CONCENTRATING SOLUTIONS FOR A CLASS OF INDEFINITE SCHRÖDINGER-POISSON SYSTEMS WITH DOUBLY CRITICAL GROWTH

LIEJUN SHEN AND MARCO SQUASSINA

ABSTRACT. We study the following Schrödinger-Poisson system involving critical nonlocal term with indefinite steep potential well

$$\begin{cases} -\Delta u + (\lambda V(x) - \mu)u - \phi|u|^3 u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & x \in \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter, $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}^+)$ admits a potential well $\Omega \triangleq \text{int}V^{-1}(0)$, and $\mu > \mu_1$ is a constant such that the operator $L_\lambda \triangleq -\Delta + \lambda V - \mu$ is non-degenerate when λ is large enough with $\{\mu_j\}_{j=1}^{\infty}$ denoting the Dirichlet eigenvalues of $(-\Delta, H_0^1(\Omega))$. If f satisfies some suitable assumptions involving critical growth, with the help of a linking-type result involving the modified Pankov-Nehari manifold procedure, we establish the existence and concentrating behavior of positive solutions for the given system using variational methods.

1. INTRODUCTION

1.1. **Overview.** Due to the real physical meaning, the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

proposed by Benci-Fortunato [10] was used to describe solitary waves for nonlinear Schrödinger type equations and search for the existence of standing waves interacting with an unknown electrostatic field. We refer the reader to [10,31] and the references therein to get more physical background of it. In recent years, by classical variational methods, there exist many interesting works concerning the (non)existence of nontrivial solutions, multiple solutions, sign-changing solutions and semiclassical states to the system with some different assumptions on the potential V and the nonlinearity f, see e.g. [15,16,20,23,38,41,44,47] and the references therein.

Nevertheless, the results for the generalized Schrödinger-Poisson system below

$$\begin{cases} -\Delta u + V(x)u + \varepsilon\phi g(u) = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = 2\varepsilon G(u), & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

are not as fruitful as problem (1.1), where $\varepsilon \in \mathbb{R}$, $G(t) = \int_0^t g(z)dz$ and $|g(t)| \le C(|t| + |t|^s)$ with $s \in [1, 4)$, see [3] for instance. Given a constant $R > 0$, denoting μ_0 to be the principle eigenvalue of $(-\Delta, H_0^1(B_R(0)))$, Azzollini-d'Avenia [2] investigated the existence of positive ground state solutions for the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u = \mu u + \varepsilon\phi|u|^3 u, & x \in B_R(0), \\ -\Delta\phi = \varepsilon|u|^5, & x \in B_R(0), \\ u = \phi = 0, & \text{on } \partial B_R(0), \end{cases} \tag{1.3}$$

2010 *Mathematics Subject Classification.* 35J15, 35J20, 35B06.

Key words and phrases. Schrödinger-Poisson system, Critical nonlocal term, Indefinite variational problem, Linking, Modified Pankov-Nehari manifold, Concentrating.

L.J. Shen is partially supported by NSFC (12201565). M. Squassina is member of Gruppo Nazionale per l'Analisi Matematica, la Probabilita e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

where $\mu \in (\frac{3\mu_0}{10}, \mu_0)$ and $\varepsilon > 0$. They also considered the nonexistence of nontrivial solutions when $\mu \le 0$. Note that although the second equation can be solved by a Green function, the term $\varepsilon|u|^5$ results in a critically nonlocal nonlinearity in (1.3). After it, by introducing a cut-off function (see e.g. [22]) and utilizing a monotonic trick developed by L. Jeanjean [21], for sufficiently small $\varepsilon > 0$ and $\varepsilon = -1$, the authors in [26] proved that

$$\begin{cases} -\Delta u + V(x)u + \varepsilon\phi|u|^3u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

possesses at least a positive radially symmetric solution if $V > 0$ is a constant. When $V \equiv 1$ and $f(u) = \sigma|u|^{q-1}u$ with $\sigma \ge 0$ and $q \in [1, 5]$ in (1.4), authors in [27] contemplated the nonexistence, existence, multiplicity and asymptotically behavior of nontrivial solutions for some $\varepsilon \in \mathbb{R}$, respectively.

In [32], employing the mountain-pass theorem and the concentration-compactness principle, Liu studied the existence of positive solutions of the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)|u|^5, & x \in \mathbb{R}^3, \end{cases}$$

where V, K and f are asymptotically periodic functions with respect to the variable $x \in \mathbb{R}^3$.

In [18], Feng considered the existence of positive ground state solutions to the nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u - \phi|u|^3u = |u|^4u + g(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & x \in \mathbb{R}^3, \end{cases}$$

where $V(x) = x_1^2 + x_2^2 + 1$ is a partially periodic potential for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and g is an appropriate nonlinear function involving subcritcal growth. By assuming

$(\bar{V})$ $V_\infty \triangleq \liminf_{|x|\to\infty} V(x) > V_0 \triangleq \inf_{x\in\mathbb{R}^3} V(x) > 0,$

the author in [19] investigated the existence and concentration of ground state solutions of

$$\begin{cases} -\epsilon^2\Delta u + V(x)u - \phi|u|^3u = |u|^4u + g(u), & x \in \mathbb{R}^3, \\ -\epsilon^2\Delta\phi = |u|^5, & x \in \mathbb{R}^3, \end{cases}$$

where $\epsilon > 0$ is a parameter and $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is a subcritical nonlinearity satisfying (AR) and (Ne) (see Remark 1.3 below). There are also some other interesting works on system (1.3) or (1.4), we refer the reader to [4, 25, 48] and the references therein.

If $\varepsilon = 0$ and replacing $V(x)$ with $\lambda V(x) - \mu$ in (1.2), it comes from the Schrödinger equation

$$-\Delta u + (\lambda V(x) - \mu)u = f(x,u), \ x \in \mathbb{R}^N, \tag{1.5}$$

where $\lambda > 0$ is a parameter, $\mu \in \mathbb{R}$ is a constant such that the operator $L_\lambda \triangleq \Delta + \lambda V(x) - \mu$ is non-degenerate for λ large, $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. In the pioneer work [5], Bartsch and Wang supposed that $\mu = -1$ and V satisfies

- (V_1) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ with $V(x) \ge 0$ for all $x \in \mathbb{R}^N$;
- (V_2) there is a constant $M > 0$ such that the set $\Sigma \triangleq \{x \in \mathbb{R}^N : V(x) < M\}$ is nonempty and has finite measure;
- (V_3) $\Omega \triangleq \mathrm{int}V^{-1}(0)$ is nonempty, where $V^{-1}(0) \triangleq \{x \in \mathbb{R}^N : V(x) = 0\}$,

then established the existence and multiplicity nontrivial solutions to Eq. (1.5) under some mild subcritical conditions on f for some large λ. In particular, if $f(x,t) = |t|^{p-2}t$ with $2 < p < 2^*$ for all $(x,t) \in \times\mathbb{R}^N \times \mathbb{R}$, Bartsch and Wang [7] assumed $(V_1) - (V_2)$ and

- (V_4) $\Omega \triangleq \mathrm{int}V^{-1}(0)$ is a nonempty set with smooth boundary and $\overline{\Omega} = V^{-1}(0)$;

to obtain the existence and concentration of ground state solutions for large λ. Meanwhile, by using the Lusternik-Schnirelmann theory, they derived the multiple positive solutions to Eq. (1.5). The assumptions like $(V_1)-(V_2)$ and (V_3), or (V_4), were proposed by Bartsch *et al.* in [5–7] to study the nonlinear Schrödinger equation. Obviously, the harmonic trapping potential

$$V(x)=\begin{cases}\sum_{i=1}^{N}\omega_i|x_i|^2-\omega, & \text{if } |(\sqrt{\omega_1}x_1,\sqrt{\omega_2}x_2,\cdots,\sqrt{\omega_N}x_N)|^2\geq\omega,\\ 0, & \text{if } |(\sqrt{\omega_1}x_1,\sqrt{\omega_2}x_2,\cdots,\sqrt{\omega_N}x_N)|^2\leq\omega,\end{cases}$$

with $\omega>0$ satisfying $(V_1)-(V_4)$, where $\omega_i>0$ is called by the anisotropy factor of the trap in quantum physics and trapping frequency of the ith-direction in mathematics, see e.g. [9,12]. Indeed, the potential λV with the above hypotheses is usually known as the steep potential well.

Concerning the critical case of (1.5) with steep potential well, Clapp-Ding [14] supposed $f(x,u)=|u|^{2^*-2}u$ with $2^*=2N/(N-4)$ and $N\geq 4$ to established the existence, multiplicity and concentration of positive solutions for all $\mu>0$ small and $\lambda>0$ large enough, where $(V_1)-(V_2)$ are assumed and

(V_5) $\Omega\triangleq \mathrm{int}V^{-1}(0)$ is a nonempty and bounded set with smooth boundary and $\overline{\Omega}=V^{-1}(0)$.

Subsequently, Tang [43] improved and generalized the results in [14] by replacing (V_2) with a stronger condition

(V_6) $V_\infty\triangleq\liminf\limits_{|x|\to+\infty}V(x)>0$.

As to $N=3$ in the critical case of (1.5) with steep potential well $(V_1)-(V_2)$ and (V_4), by letting $\mu=-1$, Zhang-Zou [46] proved the existence and concentration of positive ground state solutions for $\lambda>0$ sufficiently large by means of the penalized procedure. For $\mu>0$ large enough, as an application of the modified Nehari-Pankov manifold method developed in [35], Li *et al.* [29] supposed (V_1) and $(V_5)-(V_6)$ to study the existence and concentration of ground state solutions for Eq. (1.5) with a critical convolution nonlinearity.

1.2. **Functional framework.** Before stating the main results in this article, we introduce several notations and definitions. Let $(X,\|\cdot\|_X)$ be a Banach space with its dual space $(X^{-1},\|\cdot\|_{X^{-1}})$, and Φ be its functional on X. The Cerami sequence at a level $c\in\mathbb{R}$ ($(\mathrm{C})_c$ sequence in short) corresponding to Φ means that $\Phi(x_n)\to c$ and $\|\Phi'(x_n)\|_{X^{-1}})(1+\|x_n\|_X)\to 0$ as $n\to\infty$, where $\{x_n\}\subset X$. If for each $(\mathrm{C})_c$ sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k}\to x_0$ in X for $x_0\in X$, then we can say that the functional Φ satisfies the so-called $(\mathrm{C})_c$ condition. The space $L^p(\mathbb{R}^3)$ denotes the usual Lebesgue space with the norm $|\cdot|_p$ with $1\leq p\leq+\infty$.

Throughout this paper, we denote $H^1(\mathbb{R}^3)$ by a Hilbert space equipped with the following inner product and norm

$$(u,v)_{H^1(\mathbb{R}^3)}=\int_{\mathbb{R}^3}\big(\nabla u\nabla v+uv\big)dx \text{ and } \|u\|_{H^1(\mathbb{R}^3)}=\sqrt{(u,u)_{H^1(\mathbb{R}^3)}}.$$

Taking $\mu>\mu_1$ into account, motivated by [37], we shall introduce a new work space as the operator L_λ is indefinite. Let $V_\lambda(x)\triangleq\lambda V(x)-\mu$ and $V_\lambda^\pm(x)=\max\{\pm V_\lambda(x),0\}\geq 0$, define the space

$$X_\lambda\triangleq\left\{u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}V_\lambda^+(x)|u|^2dx<+\infty\right\}$$

which is a Hilbert space with the inner product and norm

$$(u,v)_{X_\lambda}\triangleq\int_{\mathbb{R}^3}[\nabla u\nabla v+V_\lambda^+(x)uv]dx \text{ and } \|u\|_{X_\lambda}\triangleq\sqrt{(u,u)_{X_\lambda}}.$$

According to Lemma 2.1 below, we conclude that $(X_\lambda,\|\cdot\|_{X_\lambda})$ can be continuously embedded into $(H^1(\mathbb{R}^3),\|\cdot\|_{H^1(\mathbb{R}^3)})$ for some large $\lambda>0$.

On the other hand, the operator $L_\lambda=-\Delta+V_\lambda$ is self-adjoint with domain $\mathcal{D}(L_\lambda)=X_\lambda$ and as we will see in Section 2, L_λ is invertible when $\lambda>0$ is suitably large. So, it holds that $X_\lambda=X_\lambda^-\oplus X_\lambda^+$, where L_λ is positive definite on the infinite dimensional space X_λ^+ and negative definite on the finite dimensional space X_λ^-. Hereafter, let us denote $P_\lambda^\pm$ by the orthogonal projections from X_λ to $X_\lambda^\pm$ with the decomposition $X_\lambda = X_\lambda^- \oplus X_\lambda^+$, respectively. What's more, the projections P_λ^- and P_λ^+ are also orthogonal with respect to L^2-inner product. For any $u \in X_\lambda$, setting

$$\|u\|_{L_\lambda}^2 = \|P_\lambda^+ u\|_{L_\lambda}^2 + \|P_\lambda^- u\|_{L_\lambda}^2,$$

where

$$\begin{cases} \|P_\lambda^+ u\|_{L_\lambda}^2 = \int_{\mathbb{R}^3} \big[|\nabla P_\lambda^+ u|^2 + (\lambda V(x) - \mu)|P_\lambda^+ u|^2\big] dx, & \forall u \in X_\lambda, \\ \|P_\lambda^- u\|_{L_\lambda}^2 = -\int_{\mathbb{R}^3} \big[|\nabla P_\lambda^- u|^2 + (\lambda V(x) - \mu)|P_\lambda^- u|^2\big] dx, & \forall u \in X_\lambda. \end{cases}$$

Particularly, one may observe that

$$\int_{\mathbb{R}^3} \big[|\nabla u|^2 + (\lambda V(x) - \mu)|u|^2\big] dx = \|P_\lambda^+ u\|_{L_\lambda}^2 - \|P_\lambda^- u\|_{L_\lambda}^2.$$

We note that the norm $\|\cdot\|_{X_\lambda}$ is equivalent to $\|\cdot\|_{L_\lambda}$ for some large $\lambda > 0$, see Lemma 2 below.

1.3. **Main results.** In the present paper, we focus on the following Schrödinger-Poisson system involving critical nonlocal term

$$\begin{cases} -\Delta u + (\lambda V(x) - \mu)u - \phi |u|^3 u = f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $\lambda > 0$ is a parameter, $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}^+)$ admits a potential well $\Omega \triangleq \mathrm{int} V^{-1}(0)$, and $\mu > \mu_1$ is a constant such that the operator $L_\lambda \triangleq -\Delta + \lambda V(x) - \mu$ is non-degenerate when λ large. More precisely, we always suppose that $\mu > \mu_1$ and $\mu \neq \mu_j$ for every $j \in \mathbb{N}^+$, where $\{\mu_j\}_{j=1}^\infty$ denotes the Dirichlet eigenvalues of $(-\Delta, H_0^1(\Omega))$. Let $\{e_j\}_{j=1}^\infty$ be the corresponding eigenfunctions of $L_0 = -\Delta - \mu$ in $H_0^1(\Omega)$. It is clear that

$$0 < \mu_1 < \cdots < \mu_j < \mu_{j+1} < \cdots \text{ and } \mu_j \to +\infty \text{ as } j \to +\infty, \tag{1.7}$$

where each μ_j has been repeated in the sequence according to its finite multiplicity. When $\mu \notin \{\mu_j\}_{j=1}^\infty$ and $\mu > \mu_1$, under some suitable assumptions on f with critical growth, we study the existence and concentrating behavior of positive solutions for (1.6). We shall get existence and concentrating behavior of positive solutions for (1.6). From now on until the end of the paper, we would always suppose that the assumptions (V_1) and $(V_5) - (V_6)$ hold true in $\mathbb{R}^3$. As to the nonlinearity f, it is assumed that

$$f(t) = \begin{cases} g(t) + t^5, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

where $g \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ vanishes in $(-\infty, 0]$ and satisfies

- (g_1) $g(t) = o(t)$ as $t \to 0$;
- (g_2) there are $C_0 > 0$ and $p \in (2, 6)$ such that $|g(t)| \leq C_0(1 + |t|^{p-1})$ for any $t \in \mathbb{R}$;
- (g_3) there is $\eta > 2$ such that $g(t)t \geq \eta G(t) > 0$ for all $t \in \mathbb{R}$, where $G(t) = \int_0^t g(z) dz$;
- (g_4) there are $r \in (4, 6)$, $A > 0$ and $B > 0$ such that $G(t) \geq A|t|^r - B|t|^2$ for any $t \in \mathbb{R}$.

We obtain the following result.

Theorem 1.1. *Suppose (V_1) and $(V_5) - (V_6)$ as well as and $(g_1) - (g_4)$ hold true, then there is a constant $\Lambda^* > 0$ such that (1.6) has a nontrivial solution $(u_\lambda, \phi_{u_\lambda}) \in X_\lambda \times D^{1,2}(\mathbb{R}^3)$ for all $\lambda > \Lambda^*$.*

It is very natural to wonder that whether the obtained solution in Theorem 1.1 is a ground state solution. Thus, our next aim concerns this issue and try to give an affirmative answer to the question.

Theorem 1.2. *Under the assumptions of Theorem 1.1, if in addition we suppose that*

- (g_5) *if $g(t)s = g(s)t > 0$, then $G(t) - G(s) \leq \frac{1}{2}[g(t) + g(s)](t - s)$ for all $t, s \in \mathbb{R}$,* *then the solution obtained in Theorem 1.1 is a ground state for all* $\lambda > \Lambda^*$.

Remark 1.3. The term $|u|^4u$ in the nonlinearity $f(u) = g(u)+|u|^4u$ and the critically nonlocal term $\phi_u|u|^3u$ (see (2.18) below) lead to system (1.6) has doubly critical growth, see [19] for instance. The assumption (g_4) is mainly exploited to estimate the critical value by pulling it down to a particular constant which is independent of λ. As a direct byproduct, we shall combine (g_3) and (g_4) to verify that every (C) sequence at this level is uniformly bounded in X_λ. It is worthy pointing out that the assumptions (g_3) and (g_4) can be replaced with the following stronger Ambrosetti-Rabinowitz condition (which plays a vital role in [19])

(AR) there exists a constant $\eta > 4$ such that $g(t)t \geq \eta G(t) > 0$ for all $t \in \mathbb{R}$.

The reader is invited to see that (g_5) is weaker than the following Nehari type monotone condition

(Ne) the function $g(t)/|t|$ in nondecreasing for all $t \in \mathbb{R}\backslash\{0\}$.

Indeed, without loss of generality, we suppose that $t > s > 0$ in (g_5). By (Ne), if $g(t)s = g(s)t > 0$, one has

$$G(t) - G(s) = \int_s^t \frac{g(z)}{z} zdz \leq \frac{g(t)}{t} \int_s^t zdz = \frac{g(t)+g(s)}{t+s} \int_s^t zdz = \frac{[g(t)+g(s)](t-s)}{2}.$$

As a consequence, the approach adopted in [19] would be unapplicable to our problem and there is a new analytic technique to show that the (C) sequence is uniformly bounded. We highlight that the assumptions $(g_1) - (g_5)$ are suitable for the nonlinearity $g(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$ with $4 < p < 6$.

Remark 1.4. To find a ground state solution for the indefinite variational problems, it is known that the Nehari-Pankov manifold procedure introduced by Pankov in [36], later developed by Szulkin-Weth in [42], is an effective and important method, see [1,29,43] for example. The authors in [35] proposed a modified Nehari-Pankov manifold idea to investigate the concentrating behavior of ground state solution for indefinite problems with steep potential well. Alternatively, the doubly critical growth and the nonlinearity $f \in \mathcal{C}(\mathbb{R})$ appearing in system (1.6) prevent us repeating the relevant arguments in [1,29,35,42]. Hence, some interesting tricks would be proposed in this paper to conclude Theorems 1.1 and 1.2, respectively.

To derive the proofs of Theorems 1.1 and 1.2, we firstly take advantage of the arguments introduced in [33, Theorem 2.1] (see also [8,24,28,45] for example) to construct a $(C)_c$ sequence of the variational functional $J_\lambda : X_\lambda \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + V_\lambda(x)u^2\right] dx - \frac{1}{10}\int_{\mathbb{R}^3} \phi_u|u|^5 dx - \int_{\mathbb{R}^3} F(u)dx,\ \forall u \in X_\lambda. \tag{1.8}$$

In order to explain it clearly, given a Hilbert space and a variational functional $\Psi \in \mathcal{C}^1(E,\mathbb{R})$, we denote $u_n \xrightarrow{\mathcal{T}} u$ by the convergence of a sequence in topology $\mathcal{T}$ and suppose that

(A_1) Ψ is $\mathcal{T}$-upper semicontinuous, that is, $\Psi^{-1}([t,\infty))$ is $\mathcal{T}$-closed for any $t \in \mathbb{R}$;

(A_2) Ψ' is $\mathcal{T}$-to-weak* continuous in $\Psi^{-1}([0,\infty))$, that is, $\Psi'(u_n) \rightharpoonup \Psi'(u_0)$ provided that $u_n \xrightarrow{\mathcal{T}} u_0$ and $\Psi(u_n) \geq 0$ for each $n \in \mathbb{N}$;

(A_3) There exists $r > 0$ such that $m \triangleq \inf\limits_{\{u\in Z:\|u\|_E = r\}} \Psi(u) > 0$, where $E = Y \oplus Z$;

(A_4) For every $u \in E\backslash Y$, there exists an $R = R(u) > r$ such that $\sup\limits_{\partial M(u)} \Psi \leq \Psi(0) = 0$, where

$$M(u) \triangleq \{tu + v \in E : v \in Y,\ \|tu+v\|_E \leq R,\ t \geq 0\}.$$

The following result can be found in [33, Theorem 2.1].

Proposition 1.5. *If* $\Psi \in \mathcal{C}^1(E,\mathbb{R})$ *satisfies* $(A_1) - (A_4)$, *then there is a* (C) *sequence* $\{u_n\}$ *at the level*

$$c = \inf_{E\backslash Y} \inf_{h\in\Gamma(u)} \sup_{u'\in M(u)} \Psi(h(u',1)),$$

where $\Gamma(u)$ *consists of* $h \in \mathcal{C}(M(u)\times[0,1])$ *satisfying the following conditions*

(h1) h *is* $\mathcal{T}$*-continuous with respect to norm* $\|\cdot\|_{\mathcal{T}}$*;*
(h2) $h(u,0)=u$ *for all* $u\in M(u)$*;*
(h3) $\Psi(u)\geq \Psi(h(u,t))$ *for all* $(u,t)\in M(u)\times[0,1]$*;*
(h4) each $(u,t)\in M(u)\times[0,1]$ *has an open neighborhood* W *in the product topology of* $(E,\mathcal{T})$ *and* $[0,1]$ *such that the set* $\{v-h(v,s):(v,s)\in W\cap(M(u)\times[0,1])\}$ *is contained in a finite dimentional subsequence of* E*.*

Suppose in addition that

(A_5) *If* $u\in\mathcal{M}\triangleq\{u\in E\backslash Y:\Psi'(u)(u)=0$ *and* $\Psi'(u)(v)=0$ *for any* $v\in Y\}$*, then for all* $v\in Y$ *and* $t\geq 0$*, there holds* $\Psi(u)\geq\Psi(tu+v)$*,*

then it holds that

$$c\leq c_{\mathcal{M}}\triangleq\inf_{v\in\mathcal{M}}\Psi(v).$$

Moreover, if $c\geq\Psi(u)$ *for some critical point* $u\in E\backslash Y$*, then* $c=c_{\mathcal{M}}=\Psi(u)$*.*

With Proposition 1.5 in hands, we are able to exhibit the detailed proofs of Theorems 1.1 and 1.2 by setting $E=X_\lambda$, $Y=X_\lambda^-$, $Z=X_\lambda^+$ and $\Psi=J_\lambda$, respectively. Actually, if we can verify that the variational functional J_λ satisfies the (C) condition for some suitably large $\lambda>0$, then the proofs become available. Nevertheless, the above choices of Y and Z would prevent us contemplating the asymptotical behavior of the obtained solution when $\lambda\to+\infty$. As a consequence, motivated by [35] we shall reconstruct the pair Y and Z to reach our another aim in this paper.

According to (1.7), there is a $j_0\in\mathbb{N}^+$ such that $0<\mu_1<\mu_2<\cdots<\mu_{j_0}<\mu<\mu_{j_0+1}<\cdots$. We recall that $\{e_j\}_{j=1}^\infty$ are the corresponding eigenfunctions of $L_0=-\Delta-\mu$ in $H_0^1(\Omega)$ and define

$$X_0^-\triangleq\operatorname{span}\{e_1,e_2,\cdots,e_{j_0}\}\text{ and }X_0^-\triangleq\operatorname{span}\{e_{j_0+1},e_{j_0+2},\cdots\},$$

then $H_0^1(\Omega)=X_0^-\oplus X_0^+$. Moreover, one knows that L_0 is negative and positive definite on X_0^- and X_0^+, respectively. For each $e_j\in H_0^1(\Omega)$ with $j\in\{1,2,\cdots,j_0\}$, we denote by $\tilde{e}_j\in H^1(\mathbb{R}^3)$ its trivial extension, namely

$$\tilde{e}_j\triangleq\begin{cases}e_j & \text{in }\Omega,\\ 0 & \text{in }\Omega^c=\{x:x\in\mathbb{R}^3\backslash\Omega\}.\end{cases}\tag{1.9}$$

We now define

$$\tilde{X}_0^-\triangleq\operatorname{span}\{\tilde{e}_1,\tilde{e}_2,\cdots,\tilde{e}_{j_0}\}\text{ and }\tilde{X}_0^+=(\tilde{X}_0^-)^\perp$$

indicating that $X_\lambda=\tilde{X}_0^-\oplus\tilde{X}_0^+$ and we shall denote $\tilde{P}_\lambda^\pm$ by the corresponding orthogonal projections from X_λ to $\tilde{X}_0^\pm$ with such a decomposition.

At this stage, we are going to choose $(E,\|\cdot\|_E)=(X_\lambda,\|\cdot\|_{L_\lambda})$, $Y=\tilde{X}_0^-$, $Z=\tilde{X}_0^+$ and $\Psi=J_\lambda$ in Proposition 1.5, respectively. Moreover, we define the ground state energy by

$$d_\lambda\triangleq\inf_{u\in\mathcal{M}_\lambda}J_\lambda(u),$$

where and in the sequel

$$\mathcal{M}_\lambda=\{u\in X_\lambda\backslash\tilde{X}_0^-:J_\lambda'(u)(u)=0\text{ and }J_\lambda'(u)(v)=0\text{ for any }v\in\tilde{X}_0^-\}.$$

As a consequence, we are in a correct direction arrive at the proofs of Theorems 1.1 and 1.2 which enable us to contemplate the asymptotical behavior of the obtained solution as follows.

Theorem 1.6. *Under the assumptions of Theorem 1.1 above and let $(u_\lambda, \phi_{u_\lambda}) \in X_\lambda \times D^{1,2}(\mathbb{R}^3)$ be a nontrivial solution of (1.6) for all $\lambda > \Lambda^*$. Then, passing to a subsequence if necessary, it holds that $u_\lambda \to u_0$ in $H^1(\mathbb{R}^3)$ and $\phi_{u_\lambda} \to \phi_{u_0}$ in $D^{1,2}(\mathbb{R}^3)$ in usual senses, where (u_0, ϕ_{u_0}) is a solution of*

$$\begin{cases} -\Delta u - \mu u - \left(\displaystyle\int_\Omega \frac{|u(y)|^5}{|x-y|}dy\right)|u|^3u = g(u) + |u|^4u, & x \in \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.10}$$

We note that, to the best knowledge of us, the results in Theorems 1.1, 1.2 and 1.6 are new by now. It should be mentioned here that this paper should be regarded as a supplement and generalization as the works in [29, 35]. Nevertheless, we prefer to emphasize that one cannot conclude our results just by repeating the methods in these two quoted papers. On the one hand, the Lagrange multiplier theorem, which played crucial roles in [29, 35], is no longer applicable because $g \in \mathcal{C}^0$ leads to that $\mathcal{M}_\lambda$ is not a $\mathcal{C}^1$-manifold; On the other hand, because of the appearance of doubly critical terms in the system (1.6), compared with [29, Lemma 2.11], we have to take a different idea to pull down the critical value. As a matter of fact, there are some other unpleasant barriers. For instance, to derive the linking structure of J_λ, there are some deep analyses in Lemmas 3.3 and 3.4 below. Moreover, even if one can borrow the techniques in the literature to prove Lemma 3.5, we have to introduce some new tricks to deduce it because of (g_5). As one can observe later, all of the above facts prevent us applying the variational method to prove the main results in an usual way.

This paper is organized as follows. In Section 2, we shall provide some preliminaries. Section 3 is devoted to the verifications of the conditions $(A_1) - (A_5)$ associated with J_λ in our variational settings. In Section 4, we exhibit the detailed proofs of Theorems 1.1, 1.2 and 1.6, respectively.

Notations: From now on in this paper, otherwise mentioned, we ultilize the following notations:

- $C, C_1, C_2, \cdots$ denote any positive constant, whose value is not relevant and $\mathbb{R}^+ \triangleq (0, +\infty)$.
- $|\cdot|_{p,\Omega}$ stands for the usual norm of the Lebesgue space $L^p(\Omega)$ for all $p \in [1, +\infty]$.
- For any $\varrho > 0$ and every $x \in \mathbb{R}^3$, $B_\varrho(x) \triangleq \{y \in \mathbb{R}^3 : |y - x| < \varrho\}$.
- $o_n(1)$ denotes the real sequences with $o_n(1) \to 0$ as $n \to +\infty$.
- " $\to$ " and " $\rightharpoonup$ " stand for the strong and weak convergence in the related function spaces, respectively;

2. Preliminaries

In this section, we introduce some preliminary results. From now on until the end of this article, we shall always suppose that the assumptions (V_1) and $(V_5) - (V_6)$ hold just for simplicity.

As stated in the introduction, we will verify that the norms $\|\cdot\|_{X_\lambda}$ and $\|\cdot\|_{L_\lambda}$ are equivalent and clarify the decompositions of the work space X_λ associated with $X_\lambda = X_\lambda^- \oplus X_\lambda^+$ and $X_\lambda = \tilde{X}_0^- \oplus \tilde{X}_\lambda^+$.

First of all, we prove the following imbedding result.

Lemma 2.1. *There is a constant $\Lambda_0 > 0$ such that for any $\lambda > \Lambda_0$ and $u \in X_\lambda$, we have*

$$\|u\|_{H^1(\mathbb{R}^3)} \le \tilde{C}_0 \|u\|_{X_\lambda}, \tag{2.1}$$

for some $\tilde{C}_0 > 0$ independent of $\lambda > \Lambda_0$.

Proof. Motivated by [35, Lemma 2.1], choosing $M_0 = \frac{1}{2}V_\infty > 0$, then there exists an $R > 0$ such that

$$V(x) \ge M_0, \ \forall x \in B_R^c(0) \text{ and } \operatorname{supp} V_\lambda^- \subset B_R(0) \text{ whenever } \lambda > \mu {M_0}^{-1}, \tag{2.2}$$

where $\operatorname{supp} V_\lambda^-$ denotes the support set of V_λ^-. Define $\Lambda_0 \triangleq (\mu + M_0){M_0}^{-1} > 0$, then for any $u \in X_\lambda$ and $\lambda > \Lambda_0$, we apply (2.2) to obtain

$$\int_{B_R^c(0)} |u|^2 dx \le \frac{1}{M_0} \int_{B_R^c(0)} (\lambda V(x) - \mu)|u|^2 dx \le \frac{1}{M_0} \int_{B_R^c(0)} V_\lambda^+(x)|u|^2 dx \le \frac{1}{M_0}\|u\|_{X_\lambda}^2. \tag{2.3}$$

Denoting the constant

$$S \triangleq \inf\{|\nabla u|_2^2 : u \in D^{1,2}(\mathbb{R}^3) \text{ and } |u|_6 = 1\} > 0, \tag{2.4}$$

where $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$ with a norm $\|\cdot\|_{D^{1,2}(\mathbb{R}^3)} \triangleq |\nabla \cdot|_2$. Then, one has

$$\begin{aligned}\int_{B_R(0)} |u|^2 dx &\le |B_R(0)|^{\frac{2}{3}} \Big(\int_{B_R(0)} |u|^6 dx \Big)^{\frac{1}{3}} \le |B_R(0)|^{\frac{2}{3}} \Big(\int_{\mathbb{R}^3} |u|^6 dx \Big)^{\frac{1}{3}} \\ &\le S^{-1} |B_R(0)|^{\frac{2}{3}} \int_{\mathbb{R}^3} |\nabla u|^2 dx \le S^{-1} |B_R(0)|^{\frac{2}{3}} \|u\|_{X_\lambda}^2, \ \forall u \in H^1(\mathbb{R}^3).\end{aligned} \tag{2.5}$$

It follows from (2.3) and (2.5) that

$$\|u\|_{H^1(\mathbb{R}^3)}^2 \le \left(\frac{1 + M_0}{M_0} + S^{-1} |B_R(0)|^{\frac{2}{3}} \right) \|u\|_{X_\lambda}^2 \triangleq \tilde{C}_0^2 \|u\|_{X_\lambda}^2$$

whenever $\lambda > \Lambda_0 \triangleq (\mu + M_0) M_0{}^{-1} > 0$. The proof is completed. □

As a corollary of Lemma 2.1, one may find that the work space X_λ can be continuously imbedded into $L^p(\mathbb{R}^3)$ and compactly imbedded into $L^p_{\text{loc}}(\mathbb{R}^3)$ for $2 \le p < 6$ whence $\lambda > \Lambda_0$. In particular, for every $u \in X_\lambda$, there exists a constant $S_p > 0$ independent of $\lambda > \lambda_0$ such that

$$|u|_p \le S_p \|u\|_{X_\lambda}, \text{ whenever } \lambda > \Lambda_0. \tag{2.6}$$

Moreover, let $\{\mu_j(L_\lambda)\}_{j=1}^\infty$ and $\{\mu_j(L_0)\}_{j=1}^\infty$ be the class of all distinct eigenvalues of the operators L_λ in X_λ and L_0 in $H_0^1(\Omega)$, respectively. It is well known that $\mu_j(L_0) = \mu_j - \mu \to +\infty$ as $j \to \infty$, where $\{\mu_j\}_{j=1}^\infty$ is the eigenvalue sequence of $(-\Delta, H_0^1(\Omega))$. Since $\mu \in (\mu_{j_0}, \mu_{j_0+1})$ for some $j_0 \in \mathbb{N}^+$, there holds

$$\mu_1(L_0) < \mu_2(L_0) < \cdots < \mu_{j_0}(L_0) < 0 < \mu_{j_0+1}(L_0) < \cdots < \mu_{j+1}(L_0) < \cdots . \tag{2.7}$$

Let $\mathbb{V}_j(L_\lambda)$ and $\mathbb{V}_j(L_0)$ be the eigenfunction spaces of the eigenvalues $\mu_j(L_\lambda)$ and $\mu_j(L_0)$. If for every sequence $\lambda_n \to +\infty$ and normalized eigenfunction $\psi_n \in \mathbb{V}_j(L_{\lambda_n})$, there exists a normalized eigenfunction $\psi \in \mathbb{V}_j(L_0)$ such that $\psi_n \to \psi$ in $H^1(\mathbb{R}^3)$ along a subsequence, we say that $\mathbb{V}_j(L_\lambda) \to \mathbb{V}_j(L_0)$ as $\lambda \to +\infty$.

The following lemma can be proved similarly as that of [35, Lemma 2.5], hence we just present them without proof.

Lemma 2.2. *For all $j = 1, 2, \cdots$, one has*

$$\mu_j(L_\lambda) \to \mu_j(L_0) \text{ and } \mathbb{V}_j(L_\lambda) \to \mathbb{V}_j(L_0) \text{ as } \lambda \to +\infty. \tag{2.8}$$

Moreover, there exists a $\Lambda_1 > \Lambda_0$ such that for any $\lambda > \Lambda_1$ there holds

$$\mu_1(L_\lambda) < \mu_2(L_\lambda) < \cdots < \mu_{j_0}(L_\lambda) < 0 < \mu_{j_0+1}(L_\lambda) < \cdots . \tag{2.9}$$

It follows from (2.7), (2.8) and (2.9) that there exists a $\Lambda_2 > \Lambda_1$ such that

$$\mu_{j_0}(L_\lambda) < \frac{1}{2}\mu_{j_0}(L_0) < 0 < \frac{1}{2}\mu_{j_0+1}(L_0) < \mu_{j_0+1}(L_\lambda) \text{ whenever } \lambda > \Lambda_2. \tag{2.10}$$

This implies that $X_\lambda^- = \bigoplus_{i=1}^{j_0} \mathbb{V}_j(L_\lambda)$, for all $\lambda > \Lambda_2$.

Lemma 2.3. *The norms $\|\cdot\|_{X_\lambda}$ and $\|\cdot\|_{L_\lambda}$ are equivalent on X_λ for each $\lambda > \Lambda_2$, that is, there are two constants $C_1, C_2 > 0$ independent of $\lambda > \Lambda_2$ such that*

$$C_1 \|u\|_{X_\lambda} \le \|u\|_{L_\lambda} \le C_2 \|u\|_{X_\lambda}, \ \forall u \in X_\lambda.$$

Proof. For all $u \in X_\lambda$, then $u = P_\lambda^- u + P_\lambda^+ u$, where $P_\lambda^- u \in X_\lambda^-$ and $P_\lambda^+ u \in X_\lambda^+$. Since $0 \le V_\lambda^-(x) = \max\{\mu - \lambda V(x), 0\} \le \mu$ jointly with the fact that P_λ^- and P_λ^+ are orthogonal, we obtain

$$\begin{aligned}
\|u\|_{X_\lambda}^2 &= \int_{\mathbb{R}^3} \big[|\nabla P_\lambda^+ u|^2 + V_\lambda^+(x)|P_\lambda^+ u|^2\big]dx + \int_{\mathbb{R}^3} \big[|\nabla P_\lambda^- u|^2 + V_\lambda^+(x)|P_\lambda^- u|^2\big]dx \\
&\quad + 2\int_{\mathbb{R}^3} \big[\nabla(P_\lambda^+ u)\nabla(P_\lambda^- u) + V_\lambda^+(x)(P_\lambda^+ u)(P_\lambda^- u)\big]dx \\
&= \|P_\lambda^+ u\|_{X_\lambda}^2 + \|P_\lambda^- u\|_{X_\lambda}^2 + 2\int_{\mathbb{R}^3} \big[V_\lambda^-(x)(P_\lambda^+ u)(P_\lambda^- u)\big]dx \\
&\ge \|P_\lambda^+ u\|_{X_\lambda}^2 + \|P_\lambda^- u\|_{X_\lambda}^2 - 2\mu|P_\lambda^+ u|_2|P_\lambda^- u|_2 \\
&\ge \|P_\lambda^+ u\|_{X_\lambda}^2 + \|P_\lambda^- u\|_{X_\lambda}^2 - 2\mu S_2^2\|u\|_{X_\lambda}^2, \text{ provided } \lambda > \Lambda_0,
\end{aligned} \tag{2.11}$$

where $S_2 > 0$ comes from (2.6). Some simple observations give us that

$$\begin{aligned}
\|P_\lambda^+ u\|_{L_\lambda}^2 &= \int_{\mathbb{R}^3} \big[|\nabla P_\lambda^+ u|^2 + V_\lambda(x)|P_\lambda^+ u|^2\big]dx \\
&\le \int_{\mathbb{R}^3} \big[|\nabla P_\lambda^+ u|^2 + V_\lambda^+(x)|P_\lambda^+ u|^2\big]dx = \|P_\lambda^+ u\|_{X_\lambda}^2,
\end{aligned} \tag{2.12}$$

and since $V_\lambda(x) = V_\lambda^+(x) - V_\lambda^+(x) \ge -V_\lambda^-(x) \ge -\mu$, there holds

$$\begin{aligned}
\|P_\lambda^- u\|_{L_\lambda}^2 &= -\int_{\mathbb{R}^3} \big[|\nabla P_\lambda^- u|^2 + V_\lambda(x)|P_\lambda^- u|^2\big]dx \le \int_{\mathbb{R}^3} V_\lambda^-(x)|P_\lambda^- u|^2 dx \\
&\le \mu \int_{\mathbb{R}^3} |P_\lambda^- u|^2 dx \le \mu S_2^2\|P_\lambda^- u\|_{X_\lambda}^2, \text{ provided } \lambda > \Lambda_0.
\end{aligned} \tag{2.13}$$

Combining (2.11), (2.12) and (2.13), there holds

$$\begin{aligned}
\|u\|_{L_\lambda}^2 &= \|P_\lambda^+ u\|_{L_\lambda}^2 + \|P_\lambda^- u\|_{L_\lambda}^2 \le \max\big\{1, \mu S_2^2\big\}(1 + 2\mu S_2^2)\|u\|_{X_\lambda}^2 \\
&\triangleq C_2^2\|u\|_{X_\lambda}^2, \text{ whenever } \lambda > \Lambda_0.
\end{aligned} \tag{2.14}$$

On the other hand, due to the Cauchy-Schwarz inequality, we apply (2.10) to get

$$\begin{aligned}
\|u\|_{X_\lambda}^2 &= \int_{\mathbb{R}^3} \big[|\nabla u|^2 + V_\lambda^+(x)|u|^2\big]dx = \int_{\mathbb{R}^3} \big[|\nabla u|^2 + V_\lambda(x)|u|^2\big]dx + \int_{\mathbb{R}^3} V_\lambda^-(x)|u|^2 dx \\
&\le \|u\|_{L_\lambda}^2 + 2\mu(|P_\lambda^+ u|_2^2 + |P_\lambda^- u|_2^2) \le \|u\|_{L_\lambda}^2 + 2\mu\bigg(\frac{\|P_\lambda^+ u\|_{L_\lambda}^2}{\mu_{j_0+1}(L_\lambda)} + \frac{\|P_\lambda^- u\|_{L_\lambda}^2}{-\mu_{j_0}(L_\lambda)}\bigg) \\
&\le \bigg(1 + 4\mu \max\bigg\{\frac{1}{\mu_{j_0+1}(L_0)}, \frac{1}{-\mu_{j_0}(L_0)}\bigg\}\bigg)\|u\|_{L_\lambda}^2 \triangleq C_1^{-2}\|u\|_{L_\lambda}^2.
\end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15), we can arrive at the desired results. The proof is completed. □

Once the work space X_λ is built, we turn to establish the variational structure of system (1.6). Following the classic Schrödinger-Poisson system, it can reduce to be a single equation. Actually, according to the Hölder's inequality, for every $u \in H^1(\mathbb{R}^3)$ and $v \in D^{1,2}(\mathbb{R}^3)$, one has

$$\begin{aligned}
\int_{\mathbb{R}^3} |u|^5 v dx &\le \bigg(\int_{\mathbb{R}^3} |u|^6 dx\bigg)^{\frac{5}{6}}\bigg(\int_{\mathbb{R}^3} |v|^6 dx\bigg)^{\frac{1}{6}} \\
&\le S^{-\frac{1}{2}}|u|_6^5\|v\|_{D^{1,2}(\mathbb{R}^3)} \le S^{-\frac{1}{2}}S_6^5\|u\|_{H^1(\mathbb{R}^3)}^5\|v\|_{D^{1,2}(\mathbb{R}^3)},
\end{aligned} \tag{2.16}$$

where we have used the inequality (2.6).

Given $u \in H^1(\mathbb{R}^3)$, one can make use of the Lax-Milgram theorem and then there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}(-\Delta\phi_u)vdx = \int_{\mathbb{R}^3}\nabla\phi_u\nabla vdx = \int_{\mathbb{R}^3}|u|^5vdx,\ \forall v \in D^{1,2}(\mathbb{R}^3), \tag{2.17}$$

showing that ϕ_u satisfies the Poisson equation

$$-\Delta\phi_u = |u|^5,\ x \in \mathbb{R}^3.$$

In view of [30, Theorem 6.21], its integral expression can be characterized by

$$\phi_u(x) = \int_{\mathbb{R}^3}\frac{|u(y)|^5}{|x-y|}dy,\ x \in \mathbb{R}^3, \tag{2.18}$$

which is called by the Riesz potential. It follows from (2.18) that $\phi_u(x) \geq 0$ for all $x \in \mathbb{R}^3$. Taking $v = \phi_u$ in (2.16) and (2.17), we derive

$$\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \leq S^{-1}S_6^{10}\|u\|_{H^1(\mathbb{R}^3)}^{10}. \tag{2.19}$$

Substituting (2.18) into (1.6), one can rewrite (1.6) in the following equivalent form

$$(-\Delta)^s u + V_\lambda(x)u + \phi_u u = g(u) + |u|^4u,\ x \in \mathbb{R}^3, \tag{2.20}$$

whose variational functional is exactly defined by (1.8) above. It would be simply verified that J_λ is well-defined in X_λ and belongs to $\mathcal{C}^1(X_\lambda, \mathbb{R})$ with derivative given by

$$J_\lambda'(u)(v) = \int_{\mathbb{R}^3}[\nabla u\nabla v + V_\lambda(x)uv]dx - \int_{\mathbb{R}^3}\phi_u|u|^3uvdx - \int_{\mathbb{R}^3}(g(u) + |u|^4u)vdx$$

for any $u, v \in X_\lambda$. It is clear to see that if u is a critical point of J_λ, then the pair (u, ϕ_u) is a solution of system (1.6).

Definition 2.4. The solutions of (1.6) and (2.20) have the following relationships:

(l) We call $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a weak solution of problem (1.6) if u is a weak solution of problem (2.20).

(2) We call $u \in H^1(\mathbb{R}^3)$ is a weak solution of (2.20) if

$$\int_{\mathbb{R}^3}\left[\nabla u\nabla v + V_\lambda(x)uv + \phi_u|u|^3uv - g(u)v - |u|^4uv\right]dx = 0,$$

for any $v \in H^1(\mathbb{R}^3)$.

(3) If $u \in H^1(\mathbb{R}^3)$ is a positive solution of (2.20), then we call $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ to be a positive solution of (1.6). If u is a ground state of (2.20), then so is $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of (1.6).

Let us define the variational functional $N : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$N(u) = \frac{1}{10}\int_{\mathbb{R}^3}\phi_u|u|^5dx \triangleq \frac{1}{10}\int_{\mathbb{R}^3}(I_{\mathrm{e}} * |u|^5)|u|^5dx,$$

where $I_{\mathrm{e}}(x) = |x|^{-1}$ for all $x \in \mathbb{R}^3\backslash\{0\}$ and $*$ denotes the convolution operator.

In the following, we collect some important properties for N as follows.

Lemma 2.5. *For every $u \in H^1(\mathbb{R}^3)$, one has the following conclusions:*

(i) *For any $t > 0$, there holds $\phi_{tu} = t^5\phi_u$.*

(ii) *$N(u) \geq 0$ and $N(tu) = t^{10}N(u)$ for any $t > 0$.*

(iii) *If* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$*, then, going to a subsequence if necessary,*

$$N(u_n) - N(u_n - u) - N(u) \to 0. \tag{2.21}$$

The derivative of N *is given by* $N'(u)(v) = \int_{\mathbb{R}^3} \phi_u |u|^3 uv dx$ *for all* $u, v \in H^1(\mathbb{R}^3)$ *and so*

$$N'(u_n)(\varphi) - N'(u)(\varphi) \to 0, \ \forall \varphi \in C_0^\infty(\mathbb{R}^3). \tag{2.22}$$

(iv) $N(u) = \frac{1}{10}|I_{e/2} * |u|^5|_2^2$*, where* $I_{e/2}(x) = |x|^{-2}$ *for all* $x \in \mathbb{R}^3 \backslash \{0\}$*.*

(v) *Let* $w = tu + v$ *for all* $v \in H^1(\mathbb{R}^3)$ *and* $t > 0$*, then*

$$N'(u)\left(\frac{t^2-1}{2}u + tv\right) - N(w) + N(u) \le 0.$$

Proof. For the proofs of points (i)-(iii), we refer the reader to the counterparts in [26,27] and so the details are omitted.

Then, we begin verifying the point (iv). Thanks to [30, Theorem 5.10], it holds that $I_{\mathrm{e}} = I_{\mathrm{e}/2} * I_{\mathrm{e}/2}$ and so

$$N(u) = \frac{1}{10}\int_{\mathbb{R}^3} (I_{\mathrm{e}} * |u|^5)|u|^5 dx = \frac{1}{10}\int_{\mathbb{R}^3} (I_{\mathrm{e}/2} * I_{\mathrm{e}/2} * |u|^5)|u|^5 dx = \frac{1}{10}\int_{\mathbb{R}^3} (I_{\mathrm{e}/2} * |u|^5)^2 dx$$

which is the result in point (iv).

Finally, we show the verification of point (v). Clearly, it follows from the Holder's inequality that

$$\left(I_{\mathrm{e}/2} * |u|^3 uw\right)^2 \le \left(I_{\mathrm{e}/2} * |u|^5\right)^{\frac{8}{5}} \left(I_{\mathrm{e}/2} * |w|^5\right)^{\frac{2}{5}}.$$

From which, we are able to take advantage of point (iv) to derive

$$\begin{aligned}
&\int_{\mathbb{R}^3} \phi_u |u|^3 u \left(\frac{t^2-1}{2}u + tv\right) dx - \frac{1}{10}\int_{\mathbb{R}^3} \phi_w |w|^5 dx + \frac{1}{10}\int_{\mathbb{R}^3} \phi_u |u|^5 dx \\
&\quad = \int_{\mathbb{R}^3} \left(I_{\mathrm{e}/2} * |u|^5\right)\left(tI_{\mathrm{e}/2} * |u|^3 uw - \frac{t^2}{2} I_{\mathrm{e}/2} * |u|^5\right) dx - \frac{1}{10}\int_{\mathbb{R}^3} \left(I_{\mathrm{e}/2} * |w|^5\right)^2 dx \\
&\qquad - \frac{2}{5}\int_{\mathbb{R}^3} \left(I_{\mathrm{e}/2} * |u|^5\right)^2 dx \\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^3} \left(tI_{\mathrm{e}/2} * |u|^5 - I_{\mathrm{e}/2} * |u|^3 uw\right)^2 dx \\
&\qquad - \frac{1}{10}\int_{\mathbb{R}^3} \left\{\left(I_{\mathrm{e}/2} * |w|^5\right)^2 + 4\left(I_{\mathrm{e}/2} * |u|^5\right)^2 - 5\left(I_{\mathrm{e}/2} * |u|^3 uw\right)^2\right\} dx \\
&\quad \le -\frac{1}{10}\int_{\mathbb{R}^3} \left\{\left(I_{\mathrm{e}/2} * |w|^5\right)^2 + 4\left(I_{\mathrm{e}/2} * |u|^5\right)^2 - 5\left(I_{\mathrm{e}/2} * |u|^5\right)^{\frac{8}{5}}\left(I_{\mathrm{e}/2} * |w|^5\right)^{\frac{2}{5}}\right\} dx \\
&\quad = -\frac{1}{10}\int_{\mathbb{R}^3} \left(I_{\mathrm{e}/2} * |u|^5\right)^2 \left\{\left(\frac{I_{\mathrm{e}/2} * |w|^5}{I_{\mathrm{e}/2} * |u|^5}\right)^2 - 5\left(\frac{I_{\mathrm{e}/2} * |w|^5}{I_{\mathrm{e}/2} * |u|^5}\right)^{\frac{2}{5}} + 4\right\} dx.
\end{aligned}$$

According to the obvious fact that $t^2 - 5t^{\frac{2}{5}} + 4 \ge 0$ for each $t \in (0, +\infty)$, we shall immediately reach the desired result. The proof is completed. □

3. Topological and linking structures

In this section, we shall mainly pay our attentions to the verifications of the conditions $(A_1) - (A_5)$ associated with J_λ in our variational settings

First of all, it follows from $(g_1) - (g_2)$ that for each $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^3} \max\left\{|G(u)|, |g(u)u|\right\} dx \le \varepsilon |u|_2^2 + C_\varepsilon |u|_p^p, \text{ where } p \in (2, 6). \tag{3.1}$$

Then, let us prove the $\mathcal{T}$-topology properties for J_λ.

Lemma 3.1. *Let $\lambda > \Lambda_2$ with $\Lambda_2 > 0$ given by Lemma 2.3 and suppose that $(g_1)-(g_2)$, then the variational functional J_λ defined by (1.8) is $\mathcal{T}$-upper semicontinuous.*

Proof. For each $t \in \mathbb{R}$, we shall suppose that $\{u_n\} \subset J_\lambda^{-1}([t,\infty))$ satisfies $u_n \xrightarrow{\mathcal{T}} u$. According to the definition of $\mathcal{T}$-norm, we have that $\{\|P_\lambda^+ u_n\|_{L_\lambda}\}$ is uniformly bounded and $P_\lambda^+ u_n \to P_\lambda^+ u$. Since $G(s) \geq 0$ for all $s \in \mathbb{R}$ and $N \geq 0$ by Lemma 2.5-(ii), one can find that $\{\|P_\lambda^- u_n\|_{L_\lambda}\}$ is uniformly bounded and $P_\lambda^- u_n \rightharpoonup P_\lambda^- u$. It follows from the Fatou's lemma that

$$\begin{aligned}
-J_\lambda(u) &= \frac{1}{2}\left(\|P_\lambda^- u\|_{L_\lambda}^2 - \|P_\lambda^+ u\|_{L_\lambda}^2\right) + \frac{1}{10}\int_{\mathbb{R}^3} \phi_u |u|^5 dx + \int_{\mathbb{R}^3} F(u)dx \\
&\leq \liminf_{n\to\infty} \left\{\frac{1}{2}\left(\|P_\lambda^- u_n\|_{L_\lambda}^2 - \|P_\lambda^+ u_n\|_{L_\lambda}^2\right) + \frac{1}{10}\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx + \int_{\mathbb{R}^3} F(u_n)dx\right\} \\
&= \liminf_{n\to\infty} -J_\lambda(u_n) \leq -t
\end{aligned}$$

implying that $J_\lambda(u) \geq t$. The proof is completed. □

Lemma 3.2. *Let $\lambda > \Lambda_2$ and assume $(g_1)-(g_2)$, then J_λ is $\mathcal{T}$-to-weak* continuous in $\Psi^{-1}([0,\infty))$.*

Proof. Given a sequence $\{u_n\} \subset X_\lambda$ satisfying $\{u_n\} \subset J_\lambda^{-1}([0,\infty))$ and $u_n \xrightarrow{\mathcal{T}} u$, then, proceeding as the proof of Lemma 3.1, one sees that $\{\|u_n\|_{L_\lambda}\}$ is uniformly bounded. Moreover, $P_\lambda^+ u_n \to P_\lambda^+ u$ and $P_\lambda^- u_n \rightharpoonup P_\lambda^- u$. Becasue $\dim X_\lambda^- = j_0 < +\infty$, one has that $P_\lambda^- u_n \to P_\lambda^- u$. As a consequence, it holds that $u_n \to u$ in X_λ. Recalling Lemma 2.3, we can make use of Lemma 2.5-(iii) and (3.1) finish the proof of this lemma immediately. □

Finally, we begin verifying the geometry structures of J_λ.

Lemma 3.3. *Suppose that $(g_1)-(g_4)$, then there is a constant $\Lambda_3 > \Lambda_2$ such that, for all $\lambda > \Lambda_3$, there holds*

$$\inf_{\{u \in \tilde{X}_0^+ :\ \|u\|_{L_\lambda} = r\}} J_\lambda(u) \geq \varrho, \tag{3.2}$$

where $r, \varrho > 0$ are some constants independent of $\lambda > \Lambda_3$.

Proof. First of all, we claim that there exists a $\Lambda_3 > \Lambda_2$ such that for any $\lambda > \Lambda_3$ and $u \in \tilde{X}_0^+$, we have the following inequality

$$\int_{\mathbb{R}^3} \left[|\nabla u|^2 + V_\lambda(x)u^2\right] dx \geq C_0 \int_{\mathbb{R}^3} \left[|\nabla u|^2 + V_\lambda^+(x)u^2\right] dx, \tag{3.3}$$

where $C_0 > 0$ is a constant independent of $\lambda > \Lambda_3$. Since $u \in X_\lambda$, one has that

$$u = P_\lambda^+ u + P_\lambda^- u,$$

where $P_\lambda^+ u \in X_\lambda^+$ and $P_\lambda^- u \in X_\lambda^-$. Recalling $V_\lambda(x) = V_\lambda^+(x) - V_\lambda^-(x)$ for all $x \in \mathbb{R}^3$ and $V_\lambda^-(x) \leq \mu$ in $\mathbb{R}^3$, it follows from some simple calculations that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[|\nabla P_\lambda^+ u|^2 + V_\lambda^+(x)|P_\lambda^+ u|^2\right] dx \\
&= \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^+ u|^2 + V_\lambda(x)|P_\lambda^+ u|^2\right] dx + \int_{\mathbb{R}^3} V_\lambda^-(x)|P_\lambda^+ u|^2 dx \\
&\leq \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^+ u|^2 + V_\lambda(x)|P_\lambda^+ u|^2\right] dx + \mu \int_{\mathbb{R}^3} |P_\lambda^+ u|^2 dx \\
&\leq \left(1 + \frac{\mu}{\mu_{j_0+1}(L_\lambda)}\right) \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^+ u|^2 + V_\lambda(x)|P_\lambda^+ u|^2\right] dx,
\end{aligned} \tag{3.4}$$

where we have used the fact that $\mu_{j_0+1}(L_\lambda)$ is the smallest eigenvalue in the space X_λ^+.

Then, due to (3.4) and $V_\lambda(x) = V_\lambda^+(x) - V_\lambda^-(x)$ for all $x \in \mathbb{R}^3$, some elementary computations give us that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left[|\nabla u|^2 + V_\lambda(x)u^2\right] dx \\
&= \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^+ u|^2 + V_\lambda(x)|P_\lambda^+ u|^2\right] dx + \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^- u|^2 + V_\lambda(x)|P_\lambda^- u|^2\right] dx \\
&\geq \frac{\mu_{j_0+1}(L_\lambda)}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^+ u|^2 + V_\lambda^+(x)|P_\lambda^+ u|^2\right] dx + \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^- u|^2 + V_\lambda(x)|P_\lambda^- u|^2\right] dx \\
&= \frac{\mu_{j_0+1}(L_\lambda)}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^+ u|^2 + V_\lambda^+(x)|P_\lambda^+ u|^2\right] dx \\
&\quad + \frac{\mu_{j_0+1}(L_\lambda)}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^- u|^2 + V_\lambda^+(x)|P_\lambda^- u|^2\right] dx - \frac{\mu_{j_0+1}(L_\lambda)}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} V_\lambda^-(x)|P_\lambda^- u|^2 dx \\
&\quad + \frac{\mu}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^- u|^2 + V_\lambda(x)|P_\lambda^- u|^2\right] dx \\
&= \frac{\mu_{j_0+1}(L_\lambda)}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} \left[|\nabla u|^2 + V_\lambda^+(x)|u|^2\right] dx + \frac{\mu}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^- u|^2 + V_\lambda(x)|P_\lambda^- u|^2\right] dx \\
&\quad - \frac{\mu_{j_0+1}(L_\lambda)}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} V_\lambda^-(x)\left(|P_\lambda^- u|^2 + 2P_\lambda^+ u P_\lambda^- u\right) dx.
\end{aligned}
\tag{3.5}
$$

We denote e_j and $e_{\lambda,j}$ by the eigenfunctions of L_0 and L_λ corresponding to $\mu_j(L_0)$ and $\mu_j(L_\lambda)$ for $j \in \{1, 2, \cdots\}$, respectively. Moreover, without loss of generality, we would suppose that $|e_j|_{2,\Omega} = 1$ and $|e_{\lambda,j}|_2 = 1$ for $j \in \{1, 2, \cdots\}$. Then, for all $u \in \tilde{X}_0^+$ and so $(u, \tilde{e}_j)_{L_\lambda} = 0$, we obtain

$$P_\lambda^- u = \sum_{j=1}^{j_0} \|e_{\lambda,j}\|_{L_\lambda}^{-2}(u, e_{\lambda,j})_{L_\lambda} e_{\lambda,j} = \sum_{j=1}^{j_0} \|e_{\lambda,j}\|_{L_\lambda}^{-2}(u, e_{\lambda,j} - \tilde{e}_j)_{L_\lambda} e_{\lambda,j}$$

which together with (2.6), (2.8) and Lemma 2.3 implies that

$$
\begin{aligned}
\int_{\mathbb{R}^3} |P_\lambda^- u|^2 dx &= \sum_{j=1}^{j_0} \|e_{\lambda,j}\|_{L_\lambda}^{-4} |(u, e_{\lambda,j} - \tilde{e}_j)_{L_\lambda}|^2 |e_{\lambda,j}|_2^2 \leq S_2^2 C_1^{-2} \sum_{j=1}^{j_0} \|e_{\lambda,j}\|_{L_\lambda}^{-2} |(u, e_{\lambda,j} - \tilde{e}_j)_{L_\lambda}|^2 \\
&\leq -j_0 \mu_{j_0}(L_\lambda) S_2^2 \|u\|_{X_\lambda}^2 \sum_{j=1}^{j_0} \|e_{\lambda,j} - \tilde{e}_j\|_{L_\lambda}^2 = o_\lambda(1)\|u\|_{X_\lambda}^2.
\end{aligned}
$$

Adopting (2.8) again, there is a $\Lambda_3 > \Lambda_2$ such that, for all $\lambda > \Lambda_3$,

$$\frac{\mu_{j_0+1}(L_\lambda)}{\mu_{j_0+1}(L_\lambda)+\mu} \geq \frac{\mu_{j_0+1}(L_0)}{2(\mu_{j_0+1}(L_0)+\mu)}$$

and

$$\frac{\mu_{j_0+1}(L_\lambda)}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} V_\lambda^-(x)\left(|P_\lambda^- u|^2 + 2P_\lambda^+ u P_\lambda^- u\right) dx \leq \frac{\mu_{j_0+1}(L_0)}{4(\mu_{j_0+1}(L_0)+\mu)} \|u\|_{X_\lambda}^2.$$

It is clear that

$$\frac{\mu}{\mu_{j_0+1}(L_\lambda)+\mu} \int_{\mathbb{R}^3} \left[|\nabla P_\lambda^- u|^2 + V_\lambda(x)|P_\lambda^- u|^2\right] dx \leq 0.$$

Inserting the above formulas into (3.5), we obtain the desired result by choosing $C_0 = \frac{\mu_{j_0+1}(L_0)}{4(\mu_{j_0+1}(L_0)+\mu)}$.

With (3.3) in hands, for all $u \in \tilde{X}_0^+$, we combine (2.1), (2.6), (2.19) and (3.1) to derive

$$J_\lambda(u) \geq \frac{C_0}{2}\|u\|_{X_\lambda}^2 - \frac{1}{10}\int_{\mathbb{R}^3} \phi_u |u|^5 dx - \int_{\mathbb{R}^3} G(u)dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx$$
$$\geq \frac{C_0}{4}\|u\|_{X_\lambda}^2 - \hat{C}_0\|u\|_{X_\lambda}^{10} - \hat{C}_0\|u\|_{X_\lambda}^p - \hat{C}_0\|u\|_{X_\lambda}^6.$$

Since $p > 2$, there exists a constant $r > 0$ small to reach the desired result. The proof is completed. □

Lemma 3.4. *Let $\lambda > \Lambda_2$ and suppose that $(g_1) - (g_4)$, then for all $u_0 \in X_\lambda \backslash \tilde{X}_0^-$, there is an $R_0 > r$ such that*

$$\max_{u \in \partial M(u_0)} J_\lambda(u) \leq 0,$$

where $M(u_0) = \{tu_0 + v \in X_\lambda : v \in \tilde{X}_0^-,\ \|tu_0 + v\|_{L_\lambda} \leq R_0$ and $t \geq 0\}$.

Proof. First of all, we claim that

$$\int_{\mathbb{R}^3} \left[|\nabla v|^2 + V_\lambda(x)v^2\right] dx < 0,\ \forall v \in \tilde{X}_0^- \backslash \{0\}. \tag{3.6}$$

Actually, one can conclude that $v = \sum_{j=1}^{j_0} \beta_j \tilde{e}_j$ for some $\beta_j \in \mathbb{R}$ with $j \in \{1, 2, \cdots, j_0\}$ according to the definition of $\tilde{X}_0^-$, where e_j comes from (1.9). So, we have that

$$\int_{\mathbb{R}^3} \left[|\nabla v|^2 + V_\lambda(x)v^2\right] dx = \sum_{j=1}^{j_0} \int_\Omega \left[|\nabla e_j|^2 - \mu e_j^2\right] dx = \sum_{j=1}^{j_0} (\mu_j - \mu) \int_\Omega e_j^2 dx < 0$$

showing (3.6). Since $G(t) \geq 0$ for all $t \in \mathbb{R}$, we are able to gather Lemma 2.5-(ii) and (3.6) to obtain that $J_\lambda(v) \leq 0$ for all $v \in \tilde{X}_0^-$.

Combining $G(t) \geq 0$ for all $t \in \mathbb{R}$ and Lemma 2.5-(ii) again, one has

$$J_\lambda(u) \leq \frac{1}{2}\left(\|P_\lambda^+ u\|_{L_\lambda}^2 - \|P_\lambda^- u\|_{L_\lambda}^2\right) - \frac{1}{6}|w|_6^6 \|u\|_{L_\lambda}^6,\ \forall u = P_\lambda^+ u + P_\lambda^- u \in X_\lambda. \tag{3.7}$$

where $w = u/\|u\|_{L_\lambda}$, which reveals that $w \in \tilde{X}_0^- \bigoplus \mathbb{R}^+ u_0$ if $u \in \partial M(u_0)$. Motivated by [40, Lemma 2.2], we claim that for each fixed constant $\alpha \in (0, 1)$, there exists a constant $C_\alpha > 0$ such that

$$\|P_\lambda^+ w\|_{L_\lambda} \geq \sin(\arctan \alpha) \Longrightarrow |w|_6^6 \geq C_\alpha. \tag{3.8}$$

In fact, let $\gamma \triangleq \sin(\arctan \alpha) \in (0, 1)$ and define

$$\mathfrak{F}^\alpha \triangleq \left\{v \in \tilde{X}_0^- \bigoplus \mathbb{R}^+ u_0 : \|v\|_{L_\lambda} = 1 \text{ and } \|P_\lambda^+ v\|_{L_\lambda} \geq \gamma\right\}.$$

Obviously, $|w|_6^6 \geq \inf_{v \in \mathfrak{F}^\alpha} |v|_6^6 \triangleq C_\alpha \geq 0$. Arguing it by contradiction, we could suppose that $C_\alpha = 0$. Thereby, there exists a sequence $\{v_n\} \subset \mathfrak{F}^\alpha$ such that $|v_n|_6^6 \to 0$ as $n \to \infty$. Passing to a subsequence if necessary, there is $v \in X_\lambda$ such that $v_n \rightharpoonup v$ in X_λ and $P_\lambda^+ v_n \to P_\lambda^+ v$ because $\{P_\lambda^+ v_n\} \subset \mathbb{R}^+ u_0$. Therefore, $\|P_\lambda^+ v\|_{L_\lambda} \geq \gamma > 0$ and $|v|_6^6 \leq \liminf_{n \to \infty} |v_n|_6^6 = 0$, a contradiction. Hence, (3.8) holds true. To proceeed the proof, we distinguish it two cases:

$$\text{(I) } \|P_\lambda^+ u\|_{L_\lambda}/\|P_\lambda^- u\|_{L_\lambda} < \alpha \text{ and (II) } \|P_\lambda^+ u\|_{L_\lambda}/\|P_\lambda^- u\|_{L_\lambda} \geq \alpha.$$

If (I) occurs, then $\|u\|_{L_\lambda}^2 < (1 + \alpha^2)\|P_\lambda^- u\|_{L_\lambda}^2$. By (3.7), we obtain

$$J_\lambda(u) \leq \frac{1}{2}\|u\|_{L_\lambda}^2 - \|P_\lambda^- u\|_{L_\lambda}^2 \leq -\frac{1 - \alpha^2}{2(1 + \alpha^2)}\|u\|_{L_\lambda}^2 \to -\infty \text{ as } \|u\|_{L_\lambda} \to \infty \text{ and } u \in \partial M_0.$$

If (II) occurs, then $\|P_\lambda^+ w\|_{L_\lambda} \geq \sin(\arctan\alpha)$. Combining (3.7) and (3.8), we have

$$J_\lambda(u) \leq \frac{1}{2}\|u\|_{L_\lambda}^2 - \frac{C_\alpha}{6}\|u\|_{L_\lambda}^6 \to -\infty \text{ as } \|u\|_{L_\lambda} \to \infty \text{ and } u \in \partial M_0.$$

In summary, there is a sufficiently large $R_0 > r$ to reach the desired result. The proof is completed. □

Lemma 3.5. *Let $\lambda > \Lambda_2$ and suppose that $(g_1) - (g_5)$, then for any $u \in X_\lambda \backslash \tilde{X}_0^-$ and $v \in \tilde{X}_0^-$ with $t \geq 0$, there holds*

$$J_\lambda(u) \geq J_\lambda(tu + v) - J_\lambda'(u)\left(\frac{t^2-1}{2}u + tv\right). \tag{3.9}$$

In particular, the condition (A_5) holds true.

Proof. Some elementary calculations provide us that

$$\begin{aligned} &J_\lambda(tu+v) - J_\lambda(u) - J_\lambda'(u)\left(\frac{t^2-1}{2}u + tv\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla v|^2 + V_\lambda(x)v^2\right]dx + N'(u)\left(\frac{t^2-1}{2}u + tv\right) - N(w) + N(u) \\ &\quad + \int_{\mathbb{R}^3}\left\{f(u)\left(\frac{t^2-1}{2}u + tv\right) + F(u) - F(tu+v)\right\}dx. \end{aligned}$$

In view of Lemma 2.5-(v) and (3.6), to derive (3.9), it suffices to show that

$$\Phi(t,\psi) \triangleq f(u)\left(\frac{t^2-1}{2}u + t\psi\right) + F(u) - F(tu+\psi) \leq 0$$

for all $(t,\psi) \in \mathbb{R}^+ \times \mathbb{R}$. We shall follow the idea used in [34, Proposition 4.1] to achieve this goal.

It is simple to observe that the nonlinearity f satisfies the assumptions (g_1), (g_3) and (g_5). By (g_3), we see $\Phi(0,\psi) \leq 0$ and

$$\begin{aligned} \Phi(t,\psi) &\leq f(u)\left(\frac{t^2-1}{2}u + t\psi\right) + \frac{1}{2}f(u)u - F(tu+\psi) \\ &= \left(-\frac{t^2}{2}f(u)u + tf(u)\zeta - A_0|\zeta|^2\right) + \left(A_0|\zeta|^2 - F(\zeta)\right) \\ &\triangleq \Phi^1(t,\psi) + \Phi^2(t,\psi), \end{aligned}$$

where $\zeta = tu + \psi$ and $A_0 > 0$ is a sufficiently large constant satisfying Φ^1 is negative definite with respect to $(t,\psi) \in \mathbb{R}^+ \times \mathbb{R}$ and so Φ^1 is bounded from above for all $(t,\psi) \in \mathbb{R}^+ \times \mathbb{R}$. Then, it cleanly knows that $\Phi(t,\psi) \to -\infty$ as $|(t,\psi)| = (t^2 + |\psi|^2)^{1/2} \to \infty$ and there exists a (t_0,ψ_0) corresponding to the maximum of Φ. Since $\Phi(0,\psi) \leq 0$, we can assume that $t_0 > 0$. Moreover, one has that

$$\begin{cases} \partial_t\Phi(t_0,\psi_0) = f(u)(t_0u + \psi_0) - f(t_0u+\psi_0)u = 0, \\ \partial_\psi\Phi(t_0,\psi_0) = t_0f(u) - f(t_0u+\psi_0) = 0, \end{cases} \tag{3.10}$$

yielding that

$$f(u)\psi_0 = 0. \tag{3.11}$$

Inserting (3.11) into the first equality in (3.10), by $t_0 > 0$ and (g_3), we reach

$$f(t_0u+\psi_0)u = f(u)(t_0u+\psi_0) = t_0f(u)u > 0. \tag{3.12}$$

Let $t \triangleq u$ and $s \triangleq t_0u + \psi_0$ in (g_5), it follows from (3.10), (3.11) and (3.12) that

$$\Phi(t,\psi) \leq \Phi(t_0,\psi_0) = \frac{t_0^2-1}{2}f(u)u + F(u) - F(t_0u+\psi_0)$$

$$
\begin{aligned}
&\le \frac{t_0^2-1}{2}f(u)u+\frac{\big[f(u)+f(t_0u+\psi_0)\big]\big[(1-t_0)u-\psi_0\big]}{2}\\
&\le \frac{t_0^2-1}{2}f(u)u+\frac{(1+t_0)f(u)\big[(1-t_0)u-\psi_0\big]}{2}\\
&=0.
\end{aligned}
$$

The proof is completed. □

4. Proofs of the main results

In this section, we are going to focus on showing the detailed proofs of Theorems 1.1, 1.2 and 1.6, respectively.

According to Section 3, under the assumptions (V_1) and $(V_5)-(V_6)$ as well as $(g_1)-(g_4)$, by the settings $(E,\|\cdot\|_E)=(X_\lambda,\|\cdot\|_{L_\lambda})$, $Y=\tilde{X}_0^-$, $Z=\tilde{X}_0^+$ and $\Psi=J_\lambda$ in Proposition 1.5, there is a (C) sequence $\{u_n\}\subset X_\lambda$ at the level

$$c_\lambda \triangleq \inf_{X_\lambda\backslash \tilde{X}_0^-}\inf_{h\in\Gamma(u)}\sup_{u'\in M(u)} J_\lambda(h(u',1)). \tag{4.1}$$

Speaking it clearly, the sequence $\{u_n\}$ satisfies

$$J_\lambda(u_n)\to c_\lambda \text{ and } (1+\|u_n\|_{L_\lambda})\|J_\lambda'(u_n)\|_{X_\lambda^{-1}}\to 0 \text{ as } n\to\infty. \tag{4.2}$$

First of all, we shall show that the mountain-pass value c_λ can be controlled by a positive constant which is independent of $\lambda>\Lambda_3$.

Lemma 4.1. *Soppose $\mu\in(\mu_{j_0},\mu_{j_0+1})$ and $(g_1)-(g_4)$, then*

$$\sup_{\lambda>\Lambda_3} c_\lambda < c^*\triangleq \frac{13-\sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac12} S^{\frac32},$$

where $\Lambda_3>0$ is determined by Lemma 3.3.

Proof. Without loss of generality, we are assuming that $0\in\Omega$ and $\bar{B}_2(0)\subset\Omega$. Let $\psi\in C_0^\infty(\mathbb{R}^3)$ be a cutoff function with its support set located in $B_2(0)$ such that $0\le\psi(x)\le 1$ and $\psi(x)\equiv 1$ on $B_1(0)$. It is well-known that $S>0$ in (2.4) is achieved by $U_\varepsilon(x)=(3\varepsilon)^{1/4}(\varepsilon+|x|^2)^{-1/2}$, where $\varepsilon>0$. Then, set $u_\varepsilon(x)=\psi(x)U_\varepsilon(x)\in H_0^1(\Omega)$, inspired by [11], one can obtain the following estimates

$$|\nabla u_\varepsilon|_{2,\Omega}^2=S^{\frac32}+O(\varepsilon^{\frac12}),\ |u_\varepsilon|_{6,\Omega}^6=S^{\frac32}+O(\varepsilon^{\frac32}),\ |u_\varepsilon|_{2,\Omega}^2=O(\varepsilon^{\frac12}), \tag{4.3}$$

and for some constant $K_s>0$ with $s\in(3,6)$ such that

$$|u_\varepsilon|_{s,\Omega}^s=K_s\varepsilon^{\frac{6-s}{4}}. \tag{4.4}$$

Thanks to the Cauchy inequality and Diamagnetic inequality (see e.g. [30, Theorem 7.21]),

$$\int_\Omega |u_\varepsilon|^6dx=\int_\Omega \nabla\phi_{u_\varepsilon}\nabla|u_\varepsilon|dx\le\frac12\int_\Omega|\nabla\phi_{u_\varepsilon}|^2dx+\frac12\int_\Omega|\nabla u_\varepsilon|^2dx$$

which together with (4.3) gives that

$$
\begin{aligned}
\int_\Omega|\nabla\phi_{u_\varepsilon}|^2dx &\ge 2\int_\Omega|u_\varepsilon|^6dx-\int_\Omega|\nabla u_\varepsilon|^2dx\\
&=S^{\frac32}+O(\varepsilon^{\frac12}).
\end{aligned}
\tag{4.5}
$$

To continue the proof, we define

$$I_\Omega(u)=\frac12\int_\Omega|\nabla u|^2dx-\frac{1}{10}\int_\Omega\int_\Omega\frac{|u(x)|^5|u(y)|^5}{|x-y|}dxdy-\int_\Omega G(u)dx-\frac16\int_\Omega|u|^6dx,\ \forall u\in H_0^1(\Omega).$$

We claim that $\max_{t>0} I_\Omega(tu_\varepsilon) < c^*$ if $\varepsilon > 0$ small enough. Indeed, it is standard to find a constant $t_\varepsilon \in [T_0, T^0]$ such that $I_\Omega(t_\varepsilon u_\varepsilon) = \max_{t>0} I_\Omega(tu_\varepsilon)$, where $T_0, T^0 \in (0, +\infty)$ are two constants independent of $\varepsilon > 0$. As a consequence, it follows from (4.3)-(4.4) and (g_4) that

$$\begin{aligned} I_\Omega(t_\varepsilon u_\varepsilon) &\le \frac{t_\varepsilon^2}{2}|\nabla u_\varepsilon|_{2,\Omega}^2 - \frac{t_\varepsilon^{10}}{10}|\nabla \phi_{u_\varepsilon}|_{2,\Omega}^2 - \frac{t_\varepsilon^6}{6}|u_\varepsilon|_{6,\Omega}^6 - \left(A|t_\varepsilon|^r|u_\varepsilon|_{r,\Omega}^r - B|t_\varepsilon|^2|u_\varepsilon|_{2,\Omega}^2\right) \\ &\le \max_{t>0}\left\{\frac{t^2}{2}|\nabla u_\varepsilon|_2^2 - \frac{t^{10}}{10}|\nabla \phi_{u_\varepsilon}|_2^2 - \frac{t^6}{6}|u_\varepsilon|_6^6\right\} - C\left(O(\varepsilon^{\frac{6-r}{4}}) - O(\varepsilon^{\frac{1}{2}})\right). \end{aligned}$$

In view of [18, Lemma 3.4], we can apply (4.3), (4.4) and (4.5) to conclude that

$$\max_{t>0} I_\Omega(tu_\varepsilon) \le c^* + O(\varepsilon^{\frac{1}{2}}) - CO(\varepsilon^{\frac{6-r}{4}}) < c^* \tag{4.6}$$

if $\varepsilon > 0$ is sufficiently small since $r > 4$. So, the claim is true.

Finally, for any $v \in X_0^- \subset H_0^1(\Omega)$ and all $t > 0$, it easily concludes from (V_5) that

$$\begin{aligned} J_\lambda(tu_\varepsilon + v) &= \frac{1}{2}\int_\Omega \left(|\nabla(tu_\varepsilon + v)|^2 - \mu|tu_\varepsilon + v|^2\right) dx - \frac{1}{10}\int_\Omega\int_\Omega \frac{|tu_\varepsilon + v|^5|tu_\varepsilon + v|^5}{|x-y|}dxdy \\ &\quad - \int_\Omega G(tu_\varepsilon + v)dx - \frac{1}{6}\int_\Omega |tu_\varepsilon + v|^6 dx \\ &\le \frac{t^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{t^{10}}{10}\int_\Omega\int_\Omega \frac{|u_\varepsilon|^5|u_\varepsilon|^5}{|x-y|}dxdy - \int_\Omega G(tu_\varepsilon)dx - \frac{t^6}{6}\int_\Omega |u_\varepsilon|^6 dx \\ &\le \max_{t>0} I_\Omega(tu_\varepsilon) = I_\Omega(t_\varepsilon u_\varepsilon) < c^*, \end{aligned}$$

where we have made use of [13, Proposition 4.2] in the first inequality and (4.6) in the last inequality, respectively. In view of the definition of c_λ in (4.1), the proof concludes. □

Next, we mainly focus on verifying that the variational functional J_λ satisfies the $(C)_{c_\lambda}$ for some suitably large $\lambda > 0$. Before proceeding it, we shall deduce that any $(C)_{c_\lambda}$ sequence of J_λ is uniformly bounded for all $\lambda > \Lambda_3$.

Lemma 4.2. *If $\mu \in (\mu_{j_0}, \mu_{j_0+1})$ and $(g_1) - (g_4)$ hold, suppose that $\{u_n\} \subset X_\lambda$ is a $(C)_{c_\lambda}$ sequence of J_λ for any $\lambda > \Lambda_3$, then $\{u_n\}$ is uniformly bounded in X_λ.*

Proof. Suppose it by contradiction, we would suppse that $\|u_n\|_{X_\lambda} \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_{X_\lambda}}$, passing to a subsequence if necessary, there exists a $v \in X_\lambda$ such that $v_n \rightharpoonup v$ in X_λ, $v_n \to v$ in $L_{\rm loc}^p(\mathbb{R}^3)$ with $p \in [2, 6)$ and $v_n \to v$ a.e. in $\mathbb{R}^3$. Firstly, we claim that $v \equiv 0$ a.e. in $\mathbb{R}^3$. Otherwise, the set $\Upsilon \triangleq \{x \in \mathbb{R}^3 : |v(x)| > 0\}$ possesses a positive Lebesgue measure and so $|u_n| \to +\infty$ on Υ as $n \to \infty$. It follows from (g_3) and (g_4) that

$$\liminf_{n\to\infty} \int_\Upsilon \frac{g(u_n)u_n}{|u_n|^2}|v_n|^2 dx = +\infty. \tag{4.7}$$

Since $\{u_n\} \subset X_\lambda$ is a $(C)_{c_\lambda}$ sequence of J_λ, we apply (g_3) to obtain

$$\begin{aligned} c_\lambda &= \limsup_{n\to\infty}\left[J_\lambda(u_n) - \frac{1}{2}J_\lambda'(u_n)(u_n)\right] \\ &= \limsup_{n\to\infty}\left\{\frac{2}{5}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5 dx + \frac{1}{2}\int_{\mathbb{R}^3}\left[g(u_n)u_n - 2G(u_n)\right]dx + \frac{1}{3}\int_{\mathbb{R}^3}|u_n|^6 dx\right\} \\ &\ge \frac{1}{3}\limsup_{n\to\infty}\left[N(u_n) + |u_n|_6^6\right] \end{aligned}$$

which together with Lemma 4.1 indicates that $\{N(u_n)\}$ and $\{|u_n|_6\}$ are uniformly bounded for every $\lambda > \Lambda_2$. Consequently, we have

$$0 = \limsup_{n\to\infty} \frac{J_\lambda'(u_n)(u_n)}{\|u_n\|_{X_\lambda}^2} \leq C_2^2 - \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{g(u_n)u_n}{|u_n|^2} |v_n|^2 dx,$$

a contradiction to (4.7), where $C_2 > 0$ comes from Lemma 2.3. Thereby, $v = 0$ a.e. in $\mathbb{R}^3$. Because $\dim X_\lambda^- < +\infty$, $v_n^- \to 0$, where $v_n^- = P_\lambda^- v_n \in X_\lambda^-$ and $v_n^+ = P_\lambda^+ v_n \in X_\lambda^+$. According to Lemma 2.3, without loss of generality, we can suppose that $\|v_n^+\|_{L_\lambda}^2 + \|v_n^-\|_{L_\lambda}^2 \equiv 1$, then it permits us to see that $\|v_n^+\|_{L_\lambda}^2 - \|v_n^-\|_{L_\lambda}^2 \geq \frac{1}{2}$ for some sufficiently large $n \in \mathbb{N}$. Taking (g_3) into account, then

$$\begin{aligned} c_\lambda &= \limsup_{n\to\infty} \left[J_\lambda(u_n) - \frac{1}{\eta} J_\lambda'(u_n)(u_n)\right] \\ &= \limsup_{n\to\infty} \left\{ \frac{\eta-2}{2\eta} \|u_n\|_{L_\lambda}^2 (\|v_n^+\|_{L_\lambda}^2 - \|v_n^-\|_{L_\lambda}^2) + 10\left(\frac{1}{\eta} - \frac{1}{10}\right) N(u_n) \right. \\ &\qquad \left. + \frac{1}{\eta} \int_{\mathbb{R}^3} \left[g(u_n)u_n - \eta G(u_n)\right] dx + \left(\frac{1}{\eta} - \frac{1}{6}\right) |u_n|_6^6 \right\} \\ &\geq \limsup_{n\to\infty} \left\{ \frac{\eta-2}{4\eta} \|u_n\|_{L_\lambda}^2 + 10\left(\frac{1}{\eta} - \frac{1}{10}\right) N(u_n) + \left(\frac{1}{\eta} - \frac{1}{6}\right) |u_n|_6^6 \right\} \\ &= +\infty \end{aligned}$$

which is impossible, where we depend on Lemmas 2.3 and 4.1 as well as the facts that $\{N(u_n)\}$ and $\{|u_n|_6\}$ are uniformly bounded for all $\lambda > \Lambda_3$. The proof is completed. □

Lemma 4.3. *If $\mu \in (\mu_{j_0}, \mu_{j_0+1})$ and $(g_1) - (g_4)$ hold. Let $\lambda > \Lambda_3$ and $\{u_n\} \subset X_\lambda$ be a $(C)_{c_\lambda}$ sequence, then there exist $r \in (2, 6)$ and $\sigma_0 > 0$, independent of λ, such that $|u_n|_r \geq \sigma_0$, for all $n \geq 1$.*

Proof. First of all, by Lemma 4.2, the sequence $\{u_n\}$ is uniformly bounded in $n \in \mathbb{N}$ for all $\lambda > \Lambda_3$. Motivated by the counterparts in [37, 39], let us divide the proof into intermediate steps.

Step I: Let $\lambda > \Lambda_3$ and $\{u_n\} \subset X_\lambda$ be a $(C)_{c_\lambda}$ sequence, then there are $r \in (2, 6)$ and $\sigma = \sigma(\lambda) > 0$ such that $|u_n|_r \geq \sigma$, for all $n \geq 1$.

Suppose, by contradiction, that $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for every $r \in (2, 6)$. Owing to the boundedness of $\{u_n\}$ in X_λ, we conclude that $\{u_n\}$ is uniformly bounded in $L^q(\mathbb{R}^2)$ for all $q \in [2, 6)$, too. As a consequence, one simply invokes from (3.1) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} g(u_n)u_n dx = 0 \text{ and } \lim_{n\to\infty} \int_{\mathbb{R}^3} G(u_n) dx = 0. \tag{4.8}$$

In view of (2.2), by using the Hölder's inequality, we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^3} V_\lambda^-(x)|u_n|^2 dx = \int_{B_R(0)} V_\lambda^-(x)|u_n|^2 dx \leq \mu \int_{B_R(0)} |u_n|^2 dx \\ &\leq \mu |B_R(0)|^{\frac{s-2}{s}} \left(\int_{B_R(0)} |u_n|^s dx\right)^{\frac{2}{s}} \to 0, \text{ as } n \to \infty. \end{aligned} \tag{4.9}$$

According to (4.8)-(4.9) and $\lim_{n\to\infty} J_\lambda'(u_n)(u_n) = 0$, it holds that

$$\|u_n\|_{X_\lambda}^2 - 10N(u_n) - |u_n|_6^6 = o_n(1). \tag{4.10}$$

Without loss of generality, we could suppose that $\|u_n\|_{X_\lambda}^2 \to l$ as $n \to \infty$. Obviously, we conclude $l > 0$. Otherwise, $\|u_n\|_{X_\lambda}^2 \to 0$ which indicates $N(u_n) \to 0$ and $|u_n|_6^6 \to 0$ by Lemma 2.5-(ii) and

(2.4), respectively. Combining these facts and (4.8), it reveals that $c_\lambda = \lim_{n\to\infty} J_\lambda(u_n) = 0$, which is absurd because of Lemma 3.3. Now, we suppose that $10N(u_n) \to l_1$ and $|u_n|_6^6 \to l_2$. By the Cauchy inequality and Diamagnetic inequality (see e.g. [30, Theorem 7.21]), one has

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n|^6 dx &= \int_{\mathbb{R}^3} \nabla\phi_{u_n}\nabla|u_n| dx \leq \frac{\sqrt{5}+1}{4}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^2dx + \frac{\sqrt{5}-1}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\\ &= \frac{\sqrt{5}+1}{4}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5dx + \frac{\sqrt{5}-1}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2dx\end{aligned}$$

which implies that

$$l_2 \leq \frac{\sqrt{5}+1}{4} l_1 + \frac{\sqrt{5}-1}{4} l. \tag{4.11}$$

Recalling $l = l_1 + l_2$, we apply (4.11) to reach $l_1 \geq \frac{3-\sqrt{5}}{2} l$. So, by (4.11), there holds

$$c_\lambda = \lim_{n\to\infty} J_\lambda(u_n) = \frac{1}{2}l - \frac{1}{10}l_1 - \frac{1}{6}l_2 = \frac{1}{3}l + \frac{1}{15}l_1 \geq \frac{13-\sqrt{5}}{30}l. \tag{4.12}$$

In view of (2.4) and (2.16), by (4.10), one has

$$l \leq S^{-6} l^5 + S^{-3} l^3$$

yielding that $l^2 \geq \frac{\sqrt{5}-1}{2} S^3$ since $l > 0$. Whereas, with the help of (4.12), we reach

$$c_\lambda \geq \frac{13-\sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{2}} S^{\frac{3}{2}}.$$

which contradicts with Lemma 4.1. The proof of this step is done.

Step II: Conclusion.

Let $r \in (2,6)$ be as in Step I. Suppose by contradiction that the uniform control from below of $L^r(\mathbb{R}^3)$-norm is false. Then, for all $k \in \mathbb{N}$, $k \neq 0$, there exist $\lambda_k > \Lambda_3$ and a $(C)_{c_{\lambda_k}}$ sequence $\{u_{k,n}\}$ such that

$$|u_{k,n}|_r < \frac{1}{k}, \text{ definitely.}$$

Then, by a diagonalization argument, for any $k \geq 1$, we can find an increasing sequence $\{n_k\}$ in $\mathbb{N}$ and $u_{n_k} \in X_{\lambda_{n_k}}$ such that

$$J_{\lambda_{n_k}}(u_{n_k}) = c_{\lambda_{n_k}} + o_k(1),\ (1 + \|u_n\|_{L_{\lambda_{n_k}}})\|J'_{\lambda_{n_k}}(u_n)\|_{X^{-1}_{\lambda_{n_k}}} \to 0,\ |u_{n_k}|_r = o_k(1),$$

where $o_k(1)$ is a positive quantity which goes to zero as $k \to +\infty$. Then, we are able to arrive at a same contradiction in the Step I with Lemma 4.1, again. The proof is completed. □

Lemma 4.4. *If $\mu \in (\mu_{j_0}, \mu_{j_0+1})$ and $(g_1)-(g_4)$ hold. Assume $\lambda > \Lambda_3$ and $\{u_n\} \subset X_\lambda$ is a $(C)_{c_\lambda}$ sequence, then there is a constant $\Lambda^* > \Lambda_3$ such that J_λ satisfies the $(C)_{c_\lambda}$ condition for any $\lambda > \Lambda^*$.*

Proof. Let $\{u_n\}$ be a $(C)_{c_\lambda}$ sequence of J_λ, then $\{\|u_n\|_{X_\lambda}\}$ is uniformly bounded by Lemma 4.2 for each $\lambda > \Lambda_3$. Passing to a subsequence if necessary, there exists a $u \in X_\lambda$ such that $u_n \rightharpoonup u$ in X_λ, $u_n \to u$ in $L^p_{\text{loc}}(\mathbb{R}^3)$ with $p \in [2,6)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. To show the proof clearly, we shall split it into several steps:

Step 1: $J'_\lambda(u) = 0$ and $J_\lambda(u) \geq 0$.

To show $J'_\lambda(u)=0$, since $C_0^\infty(\mathbb{R}^3)$ is dense in X_λ, then it suffices to exhibit that $J'_\lambda(u)(\varphi)=0$ for every $\varphi\in C_0^\infty(\mathbb{R}^3)$. Thanks to Lemma 2.5-(iii), it is a direct conclusion. Because u is a critical point of J_λ, according to (g_3), one has that

$$J_\lambda(u)=J_\lambda(u)-\frac{1}{2}J'_\lambda(u)(u)\geq 0.$$

Step 2: Define $v_n\triangleq u_n-u$, then there is a $\Lambda^*>\Lambda_3$ such that $v_n\to 0$ in $L^q(\mathbb{R}^3)$ for all $q\in(2,6)$ along a subsequence as $n\to\infty$ when $\lambda>\Lambda^*$.

Actually, since $\{v_n\}$ is uniformly bounded in $n\in\mathbb{N}$ for all $\lambda>\Lambda_3$, then we have one of the following two possibilities for some $r>0$:

$$\begin{cases}\text{(i)}\ \lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^3}\displaystyle\int_{B_r(y)}|v_n|^2dx>0,\\ \text{(ii)}\ \lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^3}\displaystyle\int_{B_r(y)}|v_n|^2dx=0.\end{cases}$$

As a consequence, the conclusion would be clear if we could demonstrate that the case (i) cannot occur for sufficiently large $\lambda>0$. Now, we suppose, by contradiction, that (i) was true. Proceeding as the very similar way in Lemma 4.3, there is a constant $\hat{\delta}>0$ independent of $\lambda>\Lambda_3$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|v_n|^2dx\geq\hat{\delta}$$

for some $r>0$. Since $\{u_n\}$ is uniformly bounded in X_λ, without loss of generality, we can assume that $\lim\limits_{n\to\infty}\|u_n\|^2_{X_\lambda}\leq\Theta$ for some $\Theta\in(0,+\infty)$. Clearly, there holds $\lim\limits_{n\to\infty}\|v_n\|^2_{X_\lambda}\leq 4\Theta$. Recalling $v_n\to 0$ in $L^q_{\text{loc}}(\mathbb{R}^3)$ with $q\in(2,6)$ and $|\mathcal{A}_\rho|\to 0$ as $\rho\to+\infty$ by (2.2), where $\mathcal{A}_\rho\triangleq\{x\in\mathbb{R}^3\backslash B_\rho(0):V(x)<M_0\}$, we can determine a sufficiently large but fixed $\rho>0$ to satisfy

$$\limsup_{n\to\infty}\int_{B_\rho(0)}|v_n|^2dx<\frac{\hat{\delta}}{4}\tag{4.13}$$

and

$$|\mathcal{A}_\rho|<\left(\frac{\hat{\delta}}{16\Theta S_q^2}\right)^{\frac{q}{q-2}},\tag{4.14}$$

where $S_q>0$ comes from (2.6). Combining (2.6) and (4.14), one sees that

$$\limsup_{n\to\infty}\int_{\mathcal{A}_\rho}|v_n|^2dx\leq\limsup_{n\to\infty}\left(\int_{\mathcal{A}_R}|v_n|^qdx\right)^{\frac{2}{q}}|\mathcal{A}_\rho|^{\frac{q-2}{q}}\leq 4\Theta S_q^2|\mathcal{A}_\rho|^{\frac{q-2}{q}}<\frac{\hat{\delta}}{4}.\tag{4.15}$$

Let us choose $\Lambda^*=\max\left\{1,\Lambda_3,\frac{16\Theta}{\hat{\delta}M_0}\right\}$, then for all $\lambda>\hat{\Lambda}$, we reach

$$\limsup_{n\to\infty}\int_{\mathcal{B}_\rho}|v_n|^2dx\leq\limsup_{n\to\infty}\frac{1}{\lambda M_0}\int_{\mathcal{B}_\rho}[\lambda V(x)-\mu]\,|v_n|^2dx\leq\frac{4\Theta}{\lambda M_0}<\frac{\hat{\delta}}{4},\tag{4.16}$$

where $\mathcal{B}_\rho\triangleq\{x\in\mathbb{R}^3\backslash B_\rho(0):V(x)\geq M_0\}$. We gather (4.13), (4.15) and (4.16) to derive

$$\begin{aligned}\hat{\delta}&\leq\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|v_n|^2dx\leq\limsup_{n\to\infty}\int_{\mathbb{R}^3}|v_n|^2dx\\&=\limsup_{n\to\infty}\left(\int_{\mathbb{R}^3\backslash B_\rho(0)}|v_n|^2dx+\int_{B_\rho(0)}|v_n|^2dx\right)\leq\frac{3\hat{\delta}}{4}\end{aligned}$$

which is impossible. The proof of this step is done.

Step 3: Passing to a subsequence if necessary, $u_n \to u$ in X_λ as $n \to \infty$. Since $v_n \triangleq u_n - u$, by Lemma 2.5-(iii) and the Brézis-Lieb lemma, one has

$$J_\lambda(v_n) = J_\lambda(u_n) - J_\lambda(u) + o_n(1) \text{ and } J'_\lambda(v_n) = J'_\lambda(u_n) + o_n(1). \tag{4.17}$$

According to Step 2, we take advantage of (3.1) to deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} g(v_n)v_n dx = 0 \text{ and } \lim_{n\to\infty} \int_{\mathbb{R}^3} G(v_n)dx = 0.$$

A similar argument in (4.9) provides us that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} V_\lambda^-(x)|v_n|^2 dx = 0.$$

The above two formulas together with Lemma 2.5-(iii) and the Brézis-Lieb lemma indicate that

$$o_n(1) = J'_\lambda(u_n)(u_n - u) - J'_\lambda(u)(u_n - u) = \|v_n\|_{X_\lambda}^2 - 10N(v_n) - |v_n|_6^6.$$

If $\|v_n\|_{X_\lambda} \to \bar{l} > 0$, proceeding as the very similar calculations in Step I in the proof of Lemma 4.3, we are able to apply the Step 1 and (4.17) to get

$$c_\lambda \geq c_\lambda - J_\lambda(u) = \lim_{n\to\infty} J_\lambda(v_n) \geq \frac{13 - \sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{2}} S^{\frac{3}{2}}, \tag{4.18}$$

which is absurd because of Lemma 4.1. Therefore, $\bar{l} = 0$ and it is the desired result. The proof is completed. □

Now, we are in a position to show the detailed proofs of the main results in this paper.

4.1. **Proof of Theorem 1.1.** First of all, due to the discussions in Section 3, there is a sequence $\{u_n\} \subset X_\lambda$ satisfying (4.2) for all $\lambda > \Lambda_3$. Then, we are derived from Lemma 4.2 that $\{u_n\} \subset X_\lambda$ is uniformly bounded in X_λ for all $\lambda > \Lambda_3$. So, passing to a subsequence of necessary, there is a $u \in X_\lambda$ such that $u_n \rightharpoonup u$ in X_λ, $u_n \to u$ in $L^p_{\text{loc}}(\mathbb{R}^3)$ for all $2 < p < 6$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Recalling Lemma 4.4, there is a $\Lambda^* > \Lambda_3$ such that $u_n \to u$ in X_λ for all $\lambda > \Lambda^*$. As a consequence, we have that $J'_\lambda(u) = 0$ and $J_\lambda(u) = c_\lambda$, with c_λ given by (4.1). As to the positivity of u, it is trivial and we omit it here. The proof is completed.

4.2. **Proof of Theorem 1.2.** If in addition (g_5) is supposed, we conclude that (A_5) holds true by Lemma 3.5 and so $c_\lambda \leq d_\lambda$. The proof would be done if $c_\lambda \geq J_\lambda(u)$. Actually, since $u_n \to u$ in X_λ for all $\lambda > \Lambda^*$, we see that $\|u_n - u\|_{L_\lambda} \to 0$ by Lemma 2.3. Thereby, for $u_n = P_\lambda^+ u_n + P_\lambda^- u_n$ with $P_\lambda^+ u_n \in X_\lambda^+$ and $P_\lambda^- u_n \in X_\lambda^-$, it holds that $\|P_\lambda^+ u_n - P_\lambda^+ u\|_{L_\lambda} \to 0$ and $\|P_\lambda^- u_n - P_\lambda^- u\|_{L_\lambda} \to 0$. As a consequence, combining (g_3) and the Fatou's lemma, it holds that

$$\begin{aligned}
c_\lambda &= \liminf_{n\to\infty} \left[J_\lambda(u_n) - \frac{1}{\eta} J'_\lambda(u_n)(u_n)\right] \\
&= \liminf_{n\to\infty} \left\{\frac{\eta - 2}{2\eta}(\|u_n^+\|_{L_\lambda}^2 - \|u_n^-\|_{L_\lambda}^2) + 10\left(\frac{1}{\eta} - \frac{1}{10}\right)N(u_n) \right. \\
&\quad \left. + \frac{1}{\eta}\int_{\mathbb{R}^3} \big[g(u_n)u_n - \eta G(u_n)\big]dx + \left(\frac{1}{\eta} - \frac{1}{6}\right)|u_n|_6^6\right\} \\
&\geq \frac{\eta - 2}{2\eta}(\|u^+\|_{L_\lambda}^2 - \|u^-\|_{L_\lambda}^2) + 10\left(\frac{1}{\eta} - \frac{1}{10}\right)N(u) + \frac{1}{\eta}\int_{\mathbb{R}^3} \big[g(u)u - \eta G(u)\big]dx \\
&\quad + \left(\frac{1}{\eta} - \frac{1}{6}\right)|u|_6^6
\end{aligned}$$

$$\geq J_\lambda(u) - \frac{1}{\eta}J'_\lambda(u)(u) = J_\lambda(u)$$

finishing the proof.

4.3. **Proof of Theorem 1.6.** Suppose $\{u_{\lambda_n}\} \subset X_{\lambda_n}$ to be a sequence satisfying $J'_{\lambda_n}(u_{\lambda_n}) = 0$ and $J_{\lambda_n}(u_{\lambda_n}) = \bar{c}_{\lambda_n}$. Passing to a subsequence if necessary, we are derived from Lemmas 3.3 and 4.1 that

$$0 < \varrho \leq \lim_{\lambda_n\to\infty} J_{\lambda_n}(u_{\lambda_n}) = \lim_{\lambda_n\to\infty} \bar{c}_{\lambda_n} \triangleq \bar{c}_\Omega < c^*. \tag{4.19}$$

Adopting (4.19), we can argue as the proof of Lemma 4.2 to prove that $\{u_{\lambda_n}\}$ is uniformly bounded in X_{λ_n}. Up to a subsequence if necessary, there exists a $u \in H^1(\mathbb{R}^3)$ such that $u_{\lambda_n} \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, $u_{\lambda_n} \to u$ in $L^p_{\text{loc}}(\mathbb{R}^3)$ with $2 \leq p < 6$ and $u_{\lambda_n} \to u$ a.e. in $\mathbb{R}^3$. Then, we are ready to prove $u \equiv 0$ in $\Omega^c = \{x : x \in \mathbb{R}^3\backslash\Omega\}$. Otherwise, there is a compact subset $\Sigma_u \subset \Omega^c$ with $\text{dist}(\Sigma_u, \partial\Omega^c) > 0$ such that $u \neq 0$ on Σ_u and by Fatou's lemma

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} u_n^2 dx \geq \int_{\Sigma_u} u^2 dx > 0. \tag{4.20}$$

Moreover, there exists $\varepsilon_0 > 0$ such that $V(x) \geq \varepsilon_0$ for any $x \in \Sigma_u$ by the assumptions (V_1) and (V_6). Combining (g_3), (4.19) and(4.20),

$$\begin{aligned}
\bar{c}_\Omega &\geq \liminf_{n\to\infty} J_{\lambda_n}(u_{\lambda_n}) = \liminf_{n\to\infty}\left[J_{\lambda_n}(u_{\lambda_n}) - \frac{1}{\eta}J'_{\lambda_n}(u_{\lambda_n})(u_{\lambda_n})\right] \\
&\geq \liminf_{n\to\infty}\left[\frac{\eta-2}{2\eta}\int_{\mathbb{R}^3}[\lambda_n V(x) - \mu]u_{\lambda_n}^2 dx - \frac{\eta-10}{\eta}N(u_{\lambda_n}) - \frac{\eta-6}{6\eta}|u_{\lambda_n}|_6^6\right] \\
&\geq \left(\frac{(\theta-2)\varepsilon_0}{2\theta}\int_{\Sigma_u} u^2 dx\right)\liminf_{n\to\infty}\lambda_n - C = +\infty,
\end{aligned}$$

a contradiction because $\{N(u_{\lambda_n})\}$ and $|u_{\lambda_n}|_6^6$ are uniformly bounded in X_{λ_n}. Hence, by the fact that $\partial\Omega$ is smooth, one can conclude that $u \in H_0^1(\Omega)$.

For the above $u \in H_0^1(\Omega)$, we denote by $\tilde{u} \in H^1(\mathbb{R}^3)$ its trivial extension, namely

$$\tilde{u} \triangleq \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \Omega^c = \{x : x \in \mathbb{R}^3\backslash\Omega\}. \end{cases}$$

We now define $J_0|_\Omega : H_0^1(\Omega) \to \mathbb{R}$ as

$$J_0|_\Omega(v) = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 - \mu v^2\right)dx - \frac{1}{10}\int_\Omega\int_\Omega \frac{|v(x)|^5|v(y)|^5}{|x-y|}dxdy - \int_\Omega G(v)dx - \frac{1}{6}\int_\Omega |v|^6 dx.$$

Let us claim that $J_0|_\Omega(u) = 0$. In fact, by using (2.22), we have

$$\begin{aligned}
0 = J'_{\lambda_n}(u_{\lambda_n})(\varphi) &= \int_{\mathbb{R}^3}[\nabla u_{\lambda_n}\nabla\varphi + V_{\lambda_n}(x)u_{\lambda_n}\varphi]dx - 10N'(u_n)[\varphi] - \int_{\mathbb{R}^3}[g(u_n) + |u_n|^4 u_n]\varphi dx \\
\to \int_\Omega[\nabla u\nabla\varphi - \mu u\varphi]dx &- 10N'_\Omega(u)[\varphi] - \int_\Omega[g(u) + |u|^4 u]\varphi dx = J'_0|_\Omega(u)(u)(\varphi), \forall\varphi \in C_0^\infty(\Omega),
\end{aligned}$$

which shows that $J'_0|_\Omega(u) = 0$. Next, we define $\bar{u}_n \triangleq u_n - u \rightharpoonup 0$, then

$$\begin{aligned}
\int_{\mathbb{R}^3} V_{\lambda_n}(x)|u_n|^2 dx &= \int_{\mathbb{R}^3} V_{\lambda_n}(x)|\bar{u}_n|^2 dx + 2\int_{\mathbb{R}^3} V_{\lambda_n}(x)\bar{u}_n u dx + \int_{\mathbb{R}^3} V_{\lambda_n}(x)|u|^2 dx \\
&= \int_{\mathbb{R}^3} V_{\lambda_n}(x)|\bar{u}_n|^2 dx - 2\mu\int_\Omega \bar{u}_n u dx - \mu\int_\Omega |u|^2 dx \\
&= \int_{\mathbb{R}^3} V_{\lambda_n}(x)|\bar{u}_n|^2 dx - \mu\int_\Omega |u|^2 dx + o_n(1)
\end{aligned}$$

which together with the Brézis-Lieb lemma and (2.21) gives that

$$J_{\lambda_n}(u_n) = J_0|_\Omega(u) + J_{\lambda_n}(\bar{u}_n) + o(1) \text{ and } J'_{\lambda_n}(u_n)(u_n) = J'_{\lambda_n}(\bar{u}_n)(\bar{u}_n) + J'_0|_\Omega(u)(u) + o(1).$$

On the other hand, since $\{\bar{u}_n\}$ is uniformly bounded in $H^1(\mathbb{R}^3)$, by (2.2), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\bar{u}_n|^2dx = \lim_{n\to\infty}\int_{\mathbb{R}^3\setminus B_R(0)}|\bar{u}_n|^2dx \le \lim_{n\to\infty}\frac{1}{\lambda_n M_0-\mu}\int_{\mathbb{R}^3\setminus B_R(0)}V_{\lambda_n}(x)|\bar{u}_n|^2dx$$

$$\le \lim_{n\to\infty}\frac{1}{\lambda_n M_0-\mu}\int_{\mathbb{R}^3}V^+_{\lambda_n}(x)|\bar{u}_n|^2dx \le \lim_{n\to\infty}\frac{\|\bar{u}_n\|^2_{X_{\lambda_n}}}{\lambda_n M_0-\mu} \le \lim_{n\to\infty}\frac{C}{\lambda_n M_0-\mu} = 0,$$

which indicates that $\bar{u}_n \to 0$ in $L^p(\mathbb{R}^3)$ with $2 \le p < 6$ as $n\to\infty$. Hence, (4.8) and (4.8) hold true for $\{\bar{u}_n\}$. Combining $\lim\limits_{n\to\infty} J'_{\lambda_n}(u_{\lambda_n})(u_{\lambda_n}) = 0$ and $J'_0|_\Omega(u)(u) = 0$, we obtain

$$\bar{c}_\Omega - J_0|_\Omega(u) = \lim_{n\to\infty} J_{\lambda_n}(\bar{u}_{\lambda_n}) = \lim_{n\to\infty}\left(\frac{1}{2}\|\bar{u}_{\lambda_n}\|^2_{X_{\lambda_n}} - N(\bar{u}_{\lambda_n}) - \frac{1}{6}|\bar{u}_{\lambda_n}|_6^6\right)$$

and

$$0 = \lim_{n\to\infty} J'_{\lambda_n}(\bar{u}_{\lambda_n})\bar{u}_{\lambda_n} = \lim_{n\to\infty}\left(\|\bar{u}_n\|^2_{X_{\lambda_n}} - 10N(\bar{u}_{\lambda_n}) - |\bar{u}_{\lambda_n}|_6^6\right).$$

Due to (g_3) again, one easily sees that $J_0|_\Omega(u) = J_0|_\Omega(u) - \frac{1}{2}J'_0|_\Omega(u)(u) \ge 0$. Repeating the arguments explored in Lemma 4.4, we immediately derive that $\bar{u}_{\lambda_n} \to u$ in $H^1(\mathbb{R}^3)$.

If in addition (g_5) is supposed, we can further conclude that $\bar{c}_\Omega = c_\Omega$, where

$$c_\Omega \triangleq \inf_{w\in\mathcal{M}_\Omega} J_0|_\Omega(w)$$

with $\mathcal{M}_\Omega = \{u \in H^1_0(\Omega)\setminus X_0^- : J'_0|_\Omega(u)(u) = 0 \text{ and } J'_0|_\Omega(u)(v) = 0 \text{ for any } v \in X_0^-\}$. Since the proof would be easier, we omit the details. The proof is completed.

Acknowledgements. Both the authors are very grateful to the anonymous referee for his/her careful reading and valuable comments which improved this manuscript greatly.

References

[1] C.O. Alves, G.F. Germano, Ground state solution for a class of indefinite variational problems with critical growth, *J. Differential Equations,* **265** (2018), 444–437. 5

[2] A. Azzollini, P. d'Avenia, On a system involving a critically growing nonlinearity, *J. Math. Anal. Appl.,* **387** (2012), 433–438. 1

[3] A. Azzollini, P. d'Avenia, V. Luisi, Generalized Schrödinger-Poisson type systems, *Commun. Pure Appl. Anal.,* **12** (2013), 867–879. 1

[4] A. Azzollini, P. d'Avenia, G. Vaira, Generalized Schrödinger-Newton system in dimension $N \ge 3$: Critical case, *J. Math. Anal. Appl.,* **449** (2017), 531–552. 2

[5] T. Bartsch, Z.-Q. Wang, Existence and multiplicity results for superlinear elliptic problems on $\mathbb{R}^N$, *Commun. Partial Differential Equations,* **20** (1995), 1725–1741. 2, 3

[6] T. Bartsch, Z.-Q. Wang, Multiple positive solutions for a nonlinear Schrödinger equation, *Z. Angew. Math. Phys.,* **51** (2000), 366–384. 3

[7] T. Bartsch, A. Pankov, Z.-Q. Wang, Nonlinear Schrödinger equations with steep potential well, *Commun. Contemp. Math.,* **3** (2001), 549–569. 2, 3

[8] T. Bartsch, Y. Ding, Deformation theorems on non-metrizable vector spaces and applications to critical point theory, *Math. Nachr.,* **279** (2006), 1267–1288. 5

[9] J. Bellazzini, L. Jeanjean, On dipolar quantum gases in the unstable regime, *SIAM J. Math. Anal.,* **48** (2017), 2028–2058. 3

[10] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods. Nonlinear Anal.,* **11** (1998), 283–293. 1

[11] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Commun. Pure Appl. Math.,* **36** (1983), 437–477. 16

[12] R. Carles, P. Markowich, C. Sparber, On the Gross-Pitaevskii equation for trapped dipolar quantum gases, *Nonlinearity,* **21** (2008), 2569–2590. 3

[13] J. Chabrowski, A. Szulkin, On a semilinear Schrödinger equation with critical Sobolev exponent, *Proc. Amer. Math. Soc.,* **130** (2001), 85–93. 17

[14] M. Clapp, Y. Ding, Positive solutions of a Schrödinger equation with critical nonlinearity, *Z. Angew. Math. Phys.,* **55** (2004), 592–605. 3

[15] T. d'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. A,* **134** (2004), 893–906. 1

[16] T. d'Aprile, D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, *Adv. Nonlinear Stud.,* **4** (2004), 307–322. 1

[17] M. Dong, G. Lu, Best constants and existence of maximizers for weighted Trudinger-Moser inequalities, *Calc. Var. Partical Differential Equations,* (2016), **55**:88.

[18] X. Feng, Ground state solution for a class of Schrödinger-Poisson-type systems with partial potential, *Z. Angew. Math. Phys.,* (2020), **71**:37. 2, 17

[19] X. Feng, Existence and concentration of ground state solutions for doubly critical Schrödinger-Poisson-type systems, *Z. Angew. Math. Phys.,* (2020), **71**:157. 2, 5

[20] L. Huang, E. Rocha, J. Chen, Two positive solutions of a class of Schrödinger-Poisson system with indefinite nonlinearity, *J. Differential Equations,* **255** (2013), 2463–2483. 1

[21] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A,* **129** (1999), 787–809. 2

[22] L. Jeanjean, S. Le Coz, An existence and stability result for standing waves of nonlinear Schrödinger equations, *Adv. Differential Equations,* **11** (2006), 813–840. 2

[23] Y. Jiang, H. Zhou, Schrödinger-Poisson system with steep potential well, *J. Differential Equations,* **251** (2011), 582–608. 1

[24] W. Kryszewski, A. Szulkin, Generalized linking theorem with an application to a semilinear Schrödinger equation, *Adv. Differential Equations,* **3** (1998), 441–472. 5

[25] C. Lei, G. Liu, Multiple positive solutions for a Schrödinger-Newton system with sign-changing potential, *Comput. Math. Appl.,* **77** (2019), 631–640. 2

[26] F. Li, Y. Li, J. Shi, Existence of positive solutions to Schrödinger-Poisson type systems with critical exponent, *Commun. Contemp. Math.,* **10** (2014), 1450036. 2, 11

[27] F. Li, Y. Li, J. Shi, Existence and multiplicity of positive solutions to Schrödinger-Poisson type systems with critical nonlocal term, *Calc. Var. Partial Differential Equations,* **56** (2017), Paper No. 134. 2, 11

[28] G. Li, A. Szulkin, An asymptotically periodic Schrödinger equation with indefinite linear part, *Commun. Contemp. Math.,* **4** (2002), 763–776. 5

[29] Y.Y. Li, G.D. Li, C.L. Tang, Existence and concentration of ground state solutions for Choquard equations involving critical growth and steep potential well, *Nonlinear Anal.,* **200** (2020), 111997. 3, 5, 7

[30] E.H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997. 10, 11, 16, 19

[31] P.-L. Lions, Solutions of Hartree-Fock equations for Coulomb systems, *Commun. Math. Phys.,* **109** (1987), 33–97. 1

[32] H. Liu, Positive solutions of an asymptotically periodic Schrödinger-Poisson system with critical exponent, *Nonlinear Anal.,* **32** (2016), 198–212. 2

[33] J. Mederski, Ground states of a system of nonlinear Schrödinger equations with periodic potentials, *Comm. Partial Differential Equations,* **41** (2016), 1426–1440. 5

[34] J. Mederski, J. Schino, A. Szulkin, Multiple solutions to a nonlinear curl-curl problem in $\mathbb{R}^3$, *Arch. Rational Mech. Anal.,* **236** (2020), 253–288. 15

[35] M. Niu, Z. Tang, L. Wang, Least energy solutions for indefinite biharmonic problems via modified Nehari-Pankov manifold, *Commun. Contemp. Math.,* **20** (2018), 1750047. 3, 5, 6, 7, 8

[36] A. Pankov, Periodic nonlinear Schrödinger equation with application to photonic crystals, *Milan J. Math.,* **73** (2005), 259–287. 5

[37] A. Pomponio, L. Shen, X. Zeng, Y. Zhang, Generalized Chern-Simons-Schrödinger system with sign-changing steep potential well: critical and subcritical exponential case, *J. Geom. Anal.,* **33** (2023), no. 6, Paper No. 185, 34 pp. 3, 18

[38] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.,* **237** (2006), 655–674. 1

[39] L. Shen, M. Squassina, Planar Schrödinger-Poisson system with steep potential well: supercritical exponential case, arXiv:2401.10663. 18

[40] L. Shen, M. Squassina, X. Zeng, Infinitely many solutions for a class of fractional Schrödinger equations coupled with neutral scalar field, *Discrete Contin. Dyn. Syst. S,* (2024), https://doi.org/10.3934/dcdss.2024084. 14

[41] J. Sun, S. Ma, Ground state solutions for some Schrödinger-Poisson systems with periodic potentials, *J. Differential Equations,* **260** (2016), 2119–2149. 1

[42] A. Szulkin, T. Weth, Ground state solutions for some indefinite variational problems, *J. Funct. Anal.,* **257** (2009), 3802–3822. 5

[43] Z. Tang, Least energy solutions for semilinear Schrödinger equations involving critical growth and indefinite potentials, *Commun. Pure Appl. Anal.,* **13** (2014), 237–248. 3, 5

[44] Z. Wang, H. Zhou, Sign-changing solutions for the nonlinear Schrödinger-Poisson system in $\mathbb{R}^3$, *Calc. Var. Partial Differential Equations,* **52** (2015), 927–943. 1

[45] M. Willem, W. Zou, On a Schrödinger equation with periodic potential and spectrum point zero, *Indiana Univ. Math. J.,* **52** (2003), 109–132. 5

[46] J. Zhang, W. Zou, Existence and concentrate behavior of Schrödinger equations with critical exponential growth in $\mathbb{R}^N$, *Topol. Methods Nonlinear Anal.,* **48** (2016), 345–370. 3

[47] L. Zhao, H. Liu, F. Zhao, Existence and concentration of solutions for Schrödinger-Poisson equations with steep well potential, *J. Differential Equations,* **255** (2013), 1–23. 1

[48] Y. Zhou, J. Lei, Y. Wang, Z. Xiong, Positive solutions of a Kirchhoff-Schrödinger-Newton system with critical nonlocal term, *Electron. J. Qual. Theory Differ. Equ.,* (2022), Paper No. 50, 12 pp. 2

Liejun Shen,
Department of Mathematics, Zhejiang Normal University,
Jinhua, Zhejiang, 321004, People's Republic of China
Email address: ljshen@zjnu.edu.cn.

Marco Squassina,
Dipartimento di Matematica e Fisica
Università Cattolica del Sacro Cuore,
Via della Garzetta 48, 25133, Brescia, Italy
Email address: marco.squassina@unicatt.it.