our direction in this section.

To the aim, we must first prove that the constants  $\lambda_0(R), \lambda'_0(R)$ , and  $\xi_0(R)$ appearing in Theorem 1.2 do not depend on R. Let us recall the two cases in Theorem 1.2. To proceed clearly, we split it into two subsections: (I)  $\delta = \delta \in (0, 2)$ , and (II)  $\bar{\delta} = 2$ .

In Cases (I) and (II), we shall choose  $\alpha^* = \frac{1}{R^{\tau-\delta}} > 0$  and  $\tau_* = 2 + \frac{1}{R} > 0$ , respectively.

4.1. The Case (I) in (10):  $\bar{\delta} = \delta \in (0, 2)$ . In this subsection, we shall suppose that

V satisfies  $(V_1) - (V_3)$ , and the nonlinearity f defined in (2) requires  $(h_1) - (h_3)$ . With the choice of  $\alpha^* = \frac{1}{R^{\tau-\delta}} > 0$  in this subsection, we improve (23) in the sense that, for each  $\varepsilon > 0$ , there is a constant  $C_{\varepsilon} > 0$  independent of R > 0 such that

$$|f^{R,\delta}(t)| \le \varepsilon |t| + C_{\varepsilon} |t|^{q-1} [e^{(K+1)^{-1}|t|^2} - 1], \ \forall t \in \mathbb{R},$$
(44)

where q > 1 is arbitrary and K > 0 is independent of R > 0, which is determined later. Via the two facts in the proof of Lemma 2.6, for all  $\alpha \in (0, \alpha^*)$  and  $|t| \ge 1$ , there are  $M_2 \ge M_1 \ge M$  independent of R > 0 such that

$$|f^{R,\delta}(t)| \leq M(e^{\gamma|t|^{\delta}} - 1)e^{\alpha^* R^{\tau-\delta}|t|^{\delta}}$$
  
$$\leq Me^{(\gamma+1)|t|^{\delta}}$$
  
$$\leq M_1 e^{(K+1)^{-1}|t|^2}$$
  
$$\leq M_2 [e^{(K+1)^{-1}|t|^2} - 1]$$
  
$$\leq M_2 |t|^{q-1} [e^{(K+1)^{-1}|t|^2} - 1]$$

jointly with  $f^{R,\delta}(t) = o(t)$  uniformly in R as  $t \to 0$ , we have (44) at once.

**Lemma 4.1.** If  $\lambda \geq 1$ , and we let  $\{u_n\} \subset X_{\lambda}$  be a (C) sequence of  $J_{\lambda}^{R,\delta}$  at the level  $c_{\lambda}^{R,\delta}$ , then  $\{u_n\}$  is is uniformly bounded in  $n \in \mathbb{N}$  and R > 0, that is, there is a constant K > 0 independent of  $n \in \mathbb{N}$  and R > 0 such that

$$\sup_{n \in \mathbb{N}} \|u_n\|_{H^1(\mathbb{R}^2)}^2 < \frac{2\pi K}{q} < +\infty.$$
(45)

*Proof.* We claim that there are constants  $A_0 > 0$  and c > 0 independent of R > 0 and  $\lambda \ge 1$  such that

$$A_0 < c_{\lambda}^{R,\delta} \le c < +\infty, \ \forall R > 0 \ \text{and} \ \lambda \ge 1.$$
(46)

Indeed, recalling the definition of  $f^{R,\delta}$ , one has that  $F^{R,\delta}(t) \ge H(t)$  for all  $t \in \mathbb{R}$ , and so  $J_{\lambda}^{R,\delta}(u) \ge I_{\lambda}(u)$  for all  $u \in X_{\lambda}$ , where the variational functional  $I_{\lambda} : X_{\lambda} \to \mathbb{R}$ is defined by

$$I_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] dx + \frac{1}{4} V_0(u) - \int_{\mathbb{R}^2} H(u) dx.$$
(47)

Let us choose the constant  $c_{\lambda} > 0$  to be a mountain-pass level associated with  $I_{\lambda}$ , the existence of such number follows Lemma 3.2. In light of  $\psi$  as in Lemma 3.5, one has

$$I_{\lambda}(t\psi) = \frac{t^2}{2} \int_{\Omega} |\nabla \psi|^2 dx + \frac{t^4}{4} V_0(\psi) - \int_{\Omega} H(t\psi) dx.$$

Then, using the same arguments in Lemma 3.5, we could find a c > 0 independent of  $\lambda \ge 1$  such that  $c_{\lambda} \le c$ . Besides, by exploiting (44), we proceed as with (26) to derive such a constant  $A_0 > 0$  satisfying  $c_{\lambda}^{R,\delta} \ge A_0$ . So, (46) holds true. Combining (33) and (46) as well as Lemma 2.4, we would receive the desired result (45). The proof is complete.

**Remark 4.2.** In view of (38), due to (46), one can see that we can determine the constant  $\lambda_0(R)$  to be independent of R > 0. Moreover, by using (44) and (45), one would conclude that (21) holds independently with respect to R > 0.

**Lemma 4.3.** Let  $u_R \in X_{\lambda}$  be a nonnegative solution of Eq. (14) with  $\overline{\delta} = \delta$  established by Theorem 1.2 for all fixed R > 0. If  $\alpha^* = \frac{1}{R^{\tau-\delta}} > 0$ , then for all  $\alpha \in (0, \alpha^*)$  and  $\tau > 2$ , we have

$$|u_R|_{\infty} \le C_0, \ \forall R > 0,$$

for some  $C_0 > 0$  independent of R > 0 and  $\lambda > \lambda_0(R)$ .

*Proof.* In view of Section 3, we know that the nonnegative solution  $u_R \neq 0$  of Eq. (14) is established by looking for the weak limit of  $\{u_n\} \subset X_\lambda$ , which is a (C) sequence of  $J_\lambda^{R,\delta}$  at the level  $c_\lambda^{R,\delta}$ . So, combining the Fatou's lemma and (45),  $||u_R||_{H^1(\mathbb{R}^2)}^2 \leq \frac{2\pi K}{q}$  for all R > 0, where K > 0 is a constant independent of R > 0. We claim that there is a constant  $C_1 > 0$  independent of R > 0

$$|f^{R,\delta}(u_R)|_2 \le C_1 < +\infty.$$
 (48)

In fact, due to (44), by applying (12), then it suffices to show that

$$\int_{\mathbb{R}^2} |u_R|^{2(q-1)} [e^{2(K+1)^{-1}|u_R|^2} - 1] dx$$
  
$$\leq \left( \int_{\mathbb{R}^2} |u_R|^{2q} dx \right)^{\frac{q-1}{q}} \left( \int_{\mathbb{R}^2} [e^{2q(K+1)^{-1}|u_R|^2} - 1] dx \right)^{\frac{1}{q}}$$

$$= \left(\int_{\mathbb{R}^2} |u_R|^{2q} dx\right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^2} \left[e^{\frac{2q ||u_R||^2_{H^1(\mathbb{R}^2)}}{K+1} \left(|u_R|/||u_R||_{H^1(\mathbb{R}^2)}\right)^2} - 1\right] dx\right)^{\frac{1}{q}} \le C$$

for some C > 0 independent of R > 0.

Then, proceeding as with the proof of [5, Lemma 4.10], we shall show that  $|u_R|_{\infty} \leq C_0$ , where  $C_0 > 0$  is independent of R > 0. Taking the convenience of the reader into account, it should be done in detail. Since  $u_R$  is a nonnegative solution of Eq. (14), by  $(V_1)$ ,  $u_R$  must satisfy the inequality

$$-\Delta u_R + u_R \le [1 + g_{u_R}(x)]u_R + f^{R,\delta}(u_R), \ x \in \mathbb{R}^2,$$

where  $g_{u_R}(x)$  is defined as in Lemma 2.1. Set  $\Upsilon := 1 + |g_{u_R}|_{\infty}$ . Then,  $\Upsilon \in (1, +\infty)$  does not depend on R > 0 by Lemma 2.1 and (45). From this and (48), there is a  $v_R \in H^2(\mathbb{R}^2)$  such that

$$-\Delta v_R + v_R = \Upsilon u_R + f^{R,\delta}(u_R)$$
 in  $\mathbb{R}^2$ .

Next, we fix the test function

$$z_r(x) = \phi(x/r)(u_R - v_R)^+(x) \in X_\lambda,$$

where  $\phi \in C_0^{\infty}(\mathbb{R}^2)$  satisfies

 $0 \leq \phi(x) \leq 1 \quad \forall x \in \mathbb{R}^2, \quad \phi(x) = 1 \quad \forall x \in B_1(0) \text{ and } \phi(x) = 0 \quad \forall x \in \mathbb{R}^2 \setminus B_2(0).$ Using the function test  $z_r$  on  $-\Delta(u_R - v_R) + (u_R - v_R) \leq 0$  in  $\mathbb{R}^2$ , we get the inequality below

$$\int_{\mathbb{R}^2} \left[ \nabla (u_R - v_R) \nabla z_r + (u_R - v_R) z_r \right] dx \le 0, \quad \forall r > 0.$$

Since

$$z_r \to (u_R - v_R)^+$$
 as  $r \to +\infty$  in  $H^1(\mathbb{R}^2)$ ,

by Lebesgue's dominated convergence theorem, we arrive at

$$\int_{\mathbb{R}^2} |\nabla (u_R - v_R)^+|^2 + |(u_R - v_R)^+|^2 \, dx \le 0,$$

implying that

$$0 \le u_R(x) \le v_R(x), \quad \forall x \in \mathbb{R}^2.$$

By using the continuous Sobolev embedding  $H^2(\mathbb{R}^2) \hookrightarrow L^{\infty}(\mathbb{R}^2)$ , there is  $C_2 > 0$ independent of R > 0 such that

$$|v_R|_{\infty} \le C_2 ||v_R||_{H^2(\mathbb{R}^2)}, \quad \forall R > 0$$

which, together with the last fact, gives that

$$|u_R|_{\infty} \le C_3 ||v_R||_{H^2(\mathbb{R}^2)}, \quad \forall R > 0$$

On the other hand, by Brézis [15, Theorem 9.25], there is  $C_4 > 0$  independent of R > 0 such that

$$||v_R||_{H^2(\mathbb{R}^2)} \le C_4 |\Upsilon u_R + f^{R,\delta}(u_R)|_2, \quad \forall R > 0,$$

where from (45) as well as (48), it follows that

$$\|v_R\|_{H^2(\mathbb{R}^2)} \le C_5, \quad \forall R > 0$$

for some  $C_5 > 0$  independent of R > 0. With this in mind, we must have

$$|u_R|_{\infty} \le C_0, \quad \forall R > 0$$

for some  $C_0 > 0$  independent of R > 0, showing the desired result.

4.2. The Case (II) in (10):  $\overline{\delta} = 2$ . In this subsection, we shall suppose that V satisfies  $(V_1) - (V_3)$  and the nonlinearity f defined in (2) requires  $(h_1) - (h_3)$  as well as  $(h_4)$ .

With the choice of  $\tau_* = 2 + \frac{1}{R} > 0$  in this subsection, we improve (24) in the sense that, for all  $\varepsilon > 0$ , there is a constant  $C'_{\varepsilon} > 0$  independent of R > e such that

$$|f^{R,2}(t)| \le \varepsilon |t| + C'_{\varepsilon} |t|^{q-1} [e^{(\gamma + \alpha e^{\frac{1}{e}})|t|^2} - 1], \ \forall t \in \mathbb{R}.$$
(49)

First, one can observe that  $\lim_{R \to +\infty} R^{\frac{1}{R}} = 1$ , and the function  $R^{\frac{1}{R}}$  is strictly decreasing in  $R \in (e, +\infty)$ . Then,  $0 < R^{\frac{1}{R}} \le e^{\frac{1}{e}}$  for each  $R \in (e, +\infty)$ . For all  $|t| \ge 1$ , by using  $(h_3)$ , there is  $M_1 > M$  independent of R > e such that

$$|f^{R,2}(t)| \leq M(e^{\gamma|t|^{\delta}} - 1)e^{\alpha R^{\tau^* - 2}|t|^2}$$
  
=  $Me^{\gamma|t|^{\delta}}e^{\alpha R^{\frac{1}{R}}|t|^2}$   
 $\leq Me^{\gamma|t|^{\delta}}e^{\alpha e^{\frac{1}{e}}|t|^2}$   
 $\leq M_1[e^{\gamma|t|^2} - 1][e^{\alpha e^{\frac{1}{e}}|t|^2} - 1]$   
 $\leq M_1[e^{(\gamma + \alpha e^{\frac{1}{e}})|t|^2} - 1]$   
 $\leq M_1|t|^{q-1}[e^{(\gamma + \alpha e^{\frac{1}{e}})|t|^2} - 1]$ 

together with  $f^{R,2}(t) = o(t)$  uniformly in R > 0 as  $t \to 0$ .

**Lemma 4.4.** If  $\lambda \ge 1$ , there are some constants  $\bar{A}_0 > 0$  and  $\xi_0 > 0$  independent of R > e such that, for all  $\xi > \xi_0$ , there holds

$$\bar{A}_0 \le c_{\lambda}^{R,2} < \frac{\pi}{4C_{\Xi,b}(1+\gamma+\alpha e^{\frac{1}{e}})}, \ \forall R > e.$$

$$(50)$$

*Proof.* Applying (49) to (26), one can find such  $\bar{A}_0 > 0$ , and the details are omitted. By the definition of  $f^{R,2}$ , then  $c_{\lambda}^{R,2} \leq c_{\lambda}$ , where  $c_{\lambda}$  is a mountain-pass level corresponding to the variational functional  $I_{\lambda}$  defined by (47). Let  $\varphi_0$  be as in Lemma 3.5. Then, according to  $(h_3)$  and Lemma 2.1-(ii),

$$\begin{split} I_{\lambda}(\varphi_{0}) &\leq \frac{1}{2} \int_{B_{1}(0)} |\nabla \varphi_{0}|^{2} dx \\ &+ \frac{1}{2} \int_{B_{1}(0)} |\varphi_{0}|^{2} dx \int_{B_{1}(0)} \log(1+|x|) |\varphi_{0}|^{2} dx - \int_{B_{1}(0)} H(\varphi_{0}) dx \\ &< \frac{1}{2} (1+\pi \log 2)\pi - \xi \int_{B_{1/2}(0)} |\varphi_{0}|^{p} dx \\ &\leq \frac{1}{2} (1+\pi \log 2)\pi - \frac{\xi_{1}}{4}\pi \\ &= 0, \end{split}$$
(51)

where  $\xi_1 = 2(1 + \pi \log 2)$ . In particular, invoking (51), we have that

$$\frac{1}{2} \int_{B_1(0)} \left[ |\nabla \varphi_0|^2 + \lambda V(x) |\varphi_0|^2 \right] dx + \frac{1}{4} V_0(\varphi_0) < \xi_1 \int_{B_{1/2}(0)} |\varphi_0|^p dx.$$
(52)

Defining  $\gamma_0(t) = t\varphi_0$ , one deduces that  $\gamma_0 \in \Gamma_{\lambda} = \{\gamma \in \mathcal{C}([0,1], X_{\lambda}) : \gamma(0) = 0, I_{\lambda}(\gamma(1)) < 0\}$  by (31). Therefore, we have that

$$\max_{t \in [0,1]} I_{\lambda}(t) \le \max_{t \in [0,1]} \left\{ \frac{t^2}{2} (1 + \pi \log 2) \pi - \xi t^p \int_{B_{1/2}(0)} |\varphi_0|^p dx \right\}$$
$$\le \max_{t \ge 0} \left\{ \frac{t^2}{2} (1 + \pi \log 2) - \frac{\pi \xi}{4} t^p \right\}$$
$$= \frac{\pi (p-1)(1 + \pi \log 2)}{2p} \left[ \frac{2(1 + \pi \log 2)}{p\xi} \right]^{\frac{2}{p-2}}$$

As a consequence, the constant  $\xi_0$  independent of R > e can be chosen as

$$\xi_0 = \max\left\{\xi_1, \frac{2(1+\pi\log 2)}{p} \left[\frac{4C_{\Xi,b}(p-1)(1+\pi\log 2)(1+\gamma+\alpha e^{\frac{1}{e}})}{2p}\right]^{\frac{p-2}{2}}\right\}$$

showing that  $c_{\lambda} < \frac{\pi}{4C_{\Xi,b}(1+\gamma+\alpha e^{\frac{1}{e}})}$  for all R > e. By the fact  $c_{\lambda}^{R,2} \leq c_{\lambda}$  for every R > 0, we get the desired result immediately. The proof is completed.  $\Box$ 

**Remark 4.5.** With Lemma 4.4 in hand, for all  $\lambda \geq 1$ , we obtain that

$$\limsup_{n \to \infty} \|u_n\|_{H^1(\mathbb{R}^2)}^2 < \frac{\pi}{\gamma + \alpha e^{\frac{1}{e}}} \ \forall R > e, \tag{53}$$

where  $\{u_n\} \subset X_\lambda$  is a (C) sequence of  $J_\lambda^{R,2}$  at the level  $c_\lambda^{R,2}$ . Indeed, using (33) and (50) together with Lemma 2.4, it is obvious. With the help of (38) and (50), we can also conclude that the constant  $\lambda'_0 > 0$  is independent of R > e. Moreover, it follows from (49) and (53) that (22) holds independently with respect to R > e.

**Lemma 4.6.** Let  $u_R \in X_{\lambda}$  be a nonnegative solution of Eq. (14) with  $\overline{\delta} = 2$  established by Theorem 1.2 for all fixed R > e. If  $\tau_* = 2 + \frac{1}{R} > 0$ , then for all  $\alpha > 0$  and  $\tau \in [2, \tau_*)$ , we have

 $|u_R|_{\infty} \le C'_0, \ \forall R > e.$ 

for some  $C'_0 > 0$  independent of R > e and  $\lambda > \lambda'_0(R)$ .

*Proof.* In light of Lemma 4.3, it suffices to prove that

$$|f^{R,2}(u_R)|_2 \le C_1' \tag{54}$$

for some  $C'_1 > 0$  independent of R > e. First, due to the Fatou's lemma,  $||u_R||^2_{H^1(\mathbb{R}^2)} < \frac{\pi}{\gamma + \alpha e^{\frac{1}{e}}}$  by (53) for all R > e. Hence, using (49) with q = 4, we exploit (12) to have

$$\begin{split} \int_{\mathbb{R}^2} |f^{R,2}(u_R)|^2 dx &\leq \int_{\mathbb{R}^2} |u_R|^2 dx + C' \int_{\mathbb{R}^2} |u_R|^{2(q-1)} [e^{(\gamma + \alpha e^{\frac{1}{e}})|u_R|^2} - 1] dx \\ &\leq \int_{\mathbb{R}^2} |u_R|^2 dx + C' \bigg( \int_{\mathbb{R}^2} |u_R|^8 dx \bigg)^{\frac{3}{4}} \\ &\qquad \times \bigg( \int_{\mathbb{R}^2} [e^{4(\gamma + \alpha e^{\frac{1}{e}})||u_R||^2_{H^1(\mathbb{R}^2)} (|u_R|/||u_R||_{H^1(\mathbb{R}^2)})^2} - 1] dx \bigg)^{\frac{1}{4}} \\ &\leq C'_1 \end{split}$$

as required.

Now, we can present the proof of Theorem 1.1 below.

Proof of Theorem 1.1. Recalling Theorem 1.2, we have established a nonnegative ground state solution for Eq. (14) under the suitable assumptions. Let us denote the obtained ground state solution by  $u_R$ . Thanks to the explanations in Remarks 4.2 and 4.5, the constants  $\lambda_0$  for  $\bar{\delta} = \delta \in (0, 2)$  and  $\lambda'_0$  and  $\xi_0$  for  $\bar{\delta} = 2$  are independent of R > 0 and R > e, respectively. It follows from Lemmas 4.4 and 4.6 that we can choose  $R = C_0$  and  $R = \max\{C'_0, e\}$  for  $\bar{\delta} = \delta \in (0, 2)$  and  $\bar{\delta} = 2$ , respectively. In this situation,  $\alpha^* = \frac{1}{C_0^{\tau-\delta}}$  and  $\tau_* = 2 + \frac{1}{\max\{C'_0, e\}}$ . So,  $u_R$  is a nonnegative ground state solution of Eq. (1). The proof is completed.

Finally, we are concerned with the asymptotical behavior of ground state solutions of Eq. (1) obtained in Theorem 1.1 as  $\lambda \to +\infty$ .

Before showing the proof of Theorem 1.4, via the same constant R > 0 determined in the proof of Theorem 1.2, we need the variational functionals below:

$$\begin{cases} J_{\Omega}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4} V_0|_{\Omega}(u) - \int_{\Omega} F(u) dx, & \forall u \in H_0^1(\Omega), \\ J_{\Omega}^{R,\bar{\delta}}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4} V_0|_{\Omega}(u) - \int_{\Omega} F^{R,\bar{\delta}}(u) dx, & \forall u \in H_0^1(\Omega), \end{cases}$$

where the functional  $V_0|_{\Omega}: H_0^1(\Omega) \to \mathbb{R}$  is defined by  $V_0|_{\Omega} = V_1|_{\Omega} - V_2|_{\Omega}$  with

$$V_1|_{\Omega}(u) := \int_{\Omega} \int_{\Omega} \log(1 + |x - y|) u^2(x) u^2(y) dx dy, \ \forall u \in H_0^1(\Omega).$$

and

$$V_2|_{\Omega}(u) := \int_{\Omega} \int_{\Omega} \log\left(1 + \frac{1}{|x-y|}\right) u^2(x) u^2(y) dx dy, \ \forall u \in H^1_0(\Omega),$$

Since meas( $\Omega$ ) < + $\infty$ , there is a constant  $\rho > 0$  such that  $\Omega \subset B_{\rho}(0)$ , and so

$$0 \le \log(1 + |x - y|) \le \log(1 + 2\varrho), \ \forall x, y \in \Omega$$

indicating that  $V_1|_{\Omega}$  is well-defined and of class of  $C^1$  in  $H_0^1(\Omega)$  endowed with its usual norm.

Moreover, we define the ground state  $c_0$  associated with (60) by

$$m_{\Omega} := \inf_{u \in \mathcal{N}_{\Omega}^{R,\bar{\delta}}} J_{\Omega}^{R,\bar{\delta}}(u), \text{ where } \mathcal{N}_{\Omega}^{R,\bar{\delta}} = \{u \in H_0^1(\Omega) \setminus \{0\} : u \neq 0, \ (J_{\Omega}^{R,\bar{\delta}})'(u)[u] = 0\}.$$

Now, we are ready to prove Theorem 1.4 as follows.

Proof of Theorem 1.4. Let  $u_{\lambda} \in X_{\lambda}$  be a ground state solution for Eq. (1), choosing a subsequence  $\lambda_n \to +\infty$  as  $n \to \infty$ , we denoted  $\{u_{\lambda_n}\}$  by the subsequence of  $\{u_{\lambda}\}$ . In view of the proof of Theorem 1.1, we know that

$$\sup_{n \in \mathbb{N}} \|u_{\lambda_n}\|_{H^1(\mathbb{R}^2)}^2 < \frac{2\pi K}{q} \text{ and } \sup_{n \in \mathbb{N}} |u_{\lambda_n}|_{\infty} \le C_0$$
(55)

and

$$\limsup_{n \to \infty} \|u_{\lambda_n}\|_{H^1(\mathbb{R}^2)}^2 < \frac{\pi}{\gamma + \alpha e^{\frac{1}{e}}} \text{ and } \sup_{n \in \mathbb{N}} |u_{\lambda_n}|_{\infty} \le C'_0$$
(56)

for  $\bar{\delta} = \delta \in (0, 2)$  and  $\bar{\delta} = 2$ , respectively. Moreover, due to the proof of Lemma 3.10, we deduce that  $||u_{\lambda_n}||_{X_{\lambda_n}}$  is uniformly bounded in  $n \in \mathbb{N}$  since we have shown the facts that (46) and (50) are true for all  $\lambda \geq \lambda_0$  if  $\bar{\delta} = \delta \in (0, 2)$ , or  $\lambda \geq \lambda'_0$  if  $\bar{\delta} = 2$ . Going to a subsequence if necessary, there is a  $u \in X$  such that  $u_{\lambda_n} \rightharpoonup u$  in  $X, u_{\lambda_n} \rightarrow u$  in  $L^s(\mathbb{R}^2)$  for each  $2 \leq s < \infty$  by Lemma 2.1-(i), and  $u_{\lambda_n} \rightarrow u$  a.e. in  $\mathbb{R}^2$  as  $n \rightarrow \infty$ . We claim that  $u \equiv 0$  in  $\Omega^c$ . Otherwise, there is a compact subset  $\Theta_u \subset \Omega^c$  with  $dist(\Theta_u, \partial \Omega^c) > 0$  such that  $u \neq 0$  on  $\Theta_u$ , and by Fatou's lemma,

$$\liminf_{n \to \infty} \int_{\mathbb{R}^2} u_{\lambda_n}^2 dx \ge \int_{\Theta_u} u^2 dx > 0.$$
(57)

25

Moreover, there exists  $\varepsilon_0 > 0$  such that  $V(x) \ge \varepsilon_0$  for any  $x \in \Theta_u$  by assumptions  $(V_1)$  and  $(V_2)$ . Combining (28) and (57), one has

$$\begin{aligned} z_{\lambda_n}^{R,\delta} &= \liminf_{n \to \infty} J_{\lambda_n}^{R,\delta}(u_{\lambda_n}) \\ &= \liminf_{n \to \infty} \left[ J_{\lambda_n}^{R,\bar{\delta}}(u_{\lambda_n}) - \frac{1}{4} (J_{\lambda_n}^{R,\bar{\delta}})'(u_{\lambda_n})[u_{\lambda_n}] \right] \\ &\geq \frac{1}{4} \liminf_{n \to \infty} \int_{\mathbb{R}^2} \lambda_n V(x) |u_{\lambda_n}|^2 dx \\ &\geq \frac{\varepsilon_0}{4} \left( \int_{\Theta_u} u^2 dx \right) \liminf_{n \to \infty} \lambda_n \\ &= +\infty, \end{aligned}$$

violating (46) and (50). Consequently,  $u \in H_0^1(\Omega)$  by the fact that  $\partial\Omega$  is smooth.

Recalling  $(J_{\lambda_n}^{R,\bar{\delta}})'(u_{\lambda_n}) = 0$  for each fixed  $n \in \mathbb{N}$ , we now claim that  $(J_{\Omega}^{R,\bar{\delta}})'(u) = 0$ . In fact, for every  $\psi \in C_0^{\infty}(\Omega)$ , it is very similar to (41) and (42) in that

$$V_{i}'(u_{\lambda_{n}})[\psi] - V_{i}'(u)[\psi] = 4B_{i}(u_{\lambda_{n}}^{2}, u_{\lambda_{n}}\psi) - 4B_{i}(u^{2}, u\psi)$$
  
=  $4B_{i}(u_{\lambda_{n}}^{2}, (u_{\lambda_{n}} - u)\psi) + 4B_{i}(u\psi, u_{\lambda_{n}}^{2} - u^{2})$   
=  $o_{n}(1).$  (58)

By means of the Vitali's dominated convergence theorem, one can easily verify that

$$\int_{\mathbb{R}^2} f^{R,\bar{\delta}}(u_{\lambda_n})\psi dx = \int_{\mathbb{R}^2} f^{R,\bar{\delta}}(u)\psi dx + o_n(1).$$
(59)

With the above two formulas in hand, by using  $(V_2)$ , we derive

$$\begin{aligned} 0 &= (J_{\lambda_n}^{R,\bar{\delta}})'(u_{\lambda_n})[\psi] \\ &= \int_{\mathbb{R}^2} \nabla u_{\lambda_n} \nabla \psi dx + V_1'(u_{\lambda_n})[\psi] - V_2'(u_{\lambda_n})[\psi] - \int_{\mathbb{R}^2} f^{R,\bar{\delta}}(u_{\lambda_n})\psi dx \\ &= \int_{\Omega} \nabla u \nabla \psi dx + V_1'|_{\Omega}(u)[\psi] - V_2|_{\Omega}'(u)[\psi] - \int_{\mathbb{R}^2} f^{R,\bar{\delta}}(u)\psi dx + o_n(1) \\ &= (J_{\Omega}^{R,\bar{\delta}})'(u)[\psi] + o_n(1), \end{aligned}$$

which is the claim. Next, we begin by showing that  $u \neq 0$ . Arguing it indirectly and supposing that  $u \equiv 0$ , since  $\{u_{\lambda_n}\}$  is uniformly bounded in  $L^2(\mathbb{R}^2)$ , recalling  $(V_3)$ , we can proceed as with the calculations in the proof of Lemma 3.10 to find a sufficiently large constant  $\varrho > 0$  such that

$$\int_{B_{\varrho}^{c}(0)\cap\Xi}|u_{\lambda_{n}}|^{2}dx\leq\frac{\beta_{0}}{4},\text{ for }n\text{ sufficiently large}.$$

Adopting  $V(x) \ge b$  on  $\Xi^c$  by  $(V_3)$ , and since  $||u_{\lambda_n}||_{X_{\lambda_n}}$  is uniformly bounded, there holds

$$\int_{B^c_{\varrho}(0)\cap\Xi^c} |u_{\lambda_n}|^2 dx \leq \frac{1}{\lambda_n b} \int_{B^c_{\varrho}(0)\cap\Xi^c} \lambda_n V(x) |u_{\lambda_n}|^2 dx \leq \frac{\beta_0}{4}, \text{ for } n \text{ sufficiently large.}$$

Let us recall that  $u_{\lambda_n} \to 0$  along a subsequence in  $L^2(B_\rho(0))$ . Then,

$$\int_{B_{\varrho}(0)} |u_{\lambda_n}|^2 dx \leq \frac{\beta_0}{4}, \text{ for } n \text{ sufficiently large.}$$

Combining the above three formulas, there would be a contradiction to Lemma 3.9. So,  $u \neq 0$ .

Finally, the remaining part is to verify  $J_{\Omega}^{R,\bar{\delta}}(u) = c_{\Omega}^{R,\bar{\delta}}$  because each of (55) and (56) indicate that  $F^{R,\bar{\delta}} = F$ , which yields that  $J_{\Omega}^{R,\bar{\delta}} = J_{\Omega}$ . Obviously,  $\mathcal{N}_{\Omega}^{R,\bar{\delta}} \subset \mathcal{N}_{\lambda_n}^{R,\bar{\delta}}$  for all fixed  $n \in \mathbb{N}$ , and so we have that

$$\begin{split} c_{\Omega}^{R,\bar{\delta}} &\geq \liminf_{n \to \infty} \left( J_{\lambda_n}^{R,\bar{\delta}}(u_{\lambda_n}) - \frac{1}{4} (J_{\lambda_n}^{R,\bar{\delta}})'(u_{\lambda_n})[u_{\lambda_n}] \right) \\ &\geq J_{\Omega}^{R,\bar{\delta}}(u) - \frac{1}{4} (J_{\Omega}^{R,\bar{\delta}})'(u)[u] \\ &= J_{\Omega}^{R,\bar{\delta}}(u) \\ &\geq c_{\Omega}^{R,\bar{\delta}} \end{split}$$

indicating that  $u_{\lambda_n} \to u$  in X and  $J_{\Omega}^{R,\bar{\delta}}(u) = c_{\Omega}^{R,\bar{\delta}}$ , where we have exploited Fatou's lemma together with  $(J_{\Omega}^{R,\bar{\delta}})'(u) = 0$  and  $u \neq 0$ . The proof is finished.  $\Box$ 

5. Some further remarks. In this section, we tend to present some further results on Eq. (1) with steep potential well and supercritical exponential growth. Inspired by [54, Theorem 1.3], one could prove that assumption  $(h_2)$  is unnecessary to manipulate Theorems 1.1 and 1.4. Speaking simply, there are a wider class of nonlinearities which are suitable for the main results in this article. Hence, we shall mainly discuss how to relax the restriction associated with  $(h_2)$ . For this purpose, we suppose that:

 $(h_{21})$  There is a constant  $\theta > 3$  such that  $h(t)t \ge \theta H(t)$  for all  $t \in \mathbb{R}^+$ ;

 $(h_{22})$  The function  $h(t)/t^2$  is nondecreasing on  $t \in \mathbb{R}^+$ .

Concerning the potential V, we need to put forward some additional conditions below:

 $(V_4^1)$   $V \in C^1(\mathbb{R}^2, \mathbb{R})$  and  $2V(x) + (\nabla V, x) \ge 0$  for all  $x \in \mathbb{R}^2$ ;

 $(V_4^2)$   $V \in C^1(\mathbb{R}^2, \mathbb{R})$  and  $t \mapsto t^2[2V(tx) - (\nabla V(tx), tx)]$  is nondecreasing on  $t \in \mathbb{R}^+$  for all  $x \in \mathbb{R}^2$  as well as  $V(x) - (\nabla V(x), x) \ge 0$  for all  $x \in \mathbb{R}^2$ .

The main results in this section are as follows.

**Theorem 5.1.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1), (h_3)$ , and  $(h_{21})$ . Then, for all  $\tau > 2$ , there are  $\alpha_1^* = \alpha_1^*(\tau) > 0$  and  $\lambda_0^1 > 0$  such that Eq. (1) has a nonnegative nontrivial solution in  $X_{\lambda}$  for all  $\alpha \in (0, \alpha_1^*)$  and  $\lambda > \lambda_0^1$ . Moreover, if we suppose that

(h<sub>4</sub>) there are constants  $\xi > 0$  and p > 4 such that  $H(t) = \int_0^t h(s) ds \ge \xi t^p$  for all  $t \in [0, 1]$ ,

then for every  $\alpha > 0$ , there exist  $\tau_*^1 = \tau_*^1(\alpha) > 2$ ,  $\bar{\lambda}'_0 > 0$ , and  $\xi_0^1 > 0$  such that Eq. (1) possesses a nonnegative nontrivial solution in  $X_{\lambda}$  for every  $\tau \in [2, \tau_*^1)$ ,  $\lambda > \bar{\lambda}'_0$ , and  $\xi > \xi_0^1$ .

**Theorem 5.2.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1) - (V_4^2)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1), (h_3)$ , and  $(h_{21}) - (h_{22})$ . Then, for all  $\tau > 2$ ,

27

there are  $\alpha_2^* = \alpha_2^*(\tau) > 0$  and  $\lambda_0^2 > 0$  such that Eq. (1) has a nonnegative ground state solution in  $X_{\lambda}$  for all  $\alpha \in (0, \alpha_2^*)$  and  $\lambda > \lambda_0^2$ . Moreover, if we suppose that

(h<sub>4</sub>) there are constants  $\xi > 0$  and p > 4 such that  $H(t) = \int_0^t h(s) ds \ge \xi t^p$  for all  $t \in [0, 1]$ ,

then for every  $\alpha > 0$ , there exist  $\tau_*^2 = \tau_*^2(\alpha) > 2$ ,  $\tilde{\lambda}'_0 > 0$ , and  $\xi_0^2 > 0$  such that Eq. (1) possesses a nonnegative ground state solution in  $X_{\lambda}$  for every  $\tau \in [2, \tau_*^2)$ ,  $\lambda > \tilde{\lambda}'_0$ , and  $\xi > \xi_0^2$ .

**Remark 5.3.** It is simple to observe that  $(h_2)$  is definitely stronger than  $(h_{21})$  and  $(h_{22})$ . Chen *et al.* [23] obtained the existence of nontrivial solutions for Eq. (14) with  $\lambda \equiv 0$  and  $\theta = 3$  in  $(h_2)$ , but the novelty is that we consider  $\lambda > 0$ , and f can be admitted a supercritical exponential growth.

**Remark 5.4.** There are a lot of functions V satisfying  $(V_1) - (V_3)$  and  $(V_4^1) - (V_4^2)$ . For example, in the sense of ignoring a zero measure set, we define V(x) = V(|x|) for a.e.  $x \in \mathbb{R}^2$  given by

$$V(x) = \begin{cases} 0, & 0 \le |x| \le 1.\\ |x|, & |x| \ge 1. \end{cases}$$

Due to the discussions in the introduction, to accomplish the proofs of Theorems 5.1 and 5.2, we have to establish the following two results.

**Theorem 5.5.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1), (h_3)$ , and  $(h_{21})$ . Then, for each fixed R > 0, there is a  $\overline{\lambda}_0(R) > 0$  dependent of R such that Eq. (14) with  $\overline{\delta} = \delta$  has a nonnegative nontrivial solution in  $X_{\lambda}$  for all  $\lambda > \overline{\lambda}_0(R)$ . Moreover, if in addition we suppose that  $(h_4)$ , then there exist  $\overline{\lambda}'_0(R) > 0$  and  $\overline{\xi}_0(R) > 0$  such that Eq. (14) with  $\overline{\delta} = 2$ possesses a nonnegative nontrivial solution in  $X_{\lambda}$  for all  $\lambda > \overline{\lambda}'_0(R)$  and  $\xi > \overline{\xi}_0(R)$ .

**Theorem 5.6.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1) - (V_4^2)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1), (h_3)$ , and  $(h_{21}) - (h_{22})$ . Then, for each fixed R > 0, there is a  $\tilde{\lambda}_0(R) > 0$  dependent on R such that Eq. (14) with  $\bar{\delta} = \delta$  has a nonnegative ground state solution in  $X_{\lambda}$  for all  $\lambda > \tilde{\lambda}_0(R)$ . Moreover, if in addition we suppose that  $(h_4)$ , then there exist  $\tilde{\lambda}'(R) > 0$  and  $\tilde{\xi}_0(R) > 0$  such that Eq. (14), where  $\bar{\delta} = 2$  admits a nonnegative ground state solution in  $X_{\lambda}$  for all  $\lambda > \tilde{\lambda}'_0(R)$ and  $\xi > \tilde{\xi}_0(R)$ .

**Remark 5.7.** Without considering the computations in Theorem 5.5 or 5.6, combining the arguments in Section 3 and [24, Theorem 1.1], one can get the desired results. Nevertheless, the most challenging difficulty is how to investigate the  $L^{\infty}$ estimate of the solution obtained by Theorem 5.5 or 5.6. To be more eloquent, because of the lack of (33), we must come up with some new analytic tricks to receive the analogous estimate in (45) or (53). Let us point out here that, although it can be obtained easily with the help of the method in [49], which should make the nonlinearity f have to be imposed on some more stronger restrictions, this is not what we expected in this section.

Finally, we invite the reader to note that the so-called ground state in Theorems 5.2 and 5.6 is of class Pohožaaev-Nehari type, instead of the Nehari type in Theorems 1.1 and 1.2.

Following very similar calculations as in Theorem 1.4, we can conclude the asymptotical behavior of  $u_{\lambda}$ , which is the solution established by Theorem 5.1 (or Theorem 5.2). Let us state it without the detailed proof as follows.

**Theorem 5.8.** Under the assumptions in Theorem 5.1 (or Theorem 5.2), passing to a subsequence,  $u_{\lambda} \to u_0$  in X as  $\lambda \to +\infty$ , where  $u_0$  is a nontrivial (or ground state) solution for the Schrödinger-Poisson equation

$$\begin{cases} -\Delta u + \left(\int_{\Omega} \log |x - y| u^2(y) dy\right) u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$
(60)

Now, we start with some brief proofs of the main results in Theorems 5.1, 5.2, 5.5, and 5.6 in this section.

5.1. Proofs of Theorems 5.5 and 5.6. The proofs are divided into two parts below.

5.1.1. Proof of Theorem 5.5. First, we recall the following result introduced by Jeanjean [34,35], which is crucial for finding a bounded (PS) sequence without the Ambrosetti-Rabinowitz condition, namely  $\theta \ge 4$  in  $(h_2^1)$ .

**Proposition 5.9.** Let  $(Z, \|\cdot\|_Z)$  be a Banach space, and let  $\Lambda \subset \mathbb{R}^+$  be an interval. Consider a family  $\{\Phi_{\mu}\}_{\mu\in\Lambda}$  on Z of  $\mathcal{C}^1$ -functional having the form

$$\Phi_{\mu}(u) = A(u) - \mu B(u), \ \forall \mu \in \Lambda,$$

with  $B(u) \geq 0$  for all  $u \in Z$  and either  $A(u) \to +\infty$  or  $B(u) \to +\infty$  as  $||u||_Z \to 0$  $+\infty$ . Assume that there are two points  $v_1, v_2 \in Z$  such that

$$c_{\mu} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_{\mu}(\gamma(t)) > \max\{\Phi_{\mu}(v_1), \Phi_{\mu}(v_2)\},\$$

where  $\Gamma = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = 0, \gamma(1) = v_2\}$ . Then, for a.e.  $\mu \in \Lambda$ , there is a bounded  $(PS)_{c_{\mu}}$  sequence for  $\Phi_{\mu}$  in Z, namely a sequence  $\{u_n(\lambda)\} \subset X$  satisfying

- (i)  $\{u_n(\lambda)\}$  is bounded in Z;
- (ii)  $\Phi_{\mu}(u_n(\lambda)) \to c_{\mu} \text{ as } n \to \infty;$ (iii)  $\Phi'_{\mu}(u_n(\lambda)) \to 0 \text{ in } Z^{-1}.$

Moreover,  $c_{\mu}$  is non-increasing on  $\mu \in \Lambda$ .

To apply Proposition 5.9 successfully, inspired by [24], we have to modify the work space  $(X_{\lambda}, \|\cdot\|_{X_{\lambda}})$  mildly. Speaking clearly, according to Remark 2.5, for all  $\lambda \geq 1$ , we redefine the space  $X_{\lambda}$  by

$$X_{\lambda} := \left\{ u \in E_{\lambda} : \int_{\mathbb{R}^2} \log(2 + |x|) |u|^2 dx < +\infty \right\}$$

which is a Hilbert space equipped with the inner product and norm

$$\begin{aligned} (u,v)_{X_{\lambda}} &= \int_{\mathbb{R}^2} \left[ \nabla u \nabla v + (\lambda V(x) + \log(2 + |x|)) uv \right] dx \text{ and } \|u\|_{X_{\lambda}} \\ &= \sqrt{(u,u)_{X_{\lambda}}}, \ \forall u, v \in X_{\lambda}. \end{aligned}$$

So,  $\|\cdot\|_{X_{\lambda}} = \sqrt{\|\cdot\|_{E_{\lambda}}^{2} + \|\cdot\|_{*}^{2}}$ , where  $\|u\|_{*} = (\int_{\mathbb{R}^{2}} \log(2 + |x|)|u|^{2} dx)^{\frac{1}{2}}$  for all  $u \in X$ , where

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(2+|x|) u^2 dx < +\infty \right\}$$

With this space X, we rewrite  $V_1$  and  $V_2$  on X in this subsection below:

$$V_1(u) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x - y|) u^2(x) u^2(y) dx dy, \ \forall u \in X,$$

and

$$V_2(u) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1 + \frac{2}{|x-y|}\right) u^2(x) u^2(y) dx dy, \ \forall u \in X.$$

As one could observe that there is no essential difference between the above definitions and those in Sections 1-2. Thereby, we keep the same notations in this subsection just for simplicity when there is no misunderstanding.

Setting  $\Phi_{\mu}(u) = J_{\lambda,\mu}^{R,\overline{\delta}}(u)$  on the work space  $(Z, \|\cdot\|_Z) = (X_{\lambda}, \|\cdot\|_{X_{\lambda}})$  with  $\lambda \ge 1$ , where

$$J_{\lambda,\mu}^{R,\bar{\delta}}(u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] dx + \frac{1}{2} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{2} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{2} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{2} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{2} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{4} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{4} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{4} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{4} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{4} ||u||_*^2 + \frac{1}{4} V_0(u) - \mu \left(\int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} ||u||_*^2\right) dx + \frac{1}{4} ||u||_*^2 + \frac{1}{4} V_0(u) - \frac{1}{4} ||u||_*^2 + \frac{1}{4} ||u||_*^2$$

for all  $\mu \in [\frac{1}{2}, 1]$ , let us rewrite  $J_{\lambda,\mu}^{R,\bar{\delta}}(u) = A(u) - \mu B(u)$  on  $X_{\lambda}$  with

$$\begin{split} A(u) &:= \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] dx + \frac{1}{2} \|u\|_*^2 + \frac{1}{4} V_0(u), \\ B(u) &:= \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx + \frac{1}{2} \|u\|_*^2. \end{split}$$

**Lemma 5.10.** Let V satisfy  $(V_1) - (V_3)$ . Suppose that f defined by (2) satisfies  $(h_1)$  and  $(h_3)$ . Then, for all fixed R > 0 and  $\lambda \ge 1$ , then  $B(u) \ge 0$  for any  $u \in X_{\lambda}$ . Moreover, either  $A(u) \to +\infty$  or  $B(u) \to +\infty$  as  $||u||_{X_{\lambda}} \to +\infty$ .

*Proof.* By  $(h_3)$ , one observes that  $B(u) \ge 0$  for each  $u \in X_{\lambda}$ . In consideration of the completeness, we borrow the ideas in [24, Lemma 3.2] to conclude the proof. Arguing it indirectly, we could suppose that, up to a subsequence if necessary, there is a sequence  $\{u_n\} \subset X_{\lambda}$  such that

$$||u_n||_{X_{\lambda}} \to +\infty, \ A(u_n) \le C \text{ and } B(u_n) \le C.$$
 (61)

So, recalling the definition of  $\|\cdot\|_*$  in this subsection and (61),

$$|u_n|_2^2 \le \frac{1}{\log 2} ||u_n||_*^2 \le \frac{1}{\log 2} B(u_n) \le C$$

which, together with (17) and the Gagliardo-Nirenberg inequality, indicates that

$$V_2(u_n) \le K_0 |u_n|_{\frac{8}{3}}^4 \le C |u_n|_2^3 |\nabla u_n|_2 \le C ||u_n||_{X_\lambda}.$$

As a consequence of the above formula and (61), there holds

$$C \ge A(u_n) \ge \frac{1}{2} \|u_n\|_{X_{\lambda}}^2 - \frac{1}{4} V_2(u_n) \ge \frac{1}{2} \|u_n\|_{X_{\lambda}}^2 - \frac{C}{4} \|u_n\|_{X_{\lambda}}$$

which contradicts with  $||u_n||_{X_{\lambda}} \to +\infty$ . The proof is complete.

**Lemma 5.11.** Let V satisfy  $(V_1) - (V_3)$ . Suppose that f defined by (2) satisfies  $(h_1)$  and  $(h_3)$ . Then, for all fixed R > 0 and  $\lambda \ge 1$ , we have that:

- (i) There exists a  $v_0 \in X_{\lambda} \setminus \{0\}$  independent of  $\mu$  and  $\lambda$  such that  $J^{R,\delta}_{\lambda,\mu}(v_0) \leq 0$ for all  $\mu \in [\frac{1}{2}, 1]$ ;
- for all  $\mu \in [\frac{1}{2}, 1]$ ; (ii) Denoting  $\Gamma_{\lambda,\mu}^{R,\bar{\delta}} = \{\gamma \in \mathcal{C}([0, 1], X_{\lambda}) : \gamma(0) = 0, \gamma(1) = v_0\}$ , then

$$c_{\lambda,\mu}^{R,\bar{\delta}} := \inf_{\gamma \in \Gamma_{\lambda,\mu}^{R,\bar{\delta}}} \max_{t \in [0,1]} J_{\lambda,\mu}^{R,\bar{\delta}}(\gamma(t)) \ge A_1 > \max\{J_{\lambda,\mu}^{R,\bar{\delta}}(0), J_{\lambda,\mu}^{R,\bar{\delta}}(v_0)\}, \ \forall \mu \in [\frac{1}{2}, 1];$$

(iii) There exists a constant  $M^{R,\delta} > 0$  independent of  $\mu$  and  $\lambda$  such that  $c_{\lambda,\mu}^{R,\delta} \leq M^{R,\delta}$ .

29

*Proof.* (i) Without loss of generality, we suppose that  $0 \in \Omega$ , and then there exists an  $\varepsilon_0 > 0$  such that  $B_{\varepsilon_0}(0) \subset \Omega$  by  $(V_2)$ . Let  $\psi \in C_0^{\infty}(B_{\varepsilon_0}(0))$ , and set  $\psi_t := t^2 \psi(t)$ for all t > 1, hence  $\operatorname{supp} \psi_t \in B_{\varepsilon_0}(0)$ . By some direct computations, we have

$$\begin{cases} \int_{\mathbb{R}^{3}} |\nabla\psi_{t}|^{2} dx = t^{4} \int_{B_{\varepsilon_{0}}(0)} |\nabla\psi|^{2} dx, \ \int_{\mathbb{R}^{2}} F^{R,\bar{\delta}}(\psi_{t}) dx = t^{-2} \int_{B_{\varepsilon_{0}}(0)} F^{R,\bar{\delta}}(t\psi) dx, \\ \int_{\mathbb{R}^{2}} \log(2+|x|) |\psi_{t}|^{2} dx = t^{2} \int_{B_{\varepsilon_{0}}(0)} \log(2+t^{-1}|x|) |\psi|^{2} dx \\ \leq \log(2+\varepsilon_{0}) t^{2} \int_{B_{\varepsilon_{0}}(0)} |\psi|^{2} dx, \\ V_{0}(\psi_{t}) = t^{4} V_{0}(\psi) - t^{4} \log t |\psi|^{4}_{2} \\ \leq t^{4} \bigg[ \log(2+\varepsilon_{0}) \int_{B_{\varepsilon_{0}}(0)} |\psi|^{2} dx + \log t \bigg( \int_{B_{\varepsilon_{0}}(0)} |\psi|^{2} dx \bigg)^{2} \bigg]. \end{cases}$$
(62)

Due to  $(h_1)$  and  $(h_3)$  together with (62), there holds  $F^{R,\overline{\delta}}(t) \ge C_1 t^{\theta} - C_2 t^2$  for all t > 0, and thus

$$\frac{J_{\lambda,\frac{1}{2}}^{R,\bar{\delta}}(\psi_t)}{t^4\log t} \le \frac{1}{4} \left( \int_{B_{\varepsilon_0}(0)} |\psi|^2 dx \right)^2 - \frac{t^{2(\theta-3)}}{2\log t} \int_{\mathbb{R}^2} \frac{F^{R,\bar{\delta}}(t^2\psi)}{(t^2\psi)^\theta} \psi^\theta dx + o_t(1) \to -\infty$$
(63)

as  $t \to +\infty$  because  $\theta > 3$ . Choosing  $v_0 = \psi_t$  with t sufficiently large, we have  $J_{\lambda,\mu}^{R,\bar{\delta}}(v_0) \le J_{\lambda,\frac{1}{2}}^{R,\bar{\delta}}(v_0) < 0 \text{ for all } \mu \in [\frac{1}{2},1].$ 

(ii) We can repeat the calculations in Lemma 3.2-(i) to find such a constant

 $\begin{array}{l} A_1 > 0 \text{ independent of } \mu \in [\frac{1}{2}, 1] \text{ and } \lambda \geq 1. \\ \text{(iii) Since } c_{\lambda,\mu}^{R,\delta} \leq \max_{t>0} J_{\lambda,\mu}^{R,\bar{\delta}}(\psi_t) \leq J_{\lambda,\frac{1}{2}}^{R,\bar{\delta}}(\psi_t) \text{ for all } \mu \in [\frac{1}{2}, 1], \text{ the conclusion immediately follows from (63). Thus, we finish the proof of this lemma.} \end{array}$ 

**Lemma 5.12.** Under the assumptions of Lemma 5.11, if in addition we suppose that  $(h_4)$ , then there exists a  $\underline{\xi}_0 = \underline{\xi}_0(R) > 0$  such that, for all  $\xi > \overline{\xi}_0$ ,

$$c^{R,2}_{\lambda,\mu} < \frac{\pi(\theta-2)}{2\theta C_{\Xi,b}(1+\gamma+\alpha R^{\tau-2})}, \ \forall \lambda \geq 1, \ \mu \in [\frac{1}{2},1],$$

where  $\gamma > 2$ ,  $\theta > 3$ , and  $C_{\Xi,b} > 0$  come from  $(h_3)$ ,  $(h_{21})$ , and Lemma 2.4, respectively.

*Proof.* We postpone the proof and refer the reader to Lemma 5.15 below for the details. 

It is essentially same as the proof of [32, Lemma 2.4]. We can derive the following lemma.

**Lemma 5.13.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1)$ . Assume that the nonlinearity f defined in (2) requires  $(h_1)$  and  $(h_3)$ . For each fixed R > 0 and  $\lambda \ge 1$ , if  $\bar{u} \in X_{\lambda}$  is a critical point of  $J_{\lambda,\mu}^{R,\bar{\delta}}$ , then it satisfies the Pohožaev identity  $P_{\lambda,\mu}^{R,\bar{\delta}}(\bar{u}) \equiv 0$ , where  $P^{R,\bar{\delta}}_{\lambda,\mu}: X_{\lambda} \to \mathbb{R} \text{ is given by}$ 

$$\begin{split} P^{R,\bar{\delta}}_{\lambda,\mu}(u) &= \frac{1}{2} \int_{\mathbb{R}^2} \lambda [2V(x) + (\nabla V, x)] |u|^2 dx + \frac{1}{4} |u|_2^4 + V_0(u) \\ &+ \frac{1-\mu}{2} \bigg( 2 ||u||_*^2 + \int_{\mathbb{R}^2} \frac{|x|}{2+|x|} u^2 dx \bigg) - 2\mu \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u) dx. \end{split}$$

**Lemma 5.14.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1)$ . Assume that the nonlinearity f defined in (2) requires  $(h_1), (h_3)$ , and  $(h_{21})$ . Then, for each fixed R > 0 and  $\lambda \ge 1$ , there is a  $u_{\mu} \in X_{\lambda} \setminus \{0\}$  such that

$$(J_{\lambda,\mu}^{R,\bar{\delta}})'(u_{\mu}) = 0 \text{ and } J_{\lambda,\mu}^{R,\bar{\delta}}(u_{\mu}) \in (0, c_{\lambda,\mu}^{R,\bar{\delta}}], \ \forall a.e. \ \mu \in [\frac{1}{2}, 1],$$
(64)

where we must additionally suppose  $(h_4)$  with  $\xi > \underline{\xi}_0$  appearing in Lemma 5.12, whence  $\overline{\delta} = 2$ .

*Proof.* Combining Proposition 5.9 and Lemmas 5.10 and 5.11, for a.e.  $\mu \in [\frac{1}{2}, 1]$ , there is a sequence  $\{u_n(\mu)\} \subset X_{\lambda}$  (we denote it by  $\{u_n\}$  just for short) such that

$$\|u_n\|_{X_{\lambda}} \le C, \ J^{R,\bar{\delta}}_{\lambda,\mu}(u_n) \to c^{R,\bar{\delta}}_{\lambda,\mu} > 0 \text{ and } (J^{R,\bar{\delta}}_{\lambda,\mu})'(u_n) \to 0 \text{ in } X^{-1}_{\lambda}.$$
(65)

Since  $||u_n||_{X_{\lambda}} \leq C$ , passing to a subsequence if necessary, there is a  $u_{\mu} \in X_{\lambda}$  such that  $u_n \rightharpoonup u_{\mu}$  in  $X_{\lambda}$ ,  $u_n \rightarrow u_{\mu}$  in  $L^s(\mathbb{R}^2)$  for every  $s \in [2, +\infty)$ , and  $u_n \rightarrow u_{\mu}$  a.e. in  $\mathbb{R}^2$ . We then claim that  $u_{\mu} \neq 0$ . By contradiction, using Lemma 2.1-(ii) and (iv), we have

$$|V_0(u_n)| \le V_1(u_n) + V_2(u_n) \le 2|u_n|_2^2 ||u_n||_*^2 + K_0|u_n|_{\frac{8}{3}}^4 = o_n(1), \tag{66}$$

where we have used the fact  $||u_n||_*^2 \leq C^2$ . By means of (65) and (66) jointly with  $(h_{21})$ , there holds

$$\begin{aligned} c_{\lambda,\mu}^{R,\bar{\delta}} &= J_{\lambda,\mu}^{R,\bar{\delta}}(u_n) - \frac{1}{\theta} (J_{\lambda,\mu}^{R,\bar{\delta}})'(u_n)[u_n] + o_n(1) \\ &= \frac{2\theta}{\theta - 2} \|u_n\|_{E_{\lambda}}^2 + \frac{2\theta(1 - \mu)}{\theta - 2} \|u_n\|_*^2 \\ &+ \frac{\mu}{\theta} \int_{\mathbb{R}^2} [f^{R,\bar{\delta}}(u_n)u_n - \theta F^{R,\bar{\delta}}(u_n)] dx + o_n(1) \\ &\geq \frac{2\theta}{\theta - 2} \|u_n\|_{E_{\lambda}}^2 + o_n(1) \end{aligned}$$

which, together with Lemma 5.11-(iii) and Lemma 5.12, reveals that all the assumptions in Lemma 2.6 holds true. Consequently, we deduce that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} f^{R,\bar{\delta}}(u_n) u_n dx = 0 \text{ and } \lim_{n \to \infty} \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx = 0$$
(67)

With (66) and (67) in hand, we could derive  $||u_n||_{X_{\lambda}} \to 0$  by  $(J_{\lambda,\mu}^{R,\bar{\delta}})'(u_n)[u_n] \to 0$ . As a consequence, it holds that  $0 = \lim_{n \to \infty} J_{\lambda,\mu}^{R,\bar{\delta}}(u_n) = c_{\lambda,\mu}^{R,2}$ , which is absurd since Lemma 5.11-(ii). So,  $u_{\mu} \neq 0$  is true.

Next, we shall show that  $(J_{\lambda,\mu}^{R,\bar{\delta}})'(u_{\mu}) = 0$ . To see it, for all  $\phi \in C_0^{\infty}(\mathbb{R}^2)$ , proceeding as (58) and (59), there holds  $(J_{\lambda,\mu}^{R,\bar{\delta}})'(u_{\mu})[\psi] = \lim_{n \to \infty} (J_{\lambda,\mu}^{R,\bar{\delta}})'(u_n)[\psi] = 0$ , which indicates the desired result. Using Lemma 5.13, we obtain  $P_{\lambda,\mu}^{R,\bar{\delta}}(u_{\mu}) = 0$ , and so

$$\begin{split} J_{\lambda,\mu}^{R,\bar{\delta}}(u_{\mu}) &= J_{\lambda,\mu}^{R,\bar{\delta}}(u_{\mu}) - \frac{1}{4} \Big[ 2(J_{\lambda,\mu}^{R,\bar{\delta}})'(u_{\mu})[u_{\mu}] - P_{\lambda,\mu}^{R,\bar{\delta}}(u_{\mu}) \Big] \\ &= \frac{1}{8} \int_{\mathbb{R}^{2}} \lambda [2V(x) + (\nabla V, x)] |u_{\mu}|^{2} dx + \frac{1-\mu}{8} \Big( 2||u_{\mu}||_{*}^{2} + \int_{\mathbb{R}^{2}} \frac{|x|}{2+|x|} u_{\mu}^{2} dx \Big) \\ &+ \frac{1}{16} |u_{\mu}|_{2}^{4} + \frac{\mu}{2} \int_{\mathbb{R}^{2}} [f^{R,\bar{\delta}}(u_{\mu})u_{\mu} - 3F^{R,\bar{\delta}}(u_{\mu})] dx \end{split}$$

implying that  $J_{\lambda,\mu}^{R,\bar{\delta}}(u_{\mu}) > 0$ , where we have exploited  $(h_{2}^{1})$  and  $(V_{4}^{1})$ . Finally, the remaining part is to verify  $J_{\lambda,\mu}^{R,\bar{\delta}}(u_{\mu}) \leq c_{\lambda,\mu}^{R,\bar{\delta}}$ . To this aim, we claim that  $V_{1}(u_{n}) \rightarrow V_{1}(u_{\mu})$  as  $n \rightarrow \infty$ . Indeed, in view of Lemma 2.1-(ii) and  $||u_{n}||_{X_{\lambda}} \leq C_{\lambda,\mu}^{R,\bar{\delta}}$ . C,

$$V_1(u_n) = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \log(2 + |x - y|) u^2(y) dy \right) u^2(x) dx$$
$$\leq 2 \|u_n\|_*^2 \int_{\mathbb{R}^2} u_n^2 dx$$
$$\leq 2C^2 \int_{\mathbb{R}^2} u_n^2 dx$$

jointly with  $u_n \to u_\mu$  in  $L^2(\mathbb{R}^2)$  yields the claim. By adopting 2.1-(v) and  $(h_{21})$ , it follows from the Fatou's lemma that

$$\begin{split} c_{\lambda,\mu}^{R,\bar{\delta}} &= \liminf_{n \to \infty} \left[ J_{\lambda,\mu}^{R,\bar{\delta}}(u_n) - \frac{1}{\theta} (J_{\lambda,\mu}^{R,\bar{\delta}})'(u_n)[u_n] \right] \\ &= \liminf_{n \to \infty} \left\{ \frac{2\theta}{\theta - 2} \|u_n\|_{E_{\lambda}}^2 + \frac{2\theta(1 - \mu)}{\theta - 2} \|u_n\|_*^2 + \frac{\theta - 4}{4\theta} [V_1(u_n) - V_2(u_n)] \\ &+ \frac{\mu}{\theta} \int_{\mathbb{R}^2} [f^{R,\bar{\delta}}(u_n)u_n - \theta F^{R,\bar{\delta}}(u_n)] dx \right\} \\ &\geq \frac{2\theta}{\theta - 2} \|u_\mu\|_{E_{\lambda}}^2 + \frac{2\theta(1 - \mu)}{\theta - 2} \|u_\mu\|_*^2 + \frac{\theta - 4}{4\theta} [V_1(u_\mu) - V_2(u_\mu)] \\ &+ \frac{\mu}{\theta} \int_{\mathbb{R}^2} [f^{R,\bar{\delta}}(u_\mu)u_\mu - \theta F^{R,\bar{\delta}}(u_\mu)] dx \\ &= J_{\lambda,\mu}^{R,\bar{\delta}}(u_\mu) - \frac{1}{\theta} (J_{\lambda,\mu}^{R,\bar{\delta}})'(u_\mu)[u_\mu] \\ &= J_{\lambda,\mu}^{R,\bar{\delta}}(u_\mu). \end{split}$$

So, we accomplish the proof of this lemma.

As a direct consequence of Lemma 5.14, there exist two sequences  $\{\mu_n\} \subset [\frac{1}{2}, 1]$ and  $\{u_n(\mu_n)\} \subset X_\lambda$  still denoted by  $\{u_n\}$  such that

$$\mu_n \to 1^-, \ J^{R,\bar{\delta}}_{\lambda,\mu_n}(u_n) := \bar{c}^{R,\bar{\delta}}_{\lambda,\mu_n} \in (0, c^{R,\bar{\delta}}_{\lambda,\mu_n}] \text{ and } (J^{R,\bar{\delta}}_{\lambda,\mu_n})'(u_n) = 0.$$
 (68)

Before presenting the proof of Theorem 5.5, we need to pull  $\bar{c}_{\lambda,\mu_n}^{R,\bar{\delta}}$  down to some critical threshold value, whence  $\bar{\delta} = 2$ , namely the following lemma.

Lemma 5.15. Under the assumptions of Lemma 5.11, if in addition we suppose that  $(h_4)$ , then there exists a  $\overline{\xi}_0 = \overline{\xi}_0(R) (\geq \underline{\xi}_0(R)) > 0$  such that for all  $\xi > \overline{\xi}_0$ , there holds

$$\frac{8(\theta-3)}{\theta-2}c_{\lambda,\mu}^{R,2} + 16K_0^2\kappa_{GN}^3(c_{\lambda,\mu}^{R,2})^{\frac{3}{2}} < \frac{\pi}{C_{\Xi,b}(1+\gamma+\alpha R^{\tau-2})}, \ \forall \lambda \ge 1, \ \mu \in [\frac{1}{2},1], \ (69)$$

where  $\gamma > 2$ ,  $\theta > 3$ ,  $K_0 > 0$ , and  $C_{\Xi,b} > 0$  are from  $(h_3)$ ,  $(h_{21})$ , and Lemmas 2.1-(iv) and 2.4, respectively. Moreover,  $\kappa_{GN} > 0$  denotes the best constant associated with the Gagliardo-Nirenberg inequality.

*Proof.* Since  $c_{\lambda,\mu}^{R,2}$  is non-increasing on  $\mu \in [\frac{1}{2}, 1]$  by Proposition 5.9, it suffices to deduce that  $c_{\lambda,\frac{1}{2}}^{R,2}$  satisfies (69). On the one hand, we could claim that  $c_{\lambda,\frac{1}{2}}^{R,2} \leq$ 

 $\inf_{u \in X_{\lambda} \setminus \{0\}} \max_{t>0} J_{\lambda,\frac{1}{2}}^{R,2}(tu).$  Indeed, this is a direct corollary of Lemma 5.11-(i) and (ii). On the other hand, we can follow the calculations in Lemma 3.5 to determine some suitable  $\underline{\xi}_0 = \underline{\xi}_0(R) > 0$   $\overline{\xi}_0 = \overline{\xi}_0(R) > 0$  to ensure Lemma 5.12, and this lemma holds true. The proof is finished.

Now, we are ready to show the proof of Theorem 5.5 as follows.

Proof of Theorem 5.5. First, we fix the constants R > 0 and  $\lambda \geq \overline{\lambda}'_0(R) \geq 1$ . Depending on the above results, we obtain two sequences  $\{\mu_n\} \subset [\frac{1}{2}, 1]$  and  $\{u_n\} \subset X_{\lambda}$  satisfying (68). Obviously, we have that  $P^{R,\overline{\delta}}_{\lambda,\mu_n}(u_n) = 0$  by Lemma 5.13 and (68). So, taking account of  $(V_4^1)$  and  $(h_{21})$ , it holds that

$$\begin{split} \bar{c}_{\lambda,\mu_n}^{R,\bar{\delta}} &= J_{\lambda,\mu_n}^{R,\bar{\delta}}(u_n) \\ &= J_{\lambda,\mu_n}^{R,\bar{\delta}}(u_n) - \frac{1}{4} \Big[ 2(J_{\lambda,\mu_n}^{R,\bar{\delta}})'(u_n)[u_n] - P_{\lambda,\mu_n}^{R,\bar{\delta}}(u_n) \Big] \\ &= \frac{1}{8} \int_{\mathbb{R}^2} \lambda [2V(x) + (\nabla V, x)] |u_n|^2 dx + \frac{1-\mu_n}{8} \bigg( 2 ||u_n||_*^2 + \int_{\mathbb{R}^2} \frac{|x|}{2+|x|} u_n^2 dx \bigg) \\ &\quad + \frac{1}{16} |u_n|_2^4 + \frac{\mu_n}{2} \int_{\mathbb{R}^2} [f^{R,\bar{\delta}}(u_n)u_n - 3F^{R,\bar{\delta}}(u_n)] dx \\ &\geq \frac{1}{16} |u_n|_2^4 + \frac{\theta-3}{4} \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx \end{split}$$

which indicates that

$$|u_n|_2^2 \le 4\sqrt{\bar{c}_{\lambda,\mu_n}^{R,\bar{\delta}}} \text{ and } \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx \le \frac{4\bar{c}_{\lambda,\mu_n}^{R,\delta}}{\theta-3}.$$
 (70)

In light of (68) and (70), we apply Lemma 2.1-(iv) and the Gagliardo-Nirenberg inequality to get

$$\begin{split} \bar{c}_{\lambda,\mu_n}^{R,\bar{\delta}} &= \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u_n|^2 + \lambda V(x)|u_n|^2] dx + \frac{1}{2} ||u_n||_*^2 + \frac{1}{4} V_0(u_n) \\ &- \mu_n \bigg( \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx + \frac{1}{2} ||u_n||_*^2 \bigg) \\ &\geq \frac{1}{2} ||u_n||_{E_{\lambda}}^2 + \frac{1}{4} ||u_n||_*^2 - \frac{1}{4} V_2(u_n) - \frac{1}{2} \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx \\ &\geq \frac{1}{2} ||u_n||_{E_{\lambda}}^2 + \frac{1}{4} ||u_n||_*^2 - \frac{K_0 \kappa_{GN}^3}{4} |u_n|_2^3 |\nabla u_n|_2 - \frac{1}{2} \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx \\ &\geq \frac{1}{4} ||u_n||_{E_{\lambda}}^2 + \frac{1}{4} ||u_n||_*^2 - 4K_0^2 \kappa_{GN}^3 (\bar{c}_{\lambda,\mu_n}^{R,\bar{\delta}})^{\frac{3}{2}} - \frac{2\bar{c}_{\lambda,\mu_n}^{R,\bar{\delta}}}{\theta - 3}. \end{split}$$

In this situation, we conclude that

$$\|u_n\|_{X_{\lambda}}^2 = \|u_n\|_{E_{\lambda}}^2 + \|u_n\|_*^2 \le \frac{8(\theta - 3)}{\theta - 2} \bar{c}_{\lambda,\mu_n}^{R,\bar{\delta}} + 16K_0^2 \kappa_{\mathrm{GN}}^3 (\bar{c}_{\lambda,\mu_n}^{R,\bar{\delta}})^{\frac{3}{2}}.$$
 (71)

Combining Lemma 5.11-(iii) and Lemma 5.15,  $\{||u_n||_{X_{\lambda}}\}$  is uniformly bounded in  $n \in \mathbb{N}$ . Moreover, due to Lemma 2.4, we deduce that  $\{u_n\}$  satisfies all assumptions in Lemma 2.6. Repeating the proof of Theorem 1.2, we could verify that, going to

a subsequence if necessary,  $u_n \to u$  in  $X_{\lambda}$  as  $n \to \infty$ . Besides,  $u \neq 0$  by (68), and  $\mu_n \to 1^-$  implies that

$$(J_{\lambda}^{R,\bar{\delta}}(u))'[\phi] = \lim_{n \to \infty} (J_{\lambda,\mu_n}^{R,\bar{\delta}}(u_n))'[\phi] = 0, \ \forall \phi \in C_0^{\infty}(\mathbb{R}^2)$$

finishing the proof.

5.1.2. Proof of Theorem 5.6. Let us continue to use the work space  $(X_{\lambda}, \|\cdot\|_{X_{\lambda}})$  in this subsection.

Motivated by [24, Theorem 1.3], we shall take advantage of the Nehari-Pohožaev manifold method to consider Theorem 5.6, that is, looking for a minimizer of the minimization problem

$$m_{\lambda,V}^{R,\bar{\delta}} := \inf_{u \in \mathcal{M}_{\lambda,V}^{R,\bar{\delta}}} J_{\lambda}^{R,\bar{\delta}}(u) \text{ with } \mathcal{M}_{\lambda,V}^{R,\bar{\delta}} = \{ u \in X_{\lambda} \setminus \{0\} : G_{\lambda}^{R,\bar{\delta}}(u) = 0 \},$$
(72)

where  $G_{\lambda}^{R,\bar{\delta}}(u) = 2(J_{\lambda}^{R,\bar{\delta}})'(u)[u] - P_{\lambda}^{R,\bar{\delta}}(u)$  for each  $u \in X_{\lambda}$  and  $P_{\lambda}^{R,\bar{\delta}} = P_{\lambda,1}^{R,\bar{\delta}}$  in Lemma 5.13. Recalling that every critical point  $u \in X_{\lambda} \setminus \{0\}$  satisfies  $(J_{\lambda}^{R,\bar{\delta}})'(u)[u] = 0$  and  $P_{\lambda}^{R,\bar{\delta}}(u) = 0$ , it is natural to minimize  $J_{\lambda}^{R,\bar{\delta}}$  over  $\mathcal{M}_{\lambda,V}^{R,\bar{\delta}}$ .

**Lemma 5.16.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1) - (V_4^2)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1), (h_3)$ , and  $(h_{21}) - (h_{22})$ . Then, for each  $u \in X_{\lambda} \setminus \{0\}$ , there exists a unique  $t_u > 0$  such that  $t_u^2 u(t_u \cdot) \in \mathcal{M}_{\lambda,V}^{R,\bar{\delta}}$  for each fixed R > 0 and  $\lambda \geq 1$ . Moreover,

$$m_{\lambda,V}^{R,\bar{\delta}} = \inf_{u \in X_{\lambda} \setminus \{0\}} \max_{t>0} J_{\lambda,V}^{R,\bar{\delta}}(t^{2}u(t \cdot)).$$

*Proof.* According to the definition of  $f^{R,\bar{\delta}}$  defined in (13), we deduce that  $f^{R,\bar{\delta}}(t)/t^2$  is increasing on  $t \in \mathbb{R}^+$ , and so  $[f^{R,\bar{\delta}}(t)t - F^{R,\bar{\delta}}(t)]/t^3$  is increasing on  $t \in \mathbb{R}^+$  on  $t \in \mathbb{R}^+$  either. Proceeding as in the proofs of [24, Lemmas 4.3, 4.6, and 4.7], we can get the desired result.

**Lemma 5.17.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1) - (V_4^2)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1), (h_3)$  and  $(h_{21}) - (h_{22})$ , then for each fixed R > 0 and  $\lambda \ge 1$ , we conclude that

- (i) There is a constant  $\varrho > 0$  independent  $\lambda \ge 1$  such that  $||u||_{H^1(\mathbb{R}^2)} \ge \varrho$  for all  $u \in \mathcal{M}_{\lambda,V}^{R,\bar{\delta}}$ ;
- (ii) The minimum  $m_{\lambda,V}^{R,\bar{\delta}} > 0$ , where it is defined in (72).

*Proof.* (i) We suppose it by a contradiction, namely there exists a sequence  $\{u_n\} \subset \mathcal{M}^{R,\overline{\delta}}_{\lambda,V}$  such that  $||u_n||_{H^1(\mathbb{R}^2)} \to 0$ . It follows from Lemma 2.4,  $(V_4^2)$ , (17), and (25) that

$$C_1 \|u_n\|_{H^1(\mathbb{R}^2)}^2 \le C_2 \|u_n\|_{H^1(\mathbb{R}^2)}^4 + \frac{C_1}{2} \|u_n\|_{H^1(\mathbb{R}^2)}^2 + C_2(R) \|u_n\|_{H^1(\mathbb{R}^2)}^q$$

yielding a contradiction if we choose q > 2.

(ii) Arguing it indirectly, we suppose that  $m_{\lambda,V}^{R,\bar{\delta}} = 0$ . Let  $\{u_n\} \subset X_{\lambda}$  be a minimizing sequence of  $m_{\lambda,V}^{R,\bar{\delta}}$ , that is,  $\{u_n\} \subset \mathcal{M}_{\lambda,V}^{R,\bar{\delta}}$  and  $J_{\lambda}^{R,\bar{\delta}}(u_n) \to 0$  as  $n \to \infty$ . Due to  $(h_2^1)$  and  $(V_4^1)$ , there holds

$$o_n(1) = J_{\lambda}^{R,\bar{\delta}}(u_n) - \frac{1}{4}G_{\lambda}^{R,\bar{\delta}}(u_n)$$

$$\begin{split} &= \frac{1}{8} \int_{\mathbb{R}^2} \lambda [2V(x) + (\nabla V, x)] |u_n|^2 dx + \frac{1}{16} |u_n|_2^4 \\ &+ \frac{1}{2} \int_{\mathbb{R}^2} [f^{R,\bar{\delta}}(u_n) u_n - 3F^{R,\bar{\delta}}(u_n)] dx \\ &\geq \frac{1}{16} |u_n|_2^4 + \frac{\theta - 3}{2\theta} \int_{\mathbb{R}^2} f^{R,\bar{\delta}}(u_n) u_n dx \\ &\geq \frac{1}{16} |u_n|_2^4 + \frac{\theta - 3}{2} \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx \\ &\geq 0 \end{split}$$

which reveals that

$$|u_n|_2 = o_n(1), \ \int_{\mathbb{R}^2} f^{R,\bar{\delta}}(u_n)u_n dx = o_n(1) \text{ and } \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n)dx = o_n(1).$$

Using  $G_{\lambda}^{R,\delta}(u_n) = 0$  again, via the above formula, we obtain

$$2\int_{\mathbb{R}^2} |\nabla u_n|^2 dx \le V_2(u_n) + \frac{1}{4} |u_n|_2^4 + 2\int_{\mathbb{R}^2} [f^{R,\bar{\delta}}(u_n)u_n - F^{R,\bar{\delta}}(u_n)] dx$$
  
$$= V_2(u_n) + o_n(1)$$
  
$$\le K_0 \kappa_{GN}^{\frac{3}{2}} |u_n|_2^3 |\nabla u_n|_2 + o_n(1)$$
  
$$\le \frac{1}{2} |\nabla u_n|_2^2 + \frac{1}{2} K_0^2 \kappa_{GN}^3 |u_n|_2^6 + o_n(1)$$
  
$$= \frac{1}{2} |\nabla u_n|_2^2 + o_n(1)$$

which is impossible by Point-(i). The proof of this lemma is finished.

35

**Lemma 5.18.** Let V satisfy  $(V_1) - (V_3)$  and  $(V_4^1) - (V_4^2)$ . Suppose that the nonlinearity f defined in (2) requires  $(h_1), (h_3)$ , and  $(h_{21}) - (h_{22})$ . Then, for each fixed R > 0 and  $\lambda \ge 1$ , any minimizing sequence  $\{u_n\} \subset X_{\lambda}$  satisfies

$$\|u_n\|_{E_{\lambda}}^2 \le \frac{8(\theta-3)}{\theta-2} m_{\lambda,V}^{R,\bar{\delta}} + 16K_0^2 \kappa_{GN}^3 (m_{\lambda,V}^{R,\bar{\delta}})^{\frac{3}{2}} + o_n(1).$$
(73)

*Proof.* Let  $\{u_n\} \subset X_{\lambda}$  be a minimizing sequence of  $m_{\lambda,V}^{R,\bar{\delta}}$ , that is,  $\{u_n\} \subset \mathcal{M}_{\lambda,V}^{R,\bar{\delta}}$  and  $J_{\lambda}^{R,\bar{\delta}}(u_n) \to m_{\lambda,V}^{R,\bar{\delta}}$  as  $n \to \infty$ . Recalling  $G_{\lambda}^{R,\bar{\delta}}(u_n) = 2(J_{\lambda}^{R,\bar{\delta}})'(u_n)[u_n] - P_{\lambda}^{R,\bar{\delta}}(u_n)$ , it is very similar to (70) that

$$|u_n|_2^2 \le 4\sqrt{m_{\lambda,V}^{R,\bar{\delta}} + o_n(1)} \text{ and } \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx \le \frac{2m_{\lambda,V}^{R,\delta}}{\theta - 3} + o_n(1).$$

According to this formula, using Lemma  $2.1\mathchar`-(iv)$  and the Gagliardo-Nirenberg inequality,

$$\begin{split} m_{\lambda,V}^{R,\bar{\delta}} + o_n(1) &\geq \frac{1}{2} \|u_n\|_{E_{\lambda}}^2 - \frac{1}{4} V_2(u_n) - \int_{\mathbb{R}^2} F^{R,\bar{\delta}}(u_n) dx \\ &\geq \frac{1}{2} \|u_n\|_{E_{\lambda}}^2 - \frac{K_0 \kappa_{GN}^{\frac{3}{2}}}{4} \|u_n\|_2^3 |\nabla u_n|_2 - \frac{2}{\theta - 3} [m_{\lambda,V}^{R,\bar{\delta}} + o_n(1)] \\ &\geq \frac{1}{4} \|u_n\|_{E_{\lambda}}^2 - 4K_0^2 \kappa_{GN}^3 \left[m_{\lambda,V}^{R,\bar{\delta}} + o_n(1)\right]^{\frac{3}{2}} - \frac{2}{\theta - 3} [m_{\lambda,V}^{R,\bar{\delta}} + o_n(1)] \end{split}$$

showing the desired result. The proof of this lemma is complete.

Arguing as before, we have to verify that the minimum  $m_{\lambda,V}^{R,\bar{\delta}}$  can be controlled suitably.

**Lemma 5.19.** Let V satisfy  $(V_1) - (V_3)$ . Suppose that f defined by (2) satisfies  $(h_1)$  and  $(h_3)$ . Then, for all fixed R > 0 and  $\lambda \ge 1$ , there is a constant  $\overline{M}^{R,\delta} > 0$  such that  $m_{\lambda,V}^{R,\delta} \le \overline{M}^{R,\delta}$ . Moreover, if in addition we suppose that  $(h_4)$ , then there exists a  $\tilde{\xi}_0 = \tilde{\xi}_0(R) > 0$  such that, for all  $\xi > \tilde{\xi}_0$ , there holds

$$\frac{8(\theta-3)}{\theta-2}m_{\lambda,V}^{R,2} + 16K_0^2\kappa_{GN}^3 \left(m_{\lambda,V}^{R,2}\right)^{\frac{3}{2}} < \frac{\pi}{C_{\Xi,b}(1+\gamma+\alpha R^{\tau-2})}, \ \forall \lambda \ge 1.$$

*Proof.* The proof is very similar to that of Lemma 5.15, so we omit it here.  $\Box$ 

Proof of Theorem 5.6. We divide the proof into intermediate steps.

**Step 1.** The minimization problem  $m_{\lambda,V}^{R,\bar{\delta}}$  in (72) can be attained.

Let  $\{u_n\} \subset X_{\lambda}$  be a minimizing sequence of  $m_{\lambda,V}^{R,\bar{\delta}}$ , that is,  $\{u_n\} \subset \mathcal{M}_{\lambda,V}^{R,\bar{\delta}}$  and  $J_{\lambda}^{R,\bar{\delta}}(u_n) \to 0$  as  $n \to \infty$ . In view of Lemma 5.18,  $\{||u_n||_{E_{\lambda}}\}$  is uniformly bounded in  $n \in \mathbb{N}$ . Using Lemma 5.18 again, thanks to Lemma 5.19, we can follow step by step in Lemmas 3.8, 3.9, and 3.10 to find  $\tilde{\lambda}(R)$  if  $\bar{\delta} = \delta$ , or  $\tilde{\lambda}'(R)$  if  $\delta = 2$ , to assure that  $\{||u_n||_{X_{\lambda}}\}$  is uniformly bounded in  $n \in \mathbb{N}$  for all  $\lambda > \tilde{\lambda}(R)$  if  $\bar{\delta} = \delta$ , or  $\lambda > \tilde{\lambda}'(R)$  if  $\delta = 2$ , respectively. Repeating the proof of Theorem 1.2, up to a subsequence, there is a  $u \in X_{\lambda}$  such that  $u_n \to u$  in  $X_{\lambda}$ . So, the nontrivial function u is the desired attained function.

**Step 2.** The attained function of  $m_{\lambda,V}^{R,\bar{\delta}}$  is indeed a nontrivial nonnegative critical point of  $J_{\lambda,V}^{R,\bar{\delta}}$ .

We refer the interested reader to [24, Lemma 4.11] for the detailed proof since there is no essential difference.  $\hfill\square$ 

5.2. Proofs of Theorems 5.1 and 5.2. Let us note that, combining Lemma 5.11-(iii), Lemma 5.15, and (71), we can still derive (45) and (53) for the proof of Theorem 5.1. On the other hand, with the help of (73) and Lemma 5.19, we conclude (45) and (53) for the proof of Theorem 5.2. As a consequence, by repeating the calculations in Section 4, we would accomplish the proofs of Theorems 5.1 and 5.2 immediately.

**Conflict of interest.** The authors have no competing interests to declare related to this paper.

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