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Generalized Chern–Simons–Schrödinger system with critical exponential growth: The zero-mass case

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Abstract. We consider the existence of ground state solutions for a class of zero-mass Chern–Simons–Schrödinger systems

$$\begin{cases} -\Delta u + A_0 u + \sum_{j=1}^{2} A_j^2 u = f(u) - a(x)|u|^{p-2}u, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \qquad \partial_1 A_1 + \partial_2 A_2 = 0, \\ \partial_1 A_0 = A_2 |u|^2, \qquad \partial_2 A_0 = -A_1 |u|^2, \end{cases}$$

where $a : \mathbb{R}^2 \to \mathbb{R}^+$ is an external potential, $p \in (1, 2)$ and $f \in \mathcal{C}(\mathbb{R})$ denotes the nonlinearity that fulfills the critical exponential growth in the Trudinger–Moser sense at infinity. By introducing an improvement of the version of Trudinger–Moser inequality approached in (J. Differential Equations **393** (2024) 204–237), we are able to investigate the existence of positive ground state solutions for the given system using variational method.

Keywords: Zero-mass, Chern–Simons–Schrödinger system, Trudinger–Moser inequality, Critical exponential growth, Ground state solution, Variational method

1. Introduction and main results

In this article, we focus on establishing the existence of positive ground state solutions for the following generalized Chern–Simons–Schrödinger (**CSS** in short) system/equation with critical exponential growth

$$\begin{cases} -\Delta u + A_0 u + \sum_{j=1}^{2} A_j^2 u = f(u) - a(x)|u|^{p-2}u, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \qquad \partial_1 A_1 + \partial_2 A_2 = 0, \\ \partial_1 A_0 = A_2 |u|^2, \qquad \partial_2 A_0 = -A_1 |u|^2, \end{cases} \tag{1.1}$$

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where $a:\mathbb{R}^2\to\mathbb{R}^+$ is an external potential, $p\in(1,2)$ and $f\in\mathcal{C}(\mathbb{R})$ denotes the nonlinearity that fulfills the critical exponential growth in the Trudinger–Moser sense at infinity which would be specified later.

Recently, great attention has been paid to the time-dependent CSS system in two spatial dimension

$$\begin{cases} \mathrm{i}D_0\psi+(D_1D_1+D_2D_2)\psi+g(x,|\psi|^2)\psi=0,\\ \partial_0A_1-\partial_1A_0=-\operatorname{Im}(\overline{\psi}D_2\psi),\\ \partial_0A_2-\partial_2A_0=\operatorname{Im}(\overline{\psi}D_1\psi),\\ \partial_1A_2-\partial_2A_1=-\frac{1}{2}|\psi|^2, \end{cases}\tag{1.2}$$

where i stands for the imaginary unit, $\partial_0=\frac{\partial}{\partial t}$, $\partial_1=\frac{\partial}{\partial x_1}$, $\partial_2=\frac{\partial}{\partial x_2}$ for $(t,x_1,x_2)\in\mathbb{R}^{1+2}$, $\psi:\mathbb{R}^{1+2}\to\mathbb{C}$ acts as the complex scalar field, $A_j:\mathbb{R}^{1+2}\to\mathbb{R}$ denotes the gauge field, $D_j=\partial_j+\mathrm{i}A_j$ is the covariant derivative for $j=0,1,2$ and g is the nonlinearity. In real world, it is usually exploited to describe the non-relativistic dynamics behavior of massive number of particles in Chern–Simons gauge fields. This model plays an important role in the study of high-temperature superconductors, Aharovnov–Bohm scattering, and quantum Hall effect, we refer the reader to [16–18]. Moreover, there exist some further physical motivations for considering CSS system (1.2), see [12,14,25,26] for example.

For all $(t,x_1,x_2)\in\mathbb{R}^{1+2}$ and $j=0,1,2$, one usually considers the situation $A_j(t,x)=A_j(x)$. If the standing wave ansatz $\psi(t,x)=e^{\mathrm{i}\lambda t}u(x)$ with a given $\lambda\in\mathbb{R}$ for $u:\mathbb{R}^2\to\mathbb{R}$, then (1.2) reduces to

$$\begin{cases} -\Delta u+\lambda u+A_0u+\sum_{j=1}^2A_j^2u=f(x,u),\\ \partial_1A_2-\partial_2A_1=-\frac{1}{2}|u|^2,\\ \partial_1A_0=A_2|u|^2,\qquad \partial_2A_0=-A_1|u|^2, \end{cases}\tag{1.3}$$

where $f(x,u)=g(x,|u|^2)u$. Suppose A_j satisfies the Coulomb gauge condition $\sum_{j=0}^2\partial_jA_j=0$, then (1.3) with $\lambda\equiv0$ becomes the original CSS equation (1.1), namely

$$\begin{cases} -\Delta u+A_0u+\sum_{j=1}^2A_j^2u=f(x,u),\\ \partial_1A_0=A_2|u|^2,\qquad \partial_2A_0=-A_1|u|^2,\\ \partial_1A_2-\partial_2A_1=-\frac{1}{2}|u|^2,\qquad \partial_1A_1+\partial_2A_2=0. \end{cases}\tag{1.4}$$

It follows from $\partial_1A_0=A_2|u|^2$ and $\partial_2A_0=-A_1|u|^2$ in (1.4) that

$$\Delta A_0=\partial_1\big(A_2|u|^2\big)-\partial_2\big(A_1|u|^2\big),$$

leading to

$$A_0[u](x)=\frac{x_1}{2\pi|x|^2}*\big(A_2|u|^2\big)-\frac{x_2}{2\pi|x|^2}*\big(A_1|u|^2\big).\tag{1.5}$$

In a similar way, we depend on $\partial_1A_2-\partial_2A_1=-\frac{1}{2}|u|^2$ and $\partial_1A_1+\partial_2A_2=0$ in (1.4) to derive

$$\Delta A_1=\partial_2\left(\frac{|u|^2}{2}\right)\quad\text{and}\quad \Delta A_2=-\partial_1\left(\frac{|u|^2}{2}\right).$$

From which, the components A_j for $j = 1, 2$ in (1.4) can be represented as

$$\begin{cases} A_1[u](x) = \frac{x_2}{2\pi|x|^2} * (\frac{|u|^2}{2}) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{(x_2-y_2)u^2(y)}{|x-y|^2}\,dy, \\ A_2[u](x) = -\frac{x_1}{2\pi|x|^2} * (\frac{|u|^2}{2}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{(x_1-y_1)u^2(y)}{|x-y|^2}\,dy. \end{cases} \tag{1.6}$$

In the sequel, we shall write A_j in place of $A_j[u]$ for $j \in \{0, 1, 2\}$ for simplicity as long as there is no misunderstanding. There are some further properties of A_j for $j \in \{0, 1, 2\}$ in Section 2 below.

Indeed, CSS system (1.2) can reduce to a single equation if one studies the standing wave ansatz $\psi(t, x) = e^{\mathrm{i}\lambda t}u(x)$ with a radially symmetric u. Actually, Byeon–Huh–Seok [6] considered the standing waves of type

$$\begin{aligned} &\psi(t, x) = u(|x|)e^{\mathrm{i}\lambda t}, && A_0(t, x) = k(|x|), \\ &A_1(t, x) = \frac{x_2}{|x|^2}h(|x|), && A_2(t, x) = -\frac{x_1}{|x|^2}h(|x|), \end{aligned} \tag{1.7}$$

where k and h are real value functions depending only on $|x|$. Note that (1.7) satisfies the Coulomb gauge condition with $\varsigma = ct + n\pi$, where n is an integer and c is a real constant. To seek for solutions of CSS system (1.2) of the type (1.7), it is enough to handle the following semilinear elliptic equation

$$-\Delta u + \lambda u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s}u^2(s)\,ds + \frac{h^2(|x|)}{|x|^2}\right)u = f(x, u) \quad \text{in } \mathbb{R}^2, \tag{1.8}$$

where $h(s) = \int_0^s \frac{r}{2}u^2(r)\,dr$. As before, we continue to assume that $\lambda \equiv 0$ in Eq. (1.8).

At this stage, there are two kinds of CSS equations, (1.4) and (1.8), which could be called by the so called zero-mass ones. Generally, when $f(x, t) = \bar{f}(x, t) - V(x)t$ for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$ in the classic CSS equations, more and more interesting results have been explored by many mathematicians over the past decades for various assumptions on $\bar{f}$ and V. Speaking precisely, for $\bar{f}(x, t) = |t|^{p-2}t$ and $V \equiv 1$, by exploiting the Nehari–Pohožaev manifold argument, Byeon *et al.* [6] derived the existence of positive solutions for all $p > 6$. Particularly, with the prescribed mass constraint $\int_{\mathbb{R}^2} |u|^2\,dx = c^2$, they showed some existence results for each $c \neq 0$ if $p \in (2, 3]$ and sufficiently small $|c|$ if $p \in (3, 4)$. Afterwards, the existence, nonexistence and multiplicity of nontrivial solutions for (1.3), or (1.8), have been considerably contemplated by a lot of mathematicians, see [4,8,11,19,20,23,27,29,30,32,34,35,37] and the references therein for example even if these references are far to be exhaustive.

Next, we should turn to consider the so-called zero-mass CSS equation. Very recently, Zhang, Tang and Chen [40] handled the following zero-mass CSS equation

$$-\Delta u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s}u^2(s)\,ds + \frac{h^2(|x|)}{|x|^2}\right)u = f(u) - \bar{a}|u|^{p-2}u \quad \text{in } \mathbb{R}^2, \tag{1.9}$$

where $\bar{a} > 0$ is a constant, $p \in (3, 4)$ and the nonlinearity f admits the critical exponential growth in the Trudinger–Moser sense at infinity. In fact, we say that a function f possesses the *critical exponential growth* at infinity if there exists a constant $\alpha_0 > 0$ such that

$$\lim_{t \to +\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases} \tag{1.10}$$

The above definition was introduced by Adimurthi and Yadava in [1], see also de Figueiredo, Miyagaki and Ruf [9] for example.

In [40], the authors depended on the work space below

$$E \triangleq \left\{u : u(x) \text{ is Lebesgue measurable s.t. } \int_{\mathbb{R}^2} |\nabla u|^2\,dx < +\infty \text{ and } \int_{\mathbb{R}^2} |u|^p\,dx < +\infty\right\}$$

which is the completion of $C_0^\infty(\mathbb{R}^2)$ under the norm

$$\|u\| = \sqrt{|\nabla u|_2^2 + |u|_p^2}, \quad \forall u \in E,$$

where $|\cdot|_q$ denotes the usual norm corresponding to the Lebesgue space $L^q(\mathbb{R}^2)$ for every $1 \leqslant q \leqslant \infty$. In order to treat the problem variationally, proceeding as [1,2,7,9,10,22,39], they established the following version of Trudinger–Moser inequality

Proposition 1.1. *Suppose that* $3 < p < 4$, *then* $(e^{\alpha u^2} - 1 - \alpha u^2) \in L^1(\mathbb{R}^2)$ *for all* $\alpha > 0$ *and* $u \in E$. *Moreover, if* $u \in E$, $|\nabla u|_2^2 \leqslant 1$, $|u|_p^p \leqslant M > +\infty$ *and* $\alpha < 4\pi$, *then there exists a constant* $C(M,\alpha) > 0$, *which depends only on* M *and* α, *such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1 - \alpha u^2\right) dx \leqslant C(M,\alpha). \tag{1.11}$$

With the help of Proposition 1.1, they concluded the existence of mountain-pass solutions for Eq. (1.9) with a nonlinearity f involving the critical exponential growth. Actually, to search for the nontrivial solutions, they Actually, to look for the nontrivial solutions, they restricted themselves in the radially symmetric subspace of E, namely $E_r = \{u \in E : u(x) = u((|x|)\}$. In this situation, they immediately have the compact imbedding $E_r \hookrightarrow L^s(\mathbb{R}^2)$ for all $p < s < +\infty$.

Afterwards, Shen [33] generalized and improved the results in [40] to the case that $1 < p < 2$ and the nonlinearity f having supercritical exponential growth. Precisely, by contemplating the work space E above and introducing the Young function defined by

$$\Phi_{\alpha,j_0}(t) = e^{\alpha t^2} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!}|t|^{2j}, \quad \forall t \in \mathbb{R}, \tag{1.12}$$

where $\alpha > 0$ appearing in (1.10) and $j_0 \triangleq \inf\{j \in \mathbb{N}^+ : 2j \geqslant p^*\}$ with $p^* = \frac{2p}{2-p} > 2$, Shen [33] firstly established the Trudinger–Moser inequality below

Proposition 1.2. *Suppose that* $1 < p < 2$, *then* $\Phi_{\alpha,j_0}(u) \in L^1(\mathbb{R}^2)$ *for all* $\alpha > 0$ *and* $u \in E$. *Moreover*

$$\mathbb{S}(\alpha) \triangleq \sup_{u\in E, \|u\|\leqslant 1} \int_{\mathbb{R}^2} \Phi_{\alpha,j_0}(u)\,dx < +\infty \tag{1.13}$$

for all $0 < \alpha < 4\pi$. *Finally, if* $\alpha > 4\pi$, *then* $\mathbb{S}(\alpha) = +\infty$.

Then, combining the minimax procedure and elliptic regular theory, Shen investigated the existence of a nontrivial solution with the mountain-pass energy in [33], where the subspace E_r was still considered.

Motivated by all of the quoted papers above, particular by [33,40], it is quite natural to ask some interesting questions. For example,

(I) As pointed out in [33], either (1.11) or (1.13) is a subcritical Trudinger–Moser type inequality in the whole space $\mathbb{R}^2$, namely there is no information when α is exactly equal to 4π. Thereby, can we given an affirmative answer that whether $\mathbb{S}(4\pi) < +\infty$ or $\mathbb{S}(4\pi) = +\infty$.

(II) Owing to the compact imbedding $E_r \hookrightarrow L^s(\mathbb{R}^2)$ for each $p < s < +\infty$, although the nonlinearity f possesses the (super)critical exponential growth in [33,40], it is simple to recover the compactness to some extent. Hence, can we find nontrivial solutions for zero-mass CSS equation (1.9). In other words, whether the existence results in [33,40] remain true for Eq. (1.4) with $f(x,t) = \bar{a}|t|^{p-2}t$ for all $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$, where $p \in (1,2)$, or $p \in (3,4)$.

(III) The reader is invited to observe that Eq. (1.9) is an autonomous one because $\bar{a} > 0$ is just a constant. Thus, can we improve this constant to a general potential function. Moreover, if it was true, whether the obtained nontrivial solution is indeed a ground state solution.

As a consequence, we shall try our best to introduce some new analytic tricks and then contemplate the above Questions.

First of all, we focus on the Question (I). Let us continue to use the Young function Φ_{α,j_0} defined in (1.12), we shall prove the following result.

Theorem 1.3. *Suppose that $1 < p < 2$, then $\Phi_{\alpha,j_0}(u) \in L^1(\mathbb{R}^2)$ for all $\alpha > 0$ and $u \in E$. Moreover*

$$\mathbb{S}(\alpha) \triangleq \sup_{u\in E, \|u\|\leqslant 1} \int_{\mathbb{R}^2} \Phi_{\alpha,j_0}(u)\,dx < +\infty \tag{1.14}$$

for all $\alpha \in (0, 4\pi]$. Moreover, $\mathbb{S}(\alpha) = +\infty$ if $\alpha > 4\pi$.

Remark 1.4. Due to Theorem 1.3, we can make sure that $\mathbb{S}(4\pi) < +\infty$, and so it solves the Question (I) completely. Moreover, we do believe that the technique for the proof of Theorem 1.3 can be also adapted to Proposition 1.1.

Next, in order to solve Questions (II) and (III), we are ready to introduce some technical assumptions on the potential $a : \mathbb{R}^2 \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ in Eq. (1.1) as follows.

- (A_1) $a \in \mathcal{C}^0(\mathbb{R}^2)$ with $\inf_{x\in\mathbb{R}^2} a(x) > 0$;
- (A_2) for almost every $x \in \mathbb{R}^2$, $a(x) \leqslant \lim_{|x|\to\infty} a(x) \triangleq a_\infty < +\infty$ and this inequality is strict in a subset of positive Lebesgue measure

Concerning the nonlinearity f, we suppose that

- (f_1) $f \in \mathcal{C}(\mathbb{R})$ with $f(t) \equiv 0$ for all $t \in (-\infty, 0]$ and $f(t) = o(t)$ as $t \to 0^+$;
- (f_2) the map $t \mapsto f(t)/t^5$ is strictly increasing on $t \in (0, +\infty)$;
- (f_3) there exist constants $t_0 > 0$, $M_0 > 0$ and $\vartheta \in [0, 1)$ such that

$$0 < t^{\vartheta} F(t) \leqslant M_0 f(t), \quad \forall t > t_0,$$

  where and in the sequel $F(t) = \int_0^t f(s)\,ds$ for all $t > 0$;

(f_4) $\lim_{t\to+\infty} F(t)e^{-\alpha_0 t^2} \triangleq \beta_0 > 0$, where $\alpha_0 > 0$ comes from (1.10).

We are now in a position to state the second main result in this article.

Theorem 1.5. *Let $1 < p < 2$ and suppose (A_1)–(A_2). If f satisfies (1.10) and (f_1)–(f_4), then Eq. (1.1) admits at least a positive ground state solution in E.*

Remark 1.6. It is obvious that Questions (II) and (III) are uncovered by Theorem 1.5 which in turn indicates that our results improve and replenish the counterparts in [33,40]. It should be mentioned here that both the assumptions on the potential a and the nonlinearity f are standard. On the one hand, the function a equipping with (A_1)–(A_2) is usually called by the well-known Rabinowitz's type potential introduced in [31] and it was later exploited by Wan and Tan in [37]. On the other hand, as to the function f having critical exponential growth and satisfying (f_1)–(f_4), we prefer to refer the reader to [30,34,40] and their references therein.

Finally, we shall exhibit the main idea for the proof of Theorem 1.5. The reader is invited to see that the work space

$$E_a \triangleq \left\{u : u(x) \text{ is Lebesgue measurable s.t. } \int_{\mathbb{R}^2} |\nabla u|^2\,dx < +\infty \text{ and } \int_{\mathbb{R}^2} a(x)|u|^p\,dx < +\infty\right\}$$

endowed with the norm

$$\|u\|_a = \left(\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \left(\int_{\mathbb{R}^2} a(x)|u|^p\,dx\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}, \quad \forall u \in E_a,$$

is equivalent to $(E, \|\cdot\|)$ because a is a positive and bounded function in $\mathbb{R}^2$. Thus, we will exploit the work space $(E, \|\cdot\|)$, instead of $(E_a, \|\cdot\|_a)$, just for simplicity. Due to the lack of compactness caused by the critical exponential growth and the absence of the compact imbedding $E_r \hookrightarrow L^s(\mathbb{R}^2)$ for every $p < s < +\infty$, the foremost point of the proof of Theorem 1.5 is to restore the compactness. Inspired by [37], we need to investigate the existence of ground state solutions of the associated "limit problem" of (1.1), which is given as

$$\begin{cases} -\Delta u + a_\infty |u|^{p-2}u + A_0 u + \sum_{j=1}^2 A_j^2 u = f(u), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \qquad \partial_1 A_1 + \partial_2 A_2 = 0, \\ \partial_1 A_0 = A_2|u|^2, \qquad \partial_2 A_0 = -A_1|u|^2. \end{cases} \tag{1.15}$$

We obtain the following result.

Theorem 1.7. *Let $1 < p < 2$. If f satisfies (1.10) and (f_1)–(f_4), then Eq. (1.15) possesses a positive ground state solution in E.*

Remark 1.8. Obviously, one realizes that Theorem 1.7 also provides a positive answer to the Question (II) above. In the proof of Theorem 1.7, the most striking point is that we success in establishing the Vanishing lemma corresponding to the work space $(E, \|\cdot\|)$, see Theorem 3.8 below. Although the essential idea originates from its classic version due to Lions, c.f. [38, Lemma 1.21], we have to make some efforts to prove it and it may prompt some further studies for zero-mass Schrödinger equation.

With Theorem 1.7 in hands, the solvability of Theorem 1.5 becomes available so far, but we emphasize here that the condition (f_2) plays a crucial role in restoring the compactness, see Lemma 3.5 and (3.17) for instance. As a consequence, one naturally wonders that whether there still exists a mountain-pass type solution for Eq. (1.15) when (f_2) is replaced with a weak type condition below

(f_2') for all $t > 0$, there holds $f(t)t - 6F(t) \geqslant 0$.

Actually, we are going to conclude the existence result as follows.

Theorem 1.9. *Let $1 < p < 2$ and suppose (A_1)–(A_2). If f satisfies (1.10) and (f_1)–(f_2') as well as (f_3)–(f_4), then Eq. (1.15) has a positive mountain-pass type solution in E.*

Remark 1.10. We note that Theorem 1.9 solves Question (II) and (III) partially. Let us point out here that we will borrow some idea adopted in [33] to reach the proof. Nevertheless, there are some new challenges that prevent us repeating the arguments simply. For example, as to the critical exponential case in [33], the author strongly relied on the following condition of type

(f_4') there are $\gamma > 0$ and $s > 6$ such that $F(t) \geqslant \gamma t^s$ for all $t \geqslant 0$,

to restore the compactness, where $\gamma > 0$ is sufficiently large. One would easily deduce that the condition (f_4) in the present article is weaker than (f_4') which is a global one that does never reveal the essential feature of the critical exponential growth in (1.10).

The outline of the paper is organized as follows. In Section 2, we mainly present some preliminary results and show the proofs of Theorem 1.3. Sections 3 and 4 are devoted to the proofs of Theorems 1.7 and 1.5, respectively. The proof of Theorem 1.9 shall be presented in Section 5.

Notations: From now on in this paper, otherwise mentioned, we ultilize the following notations:

- $C, C_1, C_2, \ldots$ denote any positive constant, whose value is not relevant and $\mathbb{R}^+ \triangleq (0, +\infty)$.
- Let $(X, \|\cdot\|_X)$ be a Banach space with dual space $(X^{-1}, \|\cdot\|_{X^{-1}})$, and Ψ be functional on X.
- The (C) sequence at a level $c \in \mathbb{R}$ ($(C)_c$ sequence in short) corresponding to Ψ means that $\Psi(x_n) \to c$ and $(1 + \|x_n\|_X)\|\Psi'(x_n)\|_{X^{-1}} \to 0$ in $\mathbb{R}$ as $n \to \infty$, where $\{x_n\} \subset X$.
- $|\cdot|_p$ stands for the usual norm of the Lebesgue space $L^p(\mathbb{R}^2)$ for all $p \in [1, +\infty]$.
- For any $\varrho > 0$ and every $x \in \mathbb{R}^2$, $B_\varrho(x) \triangleq \{y \in \mathbb{R}^2 : |y - x| < \varrho\}$.
- $o_n(1)$ denotes the real sequences with $o_n(1) \to 0$ as $n \to +\infty$.
- "$\to$" and"$\rightharpoonup$" stand for the strong and weak convergence in the related function spaces, respectively.

2. Variational framework and preliminaries

In this section, we are going to exhibit some preliminary results which enable us to treat the problems variationally.

First of all, let us recall some imbedding results which would play a foremost role in formulating the variational structure. The following results can be found in [33, Lemmas 2.1 and 2.2], so we shall omit the detailed proofs.

Lemma 2.1. *Assume $1 < p < 2$, then the imbedding $E \hookrightarrow L^s(\mathbb{R}^2)$ is continuous and $E \hookrightarrow L^s_{\text{loc}}(\mathbb{R}^2)$ is compact for all $p \leqslant s < +\infty$, respectively. Moreover, $(E, \|\cdot\|)$ is a reflexive Banach space.*

Then, we turn to contemplate the so called Chern–Simons term in Eq. (1.1). To begin with, there exist some meaningful and significant observations. According to the second equation and the last two equations in Eq. (1.1), for each $u \in E$, one has

$$\begin{aligned}\int_{\mathbb{R}^2} A_0|u|^2\,dx &= 2\int_{\mathbb{R}^2} A_0(\partial_2 A_1 - \partial_1 A_2)\,dx\\ &= 2\int_{\mathbb{R}^2}(A_2\partial_1 A_0 - A_1\partial_2 A_0)\,dx = 2\int_{\mathbb{R}^2}\big(A_1^2 + A_2^2\big)|u|^2\,dx. \end{aligned}\tag{2.1}$$

As a by-product of the well-known Hardy–Littlewood–Sobolev inequality [24, Theorem 4.3], we could conclude the following estimates to the gauge fields A_j for $j \in \{0, 1, 2\}$.

Lemma 2.2 (see [15, Propositions 4.2-4.3]). *Assume $1 < r < 2$ and $\frac{1}{r} - \frac{1}{\hat{r}} = \frac{1}{2}$, then*

$$|A_j|_{\hat{r}} \leqslant C_r|u|_{2r}^2 \quad \textit{for } j = 1, 2, \; |A_0|_{\hat{r}} \leqslant C_r|u|_{2r}^2|u|_4^2,$$

where $C_r > 0$ is a constant dependent of r.

Combining Lemmas 2.1 and 2.2, one can easily see that

$$|A_j u|_2 \leqslant |A_j|_{\hat{r}}|u|_{\frac{r}{r-1}} \leqslant C_r|u|_{2r}^2|u|_{\frac{r}{r-1}} \leqslant \bar{C}_r\|u\|^3, \quad \text{for } j = 1, 2,\tag{2.2}$$

because $2r > 2$ and $r/(r-1) > 2$, where $\bar{C}_r > 0$ depends only on $r > 1$. We also need the following Brézis-Lieb type lemma for the Chern–Simons term.

Lemma 2.3 (see [13, Lemma 2.4]). *If $u_n \rightharpoonup u$ in E and $u_n \to u$ a.e. in $\mathbb{R}^2$, then one has $A_j[u_n] \to A_j[u]$ a.e. for $j = 1, 2$,*

$$\begin{cases}\lim_{n\to\infty}\int_{\mathbb{R}^2} A_0[u_n]u_n\psi\,dx = \int_{\mathbb{R}^2} A_0[u]u\psi\,dx, & \forall\psi \in E,\\ \lim_{n\to\infty}\int_{\mathbb{R}^2} A_j^2[u_n]u_n\psi\,dx = \int_{\mathbb{R}^2} A_j^2[u]u\psi\,dx, & \forall\psi \in E \textit{ with } j = 1, 2,\end{cases}\tag{2.3}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\big[A_j^2[u_n]|u_n|^2 - A_j^2[u_n - u]|u_n - u|^2\big]\,dx = \int_{\mathbb{R}^2} A_j^2[u]|u|^2\,dx, \quad \textit{for } j = 1, 2.\tag{2.4}$$

Finally, we shall focus on the nonlinearity f. Whereas, it possesses the critical exponential growth at infinity in the Trudinger–Moser sense at infinity, we have to derive the Trudinger–Moser type inequality associated with the work space E in this article.

Proof of Theorem 1.3. Let $\alpha \in (0, 4\pi]$ and $u \in E$ with $\|u\| \leqslant 1$. We denote by u^* the Schwarz symmetrization of u, then u^* is radial and non-increasing. Thanks to the results in [21],

$$\int_{\mathbb{R}^2}|\nabla u^*|^2\,dx \leqslant \int_{\mathbb{R}^2}|\nabla u|^2\,dx, \int_{\mathbb{R}^2}|u^*|^p\,dx = \int_{\mathbb{R}^2}|u|^p\,dx$$

and

$$\int_{\mathbb{R}^2} \Phi_{\alpha,j_0}\big(u^*\big)\,dx = \int_{\mathbb{R}^2} \Phi_{\alpha,j_0}(u)\,dx.$$

So, without loss of generality, we can suppose that $u \in E$ is non-increasing. Given an $R > 0$ which will be determined later, due to Lemma 2.1, we could deduce that the function $v(x) \triangleq u(x) - u(R)$ belongs to $H_0^1(B_R(0))$. Adopting the Young's inequality, there holds

$$\begin{aligned} u^2(x) &= v^2(x) + 2v(x)u(R) + u^2(R) \leqslant v^2(x) + \big[1 + v^2(x)u^2(R)\big] + u^2(R) \\ &= \big[1 + u^2(R)\big]v^2(x) + 1 + u^2(R) \triangleq w^2(x) + 1 + u^2(R), \quad \forall x \in B_R(0). \end{aligned}$$

Obviously, $w \triangleq \sqrt{1+u^2(R)}v \in H_0^1(B_R(0))$ and $\nabla w = \sqrt{1+u^2(R)}\nabla v = \sqrt{1+u^2(R)}\nabla u$. Moreover, we recall from [5, Lemma A.IV], because $u \in L^p(\mathbb{R}^2)$ by Lemma 2.1, there is a constant $C_p > 0$ which is independent of u such that

$$|u(x)| \leqslant C_p|x|^{-\frac{2}{p}}|u|_p \quad \text{for all } x \neq 0. \tag{2.5}$$

Consequently, with the help of (2.5), we are able to see that

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla w|^2\,dx &= \big[1 + u^2(R)\big]\int_{\mathbb{R}^2} |\nabla u|^2\,dx \leqslant \big[1 + u^2(R)\big]\bigg[1 - \bigg(\int_{\mathbb{R}^2} |u|^p\,dx\bigg)^{\frac{2}{p}}\bigg] \\ &\leqslant 1 - \bigg(\int_{\mathbb{R}^2} |u|^p\,dx\bigg)^{\frac{2}{p}} + u^2(R) \leqslant 1 - \big(1 - C_p^2 R^{-\frac{4}{p}}\big)\bigg(\int_{\mathbb{R}^2} |u|^p\,dx\bigg)^{\frac{2}{p}}. \end{aligned}$$

We then choose the constant R to be sufficiently large such that $1 - C_p^2 R^{-\frac{4}{p}} \geqslant 0$ and so $|\nabla w|_2^2 \leqslant 1$. As a consequence, we can apply the classic Trudinger–Moser inequality explored in [7,10] to arrive at

$$\int_{B_R(0)} \Phi_{4\pi,j_0}(u)\,dx \leqslant \int_{B_R(0)} e^{4\pi u^2}\,dx \leqslant e^{4\pi(1+u^2(R))}\int_{B_R(0)} e^{4\pi w^2}\,dx \leqslant C.$$

From which, one concludes that

$$\sup_{u \in E: \|u\| \leqslant 1} \int_{B_R(0)} \Phi_{4\pi,j_0}(u)\,dx \leqslant C < +\infty. \tag{2.6}$$

On the other hand, we can follow the proof of [33, Theorem 1.1] to verify that the integral on the complement of $B_R(0)$ is uniformly bounded. For the sake of the reader's convenience, we shall show the proof in detail. Using [5, Lemma A.IV] and Lemma 2.1 again, for each $u \in L^{p^*}(\mathbb{R}^2)$, there is a constant $\bar{C}_p > 0$ independent of u such that

$$|u(x)| \leqslant \bar{C}_p|x|^{-\frac{2-p}{p}}|u|_{p^*} \quad \text{for all } x \neq 0. \tag{2.7}$$

Since $j_0 = \inf\{j \in \mathbb{N} : 2j \geqslant p^*\}$, then one sees $2j \geqslant p^*$ for all $j \geqslant j_0$ and so applying (2.7) and $R > 1$,

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R(0)} \Phi_{4\pi,j_0}(u)\,dx &= \sum_{j=j_0}^{\infty} \frac{(4\pi)^j}{j!} \int_{\mathbb{R}^2\setminus B_R(0)} |u|^{2j}\,dx = \sum_{j=j_0}^{\infty} \frac{(4\pi)^j}{j!} \int_{\mathbb{R}^2\setminus B_R(0)} |u|^{2j-p^*}|u|^{p^*}\,dx \\ &\leqslant \sum_{j=j_0}^{\infty} \frac{(4\pi)^j}{j!} \int_{\mathbb{R}^2\setminus B_R(0)} (\bar{C}_p|u|_{p^*})^{2j-p^*}|u|^{p^*}\,dx \\ &= \frac{1}{\bar{C}_p^{p^*}|u|_{p^*}^{p^*}} \sum_{j=j_0}^{\infty} \frac{(4\pi)^j}{j!} \int_{\mathbb{R}^2\setminus B_\varrho(0)} (\bar{C}_p|u|_{p^*})^{2j}|u|^{p^*}\,dx \\ &\leqslant \frac{e^{4\pi(\bar{C}_p|u|_{p^*})^2}}{\bar{C}_p^{p^*}|u|_{p^*}^{p^*}} \int_{\mathbb{R}^2\setminus B_\varrho(0)} |u|^{p^*}\,dx \leqslant \frac{e^{4\pi(\bar{C}_p|u|_{p^*})^2}}{\bar{C}_p^{p^*}} \leqslant \frac{e^{4\pi(\tilde{C}_p\|u\|)^2}}{\bar{C}_p^{p^*}} = \frac{e^{4\pi\tilde{C}_p^2}}{\bar{C}_p^{p^*}}.\end{aligned}$$

From this inequality and (2.6), we obtain that

$$\mathbb{S}(\alpha) \leqslant C + e^{4\pi\tilde{C}_p^2}\bar{C}_p^{-p^*} < +\infty, \quad \forall \alpha \in (0, 4\pi].$$

The remaining parts are totally same as in [33, Theorem 1.1], so we omit them here. □

Now, we are able to verify that the variational functional $J_a : E \to \mathbb{R}$ defined by

$$J_a(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + \left(A_1^2 + A_2^2\right)u^2\right]dx + \frac{1}{p}\int_{\mathbb{R}^2} a(x)|u|^p\,dx - \int_{\mathbb{R}^2} F(u)\,dx,$$

is well-defined and of class $\mathcal{C}^1(E, \mathbb{R})$. Actually, due to (1.10) and (f_1), for all $\varepsilon > 0$ and $\alpha > \alpha_0$, there is a constant $C_\varepsilon > 0$ such that

$$|f(s)| \leqslant \varepsilon|s| + C_\varepsilon|s|^{q-1}\Phi_{\alpha,j_0}(s), \quad \forall s \in \mathbb{R}, \tag{2.8}$$

where Φ_{α,j_0} is defined by (1.12) and $q > 2$ can be arbitrarily chosen later. Using (f_2), there holds

$$|F(s)| \leqslant \varepsilon|s|^2 + C_\varepsilon|s|^q\Phi_{\alpha,j_0}(s), \quad \forall s \in \mathbb{R}. \tag{2.9}$$

Moreover, without mentioned any longer, let us exploit directly the following inequality (see e.g. [39, Lemma 2.1]):

$$\left(\Phi_{\alpha,j_0}(s)\right)^m \leqslant \Phi_{m\alpha,j_0}(s), \quad \forall s \in \mathbb{R}, \alpha > 0 \text{ and } m > 1.$$

With (2.8) and (2.9) in hands, exploiting Theorem 1.3, we could proceed as the calculations in [32,33] to deduce that the variational functional J_a associated with (1.1) is well-defined and belongs to $\mathcal{C}^1(E, \mathbb{R})$ such that

$$J_a'(u)[v] = \int_{\mathbb{R}^2} \left[\nabla u\nabla v + \left(A_1^2 + A_2^2 + A_0\right)uv\right]dx + \int_{\mathbb{R}^2} a(x)|u|^{p-2}uv\,dx - \int_{\mathbb{R}^2} f(u)v\,dx, \quad \forall v \in E.$$

In particular, it follows from (2.1) that

$$J_a'(u)[u] = \int_{\mathbb{R}^2} \big[|\nabla u|^2 + 3\big(A_1^2 + A_2^2\big)u^2\big]\,dx + \int_{\mathbb{R}^2} a(x)|u|^p\,dx - \int_{\mathbb{R}^2} f(u)u\,dx.$$

Hence, any (weak) solution of Eq. (1.1) corresponds to a critical point of J_a. In order to search for the critical points of J_a, we introduce the following results.

Lemma 2.4. *Let $1 < p < 2$ and f satisfies (1.10) and (f_1)–(f_4). Suppose there exists a sequence $\{u_n\} \subset E$ such that $u_n \rightharpoonup u$ in E and $u_n \to u$ a.e. in $\mathbb{R}^2$. If in addition, we assume that*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} f(u_n)u_n\,dx \leqslant K_0 \tag{2.10}$$

for some $K_0 \in (0, +\infty)$ independent of $n \in \mathbb{N}$, then, going to a subsequence if necessary,

$$\lim_{n\to\infty} \int_\Omega F(u_n)\,dx = \int_\Omega F(u)\,dx \quad \text{for any compact set } \Omega \subset \mathbb{R}^2. \tag{2.11}$$

Moreover, passing to a subsequence if necessary, there holds

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f(u_n)\psi\,dx = \int_{\mathbb{R}^2} f(u)\psi\,dx \quad \text{for all } \psi \in C_0^\infty\big(\mathbb{R}^2\big). \tag{2.12}$$

Proof. We can follow the essential ideas adopted in [9, Lemma 2.1] and the details will be omitted. □

3. The limit problem (1.15)

The main objective of this section is to investigate the existence of positive ground state solutions for the CSS equation (1.15) which acts as the "limit problem" of Eq. (1.1).

In order to solve Eq. (1.15), we are going to look for the critical points of its corresponding variational functional $J_\infty : E \to \mathbb{R}$ below

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^2} \big[|\nabla u|^2 + \big(A_1^2 + A_2^2\big)u^2\big]\,dx + \frac{a_\infty}{p}\int_{\mathbb{R}^2} |u|^p\,dx - \int_{\mathbb{R}^2} F(u)\,dx, \quad \forall u \in E. \tag{3.1}$$

Arguing as before, it is simple to show that J_∞ is well-defined and it is of class $\mathcal{C}^1(E, \mathbb{R})$ satisfying

$$J_\infty'(u)[v] = \int_{\mathbb{R}^2} \big[\nabla u \nabla v + \big(A_1^2 + A_2^2 + A_0\big)uv\big]\,dx + \frac{a_\infty}{p}\int_{\mathbb{R}^2} |u|^{p-2}uv\,dx - \int_{\mathbb{R}^2} f(u)v\,dx, \quad \forall v \in E.$$

Moreover, we are derived from (2.1) that

$$J_\infty'(u)[u] = \int_{\mathbb{R}^2} \big[|\nabla u|^2 + 3\big(A_1^2 + A_2^2\big)u^2\big]\,dx + a_\infty\int_{\mathbb{R}^2} |u|^p\,dx - \int_{\mathbb{R}^2} f(u)u\,dx.$$

In what follows, we shall denote the Nehari manifold associated with J_∞ by

$$\mathcal{N}_\infty \triangleq \big\{u \in E\backslash\{0\} : J'_\infty(u)[u] = 0\big\}$$

and the corresponding ground state energy level on $\mathcal{N}_\infty$ is defined by

$$m_\infty \triangleq \min_{u\in\mathcal{N}_\infty} J_\infty(u). \tag{3.2}$$

Our main result concerning the autonomous CSS equation (1.15) is the following:

Theorem 3.1. *Let $1 < p < 2$ and suppose that f satisfies (1.10) and (f_1)–(f_5), then Eq. (1.15) admits a positive ground state solution $u_\infty \in E$ such that*

$$J_\infty(u_\infty) = m_\infty = \inf_{u\in E\backslash\{0\}} \max_{t\geqslant 0} J_\infty(tu).$$

The proof of the above theorem will be divided into several lemmas. For simplicity, we shall always suppose that the nonlinearity f satisfies (1.10) and (f_1)–(f_4) and do not mention them unless needed.

First of all, let us give some key observations on the shape of the functional J_∞.

Lemma 3.2. *Let $1 < p < 2$, then there exists a constant $\zeta > 0$ such that*

$$m_\rho \triangleq \inf\big\{J_\infty(u) : u \in E, \|u\| = \rho\big\} > 0, \quad \forall\rho \in (0, \zeta], \tag{3.3}$$

and

$$n_\rho \triangleq \inf\big\{J'_\infty(u)[u] : u \in E, \|u\| = \rho\big\} > 0, \quad \forall\rho \in (0, \zeta]. \tag{3.4}$$

Proof. It follows from (2.9) that

$$J_\infty(u) \geqslant \frac{1}{2}\int_{\mathbb{R}^2}\big(|\nabla u|^2 + a_\infty|u|^p\big)\,dx - \varepsilon\int_{\mathbb{R}^2}|u|^2\,dx - C(\varepsilon, q, \alpha)\int_{\mathbb{R}^2}|u|^q\Phi_{\alpha,j_0}(u)\,dx.$$

Using Lemma 2.1 and letting $\varepsilon > 0$ be suitably small, with the help of (1.14), there exists a constant $\overline{\zeta} \in (0, 1)$ such that

$$J_\infty(u) \geqslant \frac{1}{4}\|u\|^2 - C\|u\|^q \quad \text{when } \|u\| \leqslant \overline{\zeta}.$$

In light of $q > 2$, we can determine a constant $\zeta \in (0, \overline{\zeta})$ such that (3.3) holds true. According to the definition of J'_∞, it is easy to reach (3.4) as before. The proof is completed. □

Lemma 3.3. *Let $1 < p < 2$ and suppose that $u \in E\backslash\{0\}$, then for all $t > 0$, there holds*

$$J_\infty(tu) \to -\infty \quad \textit{as } t \to +\infty.$$

In particular, the functional J_∞ is not bounded from below.

Proof. For any fixed positive function $u \in E\backslash\{0\}$ and $t > 1$, we have that

$$\frac{J_\infty(tu)}{t^6} \leqslant \frac{1}{p}\int_{\mathbb{R}^2}\big[|\nabla u|^2 + \big(A_1^2 + A_2^2\big)u^2 + a_\infty|u|^p\big]\,dx - \frac{1}{t^6}\int_{\mathbb{R}^2} F(tu)\,dx.$$

Due to (f_4), one sees that $F(t)t^{-6} \to +\infty$ as $t \to +\infty$. Thereby, using the Fatou's lemma, we arrive at $J_\infty(tu)/t^6 \to -\infty$ as $t \to +\infty$, and the claim follows. □

Relying on Lemmas 3.2 and 3.3, we shall exploit the following critical point theorem without the (C) condition introduced in [28] to find a (C) sequence for J_∞.

Proposition 3.4. *Let X be a Banach space and $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ Gateaux differentiable for all $v \in X$, with G-derivative $\varphi'(v) \in X^{-1}$ continuous from the norm topology of X to the weak $*$ topology of X^{-1} and $\varphi(0) = 0$. Let S be a closed subset of X which disconnects (archwise) X. Let $v_0 = 0$ and $v_1 \in X$ be points belonging to distinct connected components of $\bar{X}\backslash X$. Suppose that*

$$\inf_S \varphi \geqslant \varrho > 0 \quad \textit{and} \quad \varphi(v_1) \leqslant 0$$

and let $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0)$ and $\gamma(1) = v_1\}$. Then

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \varphi\big(\gamma(t)\big) \geqslant \varrho > 0$$

and there is a $(C)_c$ sequence for φ.

Combining Lemmas 3.2 and 3.3 as well as Proposition 3.4, there is a sequence $\{u_n\} \subset E$ such that

$$J_\infty(u_n) \to c_\infty \quad \text{and} \quad \big(1 + \|u_n\|\big)\big\|J'_\infty(u_n)\big\|_{E^{-1}} \to 0, \tag{3.5}$$

where

$$c_\infty \triangleq \inf_{\gamma\in\Gamma_\infty}\max_{t\in[0,1]} J_\infty\big(\gamma(t)\big) > 0 \tag{3.6}$$

with $\Gamma_\infty = \{\gamma \in \mathcal{C}([0, 1], E) : \gamma(0) = 0 \text{ and } J_\infty(\gamma(1)) < 0\}$.

Lemma 3.5. *Let $1 < p < 2$, then for every $u \in E\backslash\{0\}$, there exists a unique constant $t_u > 0$ such that $J_\infty(t_u u) = \max_{t\geqslant 0} J_\infty(tu)$ and $t_u u \in \mathcal{N}_\infty$. In particular, we could conclude that $c_\infty = m_\infty = d_\infty$, where*

$$d_\infty \triangleq \inf_{u\in E\backslash\{0\}}\max_{t\geqslant 0} J_\infty(tu).$$

Proof. For any $u \in E\backslash\{0\}$ and $t > 0$, we define $\xi(t) = J_\infty(tu)$ and so

$$\begin{aligned}\xi'(t) = 0 &\Longleftrightarrow \int_{\mathbb{R}^2}\big[t|\nabla u|^2 + 3t^5\big(A_1^2 + A_2^2\big)u^2\big]\,dx + a_\infty t^{p-1}\int_{\mathbb{R}^2}|u|^p\,dx - \int_{\mathbb{R}^2} f(tu)u\,dx = 0\\ &\Longleftrightarrow J'_\infty(tu)[tu]/t = 0 \Longleftrightarrow J'_\infty(tu)[tu] = 0 \Longleftrightarrow tu \in \mathcal{N}_\infty.\end{aligned}$$

Proceeding as in the proofs of Lemmas 3.2 and 3.3, $\xi(t)$ possesses a critical point which corresponds to its maximum, that is, there exists a constant $t_u > 0$ such that $\xi'(t_u) = 0$. Next, we verify that t_u is unique. Arguing it indirectly, we would assume that there exist two constants $t_1, t_2 > 0$ with $t_1 \neq t_2$ such that $u_{t_i} \in \mathcal{N}_\infty$ for $i \in \{1, 2\}$. It follows from some elementary computations that

$$\begin{aligned} &J_\infty(t_1u) - J_\infty(t_2u) - \frac{t_1^6 - t_2^6}{6t_1^6} J'_\infty(t_1u)[t_1u] \\ &\quad = \frac{t_1^2}{6}\left[2 - 3\left(\frac{t_2}{t_1}\right)^2 + \left(\frac{t_2}{t_1}\right)^6\right] \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \\ &\qquad + \frac{t_1^p}{6p}\left[(6-p) - 6\left(\frac{t_2}{t_1}\right)^p + p\left(\frac{t_2}{t_1}\right)^6\right] a_\infty \int_{\mathbb{R}^2} |u|^p \, dx \\ &\qquad + \int_{\mathbb{R}^2} \left[\frac{1 - (t_1^{-1}t_2)^6}{6} f(t_1u)t_1u - F(t_1u) + F\big((t_1^{-1}t_2)t_1u\big)\right] dx \end{aligned}$$

and

$$\begin{aligned} &J_\infty(t_2u) - J_\infty(t_1u) - \frac{t_2^6 - t_1^6}{6t_2^6} J'_\infty(t_2u)[t_2u] \\ &\quad = \frac{t_2^2}{6}\left[2 - 3\left(\frac{t_1}{t_2}\right)^2 + \left(\frac{t_1}{t_2}\right)^6\right] \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \\ &\qquad + \frac{t_2^p}{6p}\left[(6-p) - 6\left(\frac{t_1}{t_2}\right)^p + p\left(\frac{t_1}{t_2}\right)^6\right] a_\infty \int_{\mathbb{R}^2} |u|^p \, dx \\ &\qquad + \int_{\mathbb{R}^2} \left[\frac{1 - (t_2^{-1}t_1)^6}{6} f(t_2u)t_2u - F(t_2u) + F\big((t_2^{-1}t_1)t_2u\big)\right] dx. \end{aligned}$$

Combining the above two formulas and $J'_\infty(t_iu)[t_iu] = 0$ for $i \in \{1, 2\}$, we arrive at a contradiction if $t_1 \neq t_2$. Next, to coincide the three numbers with each other, we shall firstly conclude that $c_\infty \leqslant d_\infty$, then $d_\infty \leqslant m_\infty$ and finally $m_\infty \leqslant c_\infty$ step by step.

(1). $c_\infty \leqslant d_\infty$. In view of Lemma 3.3, there is a sufficiently large $t_0 > 0$ such that $J_\infty(t_0u) < 0$ for every $u \in E\backslash\{0\}$. Define $\gamma_0(t) = tt_0u \in \Gamma_\infty$, then $c_\infty \leqslant \max_{t\in[0,1]} J_\infty(tt_0u) \leqslant \max_{t\geqslant 0} J_\infty(tu)$ which implies that $c_\infty \leqslant d_\infty$.

(2). $d_\infty \leqslant m_\infty$. We claim that, for all $u \in E$ and $t > 0$, there holds

$$\begin{aligned} &J_\infty(u) - J_\infty(tu) - \frac{1 - t^6}{6} J'_\infty(u)[u] \\ &\quad = \frac{1}{6}\big(2 - 3t^2 + t^6\big) \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \\ &\qquad + \frac{a_\infty}{6p}\big[(6-p) - 6t^p + pt^6\big] \int_{\mathbb{R}^2} |u|^p \, dx + \int_{\mathbb{R}^2} \left[\frac{1 - t^6}{6} f(u)u - F(u) + F(tu)\right] dx. \end{aligned} \tag{3.7}$$

Due to (f_2), it suffices to verify that

$$\tau(s,t) \triangleq \frac{1-t^6}{6} f(s)s + F(st) - F(s) \geqslant 0 \quad \text{for all } s \geqslant 0 \text{ and } t > 0.$$

Indeed, it is simple to calculate that

$$\frac{\partial}{\partial t}\tau(s,t) = f(st)s - t^5 f(s)s = t^5 s^6\left[\frac{f(st)}{(st)^5} - \frac{f(s)}{s^5}\right]$$
$$\begin{cases} \geqslant 0, & \text{if } t \in [1,+\infty), \\ \leqslant 0, & \text{if } t \in (0,1], \end{cases}$$

where we have used (f_2) in the last inequalities. Therefore, we obtain that $t \mapsto \tau(s,t)$ is decreasing in $(0,1)$ and increasing in $(1,+\infty)$ for all $s \geqslant 0$, respectively. It has that $\tau(s,t) \geqslant \min_{t\in(0,+\infty)} \tau(s,t) = \tau(s,1) = 0$ for every $s \geqslant 0$ and the claim concludes. Owing to the claim, for all $u \in \mathcal{N}_\infty$, one sees that $J_\infty(u) \geqslant J_\infty(tu)$ for all $t > 0$ yielding that $d_\infty \leqslant m_\infty$.

(3). $m_\infty \leqslant c_\infty$**.** We follow [38, Theorem 4.2] and present the details for the sake of completeness. The manifold $\mathcal{N}_\infty$ clearly separates E into two components, we say them by $\{J_\infty > 0\}$ and $\{J_\infty < 0\}$, respectively. According to Lemma 3.2, one can conclude that $\{J_\infty > 0\}$ contains the origin and a small ball around the origin. Moreover, adopting (f_2), $J_\infty(u) \geqslant J_\infty(u) - \frac{1}{6}J'_\infty(u)[u] \geqslant 0$ for all $u \in \{J_\infty > 0\}$, so one must have that $\gamma(1) \in \{J_\infty < 0\}$ for all $\gamma \in \Gamma_\infty$. Because $\gamma \in \mathcal{C}^0$, there exists a $t_0 \in (0,1)$ such that $\gamma(t_0) \in \mathcal{N}_\infty$ showing that $m_\infty \leqslant c_\infty$ by the arbitrariness of γ. The proof is completed. □

Since the nonlinearity f fulfills the critical exponential growth at infinity which leads to the lack of compactness, to recover it, we have to pull the mountain-pass level c_∞ down below a critical value. Have this aim in mind, inspired by [1,2,7,9,10,22,39], we will consider the Moser sequence functions defined by

$$\bar{w}_n(x) \triangleq \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & \text{if } 0 \leqslant |x| \leqslant \frac{1}{n}, \\ \frac{\log(\frac{1}{|x|})}{\sqrt{\log n}}, & \text{if } \frac{1}{n} < |x| \leqslant 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$

Lemma 3.6. *Let* $1 < p < 2$*, then* $0 < c_\infty < \frac{2\pi}{\alpha_0}$.

Proof. We have $c_\infty \geqslant m_\zeta > 0$ by Lemma 3.2. Taking advantage of Lemma 3.3, it is not difficult to observe that $c_\infty = \inf_{\gamma\in\Gamma} \max_{t\in(0,1]} J_\infty(\gamma(t)) \leqslant \inf_{u\in E\setminus\{0\}} \max_{t>0} J_\infty(tu)$. As a consequence, it suffices to conclude that there exists a function $w \in E\setminus\{0\}$ such that $\max_{t>0} J_\infty(tw) < \frac{2\pi}{\alpha_0}$. It follows from some elementary computations that $\overline{w}_n \in C_0^\infty(B_1(0)) \subset E$ and it satisfies

$$\int_{\mathbb{R}^2} |\nabla \bar{w}_n|^2\,dx = \frac{1}{2\pi \log n}\int_{B_1(0)\setminus B_{\frac{1}{n}}(0)} \frac{1}{|x|^2}\,dx = \frac{1}{\log n}\int_{\frac{1}{n}}^1 \frac{1}{\rho}\,d\rho = 1,$$

and

$$\int_{\mathbb{R}^2} |\bar{w}_n|\, dx = \sqrt{\frac{\log n}{2\pi}}\frac{\pi}{n^2} + \sqrt{\frac{2\pi}{\log n}}\left(\frac{1}{4} - \frac{\log n}{2n^2} - \frac{1}{4n^2}\right) = o_n(1).$$

Thanks to the interpolation inequality, there holds $|\bar{w}_n|_s^s = o_n(1)$ for all $1 \leqslant s < +\infty$.

Denoting $|\bar{w}_n|_p^p = \delta_n$ with $\delta_n \to 0$ and we then define $w_n = \bar{w}_n/(1 + \sqrt[p]{\delta_n})$ for all $n \in \mathbb{N}$. Obviously, it has that $\|w_n\| \equiv 1$ which together with $\delta_n \to 0$ indicates that

$$|\nabla w_n|_2^2 \to 1 \quad \text{and} \quad |w_n|_p^p \to 0. \tag{3.8}$$

With the help of (2.2) and (3.8), we follow [35, Lemma 3.10] to show that

$$c(w_n) \triangleq \int_{\mathbb{R}^2} \big(A_1^2[w_n] + A_2^2[w_n]\big) w_n^2\, dx \to 0. \tag{3.9}$$

We now claim that there is a $n \in \mathbb{N}^+$ such that

$$\max_{t>0} J_\infty(tw_n) < \frac{2\pi}{\alpha_0}. \tag{3.10}$$

Otherwise, for all $n \in \mathbb{N}^+$, there exists a $t_n > 0$ corresponding to the maximum point of $\max_{t>0} J(tw_n)$

$$J_\infty'(t_n w_n)[t_n w_n] = 0 \quad \text{and} \quad J_\infty(t_n w_n) = \max_{t>0} J_\infty(tw_n) \geqslant \frac{2\pi}{\alpha_0}. \tag{3.11}$$

From (f_3)–(f_4), for all $\epsilon \in (0, \beta_0)$, there exists a constant $R_\epsilon = R(\epsilon) > 0$ such that

$$f(s)s \geqslant M_0^{-1}(\beta_0 - \epsilon)s^{\vartheta+1}e^{\alpha_0|s|^2}, \quad \forall |s| \geqslant R_\epsilon.$$

According to the second formula in (3.11), $\{t_n\}$ is bounded below by some positive constant. For some sufficiently large $n \in \mathbb{N}$, one knows that $t_n w_n \geqslant R_\epsilon$ on $B_{1/n}(0)$. Using (3.11) again,

$$\begin{aligned}
&t_n^2|\nabla w_n|_2^2 + a_\infty t_n^p |w_n|_p^p + 3t_n^6 c(w_n) \\
&\quad = \int_{\mathbb{R}^2} f(t_n w_n)t_n w_n\, dx \geqslant \int_{B_{1/n}(0)} f(t_n w_n)t_n w_n\, dx \\
&\quad \geqslant \pi M_0^{-1}(\beta_0 - \epsilon)(t_n w_n)^{\vartheta+1} e^{\alpha_0|t_n w_n|^2} n^{-2} \\
&\quad = \pi M_0^{-1}(\beta_0 - \epsilon)\left[\frac{t_n\sqrt{\log n}}{(1 + \sqrt[p]{\delta_n})\sqrt{2\pi}}\right]^{\vartheta+1} \exp\left[\alpha_0 t_n^2 \frac{\log n}{2\pi(1 + \sqrt[p]{\delta_n})^2} - 2\log n\right]
\end{aligned}$$

indicating that $\{t_n\}$ is uniformly bounded in $n \in \mathbb{N}$. Up to a subsequence if necessary, there exists a constant $t_0 \in (0, +\infty)$ such that $t_n \to t_0$. Since $F(s) \geqslant 0$ for all $s \in \mathbb{R}$, we invoke from (3.8) and the second formula in (3.11) that

$$t_0^2 \geqslant \frac{4\pi}{\alpha_0}. \tag{3.12}$$

Choosing $\epsilon = \beta_0/2$, we apply (3.8)–(3.9) and $t_n \to t_0$ to get

$$(1-\vartheta)\log t_0 \geqslant C\left[1+\frac{\vartheta+1}{2}\log(\log n)+\left(\alpha_0 t_0^2(2\pi)^{-1}-2\right)\log n\right]+o_n(1),$$

where $C > 0$ is independent of $n \in \mathbb{N}$. Recalling (3.12), we would arrive at a contradiction by tending $n \to \infty$ and so (3.10) holds true. The proof is completed. □

Lemma 3.7. *Let $1 < p < 2$, then each sequence $\{u_n\} \subset E$ satisfying (3.5) is uniformly bounded in E. Moreover, there is a constant $K_0 > 0$ independent of $n \in \mathbb{N}$ such that (2.10) holds true.*

Proof. Given a sequence $\{u_n\} \subset E$ satisfying (3.5), it follows from (f_2) that

$$\begin{aligned} c_\infty + o_n(1) &= J_\infty(u_n) - \frac{1}{6}J'_\infty(u_n)[u_n] \\ &\geqslant \frac{1}{3}\int_{\mathbb{R}^2}|\nabla u_n|^2\,dx + \left(\frac{1}{p}-\frac{1}{6}\right)a_\infty\int_{\mathbb{R}^2}|u_n|^p\,dx. \end{aligned}$$

Recalling Lemma 3.6 and $1 < p < 2$, we see that $\{\|u_n\|\}$ is uniformly bounded in $n \in \mathbb{N}$. Then, we are derived from $\|u_n\|\|J'_\infty(u_n)\|_{E^{-1}} \to 0$ and (2.2) that

$$\begin{aligned} \int_{\mathbb{R}^2} f(u_n)u_n\,dx &= \int_{\mathbb{R}^2}\left[|\nabla u_n|^2 + 3\left(A_1^2+A_2^2\right)u_n^2\right]dx + a_\infty\int_{\mathbb{R}^2}|u_n|^p\,dx + o_n(1) \\ &\leqslant \int_{\mathbb{R}^2}|\nabla u_n|^2\,dx + a_\infty\int_{\mathbb{R}^2}|u_n|^p\,dx + 3\bar{C}_r^2\|u_n\|^6 + o_n(1) \end{aligned}$$

implying the desired result. The proof is completed. □

Before establishing the existence of ground state solutions for Eq. (1.15), we shall introduce a new version type of Vanishing lemma with respect to our variational setting.

Theorem 3.8. *Let $1 < p < 2$ and $r > 0$. If $\{u_n\}$ is bounded in E and suppose that*

$$\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_r(y)}|u_n|^p\,dx = 0,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for all $p < s < +\infty$.

Proof. We follow the idea adopted in [38, Lemma 1.21] to conclude the proof and exhibit it in detail for the convenience of the reader. First of all, we recall the Gagliardo–Nirenberg inequality in [3] that

$$|u|_s^s \leqslant \mathcal{C}_s|\nabla u|_2^{s-p}|u|_p^p, \quad \forall u \in E \text{ and } p < s < +\infty, \tag{3.13}$$

where the constant $\mathcal{C}_s > 0$ only depends on s. So, we are derived form (3.13) that

$$\int_{B_r(y)} |u_n|^s \, dx \leqslant \mathcal{C}_s \left(\int_{B_r(y)} |u_n|^p \, dx \right) \left(\int_{B_r(y)} |\nabla u_n|^2 \, dx \right)^{\frac{s-p}{2}}.$$

Covering $\mathbb{R}^2$ by balls of radius r in such a way that each point of $\mathbb{R}^2$ is contained in at most 3 balls, we are able to see that

$$\int_{\mathbb{R}^2} |u_n|^s \, dx \leqslant 3\mathcal{C}_s \sup_{y\in\mathbb{R}^2} \left(\int_{B_r(y)} |u_n|^p \, dx \right)^{\frac{p(1-\varpi)}{q^*}} \|u_n\|^{s-p}.$$

Under the assumption of this lemma, it holds that $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for each $p < s < +\infty$. The proof is completed. $\square$

We are now in a position to show the proof of Theorem 4.1.

Proof of Theorem 3.1. Due to Lemmas 3.2–3.3 and Proposition 3.4, there is a sequence $\{u_n\} \subset E$ satisfying (3.5). From Lemma 3.7, $\{\|u_n\|\}$ is uniformly bounded in $n \in \mathbb{N}$. Passing to a subsequence if necessary, using Lemma 2.1, there exists a $u_\infty \in E$ such that $u_n \rightharpoonup u_\infty$ in E, $u_n \to u_\infty$ in $L^s_{\text{loc}}(\mathbb{R}^2)$ with $s \in (p, +\infty)$ and $u_n \to u_\infty$ a.e. in $\mathbb{R}^2$. We claim that, there are $y \in \mathbb{R}^2$ and $r, \tau > 0$ such that

$$\int_{B_r(y)} |u_n|^p \, dx \geqslant \tau. \tag{3.14}$$

Otherwise, thanks to Theorem 3.8, we obtain that $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for every $s \in (p, +\infty)$. According to Lemma 3.7, we now take a similar calculations in (2.11) to deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} F(u_n) \, dx = 0. \tag{3.15}$$

Our next goal is to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f(u_n) u_n \, dx = 0. \tag{3.16}$$

Indeed, on the one hand, taking (3.15), $J_\infty(u_n) \to c_\infty$ and Lemma 3.6 into account that $\limsup_{n\to\infty} |\nabla u_n|_2^2 < \frac{4\pi}{\alpha_0}$. Thereby, we shall choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu > 1$ sufficiently close to 1 in such a way that $\frac{1}{\nu} + \frac{1}{\nu'} = 1$ with $\nu' > 1$ and

$$\alpha\nu |\nabla u_n|_2^2 < 4\pi(1-\epsilon) \quad \text{for some suitable } \epsilon \in (0,1).$$

We define

$$\bar{u}_n = \sqrt{\frac{\alpha\nu}{4\pi(1-\epsilon)}} u_n, \quad \forall n \in \mathbb{N}.$$

Obviously, one sees that $|\nabla \bar{u}_n|_2^2 \leqslant 1$ for all sufficiently $n \in \mathbb{N}$ and $|\bar{u}_n|_p^p$ is uniformly bounded in $n \in \mathbb{N}$. On the other hand, we apply (1.14) in (2.8) to get

$$\begin{aligned}\int_{\mathbb{R}^2} f(u_n)u_n\,dx &\leqslant \varepsilon \int_{\mathbb{R}^2} |u_n|^2\,dx + C_\varepsilon \int_{\mathbb{R}^2} |u_n|^q \Phi_{\alpha,j_0}(u_n)\,dx \\ &\leqslant \varepsilon \int_{\mathbb{R}^2} |u_n|^2\,dx + C_\varepsilon \left(\int_{\mathbb{R}^2} |u_n|^{q\nu'}\,dx\right)^{\frac{1}{\nu'}} \left(\int_{\mathbb{R}^2} \Phi_{4\pi(1-\epsilon),j_0}(\bar{u}_n)\,dx\right)^{\frac{1}{\nu}} \\ &\leqslant \varepsilon \int_{\mathbb{R}^2} |u_n|^2\,dx + C_\varepsilon \mathbb{S}(4\pi)\left(\int_{\mathbb{R}^2} |u_n|^{q\nu'}\,dx\right)^{\frac{1}{\nu'}}.\end{aligned}$$

Letting $n \to \infty$ and then tending $\varepsilon \to 0^+$, we reach the desired result (3.16). With (3.16) in hands, as a direct consequence of $J'_\infty(u_n)[u_n] \to 0$, we derive that $|\nabla u_n|_2^2 \to 0$, $|u_n|_p^p \to 0$ and $\int_{\mathbb{R}^2}(A_1^2 + A_2^2)u_n^2\,dx \to 0$. Exploiting (3.15) and $J_\infty(u_n) \to c_\infty$ again, it immediately concludes that $c_\infty \equiv 0$ which contradicts with $c_\infty > 0$ in Lemma 3.6. So, we see that (3.14) must hold true.

According to (3.14), we define $v_n = u_n(\cdot + y_n)$ for every $n \in \mathbb{N}$. Since both J_∞ and J'_∞ are translation invariant in $\mathbb{R}^2$, one knows that $\{v_n\}$ is still a (C) sequence of J_∞ at the level c_∞. Arguing as before, passing to a subsequence if necessary, $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ with $s \in (p, +\infty)$ and $v_n \to v$ a.e. in $\mathbb{R}^2$. Moreover, we can see that $v \neq 0$ by (3.14). As a consequence, without loss of generality, we consider the sequence $\{u_n\}$ instead of v_n to suppose that $u_\infty \neq 0$. In view of (2.3) and (2.12), there holds $J'_\infty(u_\infty) = 0$ and so $u_\infty \in \mathcal{N}_\infty$. We are then derived from (3.5) and the Fatou's lemma that

$$\begin{aligned}c_\infty &= \liminf_{n\to\infty} J_\infty(u_n) = \liminf_{n\to\infty}\left\{J_\infty(u_n) - \frac{1}{6}J'_\infty(u_n)[u_n]\right\} \\ &= \liminf_{n\to\infty}\left\{\frac{1}{3}\int_{\mathbb{R}^2}|\nabla u_n|^2\,dx + \left(\frac{1}{p}-\frac{1}{6}\right)a_\infty\int_{\mathbb{R}^2}|u_n|^p\,dx + \frac{1}{6}\int_{\mathbb{R}^2}\left[f(u_n)u_n - 6F(u_n)\right]dx\right\} \\ &\geqslant \frac{1}{3}\int_{\mathbb{R}^2}|\nabla u_\infty|^2\,dx + \left(\frac{1}{p}-\frac{1}{6}\right)a_\infty\int_{\mathbb{R}^2}|u_\infty|^p\,dx + \frac{1}{6}\int_{\mathbb{R}^2}\left[f(u_\infty)u_\infty - 6F(u_\infty)\right]dx \\ &= J_\infty(u_\infty) - \frac{1}{6}J'_\infty(u_\infty)[u_\infty] = J_\infty(u_\infty) \geqslant m_\infty. \end{aligned} \tag{3.17}$$

From which, it follows from Lemma 3.5 that $u_n \to u_\infty$ in E along a subsequence. In other words, we deduce that u_∞ is a solution of Eq. (1.15) with $J_\infty(u_\infty) = m_\infty$. The positivity of u_∞ is trivial, and so we omit it here. The proof is completed. □

Remark 3.9. We invite the reader to see that Theorem 1.7 is a direct corollary of Theorem 3.1.

4. Proof od Theorem 1.5

In this section, we are going to investigate the existence of positive ground state solutions for Eq. (1.1). From the view point of variational method, we search for critical points of J_a defined in (3.1). Recalling the discussions in Section 2, J_a is well-defined and of class of $C^1(E, \mathbb{R})$.

We shall prove the following result.

Theorem 4.1. *Let $1 < p < 2$ and suppose (A_1)–(A_2). If f satisfies (1.10) and (f_1)–(f_4), then Eq. (1.1) admits at least a positive ground state solution $u_a \in E$ such that*

$$J_a(u_a) = m_a = \inf_{u\in E\setminus\{0\}} \max_{t\geqslant 0} J_a(tu).$$

Associated with J_a, we have the Nehari manifold given by

$$\mathcal{N}_a = \big\{u \in E \setminus \{0\} : J_a'(u)[u] = 0\big\},$$

and define the minimization problem

$$m_a = \min_{u\in\mathcal{N}_a} J_a(u). \tag{4.1}$$

By definitions of m_∞ and m_a in (3.2) and (4.1), adopting (A_2), it is easy to check that

$$m_a < m_\infty. \tag{4.2}$$

Moreover, because a is a positive and bounded function, it permits us to repeat the arguments in Section 3 to find a (C) sequence $\{u_n\} \subset E$ of J_a at the level c_a, where

$$c_a \triangleq \inf_{\gamma\in\Gamma_a} \max_{t\in[0,1]} J_a\big(\gamma(t)\big) > 0 \tag{4.3}$$

with $\Gamma_a = \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0 \text{ and } J_a(\gamma(1)) < 0\}$. We can also deduce that

$$m_a = c_a = d_a \triangleq \inf_{u\in E\setminus\{0\}} \max_{t\geqslant 0} J_a(tu) \tag{4.4}$$

and

$$0 < c_a < \frac{2\pi}{\alpha_0}. \tag{4.5}$$

Actually, the essential, or unique, difference between the proof of Theorem 3.1 and that of Theorem 4.1 is that whether the variational functional is translation invariant in $\mathbb{R}^2$. Clearly, we realize that J_a does not have such a good property because of the appearance of the nonconstant potential a. Thereby, to reach the proof, it is enough to verify that the weak limit of $(C)_{c_a}$ sequence is nontrivial.

Lemma 4.2. *Under the assumptions of Theorem 4.1, if $\{u_n\} \subset \mathcal{N}_a$ denotes a $(C)_{c_a}$ sequence of J_a and $u_n \rightharpoonup u_a$ in E along a subsequence, then $u_a \neq 0$.*

Proof. Assume by contradiction that $u_a = 0$ and let $t_n > 0$ such that $t_n u_n \in \mathcal{N}_\infty$ by Lemma 3.5. The standard calculations show that $\{t_n\}$ is bounded, and so,

$$c_a + o_n(1) = J_a(u_n) = \max_{t\geqslant 0} J_a(tu_n) \geqslant J_a(t_n u_n) = J_\infty(t_n u_n) + \frac{t_n^p}{p}\int_{\mathbb{R}^2}\big(a(x) - a_\infty\big)|u_n|^p\,dx,$$

where we have used Lemma 3.5 again in the second equality. From this,

$$c_a + o_n(1) \geqslant m_\infty + \frac{t_n^p}{p} \int_{\mathbb{R}^2} \big(a(x) - a_\infty\big)|u_n|^p \, dx.$$

In light of the fact that $\{t_n\}$ is bounded, we can make use of (A_2) to have that

$$\int_{\mathbb{R}^2} \big(a(x) - a_\infty\big)|u_n|^p \, dx \to 0$$

loading to

$$c_a \geqslant m_\infty,$$

which contradicts with (4.2) and (4.4). The proof is completed. □

Proof of Theorem 4.1. Owing to (4.4), we just need to find a sequence $\{u_n\} \subset \mathcal{N}_a$ and it is a $(C)_{m_a}$ sequence of J_a. It is standard, we refer the reader to [30, Theorem 1.1] and so the proof is done. □

5. Proof od Theorem 1.9

In this section, we aim to derive that Eq. (1.15) admits a mountain-pass type solution whose energy is equal to the mountain-pass level.

The main result in this direction can be stated as follows.

Theorem 5.1. *Let $1 < p < 2$ and suppose (A_1)–(A_2). If f satisfies (1.10) and (f_1)–(f_2') as well as (f_3)–(f_4), then Eq. (1.15) has a positive mountain-pass type solution in $u \in E$ with $J_\infty(u) = c_\infty$, where J_∞ and c_∞ are defined by (3.1) and (3.6), respectively.*

As we have pointed out in the Introduction, when (f_2) is absence, we cannot restore the compactness as what we have done in the Sections 3 and 4. Speaking it clearly, let $\{u_n\} \subset E$ be a $(C)_{c_\infty}$ sequence of J_∞, it is impossible to conclude that $\{u_n\}$ admits a strongly convergent subsequence in E by (3.17). The existence of such a sequence is guaranteed by adopting some very similar calculations in Section 3.

Whereas, since the whole space $\mathbb{R}^2$ itself also results in the lack of compactness, we shall always restrict ourselves in the radially symmetric subspace of E. In other words, in this section, we prefer to take advantage of E_r to be the work space, instead of E.

Now, we are able to verify that the variational functional J_∞ satisfies the (C) condition at the level c_∞.

Lemma 5.2. *Under the assumptions of Theorem 5.1, if $\{u_n\} \subset E_r$ is a $(C)_{c_\infty}$ sequence of J_∞, then there is a $u_0 \in E$ such that $u_n \to u_0$ in E_r along a subsequence.*

Proof. Proceeding as the proof of Lemma 3.7, $\{\|u_n\|\}$ is uniformly bounded in $n \in \mathbb{N}$. Passing to a subsequence if necessary, there is a $u_0 \in E_r$ such that $u_n \rightharpoonup u_0$ in E_r, $u_n \to u_0$ in $L^s(\mathbb{R}^N)$ with $s > p$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$. Combining (2.3) and (2.12), one has $J_\infty'(u_0) = 0$ which implies that

$$J_\infty(u_0) = J_\infty(u_0) - \frac{1}{6} J_\infty'(u_0)[u_0] \geqslant 0. \tag{5.1}$$

Moreover, it follows from the Brézis-Lieb lemma, (2.4), (2.11) and (5.1) that

$$\begin{aligned} c_\infty &= \frac{1}{2}|\nabla u_n|_2^2 + \frac{a_\infty}{p}|u_n|_p^p + \frac{1}{2}\int_{\mathbb{R}^2}\big(A_1^2[u_n] + A_2^2[u_n]\big)u_n^2\,dx - \int_{\mathbb{R}^2} F(u_n)\,dx + o_n(1) \\ &= \frac{1}{2}|\nabla u_n - \nabla u_0|_2^2 + \frac{a_\infty}{p}|u_n - u_0|_p^p + \frac{1}{2}\int_{\mathbb{R}^2}\big(A_1^2[u_n - u_0] + A_2^2[u_n - u_0]\big)(u_n - u_0)^2\,dx \\ &\quad + \frac{1}{2}|\nabla u_0|_2^2 + \frac{a_\infty}{p}|u_0|_p^p + \frac{1}{2}\int_{\mathbb{R}^2}\big(A_1^2[u_0] + A_2^2[u_0]\big)u_0^2\,dx - \int_{\mathbb{R}^2} F(u_0)\,dx + o_n(1) \\ &\geqslant \frac{1}{2}|\nabla u_n - \nabla u_0|_2^2 + J_\infty(u_0) + o_n(1) \geqslant \frac{1}{2}|\nabla u_n - \nabla u_0|_2^2 + o_n(1). \end{aligned}$$

From which and Lemma 3.6, then $\limsup_{m\to\infty}|\nabla u_n - \nabla u_0|_2^2 < \frac{4\pi}{\alpha_0}$. Consequently, we shall choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu > 1$ sufficiently close to 1 in such a way that $\frac{1}{\nu} + \frac{1}{\nu'} = 1$ with $\nu' > 1$ and

$$\alpha\nu|\nabla u_n - \nabla u_0|_2^2 < 4\pi(1-\hat{\epsilon}) \quad \text{for some suitable } \epsilon \in (0,1).$$

We define

$$\hat{u}_n = \sqrt{\frac{\alpha\nu}{4\pi(1-\hat{\epsilon})}}(u_n - u_0), \quad \forall n \in \mathbb{N}.$$

Obviously, one sees that $|\nabla\hat{u}_n|_2^2 \leqslant 1$ for all sufficiently $n \in \mathbb{N}$ and $|\hat{u}_n|_p^p$ is uniformly bounded in $n \in \mathbb{N}$. Besides, for the above fixed $\hat{\epsilon} \in (0,1)$, we need the following two types of Young's inequality

$$|a+b|^2 \leqslant (1+\hat{\epsilon})|a|^2 + \big(1+\hat{\epsilon}^{-1}\big)|b|^2, \quad \forall a,b \in \mathbb{R}$$

and

$$e^{a+b} - d \leqslant \frac{1}{1+\hat{\epsilon}}\big[e^{(1+\hat{\epsilon})a} - d\big] + \frac{\hat{\epsilon}}{1+\hat{\epsilon}}\big[e^{(1+\hat{\epsilon}^{-1})b} - d\big], \quad \forall a,b,d \in \mathbb{R}.$$

By means of the above facts together with (1.14), we derive

$$\begin{aligned} \int_{\mathbb{R}^2} \Phi_{\alpha\nu,j_0}(u_n)\,dx &\leqslant \frac{1}{1+\bar{\epsilon}}\int_{\mathbb{R}^2}\Phi_{4\pi(1+\bar{\epsilon})^{-2},j_0}(\hat{u}_n)\,dx + \frac{\bar{\epsilon}}{1+\bar{\epsilon}}\int_{\mathbb{R}^2}\Phi_{\nu\alpha(1+\bar{\epsilon}^{-1})^2,j_0}(u_0)\,dx \\ &\leqslant \frac{\mathbb{S}(4\pi)}{1+\bar{\epsilon}} + \frac{C_2\bar{\epsilon}}{1+\bar{\epsilon}} \leqslant C_3 < +\infty, \quad \forall n \in \mathbb{N}^+. \end{aligned}$$

As a consequence, by (2.8), we obtain

$$\left|\int_{\mathbb{R}^2} f(u_n)(u_n - u_0)\,dx\right| \leqslant \left(\int_{\mathbb{R}^2}|u_n|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|u_n - u_0|^2\,dx\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&+ C|u_n|^{(q-1)}_{2(q-1)\nu'}|u_n - u_0|_{2\nu'}\left(\int_{\mathbb{R}^2} \Phi_{\alpha\nu, j_0}(u_n)\,dx\right)^{\frac{1}{\nu}} \\
&= o_n(1).
\end{aligned} \tag{5.2}
$$

It is simple to see that

$$
\int_{\mathbb{R}^2} f(u_0)(u_n - u_0)\,dx = o_n(1). \tag{5.3}
$$

For all $1 < r < 2$, thanks to the significant inequality [36, (2.2)] which can be stated as follows

$$
\big(|y_2|^{r-2}y_2 - |y_1|^{r-2}y_1\big)\cdot(y_2 - y_1) \geqslant \hat{C}_r \cdot \frac{|y_2 - y_1|^2}{(|y_2| + |y_1|)^{2-r}}.
$$

From which, using $J'_\infty(u_n) = o_n(1)$ and $J'_\infty(u_0) = 0$ as well as (2.3) and (5.2)–(5.3), it holds that

$$
\begin{aligned}
o_n(1) &= J'_\infty(u_n)[u_n - u_0] - J'_\infty(u_0)[u_n - u_0] \\
&= \int_{\mathbb{R}^2}\big[|\nabla u_n - \nabla u_0|^2 + \big(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0\big)(u_n - u_0)\big]\,dx \\
&\quad + \int_{\mathbb{R}^2}\big(A_1^2[u_n]u_n + A_2^2[u_n]u_n\big)(u_n - u)\,dx + \int_{\mathbb{R}^2}\big(A_1^2[u_0]u_0 + A_2^2[u_0]u_0\big)(u_n - u_0)\,dx \\
&\quad + \int_{\mathbb{R}^2} f(u_n)(u_n - u_0)\,dx + \int_{\mathbb{R}^2} f(u_0)(u_n - u_0)\,dx \\
&= \int_{\mathbb{R}^2}\big[|\nabla u_n - \nabla u_0|^2 + \big(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0\big)(u_n - u_0)\big]\,dx + o_n(1) \geqslant o_n(1)
\end{aligned}
$$

yielding that

$$
|\nabla u_n - \nabla u_0|_2^2 = o_n(1) \quad \text{and} \quad \int_{\mathbb{R}^2}\big(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0\big)(u_n - u_0)\,dx.
$$

At this stage, we apply the Hölder's inequality to get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |u_n - u_0|^p\,dx \\
&\quad \leqslant \hat{C}_p^{-\frac{p}{2}} \int_{\mathbb{R}^2} \big|\big(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0\big)(u_n - u_0)\big|^{\frac{p}{2}}(|u_n| + |u_0|)^{\frac{p(2-p)}{2}}\,dx \\
&\quad \leqslant \hat{C}_p^{-\frac{p}{2}}\left(\int_{\mathbb{R}^2} \big|\big(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0\big)(u_n - u_0)\big|\,dx\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^2}(|u_n| + |u_0|)^p\,dx\right)^{\frac{2-p}{2}} \\
&\quad \leqslant C\left(\int_{\mathbb{R}^2} \big|\big(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0\big)(u_n - u_0)\big|\,dx\right)^{\frac{p}{2}} = o_n(1).
\end{aligned}
$$

Thus, we can derive that $u_n \to u_0$ in E_r as $n \to \infty$. The proof is completed. □

Proof of Theorem 5.1. In view of Section 3, there is a $(C)_{c_\infty}$ sequence $\{u_n\} \subset E_r$ of J_∞. So, we can finish the proof by Lemma 5.2. This proof also concludes Theorem 1.9. □

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