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# CONCENTRATING NORMALIZED SOLUTIONS FOR 2D NONLOCAL SCHRÖDINGER EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH

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Abstract. We study the existence of solutions to nonlocal Schrödinger problems with different types of potentials

$$-\Delta u + W(x)u = \sigma u + \kappa[|x|^{-\mu} * F(u)]f(u) \quad \text{in } \mathbb{R}^2,$$
$$\int_{\mathbb{R}^2} |u|^2 dx = a^2,$$

where $a \neq 0$, $\sigma \in \mathbb{R}$ is known as the Lagrange multiplier, $\kappa > 0$ is a parameter, $W \in \mathcal{C}(\mathbb{R}^2)$ is the nonnegative external potential, $\mu \in (0,2)$, and $F$ denotes the primitive function of $f \in \mathcal{C}(\mathbb{R})$ which has critical exponential growth in the Trudinger-Moser sense at infinity. We prove that the problems admit at least a positive solution, and we analyze the concentrating behavior.

## 1. Introduction

In this article, we aim to prove existence of positive solutions to the nonlocal Schrödinger equation with different types of potentials

$$-\Delta u + W(x)u = \sigma u + \kappa[|x|^{-\mu} * F(u)]f(u) \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 dx = a^2, \tag{1.2}$$

where $a \neq 0$, $\sigma \in \mathbb{R}$ is known as the Lagrange multiplier, $\kappa > 0$ is a parameter, $W \in \mathcal{C}(\mathbb{R}^2)$ is the nonnegative external potential, $\mu \in (0,2)$ and $F$ denotes the primitive function of $f \in \mathcal{C}(\mathbb{R})$ which has critical exponential growth in the Trudinger-Moser sense at infinity.

Inspired by the well-known Trudinger-Moser type inequality, we recall that a function $f$ has the critical exponential growth at infinity if there exists a constant $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases} \tag{1.3}$$

This definition was introduced by Adimurthi and Yadava [2], see also de Figueiredo, Miyagaki and Ruf [30] for example.

Hereafter, we shall assume that the nonlinearity $f$ satisfies (1.3) and the following assumptions:

- (A1) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $f(s) \equiv 0$ for all $s \in (-\infty, 0]$;
- (A2) There is a $q \in \big(2, \frac{6-\mu}{2}\big)$ such that $f(s)/s^{q-1}$ is an increasing function of $s$ on $(0,+\infty)$.
- (A3) There is a $c_0 > 0$ such that $f(s) \geq c_0 s^{q-1}$ for all $s \in [0,+\infty)$.

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We would like to highlight here that many functions $f$ satisfy the above assumptions, with $\alpha_0 = 4\pi$ and $c_0 = 1$, for example

$$f(s) = \begin{cases} 0, & s \le 0, \\ s^{q-1} e^{4\pi s^2}, & s > 0, \end{cases}$$

where $q \in (2, \frac{6-\mu}{2})$. Similar assumptions for a nonlinearity $f$ satisfying (1.3) and (A1)–(A3) can be found in [13, 58].

Over the past few decades, a lot of attentions have been paid to the standing wave solutions to the time-dependent nonlinear Choquard equation

$$i\frac{\partial \psi}{\partial t} = \Delta\psi - W(x)\psi + [|x|^{-\mu} * F(\psi)]f(\psi) \text{ in } \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.4}$$

where $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ acts as the time-dependent wave function, $W : \mathbb{R}^N \to \mathbb{R}$ stands for the real external potential and nonlinear term $f(\psi)$ describes the interaction effect among particles. Inserting the standing wave ansatz $\psi(x,t) = \exp(-i\omega t)u(x)$ with $\omega \in \mathbb{R}$ and $x \in \mathbb{R}^N$ into (1.4), it follows that $u : \mathbb{R}^N \to \mathbb{R}$ satisfies the Choquard equation

$$-\Delta u + \bar{W}(x)u = [|x|^{-\mu} * F(u)]f(u) \quad \text{in } \mathbb{R}^N; \tag{1.5}$$

here and in the sequel $\bar{W}(x) = W(x) + \omega$ for all $x \in \mathbb{R}^N$.

There exist two directions in the studies of standing waves of the Choquard equation (1.5). On the one hand, one can choose the frequency $\omega \in \mathbb{R}$ to be fixed and investigate the existence of nontrivial solutions for (1.5) obtained as the critical points of the variational functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u|^2 + (W(x) + \omega)u^2\right] dx - \frac{1}{2}\int_{\mathbb{R}^N} [|x|^{-\mu} * F(u)]F(u)dx.$$

Actually, (1.5) is closely related to the Choquard equation arising from the studies of Bose-Einstein condensation and can be used to describe the finite-range many-body interactions between particles since $|x|^{-\mu}$ can be reviewed as the classic Riesz potential. Letting $N \ge 3$ and $f(s) = |s|^{p-2}s$ for all $s \in \mathbb{R}$, (1.5) is of the form

$$-\Delta u + u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u, \ x \in \mathbb{R}^N. \tag{1.6}$$

To describe a polaron at rest in the quantum field theory, Pekar [51] introduced the Choquard-Pekar equation which is $N = 3$, $\mu = 1$ and $p = 2$ in (1.6). Choquard adopted this equation to characterization an electron trapped in its own hole as an approximation to the Hartree-Fock theory for the one component plasma [41]. Subsequently, Lieb [40] and Lions [43] studied the existence and uniqueness of positive solutions to (1.6) by variational methods. The authors in [45, 47] concluded the regularity, positivity and radial symmetry of the ground state solutions and investigated the decay properties at infinity. It should be pointed out that (1.6) was also proposed by Morozet al. in [46] as a model for self-gravitating particles in the context as it can be regarded as the classic Schrödinger-Newton equation, see e.g. [52, 62]. Actually, (1.6) and its variants have received more and more attentions by many mathematicians because of the appearance of the convolution type nonlinearities in these years. We refer the reader to [1, 4, 5, 8, 9, 38, 11, 12, 15, 47, 48, 55] and the references therein, particularly to [49], for some meaningful review of the Choquard equations.

On the other hand, one can consider the $\omega \in \mathbb{R}$ to be unknown. In such a situation, $\omega$ is supposed to act as a Lagrange multiplier and the $L^2$-norm of the obtained solutions would be prescribed since there is a conservation of mass which is said that the wave function $\psi(x,t)$ with its corresponding Cauchy initial function $\psi(0,x)$ which preserves $L^2$-mass in the following sense

$$\int_{\mathbb{R}^N} |\psi(t,x)|^2 dx = \int_{\mathbb{R}^N} |\psi(0,x)|^2 dx, \quad \forall t \in (0,\infty).$$

From the physical point of view, this spirit of research holds particular significance as it accounts for the conservation of mass. Moreover, it provides valuable insights into the dynamic properties of the standing waves of (1.5), for instance stability or instability in [24, 28]. In this article, we shall focus primarily on this direction.

Jenajean [34] used a minimax approach and compactness argument to conclude the existence of solutions for the Schrödinger problem

$$\begin{gathered} -\Delta u+\omega u=g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2dx=a^2. \end{gathered} \tag{1.7}$$

There exist some further complements and generalizations in [36]. In [59], letting $g(t)=\tau|t|^{q-2}t+|t|^{p-2}t$ with $2<q\le 2+\frac{4}{N}\le p<2^*$, Soave obtained the existence of solutions for problem (1.7), where $2^*=\frac{2N}{N-2}$ if $N\ge 3$ and $2^*=\infty$ if $N=2$. For this type of combined nonlinearities, Soave [60] also proved the existence of ground state and excited solutions when $p=2^*$. For more results for problem (1.7), we refer the reader to [7, 20, 35, 37, 63] and the references therein.

In the spirit of [34], when $\bar{W}(x)\equiv 0$ for all $x\in\mathbb{R}^N$ in (1.5), the authors in [39] deduced the existence of nontrivial solutions solutions to the nonlocal problem of Choquard type

$$\begin{gathered} -\Delta u+\omega u=[|x|^{-\mu}*F(u)]f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2dx=a^2, \end{gathered} \tag{1.8}$$

provided $a>0$ is sufficiently small, where $f$ possesses the Sobolev subcritical growth at infinity. Afterwards, Bartsch et al. [18] investigated the existence of solutions for problem (1.8) which is simpler and more transparent than that of [39]. As to the case that the nonlinearity $f$ admits the critical growth, Ye, Shen and Yang [65] dealt with the existence of normalized ground state solutions for the Hartree problem with a perturbation. There are some other interesting results with respect to problem (1.8), see [15, 18, 29, 39] for example.

The reader may observe that the spatial dimension of problem (1.1) is two, the case therefore is very special because $2^*=\infty$ in this situation. Explaining it more specifically, the fact $H^1(\mathbb{R}^2)\not\hookrightarrow L^\infty(\mathbb{R}^2)$ shall make the problems special and quite delicate. Thus, it is not so direct to dispose of the nonlinearity involving a critical exponential growth trivially. Letting $W(x)\equiv 0$ for all $x\in\mathbb{R}^2$ in (1.1), Deng and Yu [29] supposed that the nonlinearity satisfies (1.3) and the following assumptions

- (A4) $f:\mathbb{R}\to\mathbb{R}$ is continuous;
- (A5) $f(t)=o(|t|^\tau)$ as $|t|\to 0$ for some $\tau>3$;
- (A6) there exists a positive constant $\theta>\frac{6-\mu}{2}$ such that $0<\theta F(t)\le f(t)t$ for $t\ne 0$;
- (A7) there exist constants $\sigma>\frac{6-\mu}{2}$ and $\xi>0$ such that $F(t)\ge\xi\,|t|^\sigma$ for all $t\in\mathbb{R}$.

Then, for some sufficiently small mass $a>0$ and $\sigma>0$ large enough, the authors used the arguments in [34] to investigate the existence of normalized solution. Moreover, the ground state solution was considered when $f$ in addition has some monotone type assumptions. Afterwards, Alves and Shen [15] handled the existence of nontrivial solutions to the problem

$$\begin{gathered} -\Delta u+\omega u=[|x|^{-\mu}*(|x|^{-\beta}F(u))]|x|^{-\beta}f(u) \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2}|u|^2dx=a^2, \end{gathered} \tag{1.9}$$

where $\beta>0$, $0<\mu<2$ with $0<2\beta+\mu<2$ and $f$ admits the supercritical exponential growth (see [10, 12]). It is worth pointing out here that either assumption (A6) or

(A7') $\liminf_{s\to+\infty}\frac{F(s)}{e^{\alpha_0 s^2}}>0$, where $\alpha_0>0$ comes from (1.3),

plays a pivotal role in [15]. Actually, either the assumption (A7) or (A7') is used for restoring the compactness caused by the critical exponential growth and the whole space $\mathbb{R}^2$. As a consequence, these assumptions seem indispensable to some extent in the mentioned works.

Motivated by the quoted papers above, particularly by [13, 14, 57, 58], we are going to consider the existence of normalized solutions to nonlocal Schrödinger equations with different potentials and critical exponential growth. Speaking it clearly, let us suppose that $W(x) := V(\varepsilon x)$ for all $\varepsilon > 0$ and $x \in \mathbb{R}^2$ in (1.1) with the assumption

(A8) $V \in C(\mathbb{R}^2, \mathbb{R})$ and $0 < V_0 := \inf_{x\in\mathbb{R}^2} V(x) < V_\infty := \liminf_{|x|\to\infty} V(x) < +\infty$, where $V(0) = V_0$.

Now, we can state the first main result in this article.

**Theorem 1.1.** *Assume* (A1)–(A3), (A8), (1.3) *hold and* $\mu \in (0, 2)$. *Then there exist constants* $\kappa^* > 0$, $a^* > 0$ *and* $\varepsilon^* > 0$ *such that, for every* $\kappa \in (0, \kappa^*)$, $a > a^*$ *and* $\varepsilon \in (0, \varepsilon^*)$, *the problem*

$$\begin{gathered} -\Delta u + V(\varepsilon x)u = \sigma u + \kappa[|x|^{-\mu} * F(u)]f(u) \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |u(x)|^2 dx = a^2, \end{gathered} \tag{1.10}$$

*has a pair of weak solutions* $(\bar{u}, \bar{\sigma}) \in H^1(\mathbb{R}^2) \times \mathbb{R}$ *such that* $\bar{u}(x) > 0$ *for all* $x \in \mathbb{R}^2$ *and* $\bar{\sigma} < 0$. *Moreover, if* $z_\varepsilon$ *denotes the global maximum of* $\bar{u}$, *then, up to a subsequence if necessary,*

$$\lim_{\varepsilon\to 0^+} V(\varepsilon z_\varepsilon) = V_0.$$

We shall assume that $W(x) = \lambda V(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^2$ and the function $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following conditions

(A9) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ with $V(x) \geq 0$ on $\mathbb{R}^2$;

(A10) $\Omega := \operatorname{int} V^{-1}(0)$ is nonempty and bounded with smooth boundary, and $\overline{\Omega} = V^{-1}(0)$;

(A11) there exists a $b > 0$ such that the set $\Xi := \{x \in \mathbb{R}^2 : V(x) < b\}$ is nonempty and admits finite measure.

The main result in this directionreads as follows.

**Theorem 1.2.** *Suppose* (A)–(A3), (A9)–(A11), (1.3) *hold and* $\mu \in (0, 2)$. *then there are* $\kappa_* > 0$, $a_* > 0$ *and* $\lambda_* > 1$ *such that, for all* $\kappa \in (0, \kappa_*)$, $a > a_*$ *and* $\lambda > \lambda_*$, *the problem*

$$\begin{gathered} -\Delta u + \lambda V(x)u = \sigma u + \kappa[|x|^{-\mu} * F(u)]f(u) \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |u(x)|^2 dx = a^2, \end{gathered} \tag{1.11}$$

*has a couple of weak solution* $(\underline{u}, \underline{\sigma}) \in H^1(\mathbb{R}^2) \times \mathbb{R}$ *such that* $\underline{u}(x) > 0$ *for all* $x \in \mathbb{R}^2$ *and* $\underline{\sigma} < 0$. *If we denote* $(\underline{u}_\lambda, \underline{\sigma}_\lambda)$ *by the couple of weak solutions established above for all* $\lambda > \lambda_*$, *then for all fixed* $a > a_*$, *passing to a subsequence if necessary,* $\underline{u}_\lambda \to \underline{u}_0$ *in* $H^1(\mathbb{R}^2)$ *and* $\underline{\sigma}_\lambda \to \underline{\sigma}_0$ *in* $\mathbb{R}$ *as* $\lambda \to +\infty$, *where* $\underline{\sigma}_0 < 0$ *and* $(\underline{u}_0, \underline{\sigma}_0)$ *is a couple of weak solution to the problem*

$$\begin{gathered} -\Delta u = \sigma u + \kappa\Big(\int_\Omega \frac{F(u(y))}{|x-y|^\mu} dy\Big) f(u), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \\ \int_\Omega |u|^2 dx = a^2. \end{gathered} \tag{1.12}$$

The two types of potentials $V$ appearing in Theorems 1.1 and 1.2 have been considered by many mathematicians over the past decades, see [17, 23, 54, 33, 31, 32] and [19, 21, 22, 27, 44], respectively. As a matter of fact, the former one is known as the Rabinowitz's potential, while the latter one is called by the steep potential well.

**Remark 1.3.** Concerning the existence of normalized solutions to some classes of local equations with Rabinowitz's potential, we prefer to refer the reader to [3, 6, 14, 16, 57]. The reader can find the latest paper [58] focuses on the normalized solutions to Schrödinger-Newton system with steep potential well.

It seems the first attempts to study the existence of normalized solutions to Choquard equations with the above two types of potentials in a unified way.

It should be pointed out that we could not conclude the proofs of Theorems 1.1 and 1.2 simply by repeating the approaches adopted in the previous papers mentioned in Remark 1.3. On the one hand, we successfully generalize the local case in [3, 6, 14, 16, 57] to the nonlocal one and so there are some additional difficulties. On the other hand, thanks to the special structure of the work space in [58], the key compact imbedding holds true in advance and it mainly deals with the boundedness of minimizing sequence, while we easily obtain the boundedness and there are some subtle efforts to recover the compactness in the proof of Theorem 1.2. As a consequence, we tend to believe that this article may prompt some further studies on normalized solutions to a class of nonlocal Schrödinger equations.

To conclude this section, we simply sketch the main ideas to arrive at the proofs of Theorems 1.1 and 1.2. Owing to the arguments adopted in [13, 14, 57, 58], for each fixed constant $R > 0$, we introduce the following continuous function $f_R : \mathbb{R} \to \mathbb{R}$ defined by

$$f_R(s) = \begin{cases} 0, & \text{if } s \leq 0, \\ f(s), & \text{if } 0 \leq s \leq R, \\ \frac{f(R)}{R^{q-1}} s^{q-1}, & \text{if } R \leq s < +\infty, \end{cases} \tag{1.13}$$

where the constant $q \in \big(2, \frac{6-\mu}{2}\big)$ comes from (A2). For the rest of this article, we define $F_R(s) = \int_0^s f_R(t)dt$ for each $s \in \mathbb{R}$ to be the primitive function of $f_R$. It follows from a direct computation with (A2) that

$$qF_R(s) \leq f_R(s)s, \quad \forall s \geq 0. \tag{1.14}$$

We can use the monotone assumption in (A2) again to see that

$$f_R(s) \leq \frac{f(R)}{R^{q-1}} s^{q-1}, \quad \forall s \geq 0. \tag{1.15}$$

With such a nonlinearity $f_R$ defied in (1.13), we turn to study the auxiliary problem

$$\begin{gathered} -\Delta u + W(x)u = \sigma u + \kappa[|x|^{-\mu} * F_R(u)]f_R(u) \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |u|^2 dx = a^2. \end{gathered} \tag{1.16}$$

By (1.15), we can see that Problem (1.16) involves $L^2$-subcritical growth since $q < \frac{6-\mu}{2}$. So, the solvability of Problem (1.16) becomes available. At this stage, we invite the reader to observe that if the couple $(u_R, \sigma_R)$ is a solution of Problem (1.16), then it is indeed the solution to the original Problems (1.1)-(1.2) as long as $|u_R|_\infty \leq R$ due to the definition of $f_R$ in (1.13). Having this in mind, we shall derive the proofs of Theorems 1.1 and 1.2 combining the solvability of Problem (1.16) and the $L^\infty$-estimate.

This article is organized as follows. In Section 1.2, we will introduce some preliminary results handling the convolution parts. Sections 2 and 3 we obtain existence results for the auxiliary Problem (1.16) with two different types of potentials. Finally, the detailed proofs of Theorems 1.1 and 1.2 shall be exhibited in Section 4.

1.1. **Notation.** From now on, we use the following notation:

- $B_r(x) \subset \mathbb{R}^2$ is an open ball centered at $x \in \mathbb{R}^2$ with radius $r > 0$ and $B_r = B_r(0)$.
- $C, C_1, C_2, \cdots$ denote any positive constant, whose value is not relevant.
- For all $x \in \mathbb{R}^2$, we define
$$u^+(x) := \max\{u(x), 0\} \geq 0 \quad \text{and} \quad u^-(x) := \min\{u(x), 0\} \leq 0.$$
- $|\cdot|_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^2)$, for every $p \in [1, +\infty]$. $\|\cdot\|_{H^i}$ denotes the usual norm of the Hilbert space for $i \in \{1, 2\}$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- " $\to$ " and " $\rightharpoonup$ " stand for the strong and weak convergence in the related function spaces, respectively.
- We recall the celebrated Gagliardo-Nirenberg inequality, given an $l \in [2, +\infty)$,
$$|u|_l^l \leq \mathbb{C}|u|_2^{(1-\gamma_l)l} |\nabla u|_2^{\gamma_l l} \quad \text{in } H^1(\mathbb{R}^2),\ \gamma_l = 2\Big(\frac{1}{2} - \frac{1}{l}\Big), \tag{1.17}$$
where the constant $\mathbb{C} > 0$ is just dependent of $l$.

1.2. **Two basic facts.** In this section, we exhibit some preliminary results adopted to prove the main results. From now on, we shall always suppose that $0<\mu<2$ just for simplicity. Let us first introduce the well-known Hardy-Littlewood-Sobolev inequality.

**Lemma 1.4** ([42, Theorem 4.3]). *Suppose that $s,r>1$ and $0<\mu<N$ with $\frac{1}{s}+\frac{\mu}{N}+\frac{1}{r}=2$, $\varphi\in L^s(\mathbb{R}^N)$ and $\psi\in L^r(\mathbb{R}^N)$. Then, there exists a sharp constant $C=C(s,N,\mu,r)>0$, independent of $\varphi$ and $\psi$, such that*

$$\int_{\mathbb{R}^N}[|x|^{-\mu}*\varphi(x)]\psi(x)dx\le C\|\varphi\|_s\|\psi\|_r. \tag{1.18}$$

Since it mainly concerns the whole space $\mathbb{R}^2$ in this paper, we will assume that $N=2$ in (1.18).

Let us conclude this section by introducing the celebrated Brézis-Lieb lemma for the nonlocal term of Choquard type.

**Lemma 1.5** ([47, Lemma 2.4]). . *Let $p\in[\frac{4-\mu}{2},+\infty)$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^{\frac{4p}{4-\mu}}(\mathbb{R}^2)$. If $u_n\to u$ almost everywhere on $\mathbb{R}^2$ as $n\to\infty$, then*

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^2}\left[\left(|x|^{-\mu}*|u_n|^p\right)|u_n|^p-\left(|x|^{-\mu}*|u_n-u|^p\right)|u_n-u|^p\right]dx\\&=\int_{\mathbb{R}^2}\left(|x|^{-\mu}*|u|^p\right)|u|^pdx.\end{aligned} \tag{1.19}$$

*Moreover, for all $\varphi\in C_0^\infty(\mathbb{R}^2)$, it holds that*

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(|x|^{-\mu}*|u_n|^p\right)|u_n|^{p-2}u_n\varphi dx=\int_{\mathbb{R}^2}\left(|x|^{-\mu}*|u|^p\right)|u|^{p-2}\varphi dx. \tag{1.20}$$

## 2. Truncated problem: Rabinowitz's type potential

In this section, we are going to prove the existence of positive solutions for the nonlocal Schrödinger equation

$$-\Delta u+V(\varepsilon x)u=\sigma u+\kappa[|x|^{-\mu}*F_R(u)]f_R(u)\quad\text{in }\mathbb{R}^2, \tag{2.1}$$

under the constraint

$$\int_{\mathbb{R}^2}|u|^2dx=a^2, \tag{2.2}$$

where the potential $V:\mathbb{R}^2\to\mathbb{R}$ satisfies (A8), $\varepsilon,\kappa>0$ are parameters, $a>0$, $\sigma\in\mathbb{R}$ is known as the Lagrange multiplier and the nonlinearity $f_R$ is defined in (1.13).

In general, to solve Problems (2.1)-(2.2), we look for critical points of the variational functional

$$J_{\varepsilon,R}(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u|^2+V(\varepsilon x)|u|^2\right]dx-\frac{\kappa}{2}\int_{\mathbb{R}^2}[|x|^{-\mu}*F_R(u)]F_R(u)dx \tag{2.3}$$

restricted to the sphere $S(a)$ defined by

$$S(a)=\Big\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}|u|^2dx=a^2\Big\}. \tag{2.4}$$

Taking advantage of (A8) and (1.15) together with (1.18), it is simple to verify that the functional $J_{\varepsilon,R}$ is of class $\mathcal{C}^1(H^1(\mathbb{R}^2),\mathbb{R})$ and it derivative is given by

$$J'_{\varepsilon,R}(u)v=\int_{\mathbb{R}^2}[\nabla u\nabla v+V(\varepsilon x)uv]\,dx-\kappa\int_{\mathbb{R}^2}[|x|^{-\mu}*F_R(u)]f_R(u)vdx,\quad\forall u,v\in H^1(\mathbb{R}^2).$$

We note that since $V$ is a positive and bounded function by (A8), then the work space $H^1(\mathbb{R}^2)$ with its usual norm $\|\cdot\|_{H^1}$ will be adopted for simplicity in the present section.

The existence result concerning the Problems (2.1)-(2.2) is the following.

**Theorem 2.1.** *Suppose* (A1)–(A3), (A8), (1.3) *holda and* $\mu \in (0, 2)$. *then there exists an* $R^* > 0$ *such that for all* $R > R^*$, *there exist* $a^* = a^*(R) > 0$ *and* $\varepsilon^* = \varepsilon^*(R) > 0$ *such that, for each fixed* $\kappa \in (0, 1)$, $a > a^*$ *and* $\varepsilon \in (0, \varepsilon^*)$, *the minimization problem*

$$\Upsilon_{\varepsilon,R}(a) := \min_{u \in S(a)} J_{\varepsilon,R}(u) \tag{2.5}$$

*can be attained by some function in* $H^1(\mathbb{R}^2)$. *Hence, there is* $(u_R, \sigma_R) \in H^1(\mathbb{R}^2) \times \mathbb{R}$ *such that it is a couple solution of Problems* (2.1)-(2.2), *where* $u_R(x) > 0$ *for all* $x \in \mathbb{R}^2$ *and* $\sigma_R < 0$.

The proof of the above theorem will be divided into several lemmas. Before exhibiting them, we will always suppose that the potential $V$ and the nonlinearity $f_R$ satisfy (A8) and (1.3) with (A1)–(A3) in this section.

**Lemma 2.2.** *For all fixed* $R > 0$, *the variational functional* $J_{\varepsilon,R}$ *is coercive and bounded from below on* $S(a)$ *for each* $\kappa \in (0, 1)$, $a > 0$ *and* $\varepsilon > 0$, *where* $J_{\varepsilon,R}$ *and* $S(a)$ *are appearing in* (2.3) *and* (2.4), *respectively.*

*Proof.* By (1.14)-(1.15) and (1.18), for all $u \in S(a)$, we use (1.17) with $l = \frac{4q}{4-\mu} > 2$ to obtain

$$J_{\varepsilon,R}(u) \geq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \frac{\mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}} \Big(\int_{\mathbb{R}^2} |\nabla u|^2\,dx\Big)^{q-\frac{4-\mu}{2}}.$$

As $q \in \big(2, \frac{6-\mu}{2}\big)$, clearly $q - \frac{4-\mu}{2} < 1$, then the statement follows. $\square$

As a direct consequence of Lemma 2.2, for every fixed $R > 0$, $\kappa \in (0, 1)$, $a > 0$ and $\varepsilon > 0$, the real number $\Upsilon_{\varepsilon,R}(a)$ in (2.5) is well-defined and it shall be used to look for nontrivial solutions for Problems (2.1)-(2.2). Alternatively, we need to conclude that $\Upsilon_{\varepsilon,R}(a)$ is uniformly bounded above with respect to $\kappa \in (0, 1)$ and $\varepsilon > 0$ and so there is the result below.

**Lemma 2.3.** *There exists an* $R^* > 0$ *such that for all fixed* $R > R^*$, *there is an* $a^* = a^*(R) > 0$ *satisfying for all* $a > a^*$, *there exists a constant* $\Theta_R = \Theta(R) < 0$, *independent of* $\varepsilon$, *such that* $\Upsilon_{\varepsilon,R}(a) \leq \Theta_R$ *for all* $\kappa \in (0, 1)$ *and* $\varepsilon > 0$.

*Proof.* According to the definition of $f_R$ in (1.13), it holds that

$$\frac{f_R(s)}{s^{q-1}} = \begin{cases} \frac{f(s)}{s^{q-1}}, & 0 \leq s \leq R,\\ \frac{f(R)}{R^{q-1}}, & R \leq s < +\infty. \end{cases} \tag{2.6}$$

Since $f$ satisfies (1.3), we apply (A2) to see that $\lim_{R\to+\infty} \frac{f(R)}{R^{q-1}} = +\infty$ which indicates that there exists an $R^* > 0$ such that, for all $R > R^*$, it holds that $\frac{f(R)}{R^{q-1}} \geq c_0$. As a consequence, owing to (2.6) and (A3), we arrive at

$$f_R(s) \geq c_0 s^{q-1},\ \forall s \geq 0 \text{ and } R > R^*. \tag{2.7}$$

We now fix a positive function $\psi \in C_0^\infty(\mathbb{R}^2) \cap S(1)$; combining (2.7) and (A8), it follows that

$$J_{\varepsilon,R}(t\psi) \leq \frac{t^2}{2}\int_{\mathbb{R}^2} |\nabla\psi|^2 dx + \frac{|V|_\infty}{2} t^2 - \frac{\kappa c_0^2 t^{2q}}{2q^2}\int_{\mathbb{R}^2} (|x|^{-\mu} * |\psi|^q)|\psi|^q dx \to -\infty$$

as $t \to +\infty$, where we have used that $V(\varepsilon x) \leq |V|_\infty$ for all $\varepsilon > 0$ and $x \in \mathbb{R}^2$ by (A8). Choosing a sufficiently large $t^* = t^*(R) > 0$ and letting $a^* = t^*|\psi|_2$, it permits us to look for a constant $\Theta_R = \Theta(R) < 0$, dependent of $R$, such that

$$J_{\varepsilon,R}(u) \leq \Theta_R, \quad \forall R > R^*,\ \kappa \in (0, 1),\ a > a^* \text{ and } \varepsilon > 0,$$

provided $u \in S(a)$, as asserted. The proof is complete. $\square$

Similar to [3, 6, 14, 16, 57], we have the following result in the nonlocal case of Choquard type.

**Lemma 2.4.** *Let* $a_2 > a_1 > a^*$. *Then* $\frac{\Upsilon_{\varepsilon,R}(a_2)}{a_2^2} < \frac{\Upsilon_{\varepsilon,R}(a_1)}{a_1^2}$ *for all fixed* $R > R^*$, $\kappa \in (0, 1)$ *and* $\varepsilon > 0$.

*Proof.* Let $\xi > 1$ such that $a_2 = \xi a_1$ and $(u_n) \subset S(a_1)$ be a minimizing sequence with respect to the number $\Upsilon_{\varepsilon,R}(a_1)$, that is,

$$J_{\varepsilon,R}(u_n) \to \Upsilon_{\varepsilon,R}(a_1) \quad \text{as } n \to +\infty.$$

Setting $v_n = \xi u_n$, obviously $v_n \in S(a_2)$. According to (A2), the function $t \mapsto \frac{F_R(t)}{t^q}$ is increasing on $(0, +\infty)$, we obtain the inequality

$$F_R(ts) \geq t^q F_R(s), \quad \forall s > 0 \text{ and } t \geq 1,$$

and so, by using $\Upsilon_{\varepsilon,R}(a_2) \leq J_{\varepsilon,R}(v_n) = J_{\varepsilon,R}(\xi u_n)$, we have that

$$\begin{aligned}\Upsilon_{\varepsilon,R}(a_2) &\leq \xi^2 J_{\varepsilon,R}(u_n) + \frac{\kappa}{2}\int_{\mathbb{R}^2}\left\{\xi^2[|x|^{-\mu} * F_R(u_n)]F_R(u_n) - [|x|^{-\mu} * F_R(\xi u_n)]F_R(\xi u_n)\right\} dx\\ &\leq \xi^2 J_{\varepsilon,R}(u_n) + \frac{\kappa(\xi^2 - \xi^{2q})}{2}\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]F_R(u_n) dx.\end{aligned}$$

To continue the proof, we have a claim.

**Clame 2.5.** There exist a positive constant $C > 0$, independent of $n \in \mathbb{N}$, and a positive integer $n_0 \in \mathbb{N}$ such that $\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]F_R(u_n) dx \geq C$ for all $n \geq n_0$.

Otherwise, there exists a subsequence of $(u_n) \subset S(a_1)$, still denoted by itself, such that

$$\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]F_R(u_n) dx \to 0 \quad \text{as } n \to +\infty.$$

Now, we apply Lemma 2.3 to obtain

$$\Theta_R \geq \Upsilon_{\varepsilon,R}(a_1) + o_n(1) = J_{\varepsilon,R}(u_n) \geq -\frac{\kappa}{2}\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]F_R(u_n) dx, \quad n \in \mathbb{N},$$

which is absurd and Claim 2.5 is proved. Thanks to Claim 2.5 and the fact that $\xi^2 - \xi^{2q} < 0$, we therefore reach

$$\Upsilon_{\varepsilon,R}(a_2) \leq \xi^2 J_{\varepsilon,R}(u_n) + \kappa(\xi^2 - \xi^{2q})C,$$

for $n \in \mathbb{N}$ large. Letting $n \to +\infty$, it follows that

$$\Upsilon_{\varepsilon,R}(a_2) \leq \xi^2 \Upsilon_{\varepsilon,R}(a_1) + \kappa(\xi^2 - \xi^{2q})C < \xi^2 \Upsilon_{\varepsilon,R}(a_1),$$

that is,

$$\frac{\Upsilon_{\varepsilon,R}(a_2)}{a_2^2} < \frac{\Upsilon_{\varepsilon,R}(a_1)}{a_1^2},$$

proving the lemma. □

**Lemma 2.6.** *Let $R > R^*$, $\kappa \in (0, 1)$ and $\varepsilon > 0$ be fixed, assume $(u_n) \subset H^1(\mathbb{R}^2)$ is a minimizing sequence associated with $\Upsilon_{\varepsilon,R}(a)$ for $a > a^*$. Then, there exist bounded sequence $(\sigma_n) \subset \mathbb{R}$ and $\sigma_R < 0$ such that for some subsequence, still denoted by itself, one has $\lim_{n\to+\infty} \sigma_n = \sigma_R$ and*

$$\|J'_{\varepsilon,R}(u_n) - \sigma_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^2))^{-1}} \to 0 \quad \text{as } n \to +\infty,$$

*where $\Psi : H^1(\mathbb{R}^2) \to \mathbb{R}$ is given by*

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^2}|u|^2 dx.$$

*Proof.* Setting the functional $\Psi : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^2}|u|^2 dx,$$

we see that $S(a) = \Psi^{-1}(\{a^2/2\})$. Then, by Willem [64, Proposition 5.12], there exists $(\sigma_n) \subset \mathbb{R}$ such that

$$\|J'_{\varepsilon,R}(u_n) - \sigma_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^2))^{-1}} \to 0 \quad \text{as } n \to +\infty, \tag{2.8}$$

Since $(u_n)$ is bounded in $H^1(\mathbb{R}^2)$, it concludes that $(\sigma_n)$ is also a bounded sequence, then we can assume that $\sigma_n \to \sigma_R$ as $n \to +\infty$ along a subsequence. This together with (2.8) leads to

$$J'_{\varepsilon,R}(u_n) - \sigma_R \Psi'(u_n) = o_n(1) \quad \text{in } (H^1(\mathbb{R}^2))^{-1}.$$

Now, we prove that $\sigma_R < 0$. First of all, let us recall that

$$\int_{\mathbb{R}^2}\left[|\nabla u_n|^2 + V(\varepsilon_n x)|u_n|^2\right]dx - \kappa\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]f_R(u_n)u_n dx = \sigma_R a^2 + o_n(1).$$

Since $J_{\varepsilon,R}(u_n) = \Upsilon_{\varepsilon,R}(a) + o_n(1)$, one gets

$$2\Upsilon_{\varepsilon,R}(a) + \kappa\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)][F_R(u_n) - f_R(u_n)u_n]dx = \sigma_R a^2 + o_n(1).$$

By (A2), it holds that

$$2\Upsilon_{\varepsilon,R}(a) + \Big(\frac{1}{q} - 1\Big)\kappa\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]f_R(u_n)u_n dx \geq \sigma_R a^2 + o_n(1).$$

As $f(s)s \geq 0$ for all $s \in \mathbb{R}$, $q > 2$, we obtain

$$2\Upsilon_{\varepsilon,R}(a) \geq \sigma_R a^2.$$

Now, according to $\Upsilon_{\varepsilon,R}(a) \leq \Theta_R < 0$ for every $R > R^*$, $\kappa \in (0,1)$, $a > a^*$ and $\varepsilon > 0$ by Lemma 2.3, it follows that $\sigma_R < 0$. The proof is complete. $\square$

Our next result is a compactness theorem on $S(a)$ and then it is possible to find a minimizer for $\Upsilon_{\varepsilon,R}(a)$.

**Theorem 2.7.** *Let $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ be fixed as above. Suppose that $(u_n) \subset S(a)$ is a minimizing sequence of $\Upsilon_{\varepsilon,R}(a)$ for each fixed $a > a^*$, then $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$. If $u \neq 0$, then $u_n \to u$ in $H^1(\mathbb{R}^2)$ along a subsequence as $n \to \infty$.*

*Proof.* Since $J_{\varepsilon,R}$ is coercive on $S(a)$, the sequence $(u_n)$ is bounded, and so, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ for some subsequence. If $u \neq 0$ and $|u|_2 = \hat{a} \neq a$, we must have $\hat{a} \in (0,a)$. By the Brézis-Lieb Lemma (see [64]),

$$|u_n|_2^2 = |u_n - u|_2^2 + |u|_2^2 + o_n(1).$$

Furthermore, arguing as (1.19) to see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\{[|x|^{-\mu} * F_R(u_n)]F_R(u_n) - [|x|^{-\mu} * F_R(u_n - u)]F_R(u_n - u)\}dx = \int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u)]F_R(u)dx.$$

Setting $v_n = u_n - u$, $d_n = |v_n|_2$ and supposing that $|v_n|_2 \to d$, we reach $a^2 = \hat{a}^2 + d^2$. From $d_n \in (0,a)$ for $n$ large enough,

$$\Upsilon_{\varepsilon,R}(a) + o_n(1) = J_{\varepsilon,R}(u_n) = J_{\varepsilon,R}(v_n) + J_{\varepsilon,R}(u) + o_n(1) \geq \Upsilon_{\varepsilon,R}(d_n) + \Upsilon_{\varepsilon,R}(\hat{a}) + o_n(1).$$

thereby, by Lemma 2.4,

$$\Upsilon_{\varepsilon,R}(a) + o_n(1) \geq \frac{d_n^2}{a^2}\Upsilon_{\varepsilon,R}(a) + \Upsilon_{\varepsilon,R}(\hat{a}) + o_n(1).$$

Letting $n \to +\infty$, one finds

$$\Upsilon_{\varepsilon,R}(a) \geq \frac{d^2}{a^2}\Upsilon_{\varepsilon,R}(a) + \Upsilon_{\varepsilon,R}(\hat{a}). \tag{2.9}$$

Since $\hat{a} \in (0,a)$, employing Lemma 2.4 in (2.9) again, we arrive at the following inequality

$$\Upsilon_{\varepsilon,R}(a) > \frac{d^2}{a^2}\Upsilon_{\varepsilon,R}(a) + \frac{\hat{a}^2}{a^2}\Upsilon_{\varepsilon,R}(a) = \Big(\frac{d^2}{a^2} + \frac{\hat{a}^2}{a^2}\Big)\Upsilon_{\varepsilon,R}(a) = \Upsilon_{\varepsilon,R}(a),$$

which is absurd. This asserts that $|u|_2 = a$, or equivalently, $u \in S(a)$.

As $|u_n|_2 = |u|_2 = a$, $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ is reflexive, it is well-known that

$$u_n \to u \quad \text{in } L^2(\mathbb{R}^2). \tag{2.10}$$

This combined with interpolation theorem in the Lebesgue space and (1.14)-(1.15) gives

$$\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]F_R(u_n)dx \to \int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u)]F_R(u_n)dx. \tag{2.11}$$

These limits together with $\Upsilon_{\varepsilon,R}(a) = \lim_{n\to+\infty} J_{\varepsilon,R}(u_n)$ provide

$$\Upsilon_{\varepsilon,R}(a) \geq J_{\varepsilon,R}(u).$$

As $u \in S(a)$, we infer that $J_{\varepsilon,R}(u) = \Upsilon_{\varepsilon,R}(a)$, then

$$\lim_{n\to+\infty} J_{\varepsilon,R}(u_n) = J_{\varepsilon,R}(u),$$

that combines with (2.10) and (2.11) to give

$$\|u_n\|_{H^1}^2 \to \|u\|_{H^1}^2,$$

The last limit permits to conclude that $u_n \to u$ in $H^1(\mathbb{R}^2)$. The proof is complete. □

As we can observe that it is crucial to verify that the weak limit $u \neq 0$ before exploiting the compact result established in Theorem 2.7. To arrive at it, we need to introduce the following variational functionals $J_{0,R}$ and $J_{\infty,R}$ defined by

$$\begin{aligned} J_{0,R}(u) &= \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_0|u|^2\right] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]F_R(u)dx, \\ J_{\infty,R}(u) &= \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\infty|u|^2\right] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]F_R(u)dx, \end{aligned} \tag{2.12}$$

restricted to the sphere $S(a)$ defined in (2.4). One easily sees that $J_{0,R}, J_{\infty,R} \in \mathcal{C}^1(H^1(\mathbb{R}^2), \mathbb{R})$ and

$$\begin{aligned} J'_{0,R}(u)v &= \int_{\mathbb{R}^2} (\nabla u\nabla v + V_0 uv)\, dx - \kappa\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]f_R(u)v dx, \\ J'_{\infty,R}(u)v &= \int_{\mathbb{R}^2} (\nabla u\nabla v + V_\infty uv)\, dx - \kappa\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]f_R(u)v dx, \end{aligned}$$

for all $u, v \in S(a)$. We also need to consider the minimization problems

$$\begin{aligned} \Upsilon_{0,R}(a) &= \min_{u\in S(a)} J_{0,R}(u) \\ \Upsilon_{\infty,R}(a) &= \min_{u\in S(a)} J_{0,\infty}(u). \end{aligned} \tag{2.13}$$

Owing to the definitions of $\Upsilon_{0,R}(a)$ and $\Upsilon_{\infty,R}(a)$, by (A8), it is clear to check that

$$\Upsilon_{0,R}(a) < \Upsilon_{\infty,R}(a), \quad \forall R > R^*,\ \kappa \in (0,1) \text{ and } a > a^*. \tag{2.14}$$

**Lemma 2.8.** *If $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ are fixed, then it holds that $\lim_{\varepsilon\to 0^+} \Upsilon_{\varepsilon,R}(a) = \Upsilon_{0,R}(a)$ for all $a > a^*$. Particularly, there is a small $\varepsilon^* = \varepsilon^*(R) > 0$ such that $\Upsilon_{\varepsilon,R}(a) < \Upsilon_{\infty,R}(a)$ for all $\varepsilon \in (0, \varepsilon^*)$.*

*Proof.* To begin wit a claim.

**Clame 2.9.** If $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ are fixed, there is a $U_0 \in H^1(\mathbb{R}^2)$ such that

$$U_0 \in S(a) \quad \text{and} \quad J_{0,R}(U_0) = \Upsilon_{0,R}(a), \quad \forall a > a^*.$$

Indeed, we suppose that $(U_n) \subset S(a)$ is a minimizing sequence of $\Upsilon_{0,R}(a)$. Similar to Lemma 2.2, $(U_n)$ is bounded and there is a $\bar{U}_0$ such that $U_n \rightharpoonup \bar{U}_0$ along a subsequence. It follows from the Vanishing lemma, c.f. [64, Lemma 1,21], that there are $\rho > 0$ and $(z_n) \subset \mathbb{R}^2$ such that

$$\liminf_{n\to\infty} \int_{B_\rho(y_n)} |U_n|^2 dx > 0.$$

Otherwise, $U_n \to 0$ in $L^p(\mathbb{R}^2)$ for every $p \in (2, +\infty)$ which together with (1.18) and (1.14)-(1.15) implies that $\lim_{n\to\infty} \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(U_n)]F_R(U_n)dx = 0$. So, $\Upsilon_{0,R}(a) = \lim_{n\to\infty} \int_{\mathbb{R}^2} |\nabla U_n|^2 dx \geq 0$ but it cannot occur using a similar arguments in Lemma 2.3. Now, we define $\bar{U}_n := qU_n(\cdot + z_n)$ and it is still a minimizing sequence of $\Upsilon_{0,R}(a)$. Hence, $\bar{U}_n \rightharpoonup U_0 \neq 0$ in $H^1(\mathbb{R}^2)$ along a subsequence, and then the Claim is proved by Theorem 2.7.

Since $U_0 \in S(a)$ in Claim 2.9, we arrive at

$$\Upsilon_{\varepsilon,R}(a) \leq J_{\varepsilon,R}(U_0) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla U_0|^2 + V(\varepsilon x)|U_0|^2)dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(U_0)]F_R(U_0)dx.$$

Taking the limit as $\varepsilon \to 0^+$ and recalling $V(0) = \inf_{z\in\mathbb{R}^2} V(z) = V_0$, the we use the Lebesgue's Dominated Convergence theorem as well as Claim 2.9 to obtain

$$\limsup_{\epsilon\to 0^+} \Upsilon_{\varepsilon,R}(a) \le J_{0,R}(U_0) = \Upsilon_{0,R}(a). \tag{2.15}$$

On the other hand, by (A8), one finds that

$$J_{0,R}(u) \le J_{\varepsilon,R}(u), \quad \forall u \in H^1(\mathbb{R}^2),$$

implying that

$$\Upsilon_{0,R}(a) \le \Upsilon_{\varepsilon,R}(a), \quad \forall \varepsilon > 0.$$

Therefore,

$$\Upsilon_{0,R}(a) \le \liminf_{\varepsilon\to 0^+} \Upsilon_{\varepsilon,R}(a). \tag{2.16}$$

From (2.15) and (2.16), it holds that

$$\lim_{\varepsilon\to 0^+} \Upsilon_{\varepsilon,R}(a) = \Upsilon_{0,R}(a).$$

The limit above combined with (2.14) yields that there is a $\varepsilon^* > 0$ such that $\Upsilon_{0,R}(a) < \Upsilon_{\infty,R}(a)$ for all $\varepsilon \in (0, \varepsilon^*)$. The proof is complete. □

**Lemma 2.10.** *If $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ are fixed. Assume $(u_n) \subset S(a)$ is a minimizing sequence with respect to $\Upsilon_{\varepsilon,R}(a)$ for all $a > a^*$, then there is a function $u \in H^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u$ along a subsequence in $H^1(\mathbb{R}^2)$. Moreover, we have that $u \ne 0$ provided that $\varepsilon \in (0, \varepsilon^*)$.*

*Proof.* The first part is a direct consequence of Lemma 2.2 and hence we omit it here. Suppose by the contradiction that $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$. Then

$$\Upsilon_{\varepsilon,R}(a) + o_n(1) = J_{\varepsilon,R}(u_n) = J_{\infty,R}(u_n) + \frac{1}{2}\int_{\mathbb{R}^2} [V(\varepsilon x) - V_\infty]|u_n|^2 dx.$$

From (A8), for each $\eta > 0$, there is a sufficiently large $\rho > 0$ such that

$$V(z) \ge V_\infty - \eta \text{ for } |z| \ge \rho.$$

Thereby, in view of $(u_n) \subset S(a)$,

$$\Upsilon_{\varepsilon,R}(a) + o_n(1) \ge \Upsilon_{\infty,R}(a) - \eta a^2 + \int_{\mathcal{O}_\rho} [V(\varepsilon x) - V_\infty]|u_n|^2 dx,$$

where $\mathcal{O}_\rho := \{z \in \mathbb{R}^2 : |z| < \varepsilon^{-1}\rho\}$. Letting $n \to \infty$ and then tending $\eta \to 0^+$, we derive

$$\Upsilon_{\varepsilon,R}(a) \ge \Upsilon_{\infty,R}(a),$$

which contradicts with Lemma 2.8. The proof is complete. □

At this stage, we can show the detailed proof of Theorem 2.1.

*Proof of Theorem 2.1.* Using Lemma 2.2, we can choose a minimizing sequence $(u_n) \subset S(a)$ associated with $\Upsilon_{\varepsilon,R}(a)$ and there is a $u_R \in H^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u_R$ in $H^1(\mathbb{R}^2)$. In light of the suitable $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon \in (0, \varepsilon^*)$, we can rely on Theorem 2.7 and Lemma 2.10 to see that $u_n \to u_R$ in $H^1(\mathbb{R}^2)$ and so $u_R$ is a minimizer of $\Upsilon_{\varepsilon,R}(a)$ for every $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon \in (0, \varepsilon^*)$ whenever $a > a^*$. Thanks to the Lagrange multiplier theorem, there is a $\sigma_R \in \mathbb{R}$ such that $(u_R, \sigma_R)$ is a couple of weak solutions to (2.1), where $\sigma_R < 0$ follows directly by Lemma 2.6. We clearly know that $u_R \ge 0$ by (A1); then some very similar arguments adopted in [48] reveal that $u_R(x) > 0$ for all $x \in \mathbb{R}^2$. The proof is complete. □

Let us finish this section by exhibiting the following theorem.

**Theorem 2.11.** *Let $u_R$ be given as in Theorem 2.1, if $z_\varepsilon$ denotes the global maximum of $u_R$, then, up to a subsequence if necessary,*

$$\lim_{\varepsilon\to 0^+} V(\varepsilon z_\varepsilon) = V_0.$$

*Proof.* Let $\varepsilon_n \to 0^+$, we relabel $u_R$ as $u_n$ to be a solution of the problem

$$
\begin{gathered}
-\Delta u + V(\varepsilon_n x)u = \sigma u + \kappa[|x|^{-\mu} * F(u)]f(u) \text{ in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} |u(x)|^2 dx = a^2,
\end{gathered}
\tag{2.17}
$$

for some $\sigma = \sigma_n \le \frac{2}{a^2}\Upsilon_{\varepsilon_n,R}(a)$ by Lemma 2.6. From Lemma 2.8, we know that

$$
\lim_{n\to\infty} J_{\varepsilon_n,R}(u_n) = \lim_{n\to\infty} \Upsilon_{\varepsilon_n,R}(a) = \Upsilon_{0,R}(a). \tag{2.18}
$$

**Clame 2.12.** There exists a sequence $(\bar{y}_n) \subset \mathbb{R}^2$ such that $v_n = u_n(\cdot + \bar{y}_n)$ contains a strongly convergent subsequence in $H^1(\mathbb{R}^2)$. Moreover, up to a subsequence if necessary, $y_n = \varepsilon_n \bar{y}_n \to y$ as $n \to \infty$, where $V(y) = V_0 = \inf_{z\in\mathbb{R}^2} V(x)$.

Indeed, there are $\rho > 0$, $\beta > 0$ and $(\bar{y}_n) \subset \mathbb{R}^2$ such that

$$
\int_{B_\rho(\bar{y}_n)} |u_n|^2 dx \ge \beta, \quad \forall n \in \mathbb{N}. \tag{2.19}
$$

Otherwise, one has that $u_n \to 0$ in $L^p(\mathbb{R}^2)$ for all $2 < p < +\infty$ which together with (1.14)-(1.15) and (1.18) implies that $\lim_{n\to\infty} \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)]F_R(u_n)dx = 0$. As a consequence, by means of (2.18), we derive $\Upsilon_{0,R}(a) = \lim_{n\to\infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 dx \ge 0$ violating Lemma 2.3. Thereby, (2.19) holds and we could fix $v_n = u_n(\cdot + \bar{y}_n)$. There is a $v \ne 0$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$ along a subsequence. Since $(v_n) \subset S(a)$ and $J_{\varepsilon_n,R}(u_n) \ge J_{0,R}(u_n) = J_{0,R}(v_n) \ge \Upsilon_{0,R}(a)$, then one can invoke from (2.18) that $(v_n)$ is a minimizing sequence of $\Upsilon_{0,R}(a)$. It is very similar to Theorem 2.7 that $v_n \to v$ in $H^1(\mathbb{R}^2)$ along a subsequence. Next, we shall verify that $(y_n)$ is bounded in $n \in \mathbb{N}$. Suppose, by contradiction, that $|y_n| \to +\infty$ and so

$$
\begin{aligned}
\Upsilon_{0,R}(a) &= \lim_{n\to\infty} \Big\{ \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla v_n|^2 + V(\varepsilon_n x + y_n)|v_n|^2\right] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(v_n)]F_R(v_n)dx \Big\} \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla v|^2 + V_\infty |v|^2\right) dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(v)]F_R(v)dx \\
&\ge \Upsilon_{\infty,R}(a)
\end{aligned}
$$

which is absurd by (2.14), where we have used (2.18) and $v_n \to v$ in $H^1(\mathbb{R}^2)$. Thus, passing to a subsequence if necessary, we can assume that $y_n \to y$ in $\mathbb{R}^2$. A similar argument shows that

$$
\Upsilon_{0,R}(a) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla v|^2 + V(y)|v|^2\right] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(v)]F_R(v)dx \ge \Upsilon_{V(y),R}(a).
$$

If $V(y) > V_0$, as the byproduct of Theorem 2.7, we could conclude that $\Upsilon_{V(y),R}(a) > \Upsilon_{0,R}(a)$. So, we must have that $V(y) = V_0$ proving the Claim.

Recalling (2.17), $(v_n) \subset S(a)$ is a sequence of solutions to the equation

$$
-\Delta u + V(\varepsilon_n x + y_n)u = \sigma_n u + \kappa[|x|^{-\mu} * F_R(u)]f_R(u) \text{ in } \mathbb{R}^2
$$

with

$$
\lim_{n\to\infty} \sigma_n \le \frac{2}{a^2}\Upsilon_{\varepsilon_n,R}(a) \le \Theta_R < 0.
$$

Owing to Claim 2.12, the same arguments explored in [4, Lemma 4.3] become available in this scenario to verify that

$$
\lim_{n\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.
$$

From which, given a $\tau > 0$, there are some $\rho_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$
v_n(x) \le \tau, \ \forall |x| \ge \rho_0 \text{ and } n \ge n_0.
$$

Clearly, it holds that $|v_n|_\infty \not\to 0$. In fact, we can derive from (2.19) that $|v_n|_\infty^2 \ge \beta \,\mathrm{meas}(B_\rho(0))$. At this stage, let us fix $\tau > 0$ such that $|v_n|_\infty \ge 2\tau$ and let $\hat{y}_n \in \mathbb{R}^2$ satisfy $v_n(\hat{y}_n) = |v_n|_\infty$ for all $n \in \mathbb{N}$. Therefore, according to the above discussions, it holds $|\hat{y}_n| \le \rho_0$ for all $n \in \mathbb{N}$. Furthermore, if we denote $z_n$ by $u_n(z_n) = |u_n|_\infty$ for all $n \in \mathbb{N}$, then $z_n = \hat{y}_n + \bar{y}_n$ and

$$
\lim_{n\to\infty} V(\varepsilon_n z_n) = \lim_{n\to\infty} V(\varepsilon_n \hat{y}_n + \varepsilon_n \bar{y}_n) = \lim_{n\to\infty} V(\varepsilon_n \hat{y}_n + y_n) = V(y) = V_0
$$

completing the proof. □

## 3. Truncated problem: steep potential well

In this section, we shall conclude the existence of positive solutions for the nonlocal Schrödinger equation

$$-\Delta u + \lambda V(x)u = \sigma u + \kappa[|x|^{-\mu} * F_R(u)]f_R(u) \text{ in } \mathbb{R}^2, \tag{3.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 dx = a^2, \tag{3.2}$$

where the potential $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies the assumptions (A9)–(A11), $\lambda, \kappa > 0$ are parameters, $a > 0$, $\sigma \in \mathbb{R}$ is known as the Lagrange multiplier and the nonlinearity $f_R$ is defined in (1.13).

Before solving Problems (3.1)-(3.2), we have to determine a suitable work space. Proceeding as [53, 56, 58], given a fixed $\lambda > 0$, by (A9), we define the space

$$E_\lambda := \Big\{u \in L^2_{\rm loc}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} \lambda V(x)|u|^2 dx < +\infty\Big\}$$

which is indeed a Hilbert space equipped with the inner product and norm

$$(u, v)_{E_\lambda} = \int_{\mathbb{R}^2} \big[\nabla u \nabla v + \lambda V(x)uv\big] dx, \quad \|u\|_{E_\lambda} = \sqrt{(u, u)_{E_\lambda}}, \ \forall u, v \in E_\lambda.$$

From here onwards, we shall denote $E$ and $\|\cdot\|_E$ by $E_\lambda$ and $\|\cdot\|_{E_\lambda}$ for $\lambda = 1$, respectively. It is simple to observe that $\|\cdot\|_E \le \|\cdot\|_{E_\lambda}$ for every $\lambda \ge 1$. Therefore, owing to [56, Lemma 2.4], $E_\lambda$ could be continuously imbedded into $H^1(\mathbb{R}^2)$ for all $\lambda \ge 1$.

Define the variational functional $\mathcal{J}_{\lambda,R} : E_\lambda \to \mathbb{R}$ by

$$\mathcal{J}_{\lambda,R}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \big[|\nabla u|^2 + \lambda V(x)|u|^2\big] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]F_R(u)dx. \tag{3.3}$$

Obviously, combining (1.15) and (1.18), one can easily show that $\mathcal{J}_{\lambda,R}$ belongs to $\mathcal{C}^1(E_\lambda, \mathbb{R})$ and it derivative is

$$\mathcal{J}'_{\lambda,R}(u)v = \int_{\mathbb{R}^2} [\nabla u \nabla v + \lambda V(x)uv]\, dx - \kappa \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]f_R(u)v dx, \ \forall u, v \in E_\lambda.$$

To solve Problems (3.1)-(3.2), we consider the minimization problem

$$\bar{\Upsilon}_{\lambda,R}(a) = \min_{u\in S(a)} \mathcal{J}_{\lambda,R}(u), \tag{3.4}$$

where, with $\lambda \ge 1$, the sphere in defined by

$$S(a) = \Big\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2 dx = a^2\Big\}. \tag{3.5}$$

The existence result for Problems (3.1)-(3.2) in this section can be stated as follows.

**Theorem 3.1.** *Suppose rm (A1)–(A3) , (A9)–(A11) and* (1.3) *hold, and $\mu \in (0, 2)$, then there is an $R_* > 0$ such that for every $R > R_*$, there exist some $a_* = a_*(R) > 0$ and $\lambda_* = \lambda_*(R) > 1$ such that, for all $\kappa \in (0, 1)$, $a > a_*$ and $\lambda > \lambda_*$, the minimization problem* (3.4) *can be achieved by some function in $E_\lambda$. Moreover, there is $(u_R, \sigma_R) \in H^1(\mathbb{R}^2) \times \mathbb{R}$ such that it is a couple solution of Problems* (3.1)-(3.2)*, where $u_R(x) > 0$ for all $x \in \mathbb{R}^2$ and $\sigma_R < 0$.*

Arguing as we did in Section 2, we introduce several lemmas to prove Theorem 3.1. For simplicity, when there is no misunderstanding, we will also suppose that the potential $V$ and the nonlinearity $f_R$ satisfy (A9)–(A11) and (1.3) with (A1)–(A3) in this section.

Using the same calculations as in the proof of Lemma 2.2, we have the following result.

**Lemma 3.2.** *For all fixed $R > 0$, the variational functional $\mathcal{J}_{\lambda,R}$ is coercive and bounded from below on $S(a)$ for each $\kappa \in (0, 1)$, $a > 0$ and $\lambda \ge 1$, where $\mathcal{J}_{\lambda,R}$ and $S(a)$ are appearing in* (3.3) *and* (3.5)*, respectively.*

The proof of the above lemms is the same as that of Lemma 2.2 and so we omit it here. Employing some necessary modifications in the proof of Lemma 2.3, we are able to conclude the following lemma.

**Lemma 3.3.** *There exists an $R_* > 0$ such that for all fixed $R > R_*$, there is an $a_* = a_*(R) > 0$ satisfying for all $a > a_*$, there exists a constant $\bar{\Theta}_R = \bar{\Theta}(R) < 0$, independent of $\lambda$, such that $\bar{\Upsilon}_{\lambda,R}(a) \leq \bar{\Theta}_R$ for all $\kappa \in (0,1)$ and $\lambda \geq 1$.*

*Proof.* The main idea originates from [58, Lemma 3.3], we show the details for the convenience of the reader. Without loss of generality, we are assuming that $0 \in \operatorname{int} V^{-1}(0)$. Therefore, there exists a sufficiently small $r > 0$ such that $B_r(0) \subset \operatorname{int} V^{-1}(0)$. Choose $\psi \in C_0^\infty(B_r(0))$ to be a function satisfying $\int_{B_r(0)} |\psi|^2 dx = 1$ and so $\psi \in S(1)$. Thanks to the definition of $\Omega$, it holds that

$$\int_{\mathbb{R}^2} V(x)|\psi|^2 dx = \int_{\Omega} V(x)|\psi|^2 dx + \int_{\Omega^c} V(x)|\psi|^2 dx = 0. \tag{3.6}$$

Proceeding as the proof of Lemma 2.3, we could determine a sufficiently large $t_* = t_*(R) > 0$ and then $t_* = t_*|\psi|_2$ to find a constant $\bar{\Theta}_R < 0$, dependent of $R$, such that

$$\mathcal{J}_{\lambda,R}(u) \leq \bar{\Theta}_R, \quad \forall R > R_*,\ \kappa \in (0,1),\ a > a_* \text{ and } \lambda \geq 1,$$

provided $u \in S(a)$. The proof is complete. □

Owing to the essential feature of steep potential well, there is no need to certify the similar result in Lemma 2.4. In other words, we shall conclude the counterpart of Theorem 2.7 directly.

**Theorem 3.4.** *Let $R > R_*$, $\kappa \in (0,1)$ and $\lambda \geq 1$ be fixed. Suppose $(u_n) \subset S(a)$ is a minimizing sequence of $\bar{\Upsilon}_{\lambda,R}(a)$ for all $a > a_*$, then $u_n \rightharpoonup u$ in $E_\lambda$ as $n \to \infty$. If in addition $u \neq 0$, there is a sufficiently large $\lambda'_* = \lambda'_*(R) > 1$ such that $u_n \to u$ in $E_\lambda$ along a subsequence as $n \to \infty$ for all $\lambda > \lambda'_*$.*

*Proof.* The first part is the same as its counterpart in Theorem 2.7, and we omit it here. To derive the remaining part, we define $v_n := u_n - u \rightharpoonup 0$ in $E_\lambda$. Let us recall from (A11) that the nonempty set $\Xi := \{x \in \mathbb{R}^2 : V(x) < b\}$ has finite measure, then

$$\begin{aligned}\int_{\mathbb{R}^2} |v_n|^2 dx &= \int_{\mathbb{R}^2\setminus\Xi} |v_n|^2 dx + \int_{\Xi} |v_n|^2 dx + = \int_{\mathbb{R}^2\setminus\Xi} |v_n|^2 dx + o_n(1)\\ &\leq \frac{1}{\lambda b}\int_{\mathbb{R}^2\setminus\Xi} \lambda V(x)|v_n|^2 dx + o_n(1) \leq \frac{1}{\lambda b}\|v_n\|_{E_\lambda}^2 + o_n(1)\end{aligned}$$

which together with (1.17) with $l = \frac{4q}{4-\mu} > 2$ gives that

$$\int_{\mathbb{R}^2} |v_n|^l dx \leq \mathbb{C}(\lambda b)^{-\frac{(1-\gamma_l)l}{2}}\|v_n\|_{E_\lambda}^l + o_n(1).$$

From the inequality above, combining (1.14)-(1.15) and (1.18), we see that

$$\kappa\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(v_n)]F_R(v_n)dx \leq \frac{\mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R)}{q^{4-\mu}R^{(4-\mu)(q-1)}}(\lambda b)^{-\frac{4-\mu}{2}}\|v_n\|_{E_\lambda}^{2q} + o_n(1). \tag{3.7}$$

On the other hand, obviously $|v_n|_2 \in (0,a)$, then Lemma 3.3 indicates that $\bar{\Upsilon}_{\lambda,R}(|v_n|_2) \leq 0$. Moreover, $\|v_n\|_{E_\lambda}$ is bounded, namely there exists a $\zeta = \zeta(R) > 0$ such that $\|v_n\|_{E_\lambda} \leq \zeta$ for all $n \in \mathbb{N}$. Combining these facts jointly with (3.7), it holds that

$$0 \geq \Big[\frac{1}{2} - \frac{\mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R)}{2q^{4-\mu}R^{(4-\mu)(q-1)}}(\lambda b)^{-\frac{4-\mu}{2}}\zeta^{2(q-1)}\Big]\|v_n\|_{E_\lambda}^2 + o_n(1). \tag{3.8}$$

Consequently, we shall determine a sufficiently large $\lambda'_* = \lambda'_*(R) > 1$ to satisfy $\|v_n\|_{E_\lambda}^2 = o_n(1)$ whenever $\lambda > \lambda'_*$. The proof is complete. □

To apply Theorem 3.4 successfully, we need the following lemma to show that the weak limit of $u$ is not 0.

**Lemma 3.5.** *Under the assumptions of Theorem 3.4, there exists a $\lambda_* = \lambda_*(R) > \lambda'_*$ such that $u \neq 0$ for all $\lambda > \lambda_*$.*

*Proof.* We collect the methods used in [53, 56, 58] to reach the proof. Firstly, we have a claim.

**Clame 3.6.** For some $q_0 \in (2, +\infty)$, there exists a constant $\beta_0 > 0$, independent of $\lambda \geq 1$, such that

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^2} \int_{B_\rho(z)} |u_n|^{q_0} dx = \beta_0.$$

To demonstrate this Claim, we can suppose that there exists a constant $\beta_\lambda = \beta(\lambda) > 0$ such that $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} |u_n|^{q_0} dx = \beta_\lambda$. Otherwise, $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for every $s \in (2, +\infty)$ jointly with (1.14)-(1.15) and (1.18) yields that $\lim_{n\to\infty} \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)] F_R(u_n) dx = 0$. Hence, we can conclude that $\bar{\Upsilon}_{\lambda,R}(a) = \lim_{n\to\infty} \mathcal{J}_{\lambda,R}(u_n) \geq 0$ and it is impossible because of Lemma 3.3. With such a $\beta_\lambda$, we are able to verify this Claim. Suppose, by contradiction, that the uniform control from below of $L^{q_0}(\mathbb{R}^2)$-norm is false. Consequently, for any $k \in \mathbb{N}$, $k \neq 0$, there are $\lambda_k > 1$ and a minimizing sequence $(u_{k,n})$ of $\bar{\Upsilon}_{\lambda_k,R}$ such that

$$|u_{k,n}|_{q_0} < \frac{1}{k}, \text{ definitely.}$$

Then, by a diagonalization argument, for any $k \geq 1$, it permits us to find an increasing sequence $(n_k) \subset \mathbb{N}$ and $(u_{n_k}) \subset E_{\lambda_{n_k}}$ such that

$$(u_{n_k}) \subset S(a),\ \mathcal{J}_{\lambda_{n_k},R}(u_{n_k}) = \bar{\Upsilon}_{\lambda_{n_k},R}(a) + o_k(1) \text{ and } |u_{n_k}|_{q_0} = o_k(1).$$

where $o_k(1) \to 0$ as $k \to +\infty$. In this situation, we could repeat the calculations above to reach a contradiction $\bar{\Upsilon}_{\lambda_{n_k},R}(a) \geq 0$, again. So, the Claim is proved.

Thanks to Claim 3.6, there exist a sequence $(z_n) \subset \mathbb{R}^2$ and a subsequence $(u_n)$, still denoted by itself, such that

$$\int_{B_\rho(z_n)} |u_n|^2 dx = \frac{1}{2}\beta_0. \tag{3.9}$$

**Clame 3.7.** The sequence $(z_n)$ above is uniformly bounded in $n \in \mathbb{N}$.

Otherwise, we suppose by contradiction to choose a subsequence if necessary that $|z_n| \to \infty$. Define

$$\Xi_n^1 := \{x \in B_\rho(z_n) : V(x) < b\} \quad \text{and} \quad \Xi_n^2 := \{x \in B_\rho(z_n) : V(x) \geq b\}.$$

Since the set $\Xi := \{x \in \mathbb{R}^2 : V(x) < b\}$ is nonempty and has finite measure, one concludes that

$$\text{meas}(\Xi_n^1) \leq \text{meas}(\{x \in \mathbb{R}^2 : |x| \geq |y_n| - 2, V(x) < b\}) \to 0 \quad \text{as } n \to \infty.$$

For $\lambda \geq 1$, one sees $|u_n|_r$ with $r > 2$ is uniformly bounded in $n \in \mathbb{N}$ by Lemma 3.2 and then

$$\int_{\Xi_n^1} |u_n|^2 dx \leq [\text{meas}(\Xi_n^1)]^{\frac{r-2}{r}} |u_n|_r^2 = o_n(1)$$

which together with (3.9) reveals that

$$\int_{\Xi_n^2} |u_n|^2 dx = \int_{B_\rho(z_n)} |u_n|^2 dx - \int_{\Xi_n^1} |u_n|^2 dx = \frac{1}{2}\beta_0 + o_n(1).$$

Thanks to $V(x) \geq 0$ for all $x \in \mathbb{R}^2$ by (A9), using the definition of $\Xi_n^2$, we obtain

$$\int_{\mathbb{R}^2} V(x)|u_n|^2 dx \geq \int_{\Xi_n^2} V(x)|u_n|^2 dx \geq b \int_{\Xi_n^2} |u_n|^2 dx = \frac{1}{2} b\beta_0 + o_n(1). \tag{3.10}$$

It follows from (1.14)-(1.15) and (1.18) that

$$\sup_{n\in\mathbb{N}} \Big\{ \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)] F_R(u_n) dx \Big\} \leq C, \quad \forall \lambda \geq 1, \tag{3.11}$$

where $C > 0$ is independent of $n \in \mathbb{N}$ and $\lambda \geq 1$. So, we deduce by (3.10) and (3.11) that

$$\bar{\Upsilon}_{\lambda,R}(a) \geq \frac{1}{2} \int_{\mathbb{R}^2} \lambda V(x)|u_n|^2 dx - C + o_n(1) \geq \frac{\lambda b\beta_0}{4} - C + o_n(1) \tag{3.12}$$

where the positive constants $b, \beta_0$ and $C$ are independent of $\lambda \geq 1$. Adopting Lemma 3.3 again, there is a sufficiently large $\lambda_* = \lambda_*(R) > \lambda'_*(R)$ such that (3.12) is impossible provided $\lambda > \lambda_*$. Hence, the Claim is proved.

Owing to Claim 3.7, passing to a subsequence if necessary, we suppose that $z_n \to z_0$ in $\mathbb{R}^2$. Since $u_n \to u$ in $L^2_{\rm loc}(\mathbb{R}^2)$, then we can arrive at the proof of this lemma. □

*Proof of Theorem 3.1.* There is a minimizing sequence $(u_n) \subset S(a)$ associated with $\bar{\Upsilon}_{\lambda,R}(a)$ by Lemma 3.2 and thus $u_n \rightharpoonup u_R$ in $E_\lambda$ for some $\lambda \geq 1$. As a consequence of Theorem 3.4 and Lemma 3.5, for all $R > R_*$, $\kappa \in (0,1)$, $a > a_*$ and $\lambda > \lambda_*$, we see that $u_n \to u_R$ in $E_\lambda$ and so $u_R$ is a minimizer of $\bar{\Upsilon}_{\lambda,R}(a)$. By exploiting the Lagrange multiplier theorem again, a similar argument in Lemma 2.6 makes sure a $\sigma_R < 0$ that $(u_R, \sigma_R)$ is a couple of weak solutions to (3.1). Finally, the reader can derive $u_R > 0$ as in Theorem 2.1. The proof is complete. □

As we can observe from the proof of Theorem 3.1, the couple $(u_R, \sigma_R)$ exists for all $R > R_*$, $\kappa \in (0,1)$, $a > a_*$ and $\lambda > \lambda_*$. In other words, if $R > R_*$, $\kappa \in (0,1)$ and $a \in (0, a_*)$ are fixed, the couple $(u_R, \sigma_R)$ would also rely on $\lambda > \lambda_*$. It is therefore that we shall relabel it by $(u_\lambda, \sigma_\lambda)$ when $R > R_*$, $\kappa \in (0,1)$ and $a \in (0, a_*)$ are fixed.

Letting $\lambda \to +\infty$, we have the following result.

**Theorem 3.8.** *Let $(u_\lambda, \sigma_\lambda) \in E_\lambda \times \mathbb{R}$ denote by the couple of weak solutions established above for all $\lambda > \lambda_*$, passing to a subsequence if necessary, $u_\lambda \to u_0$ in $H^1(\mathbb{R}^2)$ and $\sigma_\lambda \to \sigma_0$ in $\mathbb{R}$ as $\lambda \to +\infty$, where $\sigma_0 < 0$ and $(u_0, \sigma_0)$ is a couple of weak solution to the problem*

$$
\begin{gathered}
-\Delta u = \sigma u + \kappa\Big(\int_\Omega \frac{F_R(u(y))}{|x-y|^\mu} dy\Big) f_R(u), \ \ x \in \Omega, \\
u(x) = 0, \quad x \in \partial\Omega, \\
\int_\Omega |u|^2 dx = a^2.
\end{gathered} \tag{3.13}
$$

*Proof.* Let $\lambda_n \to +\infty$, we study the subsequence of $(u_\lambda, \sigma_\lambda) \in E_\lambda \times \mathbb{R}$, namely $(u_{\lambda_n}, \sigma_{\lambda_n})$ satisfies $(u_{\lambda_n}) \subset S(a)$ and $\mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) = \bar{\Upsilon}_{\lambda_n,R}$. By Lemma 3.3, the sequence $(u_{\lambda_n})$ is uniformly bounded in $n \in \mathbb{N}$. Similar to the proof of Lemma 2.6, it holds

$$
\sigma_{\lambda_n} = \frac{1}{a^2}\Big\{\int_{\mathbb{R}^2} \big[|\nabla u_n|^2 + \lambda_n V(x)|u_n|^2\big]\, dx - \kappa \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)] f_R(u_n) u_n dx\Big\} + o_n(1)
$$

showing that $(\sigma_{\lambda_n})$ is uniformly bounded in $n \in \mathbb{N}$. Up to a subsequence if necessary, $u_{\lambda_n} \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$ and $\sigma_{\lambda_n} \to \sigma_0$ in $\mathbb{R}$ as $n \to +\infty$. In view of Lemmas 2.6 and 3.3 again, it holds that

$$
\sigma_0 = \lim_{n\to\infty} \sigma_{\lambda_n} \leq \lim_{n\to\infty} \frac{2}{a^2} \bar{\Upsilon}_{\lambda_n,R}(a) \leq \bar{\Theta}_R < 0.
$$

**Clame 3.9.** $u_0 \equiv 0$ in $\Omega^c := \mathbb{R}^2 \backslash \Omega$ and so $u_0 \in S_\Omega(a) := \big\{u \in H^1_0(\Omega) : \int_\Omega |u|^2 dx = a^2\big\}$.

Otherwise, there exists a compact subset $\hat{\Omega}_{u_0} \subset \Omega^c$ with $\mathrm{dist}(\hat{\Omega}_{u_0}, \partial\Omega^c) > 0$ such that $u_0 \neq 0$ on $\hat{\Omega}_{u_0}$ and by Fatou's lemma

$$
a^2 = \liminf_{n\to\infty} \int_{\mathbb{R}^2} u^2_{\lambda_n} dx \geq \int_{\hat{\Theta}_{u_0}} {u_0}^2 dx > 0. \tag{3.14}
$$

Moreover, there exists $\zeta_0 > 0$ such that $V(x) \geq \zeta_0$ for every $x \in \hat{\Omega}_{u_0}$ by the assumptions (A9) and (A10). Combining Lemma 3.3, (1.14) and (3.14), we obtain

$$
\begin{aligned}
0 &\geq \liminf_{n\to\infty} \bar{\Upsilon}_{\lambda_n,R} = \liminf_{n\to\infty} \mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) \\
&= \liminf_{n\to\infty} \Big\{\mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) - \frac{1}{q}\left[\mathcal{J}'_{\lambda_n,R}(u_{\lambda_n})u_{\lambda_n} - \sigma_{\lambda_n} a^2\right]\Big\} \\
&\geq \frac{q-2}{2q}\zeta_0\Big(\int_{\hat{\Theta}_{u_0}} u_0^2 dx\Big) \liminf_{n\to\infty} \lambda_n + \frac{\sigma_0}{q} a^2 = +\infty
\end{aligned}
$$

which is impossible. Consequently, $u_0 \in H^1_0(\Omega)$ by the fact that $\partial\Omega$ is smooth. By taking some similar calculations explored in (3.8) to show $u_{\lambda_n} \to u_0$ in $H^1(\mathbb{R}^2)$ and so $u_0 \in S_\Omega(a)$.

**Clame 3.10.** $\mathcal{J}_{\Omega,R}(u_0) = \bar{\Upsilon}_{\Omega,R}(a)$, where $\bar{\Upsilon}_{\Omega,R} := \inf_{u\in S_\Omega(s)} \mathcal{J}_{\Omega,R}(u)$ and the variational functional $\mathcal{J}_{\Omega,R} : H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{J}_{\Omega,R}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\kappa}{2}\int_\Omega\int_\Omega \frac{F_R(u(x))F_R(u(y))}{|x-y|^\mu} dxdy, \quad \forall u \in H_0^1(\Omega).$$

Actually, it is simple to see that $S_\Omega(a) \subset S(a)$ and so $\bar{\Upsilon}_{\Omega,R}(a) \geq \bar{\Upsilon}_{\lambda_n,R}(a)$. As a consequence, it holds $\bar{\Upsilon}_{\Omega,R}(a) \geq \liminf_{n\to\infty} \bar{\Upsilon}_{\lambda_n,R}(a)$. On the other hand, we gather these facts together with the Fatou's lemma to obtain

$$\bar{\Upsilon}_{\Omega,R}(a) \geq \liminf_{n\to\infty} \bar{\Upsilon}_{\lambda_n,R}(a) = \liminf_{n\to\infty} \mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) \geq \mathcal{J}_{\Omega,R}(u_0) \geq \bar{\Upsilon}_{\Omega,R}(a)$$

proving the Claim.

Finally, we shall prove that $\mathcal{J}'_\Omega(u_0) - \sigma_0 u_0 = 0$ in $(H_0^1(\Omega))^{-1}$. To see it, for every $\psi \in C_0^\infty(\Omega)$, Combining (1.20) and $\sigma_{\lambda_n} \to \sigma_0$, it holds that

$$\lim_{n\to\infty}\Big\{\mathcal{J}'_{\lambda_n,R}(u_{\lambda_n})\psi - \sigma_{\lambda_n}\int_{\mathbb{R}^2} u_{\lambda_n}\psi dx\Big\} = 0, \quad \forall \psi \in C_0^\infty(\Omega),$$

we can arrive at the desired result. The proof is complete. □

## 4. Proofs of main results

In this section, we are concerned with the existence and concentrating behavior of positive solutions to the nonlocal Schrödinger equation (1.1) under the mass-constraint (1.2).

Firstly, we shall provide some growth conditions with the nonlinearity $f$ and $f_R$ which play foremost roles in this section. It can infer from (A1) and (A2) that

$$\lim_{s\to 0^+} \frac{f_R(s)}{s} = 0, \quad \lim_{s\to 0^+} \frac{f(s)}{s} = 0. \tag{4.1}$$

Actually, using (A1) and (A2) with $q > 2$ again we obtain

$$0 \leq \lim_{s\to 0^+} \frac{f_R(s)}{s} = \lim_{s\to 0^+} \frac{f(s)}{s} = \lim_{s\to 0^+} \frac{f(s)}{s^{q-1}} s^{q-2} \leq f(1) \lim_{s\to 0^+} s^{q-2} = 0.$$

Combining (1.3) and (4.1), given a fixed $\varepsilon > 0$, for every $\bar{p} > 2$ and $\nu > 1$, we are able to search for two constants such that $\tilde{b}_1 = \tilde{b}_1(\bar{p}, \alpha, \varepsilon) > 0$ and $\tilde{b}_2 = \tilde{b}_2(\bar{p}, \alpha, \varepsilon) > 0$ satisfying

$$|f(s)| \leq \varepsilon|s| + \tilde{b}_1|s|^{\bar{p}-1}(e^{4\pi\nu s^2} - 1), \quad \forall s \in \mathbb{R}, \tag{4.2}$$

$$|F(s)| \leq \varepsilon|s|^2 + \tilde{b}_2|s|^{\bar{p}}(e^{4\pi\nu s^2} - 1), \quad \forall s \in \mathbb{R}. \tag{4.3}$$

taking in to accoun that the nonlinearity $f$ has the critical exponential growth at infinity, the following Trudinger-Moser inequality found in [61, 50, 26] will play a crucial role in this section.

**Lemma 4.1.** *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1)dx < +\infty.$$

*Moreover, if $|\nabla u|_2^2 \leq 1$, $|u|_2^2 \leq M < +\infty$ and $\alpha < 4\pi$, then there exists $K_{\alpha,M} = K(M,\alpha)$ such that*

$$\int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1)dx \leq K_{\alpha,M}. \tag{4.4}$$

Now, we are ready to exhibit the detailed proofs of Theorems 1.1 and 1.2. In order to show them clearly, we shall divide into two subsections.

4.1. **Proof of Theorem 1.1.** Because of Theorem 2.1, we know that the minimization constant $\Upsilon_{\varepsilon,R}(a)$ defined in (2.5) can be attained by some nontrivial function in $H^1(\mathbb{R}^2)$ for every fixed $R > R^*$, $\kappa \in (0,1)$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$. In other words, there is a function $u_R \in H^1(\mathbb{R}^2)$ such that

$$u_R \in S(a) \text{ and } J_{\varepsilon,R}(u_R) = \Upsilon_{\varepsilon,R}(a), \quad \forall R > R^*,\ \kappa \in (0,1),\ a > a^* \text{ and } \varepsilon \in (0,\varepsilon^*). \tag{4.5}$$

Moreover, there is a $\sigma_R < 0$ such that the couple $(u_R, \mu_R)$ is a solution of Problems (2.1)-(2.2) for all $R > R^*$, $\kappa \in (0,1)$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, where $u_R(x) > 0$ for all $x \in \mathbb{R}^2$.

According to the introduction, the reader can observe that if $u_R$ in (4.5) satisfies $|u_R|_\infty \le R$, then $u_R$ is in fact a solution of the original (1.1) with $\sigma = \sigma_R$. Therefore it is posssible to arrive at the proof of Theorem 1.1. As a consequence, the foremost objection for us is to take the $L^\infty$-estimate on $u_R$. To this aim, we establish the uniform estimate on $|\nabla u_R|_2^2$ below.

**Lemma 4.2.** *Suppose that $V$ satisfies* (A8) *and $f$ meets* (1.3) *with* (A1)–(A3). *Let $u_R$ be given by* (4.5) *for each $R > R^*$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, then there exists a $\kappa^* = \kappa^*(R) \in (0,1)$ such that if $\kappa \in (0,\kappa^*)$, it holds that $|\nabla u_R|_2^2 < \frac{2-\mu}{2(2+\mu)\nu^2}$ for every $R > R^*$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, where the constant $\nu > 1$ is appearing in* (4.2) *and* (4.3).

*Proof.* Since $u_R \in S(a)$, we borrow the calculations in Lemma 2.2 to obtain

$$J_{\varepsilon,R}(u_R) \ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_R|^2\,dx - \frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}} \Big(\int_{\mathbb{R}^2} |\nabla u_R|^2\,dx\Big)^{q-\frac{4-\mu}{2}}.$$

Since $2 < q < \frac{6-\mu}{2}$, by means of the Young's inequality, there is a $C_1 > 0$ independent of $R > R^*$ such that

$$\begin{aligned}
&\frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}} \Big(\int_{\mathbb{R}^2} |\nabla u_R|^2\,dx\Big)^{q-\frac{4-\mu}{2}} \\
&\le C_1 \Big[\frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}}\Big]^{\frac{2}{6-\mu-2q}} + \frac{1}{4}\int_{\mathbb{R}^2} |\nabla u_R|^2\,dx.
\end{aligned}$$

Thereby, for every $R > R^*$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, it holds that

$$|\nabla u_R|_2^2 \le 4J_{\varepsilon,R}(u_R) + 4C_1 \Big[\frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}}\Big]^{\frac{2}{6-\mu-2q}}.$$

Assuming that

$$4C_1 \Big[\frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}}\Big]^{\frac{2}{6-\mu-2q}} \le \frac{2-\mu}{2(2+\mu)\nu^2},$$

we arrive at

$$|\nabla u_R|_2^2 \le 4J_{\varepsilon,R}(u_R) + \frac{2-\mu}{2(2+\mu)\nu^2}, \quad \forall R > R^*,\ a > a^*,\ \varepsilon \in (0,\varepsilon^*).$$

In light of $J_{\varepsilon,R}(u_R) = \Upsilon_{\varepsilon,R}(a) \le 0$ by Lemma 2.3 and (4.5), so it permits us to choose

$$\kappa^* = \kappa^*(R) := \min\Big\{\Big[\frac{2R^{(4-\mu)(q-1)} q^{4-\mu}}{\mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}\Big]\Big[\frac{2-\mu}{8C_1(2+\mu)\nu^2}\Big]^{\frac{6-\mu-2q}{2}},\ 1\Big\}$$

and then we can complete the proof. □

With Lemma 4.2 in hand, we can derive the following result.

**Lemma 4.3.** *Suppose that $V$ satisfies* (A8) *and $f$ meets* (1.3) *with* (A1)–(A3). *Then, for every fixed $R > R^*$, $\kappa \in (0,\kappa^*)$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, there exists a constant $\mathfrak{C} \in (0,+\infty)$ which is independent of $R > R^*$ and $\varepsilon \in (0,\varepsilon^*)$ such that*

$$\Gamma(x) := |x|^{-\mu} * F(u_R) \le \mathfrak{C},$$

*where $u_R$ comes from* (4.5).

*Proof.* Since $u_R \in S(a)$, adopting Lemma 4.2 and (1.17), there is a constant $\mathbb{T} \in (0, +\infty)$ which is independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$ such that

$$|u|_l^l \leq \mathbb{T}, \ \forall l \in (2, +\infty). \tag{4.6}$$

By (4.6), we find a constant $\mathbb{C}_0 \in (0, +\infty)$ which is independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$ such that

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|u_R(y)|^2}{|x-y|^\mu} dy &= \int_{|x-y|<1} \frac{|u_R(y)|^2}{|x-y|^\mu} dy + \int_{|x-y|\geq 1} \frac{|u_R(y)|^2}{|x-y|^\mu} dy \\ &\leq \bar{\mathcal{C}}_\mu \Big( \int_{\mathbb{R}^2} |u_R(y)|^{\frac{2(2+\mu)}{2-\mu}} dy \Big)^{\frac{2-\mu}{2+\mu}} + a^2 \leq \mathbb{C}_0.\end{aligned} \tag{4.7}$$

Let us define $\bar{u}_R = \nu \sqrt{\frac{2(2+\mu)}{2-\mu}} u_R$, then $u_R \in S(a)$ and Lemma 4.2 give us

$$|\bar{u}_R|_2^2 = \frac{2\nu^2 a^2 (2+\mu)}{2-\mu} \quad \text{and} \quad |\nabla \bar{u}_R|_2^2 \leq 1$$

which together with (4.4) and $\nu > 1$ implies that

$$\int_{\mathbb{R}^2} (e^{\frac{8\pi\nu(2+\mu)}{2-\mu}|u_R(y)|^2} - 1) dy = \int_{\mathbb{R}^2} (e^{4\pi\nu^{-1}|\bar{u}_R(y)|^2} - 1) dy \leq K(a, \nu, \mu). \tag{4.8}$$

The above inequality shall determine a constant $\mathbb{C}_1 \in (0, +\infty)$ which is independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$ to reach

$$\begin{aligned}&\int_{|x-y|<1} \frac{|u_R(y)|^{\bar{p}} (e^{4\pi\nu|u_R(y)|^2} - 1)}{|x-y|^\mu} dy \\ &\leq \mathcal{C}_\mu \Big( \int_{\mathbb{R}^2} |u_R(y)|^{\frac{\bar{p}(2+\mu)}{2-\mu}} (e^{\frac{4\pi\nu(2+\mu)}{2-\mu}|u_R(y)|^2} - 1) dy \Big)^{\frac{2-\mu}{2+\mu}} \\ &\leq \mathcal{C}_\mu \left( \int_{\mathbb{R}^2} |u_R(y)|^{\frac{2\bar{p}(2+\mu)}{2-\mu}} dy \right)^{\frac{2-\mu}{2(2+\mu)}} \left( \int_{\mathbb{R}^2} (e^{\frac{8\pi\nu(2+\mu)}{2-\mu}|u_R(y)|^2} - 1) dy \right)^{\frac{2-\mu}{2(2+\mu)}} \leq \mathbb{C}_1.\end{aligned}$$

Using similar calculations, we have that

$$\begin{aligned}\int_{|x-y|\geq 1} \frac{|u_R(y)|^{\bar{p}} (e^{4\pi\nu|u_R(y)|^2} - 1)}{|x-y|^\mu} dy &\leq \int_{\mathbb{R}^2} |u_R(y)|^{\bar{p}} (e^{4\pi\nu|u_R(y)|^2} - 1) dy \\ &\leq \Big( \int_{\mathbb{R}^2} |u_R(y)|^{2\bar{p}} dy \Big)^{1/2} \Big( \int_{\mathbb{R}^2} (e^{8\pi\nu|u_R(y)|^2} - 1) dy \Big)^{1/2} \leq \mathbb{C}_2.\end{aligned}$$

where $\mathbb{C}_2 \in (0, +\infty)$ is independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$. It follows from these two facts that

$$\int_{\mathbb{R}^2} \frac{|u_R(y)|^{\bar{p}} (e^{4\pi\nu|u_R(y)|^2} - 1)}{|x-y|^\mu} dy \leq \mathbb{C}_1 + \mathbb{C}_2. \tag{4.9}$$

Recalling (4.3) with (4.7) and (4.9), the proof will be done by choosing $\mathfrak{C} = \mathbb{C}_0 + \mathbb{C}_1 + \mathbb{C}_2$. □

With the help of the study made above, we can get the estimate for $|u_R|_\infty$ as follows.

**Lemma 4.4.** *Suppose that $V$ satisfies* (A8) *and $f$ meets* (1.3) *with* (A1)–(A3). *Then, for every fixed $R > R^*$, $\kappa \in (0, \kappa^*)$, $a > a^*$ and $\varepsilon \in (0, \varepsilon^*)$, there exists a constant $M \in (0, +\infty)$ which is independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$ such that $|u_R|_\infty \leq M$, where $u_R$ comes from* (4.5).

*Proof.* In view of the definition of $f_R$ in (1.13), one has $f_R(s) \leq f(s)$ and $F_R(s) \leq F(s)$ for all $R > 0$ and $s \in \mathbb{R}$. Since $(u_R, \sigma_R)$ with $u_R(x) > 0$ for all $x \in \mathbb{R}^2$ and $\sigma_R < 0$ is a couple of weak solution to (2.1), we then apply (A8) and Lemma 4.3 to arrive at

$$-\Delta u_R + u_R \leq \bar{f}(u_R) := u_R + \mathfrak{C} f(u_R) \text{ in } \mathbb{R}^2.$$

Proceeding with calculations similar to those in Lemma 4.3, we are able to prove that $|\bar{f}(u_R)|_2 \leq K$, where $K \in (0, +\infty)$ is a constant which is independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$. It then follows from the Lax-Milgram theorem that there is a $w_R \in H^1(\mathbb{R}^2)$ such that

$$-\Delta w_R + w_R = \bar{f}(u_R) \text{ in } \mathbb{R}^2.$$

Moreover, it can choose $w_R$ to be positive in $\mathbb{R}^2$. At this stage, we can follow the methods used in [10, 12, 13, 15, 56, 58] to complete the proof. For the completeness, we shall exhibit the details. To this end, we have the claim.

**Clame 4.5.** For all $R > R^*$, $\kappa \in (0, \kappa^*)$, $a > a^*$ and $\varepsilon \in (0, \varepsilon^*)$, it holds

$$0 < u_R(x) \le w_R(x), \quad \forall x \in \mathbb{R}^2.$$

Actually, we define the test function

$$\phi(x) := (u_R - w_R)^+(x) \in H^1(\mathbb{R}^2).$$

Muitiplying by $\phi$ on both sides of $-\Delta(u_R - w_R) + (u_R - w_R) \le 0$ in $\mathbb{R}^2$, we obtain the inequality

$$\int_{\mathbb{R}^2} [\nabla(u_R - w_R)\nabla\phi + (u_R - w_R)\phi]dx \le 0.$$

An elementary computation gives us

$$\int_{\mathbb{R}^2} [|\nabla(u_R - w_R)^+|^2 + |(u_R - w_R)^+|^2]dx = 0$$

yielding the claim.

Owing to Claim 4.5, the proof of this lemma becomes available. From [25, Theorem 9.25] in invokes that there is a $K_2 > 0$ independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$ such that

$$\|w_R\|_{H^2} \le K_2 |f_R(u_R)|_2, \quad \forall R > R^* \text{ and } \varepsilon \in (0, \varepsilon^*).$$

leading to

$$\|w_R\|_{H^2} \le K_3, \quad \forall R > R^* \text{ and } \varepsilon \in (0, \varepsilon^*),$$

for some $K_3 > 0$ independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$. In view of the continuous embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, there exists $K_4 > 0$ independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$ such that

$$|w_R|_\infty \le K_4, \quad \forall R > R^* \text{ and } \varepsilon \in (0, \varepsilon^*).$$

From which, we are derived from Claim 4.5 that

$$|u_R|_\infty \le M, \quad \forall R > R^* \text{ and } \varepsilon \in (0, \varepsilon^*).$$

Consequently, the proof is complete. □

*Proof.* Proof of Theorem 1.1] According to the above discussions, we can arrive at the first part of the proof of Theorem 1.1 by fixing $R > \{R^*, M\}$, because in this case the function $u_R \in S(a)$ is a positive solution of (1.1) with $\sigma = \sigma_R < 0$ for each $\kappa \in (0, \kappa^*)$, $a > a^*$ and $\varepsilon \in (0, \varepsilon^*)$. The remaining part follows Theorem 2.11 directly. The proof is complete. □

4.2. **Proof of Theorem 1.2.** Recalling Theorem 3.1, there is a couple $(u_R, \sigma_R) \in E_\lambda \times \mathbb{R}$ which is a weak solution to (3.1) with $\sigma = \sigma_R < 0$ for every $R > R_*$, $\kappa \in (0, 1)$, $a > a_*$ and $\lambda > \lambda_*$, where $u_R(x) > 0$ for each $x \in \mathbb{R}^2$. Moreover, it holds that

$$u_R \in S(a) \text{ and } \mathcal{J}_{\lambda,R}(u_R) = \bar{\Upsilon}_{\varepsilon,R}(a), \quad \forall R > R_*,\ \kappa \in (0, 1),\ a > a_* \text{ and } \lambda > \lambda_*. \tag{4.10}$$

Proceeding as in Subsection 4.1, we are able to conclude the counterparts of Lemmas 4.2, 4.3 and 4.4 as follows. Because there are no essential differences, we just present them without the detailed proofs.

**Lemma 4.6.** *Suppose that $V$ satisfies* (A9)–(A11) *and $f$ meets* (1.3) *with* (A1)–(A3). *Let $u_R$ be given by* (4.10) *for each $R > R_*$, $a > a_*$ and $\lambda > \lambda_*$, then there exists an $\kappa_* = \kappa_*(R) \in (0, 1)$ such that if $\kappa \in (0, \kappa_*)$, it holds that $|\nabla u_R|_2^2 < \frac{2-\mu}{2(2+\mu)\nu^2}$ for every $R > R_*$, $a > a_*$ and $\lambda > \lambda_*$, where the constant $\nu > 1$ is appears* (4.2) *and* (4.3).

**Lemma 4.7.** *Suppose that $V$ satisfies* (A9)–(A11) *and $f$ requires* (1.3) *with* (A1)–(A3). *Then, for every fixed $R > R_*$, $\kappa \in (0, \kappa_*)$, $a > a_*$ and $\lambda > \lambda_*$, there exists a constant $\bar{\mathfrak{C}} \in (0, +\infty)$ which is independent of $R > R_*$ and $\lambda > \lambda_*$ such that*

$$\bar{\Gamma}(x) := |x|^{-\mu} * F(u_R) \le \bar{\mathfrak{C}},$$

*where $u_R$ comes from* (4.10).

**Lemma 4.8.** *Suppose that $V$ satisfies* (A9)–(A11) *and $f$ requires* (1.3) *with* (A1)–(A3). *Then, for all fixed $R > R_*$, $\kappa \in (0, \kappa_*)$, $a > a_*$ and $\lambda > \lambda_*$, there is $\bar{M} \in (0, +\infty)$ which is independent of $R > R_*$ and $\lambda > \lambda_*$ such that $|u_R|_\infty \leq \bar{M}$, where $u_R$ comes from* (4.10).

*Proof of Theorem 1.2.* We are able to fix $R > \{R_*, M\}$, and Theorem 3.1 thereby indicates the first part of the proof of Theorem 1.2 to satisfy that $u_R \in S(a)$ is a positive solution of (1.1) with $\sigma = \sigma_R < 0$ for all $\kappa \in (0, \kappa_*)$, $a > a_*$ and $\lambda > \lambda_*$. As for the remaining part of the proof of Theorem 1.2, we refer to Theorem 3.8. The proof is complete. □

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