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Concentrating normalized solutions to planar Schrödinger–Poisson system with steep potential well: critical exponential case

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Abstract. In this paper, we study the following class of planar Schrödinger–Poisson problems

$$\begin{cases} -\Delta u + \lambda V(x)u + \gamma(\ln|\cdot| * |u|^2)u = \mu u + \kappa f(u) \text{ in } \mathbb{R}^2, \\ u(x) > 0 \text{ in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u(x)|^2 dx = a^2, \end{cases}$$

where $a > 0$, $\mu \in \mathbb{R}$ is an unknown parameter appearing as a Lagrange multiplier, $\lambda, \gamma, \kappa > 0$ are parameters, $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R}^+)$ admits a potential well $\Omega \triangleq \operatorname{int} V^{-1}(0)$ and f is a continuous function having critical exponential growth at infinity in the Trudinger-Moser sense. Owing to some technical tricks adopted in Alves and Shen (On existence of positive solutions to some classes of elliptic problems in the hyperbolic space, Submitted for publication), Shen and Squassina (Existence and concentration of normalized solutions for p-Laplacian equations with logarithmic nonlinearity, http://arxiv.org/abs/2403.09366), we are able to obtain the existence and concentrating behavior of positive normalized solutions for sufficiently large λ using variational method.

Keywords. Positive normalized solutions, Planar Schrödinger–Poisson equation, L^2-supercritical growth, Critical exponential growth, Steep potential well, Variational methods.

Contents

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1. Introduction

1.1. Some literature overview

We are concerned with the existence of positive solutions to the following planar Schrödinger-Poisson equation

$$-\Delta u+\lambda V(x)u+\gamma(\ln|\cdot|*|u|^2)u=\mu u+\kappa f(u) \text{ in } \mathbb{R}^2, \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^2}|u|^2dx=a^2, \tag{1.2}$$

where $a>0$, $\mu\in\mathbb{R}$ is an unknown parameter appearing as a Lagrange multiplier and $\lambda,\gamma,\kappa>0$ are parameters. The potential V is supposed to satisfy the following set of assumptions:

(V_1) $V\in\mathcal{C}(\mathbb{R}^2,\mathbb{R})$ with $V(x)\geq 0$ on $\mathbb{R}^2$;

(V_2) $\Omega\triangleq \mathrm{int}V^{-1}(0)$ is nonempty and bounded with smooth boundary, and $\overline{\Omega}=V^{-1}(0)$;

(V_3) there exists a $b>0$ such that the set $\Xi\triangleq\{x\in\mathbb{R}^2:V(x)<b\}$ is nonempty and admits finite measure.

As we all know, Bartsch and his collaborators firstly proposed the assumptions like $(V_1)-(V_3)$ in [16,18]. Particularly, the harmonic trapping potential

$$V(x)=\begin{cases}\omega_1|x_1|^2+\omega_2|x_2|^2-\omega, & \text{if } |(\sqrt{\omega_1}x_1,\sqrt{\omega_2}x_2)|^2\geq\omega,\\ 0, & \text{if } |(\sqrt{\omega_1}x_1,\sqrt{\omega_2}x_2)|^2\leq\omega,\end{cases}$$

with $\omega>0$ satisfies $(V_1)-(V_3)$, where $\omega_i>0$ is called by the anisotropy factor of the trap in quantum physics and trapping frequency of the ith-direction in mathematics, see e.g. [19,24,51]. Actually, the potential λV with the above hypotheses is usually denoted by the steep potential well.

Inspired by the well-known Trudinger-Moser type inequality, we recall that a function f has the critical exponential growth at infinity if there exists a constant $\alpha_0>0$ such that

$$\lim_{|s|\to+\infty}\frac{|f(s)|}{e^{\alpha s^2}}=\begin{cases}0, & \forall\alpha>\alpha_0,\\ +\infty, & \forall\alpha<\alpha_0.\end{cases} \tag{1.3}$$

This definition was introduced by Adimurthi and Yadava [2], see also de Figueiredo, Miyagaki and Ruf [36] for example.

Hereafter, we shall suppose that the nonlinearity f satisfies (1.3) and the assumptions below

(f_1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $f(s) \equiv 0$ for all $s \in (-\infty, 0]$;
(f_2) There is a $q \in (2, 4)$ such that $f(s)/s^{q-1}$ is an increasing function of s on $(0, +\infty)$,
(f_3) There is a $c_0 > 0$ such that $f(s) \geq c_0 s^{q-1}$ for all $s \in [0, +\infty)$.

We would like to highlight here that many functions f satisfy the above assumptions, with $\alpha_0 = 4\pi$ and $c_0 = 1$, for example,

$$f(s) = \begin{cases} 0, & s \leq 0, \\ s^{q-1}e^{4\pi s^2}, & 0 \leq s < +\infty, \end{cases}$$

where $q \in (2, 4)$.

In recent years, considerable attention was paid to the standing, or solitary, wave solutions of Schrödinger–Poisson systems of the type

$$\begin{cases} i\dfrac{\partial \psi}{\partial t} = \Delta\psi - W(x)\psi - m\phi\psi + \widetilde{f}(|\psi|)\psi, & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\ \Delta\phi = |\psi|^2, & \text{in } \mathbb{R}^d, \end{cases} \tag{1.4}$$

where $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ acts as the time-dependent wave function, $W : \mathbb{R}^d \to \mathbb{R}$ stands for the real external potential, $m \in \mathbb{R}$ is a parameter, ϕ represents an internal potential for a nonlocal self-interaction of wave function and nonlinear term $f(\psi) \triangleq \widetilde{f}(|\psi|)\psi$ describes the interaction effect among particles. Inserting the standing wave ansatz $\psi(x, t) = \exp(-i\omega t)u(x)$ with $\omega \in \mathbb{R}$ and $x \in \mathbb{R}^d$ into (1.4), then $u : \mathbb{R}^d \to \mathbb{R}$ satisfies the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \bar{V}(x)u + m\phi u = f(u), & \text{in } \mathbb{R}^d, \\ \Delta\phi = u^2, & \text{in } \mathbb{R}^d, \end{cases} \tag{1.5}$$

where and in the sequel $\bar{V}(x) = W(x) + \omega$ for all $x \in \mathbb{R}^d$. In view of the paper [34], the second equation in (1.5) determines $\phi : \mathbb{R}^d \to \mathbb{R}$ only up to harmonic functions. Conversely, it is natural to regard ϕ as the negative Newton potential of u^2, namely, the convolution of u^2 with the fundamental solution Φ_d of the Laplacian, which is denoted by $\Phi_d(x) = -1/(d(d-2)\omega_d)|x|^{2-d}$ if $d \geq 3$, and $\Phi_2(x) = -\frac{1}{2\pi}\log(|x|)$ if $d = 2$, here we denote by ω_d the volume of the unit ball in $\mathbb{R}^d$. With this inversion of the second equation in (1.5), one can receive the integro-differential equation

$$-\Delta u + \bar{V}(x)u + m\big(\Phi_d * u^2\big)u = f(u) \text{ in } \mathbb{R}^d. \tag{1.6}$$

In light of its physical relevance in physics, there exist a rich literature associated with (1.6) and the generalizations under the variant assumptions on $\bar{V}$ and f by using variational methods for $d \geq 3$, see [1,13,27,39,52,58] and their references therein. We prefer to mention that, when $m \neq 0$, the Poisson term $(\Phi_d * u^2)u$ causes that (1.6) is not a pointwise identity any longer such that there are some mathematical difficulties which make the study of it more interesting.

For $d = 2$, the Schrödinger–Poisson equation (1.6) can be rewritten as the form

$$- \Delta u + \bar{V}(x)u + \frac{m}{2\pi}\left[\ln(|x|) * u^2\right] u = f(u) \text{ in } \mathbb{R}^2. \tag{1.7}$$

For clarity, we shall suppose that $\bar{V}(x) \equiv -\mu \in \mathbb{R}$ for all $x \in \mathbb{R}^2$ and $m = 2\pi$.

Generally speaking, there are two ways to the studies of Eq. (1.7). On the one hand, one can choose frequency $\mu \in \mathbb{R}$ to be fixed and then focus on investigating the existence of nontrivial solutions by looking for critical points of the variational functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 - \mu u^2\right] dx + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u^2(x)u^2(y)dxdy \\ - \int_{\mathbb{R}^2} F(u)dx,$$

where and in the sequel $F(t) = \int_0^t f(s)ds$ for all $t \in \mathbb{R}$. Due to the variational method point of view, one usually requires I to be a class of $\mathcal{C}^1$-functional and the classic Hilbert space $H^1(\mathbb{R}^2)$ is an ideal candidate that acts as the work space. Nevertheless, Stubbe [63] pointed out clearly that the functional I is not even well-defined in $H^1(\mathbb{R}^2)$. In order to get around this obstacle, the author introduced a new Hilbert space

$$X = \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \ln(1+|x|)u^2 dx < +\infty\right\},$$

endowed with the inner product and norm

$$(u, v)_X = \int_{\mathbb{R}^2} \left[\nabla u \nabla v + uv + \ln(1+|x|)uv\right] dx \text{ and } \|u\|_X = \sqrt{(u,u)_X}.$$

With the work space X in hands, one can define the two variational functionals $V_1, V_2 : X \to \mathbb{R}$ by

$$\begin{cases} V_1(u) \triangleq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)u^2(x)u^2(y)dxdy, \\ V_2(u) \triangleq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right) u^2(x)u^2(y)dxdy, \end{cases} \forall u \in X$$

and they belong to $\mathcal{C}^1(X, \mathbb{R})$, see e.g. [63]. Now, because of the crucial identity

$$\ln r = \ln(1+r) - \ln\left(1+\frac{1}{r}\right), \ \forall r > 0,$$

it enables to have the following decomposition

$$V_0(u) \triangleq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u^2(x)u^2(y)dxdy = V_1(u) - V_2(u), \ \forall u \in X \tag{1.8}$$

indicating that I is of class $\mathcal{C}^1(X, \mathbb{R})$. Afterwards, for $\mu = -1$ and $f(t) = b|t|^{p-2}t$ with $b \geq 0$ and $p \geq 4$, Cingolani and Weth [34] obtained the existence and multiplicity of nontrivial solutions for Eq. (1.7), later the case $p \in (2, 4)$ was supplemented by Du and Weth in [38]. To acquaint the asymptotic and non-degeneracy of the ground state solution when $b = 0$, we refer the reader to [21]. Moreover, Chen, Shi and Tang [29] extended the main results to a general nonlinearity. It is significant to note that the above cited papers depended on the fact that the potential is constant or $\mathbb{Z}^2$-periodic. To avoid such restriction, the authors in [31,32] constructed nontrivial solutions in an axially symmetric space which is weaker than the radially symmetric one.

Very recently, by supposing $(V_1)-(V_2)$, Shen and Squassina [61] contemplated the existence and concentration of nontrivial solutions for the equation

$$-\Delta u + \lambda V(x)u + (\ln|\cdot| * |u|^2)u = f(u) \text{ in } \mathbb{R}^2,$$

where $\lambda > 0$ is sufficiently large and f admits the supercritical exponential growth (see [8,9]). Concerning some other interesting works associated with Eq. (1.7) and its variants, we suggest the reader to look at [14,37] and the references therein even if these ones are far to be exhaustive.

On the other hand, one can contemplate the case $\mu \in \mathbb{R}$ to be unknown. In such a situation, $\mu \in \mathbb{R}$ denotes a Lagrange multiplier and the L^2-norm of obtained solutions would be prescribed. From the physical point of view, this spirit of research holds particular significance as it accounts for the conservation of mass. What's more, it provides some valuable insights into the dynamic properties of the standing waves of (1.5), for instance stability or instability in [20,26]. In this paper, we shall focus primarily on this aspect.

In [43], due to a minimax approach and compactness argument, Jeanjean contemplated the existence of solutions for the following Schrödinger problem

$$\begin{cases} -\Delta u + \lambda u = g(u) \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a^2 > 0. \end{cases} \tag{1.9}$$

Subsequently, there exist some further complements and generalizations in [45]. In [64], letting $g(t) = \tau|t|^{q-2}t + |t|^{p-2}t$ with $2 < q \le 2 + \frac{4}{N} \le p < 2^*$, Soave obtained the existence of solutions for problem (1.9), where $2^* = \frac{2N}{N-2}$ if $N \ge 3$ and $2^* = \infty$ if $N = 2$. For this type of combined nonlinearities, Soave also [65] proved the existence of ground state and excited solutions when $p = 2^*$. For more interesting results for problem (1.9), we will refer the reader to [7,17,44,46,67] and the references therein.

Conversely, the appearance of the nonlocal convolution term $\left[\ln(|x|) * u^2\right] u$ in (1.7) exhibits some delicate mathematical difficulties with a local nonlinear term f. To address this trouble, Cingolani and Jeanjean used the combination of the fibration method of Pohozaev (relying on the decomposition of L^2-Pohožaev manifold used in [33]) and the strong compactness condition developed by Cingolani-Weth [34], where some new estimates of energy on the dilated function $tu(t\cdot)$ for all $u \in L^2$ and $t > 0$ belonging to $S(a)$ were given. Moreover, the reader may realize that the spatial dimension of Eq. (1.7) is two, the case therefore is very special because $2^* = \infty$ in this situation. In spirit of [7], Alves, de S. Böer and Miyagaki [4] investigated the existence of normalized solutions to (1.7), where the nonlinearity f satisfies (1.3) and

- (F_1) $f(0) = 0$ and there exists $\tau > 3$ such that $\lim\limits_{t\to 0} \frac{|f(s)|}{|s|^\tau} = 0$;
- (F_2) there exists $\theta > 6$ such that $f(s)s \ge \theta F(s) > 0$ for all $s \ne 0$, where $F(s) = \int_0^s f(t)dt$

(F_3) there exist $p > 6$ and a large enough $\vartheta > 0$ such that $F(s) \geq \vartheta |s|^p$ for all $s \in \mathbb{R}$.

Note that $\vartheta > 0$ in (F_3) is sufficiently large such that the obtained mountain-pass level can be chosen as arbitrarily small from which the compactness will be restored in the usual way. Very recently, due to same intention, the authors in [12,60,68] depended on the following condition, instead of (F_3),

(F_3') $\liminf\limits_{s \to +\infty} \frac{F(s)}{e^{\alpha_0 s^2}} > 0$, where $\alpha_0 > 0$ comes from (1.3).

In [28], by choosing $f(s) = \left(e^{s^2} - 1 - s^2\right)s$ for all $s \in \mathbb{R}$, Chen et al. investigated the existence of normalized solutions to (1.7) by virtue of the L^2-Pohožaev manifold. For some more results on (1.7) for normalized solutions, see e.g. [41,42] and their references therein.

1.2. Motivation for further advances

Motivated by all of the quoted papers above, particularly by [4,28,61], it seems quite natural to ask some interesting questions. For example,

(I) From [33], it can be observed that the L^2-Pohožaev manifold method seems unavoidable when the nonlinearity f satisfies the condition (F_2) with even $\theta > 4$, or it satisfies the critical exponential growth (1.3). Conversely, if the L^2-Pohožaev manifold method is invalid any longer when (1.7) is a non-automatous one, such as $V \in \mathcal{C}$ instead of $\mathcal{C}^1$ in (1.1), could we still conclude the existence of normalized solutions for the planar Schrödinger–Poisson equations?

(II) One may infer from [4] that the condition (F_3), or (F_3'), is indispensable to some extent. So, what happens if we weaken, even remove, them when f still satisfies (1.3)?

(III) Due to [61], could we apply the steep potential to deal with the existence of normalized solutions for a class of planar Schrödinger–Poisson equations with L^2-supercritical growth?

1.3. Main results

In the present article, we shall try our best to put forward some new analytic skills and then give the affirmative answers to three Questions above. First of all, in order to exhibit the main results legibly, let us first introduce the work space. Following as [61], given a fixed $\lambda > 0$, by (V_1), we define the space

$$E_\lambda \triangleq \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} \lambda V(x)|u|^2 dx < +\infty \right\}$$

which is indeed a Hilbert space equipped with the inner product and norm

$$(u, v)_{E_\lambda} = \int_{\mathbb{R}^2} \left[\nabla u \nabla v + \lambda V(x) uv\right] dx \text{ and } \|u\|_{E_\lambda} = \sqrt{(u,u)_{E_\lambda}}, \ \forall u, v \in E_\lambda.$$

From here onwards, we shall denote E and $\|\cdot\|_E$ by E_λ and $\|\cdot\|_{E_\lambda}$ for $\lambda = 1$, respectively. It is simple to observe that $\|\cdot\|_E \leq \|\cdot\|_{E_\lambda}$ for every $\lambda \geq 1$. Therefore, owing to [61, Lemma 2.4], E_λ could be continuously imbedded into

$H^1(\mathbb{R}^2)$ and then into X for every $\lambda \geq 1$. With these discussions, it permits us to introduce the work space

$$X_\lambda \triangleq \left\{ u \in X : \int_{\mathbb{R}^2} \lambda V(x)|u|^2 dx < +\infty \right\} \tag{1.10}$$

and X_λ is also a Hilbert space equipped with the inner product and norm

$$(u,v)_{X_\lambda} = \int_{\mathbb{R}^2} \left[\nabla u \cdot \nabla v + (\lambda V(x) + \ln(1+|x|))uv \right] dx \text{ and}$$
$$\|u\|_{X_\lambda} = \sqrt{(u,u)_{X_\lambda}}, \ \forall u, v \in X_\lambda.$$

Obviously, $\|\cdot\|_{X_\lambda} = \sqrt{\|\cdot\|_{E_\lambda}^2 + \|\cdot\|_*^2}$, where $\|u\|_* = \left(\int_{\mathbb{R}^2} \ln(1+|x|)|u|^2 dx\right)^{\frac{1}{2}}$ for all $u \in X$.

Now, we are in a position to state the first main result in this paper as follows.

Theorem 1.1. *Let V satisfy $(V_1)-(V_3)$ and f require (1.3) with $(f_1)-(f_3)$, then there exist some constants $\gamma^* > 0$, $\kappa^* > 0$, $a^* > 0$ and $\lambda^* > 1$ such that, for every $\gamma \in (0,\gamma^*)$, $\kappa \in (0,\kappa^*)$, $a > a^*$ and $\lambda > \lambda^*$, Problems (1.1)-(1.2) possess a couple of weak solution $(\bar{u},\bar{\mu}) \in X_\lambda \times \mathbb{R}$, where $\bar{u}(x) > 0$ for all $x \in \mathbb{R}^2$.*

Remark 1. Because the nonlinearity f satisfies the critical exponential growth (1.3) at infinity, one can regard the Problem (1.1) under the constraint (1.2) as a L^2-supercritical one. However, taking into account the mild assumptions $(V_1)-(V_3)$ imposed on the potential V, we are never able to follow the procedures in [15,35,53] to derive the desired result. Motivated by [11,62], we shall cut off the nonlinearity and then the original problem reduces to a L^2-subcritical one that makes the solvability available. Nevertheless, considering the negative interactions between the steep potential λV and the nonlocal term $[\ln(|\cdot|) * u^2]u$ together with the structure of the work space X_λ, there exist several unpleasant barriers in the present paper, see Lemmas 3.3 and 3.8 as well as Claim 4.5 below for instance. Finally, it is worthy mentioning here that the so-called cut-off technique of [11,62] and this paper is similar to that of [8–10,61].

Remark 2. Even if $V(x) \equiv 0$ for all $x \in \mathbb{R}^2$ in Eq. (1.1), up to the best knowledge of us, there seems very few results on the existence of normalized solution to the planar Schrödinger–Poisson equation with critical exponential growth, where the nonlinearity satisfies the very mild assumptions $(f_1)-(f_3)$ in contrast to the previous articles [4,28]. As a matter of fact, we also stress here that the argument adopted in the proof of Theorem 1.1 is adapted to the existence of solutions with free mass for a class of planar Schrödinger–Poisson equations with (super)critical exponential growth in [3,5,25,29,30,32,48–50, 59,61] and their references therein.

Let us sketch the main idea for the proof of Theorem 1.1, as explained in Remark 1, we shall heavily depend on the so-called cut-off technique. Describing it more precisely, for every fixed constant $R > 0$, we introduce the following continuous function $f_R : \mathbb{R} \to \mathbb{R}$ defined by

$$f_R(s) = \begin{cases} 0, & \text{if } s \le 0, \\ f(s), & \text{if } 0 \le s \le R, \\ \dfrac{f(R)}{R^{q-1}} s^{q-1}, & \text{if } R \le s < +\infty, \end{cases} \tag{1.11}$$

where the constant $q \in (2,4)$ comes from (f_2). From now on until the end of the present article, we define $F_R(s) = \int_0^s f_R(t)dt$ for each $s \in \mathbb{R}$ to be the primitive function of f_R. It follows from a direct computation that

$$qF_R(s) \le f_R(s)s, \ \forall s \in \mathbb{R}. \tag{1.12}$$

Moreover, we can exploit the monotone assumption in (f_2) to see that

$$f_R(s) \le \frac{f(R)}{R^{q-1}} s^{q-1}, \ \forall s \in \mathbb{R}. \tag{1.13}$$

With such a nonlinearity f_R defied in (1.11), we turn to contemplate the following auxiliary problem

$$\begin{cases} -\Delta u + \lambda V(x)u + \gamma(\ln|\cdot| * |u|^2)u = \mu u + \kappa f_R(u) \text{ in } \mathbb{R}^2, \\ u(x) > 0 \text{ in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u(x)|^2 dx = a^2. \end{cases} \tag{1.14}$$

In view of (1.13), we know from [33] that Problem (1.14) above involves L^2-subcritical growth since $q < 4$. If the term $(\ln|\cdot|*|u|^2)u$ vanishes, the existence result can be seen as a supplement to the results explored by Alves and Ji in [6]. Whereas, because of the appearance of this term, we can not borrow the arguments in it to obtain the existence result for Problem (1.14) due to the difficulties depicted in Remark 1. Anyway, it permits us to reach the existence of solutions for Problem (1.14).

When there is couple of weak solution in $X_\lambda \times \mathbb{R}$ to Problem (1.14), saying it (u_R, μ_R), the reader is invited to find that such pair is a solution to the original Problems (1.1)-(1.2) as long as $|u_R|_\infty \le R$ owing to the definition of f_R in (1.11). Having it in mind, we can receive the proof of Theorem 1.1 combining the solvability of Problem (1.14) and the L^∞-estimate.

Next, we shall contemplate the asymptotical behavior of the normalized solutions obtained in Theorem 1.1 as $\lambda \to +\infty$ when $a > a^*$ is fixed. Let $(u, \mu) \in X_\lambda \times \mathbb{R}$ be a positive normalized solution for Eq. (1.1), there is no doubt that it depends on the parameter $\lambda > \lambda^* > 1$, thereby we will relabeled the pair by $(u_\lambda, \mu_\lambda) \in X_\lambda \times \mathbb{R}$ to emphasize this dependence, where $\lambda^* > 1$ is a constant appearing in the proof of Lemma 3.8 below. Finally, we prove the following result.

Theorem 1.2. *Under the assumptions in Theorem 1.1 and let $a > a^*$ be fixed, then, passing to a subsequence if necessary, $u_\lambda \to u_0$ in X and $\mu_\lambda \to \mu_0$ in $\mathbb{R}$ as $\lambda \to +\infty$, where (u_0, μ_0) is a couple of weak solution to the planar*

Schrödinger–Poisson problem below

$$\begin{cases} -\Delta u + \gamma \left(\int_\Omega \ln(|x-y|) u^2(y) dy \right) u = \mu u + \kappa f(u), \ x \in \Omega, \\ u(x) = 0, \ x \in \partial\Omega, \\ \int_\Omega |u|^2 dx = a^2. \end{cases} \tag{1.15}$$

Remark 3. As far as we are concerned, there are currently no relevant results for the Problem (1.15). Actually, it truly belongs to a class of L^2-supercritical problems in the bounded domains and we would like to refer the interested reader to [55–57] and the references therein to acquaint this topic. In particular, the authors in [10,11,62] had investigated the existence of normalized solution for several kinds of local elliptic problems with mass-supercritical growth. Alternatively, there exist some new difficulties that caused by the nonlocal term $\left(\int_\Omega \ln|x-y| u^2(y) dy\right) u$ which prevents us repeating the approaches in [10,11,62] simply.

We remark that the essential difference between [61] and this paper is whether the obtained solution involves a prescribed mass, so some additional efforts are needed to conclude the proofs of Theorems 1.1 and 1.2 in the paper. Again these results are new for planar Schrödinger–Poisson equation up to now, although the subtle tricks have already appeared in the literatures [10,11,62].

In our opinion, one of the most significant contributions is that we succeed in taking advantage of the steep potential λV to investigate the existence and concentration of positive normalized solutions to the type of planar Schrödinger–Poisson equations with a wider class of nonlinearities that fulfill the critical exponential growth in (1.3). It is believed that the studies in the present paper would prompt some further explorations on related topics.

This paper is organized as follows. In Sect. 2, we will introduce some preliminary results dealing with the functionals $V_i : X_\lambda \to \mathbb{R}$ with $i \in \{0, 1, 2\}$. Section 3 is devoted to the existence result for the auxiliary Problem (1.14) above. Finally, the detailed proofs of Theorems 1.1 and 1.2 shall be exhibited in Sect. 4.

Notations: From now on in this paper, otherwise mentioned, we use the following notations:

- $B_r(x) \subset \mathbb{R}^2$ is an open ball centered at $x \in \mathbb{R}^2$ with radius $r > 0$ and $B_r = B_r(0)$.
- $C, C_1, C_2, \cdots$ denote any positive constant, whose value is not relevant.
- For all $x \in \mathbb{R}^2$, we define

$$u^+(x) \triangleq \max\{u(x), 0\} \geq 0 \text{ and } u^-(x) \triangleq \min\{u(x), 0\} \leq 0.$$

- $|\cdot|_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^2)$, for every $p \in [1, +\infty]$. $\|\cdot\|_{H^i}$ denotes the usual norm of the Hilbert space for $i \in \{1, 2\}$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- “ $\to$″ and “ $\rightharpoonup$″ stand for the strong and weak convergence in the related function spaces, respectively.

- We recall the celebrated Gagliardo-Nirenberg inequality, given an $l \in [2, +\infty)$,

$$|u|_l^l \le \mathbb{C}_{\mathrm{GN}} |u|_2^{(1-\gamma_l)l} |\nabla u|_2^{\gamma_l l} \text{ in } H^1(\mathbb{R}^2),\ \gamma_l = 2\left(\frac{1}{2} - \frac{1}{l}\right), \tag{1.16}$$

where the constant $\mathbb{C}_{\mathrm{GN}} > 0$ is just dependent of l.

2. Preliminary stuff

In this section, we shall present some preliminary results which will be exploited frequently in this paper. The following lemma is due to [34, Lemma 2.2] and we introduce them without the detailed proofs.

Lemma 2.1. *Let the space X and the functionals $V_i : X \to \mathbb{R}$ with $i \in \{0, 1, 2\}$ be defined as in the Introduction, then we have the following conclusions:*

(i) *The space X is compactly embedded in $L^s(\mathbb{R}^2)$, for all $s \in [2, \infty)$.*

(ii) $0 \le V_1(u) \le 2|u|_2^2 |u|_*^2 \le 2\|u\|_X^4$ *and V_1 is weakly semicontinuous in $H^1(\mathbb{R}^2)$.*

(iii) $V_i'(u)[u] = 4V_i(u)$ *for all $u \in X$ and $i \in \{0, 1, 2\}$.*

(iv) *There is a constant $K_0 > 0$ such that*

$$|V_2(u)| \le K_0 |u|_{\frac{8}{3}}^4, \ \forall u \in L^{\frac{8}{3}}(\mathbb{R}^2). \tag{2.1}$$

(v) *V_2 is completely continuous in X, that is,*

$$u_n \rightharpoonup u \text{ in } X \Longrightarrow V_2(u_n) \to V_2(u).$$

For any given Lebesgue measurable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, we introduce the following three auxiliary symmetric bilinear forms $X \times X \to \mathbb{R}$

$$(u, v) \mapsto B_1(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1 + |x - y|) u(x) v(y) dx dy,$$

$$(u, v) \mapsto B_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x - y|}\right) u(x) v(y) dx dy,$$

$$(u, v) \mapsto B(u, v) = B_1(u, v) - B_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x - y|) u(x) v(y) dx dy.$$

Obviously, $V_i'(u)[v] = 4B_i(u^2, uv)$ for $i \in \{0, 1, 2\}$. By [34, Lemma 2.6], there holds

Lemma 2.2. *Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be some bounded sequences in X such that $u_n \rightharpoonup u$ in X. Then, for all $z \in X$, it holds that $B_1(v_n w_n, z(u_n - u)) \to 0$, as $n \to +\infty$.*

Taking into account the uniform L^∞-estimate, the following lemma will plays a crucial role.

Lemma 2.3. *Define* $g_u(x) \triangleq \ln(1+|\cdot|^{-1})*|u|^2$ *for all* $u \in H^1(\mathbb{R}^2)$ *and* $x \in \mathbb{R}^2$, *then we have that* $g_u \in L^\infty(\mathbb{R}^2)$. *Moreover*

$$|g_u|_\infty \leq 4\pi \left(|u|_2^2 + \mathbb{C}^{\frac{1}{3}} |u|_2^{\frac{2}{3}} |\nabla u|_2^{\frac{4}{3}} \right). \tag{2.2}$$

Proof. We follow [61, Lemma 2.3] to conclude the proof. For all $x \in \mathbb{R}^2$, there holds

$$\begin{aligned} |g_u(x)| &= \int_{B_1(x)} \ln\left(1 + \frac{1}{|x-y|}\right) |u(y)|^2 dy \\ &\quad + \int_{\mathbb{R}^2 \setminus B_1(x)} \ln\left(1 + \frac{1}{|x-y|}\right) |u(y)|^2 dy \\ &\leq \int_{B_1(x)} \frac{|u(y)|^2}{|x-y|} dy + \ln 2 \int_{\mathbb{R}^2 \setminus B_1(x)} |u(y)|^2 dy. \end{aligned} \tag{2.3}$$

It follows from the Hölder's inequality that

$$\begin{aligned} \int_{B_1(x)} \frac{|u_R(y)|^2}{|x-y|} dy &\leq \left(\int_{B_1(x)} \frac{1}{|x-y|^{\frac{3}{2}}} dy \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^2} |u(y)|^6 dy \right)^{\frac{1}{3}} \\ &= (4\pi)^{\frac{2}{3}} \left(\int_{\mathbb{R}^2} |u(y)|^6 dy \right)^{\frac{1}{3}}. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we apply (1.16) with $l = 6$ to get the desired result (2.2). The proof is completed. □

3. On the auxiliary problem

In this section, we are going to investigate the existence of solutions for the auxiliary problem (1.14). More precisely, we shall contemplate the following planar Schrödinger–Poisson equation

$$-\Delta u + \lambda V(x)u + \gamma(\ln|\cdot| * |u|^2)u = \mu u + \kappa f_R(u) \text{ in } \mathbb{R}^2, \tag{3.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 dx = a^2, \tag{3.2}$$

where $a > 0$, $\mu \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $\lambda, \gamma, \kappa > 0$ are parameters. The potential $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies $(V_1) - (V_3)$ and the nonlinearity f_R is defined by (1.11) meeting (1.3) and $(f_1) - (f_3)$.

We recall that a solution u to the Problems (3.1)-(3.2) corresponds to a critical point of the variational functional $\mathcal{J}_{\lambda,R} : X_\lambda \to \mathbb{R}$ below

$$\mathcal{J}_{\lambda,R}(u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)|u|^2] dx + \frac{\gamma}{4} V_0(u) - \kappa \int_{\mathbb{R}^2} F_R(u) dx \tag{3.3}$$

restricted to the sphere

$$S(a) = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2 dx = a^2 \right\}. \tag{3.4}$$

Here the functional V_0 and the work space X_λ could be reviewed in (1.8) and (1.10), respectively. Recalling [61, Lemma 2.4] again, for all fixed $\lambda \geq 1$ and $R > 0$, we can infer from (f_1) that the variational functional $\mathcal{J}_{\lambda,R}$ is well-defined and belongs to $\mathcal{C}^1(X_\lambda, \mathbb{R})$ with its derivative defined by

$$\mathcal{J}'_{\lambda,R}(u)[v]=\int_{\mathbb{R}^2}[\nabla u\nabla v+\lambda V(x)uv]dx+\frac{\gamma}{4}V'_0(u)[v]-\kappa\int_{\mathbb{R}^2} f_R(u)v\,dx,\ \forall u,v\in X_\lambda.$$

The main result concerning Problems (3.1)-(3.2) is the following:

Theorem 3.1. *Let V satisfy $(V_1)-(V_3)$ and f meet (1.3) with $(f_1)-(f_3)$, then there exists an $R^* > 0$ such that for all fixed $R > R^*$ and $\kappa \in (0, 1)$, there are some constants $a^* = a^*(R) > 0$, $\gamma' = \gamma'(R) > 0$ and $\lambda^* = \lambda^*(R) > 1$ such that the Problems (3.1)-(3.2) have a couple of weak solution $(u_R, \mu_R) \in X_\lambda \times \mathbb{R}$ with $u_R(x) > 0$ for all $x \in \mathbb{R}^2$ if $a > a^*$, $\gamma \in (0, \gamma')$ and $\lambda > \lambda^*$.*

The proof of the above theorem will be divided into several lemmas. Before exhibiting them, we will always suppose that the potential V and the nonlinearity f_R do satisfy $(V_1)-(V_3)$ and (1.3) with $(f_1)-(f_3)$ in this section, respectively.

Lemma 3.2. *For all fixed $R > 0$, the variational functional $\mathcal{J}_{\lambda,R}$ is coercive and bounded from below on $S(a)$ for all $a > 0$, $\gamma \in (0, 1)$, $\kappa \in (0, 1)$ and $\lambda \geq 1$, where $\mathcal{J}_{\lambda,R}$ and $S(a)$ are appearing in (3.3) and (3.4), respectively.*

Proof. By (1.12)-(1.13) and (2.1), for all $u \in S(a)$, we use (1.16) with $l = \frac{8}{3}$ and $l = q$ to reach

$$\begin{aligned}\mathcal{J}_{\lambda,R}(u) \geq \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - K_0\mathbb{C}_{\rm GN}^{\frac{3}{2}}a^3\left(\int_{\mathbb{R}^2}|\nabla u|^2\,dx\right)^{\frac{1}{2}}\\ -\frac{\mathbb{C}_{\rm GN}f(R)a^{(1-\gamma_q)q}}{R^{q-1}q}\left(\int_{\mathbb{R}^2}|\nabla u|^2\,dx\right)^{\frac{\gamma_q q}{2}}.\end{aligned}$$

As $q \in (2, 4)$ in (f_2), clearly $\gamma_q q < 2$, then the statement concludes. □

As a direct consequence of Lemma 3.2, for each fixed $R > 0$, $\gamma \in (0, 1)$, $\kappa \in (0, 1)$, $a > 0$ and $\lambda \geq 1$, the real number

$$\Upsilon_{\lambda,R} \triangleq \min_{u\in S(a)} \mathcal{J}_{\lambda,R}(u) \tag{3.5}$$

is well-defined and it will be exploited to look for nontrivial solutions for Problems (3.1)-(3.2). Alternatively, we need to conclude that $\Upsilon_{\lambda,R}$ is uniformly bounded above with respect to $\lambda > 1$ and so there is the result below.

Lemma 3.3. *There exists an $R^* > 0$ such that for each fixed $R > R^*$ and $\kappa \in (0, 1)$, there are constants $\Theta_R = \Theta(R) < 0$, $\gamma' = \gamma'(R) > 0$ and $a^* = a^*(R) > 0$, which are independent of λ, such that $\Upsilon_{\lambda,R} \leq \Theta_R$ for all $\gamma \in (0, \gamma')$, $a > a^*$ and $\lambda \geq 1$.*

Proof. Without loss of generality, we are assuming that $0 \in {\rm int}V^{-1}(0)$. Therefore, there exists a sufficiently small $r > 0$ such that $B_r(0) \subset {\rm int}V^{-1}(0)$.

Choose $\psi \in C_0^\infty(B_r(0))$ to be a function satisfying $\int_{B_r(0)} |\psi|^2 dx = 1$ and so $\psi \in S(1)$. Due to the definition of Ω, there holds

$$\int_{\mathbb{R}^2} V(x)|\psi|^2 dx = \int_\Omega V(x)|\psi|^2 dx + \int_{\Omega^c} V(x)|\psi|^2 dx = 0. \tag{3.6}$$

Adopting (1.3), then $\lim_{R\to+\infty} \frac{f(R)}{R^{q-1}} = +\infty$ and so there is an $R^* > 0$ such that $\frac{f(R)}{R^{q-1}} \geq c_0$ for all $R > R^*$. Owing to (f_3) and the definition of f_R in (1.11), it holds that

$$f_R(s) \geq c_0 s^{q-1},\ \forall s \geq 0 \text{ and } R > R^*,$$

from where it follows that

$$\int_{\mathbb{R}^2} F_R(t\psi)\, dx \geq \frac{c_0 t^q}{q} \int_{\mathbb{R}^2} |\psi|^q\, dx,\ \forall t > 0 \text{ and } R > R^*. \tag{3.7}$$

Combining (3.6) and (3.7), for all $R > R^*$, one sees $\mathcal{I}_{\lambda,R}(t\psi) \to -\infty$ as $t \to +\infty$, where

$$\mathcal{I}_{\lambda,R}(u) \triangleq \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + \lambda V(x)|u|^2\right] dx - \kappa \int_{\mathbb{R}^2} F_R(u)dx,\ \forall u \in E_\lambda.$$

Consequently, we can determine some $t^* = t^*(R) > 0$ and $\mathcal{A}_{t^*} < 0$ to satisfy $\mathcal{I}_{\lambda,R}(t\psi) \leq \mathcal{A}_{t^*} < 0$ for all $t > t^*$. Letting $a^* = t^*$, for all $a > a^*$ and $u_0 = a\psi$, then $u_0 \in S(a)$ and so

$$\mathcal{I}_{\lambda,R}(u_0) = \mathcal{I}_{\lambda,R}(a\psi) \leq \mathcal{A}_{t^*},\ \forall R > R^* \text{ and } a > a^*. \tag{3.8}$$

On the other hand, using Lemma 2.1-(ii) and $V_0(u_0) = V_1(u_0) - V_2(u_0) \leq V_1(u_0)$, we have

$$V_1(u_0) \leq 2|u_0|_2^2 \int_{\mathbb{R}^2} \ln(1+|x|)|u_0|^2 dx \leq 2a^4 \ln(1+r). \tag{3.9}$$

As a consequence of (3.8) and (3.9), we reach

$$\Upsilon_{\lambda,R} \leq \mathcal{I}_{\lambda,R}(u_0) + \frac{\gamma}{4} V_1(u_0) \leq \mathcal{A}_{t^*} + \frac{\gamma}{2} a^4 \ln(1+r).$$

Thereby, define $\gamma' = \min\left\{-\frac{\mathcal{A}_{t^*}}{a^4 \ln(1+r)}, 1\right\} > 0$ for all $R > R^*$ and $a > a^*$. So, it permits us to choose $\Theta_R = \Theta(R) \triangleq \frac{1}{2}\mathcal{A}_{t^*} < 0$ for all $R > R^*$, $\kappa \in (0,1)$, $\gamma \in (0, \gamma')$, $a > a^*$ and $\lambda \geq 1$. The proof is completed. □

With Lemmas 3.2 and 3.3 in hands, we can just deduce that every minimizing sequence $\{u_n\}$ of $\Upsilon_{\lambda,R}$ is uniformly bounded in $H^1(\mathbb{R}^2)$. To derive such a desired result in the work space X_λ, we have to take some delicate analysis. First of all, let us introduce the following two lemmas due to Lions [47].

Lemma 3.4. *Let $\{\rho_n\} \subset L^1(\mathbb{R}^2)$ be a bounded sequence and $\rho_n \geq 0$, then there is a subsequence, still denoted by ρ_n, such that one of the following two possibilities occurs:*

(i) (Vanishing) $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} \rho_n dx = 0$ *for all* $\varrho > 0$*;*

(ii) (Non-Vanishing) *there are* $\beta > 0$ *and* $\varrho < +\infty$ *such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} \rho_n dx = \beta.$$

Lemma 3.5. *Assume* $\{|\nabla u_n|\}$ *is bounded in* $L^2(\mathbb{R}^2)$ *and* $\{u_n\}$ *is bounded in* $L^{q_0}(\mathbb{R}^2)$ *for some* $q_0 > 2$ *as well as*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} |u_n|^{q_0} dx = 0.$$

Then $u_n \to 0$ *in* $L^s(\mathbb{R}^2)$ *for* $s \in (2, +\infty)$ *if in addition* $\{u_n\} \subset S(a)$.

We now begin with the verification that $\{u_n\}$ is uniformly bounded in X_λ.

Lemma 3.6. *Let* $R > R^*$, $a > a^*$, $\kappa \in (0, 1)$, $\gamma \in (0, \gamma')$ *and* $\lambda \geq 1$ *be fixed. Suppose* $\{u_n\} \subset S(a)$ *is a minimizing sequence of* $\Upsilon_{\lambda,R}$, *then* $\{\|u_n\|_{E_\lambda}\}$ *is uniformly bounded in* $n \in \mathbb{N}$. *Moreover, for all* $\varrho > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} |u_n|^{q_0} dx = 0 \tag{3.10}$$

could never occur, where $q_0 > 2$.

Proof. Using Lemma 3.2, we know that $\{|\nabla u_n|_2\}$ is uniformly bounded in $n \in \mathbb{N}$ and so $\{|u_n|_r\}$ is uniformly bounded in $n \in \mathbb{N}$ for all $r \in [2, +\infty)$ by (1.16). Because $\mathcal{J}_{\lambda,R}(u_n) = \Upsilon_{\lambda,R} + o_n(1)$, combining Lemma 3.3, (1.13) and (2.1), we have that

$$\begin{aligned}\int_{\mathbb{R}^2} \lambda V(x)|u_n|^2 dx &\leq 2\Upsilon_{\lambda,R} + \frac{1}{2}V_2(u_n) + 2\kappa \int_{\mathbb{R}^2} F_R(u_n) + o_n(1) \\ &\leq 2\Theta_R + \frac{1}{2}K_0|u_n|^4_{\frac{8}{3}} + \frac{2\kappa F(R)}{qR^{q-1}} \int_{\mathbb{R}^2} |u_n|^q dx + o_n(1)\end{aligned}$$

showing the first part of this lemma. To reach the remainder, arguing it indirectly, we suppose that (3.10) holds true. As a consequence, $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for all $s \in (2, +\infty)$ by Lemma 3.5. Adopting (1.13) and (2.1) again, we have that

$$\int_{\mathbb{R}^2} F_R(u_n) = o_n(1) \text{ and } V_2(u_n) = o_n(1),$$

from where it follows that

$$\Upsilon_{\lambda,R} = \frac{1}{2}\int_{\mathbb{R}^2} [|\nabla u_n|^2 + \lambda V(x)|u_n|^2]dx + \frac{1}{4}V_1(u_n) + o_n(1) \geq o_n(1).$$

It is impossible because of Lemma 3.3. The proof is completed. □

Thanks to Lemma 3.4, with the help of Lemma 3.6, we are able to prove the following result which is crucial to prove that the sequence $\{u_n\} \subset X_\lambda$ is uniformly bounded in X_λ.

Lemma 3.7. *Under the assumptions in Lemma 3.6, for any $q_0 \in (2, +\infty)$, there is a constant $\beta_0 > 0$, independent of $\lambda \geq 1$, such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} |u_n|^{q_0} dx = \beta_0.$$

Proof. Let $\rho_n = |u_n|^{q_0} \in L^1(\mathbb{R}^2)$, we immediately see that only the **Non-Vanishing** in Lemma 3.4 occurs due to Lemma 3.6. Then, we divide the proof into intermediate steps.

Step 1: There exists a constant $\beta_\lambda = \beta(\lambda) > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} |u_n|^{q_0} dx = \beta_\lambda.$$

Suppose, by contradiction, that $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s \in (2, +\infty)$. It is very similar to the proof of Lemma 3.6, we can arrive at a contradiction.

Step 2: Conclusion.

Suppose by contradiction that the uniform control from below of $L^{q_0}(\mathbb{R}^2)$-norm is false. So, for any $k \in \mathbb{N}$, $k \neq 0$, there exist $\lambda_k > 1$ and a minimizing sequence $\{u_{k,n}\}$ of $\Upsilon_{\lambda_k,R}$ such that

$$|u_{k,n}|_{q_0} < \frac{1}{k}, \text{ definitely.}$$

Then, by a diagonalization argument, for any $k \geq 1$, it permits us to find an increasing sequence $\{n_k\}$ in $\mathbb{N}$ and $u_{n_k} \in X_{\lambda_{n_k}}$ such that

$$\{u_{n_k}\} \subset S(a),\ \mathcal{J}_{\lambda_{n_k},R}(u_{n_k}) = \Upsilon_{\lambda_{n_k},R} + o_k(1) \text{ and } |u_{n_k}|_{q_0} = o_k(1).$$

where $o_k(1) \to 0$ as $k \to +\infty$. In this situation, we can repeat the proof of Lemma 3.6 to reach a contradiction, again. The proof of this lemma is finished. □

At this stage, we are available to verify that the minimizing sequence $\{u_n\} \subset X_\lambda$ in Lemma 3.6 is uniformly bounded in X_λ for some sufficiently large $\lambda > 1$.

Lemma 3.8. *Let $R > R^*$, $\kappa \in (0, 1)$, $\gamma \in (0, \gamma')$ and $\lambda \geq 1$ be fixed. Suppose $\{u_n\} \subset S(a)$ is a minimizing sequence of $\Upsilon_{\lambda,R}$ for all $a > a^*$, then there exists a $\lambda^* = \lambda^*(R) > 1$ such that the sequence $\{\|u_n\|_{X_\lambda}\}$ is uniformly bounded in $n \in \mathbb{N}$ provided $\lambda > \lambda^*$.*

Proof. Although the proof originates from [61, Lemma 3.10], we exhibit the detailed proofs for the sake of reader. Combining Lemmas 3.6 and 3.7, there is a constant $\beta_0 > 0$, independent of $\lambda \geq 1$, such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} |u_n|^2 dx = \beta_0,$$

where we have just supposed that $\varrho = 1$ in Lemma 3.7 for simplicity. Exacting a subsequence if necessary, there exists a sequence $\{y_n\} \subset \mathbb{R}^2$ such that

$$\int_{B_1(y_n)} |u_n|^2 dx = \frac{1}{2}\beta_0. \tag{3.11}$$

Claim 3.9. The sequence $\{y_n\}$ above is uniformly bounded in $n \in \mathbb{N}$.

Otherwise, we could suppose that $|y_n| \to \infty$ in the sense of a subsequence. Define

$$\Xi_n^1 \triangleq \{x \in B_1(y_n) : V(x) < b\} \text{ and } \Xi_n^2 \triangleq \{x \in B_1(y_n) : V(x) \geq b\}.$$

Since the set $\Xi \triangleq \{x \in \mathbb{R}^2 : V(x) < b\}$ is nonempty and has finite measure, one concludes that

$$\text{meas}(\Xi_n^1) \leq \text{ meas}(\{x \in \mathbb{R}^2 : |x| \geq |y_n| - 2, V(x) < b\}) \to 0 \text{ as } n \to \infty. \tag{3.12}$$

In view of Lemma 3.6, $|u_n|_r$ with $r > 2$ is uniformly bounded in $n \in \mathbb{N}$, using (3.12) to get

$$\int_{\Xi_n^1} |u_n|^2 dx \leq [\text{meas}(\Xi_n^1)]^{\frac{r-2}{r}} |u_n|_r^2 = o_n(1)$$

leading to

$$\int_{\Xi_n^2} |u_n|^2 dx = \int_{B_1(y_n)} |u_n|^2 dx - \int_{\Xi_n^1} |u_n|^2 dx = \frac{1}{2}\beta_0 + o_n(1).$$

Thanks to $V(x) \geq 0$ for all $x \in \mathbb{R}^2$ by (V_1), using the definition of Ξ_n^2,

$$\int_{\mathbb{R}^2} V(x)|u_n|^2 dx \geq \int_{\Xi_n^2} V(x)|u_n|^2 dx \geq b \int_{\Xi_n^2} |u_n|^2 dx = \frac{1}{2} b\beta_0 + o_n(1). \tag{3.13}$$

Recalling the proof of Lemma 3.6 again, we have that

$$\{V_2(u_n)\} \text{ and } \left\{\int_{\mathbb{R}^2} F_R(u_n) dx\right\} \text{ are uniformly bounded in } n \in \mathbb{N} \text{ and } \lambda \geq 1. \tag{3.14}$$

So, as a consequence of (3.13) and (3.14), it holds that

$$\begin{aligned} \Upsilon_{\lambda,R} &\geq \frac{1}{2} \int_{\mathbb{R}^2} \lambda V(x)|u_n|^2 dx - \frac{1}{4} V_2(u_n) - \int_{\mathbb{R}^2} F_R(u_n) dx + o_n(1) \\ &\geq \frac{\lambda b \beta_0}{4} - C + o_n(1) \end{aligned} \tag{3.15}$$

where the positive constants b, β_0 and C are independent of $\lambda \geq 1$. According to Lemma 3.3, there exists a sufficiently large $\lambda^* = \lambda^*(R) > 1$ such that (3.15) is false provided $\lambda > \lambda^*$. Hence, the sequence $\{y_n\} \subset \mathbb{R}^2$ appearing in (3.11) is uniformly bounded in $n \in \mathbb{N}$.

Consequently, passing to a subsequence if necessary, we suppose that $y_n \to y_0$ in $\mathbb{R}^2$. Taking (3.11) into account, there holds

$$\int_{B_2(y_0)} |u_n|^2 dx \geq \frac{1}{4}\beta_0 > 0. \tag{3.16}$$

Since $\{\|u_n\|_{E_\lambda}\}$ is uniformly bounded in $n \in \mathbb{N}$ by Lemma 3.6, the proof of this lemma would be done by the following claim

Claim 3.10. The sequence $\|u_n\|_* = \left(\int_{\mathbb{R}^2} \ln(1+|x|)u_n^2 dx\right)^{\frac{1}{2}}$ is uniformly bounded in $n \in \mathbb{N}$.

Indeed, we choose a constant $\delta > 0$ large enough to satisfy $\delta > |y_0| + 2$. Moreover, one has

$$1 + |x-y| \geq 1 + \frac{|y|}{2} \geq \sqrt{1+|y|},\ \forall x \in B_\delta(0),\ \forall y \in \mathbb{R}^2 \backslash B_{2\delta}(0).$$

Due to this choice for δ implying that $B_2(y_0) \subset B_\delta(0)$, by means of (3.16),

$$\begin{aligned} V_1(u_n) &= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \ln(1+|x-y|)u_n^2(x)dx \right) u_n^2(y)dy \\ &\geq \int_{\mathbb{R}^2 \backslash B_{2\delta}(0)} \left(\int_{B_\delta(0)} \ln(1+|x-y|)u_n^2(x)dx \right) u_n^2(y)dy \\ &\geq \left(\int_{B_\delta(0)} u_n^2(x)dx \right) \left[\int_{\mathbb{R}^2 \backslash B_{2\delta}(0)} \ln\left(1+\frac{|y|}{2}\right) u_n^2(y)dy \right] \\ &\geq \frac{\beta_0}{8} \int_{\mathbb{R}^2 \backslash B_{2\delta}(0)} \ln(1+|y|)u_n^2(y)dy \\ &= \frac{\beta_0}{8} \left(\|u_n\|_*^2 - \int_{B_{2\delta}(0)} \ln(1+|y|)u_n^2(y)dy \right) \\ &\geq \frac{\beta_0}{8} (\|u_n\|_*^2 - \ln(1+2\delta)a^2). \end{aligned} \tag{3.17}$$

On the other hand, adopting (3.14) and Lemma 3.2 again,

$$0 \leq \gamma V_1(u_n) \leq 4\Theta_R + \gamma V_2(u_n) + 4\kappa \int_{\mathbb{R}^2} F_R(u_n)dx + o_n(1) \tag{3.18}$$

which together with (3.17) concludes the claim. The proof is completed. □

We now can show the proof of Theorem 3.1 in detail.

Proof of Theorem 3.1. First of all, by Lemma 3.2, we know that the minimum $\Upsilon_{\lambda,R}$ defined in (3.5) is well-defined for all fixed $R > 0$, $\gamma \in (0,1)$, $\kappa \in (0,1)$, $a > 0$ and $\lambda \geq 1$. Secondly, there exists a sequence $\{u_n\} \subset S(a)$ such that $J_{\lambda,R}(u_n) = \Upsilon_{\lambda,R} + o_n(1)$. According to Lemma 3.8, there exist $R^* > 0$, $\gamma' > 0$, $a^* > 0$ and $\lambda^* > 1$ such that, for all fixed $R > R^*$, $\gamma \in (0,\gamma')$, $a > a^*$ and $\lambda > \lambda^*$, the sequence $\{\|u_n\|\}_{X_\lambda}$ is uniformly bounded in $n \in \mathbb{N}$ for every $\kappa \in (0,1)$. Let us take Lemma 2.1-(i) into account, passing to a subsequence if necessary, there is a $u_R \in X_\lambda$ such that $u_n \rightharpoonup u_R$ in X_λ, $u_n \to u_R$ in $L^s(\mathbb{R}^2)$ for each $s \in [2,+\infty)$ and $u_n \to u_R$ a.e. in $\mathbb{R}^2$. So, one immediately concludes that $u_R \in S(a)$. Then, as a consequence of Lemma 2.1-(v) and (1.13), it holds that

$$\lim_{n\to\infty} V_2(u_n) = V_2(u_R)$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} F_R(u_n)dx = \int_{\mathbb{R}^2} F_R(u_R)dx.$$

Combining the above two facts and the Fatou's lemma, we have that

$$\begin{aligned}\Upsilon_{\lambda,R} &\le \frac{1}{2}\int_{\mathbb{R}^2}[|\nabla u_R|^2+\lambda V(x)|u_R|^2]dx+\frac{\gamma}{4}V_0(u_R)-\kappa\int_{\mathbb{R}^2}F_R(u_R)dx\\ &\le \liminf_{n\to\infty}\left\{\frac{1}{2}\int_{\mathbb{R}^2}[|\nabla u_n|^2+\lambda V(x)|u_n|^2]dx+\frac{\gamma}{4}V_0(u_n)-\kappa\int_{\mathbb{R}^2}F_R(u_n)dx\right\}\\ &= \liminf_{n\to\infty}\mathcal{J}_{\lambda,R}(u_n)=\Upsilon_{\lambda,R}\end{aligned}$$

which indicates that u_R is a minimizer of $\Upsilon_{\lambda,R}$ for every $R>R^*$, $\kappa\in(0,1)$, $\gamma\in(0,\gamma')$, $a>a^*$ and $\lambda>\lambda^*$. Thereby, thanks to the Lagrange multiplier theorem, there is a $\mu_R\in\mathbb{R}$ such that (u_R,μ_R) is a couple of weak solution to Problems (3.1)-(3.2).

Finally, to conclude the proof we are going to certify that u_R is in fact positive in the whole $\mathbb{R}^2$. To see why, we recall that (u_R,μ_R) is a couple of weak solution to Problems (3.1)-(3.2). Owing to (f_1), one sees that u_R is nonnegative in $\mathbb{R}^2$. Moreover, (u_R,μ_R) satisfies the equality below

$$\begin{aligned}-\Delta u_R+\hat{V}_{\lambda,R}(x)u_R=\gamma\left[\ln\left(1+\frac{1}{|x|}\right)*|u_R|^2\right]u_R\\ +(\mu_R+|\mu_R|)u_R+\kappa f_R(u_R)\ \text{ in }\ \mathbb{R}^2,\end{aligned}$$

where

$$\hat{V}_{\lambda,R}(x)\triangleq\lambda V(x)+\gamma\left[\ln(1+|x|)*|u_R|^2\right](x)+|\mu_R|,\ \forall x\in\mathbb{R}^2.$$

The following elementary inequality

$$\ln(1+|x-y|)\le\ln(1+|x|)+\ln(1+|y|),\ \forall x,y\in\mathbb{R}^2,$$

implies that

$$\begin{aligned}\hat{V}_1(x)\triangleq\left[\ln(1+|\cdot|)*|u_R|^2\right](x)&=\int_{\mathbb{R}^2}\ln(1+|x-y|)|u_R(y)|^2\,dy\\ &\le\ln(1+|x|)\int_{\mathbb{R}^2}|u_R(y)|^2\,dy+\int_{\mathbb{R}^2}\ln(1+|y|)|u_R(y)|^2\,dy.\end{aligned}$$

By means of the fact that $u_R\in X_\lambda$ and $|u_R|_2=a$, there is $C>0$ such that

$$0\le\hat{V}_1(x)\le a^2\ln(1+|x|)+C,\ \forall x\in\mathbb{R}^2,$$

and so, $\hat{V}_{\lambda,R}\in L^\infty_{\text{loc}}(\mathbb{R}^2)$ for all fixed $R>R^*$, $\kappa\in(0,1)$, $\gamma\in(0,\gamma^*)$, $a>a^*$ and $\lambda>\lambda^*$. Using the elliptic regularity theory, we know that $u_R\in W^{2,s}(\mathbb{R}^2)$ for all $s\in[2,+\infty)$ and

$$-\Delta u_R+\hat{V}_{\lambda,R}(x)u_R\ge0\text{ in }\mathcal{D}'(\mathbb{R}^2).$$

Since $\hat{V}_{\lambda,R}(x)\ge0$ for all $x\in\mathbb{R}^2$ and $u_R\gneqq 0$, we apply the strong maximum principle developed by Gilbarg and Trundiger [40, Theorem 8.19] to conclude that $u_R(x)>0$ for all $x\in\mathbb{R}^2$, proving the desired result. Then proof is completed.

4. Proofs of the main results

In this section, we address the existence and concentration of positive solutions to the planar Schrödinger–Poisson equation (1.1) under the mass-constraint (1.2).

4.1. Preliminary lemmas

First of all, there are some growth conditions for the nonlinearity f and f_R which play crucial roles in this section. It follows from $(f_1)-(f_2)$ that

$$\lim_{s\to 0^+}\frac{f_R(s)}{s}=0 \text{ and } \lim_{s\to 0^+}\frac{f(s)}{s}=0. \tag{4.1}$$

In fact, we are derived from (f_1) and (f_2) with $q>2$ that

$$0\leq \lim_{s\to 0^+}\frac{f_R(s)}{s}=\lim_{s\to 0^+}\frac{f(s)}{s}=\lim_{s\to 0^+}\frac{f(s)}{s^{q-1}}s^{q-2}\leq f(1)\lim_{s\to 0^+}s^{q-2}=0.$$

Combining (1.3) and (4.1), given a fixed $\varepsilon>0$, for every $\bar p>2$ and $\nu>1$, we are able to search for two constants such that $\tilde b_1=\tilde b_1(\bar p,\alpha,\varepsilon)>0$ and $\tilde b_2=\tilde b_2(\bar p,\alpha,\varepsilon)>0$ satisfying

$$|f(s)|\leq \varepsilon|s|+\tilde b|s|^{\bar p-1}(e^{4\pi\nu s^2}-1),\ \forall s\in\mathbb{R}, \tag{4.2}$$

and

$$|F(s)|\leq \varepsilon|s|^2+\tilde b|s|^{\bar p}(e^{4\pi\nu s^2}-1),\ \forall s\in\mathbb{R}. \tag{4.3}$$

Because the nonlinearity f admits the critical exponential growth at infinity, we introduce the famous Trudinger-Moser inequality found in [23,54,66].

Lemma 4.1. *If $\alpha>0$ and $u\in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2}(e^{\alpha|u|^2}-1)dx<+\infty.$$

Moreover, if $|\nabla u|_2^2\leq 1$, $|u|_2^2\leq M<+\infty$ and $\alpha<4\pi$, then there exists $K_{\alpha,M}=K(M,\alpha)$ such that

$$\int_{\mathbb{R}^2}(e^{\alpha|u|^2}-1)dx\leq K_{\alpha,M}. \tag{4.4}$$

Next, we recall from Theorem 3.1 that the minimization constant $\Upsilon_{\lambda,R}$ defined in (3.5) can be achieved by some nontrivial function in X_λ for every fixed $R>R^*$, $\kappa\in(0,1)$, $\gamma\in(0,\gamma')$, $a>a^*$ and $\lambda>\lambda^*$. In other words, there is a function $u_R\in X_\lambda$ such that

$$u_R\in S(a) \text{ and } \mathcal{J}_{\lambda,R}(u_R)=\Upsilon_{\lambda,R},\\ \forall R>R^*,\ \kappa\in(0,1),\ \gamma\in(0,\gamma'),\ a>a^* \text{ and } \lambda>\lambda^*. \tag{4.5}$$

Moreover, there is a $\mu_R\in\mathbb{R}$ such that the couple (u_R,μ_R) is a solution of Problems (3.1)-(3.2) for all $R>R^*$, $\kappa\in(0,1)$, $\gamma\in(0,\gamma')$, $a>a^*$ and $\lambda>\lambda^*$, where $u_R(x)>0$ for all $x\in\mathbb{R}^2$.

According to the discussions in the Introduction, the reader could observe that if u_R in (4.5) satisfies $|u_R|_\infty\leq R$, then u_R is in fact a solution of the original Eq. (1.1) with $\mu = \mu_R$, thereby it is available to arrive at the proof of Theorem 1.1. As a consequence, the foremost objection for us is to take the L^∞-estimate on u_R.

For the purpose above, we firstly have the uniform estimate on $|\nabla u_R|_2^2$ below.

Lemma 4.2. *Suppose that V satisfies $(V_1)-(V_3)$ and f meets (1.3) with $(f_1)-(f_2)$. Let u_R be given by (4.5) for all $R > R^*$, $a > a^*$ and $\lambda > \lambda^*$, then there exist some $\kappa^* = \kappa^*(R) > 0$ and $\gamma^* = \gamma^*(R) > 0$ such that if $\kappa \in (0, \kappa^*)$ and $\gamma \in (0, \gamma^*)$, it holds that $|\nabla u_R|_2^2 < \frac{1}{2\nu^2}$ for all $\lambda > \lambda^*$, where the constant $\nu > 1$ is appearing in (4.2) and (4.3).*

Proof. We continue to argue as in Lemma 3.2 to get

$$\mathcal{J}_{\lambda,R}(u) \geq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \gamma K_0\, \mathbb{C}_{\mathrm{GN}}^{\frac{3}{2}} a^3 \left(\int_{\mathbb{R}^2} |\nabla u|^2\,dx\right)^{\frac{1}{2}} - \frac{\kappa \mathbb{C}_{\mathrm{GN}} f(R) a^{(1-\gamma_q)q}}{R^{q-1}q}\left(\int_{\mathbb{R}^2} |\nabla u|^2\,dx\right)^{\frac{\gamma_q q}{2}}$$

for all $u \in S(a)$. Since $\gamma_q q < 2$, taking Young's inequality into account, there are two constants $c_1, c_2 > 0$, independent of $R > R^*$ and $\lambda > \lambda^*$, such that

$$\frac{\kappa \mathbb{C}_{\mathrm{GN}} f(R) a^{(1-\gamma_q)q}}{R^{q-1}q}\left(\int_{\mathbb{R}^2} |\nabla u|^2\,dx\right)^{\frac{\gamma_q q}{2}} \leq c_1\left[\frac{\kappa \mathbb{C}_{\mathrm{GN}} f(R) a^{(1-\gamma_q)q}}{R^{q-1}q}\right]^{\frac{2}{2-\gamma_q q}} + \frac{1}{8}\int_{\mathbb{R}^2} |\nabla u|^2\,dx$$

and

$$\gamma K_0 \mathbb{C}_{\mathrm{GN}}^{\frac{3}{2}} a^3 \left(\int_{\mathbb{R}^2} |\nabla u|^2\,dx\right)^{\frac{1}{2}} \leq c_2(\gamma K_0 \mathbb{C}_{\mathrm{GN}}^{\frac{3}{2}} a^3)^2 + \frac{1}{8}\int_{\mathbb{R}^2} |\nabla u|^2\,dx.$$

Therefore, for the minimizer u_R of $\Upsilon_{\lambda,R} = \min_{u\in S(a)} \mathcal{J}_{\lambda,R}(u)$ in (3.5), we obtain

$$|\nabla u_R|_2^2 \leq 4\mathcal{J}_{\lambda,R}(u_R) + 4c_1\left[\frac{\kappa \mathbb{C}_{\mathrm{GN}} f(R) a^{(1-\gamma_q)q}}{R^{q-1}q}\right]^{\frac{2}{2-\gamma_q q}} + 4c_2(\gamma K_0 \mathbb{C}_{\mathrm{GN}}^{\frac{3}{2}} a^3)^2,$$

for all $R > R^*$ and $\lambda > \lambda^*$. Assuming that

$$c_1\left[\frac{\kappa \mathbb{C}_{\mathrm{GN}} f(R) a^{(1-\gamma_q)q}}{R^{q-1}q}\right]^{\frac{2}{2-\gamma_q q}} \leq \frac{1}{16\nu^2} \text{ and } c_2(\gamma K_0 \mathbb{C}_{\mathrm{GN}}^{\frac{3}{2}} a^3)^2 \leq \frac{1}{16\nu^2}, \tag{4.6}$$

and taking advantage of Lemma 3.3 to get

$$|\nabla u_R|_2^2 \leq \frac{1}{2\nu^2},\ \forall R > R^* \text{ and } \lambda > \lambda^*.$$

Now, we can fix the constant $\kappa^* = \kappa^*(R)$ and $\gamma^* = \gamma^*(R)$ by

$$\kappa^* = \min\left\{\frac{R^{q-1}q}{\mathbb{C}_{\mathrm{GN}}f(R)a^{(1-\gamma_q)q}}\left(\frac{1}{16\nu^2 c_1}\right)^{\frac{2-\gamma_q q}{2}},1\right\} \text{ and}$$

$$\gamma^* = \min\left\{\frac{1}{4\nu\sqrt{c_2}K_0\mathbb{C}_{\mathrm{GN}}^{\frac{3}{2}}a^3},\gamma',1\right\} \tag{4.7}$$

to meet the requirement. So, the proof is done by the choices of κ^* and γ^* above. □

Taking the study made above into account, we are ready to conclude this section by showing our main estimate for $|u_R|_\infty$.

Lemma 4.3. *Suppose that V satisfies $(V_1)-(V_3)$ and f meets (1.3) with $(f_1)-(f_3)$. Let u_R be given by (4.5) for all $R > R^*$, $\kappa \in (0,1)$, $\gamma \in (0,\gamma')$, $a > a^*$ and $\lambda > \lambda^*$, then for every fixed $\kappa \in (0,\kappa^*)$ and $\gamma \in (0,\gamma^*)$, there exists a $M > 0$ independent of $R > R^*$ and $\lambda > \lambda^*$ such that $|u_R|_\infty \le M$.*

Proof. In order to show the proof clearly, we are going to divide the proof into several different parts. First of all, we have the following two claims.

Claim 4.4. There is a $K_1 > 0$ independent of $R > R^*$ and $\lambda > \lambda^*$ such that

$$|f(u_R)|_2 \le K_1,\ \forall R > R^*,\ \kappa \in (0,\kappa^*),\ \gamma \in (0,\gamma^*),\ a > a^* \text{ and } \lambda > \lambda^*.$$

Actually, according to $|u_R|_2^2 = a^2$ and (4.2), to arrive at the proof of this claim, it suffices to look for a constant $C > 0$ independent of $R > R^*$ and $\lambda > \lambda^*$ such that $|u_R^{\bar{p}-1}e^{4\pi\nu u_R^2}|_2 \le C$ for all fixed $\kappa \in (0,\kappa^*)$, $\gamma \in (0,\gamma^*)$ and $a > a^*$, where $\bar{p} > 2$. It then concludes from the Hölder's inequality together with (1.16) with $l = \bar{p}'$ that

$$\begin{aligned}\int_{\mathbb{R}^2}|u_R|^{2(\bar{p}-1)}e^{4\pi\nu u_R^2}dx &\le \left(\int_{\mathbb{R}^2}|u_R|^{4(\bar{p}-1)}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}e^{8\pi\nu u_R^2}dx\right)^{\frac{1}{2}}\\ &\le \sqrt{\mathbb{C}_{\mathrm{GN}}}|u_R|_2^{\frac{(1-\gamma_{\bar{p}'})\bar{p}'}{2}}|\nabla u_R|_2^{\frac{\gamma_{\bar{p}'}\bar{p}'}{2}}\left(\int_{\mathbb{R}^2}e^{4\pi\nu^{-1}\bar{u}_R^2}dx\right)^{\frac{1}{2}},\end{aligned}$$

where

$$\bar{p}' = 4(\bar{p}-1) \text{ and } \bar{u}_R = \sqrt{2}\nu u_R.$$

Since $|\bar{u}_R|_2^2 = 2\nu^2 a^2$ is uniformly bounded by (4.6) and $|\nabla\bar{u}_R|_2^2 = 2\nu^2|\nabla u_R|_2^2 \le 1$ for all $R > R^*$ and $\lambda > \lambda^*$ by Lemma 4.2. the claim is done by (4.4).

Claim 4.5. For any fixed $R > R^*$ and for all $a > a^*$, $\gamma \in (0,\gamma^*)$ and $\kappa \in (0,\kappa^*)$, then the Lagrange multiplier μ_R is uniformly bounded with respect to R and λ. In other words, there exists a constant $\Gamma > 0$ independent of R and λ such that $|\mu_R| \le \Gamma$ for all $\gamma \in (0,\gamma^*)$, $\kappa \in (0,\kappa^*)$ and $a > a^*$.

Indeed, recalling $\mathcal{J}'_{\lambda,R}(u_R) - \mu_R u_R = 0$ in X_λ^{-1}, we can combine Lemma 3.3, Lemma 2.1-(iii) and (1.12) with $2 < q < 4$ to have that

$$\begin{aligned}
0 > \Upsilon_{\lambda,R} &= \mathcal{J}_{\lambda,R}(u_R) \\
&= \mathcal{J}_{\lambda,R}(u_R) - \frac{1}{q}\left\{\int_{\mathbb{R}^2}[|\nabla u_R|^2 + \lambda V(x)|u_R|^2]dx + \gamma V_0(u_R)\right. \\
&\qquad \left. -\kappa\int_{\mathbb{R}^2} f_R(u_R)u_R\,dx - \mu_R a^2\right\} \\
&\geq \frac{q-4}{4q}\gamma[V_1(u_R) - V_2(u_R)] + \frac{\mu_R}{q}a^2 \geq \frac{q-4}{4q}\gamma V_1(u_R) + \frac{1}{4}\mu_R a^2,
\end{aligned}$$

from where it follows that

$$\mu_R \leq \frac{4-q}{qa^2}\gamma V_1(u_R).$$

According to the definition of u_R in Sect. 3, it is the weak limiting, actually strong limiting, of the minimizing sequence $\{u_n\} \subset S(a)$ of $\Upsilon_{\lambda,R}$, then it follows from the Fatou's lemma, (1.12) and (3.18) that

$$\begin{aligned}
\mu_R &\leq \frac{4-q}{qa^2}\liminf_{n\to\infty}\gamma V_1(u_n) \leq \frac{4-q}{qa^2}\left\{\gamma V_2(u_R) + 4\kappa\int_{\mathbb{R}^2} f_R(u_R)u_R\,dx\right\} \\
&\leq \frac{4-q}{qa^2}\left\{K_0|u_R|_{\frac{8}{3}}^4 + 4|f_R(u_R)|_2|u_R|_2\right\} \\
&\leq \frac{4-q}{qa^2}\left\{K_0\mathbb{C}_{\mathrm{GN}}^{\frac{3}{2}}|u_R|_2^3|\nabla u_R|_2 + 4|f_R(u_R)|_2|u_R|_2\right\},
\end{aligned}$$

where we have applied (2.1) and (1.16) with $l = \frac{8}{3}$ to the last inequality. Taking into account $\mathcal{J}'_{\lambda,R}(u_R) - \mu_R u_R = 0$ in X_λ^{-1} again, we easily conclude that

$$\begin{aligned}
\mu_R &= \frac{1}{a^2}\left\{\int_{\mathbb{R}^2}[|\nabla u_R|^2 + \lambda V(x)|u_R|^2]dx + \gamma V_0(u_R) - \kappa\int_{\mathbb{R}^2} f_R(u_R)u_R\,dx\right\} \\
&\geq -\frac{1}{a^2}\left\{V_2(u_R) + \int_{\mathbb{R}^2} f_R(u_R)u_R\,dx\right\} \geq -\frac{1}{a^2}\left\{K_0|u_R|_{\frac{8}{3}}^4 + |f_R(u_R)|_2|u_R|_2\right\} \\
&\geq -\frac{1}{a^2}\left\{K_0\mathbb{C}_{\mathrm{GN}}^{\frac{3}{2}}|u_R|_2^3|\nabla u_R|_2 + |f_R(u_R)|_2|u_R|_2\right\}.
\end{aligned}$$

The above two facts together with Lemma 4.2 and Claim 4.4 reveal this claim immediately.

Secondly, because $V(x) \geq 0$ for every $x \in \mathbb{R}^2$, we make full use of Lemmas 2.3 and 4.2 jointly with Claim 4.5 to determine a constant $\Pi > 0$, independent of $R > R^*$ and $\lambda > \lambda^*$, such that the function $u_R(x) > 0$ for all $x \in \mathbb{R}^2$ must satisfy

$$\begin{cases} -\Delta u_R + u_R \leq \Pi u_R + f(u_R), & \text{in } \mathbb{R}^2, \\ u_R > 0, & \text{in } \mathbb{R}^2, \end{cases} \tag{4.8}$$

Taking advantage of the Lax-Milgram theorem combined with Claim 4.4, there exists a function $w_R \in H^2(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ such that it is a solution of the problem

$$\begin{cases} -\Delta w_R + w_R \leq \Pi u_R + f(u_R), & \text{in } \mathbb{R}^2, \\ w_R > 0, & \text{in } \mathbb{R}^2, \end{cases} \tag{4.9}$$

for each $R > R^*$ and $\lambda > \lambda^*$. With problems (4.8) and (4.9) in hands, we then take the claim:

Claim 4.6. For all $R > R^*$ and $\lambda > \lambda^*$, it holds that

$$0 < u_R(x) \leq w_R(x), \ \forall x \in \mathbb{R}^2.$$

In fact, let us fix the test function

$$\phi(x) = (u_R - w_R)^+(x) \in H^1(\mathbb{R}^2).$$

Muitiplying this function ϕ on both sides of $-\Delta(u_R - w_R) + (u_R - w_R) \leq 0$ in $\mathbb{R}^2$, we shall get the following inequality

$$\int_{\mathbb{R}^2} [\nabla(u_R - w_R)\nabla\phi + (u_R - w_R)\phi]dx \leq 0.$$

An elementary computation gives us that

$$\int_{\mathbb{R}^2} [|\nabla(u_R - w_R)^+|^2 + |(u_R - w_R)^+|^2]dx = 0$$

yielding the claim.

Finally, we are ready to conclude the proof of this lemma. Thanks to the powerful theorem, c.f. [22, Theorem 9.25], there is $K_2 > 0$ independent of $R > R^*$ and $\lambda > \lambda^*$ such that

$$\|w_R\|_{H^2} \leq K_2 |f_R(u_R)|_2, \ \forall R > R^* \text{ and } \lambda > \lambda^*.$$

which leads to

$$\|w_R\|_{H^2} \leq K_3, \ \forall R > R^* \text{ and } \lambda > \lambda^*,$$

for some $K_3 > 0$ independent of $R > R^*$ and $\lambda > \lambda^*$. According to the continuous embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, there is $K_4 > 0$ independent of $R > R^*$ and $\lambda > \lambda^*$ such that

$$|w_R|_\infty \leq K_4, \ \forall R > R^* \text{ and } \lambda > \lambda^*.$$

From which, we are derived from Claim 4.6 that

$$|u_R|_\infty \leq M, \ \forall R > R^* \text{ and } \lambda > \lambda^*.$$

Thereby, we finish the proof of this lemma. □

4.2. Proof of Theorem 1.1

According to the above discussions, we derive the proof of Theorem 1.1 by fixing $R > \{R^*, M\}$, because in this case the function $u_R \in S(a)$ is a positive solution of Eq. (1.1) with $\mu = \mu_R$ for all $\kappa \in (0, \kappa^*)$, $\gamma \in (0, \gamma^*)$, $a > a^*$ and $\lambda > \lambda^*$. Thereby, the proof is completed.

At this stage, we are going to contemplate the asymptotical behavior of normalized solutions of Eq. (1.1) obtained in Theorem 1.1 as $\lambda \to +\infty$.

Before showing the proof of Theorem 1.2, using the same constant $R > 0$ determined in the proof of Theorem 3.1, we need the variational functionals below

$$\begin{cases} \mathcal{J}_\Omega(u) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \dfrac{\gamma}{4} V_0|_\Omega(u) - \kappa \int_\Omega F(u)dx, \\ \mathcal{J}_{\Omega,R}(u) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \dfrac{\gamma}{4} V_0|_\Omega(u) - \kappa \int_\Omega F_R(u)dx, \end{cases} \forall u \in H_0^1(\Omega),$$

where the functional $V_0|_\Omega : H_0^1(\Omega) \to \mathbb{R}$ which is defined by $V_0|_\Omega = V_1|_\Omega - V_2|_\Omega$ with

$$\begin{cases} V_1|_\Omega(u) \triangleq \displaystyle\int_\Omega \int_\Omega \ln(1 + |x-y|) u^2(x) u^2(y) dxdy, \\ V_2|_\Omega(u) \triangleq \displaystyle\int_\Omega \int_\Omega \ln\left(1 + \frac{1}{|x-y|}\right) u^2(x) u^2(y) dxdy, \end{cases} \forall u \in H_0^1(\Omega).$$

Since $\mathrm{meas}(\Omega) < +\infty$, there is a constant $\varrho > 0$ such that $\Omega \subset B_\varrho(0)$ and so

$$0 \le \ln(1 + |x-y|) \le \ln(1 + 2\varrho), \ \forall x, y \in \Omega$$

indicating that $V_1|_\Omega$ is well-defined and of class of $\mathcal{C}^1(H_0^1(\Omega), \mathbb{R})$ endowed with its usual norm.

Moreover, we define the minimization problems associated with (1.15) by

$$\Upsilon_{\Omega,R} \triangleq \inf_{u \in S_\Omega(a)} \mathcal{J}_{\Omega,R}(u) \text{ and } \Upsilon_\Omega \triangleq \inf_{u \in S_\Omega(a)} \mathcal{J}_\Omega(u)$$

where

$$S_\Omega(a) = \left\{ u \in H_0^1(\Omega) : \int_\Omega u^2 dx = a^2 \right\}.$$

Now, we are in a position to present the proof of Theorem 1.2.

4.3. Proof of Theorem 1.2

Let $(u_\lambda, \mu_\lambda) \in X_\lambda \times \mathbb{R}$ be a couple of weak solution to Problems (1.1)-(1.2), choosing a subsequence $\lambda_n \to +\infty$ as $n \to \infty$, we denote $\{(u_{\lambda_n}, \mu_{\lambda_n})\}$ by a subsequence of $\{(u_\lambda, \mu_\lambda)\}$. In view of the proofs of Lemmas 4.2 and 4.3, we know that

$$\sup_{n \in \mathbb{N}} |\nabla u_{\lambda_n}|_2^2 \le \frac{1}{2\nu^2} \text{ and } \sup_{n \in \mathbb{N}} |u_{\lambda_n}|_\infty \le M. \tag{4.10}$$

Moreover, due to the proof of Lemma 3.8, we conclude that $\|u_{\lambda_n}\|_{X_{\lambda_n}}$ is uniformly bounded in $n \in \mathbb{N}$ since we have showed the constant $\Theta_R < 0$ in Lemma 3.3 is independent of λ. Going to a subsequence if necessary, there is a $u_0 \in X$ such that $u_{\lambda_n} \rightharpoonup u_0$ in X, $u_{\lambda_n} \to u_0$ in $L^s(\mathbb{R}^2)$ for all $2 \le s < \infty$ by Lemma 2.1-(i) and $u_{\lambda_n} \to u_0$ a.e. in $\mathbb{R}^2$ as $n \to \infty$. Obviously, we have $u_0 \in S(a)$ since $\{u_{\lambda_n}\} \subset S(a)$. Recalling Claim 4.5, it permits to suppose that $\mu_{\lambda_n} \to \mu_0$ along a subsequence. As a consequence, we shall conclude that (u_0, μ_0) is a couple of weak solution to Problem (1.15).

Claim 4.7. $u_0 \equiv 0$ in $\Omega^c \triangleq \mathbb{R}^2 \backslash \Omega$ and so $u_0 \in S_\Omega(a)$.

Otherwise, there exists a compact subset $\hat{\Omega}_{u_0} \subset \Omega^c$ with $\mathrm{dist}(\hat{\Omega}_{u_0}, \partial\Omega^c) > 0$ such that $u_0 \neq 0$ on $\hat{\Omega}_{u_0}$ and by Fatou's lemma

$$a^2 = \liminf_{n\to\infty} \int_{\mathbb{R}^2} u_{\lambda_n}^2 dx \geq \int_{\hat{\Theta}_{u_0}} {u_0}^2 dx > 0. \tag{4.11}$$

Moreover, there exists $\varepsilon_0 > 0$ such that $V(x) \geq \varepsilon_0$ for every $x \in \hat{\Omega}_{u_0}$ by the assumptions (V_1) and (V_2). Combining Lemma 3.3, (1.12), (3.18), Claim 4.5 and (4.11), we derive

$$\begin{aligned}
0 &\geq \liminf_{n\to\infty} \Upsilon_{\lambda_n,R} = \liminf_{n\to\infty} \mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) \\
&= \liminf_{n\to\infty} \left\{ \mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) - \frac{1}{q}\left\{ \int_{\mathbb{R}^2} [|\nabla u_{\lambda_n}|^2 + \lambda_n V(x)|u_{\lambda_n}|^2] dx + \gamma V_0(u_{\lambda_n}) \right.\right. \\
&\qquad \left.\left. - \kappa \int_{\mathbb{R}^2} f_R(u_{\lambda_n}) u_{\lambda_n}\, dx - \mu_{\lambda_n} a^2 \right\}\right\} \\
&\geq \frac{q-2}{2q}\varepsilon_0 \left(\int_{\hat{\Theta}_{u_0}} u_0^2 dx \right) \liminf_{n\to\infty} \lambda_n + \liminf_{n\to\infty} \left[\frac{q-4}{4q} V_1(u_{\lambda_n}) + \frac{1}{q}\mu_{\lambda_n} a^2 \right] \\
&= +\infty
\end{aligned}$$

which is impossible. Consequently, $u_0 \in H_0^1(\Omega)$ by the fact that $\partial\Omega$ is smooth.

Claim 4.8. $\mathcal{J}_\Omega(u_0) = \Upsilon_\Omega$.

Indeed, since (4.10) gives us that $F_R = F$ and then $\mathcal{J}_{\Omega,R} = \mathcal{J}_\Omega$. Obviously, $S_\Omega(a) \subset S(a)$ and so $u_{\lambda_n} \to u_0$ in $L^s(\mathbb{R}^2)$ for each $2 \leq s < \infty$ and the Fatou's lemma provide us that

$$\begin{aligned}
\Upsilon_\Omega &\geq \liminf_{n\to\infty} \mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) \\
&\geq \liminf_{n\to\infty} \left\{ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_{\lambda_n}|^2 dx + \frac{\gamma}{4} V_0(u_{\lambda_n}) - \kappa \int_{\mathbb{R}^2} F_R(u_{\lambda_n}) dx \right\} \\
&\geq \mathcal{J}_{\Omega,R}(u_0) = \mathcal{J}_\Omega(u_0) \geq \Upsilon_\Omega
\end{aligned}$$

indicating that $u_{\lambda_n} \to u_0$ in X and $\mathcal{J}_\Omega(u_0) = \Upsilon_\Omega$.

Finally, we shall prove that $\mathcal{J}_\Omega'(u_0) - \mu_0 u_0 = 0$ in $(H_0^1(\Omega))^{-1}$. To see it, for every $\psi \in C_0^\infty(\Omega)$, it follows from Lemma 2.1-(iii) as well as Lemma 2.2 that

$$\begin{aligned}
V_1'(u_{\lambda_n})[\psi] - V_1'(u_0)[\psi] &= 4B_1(u_{\lambda_n}^2, u_n\psi) - 4B_1(u_0^2, u_0\psi) \\
&= 4B_1(u_{\lambda_n}^2, (u_n - u_0)\psi) + 4B_1(u_{\lambda_n}^2 - u_0^2, u_0\psi) \\
&= o_n(1).
\end{aligned}$$

Using Lemma 2.1-(i) and (iv),

$$\begin{aligned}
|V_2'(u_{\lambda_n})[\psi] - V_2'(u_0)[\psi]| &\leq 4\left|B_2(u_{\lambda_n}^2, (u_n - u_0)\psi)\right| + 4\left|B_2(u_{\lambda_n}^2 - u_0^2, u_0\psi)\right| \\
&\leq 4K_0 |u_{\lambda_n}|_{\frac{8}{3}}^2 |(u_n - u_0)\psi|_{\frac{8}{3}}^2 + 4K_0 |u_{\lambda_n}^2 - u_0^2|_{\frac{8}{3}}^2 |u_0|_{\frac{8}{3}} |\psi|_{\frac{8}{3}} \\
&= o_n(1).
\end{aligned}$$

As a direct byproduct of the above two facts and

$$\lim_{n\to\infty}\left\{\mathcal{J}'_{\lambda_n,R}(u_{\lambda_n})[\psi]-\mu_{\lambda_n}\int_{\mathbb{R}^2}u_{\lambda_n}\psi dx\right\}=0,\ \forall\psi\in C_0^\infty(\Omega),$$

we can arrive at the desired result. The proof is completed.

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Data availability We declare that the manuscript has no associated data.

Declarations

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