Manuscript submitted to AIMS Journal

doi:10.3934/xx.xxxxxxx

INFINITELY MANY SOLUTIONS FOR A CLASS OF FRACTIONAL SCHRÖDINGER EQUATIONS COUPLED WITH NEUTRAL SCALAR FIELD

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(Communicated by Handling Editor)

Abstract. We study the fractional Schrödinger equations coupled with a neutral scalar field

$$\begin{cases} (-\Delta)^s u + V(x)u = K(x)\phi u + g(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ (I-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $(-\Delta)^s$ and $(I-\Delta)^t$ denote the fractional Laplacian and Bessel operators with $\frac{3}{4} < s < 1$ and $0 < t < 1$, respectively. Under some suitable assumptions for the external potentials V, K and g, given $q \in (1,2)\cup(2,2_s^*)$ with $2_s^* := \frac{6}{3-2s}$, with the help of an improved Fountain theorem dealing with a class of strongly indefinite variational problems approached by Gu-Zhou [Adv. Nonlinear Stud., **17** (2017), 727–738], we show that the system admits infinitely many nontrivial solutions.

1. Introduction and main results.

1.1. General overview.

In the present paper, we are concerned with the following fractional elliptic system

$$\begin{cases} (-\Delta)^s u + V(x)u = K(x)\phi u + g(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ (I-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $(-\Delta)^s$ and $(I-\Delta)^t$ denote the classic fractional Laplacian and Bessel operators with $\frac{3}{4} < s < 1$ and $0 < t < 1$, respectively.

In light of its relevance in physics, the following nonlinear fractional Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = (-\Delta)^s \Psi + (V(x)+E)\Psi - f(x,\Psi) \quad \text{for all } x \in \mathbb{R}^N, \tag{2}$$

where $N > 2s$ with $s \in (0,1)$, $E \in \mathbb{R}$, V and f are continuous functions, has been received more and more attentions in recent years by a great many mathematicians.

2020 *Mathematics Subject Classification.* Primary: 35J60, 35Q55; Secondary: 53C35.

Key words and phrases. Fractional Schrödinger equation, Strongly indefinite, Bessel operator, Fountain theorem, Infinitely many soultions, Variational methods.

L. Shen is partially supported by NSFC (Grant No. 12201565). M. Squassina is member of Gruppo Nazionale per l'Analisi Matematica, la Probabilita e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). X. Zeng is supported by CSC and NSFC (Grant Nos. 12322106, 12171379, 12271417).

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Generally, when they are searching for a particular type of the solutions of Eq. (2), the so called *standing wave solution*, which carries a form of the type

$$\Psi(z,t) = \exp(-iEt)u(z),$$

revealing that u acts as a solution of the fractional elliptic equation

$$\begin{cases} (-\Delta)^s u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N),\ u > 0, & \text{on } \mathbb{R}^N. \end{cases} \tag{3}$$

In the local case, that is, the general semilinear elliptic equations (3) with $s = 1$, have been extensively considered, for example, we shall refer the reader to [20, 26, 3] and their references therein.

In the nonlocal case, namely when $s \in (0,1)$, the corresponding results for Eq. (3) do never seem to be as fruitful as the local ones. This occurs maybe because the techniques and arguments developed for local case cannot be adapted immediately, c.f. [23]. In order to introduce some results clearly for Eq. (2), we recall that, for any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy < \infty \right\},$$

equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}.$$

The fractional Laplacian, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$, with sufficient decay, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transformation which is

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x}\phi(x)dx := \widehat{\phi}(\xi),$$

for functions ϕ belonging to Schwartz class. In reality, according to [6, Lemma 3.2] the fractional Laplacian operator can be equivalently represented as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(N,s)\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}dy, \quad \forall x \in \mathbb{R}^N,$$

where

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{(1-\cos\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N.$$

Also, due to [6, Propostion 3.4, Propostion 3.6], it holds that

$$\|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s}|\widehat{u}|^2 d\xi = \frac{1}{2}C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy,$$

for all $u \in H^s(\mathbb{R}^N)$. Moreover, we usually identify these two quantities by omitting the normalization constant $\frac{1}{2}C(N,s)$ just for simplicity. The homogeneous fractiotal Sobolev space $D^{s,2}(\mathbb{R}^N)$ is defined by

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_s}(\mathbb{R}^N) : |(-\Delta)^{\frac{s}{2}} u| \in L^2(\mathbb{R}^N) \right\}$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{s,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^{\frac{1}{2}}.$$

For $N > 2s$, from [6, Theorem 6.5], we further know that, for any $p \in [2, 2_s^*]$, there exists a constant $C_p > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^N)} \le C_p \|u\|_{H^s(\mathbb{R}^N)}, \text{ for all } u \in H^s(\mathbb{R}^N).$$

Besides, the imbedding $H^s(\mathbb{R}^N) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N)$ is compact for all $1 \le p < 2_s^*$.

Actually, problem (3) was initially proposed by the author in [16, 17] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. According to the celebrated paper [5], Eq. (2) and its variants have been widely contemplated by many authors, specially on the existence of ground state solutions, positive solutions, sign-changing solutions and multiplicity of standing wave solutions, see e.g. [10, 4, 2, 13] and their references therein.

For $t > 0$, the so-called *Bessel function space* in $\mathbb{R}^3$ is defined by

$$L^{t,2}(\mathbb{R}^3) := \left\{ f \in L^2(\mathbb{R}^3) : f = G_t * h \text{ for some } h \in L^2(\mathbb{R}^3) \right\},$$

where the Bessel convolution kernel is

$$G_t(x) := \frac{1}{(4\pi)^{\frac{t}{2}} \Gamma(\frac{t}{2})} \int_0^\infty \exp\left(-\frac{\pi}{\delta}|x|^2\right) \exp\left(-\frac{\delta}{4\pi}\right) \delta^{\frac{t-5}{2}} d\delta. \tag{4}$$

The operator $(I - \Delta)^{-t} u = G_{2t} * u$ is generally known as Bessel operator of order t and it induces Bessel function space equipped with the norm $\|f\|_{L^{t,2}(\mathbb{R}^3)} = \|h\|_{L^2(\mathbb{R}^3)}$ if $f = G_t * h$. Owing to the view point of Fourier transformation, this same operator can also read

$$G_t = \mathcal{F}^{-1} \circ \left(\left(1 + |\xi|^2\right)^{-\frac{t}{2}} \circ \mathcal{F} \right),$$

so that

$$\|f\|_{L^{t,2}(\mathbb{R}^3)} = \left\| (I - \Delta)^{\frac{t}{2}} f \right\|_{L^2(\mathbb{R}^3)}.$$

We refer the interested reader to [1, 25] and the references therein for more detailed information concerning the Bessel operator and Bessel function space.

It should be mentioned here that authors in [9] introduced the pointwise formula

$$(I - \Delta)^t u(x) = c_s \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{\frac{3+2t}{2}}} \mathcal{K}_{\frac{3+2t}{2}}(|x - y|)\, dy + u(x)$$

for $u \in C_c^2(\mathbb{R}^3)$, where K_ν is the modified Bessel function of the second kind with order ν. Nevertheless, a closed formula for K_ν remains unknown, see e.g. [9, Remark 7.3].

Very recently, Felmer-Vergara [11] investigated the existence of positive solutions for fractional equations involving a Bessel operator

$$(I - \Delta)^s u + V(x) u = f(x, u), \; x \in \mathbb{R}^N. \tag{5}$$

Subsequently, under different technical assumptions on V and f owing to variational methods, there are some bibliographies in the study of (5), see [21, 14, 22] and their references therein for example.

In [24], the author contemplated the multiplicity and concentration of nontrivial solutions for the following fractional Schrödinger-Poisson system involving a Bessel operator

$$\begin{cases} (I-\Delta)^s u + \lambda V(x)u + \phi u = f(x,u) + g(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where λV represents as a deepening potential with $\lambda > 0$, $f \in C^0(\mathbb{R}^3 \times \mathbb{R})$ satisfies some suitable conditions, $g > 0$ is a weight function and $1 < q < 2$.

Motivated by the relevance of problem (3) and the mathematical point of view, we tend to consider a class of Schrödinger equations coupled with a Bessel operator. More precisely, we shall establish the existence of infinitely many nontrivial solutions for the system (1) under some suitable assumptions on V, K and g. Up to the best knowledge of us, it seems the first time to dispose of such type of problems. What's more, we anticipate that the results in this paper would prompt the further studies on (fractional) Schrödinger-Poisson systems.

1.2. **Assumptions and main results.** To arrive at the aim mentioned above, we shall suppose that

(H_1) $V \in C^0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and 0 lies in a spectrum gap of the operator $(-\Delta)^s + V$;

(H_2) $K \in L^{\frac{3}{4s-3}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $K(x) > 0$ a.e. for $x \in \mathbb{R}^3$;

Now, the main results in this paper can be stated as follows.

Theorem 1.1. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$ and suppose that $(H_1) - (H_2)$. If in addition*

(H_3) $g \in L^{q_0}(\mathbb{R}^3)$ with $q_0 = \frac{6}{6-q(3-2s)}$ and $1 < q < \frac{4}{3}$,

then system (1) possesses infinitely many solutions $\{u_n, \phi_{u_n}\} \subset H^s(\mathbb{R}^3) \times L^{t,2}(\mathbb{R}^3)$ satisfying $\lim\limits_{n\to\infty} \|u_n\|_{H^s(\mathbb{R}^3)} = +\infty$.

Theorem 1.2. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$ and suppose that $(H_1) - (H_2)$. If in addition*

(H_4) $g \in L^{q_0}(\mathbb{R}^3)$ with $g(x) < 0$ a.e. $x \in \mathbb{R}^3$, $q_0 = \frac{6}{6-q(3-2s)}$ and $\frac{4}{3} \le q < 2$;

then system (1) possesses infinitely many solutions $\{u_n, \phi_{u_n}\} \subset H^s(\mathbb{R}^3) \times L^{t,2}(\mathbb{R}^3)$ satisfying $\lim\limits_{n\to\infty} \|u_n\|_{H^s(\mathbb{R}^3)} = +\infty$.

Theorem 1.3. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$ and suppose that $(H_1) - (H_2)$. If in addition*

(H_5) $g \in L^{q_0}(\mathbb{R}^3)$ with $g(x) > 0$ a.e. $x \in \mathbb{R}^3$, $q_0 = \frac{6}{6-q(3-2s)}$ and $2 < q < 2_s^*$.

then system (1) possesses infinitely many solutions $\{u_n, \phi_{u_n}\} \subset H^s(\mathbb{R}^3) \times L^{t,2}(\mathbb{R}^3)$ satisfying $\lim\limits_{n\to\infty} \|u_n\|_{H^s(\mathbb{R}^3)} = +\infty$.

Remark 1.4. Although the result in Theorem 1.1 is similar to [12, Theorem 1.3], as far as we are concerned, one can never prove Theorem 1.1 by simply repeating the arguments exploited in the cited paper caused by the appearance of Bessel operator, see Lemma 2.2 below for example. On the other hand, with the help of some new analytic skills, the results are even new for the counterparts of [12].

Remark 1.5. It is worthy pointing out that we do not conclude whether the results in Theorems 1.2 and 1.3 remain valid when g is sign-changing in (H_4) and (H_5), respectively. Moreover, inspired by [26, 27], it is interesting to consider that problem (1) has a ground state solution. We postpone these two questions in a further work.

Again the results in Theorems 1.1, 1.2 and 1.3 are new up to now. To conclude this section, we sketch our proof. First of all, because the operator $L = (-\Delta)^s + V$ is strongly indefinite, we then follow the idea introduced in [19, 26, 27] to decompose the space $H^s(\mathbb{R}^3)$ suitably. In the meantime, there exist some standard arguments exhibited in Section 2 that allows us to treat the problem (1) by variational methods. Then, we shall depend heavily on a new type of Fountain theorem approached by Gu and Zhou in [12] to derive the existence of infinitely many nontrivial critical points. Finally, we concentrate on verifying the necessary properties of the corresponding variational functional, see Sections 2 and 3 in detail. So, we could derive the proofs successfully. Alternatively, owing to the Bessel operator appearing in (1), there are some unpleasant barriers in the last step and we have to take some careful and deep analysis there.

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2, we provide some preliminary results. Section 3 is devoted to the proofs of Theorems 1.1, 1.2 and 1.3.

Notations. From now on in the present article, otherwise mentioned particularly, we shall adopt the following notations:

- $C, C_1, C_2, \cdots$ denote any positive constant, whose value are not relevant.
- Let $(X, \|\cdot\|_X)$ be a Banach space with dual space $(X^{-1}, \|\cdot\|_{X^{-1}})$, and Φ be a functional on X.
- Let $\|\cdot\|_{L^p(\mathbb{R}^3)}$ denote the usual L^p-norm for any Lebesgue measurable function $u : \mathbb{R}^3 \to \mathbb{R}$, where $p \in [1, \infty]$.
- Palais-Smale sequence at level $c \in \mathbb{R}$ ($(PS)_c$ sequence in short) corresponding to a functional Φ on X means that $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ in X^{-1} as $n \to \infty$, where $\{x_n\} \subset X$.
- If for each $(PS)_c$ sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0$ in X for some $x_0 \in X$, then one says that the functional Φ satisfies the so called $(PS)_c$ condition.
- $o_n(1)$ denotes the real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- " $\to$ " and " $\rightharpoonup$ " stand for the strong and weak convergence in the related function spaces, respectively.

2. **Preliminary Results.** In this section, we introduce some preliminary results. For the potential $V \in C^0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we can define an operator $L := (-\Delta)^s + V$. Thanks to the celebrated results in [8, Theorem 4.26], one sees that L is self-disjoint with domain $\mathcal{D}(L) = H^s(\mathbb{R}^3)$. Assume $|L|$ and $|L|^{1/2}$ are the absolute values of L and the square root of $|L|$, respectively. We denote $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ by the spectral family with respective to L. Setting $U := \mathrm{id} - \mathcal{E}(0) - \mathcal{E}(0^-)$, then by virtue of [7, Theorem IV 3.3], U commutes with L, $|L|$ and $|L|^{1/2}$. Moreover, $L = U|L|$ is the polar decomposition of L. In view of [19, 26, 27], there holds

$$X = \mathcal{D}(|L|^{1/2}),\ Y = \mathcal{E}(0^-)X \text{ and } Z = [\mathrm{id} - \mathcal{E}(0)]X.$$

Via (V), one has $X = Y \bigoplus Z$. Given $u \in X$, then $u = u^+ + u^-$ with

$$u^- = \mathcal{E}(0^-)u := Pu \text{ and } u^+ = [\mathrm{id} - \mathcal{E}(0)]u := Qu.$$

Furthermore, for all $u \in X \cap \mathcal{D}(L)$, one also has that

$$Lu^- = -|L|u^- \text{ and } Lu^+ = |L|u^+. \tag{6}$$

With the above facts in hands, it permits us to introduce an inner product which could induce the norm on X as follows

$$(u,v)_X=(|L|^{1/2}u,|L|^{1/2}v)_{L^2(\mathbb{R}^3)} \text{ and } \|u\|_X=\big\||L|^{1/2}u\big\|_{L^2(\mathbb{R}^3)},$$

where $(\cdot,\cdot)_{L^2(\mathbb{R}^3)}$ stands for the usual inner product of $L^2(\mathbb{R}^3)$. Besides, by (6), we have that

$$\int_{\mathbb{R}^3}[|(-\Delta)^{\frac{s}{2}}u|^2+V(x)|u|^2]dx=\|Qu\|_X^2-\|Pu\|_X^2,\ \forall u\in X. \tag{7}$$

In particular, it holds that

$$\begin{cases}\displaystyle\int_{\mathbb{R}^3}[|\nabla Qu|^2+V(x)|Qu|^2]dx=\|Qu\|_X^2,\\ \displaystyle\int_{\mathbb{R}^3}[|\nabla Pu|^2+V(x)|Pu|^2]dx=-\|Pu\|_X^2.\end{cases}$$

Since $\|\cdot\|_X$ and $\|\cdot\|_{H^s(\mathbb{R}^3)}$ are equivalent by (H_1) (see [15] for example), there exists a constant $S_p>0$ such that

$$\|u\|_{L^p(\mathbb{R}^3)}\le S_p\|u\|_X,\ \forall p\in[2,2_s^*], \tag{8}$$

and by [28], there holds

$$\|u\|_{L^2(\mathbb{R}^3)}\le C_t\|u\|_{L^{t,2}(\mathbb{R}^3)}. \tag{9}$$

Moreover, one knows that X and $L^{t,2}(\mathbb{R}^3)$ can be compactly imbedded into $L^p_{\rm loc}(\mathbb{R}^3)$ with $2\le p<2_s^*:=\frac{6}{3-2s}$ and $L^2_{\rm loc}(\mathbb{R}^3)$, respectively.

2.1. **Formulation of problem (1).** In this subsection, we assume that $\frac{3}{4}<s<1$ and $0<t<1$. Considering a fixed $u\in X$, the linear functional $\mathcal{L}_u:L^{t,2}(\mathbb{R}^3)\to\mathbb{R}$ is defined by

$$\mathcal{L}_u(v):=\int_{\mathbb{R}^3}K(x)u^2vdx.$$

It is bounded since the Hölder's inequality, (8)-(9) and (H_2) show that

$$\begin{aligned}|\mathcal{L}_u(v)|&\le\left(\int_{\mathbb{R}^3}K(x)|u|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^3}K(x)|v|^2dx\right)^{\frac12}\\&\le\|K\|^{\frac12}_{L^\infty(\mathbb{R}^3)}C_t\|K\|^{\frac12}_{L^{\frac{3}{4s-3}}(\mathbb{R}^3)}S^2_{2_s^*}\|u\|_X^2\|v\|_{L^{t,2}(\mathbb{R}^3)},\ \forall v\in L^{t,2}(\mathbb{R}^3).\end{aligned}$$

So, due to the Lax-Milgram theorem, there is a unique $\phi_u\in L^{t,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}K(x)u^2vdx=\mathcal{L}_u(v)=(\phi_u,v)_{L^{t,2}(\mathbb{R}^3)}. \tag{10}$$

From which, we are derived from the Plancherel theorem [18, Theorem 5.3] that ϕ_u is a weak solution of $(I-\Delta)^t\phi=K(x)u^2$ and so, $\phi_u(x)=G_{2t}*(K(x)u^2)$.

Letting $v=\phi_u$ in (10), one has

$$\|\phi_u\|^2_{L^{t,2}(\mathbb{R}^3)}=\int_{\mathbb{R}^3}K(x)\phi_uu^2dx\le\|K\|^{\frac12}_{L^\infty(\mathbb{R}^3)}C_t\|K\|^{\frac12}_{L^{\frac{3}{4s-3}}(\mathbb{R}^3)}S^2_{2_s^*}\|u\|_X^2\|\phi_u\|_{L^{t,2}(\mathbb{R}^3)},$$

which in turn implies that

$$\int_{\mathbb{R}^3}K(x)\phi_uu^2dx\le\|K\|_{L^\infty(\mathbb{R}^3)}C_t^2\|K\|_{L^{\frac{3}{4s-3}}(\mathbb{R}^3)}S^4_{2_s^*}\|u\|_X^4,\ \forall u\in X. \tag{11}$$

Inserting ϕ_u into (1), there holds

$$(-\Delta)^su+V(x)u=K(x)\phi_uu+g(x)|u|^{q-2}u,\ x\in\mathbb{R}^3, \tag{12}$$

and its corresponding Euler-Lagrange functional $\varphi : X \to \mathbb{R}$ is defined by

$$\varphi(u) = \frac{1}{2}\|Qu\|_X^2 - \frac{1}{2}\|Pu\|_X^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \frac{1}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx. \tag{13}$$

In view of **one** of $(H_3) - (H_5)$, (7) and (11), it therefore would be standard to show that φ is well-defined on X and belongs to the class of $C^1(X, \mathbb{R})$ such that

$$\varphi'(u)[v] = \int_{\mathbb{R}^3} \left[(-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v + V(x)uv - K(x)\phi_u uv - g(x)|u|^{q-2}uv\right] dx, \ v \in X.$$

Obviously, the critical points of φ are the weak solutions of problem (12).

Definition 2.1. Let $\frac{3}{4} < s < 1$ and $0 < t < 1$.

(l) We call $(u, \phi) \in H^s(\mathbb{R}^3) \times L^{t,2}(\mathbb{R}^3)$ is a weak solution of problem (1) if u is a weak solution of problem (12).

(2) We call $u \in H^s(\mathbb{R}^3)$ is a weak solution of (12) if

$$\int_{\mathbb{R}^3} \left[(-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v + V(x)uv + K(x)\phi_u uv - g(x)|u|^{q-2}uv\right] dx = 0,$$

for any $v \in H^s(\mathbb{R}^3)$.

Let us define the variational functional $\Psi : X \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx = \int_{\mathbb{R}^3} \left[G_{2t} * (K(x)u^2)\right] K(x)u^2 dx.$$

In the following, we collect some important properties for Ψ as follows.

Lemma 2.2. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$ as well as $(H_1) - (H_2)$, then the following conclusions hold true:*

(i) *$\Psi(u) \geq 0$ and $\Psi(\theta u) = \theta^4 \Psi(u)$ for any $u \in X$ and $\theta > 0$;*

(ii) *If $u_n \rightharpoonup u$ in X, $u_n \to u$ in $L^p_{\rm loc}(\mathbb{R}^3)$ with $2 < p < 2_s^*$ and $u_n \to u$ a.e. in $\mathbb{R}^3$ as $n \to \infty$, then, going to a subsequence if necessary,*

$$\Psi(u_n) \to \Psi(u) \text{ and } \Psi'(u_n)[\psi] \to \Psi'(u)[\psi]$$

for every $\psi \in X$ as $n \to \infty$;

(iii) *For any fixed constant $\alpha \in (0, 1)$ and every finite dimension subspace $Z_0 \subset Z$, there exists a constant $C_\alpha > 0$ such that for each $u \in Y \bigoplus Z_0$ with $\|u\|_X = 1$ there holds*

$$\|Qu\|_X \geq \sin(\arctan \alpha) \Longrightarrow \Psi(u) \geq C_\alpha.$$

Proof. (i) In consideration of $K(x) > 0$ and $G_{2t}(x) > 0$ for all in $x \in \mathbb{R}^3$ by (H_2) and (4), respectively, we immediately derive that $\Psi(u) \geq 0$ according to its definition. Moreover, for all $\theta > 0$, it is simple to see that

$$\begin{aligned}\Psi(\theta u) &= \int_{\mathbb{R}^3} \left[G_{2t} * (K(x)(\theta u)^2)\right] K(x)(\theta u)^2 dx \\ &= \theta^4 \int_{\mathbb{R}^3} \left[G_{2t} * (K(x)u^2)\right] K(x)u^2 dx = \theta^4 \Psi(u), \ \forall u \in X,\end{aligned}$$

showing the Point-(i).

(ii) We can rewrite $\Psi(u_n) - \Psi(u)$ by

$$\begin{aligned}\Psi(u_n) - \Psi(u) &= \int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u_n^2 dx + \int_{\mathbb{R}^3} K(x)\phi_u(u_n^2 - u^2)dx \\ &:= I_n^1 + I_n^2.\end{aligned}$$

Recalling $\phi_{u_n} - \phi_{u_n} = G_{2t} * (K(x)(u_n^2 - u^2))$ and $\|G_{2t}\|_{L^1(\mathbb{R}^3)} = 1$, we then apply the Young's inequality with respect to the convolution operator and $K \in L^{\frac{3}{4s-3}}(\mathbb{R}^3)$ in (H_2) to get

$$\begin{aligned}\|\phi_{u_n} - \phi_{u_n}\|_{L^2(\mathbb{R}^3)} &\leq \left(\int_{\mathbb{R}^3} K^2(x)|u_n - u|^2|u_n + u|^2 dx\right)^{\frac{1}{2}} \\ &\leq \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} K(x)|u_n - u|^4 dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}^3} K(x)|u_n + u|^4 dx\right)^{\frac{1}{4}} \\ &\leq \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{3}{4}} S_4 \|u_n\|_X \left(\int_{\mathbb{R}^3} K(x)|u_n - u|^4 dx\right)^{\frac{1}{4}} = o_n(1),\end{aligned}$$

where the last equality follows the generalized Vitali's Convergence theorem. Thus,

$$\begin{aligned}|I_n^1| &\leq \left(\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} K(x)|u_n|^4 dx\right)^{\frac{1}{2}} \\ &\leq \|K\|_{L^\infty(\mathbb{R}^3)} S_4^2 \|u_n\|_X^2 \|\phi_{u_n} - \phi_{u_n}\|_{L^2(\mathbb{R}^3)} = o_n(1).\end{aligned}$$

Similarly, we can also obtain that

$$\begin{aligned}|I_n^2| &\leq \left(\int_{\mathbb{R}^3} K(x)\phi_u^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} K(x)|u_n + u|^2|u_n - u|^2 dx\right)^{\frac{1}{2}} \\ &\leq \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{3}{4}} C_t \|\phi_u\|_{L^{t,2}(\mathbb{R}^3)}^2 S_4 \|u_n + u\|_X \left(\int_{\mathbb{R}^3} K(x)|u_n - u|^4 dx\right)^{\frac{1}{4}} \\ &= o_n(1).\end{aligned}$$

Combining the above two facts, we derive the proof of the first part. The remaining part is easier, so we omit it here.

(iii) Let us define a constant $\gamma := \sin(\arctan\alpha) \in (0, 1)$ and a set

$$\Upsilon^\alpha := \left\{v \in Y \bigoplus Z_0 : \|v\|_X = 1 \text{ and } \|Qv\|_X \geq \gamma\right\}.$$

Due to the definition of u, one sees that $\Psi(u) \geq \inf_{v \in \Upsilon^\alpha} \Psi(v) := C_\alpha$. Using Point-(i), there holds that $C_\alpha \geq 0$. So, to finish the proof, it suffices to conclude that $C_\alpha > 0$. Suppose, by contradiction, that $C_\alpha = 0$. Then, there exists a sequence $\{v_n\} \subset \Upsilon^\alpha$ such that $\Psi(v_n) \to 0$ as $n \to \infty$. Since $\|v_n\|_X \equiv 1$, up to a subsequence if necessary, there is $v \in X$ such that $v_n \rightharpoonup v$ in X and $Qv_n \to Qv$ because $\{Qv_n\} \subset Z_0$ with $\dim Z_0 < +\infty$. Hence, $\|Qv\|_X \geq \gamma > 0$ and $\Psi(v) \leq \liminf_{n\to\infty} \Psi(v_n) = 0$ yielding that $v \equiv 0$, a contradiction to (ii). The proof is completed. □

Next, we prove that the variational functional φ satisfies the $(PS)_c$ condition.

Lemma 2.3. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$. Suppose that $(H_1) - (H_2)$ and **one** of $(H_3) - (H_5)$ hold, then φ satisfies the $(PS)_c$ condition.*

Proof. Assume that there is a sequence $\{u_n\} \subset X$ satisfies $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ as $n \to \infty$, then

$$\begin{aligned}c + 1 + \|u_n\|_X &\geq \varphi(u_n) - \frac{1}{2}\varphi'(u_n)[u_n] \\ &= \frac{1}{4}\Psi(u_n) + \frac{q-2}{2q}\int_{\mathbb{R}^3} g(x)|u_n|^q dx. \qquad (14)\end{aligned}$$

To show that $\|u_n\|_X$ is uniformly bounded in $n \in \mathbb{N}$, we split it into two cases.

Case 1: the assumption (H_3) holds.
In this case, that is, $g \in L^{q_0}(\mathbb{R}^3)$ with $q_0 = \frac{6}{6-q(3-2s)}$ and $1 < q < \frac{4}{3}$, we can adopt (14) together with (8) to have that

$$\begin{aligned}\Psi(u_n) &\leq 4(c+1+\|u_n\|_X) + \frac{q}{2(2-q)}\left(\int_{\mathbb{R}^3}|g|^{q_0}dx\right)^{\frac{1}{q_0}}\left(\int_{\mathbb{R}^3}|u_n|^{2^*}dx\right)^{\frac{q}{2_s^*}}\\ &\leq C(1+\|u_n\|_X+\|u_n\|_X^q).\end{aligned}$$

Denoting $u_n = Pu_n + Qu_n := y_n + z_n$ with $y_n \in Y$ and $z_n \in Z$, then

$$\begin{aligned}\left|\Psi'(u_n)[y_n]\right| = \left|\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_ny_ndx\right| &\leq \Psi^{\frac{1}{2}}(u_n)\bigg(\int_{\mathbb{R}^3}K(x)\phi_{u_n}y_n^2dx\bigg)^{\frac{1}{2}}\\ &\leq \Psi^{\frac{1}{2}}(u_n)\bigg(\int_{\mathbb{R}^3}K(x)\phi_{u_n}^2dx\bigg)^{\frac{1}{4}}\bigg(\int_{\mathbb{R}^3}K(x)y_n^4dx\bigg)^{\frac{1}{4}}\\ &\leq \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}}C_t^{\frac{1}{2}}S_4\Psi^{\frac{3}{4}}(u_n)\|y_n\|_X\\ &\leq C(1+\|u_n\|_X+\|u_n\|_X^q)^{\frac{3}{4}}\|y_n\|_X\\ &\leq C(1+\|u_n\|_X+\|u_n\|_X^q)^{\frac{3}{4}}\|u_n\|_X. \qquad (15)\end{aligned}$$

Thereby, for $n \in \mathbb{N}$ large, combining $\|u_n\|_X \geq \|y_n\|_X \geq -\varphi'(u_n)[y_n]$ and (15), we obtain

$$\begin{aligned}\|y_n\|_X^2 &= -\varphi'(u_n)[y_n] - \Psi'(u_n)[y_n] - \int_{\mathbb{R}^3}g(x)|u_n|^{q-2}u_ny_ndx\\ &\leq \|u_n\|_X + C(1+\|u_n\|_X+\|u_n\|_X^q)^{\frac{3}{4}}\|u_n\|_X + C\|u_n\|_X^{q-1}\|y_n\|_X.\end{aligned}$$

Similarly, we deduce that

$$\|z_n\|_X^2 \leq \|u_n\|_X + C(1+\|u_n\|_X+\|u_n\|_X^q)^{\frac{3}{4}}\|u_n\|_X + C\|u_n\|_X^{q-1}\|z_n\|_X.$$

Recalling the fact that $\|u_n\|_X^2 = \|y_n\|_X^2 + \|z_n\|_X^2$, we know that $\|u_n\|_X$ is uniformly bounded since $2 > 1 + \frac{3}{4}q$ which is $q < \frac{4}{3}$ by (H_3).

Case 2: either the assumption (H_4) or (H_5) holds.
Obviously, both (H_4) and (H_5) indicate that $(q-2)g(x) > 0$ a.e. in $\mathbb{R}^3$. Therefore, it follows from (14) that

$$\Psi(u_n) \leq C(1+\|u_n\|_X)$$

and

$$0 \leq (q-2)\int_{\mathbb{R}^3}g(x)|u_n|^qdx \leq C(1+\|u_n\|_X).$$

It is similar to (15) that $|\Psi'(u_n)[y_n]| \leq C(1+\|u_n\|_X)^{\frac{3}{4}}\|u_n\|_X$ and then

$$\|y_n\|_X^2 \leq \|u_n\|_X + C(1+\|u_n\|_X)^{\frac{3}{4}}\|u_n\|_X + C(1+\|u_n\|_X)^{\frac{q-1}{q}}\|y_n\|_X.$$

Analogously, one has

$$\|z_n\|_X^2 \leq \|u_n\|_X + C(1+\|u_n\|_X)^{\frac{3}{4}}\|u_n\|_X + C(1+\|u_n\|_X)^{\frac{q-1}{q}}\|z_n\|_X.$$

So, we still can derive $\|u_n\|_X$ is uniformly bounded since $2 > 1 + \frac{3}{4}$.

Based on the above discussions, we can conclude that $\|u_n\|_X$ is uniformly bounded in $n \in \mathbb{N}$. Moreover, one deduces that $\|y_n\|_X$ and $\|z_n\|_X$ are uniformly bounded in $n \in \mathbb{N}$. Passing to subsequences if necessary, there exist two functions $y \in Y$ and $z \in Z$ such that, as $n \to \infty$,

$$\begin{cases} y_n \rightharpoonup y \text{ and } z_n \rightharpoonup z, & \text{in } X, \\ y_n \to y \text{ and } z_n \rightharpoonup z, & \text{in } L^p_{\rm loc}(\mathbb{R}^3) \text{ with } 2 \le p < 2^*_s, \\ y_n \to y \text{ and } z_n \rightharpoonup z, & \text{a.e. in } \mathbb{R}^3 \end{cases}$$

Define $u := y + z \in X$, then we aim to show that $y_n \to y$ in X along a subsequence. Actually, we can argue as the calculations in (15) to obtain that

$$(\Psi'(u_n) - \Psi'(u))[y_n - y] = o_n(1)$$

as $n \to \infty$. Since $g \in L^{q_0}(\mathbb{R}^3)$, it follows from the generalized Vitali's Convergence theorem again that

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} g(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(y_n - y)dx \right| \\ &\le \left(\int_{\mathbb{R}^N} |g(x)| \left| |u_n|^{q-2}u_n - |u|^{q-2}u \right|^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^N} |g(x)||y_n - y|^q dx \right)^{\frac{1}{q}} \\ &= o_n(1) \end{aligned}$$

as $n \to \infty$. Combining the above two formulas and $\varphi'(u_n) \to 0$, there holds

$$\begin{aligned} o_n(1) &= \big(\varphi'(u_n) - \varphi'(u)\big)[y_n - y] \\ &= -\|y_n - y\|_X^2 - \big(\Psi'(u_n) - \Psi'(u)\big)[y_n - y] \\ &\quad - \int_{\mathbb{R}^N} g(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(y_n - y)dx \\ &= -\|y_n - y\|_X^2 = o_n(1) \end{aligned}$$

which indicates the desired result. Analogously, we can also derive that $\|z_n - z\|_X = o_n(1)$. So, one sees that $u_n \to u$ in X finishing the proof. □

3. **Proofs of the main results.**

In this section, we are going to exhibit the proofs of Theorems 1.1, 1.2 and 1.3 in detail. For this goal, we need to introduce the following improved Fountain theorem developed by Gu and Zhou in [12], that is,

Proposition 3.1. *Let $\Phi \in C^1(X, \mathbb{R})$ be an even functional satisfying $(PS)_c$ condition and $\nabla\Phi$ be weakly sequentially continuous, for all $k \in \mathbb{N}$ with $X = Y_k \bigoplus Z_k$, if there exist two constants $\rho_k > r_k > 0$ such that*

(A_1) $d_k := \sup\limits_{u \in Y_k, \|u\|_X \le \rho_k} \Phi(u) < +\infty$;

(A_2) $a_k := \sup\limits_{u \in Y_k, \|u\|_X = \rho_k} \Phi(u) < \inf\limits_{u \in Z_k, \|u\|_X \le r_k} \Phi(u)$;

(A_3) $b_k := \inf\limits_{u \in Z_k, \|u\|_X = r_k} \Phi(u) \to +\infty$ *as* $k \to +\infty$;

(A_4) *For any $\sigma > 0$, there is a constant $C_\sigma > 0$ such that* $\sup\limits_{\|u\|_\tau < \sigma} \Phi(u) \le C_\sigma < +\infty$, *where*

$$\|u\|_\tau := \max\left\{ \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle Pu, e_j\rangle|, \|Qu\|_X \right\} \quad \text{for } u \in X, \tag{16}$$

with $\{e_j\}_{j=1}^\infty$ denoting the normal orthogonal basis of Y.

Then, the functional Φ has a sequence of critical points $\{u^{k_m}\}$ such that

$$\lim_{m\to\infty} \|u^{k_m}\|_X = +\infty.$$

Let us recall the decomposition on $X = Y \bigoplus Z$ in Section 2, we could suppose that the two sequences $\{e_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ are normal orthogonal basis of Y and Z, respectively. For any $k \in \mathbb{N}$, define

$$Y_k := Y \bigoplus \left(\bigoplus_{j=1}^{k} \mathbb{R} f_j \right) \text{ and } Z_k := \overline{\bigoplus_{j=k+1}^{\infty} \mathbb{R} f_j}.$$

In order to apply the Proposition 3.1 successfully, we shall regard the variational functional Φ as φ which corresponds to Eq. (13). From now on until the end of this section, we shall always suppose that $\frac{3}{4} < s < 1$ and $0 < t < 1$ and do not mention them any longer.

First of all, we show that the functional φ satisfies the condition (A_1).

Lemma 3.2. *Assume $(H_1) - (H_2)$ and* **one** *of $(H_3) - (H_5)$ hold, then there exists $\rho_k > 0$ such that*

$$d_k := \sup_{u \in Y_k, \|u\|_X \le \rho_k} \Phi(u) < +\infty.$$

Proof. The proof is standard since φ maps a bounded set into a bounded set, so we omit it here. □

Then, we verify that the functional φ satisfies the conditions (A_2) and (A_3).

Lemma 3.3. *Assume $(H_1) - (H_2)$ and* **one** *of $(H_3) - (H_5)$ hold, then there exist $\rho_k > r_k > 0$ such that $a_k < \inf\limits_{u \in Z_k, \|u\|_X \le r_k} \varphi(u)$ and $b_k \to +\infty$ as $k \to +\infty$.*

Proof. Firstly, to derive the first part, we just need to show that

$$a_k := \sup_{u \in Y_k, \|u\|_X = \rho_k} \varphi(u) \to -\infty$$

as $\rho_k \to +\infty$. Either (H_3) or (H_4) holds true, for all $u \in X$, there holds

$$-\frac{1}{q} \int_{\mathbb{R}^3} g(x)|u|^q dx \le C\|u\|_X^q, \ 1 < q < 2,$$

for some positive constant C which is independent of u. If (H_5) holds true, because $g(x) > 0$ for all $x \in \mathbb{R}^3$, given $u \in X$, it has that

$$-\frac{1}{q} \int_{\mathbb{R}^3} g(x)|u|^q dx \le 0.$$

In summary, for any fixed $u = Qu + Pu \in X$, we can always deduce that

$$\varphi(u) \le \frac{1}{2}\|Qu\|_X^2 - \frac{1}{2}\|Pu\|_X^2 - \frac{1}{4}\Psi(u) + C\|u\|_X^q, \text{ where } 1 < q < 2. \tag{17}$$

Now, we begin verifying that $a_k \to -\infty$ as $\rho_k \to +\infty$. Given a $u \in Y_k = Y \bigoplus \left(\bigoplus_{j=1}^{k} \mathbb{R} f_j \right)$ with $\|u\|_X = \rho_k$.

If $u \in Y$, then $Qu = 0$ and $Pu = u$, it follows from (17) that

$$\varphi(u) \le -\frac{1}{2}\rho_k^2 + C\rho_k^q \to -\infty$$

as $\rho_k \to +\infty$ since $q < 2$ in (17).

If $u = y + z$ with $y \in Y$ and $z \in \bigoplus_{j=1}^{k} \mathbb{R}f_j$, for the constant $\alpha \in (0,1)$ given by Lemma 2.2-(iii), we shall distinguish the proof two cases:

$$\text{(i) } \|Qu\|_X/\|Pu\|_X < \alpha \text{ and (ii) } \|Qu\|_X/\|Pu\|_X \geq \alpha.$$

If (i) occurs, then $\|u\|_X^2 = \|Qu\|_X^2 + \|Pu\|_X^2 < (1+\alpha^2)\|Pu\|_X^2$ which in turn shows that

$$\|Qu\|_X^2 = \|u\|_X^2 - \|Pu\|_X^2 < \frac{\alpha^2}{1+\alpha^2}\|u\|_X^2.$$

Combining $\alpha \in (0,1)$ in Lemma 2.2-(iii) and (17) with $q < 2$, we obtain

$$\begin{aligned}\varphi(u) &\leq \frac{\alpha^2}{2(1+\alpha^2)}\|u\|_X^2 - \frac{1}{2(1+\alpha^2)}\|u\|_X^2 + C\|u\|_X^q \\ &= -\frac{1-\alpha^2}{2(1+\alpha^2)}\rho_k^2 + C\rho_k^q \\ &\to -\infty \text{ as } \|u\|_X = \rho_k \to +\infty.\end{aligned}$$

If (ii) occurs, we set $v := \frac{u}{\|u\|_X}$ and so $\|v\|_X = 1$. Moreover, one has that

$$\|Qv\|_X = \frac{\|Qu\|_X}{\|u\|_X} = \sin\left(\arctan\frac{\|Qu\|_X}{\|Pu\|_X}\right).$$

In this case, one deduces $\|Qv\|_X \geq \sin(\arctan\alpha)$. Accordingly, $v \in Y \bigoplus \left(\bigoplus_{j=1}^{k} \mathbb{R}f_j\right)$ with $\dim\left(\bigoplus_{j=1}^{k} \mathbb{R}f_j\right) < +\infty$, as a consequence of Lemma 2.2-(i) and (iii), we obtain $\Psi(u)\|u\|_X^{-4} = \Psi(v) \geq C_\alpha > 0$. Then, adopting (17) again, we arrive at

$$\begin{aligned}\varphi(u) &\leq \frac{1}{2}\|u\|_X^2 - \frac{C_\alpha}{4}\|u\|_X^4 + C\|u\|_X^q \\ &= \frac{1}{2}\rho_k^2 - \frac{C_\alpha}{4}\rho_k^4 + C\rho_k^q \\ &\to -\infty \text{ as } \|u\|_X = \rho_k \to +\infty.\end{aligned}$$

Therefore, we always have $a_k \to -\infty$ as $\rho_k \to +\infty$ and the first part concludes.

Secondly, we claim that

$$\beta_k^1 := \sup_{u \in Z_k : \|u\|_X = 1} \Psi(u) \to 0$$

as $k \to +\infty$. Indeed, according to the definition of Z_k, one simply has $0 < \beta_{k+1}^1 \leq \beta_k^1$ for any $k \in \mathbb{N}$. Then, there exists a constant $\beta \geq 0$ such that $\beta_k^1 \to \beta$ as $k \to +\infty$. By the definition of β_k^1, there is a $u_k \in Z_k$ with $\|u_k\|_X = 1$ such that $\Psi(u_k) \geq \beta_k^1/2$. Since $\{u_k\} \subset Z_k = \overline{\bigoplus_{j=k+1}^{\infty} \mathbb{R}f_j}$, one concludes that $u_k \rightharpoonup 0$ in X as $k \to +\infty$. In view of Lemma 2.2-(ii), passing to a subsequence if necessary, we derive $\Psi(u_k) \to 0$ as $k \to +\infty$. Hence, we must have $\beta = 0$ and the claim holds true.

Finally, to prove $b_k \to +\infty$ as $k \to +\infty$, we split it into two cases.

Case 1: either the assumption (H_3) or (H_4) holds true.

In this case, combining $q < 2$ and $\beta_k^1 \to 0$ as $k \to +\infty$, there holds

$$\frac{1}{4}(\beta_k^1)^{\frac{q-2}{2}} - \frac{C}{q} \geq \frac{1}{8} \text{ for some sufficiently large } k \in \mathbb{N},$$

where $C>0$ is a constant independent of k. As a consequence, for any $u\in Z_k$ with $\|u\|_X=r_k=(\beta_k^1)^{-\frac{1}{2}}$, by means of $g\in L^{q_0}(\mathbb{R}^3)$, we have

$$\begin{aligned}\varphi(u)&\geq \frac{1}{2}\|u\|_X^2-\frac{\beta_k^1}{4}\|u\|_X^4-\frac{C}{q}\|u\|_X^q\\&=\left(\frac{1}{4}\|u\|_X^{2-q}-\frac{C}{q}\right)\|u\|_X^q+\frac{1}{4}(1-\beta_k^1\|u\|_X^2)\|u\|_X^2\\&=\left(\frac{1}{4}\|u\|_X^{2-q}-\frac{C}{q}\right)\|u\|_X^q\geq\frac{1}{8}\|u\|_X^q=\frac{1}{8}(\beta_k^1)^{-\frac{q}{2}}:=\frac{1}{8}r_k^q,\end{aligned}$$

for some sufficiently large $k\in\mathbb{N}$. Clearly, $r_k=(\beta_k^1)^{-\frac{1}{2}}\to+\infty$ as $k\to+\infty$. Hence, it holds that $b_k\to+\infty$ as $k\to+\infty$.

Case 2: the assumption (H_5) holds true.
Since $g(x)>0$ a.e. for $x\in\mathbb{R}^3$ and $g\in L^{q_0}(\mathbb{R}^3)$, for any $k\in\mathbb{N}$, we can verify that

$$\beta_k^2:=\sup_{u\in Z_k:\|u\|_X=1}\int_{\mathbb{R}^3}g(x)|u|^q dx\to 0.$$

Indeed, one can easily get that $0<\beta_{k+1}^2\leq\beta_k^2$ for any $k\in\mathbb{N}$ and so there is a constant $\hat\beta\geq 0$ such that $\beta_k^2\to\hat\beta$ as $k\to+\infty$. By the definition of β_k^2, there is a $u_k\in Z_k$ with $\|u_k\|_X=1$ such that $\int_{\mathbb{R}^3}g(x)|u_k|^q dx\geq\beta_k^2/2$. Recalling $\{u_k\}\subset Z_k=\overline{\bigoplus_{j=k+1}^{\infty}\mathbb{R}f_j}$, one concludes that $u_k\rightharpoonup 0$ in X as $k\to+\infty$. Due to $g\in L^{q_0}(\mathbb{R}^3)$, one immediately has that $\int_{\mathbb{R}^3}g(x)|u_k|^q dx\to 0$ and so $\hat\beta=0$. Therefore, for all $u\in Z_k$ with $\|u\|_X=r_k=\min\left\{\sqrt{\frac{1}{2\beta_k^1}},\left(\frac{q}{8\beta_k^2}\right)^{\frac{1}{q-2}}\right\}$, we have

$$\begin{aligned}\varphi(u)&\geq\frac{1}{4}\|u\|_X^2+\left(\frac{1}{8}-\frac{\beta_k^1}{4}\|u\|_X^2\right)\|u\|_X^2+\left(\frac{1}{8}-\frac{\beta_k^2}{q}\|u\|_X^{q-2}\right)\|u\|_X^2\\&\geq\frac{1}{4}\|u\|_X^2=\frac{1}{4}\min\left\{\frac{1}{2\beta_k^1},\left(\frac{q}{8\beta_k^2}\right)^{\frac{2}{q-2}}\right\}\\&:=\frac{1}{4}r_k^2.\end{aligned}$$

It is easy to see that $r_k\to+\infty$ as $k\to+\infty$ since $q>2$ in this case. So, $b_k\to+\infty$ as $k\to+\infty$. The proof of this lemma is completed. □

Finally, the condition (A_4) will be proved for the functional φ as follows.

Lemma 3.4. *Assume $(H_1)-(H_2)$ and* **one** *of $(H_3)-(H_5)$ hold, then for any $\sigma>0$, there is a constant $C_\sigma>0$ such that $\sup_{\|u\|_\tau<\sigma}\varphi(u)\leq C_\sigma<+\infty$.*

Proof. Since $u\in X=Y\bigoplus Z$ with $Z=Y^\perp$, we may set $u=Qu+Pu$ with $Qu\in Z$ and $Pu\in Y$. Then, using a very similar calculations in (17), we reach

$$\begin{aligned}\varphi(u)&=\frac{1}{2}\|Qu\|_X^2-\frac{1}{2}\|Pu\|_X^2-\frac{1}{4}\Psi(u)-\frac{1}{q}\int_{\mathbb{R}^N}g(x)|u|^q\,dx\\&\leq\frac{1}{2}\|Qu\|_X^2-\frac{1}{2}\|Pu\|_X^2+C\|u\|_X^q\\&\leq\frac{1}{2}\|Qu\|_X^2+C\|Qu\|_X^q-\frac{1}{2}\|Pu\|_X^2+C\|Pu\|_X^q.\end{aligned}$$

Since $q < 2$ in (17), $-\frac{1}{2}\|Pu\|_X^2 + C\|Pu\|_X^q$ is bounded from above. By (16), we have $\|Qu\|_X \le \|u\|_\tau \le \sigma$, thus there exists a $C_\sigma < \infty$ such that $\sup\limits_{\|u\|_\tau \le \sigma} \varphi(u) < C_\sigma$. The proof is completed □

With Lemmas 3.2, 3.3 and 3.4 in hands, we derive that the variational functional φ has infinitely many nontrivial critical points.

Lemma 3.5. *Assume* (H_1)–(H_2) *and* **one** *of* (H_3)–(H_5) *hold, then* φ *has infinitely many nontrivial critical points* $\{u_n\} \subset H^s(\mathbb{R}^3)$ *such that* $\lim\limits_{n\to\infty} \|u_n\|_{H^s(\mathbb{R}^3)} = +\infty$.

Proof. Let $\varphi = \Phi$ be as in Proposition 3.1, and clearly φ is even. Owing to Lemmas 2.2-(ii) and 2.3, we clearly know that φ satisfies $(PS)_c$ condition and $\nabla\varphi$ is weakly sequentially continuous. On the other hand, we have validated the conditions (A_1), (A_2), (A_3) and (A_4) for φ in Lemmas 3.2, 3.3 and 3.4, respectively. As a consequence of Proposition 3.1, we can finish the proof of this lemma. □

At this stage, we can conclude the proofs of Theorems 1.1, 1.2 and 1.3 depending on Lemma 3.5.

Acknowledgments. The work was partially carried out while the third author was visiting Professor Juncheng Wei at University of British Columbia and he acknowledges the warm hospitality from Department of Mathematics.

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Received xxxx 20xx; revised xxxx 20xx; early access xxxx 20xx.