INFINITELY MANY SOLUTIONS FOR A CLASS OF FRACTIONAL SCHRÖDINGER EQUATIONS COUPLED WITH NEUTRAL SCALAR FIELD

Liejun Shen[⊠1], Marco Squassina[⊠*2] and Xiaoyu Zeng[⊠3]

[1]Department of Mathematics, Zhejiang Normal University
Jinhua, Zhejiang 321004, China

[2]Dipartimento di Matematica e Fisica Università Cattolica del Sacro Cuore
Via della Garzetta 48, 25133, Brescia, Italy

[3]Centre for Mathematical Sciences, Wuhan University of Technology
Wuhan, Hubei 430070, China

Abstract. We study the fractional Schrödinger equations coupled with a neutral scalar field

$$\begin{cases} (-\Delta)^s u + V(x)u = K(x)\phi u + g(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ (I-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $(-\Delta)^s$ and $(I-\Delta)^t$ denote the fractional Laplacian and Bessel operators with $\frac{3}{4} < s < 1$ and $0 < t < 1$, respectively. Under some suitable assumptions for the external potentials V, K, and g, given $q \in (1,2) \cup (2, 2_s^*)$ with $2_s^* := \frac{6}{3-2s}$, with the help of an improved Fountain theorem dealing with a class of strongly indefinite variational problems approached by Gu-Zhou [Adv. Nonlinear Stud., **17** (2017), 727–738], we show that the system admits infinitely many nontrivial solutions.

1. Introduction and main results.

1.1. General overview.
In the present paper, we are concerned with the following fractional elliptic system

$$\begin{cases} (-\Delta)^s u + V(x)u = K(x)\phi u + g(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ (I-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $(-\Delta)^s$ and $(I-\Delta)^t$ denote the classic fractional Laplacian and Bessel operators with $\frac{3}{4} < s < 1$ and $0 < t < 1$, respectively.

In light of its relevance in physics, the nonlinear fractional Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = (-\Delta)^s \Psi + (V(x)+E)\Psi - f(x,\Psi) \quad \text{for all } x \in \mathbb{R}^N, \tag{2}$$

where $N > 2s$ with $s \in (0,1)$, $E \in \mathbb{R}$, and V and f are continuous functions, has been received more and more attentions in recent years by a great many mathematicians. Generally, they are searching for a particular type of the solution to Eq. (2), the so called *standing wave solution*, which carries a form of the type

$$\Psi(z,t) = \exp(-iEt)u(z),$$

revealing that u acts as a solution of the fractional elliptic equation

$$\begin{cases} (-\Delta)^s u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ \quad u \in H^s(\mathbb{R}^N), \ u > 0, & \text{on } \mathbb{R}^N. \end{cases} \tag{3}$$

The local case, that is, the general semilinear elliptic equations (3) with $s = 1$, has been extensively considered; for example, we shall refer the reader to [20, 26, 3] and the references therein.

In the nonlocal case, namely when $s \in (0,1)$, the corresponding results for Eq. (3) never seem to be as fruitful as the local ones. This potentially occurs because the techniques and arguments developed for the local case cannot be adapted immediately, c.f. [23]. In order to introduce some results for Eq. (2), we recall that for any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy < \infty \right\},$$

equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy \right)^{1/2}.$$

The fractional Laplacian, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$ with sufficient decay is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transformation, which is

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \phi(x) dx := \widehat{\phi}(\xi),$$

for functions ϕ belonging to the Schwartz class. In reality, according to [6, Lemma 3.2], the fractional Laplacian operator can be equivalently represented as

$$(-\Delta)^s u(x) = -\frac{1}{2} C(N,s) \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N,$$

where

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{(1 - \cos \xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N.$$

Also, due to [6, Propostion 3.4, Propostion 3.6], it holds that

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}|^2 d\xi = \frac{1}{2} C(N,s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy,$$

for all $u \in H^s(\mathbb{R}^N)$. Moreover, we usually identify these two quantities by omitting the normalization constant $\frac{1}{2} C(N,s)$ for simplicity. The homogeneous fractiotal Sobolev space $D^{s,2}(\mathbb{R}^N)$ is defined by

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2_s^*}(\mathbb{R}^N) : |(-\Delta)^{\frac{s}{2}} u| \in L^2(\mathbb{R}^N) \right\}$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{s,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx\right)^{\frac{1}{2}}.$$

For $N > 2s$, from [6, Theorem 6.5], we further know that for any $p \in [2, 2_s^*]$ there exists a constant $C_p > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^N)} \leq C_p \|u\|_{H^s(\mathbb{R}^N)}, \text{ for all } u \in H^s(\mathbb{R}^N).$$

Besides, the imbedding $H^s(\mathbb{R}^N) \hookrightarrow L_{\mathrm{loc}}^p(\mathbb{R}^N)$ is compact for all $1 \leq p < 2_s^*$.

Actually, problem (3) was initially proposed by the author in [16, 17] as a result of expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. According to the celebrated paper [5], Eq. (2) and its variants have been widely contemplated by many authors, especially on the existence of ground state solutions, positive solutions, sign-changing solutions and multiplicity of standing wave solutions, see e.g. [10, 4, 2, 13] and their references therein.

For $t > 0$, the so-called *Bessel function space* in $\mathbb{R}^3$ is defined by

$$L^{t,2}(\mathbb{R}^3) := \left\{f \in L^2(\mathbb{R}^3) : f = G_t * h \text{ for some } h \in L^2(\mathbb{R}^3)\right\},$$

where the Bessel convolution kernel is

$$G_t(x) := \frac{1}{(4\pi)^{\frac{t}{2}}\Gamma(\frac{t}{2})} \int_0^\infty \exp\left(-\frac{\pi}{\delta}|x|^2\right) \exp\left(-\frac{\delta}{4\pi}\right) \delta^{\frac{t-5}{2}} d\delta. \qquad (4)$$

The operator $(I-\Delta)^{-t}u = G_{2t} * u$ is generally known as the Bessel operator of order t, and it induces the Bessel function space equipped with the norm $\|f\|_{L^{t,2}(\mathbb{R}^3)} = \|h\|_{L^2(\mathbb{R}^3)}$ if $f = G_t * h$. Owing to the view point of the Fourier transformation, this same operator can also read

$$G_t = \mathcal{F}^{-1} \circ \left(\left(1 + |\xi|^2\right)^{-\frac{t}{2}} \circ \mathcal{F}\right),$$

so that

$$\|f\|_{L^{t,2}(\mathbb{R}^3)} = \left\|(I-\Delta)^{\frac{t}{2}}f\right\|_{L^2(\mathbb{R}^3)}.$$

We refer the interested reader to [1, 25] and the references therein for more detailed information concerning the Bessel operator and Bessel function space.

It should be mentioned here that authors in [9] introduced the pointwise formula

$$(I-\Delta)^t u(x) = c_s \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x-y|^{\frac{3+2t}{2}}} \mathcal{K}_{\frac{3+2t}{2}}(|x-y|)\, dy + u(x)$$

for $u \in C_c^2(\mathbb{R}^3)$, where K_ν is the modified Bessel function of the second kind with order ν. Nevertheless, a closed formula for K_ν remains unknown, see e.g. [9, Remark 7.3].

Very recently, Felmer-Vergara [11] investigated the existence of positive solutions for fractional equations involving a Bessel operator

$$(I-\Delta)^s u + V(x)u = f(x,u), \ x \in \mathbb{R}^N. \qquad (5)$$

Subsequently, under different technical assumptions on V and f owing to variational methods, there are some bibliographies in the study of (5), see [21, 14, 22] and their references therein for example.

In [24], the author contemplated the multiplicity and concentration of nontrivial solutions for the following fractional Schrödinger-Poisson system involving a Bessel operator

$$\begin{cases} (I - \Delta)^s u + \lambda V(x)u + \phi u = f(x, u) + g(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where λV represents a deepening potential with $\lambda > 0$, $f \in C^0(\mathbb{R}^3 \times \mathbb{R})$ satisfies some suitable conditions, $g > 0$ is a weight function, and $1 < q < 2$.

Motivated by the relevance of problem (3) and the mathematical point of view, we consider a class of Schrödinger equations coupled with a Bessel operator. More precisely, we shall establish the existence of infinitely many nontrivial solutions for system (1) under some suitable assumptions on V, K, and g. To the best of our knowledge, it seems that this is the first time such type of problems have been considered. What is more, we anticipate that the results in this paper will prompt further studies on (fractional) Schrödinger-Poisson systems.

1.2. **Assumptions and main results.** To arrive at the aim mentioned above, we shall suppose that

(H_1) $V \in C^0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and 0 lies in a spectrum gap of the operator $(-\Delta)^s + V$;

(H_2) $K \in L^{\frac{3}{4s-3}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $K(x) > 0$ a.e. for $x \in \mathbb{R}^3$;

Now, the main results in this paper can be stated as follows.

Theorem 1.1. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$ and suppose that $(H_1) - (H_2)$. If in addition*

(H_3) $g \in L^{q_0}(\mathbb{R}^3)$ with $q_0 = \frac{6}{6 - q(3 - 2s)}$ and $1 < q < \frac{4}{3}$,

then system (1) possesses infinitely many solutions $\{u_n, \phi_{u_n}\} \subset H^s(\mathbb{R}^3) \times L^{t,2}(\mathbb{R}^3)$ satisfying $\lim_{n \to \infty} \|u_n\|_{H^s(\mathbb{R}^3)} = +\infty$.

Theorem 1.2. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$ and suppose that $(H_1) - (H_2)$. If in addition*

(H_4) $g \in L^{q_0}(\mathbb{R}^3)$ with $g(x) < 0$ a.e. $x \in \mathbb{R}^3$, $q_0 = \frac{6}{6 - q(3 - 2s)}$ and $\frac{4}{3} \leq q < 2$,

then system (1) possesses infinitely many solutions $\{u_n, \phi_{u_n}\} \subset H^s(\mathbb{R}^3) \times L^{t,2}(\mathbb{R}^3)$ satisfying $\lim_{n \to \infty} \|u_n\|_{H^s(\mathbb{R}^3)} = +\infty$.

Theorem 1.3. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$ and suppose that $(H_1) - (H_2)$. If in addition*

(H_5) $g \in L^{q_0}(\mathbb{R}^3)$ with $g(x) > 0$ a.e. $x \in \mathbb{R}^3$, $q_0 = \frac{6}{6 - q(3 - 2s)}$ and $2 < q < 2_s^*$,

then system (1) possesses infinitely many solutions $\{u_n, \phi_{u_n}\} \subset H^s(\mathbb{R}^3) \times L^{t,2}(\mathbb{R}^3)$ satisfying $\lim_{n \to \infty} \|u_n\|_{H^s(\mathbb{R}^3)} = +\infty$.

Remark 1.4. Although the result in Theorem 1.1 is similar to [12, Theorem 1.3], as far as we are concerned, one can never prove Theorem 1.1 by simply repeating the arguments exploited in the cited paper caused by the appearance of the Bessel operator; see Lemma 2.2 below for example. On the other hand, with the help of some new analytic skills, the results are even new for the counterparts of [12].

Remark 1.5. It is worthy pointing out that we do not conclude whether the results in Theorems 1.2 and 1.3 remain valid when g is sign-changing in (H_4) and (H_5), respectively. Moreover, inspired by [26, 27], it is interesting to consider that problem (1) has a ground state solution. We postpone these two questions to a further work.

Again, up to now the results in Theorems 1.1, 1.2, and 1.3 are new. To conclude this section, we sketch our proof. First of all, because the operator $L = (-\Delta)^s + V$ is strongly indefinite, we then follow the idea introduced in [19, 26, 27] to decompose the space $H^s(\mathbb{R}^3)$ suitably. In the meantime, there exist some standard arguments exhibited in Section 2 that allow us to treat problem (1) by variational methods. Then, we shall depend heavily on a new type of Fountain theorem approached by Gu and Zhou in [12] to derive the existence of infinitely many nontrivial critical points. Finally, we concentrate on verifying the necessary properties of the corresponding variational functional, see Sections 2 and 3 in detail. So, we can derive the proofs successfully. However, owing to the Bessel operator appearing in (1), there are some unpleasant barriers in the last step and we have to take some careful and deep analysis there.

1.3. **Organization of the paper.** This paper is organized as follows. In Section 2, we provide some preliminary results. Section 3 is devoted to the proofs of Theorems 1.1, 1.2, and 1.3.

Notations. From now on in the present article, unless stated otherwise, we shall adopt the following notations:

- $C, C_1, C_2, \cdots$ denote any positive constants whose values are not relevant.
- Let $(X, \|\cdot\|_X)$ be a Banach space with dual space $(X^{-1}, \|\cdot\|_{X^{-1}})$, and Φ be a functional on X.
- Let $\|\cdot\|_{L^p(\mathbb{R}^3)}$ denote the usual L^p-norm for any Lebesgue measurable function $u : \mathbb{R}^3 \to \mathbb{R}$, where $p \in [1, \infty]$.
- Palais-Smale sequence at level $c \in \mathbb{R}$ ($(PS)_c$ sequence in short) corresponding to a functional Φ on X means that $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ in X^{-1} as $n \to \infty$, where $\{x_n\} \subset X$.
- If for each $(PS)_c$ sequence $\{x_n\}$ in X there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0$ in X for some $x_0 \in X$, then one says that the functional Φ satisfies the so called $(PS)_c$ condition.
- $o_n(1)$ denotes the real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- " $\to$ " and " $\rightharpoonup$ " stand for the strong and weak convergence in the related function spaces, respectively.

2. **Preliminary results.** In this section, we introduce some preliminary results. For the potential $V \in C^0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we can define an operator $L := (-\Delta)^s + V$. Thanks to the celebrated results in [8, Theorem 4.26], one sees that L is self-disjoint with domain $\mathcal{D}(L) = H^s(\mathbb{R}^3)$. Assume $|L|$ and $|L|^{1/2}$ are the absolute values of L and the square root of $|L|$, respectively. We denote $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ by the spectral family with respective to L. Setting $U := \mathrm{id} - \mathcal{E}(0) - \mathcal{E}(0^-)$, then by virtue of [7, Theorem IV 3.3], U commutes with L, $|L|$, and $|L|^{1/2}$. Moreover, $L = U|L|$ is the polar decomposition of L. In view of [19, 26, 27], it holds that

$$X = \mathcal{D}(|L|^{1/2}), \ Y = \mathcal{E}(0^-)X \text{ and } Z = [\mathrm{id} - \mathcal{E}(0)]X.$$

Via (V), one has $X = Y \bigoplus Z$. Given $u \in X$, then $u = u^+ + u^-$ with

$$u^- = \mathcal{E}(0^-)u := Pu \text{ and } u^+ = [\mathrm{id} - \mathcal{E}(0)]u := Qu.$$

Furthermore, for all $u \in X \cap \mathcal{D}(L)$, one also has that

$$Lu^- = -|L|u^- \text{ and } Lu^+ = |L|u^+. \tag{6}$$

With the above facts in hand, we may introduce an inner product which can induce the norm on X as follows:

$$(u,v)_X = (|L|^{1/2}u, |L|^{1/2}v)_{L^2(\mathbb{R}^3)} \text{ and } \|u\|_X = \left\||L|^{1/2}u\right\|_{L^2(\mathbb{R}^3)},$$

where $(\cdot,\cdot)_{L^2(\mathbb{R}^3)}$ stands for the usual inner product of $L^2(\mathbb{R}^3)$. By (6), we have that

$$\int_{\mathbb{R}^3} [|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)|u|^2]dx = \|Qu\|_X^2 - \|Pu\|_X^2, \ \forall u \in X. \tag{7}$$

In particular, it holds that

$$\begin{cases} \int_{\mathbb{R}^3}[|\nabla Qu|^2 + V(x)|Qu|^2]dx = \|Qu\|_X^2, \\ \int_{\mathbb{R}^3}[|\nabla Pu|^2 + V(x)|Pu|^2]dx = -\|Pu\|_X^2. \end{cases}$$

Since $\|\cdot\|_X$ and $\|\cdot\|_{H^s(\mathbb{R}^3)}$ are equivalent by (H_1) (see [15] for example), there exists a constant $S_p > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^3)} \le S_p\|u\|_X, \ \forall p \in [2, 2_s^*], \tag{8}$$

and by [28], it holds that

$$\|u\|_{L^2(\mathbb{R}^3)} \le C_t\|u\|_{L^{t,2}(\mathbb{R}^3)}. \tag{9}$$

Moreover, one knows that X and $L^{t,2}(\mathbb{R}^3)$ can be compactly imbedded into $L^p_{\text{loc}}(\mathbb{R}^3)$ with $2 \le p < 2_s^* := \frac{6}{3-2s}$ and $L^2_{\text{loc}}(\mathbb{R}^3)$, respectively.

2.1. **Formulation of problem** (1). In this subsection, we assume that $\frac{3}{4} < s < 1$ and $0 < t < 1$. Considering a fixed $u \in X$, the linear functional $\mathcal{L}_u : L^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$\mathcal{L}_u(v) := \int_{\mathbb{R}^3} K(x)u^2v\,dx.$$

It is bounded since Hölder's inequality, (8)-(9), and (H_2) show that

$$|\mathcal{L}_u(v)| \le \left(\int_{\mathbb{R}^3} K(x)|u|^4 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} K(x)|v|^2 dx\right)^{\frac{1}{2}}$$

$$\le \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} C_t \|K\|_{L^{\frac{3}{4s-3}}(\mathbb{R}^3)}^{\frac{1}{2}} S_{2_s^*}^2 \|u\|_X^2 \|v\|_{L^{t,2}(\mathbb{R}^3)}, \ \forall v \in L^{t,2}(\mathbb{R}^3).$$

So, due to the Lax-Milgram theorem, there is a unique $\phi_u \in L^{t,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} K(x)u^2v\,dx = \mathcal{L}_u(v) = (\phi_u, v)_{L^{t,2}(\mathbb{R}^3)}. \tag{10}$$

From this, we can derive from the Plancherel theorem [18, Theorem 5.3] that ϕ_u is a weak solution of $(I - \Delta)^t\phi = K(x)u^2$, and so $\phi_u(x) = G_{2t} * (K(x)u^2)$.

Letting $v = \phi_u$ in (10), one has

$$\|\phi_u\|_{L^{t,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \le \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} C_t \|K\|_{L^{\frac{3}{4s-3}}(\mathbb{R}^3)}^{\frac{1}{2}} S_{2_s^*}^2 \|u\|_X^2 \|\phi_u\|_{L^{t,2}(\mathbb{R}^3)},$$

which in turn implies that

$$\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \le \|K\|_{L^\infty(\mathbb{R}^3)} C_t^2 \|K\|_{L^{\frac{3}{4s-3}}(\mathbb{R}^3)} S_{2_s^*}^4 \|u\|_X^4, \ \forall u \in X. \tag{11}$$

Inserting ϕ_u into (1), it holds that

$$(-\Delta)^s u + V(x)u = K(x)\phi_u u + g(x)|u|^{q-2}u, \ x \in \mathbb{R}^3, \tag{12}$$

and its corresponding Euler-Lagrange functional $\varphi : X \to \mathbb{R}$ is defined by

$$\varphi(u) = \frac{1}{2}\|Qu\|_X^2 - \frac{1}{2}\|Pu\|_X^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \frac{1}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx. \qquad (13)$$

In view of **one** of $(H_3)-(H_5)$, (7), and (11), it therefore would be standard to show that φ is well-defined on X and belongs to the class of $C^1(X,\mathbb{R})$ such that

$$\varphi'(u)[v] = \int_{\mathbb{R}^3} \left[(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + V(x)uv - K(x)\phi_u uv - g(x)|u|^{q-2}uv\right] dx, \ v \in X.$$

Obviously, the critical points of φ are the weak solutions of problem (12).

Definition 2.1. Let $\frac{3}{4} < s < 1$ and $0 < t < 1$.

(1) We call $(u,\phi) \in H^s(\mathbb{R}^3) \times L^{t,2}(\mathbb{R}^3)$ a weak solution of problem (1) if u is a weak solution of problem (12).

(2) We call $u \in H^s(\mathbb{R}^3)$ a weak solution of (12) if

$$\int_{\mathbb{R}^3} \left[(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + V(x)uv + K(x)\phi_u uv - g(x)|u|^{q-2}uv\right] dx = 0,$$

for any $v \in H^s(\mathbb{R}^3)$.

Let us define the variational functional $\Psi : X \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx = \int_{\mathbb{R}^3} \left[G_{2t} * (K(x)u^2)\right]K(x)u^2 dx.$$

In the following, we collect some important properties for Ψ.

Lemma 2.2. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$ as well as $(H_1)-(H_2)$, then the following conclusions hold true:*

(i) *$\Psi(u) \geq 0$ and $\Psi(\theta u) = \theta^4 \Psi(u)$ for any $u \in X$ and $\theta > 0$;*

(ii) *If $u_n \rightharpoonup u$ in X, $u_n \to u$ in $L_{loc}^p(\mathbb{R}^3)$ with $2 < p < 2_s^*$ and $u_n \to u$ a.e. in $\mathbb{R}^3$ as $n \to \infty$, then, going to a subsequence if necessary,*

$$\Psi(u_n) \to \Psi(u) \text{ and } \Psi'(u_n)[\psi] \to \Psi'(u)[\psi]$$

for every $\psi \in X$ as $n \to \infty$;

(iii) *For any fixed constant $\alpha \in (0,1)$ and every finite dimension subspace $Z_0 \subset Z$, there exists a constant $C_\alpha > 0$ such that for each $u \in Y \bigoplus Z_0$ with $\|u\|_X = 1$ it holds that*

$$\|Qu\|_X \geq \sin(\arctan\alpha) \implies \Psi(u) \geq C_\alpha.$$

Proof. (i) In consideration of $K(x) > 0$ and $G_{2t}(x) > 0$ for all in $x \in \mathbb{R}^3$ by (H_2) and (4), respectively, we immediately derive that $\Psi(u) \geq 0$ according to its definition. Moreover, for all $\theta > 0$, it is simple to see that

$$\Psi(\theta u) = \int_{\mathbb{R}^3} \left[G_{2t} * (K(x)(\theta u)^2)\right]K(x)(\theta u)^2 dx$$

$$= \theta^4 \int_{\mathbb{R}^3} \left[G_{2t} * (K(x)u^2)\right]K(x)u^2 dx$$

$$= \theta^4 \Psi(u), \ \forall u \in X,$$

proving point (i).

(ii) We can rewrite $\Psi(u_n) - \Psi(u)$ as

$$\Psi(u_n) - \Psi(u) = \int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u_n^2 dx + \int_{\mathbb{R}^3} K(x)\phi_u(u_n^2 - u^2)dx$$

$$:= I_n^1 + I_n^2.$$

Recalling $\phi_{u_n} - \phi_{u_n} = G_{2t} * (K(x)(u_n^2 - u^2))$ and $\|G_{2t}\|_{L^1(\mathbb{R}^3)} = 1$, we then apply Young's inequality with respect to the convolution operator and $K \in L^{\frac{3}{4s-3}}(\mathbb{R}^3)$ in (H_2) to get

$$\|\phi_{u_n} - \phi_{u_n}\|_{L^2(\mathbb{R}^3)}$$

$$\leq \left(\int_{\mathbb{R}^3} K^2(x)|u_n - u|^2 |u_n + u|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} K(x)|u_n - u|^4 dx \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}^3} K(x)|u_n + u|^4 dx \right)^{\frac{1}{4}}$$

$$\leq \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{3}{4}} S_4 \|u_n\|_X \left(\int_{\mathbb{R}^3} K(x)|u_n - u|^4 dx \right)^{\frac{1}{4}}$$

$$= o_n(1),$$

where the last equality follows the generalized Vitali's Convergence theorem. Thus,

$$|I_n^1| \leq \left(\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} K(x)|u_n|^4 dx \right)^{\frac{1}{2}}$$

$$\leq \|K\|_{L^\infty(\mathbb{R}^3)} S_4^2 \|u_n\|_X^2 \|\phi_{u_n} - \phi_{u_n}\|_{L^2(\mathbb{R}^3)}$$

$$= o_n(1).$$

Similarly, we can also obtain that

$$|I_n^2| \leq \left(\int_{\mathbb{R}^3} K(x)\phi_u^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} K(x)|u_n + u|^2 |u_n - u|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{3}{4}} C_t \|\phi_u\|_{L^{t,2}(\mathbb{R}^3)}^2 S_4 \|u_n + u\|_X \left(\int_{\mathbb{R}^3} K(x)|u_n - u|^4 dx \right)^{\frac{1}{4}}$$

$$= o_n(1).$$

Combining the above two facts, we derive the proof of the first part. The remaining part is easier, so we omit it here.

(iii) Let us define a constant $\gamma := \sin(\arctan \alpha) \in (0,1)$ and a set

$$\Upsilon^\alpha := \left\{ v \in Y \bigoplus Z_0 : \|v\|_X = 1 \text{ and } \|Qv\|_X \geq \gamma \right\}.$$

Due to the definition of u, one sees that $\Psi(u) \geq \inf_{v \in \Upsilon^\alpha} \Psi(v) := C_\alpha$. Using point (i), it holds that $C_\alpha \geq 0$. So, to finish the proof, it suffices to conclude that $C_\alpha > 0$. Suppose, by contradiction, that $C_\alpha = 0$. Then, there exists a sequence $\{v_n\} \subset \Upsilon^\alpha$ such that $\Psi(v_n) \to 0$ as $n \to \infty$. Since $\|v_n\|_X \equiv 1$, up to a subsequence if necessary, there is $v \in X$ such that $v_n \rightharpoonup v$ in X and $Qv_n \to Qv$ because $\{Qv_n\} \subset Z_0$ with $\dim Z_0 < +\infty$. Hence, $\|Qv\|_X \geq \gamma > 0$ and $\Psi(v) \leq \liminf_{n \to \infty} \Psi(v_n) = 0$, yielding that $v \equiv 0$, a contradiction to (ii). The proof is completed. $\qquad\square$

Next, we prove that the variational functional φ satisfies the $(PS)_c$ condition.

Lemma 2.3. *Let $\frac{3}{4} < s < 1$ and $0 < t < 1$. Suppose that $(H_1) - (H_2)$ and **one** of $(H_3) - (H_5)$ hold. Then φ satisfies the $(PS)_c$ condition.*

Proof. Assume that there is a sequence $\{u_n\} \subset X$ that satisfies $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ as $n \to \infty$. Then

$$c + 1 + \|u_n\|_X \geq \varphi(u_n) - \frac{1}{2}\varphi'(u_n)[u_n]$$

$$= \frac{1}{4}\Psi(u_n) + \frac{q-2}{2q}\int_{\mathbb{R}^3} g(x)|u_n|^q dx. \tag{14}$$

To show that $\|u_n\|_X$ is uniformly bounded in $n \in \mathbb{N}$, we split it into two cases.

Case 1. Assumption (H_3) holds.

In this case, that is, $g \in L^{q_0}(\mathbb{R}^3)$ with $q_0 = \frac{6}{6-q(3-2s)}$ and $1 < q < \frac{4}{3}$, we can adopt (14) together with (8) to have that

$$\Psi(u_n) \leq 4(c+1+\|u_n\|_X) + \frac{q}{2(2-q)}\left(\int_{\mathbb{R}^3}|g|^{q_0}dx\right)^{\frac{1}{q_0}}\left(\int_{\mathbb{R}^3}|u_n|^{2^*}dx\right)^{\frac{q}{2_s^*}}$$

$$\leq C(1 + \|u_n\|_X + \|u_n\|_X^q).$$

Denoting $u_n = Pu_n + Qu_n := y_n + z_n$ with $y_n \in Y$ and $z_n \in Z$, then

$$\left|\Psi'(u_n)[y_n]\right| = \left|\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_ny_ndx\right|$$

$$\leq \Psi^{\frac{1}{2}}(u_n)\left(\int_{\mathbb{R}^3}K(x)\phi_{u_n}y_n^2dx\right)^{\frac{1}{2}}$$

$$\leq \Psi^{\frac{1}{2}}(u_n)\left(\int_{\mathbb{R}^3}K(x)\phi_{u_n}^2dx\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}^3}K(x)y_n^4dx\right)^{\frac{1}{4}}$$

$$\leq \|K\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}}C_t^{\frac{1}{2}}S_4\Psi^{\frac{3}{4}}(u_n)\|y_n\|_X$$

$$\leq C(1 + \|u_n\|_X + \|u_n\|_X^q)^{\frac{3}{4}}\|y_n\|_X$$

$$\leq C(1 + \|u_n\|_X + \|u_n\|_X^q)^{\frac{3}{4}}\|u_n\|_X. \tag{15}$$

Thereby, for $n \in \mathbb{N}$ large, combining $\|u_n\|_X \geq \|y_n\|_X \geq -\varphi'(u_n)[y_n]$ and (15), we obtain

$$\|y_n\|_X^2 = -\varphi'(u_n)[y_n] - \Psi'(u_n)[y_n] - \int_{\mathbb{R}^3}g(x)|u_n|^{q-2}u_ny_ndx$$

$$\leq \|u_n\|_X + C(1 + \|u_n\|_X + \|u_n\|_X^q)^{\frac{3}{4}}\|u_n\|_X + C\|u_n\|_X^{q-1}\|y_n\|_X.$$

Similarly, we deduce that

$$\|z_n\|_X^2 \leq \|u_n\|_X + C(1 + \|u_n\|_X + \|u_n\|_X^q)^{\frac{3}{4}}\|u_n\|_X + C\|u_n\|_X^{q-1}\|z_n\|_X.$$

Recalling the fact that $\|u_n\|_X^2 = \|y_n\|_X^2 + \|z_n\|_X^2$, we know that $\|u_n\|_X$ is uniformly bounded since $2 > 1 + \frac{3}{4}q$, which is $q < \frac{4}{3}$ by (H_3).

Case 2. Either assumption (H_4) or (H_5) holds.

Obviously, both (H_4) and (H_5) indicate that $(q-2)g(x) > 0$ a.e. in $\mathbb{R}^3$. Therefore, it follows from (14) that

$$\Psi(u_n) \leq C(1 + \|u_n\|_X)$$

and

$$0 \leq (q-2)\int_{\mathbb{R}^3}g(x)|u_n|^qdx \leq C(1 + \|u_n\|_X).$$

This is similar to (15) in that $|\Psi'(u_n)[y_n]| \leq C(1 + \|u_n\|_X)^{\frac{3}{4}}\|u_n\|_X$ and then

$$\|y_n\|_X^2 \leq \|u_n\|_X + C(1 + \|u_n\|_X)^{\frac{3}{4}}\|u_n\|_X + C(1 + \|u_n\|_X)^{\frac{q-1}{q}}\|y_n\|_X.$$

Analogously, one has

$$\|z_n\|_X^2 \leq \|u_n\|_X + C(1 + \|u_n\|_X)^{\frac{3}{4}}\|u_n\|_X + C(1 + \|u_n\|_X)^{\frac{q-1}{q}}\|z_n\|_X.$$

So, we still can derive that $\|u_n\|_X$ is uniformly bounded since $2 > 1 + \frac{3}{4}$.

Based on the above discussions, we can conclude that $\|u_n\|_X$ is uniformly bounded in $n \in \mathbb{N}$. Moreover, one deduces that $\|y_n\|_X$ and $\|z_n\|_X$ are uniformly bounded in $n \in \mathbb{N}$. Passing to subsequences if necessary, there exist two functions $y \in Y$ and $z \in Z$ such that, as $n \to \infty$,

$$\begin{cases} y_n \rightharpoonup y \text{ and } z_n \rightharpoonup z, & \text{in } X, \\ y_n \to y \text{ and } z_n \rightharpoonup z, & \text{in } L^p_{\text{loc}}(\mathbb{R}^3) \text{ with } 2 \leq p < 2^*_s, \\ y_n \to y \text{ and } z_n \rightharpoonup z, & \text{a.e. in } \mathbb{R}^3 \end{cases}$$

Define $u := y + z \in X$. Then, we aim to show that $y_n \to y$ in X along a subsequence. Actually, we can argue as in the calculations of (15) to obtain that

$$(\Psi'(u_n) - \Psi'(u))[y_n - y] = o_n(1)$$

as $n \to \infty$. Since $g \in L^{q_0}(\mathbb{R}^3)$, it follows from the generalized Vitali's Convergence theorem again that

$$\left| \int_{\mathbb{R}^N} g(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(y_n - y)dx \right|$$
$$\leq \left(\int_{\mathbb{R}^N} |g(x)| \left| |u_n|^{q-2}u_n - |u|^{q-2}u \right|^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^N} |g(x)||y_n - y|^q dx \right)^{\frac{1}{q}}$$
$$= o_n(1)$$

as $n \to \infty$. Combining the above two formulas and $\varphi'(u_n) \to 0$, it holds that

$$\begin{aligned} o_n(1) &= \left(\varphi'(u_n) - \varphi'(u) \right)[y_n - y] \\ &= -\|y_n - y\|_X^2 - \left(\Psi'(u_n) - \Psi'(u) \right)[y_n - y] \\ &\quad - \int_{\mathbb{R}^N} g(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(y_n - y)dx \\ &= -\|y_n - y\|_X^2 = o_n(1) \end{aligned}$$

which indicates the desired result. Analogously, we can also derive that $\|z_n - z\|_X = o_n(1)$. So, one sees that $u_n \to u$ in X, finishing the proof. $\qquad\square$

3. **Proofs of the main results.** In this section, we give the proofs of Theorems 1.1, 1.2, and 1.3 in detail. Toward this goal, we need to introduce the following improved Fountain theorem developed by Gu and Zhou in [12], that is,

Proposition 3.1. *Let $\Phi \in C^1(X, \mathbb{R})$ be an even functional satisfying the $(PS)_c$ condition and $\nabla\Phi$ be weakly sequentially continuous, for all $k \in \mathbb{N}$ with $X = Y_k \bigoplus Z_k$, if there exist two constants $\rho_k > r_k > 0$ such that:*

(A_1) $d_k := \sup\limits_{u \in Y_k, \|u\|_X \leq \rho_k} \Phi(u) < +\infty;$

(A_2) $a_k := \sup\limits_{u \in Y_k, \|u\|_X = \rho_k} \Phi(u) < \inf\limits_{u \in Z_k, \|u\|_X \leq r_k} \Phi(u);$

(A_3) $b_k := \inf\limits_{u \in Z_k, \|u\|_X = r_k} \Phi(u) \to +\infty$ *as* $k \to +\infty;$

(A_4) *For any $\sigma > 0$, there is a constant $C_\sigma > 0$ such that $\sup\limits_{\|u\|_\tau < \sigma} \Phi(u) \le C_\sigma < +\infty$,*

where

$$\|u\|_\tau := \max\left\{ \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle Pu, e_j \rangle|, \|Qu\|_X \right\} \quad \text{for } u \in X, \tag{16}$$

with $\{e_j\}_{j=1}^{\infty}$ denoting the normal orthogonal basis of Y.

Then, the functional Φ has a sequence of critical points $\{u^{k_m}\}$ such that

$$\lim_{m \to \infty} \|u^{k_m}\|_X = +\infty.$$

Let us recall the decomposition on $X = Y \bigoplus Z$ in Section 2, and suppose that the two sequences $\{e_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ are normal orthogonal basis of Y and Z, respectively. For any $k \in \mathbb{N}$, define

$$Y_k := Y \bigoplus \left(\bigoplus_{j=1}^{k} \mathbb{R}f_j \right) \text{ and } Z_k := \overline{\bigoplus_{j=k+1}^{\infty} \mathbb{R}f_j}.$$

In order to apply Proposition 3.1 successfully, we shall regard the variational functional Φ as φ, which corresponds to Eq. (13). From now until the end of this section, we shall always suppose that $\frac{3}{4} < s < 1$ and $0 < t < 1$, and thus do not mention them any longer.

First of all, we show that the functional φ satisfies condition (A_1).

Lemma 3.2. *Assume $(H_1) - (H_2)$ and **one** of $(H_3) - (H_5)$ hold. Then there exists $\rho_k > 0$ such that*

$$d_k := \sup_{u \in Y_k, \|u\|_X \le \rho_k} \Phi(u) < +\infty.$$

Proof. The proof is standard since φ maps a bounded set into a bounded set, so we omit it here. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we verify that the functional φ satisfies conditions (A_2) and (A_3).

Lemma 3.3. *Assume $(H_1) - (H_2)$ and **one** of $(H_3) - (H_5)$ hold. Then there exist $\rho_k > r_k > 0$ such that $a_k < \inf\limits_{u \in Z_k, \|u\|_X \le r_k} \varphi(u)$ and $b_k \to +\infty$ as $k \to +\infty$.*

Proof. First, to derive the first part, we just need to show that

$$a_k := \sup_{u \in Y_k, \|u\|_X = \rho_k} \varphi(u) \to -\infty$$

as $\rho_k \to +\infty$. Either (H_3) or (H_4) holds true, for all $u \in X$, and it holds that

$$-\frac{1}{q} \int_{\mathbb{R}^3} g(x)|u|^q dx \le C\|u\|_X^q, \ 1 < q < 2,$$

for some positive constant C which is independent of u. If (H_5) holds true, because $g(x) > 0$ for all $x \in \mathbb{R}^3$, given $u \in X$, it holds that

$$-\frac{1}{q} \int_{\mathbb{R}^3} g(x)|u|^q dx \le 0.$$

In summary, for any fixed $u = Qu + Pu \in X$, we can always deduce that

$$\varphi(u) \le \frac{1}{2}\|Qu\|_X^2 - \frac{1}{2}\|Pu\|_X^2 - \frac{1}{4}\Psi(u) + C\|u\|_X^q, \text{ where } 1 < q < 2. \tag{17}$$

Now, we begin verifying that $a_k \to -\infty$ as $\rho_k \to +\infty$. Given a $u \in Y_k = Y \bigoplus \left(\bigoplus_{j=1}^{k} \mathbb{R} f_j \right)$ with $\|u\|_X = \rho_k$.

If $u \in Y$, then $Qu = 0$ and $Pu = u$, and it follows from (17) that

$$\varphi(u) \le -\frac{1}{2}\rho_k^2 + C\rho_k^q \to -\infty$$

as $\rho_k \to +\infty$ since $q < 2$ in (17).

If $u = y + z$ with $y \in Y$ and $z \in \bigoplus_{j=1}^{k} \mathbb{R} f_j$, for the constant $\alpha \in (0,1)$ given by Lemma 2.2-(iii), we shall distinguish the proof with two cases:

$$\text{(i) } \|Qu\|_X/\|Pu\|_X < \alpha \text{ and (ii) } \|Qu\|_X/\|Pu\|_X \ge \alpha.$$

If (i) occurs, then $\|u\|_X^2 = \|Qu\|_X^2 + \|Pu\|_X^2 < (1+\alpha^2)\|Pu\|_X^2$ which in turn shows that

$$\|Qu\|_X^2 = \|u\|_X^2 - \|Pu\|_X^2 < \frac{\alpha^2}{1+\alpha^2}\|u\|_X^2.$$

Combining $\alpha \in (0,1)$ in Lemma 2.2-(iii) and (17) with $q < 2$, we obtain

$$\varphi(u) \le \frac{\alpha^2}{2(1+\alpha^2)}\|u\|_X^2 - \frac{1}{2(1+\alpha^2)}\|u\|_X^2 + C\|u\|_X^q$$

$$= -\frac{1-\alpha^2}{2(1+\alpha^2)}\rho_k^2 + C\rho_k^q$$

$$\to -\infty \text{ as } \|u\|_X = \rho_k \to +\infty.$$

If (ii) occurs, we set $v := \frac{u}{\|u\|_X}$ and so $\|v\|_X = 1$. Moreover, one has that

$$\|Qv\|_X = \frac{\|Qu\|_X}{\|u\|_X} = \sin\left(\arctan\frac{\|Qu\|_X}{\|Pu\|_X}\right).$$

In this case, one deduces $\|Qv\|_X \ge \sin(\arctan\alpha)$. Accordingly, $v \in Y \bigoplus \left(\bigoplus_{j=1}^{k} \mathbb{R} f_j \right)$ with $\dim \left(\bigoplus_{j=1}^{k} \mathbb{R} f_j \right) < +\infty$, and as a consequence of Lemma 2.2-(i) and (iii), we obtain $\Psi(u)\|u\|_X^{-4} = \Psi(v) \ge C_\alpha > 0$. Then, adopting (17) again, we arrive at

$$\varphi(u) \le \frac{1}{2}\|u\|_X^2 - \frac{C_\alpha}{4}\|u\|_X^4 + C\|u\|_X^q$$

$$= \frac{1}{2}\rho_k^2 - \frac{C_\alpha}{4}\rho_k^4 + C\rho_k^q$$

$$\to -\infty \text{ as } \|u\|_X = \rho_k \to +\infty.$$

Therefore, we always have $a_k \to -\infty$ as $\rho_k \to +\infty$ and the first part is concluded.

Second, we claim that

$$\beta_k^1 := \sup_{u \in Z_k : \|u\|_X = 1} \Psi(u) \to 0$$

as $k \to +\infty$. Indeed, according to the definition of Z_k, one simply has $0 < \beta_{k+1}^1 \le \beta_k^1$ for any $k \in \mathbb{N}$. Then there exists a constant $\beta \ge 0$ such that $\beta_k^1 \to \beta$ as $k \to +\infty$. By the definition of β_k^1, there is a $u_k \in Z_k$ with $\|u_k\|_X = 1$ such that $\Psi(u_k) \ge \beta_k^1/2$.

Since $\{u_k\} \subset Z_k = \overline{\bigoplus_{j=k+1}^{\infty} \mathbb{R}f_j}$, one concludes that $u_k \rightharpoonup 0$ in X as $k \to +\infty$. In view of Lemma 2.2-(ii), passing to a subsequence if necessary, we derive $\Psi(u_k) \to 0$ as $k \to +\infty$. Hence, we must have $\beta = 0$ and the claim holds true.

Finally, to prove $b_k \to +\infty$ as $k \to +\infty$, we split it into two cases.

Case 1. Either assumption (H_3) or (H_4) holds true.

In this case, combining $q < 2$ and $\beta_k^1 \to 0$ as $k \to +\infty$, it holds that

$$\frac{1}{4}(\beta_k^1)^{\frac{q-2}{2}} - \frac{C}{q} \geq \frac{1}{8} \text{ for some sufficiently large } k \in \mathbb{N},$$

where $C > 0$ is a constant independent of k. As a consequence, for any $u \in Z_k$ with $\|u\|_X = r_k = (\beta_k^1)^{-\frac{1}{2}}$, by means of $g \in L^{q_0}(\mathbb{R}^3)$ we have

$$\begin{aligned}
\varphi(u) &\geq \frac{1}{2}\|u\|_X^2 - \frac{\beta_k^1}{4}\|u\|_X^4 - \frac{C}{q}\|u\|_X^q \\
&= \left(\frac{1}{4}\|u\|_X^{2-q} - \frac{C}{q}\right)\|u\|_X^q + \frac{1}{4}(1 - \beta_k^1\|u\|_X^2)\|u\|_X^2 \\
&= \left(\frac{1}{4}\|u\|_X^{2-q} - \frac{C}{q}\right)\|u\|_X^q \\
&\geq \frac{1}{8}\|u\|_X^q \\
&= \frac{1}{8}(\beta_k^1)^{-\frac{q}{2}} \\
&:= \frac{1}{8}r_k^q,
\end{aligned}$$

for some sufficiently large $k \in \mathbb{N}$. Clearly, $r_k = (\beta_k^1)^{-\frac{1}{2}} \to +\infty$ as $k \to +\infty$. Hence, it holds that $b_k \to +\infty$ as $k \to +\infty$.

Case 2. Assumption (H_5) holds true.

Since $g(x) > 0$ a.e. for $x \in \mathbb{R}^3$ and $g \in L^{q_0}(\mathbb{R}^3)$, for any $k \in \mathbb{N}$, we can verify that

$$\beta_k^2 := \sup_{u \in Z_k : \|u\|_X = 1} \int_{\mathbb{R}^3} g(x)|u|^q dx \to 0.$$

Indeed, one can easily get that $0 < \beta_{k+1}^2 \leq \beta_k^2$ for any $k \in \mathbb{N}$, and so there is a constant $\hat{\beta} \geq 0$ such that $\beta_k^2 \to \hat{\beta}$ as $k \to +\infty$. By the definition of β_k^2, there is a $u_k \in Z_k$ with $\|u_k\|_X = 1$ such that $\int_{\mathbb{R}^3} g(x)|u_k|^q dx \geq \beta_k^2/2$. Recalling $\{u_k\} \subset Z_k = \overline{\bigoplus_{j=k+1}^{\infty} \mathbb{R}f_j}$, one concludes that $u_k \rightharpoonup 0$ in X as $k \to +\infty$. Due to $g \in L^{q_0}(\mathbb{R}^3)$, one immediately has that $\int_{\mathbb{R}^3} g(x)|u_k|^q dx \to 0$, and so $\hat{\beta} = 0$. Therefore, for all $u \in Z_k$ with $\|u\|_X = r_k = \min\left\{\sqrt{\frac{1}{2\beta_k^1}}, \left(\frac{q}{8\beta_k^2}\right)^{\frac{1}{q-2}}\right\}$, we have

$$\begin{aligned}
\varphi(u) &\geq \frac{1}{4}\|u\|_X^2 + \left(\frac{1}{8} - \frac{\beta_k^1}{4}\|u\|_X^2\right)\|u\|_X^2 + \left(\frac{1}{8} - \frac{\beta_k^2}{q}\|u\|_X^{q-2}\right)\|u\|_X^2 \\
&\geq \frac{1}{4}\|u\|_X^2 \\
&= \frac{1}{4}\min\left\{\frac{1}{2\beta_k^1}, \left(\frac{q}{8\beta_k^2}\right)^{\frac{2}{q-2}}\right\}
\end{aligned}$$

$$:= \frac{1}{4} r_k^2.$$

It is easy to see that $r_k \to +\infty$ as $k \to +\infty$ since $q > 2$ in this case. So, $b_k \to +\infty$ as $k \to +\infty$. The proof of this lemma is completed. $\square$

Finally, condition (A_4) will be proved for the functional φ as follows.

Lemma 3.4. *Assume* $(H_1) - (H_2)$ *and* **one** *of* $(H_3) - (H_5)$ *hold. Then, for any* $\sigma > 0$, *there is a constant* $C_\sigma > 0$ *such that* $\sup\limits_{\|u\|_\tau < \sigma} \varphi(u) \le C_\sigma < +\infty.$

Proof. Since $u \in X = Y \bigoplus Z$ with $Z = Y^\perp$, we may set $u = Qu + Pu$ with $Qu \in Z$ and $Pu \in Y$. Then, using very similar calculations as in (17), we reach

$$\varphi(u) = \frac{1}{2}\|Qu\|_X^2 - \frac{1}{2}\|Pu\|_X^2 - \frac{1}{4}\Psi(u) - \frac{1}{q}\int_{\mathbb{R}^N} g(x)|u|^q \, dx$$

$$\le \frac{1}{2}\|Qu\|_X^2 - \frac{1}{2}\|Pu\|_X^2 + C\|u\|_X^q$$

$$\le \frac{1}{2}\|Qu\|_X^2 + C\|Qu\|_X^q - \frac{1}{2}\|Pu\|_X^2 + C\|Pu\|_X^q.$$

Since $q < 2$ in (17), $-\frac{1}{2}\|Pu\|_X^2 + C\|Pu\|_X^q$ is bounded from above. By (16), we have $\|Qu\|_X \le \|u\|_\tau \le \sigma$, and thus there exists a $C_\sigma < \infty$ such that $\sup\limits_{\|u\|_\tau \le \sigma} \varphi(u) < C_\sigma.$

The proof is completed. $\square$

With Lemmas 3.2, 3.3, and 3.4 in hand, we derive that the variational functional φ has infinitely many nontrivial critical points.

Lemma 3.5. *Assume* $(H_1) - (H_2)$ *and* **one** *of* $(H_3) - (H_5)$ *hold. Then* φ *has infinitely many nontrivial critical points* $\{u_n\} \subset H^s(\mathbb{R}^3)$ *such that* $\lim\limits_{n\to\infty} \|u_n\|_{H^s(\mathbb{R}^3)} = +\infty.$

Proof. Let $\varphi = \Phi$ be as in Proposition 3.1. Then clearly φ is even. Owing to Lemmas 2.2-(ii) and 2.3, we clearly know that φ satisfies $(PS)_c$ condition and $\nabla\varphi$ is weakly sequentially continuous. On the other hand, we have validated conditions (A_1), (A_2), (A_3), and (A_4) for φ in Lemmas 3.2, 3.3, and 3.4, respectively. As a consequence of Proposition 3.1, we can finish the proof of this lemma. $\square$

At this stage, we can conclude the proofs of Theorems 1.1, 1.2, and 1.3 depending on Lemma 3.5.

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