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Ground states for fractional Kirchhoff equations with critical nonlinearity in low dimension

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Abstract. We study the existence of ground states to a nonlinear fractional Kirchhoff equation with an external potential V. Under suitable assumptions on V, using the monotonicity trick and the profile decomposition, we prove the existence of ground states. In particular, the nonlinearity does not satisfy the Ambrosetti–Rabinowitz type condition or monotonicity assumptions.

Mathematics Subject Classification. 35Q55, 35Q51, 53C35.

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Contents

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1. Introduction and results

1.1. Overview

In this paper we are concerned with the existence of positive ground state solutions to the following nonlinear fractional Kirchhoff equation

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u|^2dx\right)(-\Delta)^\alpha u+V(x)u=f(u) \quad \text{in } \mathbb{R}^N, \\ u\in H^\alpha(\mathbb{R}^N), \quad u>0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{K}$$

where a,b are positive constants, $\alpha\in(0,1)$ and $N>2\alpha$. The operator $(-\Delta)^\alpha$ is the fractional Laplacian defined as $\mathscr{F}^{-1}(|\xi|^{2\alpha}\mathscr{F}(u))$, where $\mathscr{F}$ denotes the Fourier transform on $\mathbb{R}^N$. When $a=1$ and $b=0$, then (K) reduces to the following fractional Schrödinger equation

$$(-\Delta)^\alpha u+V(x)u=f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

which has been proposed by Laskin [20] in fractional quantum mechanics as a result of extending the Feynman integrals from the Brownian like to the Lévy like quantum mechanical paths. For such a class of fractional and nonlocal problems, Caffarelli and Silvestre [8] expressed $(-\Delta)^\alpha$ as a Dirichlet–Neumann map for a certain local elliptic boundary value problem on the half-space. This method is a valid tool to deal with equations involving fractional operators to get regularity and handle variational methods. We refer the readers to [16, 34] and to the references therein. Investigated first in [12,13] via variational methods, there has been a lot of interest in the study of the existence and multiplicity of solutions for (1.1) when V and f satisfy general conditions. We cite [11,33,36] with no attempts to provide a complete list of references.

If $\alpha=1$, then problem (K) formally reduces to the well-known Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+V(x)u=f(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

related to the stationary analogue of the Kirchhoff–Schrödinger type equation

$$\frac{\partial^2 u}{\partial t^2}-\left(a+b\int_\Omega|\nabla u|^2dx\right)\Delta u=f(t,x,u),$$

where Ω is a bounded domain in $\mathbb{R}^N$, u denotes the displacement, f is the external force, b is the initial tension and a is related to the intrinsic properties of the string. Equations of this type were first proposed by Kirchhoff [19] to describe the transversal oscillations of a stretched string. Besides, we also point out that such nonlocal problems appear in other fields like biological systems, where u describes a process depending on the average of itself. We refer readers to Chipot and Lovat [10], Alves and Corrêa [1]. However, the solvability of the Kirchhoff type equations has been well studied in a general dimension by various authors only after Lions [23] introduced an abstract framework to such problems. For more recent results concerning Kirchhoff-type equations we refer e.g. to [4,17,26,28].

In [22], by using a monotonicity trick and a global compactness lemma, Li and Ye proved that for $f(u) = |u|^{p-2}u$ and $p \in (3, 2N/(N-2))$, problem (1.2) has a positive ground state. Subsequently, Liu and Guo [25] extended the above result to $p \in (2, 2N/(N-2))$. Fiscella and Valdinoci [14], proposed the following stationary Kirchhoff variational equation with critical growth

$$\begin{cases} M\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u|^2dx\right)(-\Delta)^\alpha u = \lambda f(x,u) + |u|^{2^*_\alpha-2}u & \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

which models nonlocal aspects of the tension arising from measurements of the fractional length of the string. They obtained the existence of non-negative solutions when M and f are continuous functions satisfying suitable assumptions. Autuori et al. [3] considered the existence and the asymptotic behavior of non-negative solutions of (1.3). Pucci and Saldi [30] established multiplicity of nontrivial solutions. Via a three critical points theorem, Nyamoradi [27] studied the subcritical case of (1.3) and obtained three solutions. See also [9,15,29,31,39] for related results.

To the best of our knowledge, there are few papers in the literature on fractional Kirchhoff equations in $\mathbb{R}^N$. Recently, Ambrosio and Isernia [2] considered the fractional Kirchhoff problem

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u|^2dx\right)(-\Delta)^\alpha u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where f is an odd subcritical nonlinearity satisfying the well known Berestycki and Lions [6] assumptions. By minimax arguments, the authors establish a multiplicity result in the radial space $H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$ when the parameter b is *sufficiently small*. As in [22], Teng [37] also searched for ground state solutions for the fractional Schrödinger–Poisson system in $\mathbb{R}^3$ with critical growth

$$\begin{cases} (-\Delta)^\alpha u + V(x)u + \phi u = \mu|u|^{q-2}u + |u|^{2^*_\alpha-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

We point out that, in [22,37] the corresponding limit problems play an important role. In order to get the existence of ground state solutions of the limit problems, the authors used a constrained minimization on a manifold $\mathcal{M}$ obtained by combining the Nehari and Pohožaev manifolds.

1.2. Main results

Motivated by the works above, in this paper we aim to study the existence of positive ground state solutions to the fractional Kirchhoff equation with the Berestycki–Lions type conditions of *critical* type, firstly introduced in [40].

1.2.1. Assumptions on V. On the external potential we assume the following:

(V_1) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and, setting $W(x) := \max\{x \cdot \nabla V(x), 0\}$, we assume

$$\|W\|_{L^{\frac{N}{2\alpha}}(\mathbb{R}^N)} < 2a\alpha S_\alpha, \qquad S_\alpha := \inf_{\substack{u \in D^{\alpha,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*_\alpha} dx\right)^{2/2^*_\alpha}},$$

$$2^*_\alpha := \frac{2N}{N - 2\alpha};$$

(V_2) there exists $V_\infty \in \mathbb{R}$ such that

$$V(x) \le \lim_{|y| \to \infty} V(y) = V_\infty, \quad \text{for all } x \in \mathbb{R}^N;$$

(V_3) the operator $a(-\Delta)^\alpha + V(x) : H^\alpha(\mathbb{R}^N) \to H^{-\alpha}(\mathbb{R}^N)$ satisfies

$$\inf_{\substack{u \in H^\alpha(\mathbb{R}^N) \\ u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} \left(a|(-\Delta)^{\frac{\alpha}{2}} u|^2 + V(x)u^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} |u|^2 dx} > 0.$$

1.2.2. Assumptions on f. We assume that $f(t) = 0$ for all $t \le 0$ and

(f_1) $f \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\lim_{t \to 0} \frac{f(t)}{t} = 0$;

(f_2) $\lim_{t \to \infty} \frac{f(t)}{t^{2^*_\alpha - 1}} = 1$;

(f_3) there are $D > 0$ and $2 < q < 2^*_\alpha$ such that $f(t) \ge t^{2^*_\alpha - 1} + Dt^{q-1}$ for any $t \ge 0$.

Now we state our first result.

Theorem 1.1. *Assume* (V_1)–(V_3), (f_1)–(f_3) *and* $N = 2$ *with* $\alpha \in (\frac{1}{2}, 1)$ *or* $N = 3$ *with* $\alpha \in (\frac{3}{4}, 1)$.

(i) *If* $q \in (2, 2^*_\alpha)$, *there is* $D_1 > 0$ *such that, for* $D \ge D_1$, (K) *admits a positive ground state solution.*

(ii) *If* $q \in (\frac{4\alpha}{N-2\alpha}, 2^*_\alpha)$, *for any* $D > 0$, (K) *admits a positive ground state solution.*

Remark 1.2. It is worth pointing out that we have to restrict $\alpha \in (\frac{1}{2}, 1)$ when $N = 2$ or $\alpha \in (\frac{3}{4}, 1)$ if $N = 3$ in the process of proving the Mountain-Pass geometry for the corresponding energy functional. Moreover, in order to get a positive ground state solution, we construct a perturbed functional whose nontrivial critical points can be proved in such restrictions on α. However, when α is small, the argument in proving the two statements above does not work any more. For the details, see Lemmas 3.3 and 6.1.

We point out that without any symmetry assumption on V, the ground state solution obtained above maybe is not radially symmetric. In the following, we impose a monotonicity assumption of V and show that (K) admits a radially symmetric solution.

Assume now that V is radially symmetric and increasing, that is

$$\text{for all } x, y \in \mathbb{R}^N: \quad |x| \le |y| \ \Rightarrow \ V(x) \le V(y). \tag{V_4}$$

Theorem 1.3. *Under the assumptions of Theorem 1.1 and (V_4), (K) admits a radially symmetric positive solution at the global (unrestricted to radial paths) mountain pass energy level.*

As a main tool to prove Theorem 1.1 we shall give the profile decomposition of the Palais–Smale sequences by which we can derive some compactness and get a positive ground states for (K). The main tool for the proof of Theorem 1.3 is a symmetric version of the monotonicity trick [35]. We recall that Zhang and Zou [41] studied the critical case for Berestycki–Lions theorem of the Schrödinger equation $-\Delta u+V(x)u = f(u)$. They obtained positive ground state solutions when V satisfies similar assumptions as (V_1)–(V_3), f satisfies (f_1)–(f_3) and

(f_4) $|f'(t)| \leq C(1 + |t|^{\frac{4}{N-2}})$, for $t > 0$ and some $C > 0$.

We should mention that in the present paper, (f_4) is removed.

1.3. Main difficulties

We mention the difficulties and the idea in proving Theorem 1.1.

Firstly, without the Ambrosetti–Rabinowitz condition, it is difficult to get the *boundedness of Palais–Smale sequences*. In order to overcome this difficulty, inspired by [22], we will use the monotonicity trick developed by Jeanjean [18], introduce a family of functionals I_λ and obtain a bounded $(PS)_{c_\lambda}$ sequence for I_λ for almost all λ in an interval J, where c_λ is given in Sect. 3.

Secondly, by the presence of the Kirchhoff term, one obstacle arises in getting the compactness of I_λ, even in the subcritical case. Precisely, this does not hold in general: for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u_n (-\Delta)^{\frac{\alpha}{2}} \varphi dx \\ \to \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} \varphi dx,$$

where $\{u_n\}_{n\in\mathbb{N}}$ is a (PS)-sequence of I_λ satisfying $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R}^N)$. Then, even in the subcritical case, it is not clear that weak limits are critical points of I_λ. In [2], for (1.4) the compactness was recovered by restricting I_λ to the radial space $H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$, which is compactly embedded $L^s(\mathbb{R}^N)$ for all $s \in (2, 2^*_\alpha)$. For the related works in the bounded domains, see e.g. [14,30,39].

In the present paper, we do not impose any symmetry and just consider (K) in $H^\alpha(\mathbb{R}^N)$. So the arguments mentioned above cannot be applied. Inspired by [22], in place of I_λ, we consider a family of related functionals J_λ, whose corresponding problem is a *non Kirchhoff* equation.

Thirdly, the critical exponent makes the problem rather tough. The (PS)-condition does *not* hold in general and to overcome this difficulty, we show that the mountain pass level c_λ is strictly less than some critical level c^*_λ. For $-\Delta u + V(x)u = \lambda f(u)$ with critical growth, if S is the best constant of $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, one can show that [7]

$$c^*_\lambda = \frac{1}{N} S^{\frac{N}{2}} \lambda^{\frac{2-N}{2}}.$$

For $-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=\lambda f(u)$ in $\mathbb{R}^3$ involving critical growth [21,24]

$$c_\lambda^*=\frac{ab}{4\lambda}S^3+\frac{[b^2S^4+4\lambda aS]^{\frac{3}{2}}}{24\lambda^2}+\frac{b^3S^6}{24\lambda^2}.$$

However, for fractional Kirchhoff equations, to give the *exact value* of c_λ^* is complicated, since one cannot solve precisely a fractional order algebra equation. A careful analysis is needed at this stage. With an estimate of c_λ, inspired by [41], we establish a profile decomposition of the Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ (Lemma 5.4) related to J_λ. Thanks to this result, for almost every $\lambda\in[1/2,1]$ we obtain a nontrivial critical point u_λ of I_λ at the level c_λ. Finally, choosing a sequence $\lambda_n\subset[1/2,1]$ with $\lambda_n\to 1$, thanks to the Pohožaev identity we obtain a bounded $(PS)_{c_1}$-sequence of the original functional I. Using the idea above again, we obtain a nontrivial solution of problem (K).

Throughout this paper, C will denote a generic positive constant.

The paper is organized as follows.

In Sect. 2, the variational setting and some preliminary lemmas are presented.

In Sect. 3, we consider a perturbation of the original problem (K). Then using the monotonicity trick developed by Jeanjean, we obtain the bounded $(\mathrm{PS})_{c_\lambda}$-sequence $\{u_n\}_{n\in\mathbb{N}}$ for almost all λ. In Sect. 4, an upper estimate of the mountain pass value is obtained and the limit problem is discussed. In Sect. 5, we give the profile decomposition of $\{u_n\}_{n\in\mathbb{N}}$. In Sect. 6, Theorem 1.1 and 1.3 are finally proved.

2. Variational setting

In this section we outline the variational framework for (K) and recall some preliminary lemmas. For any $\alpha\in(0,1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ is defined by

$$H^\alpha(\mathbb{R}^N):=\left\{u\in L^2(\mathbb{R}^N):\ \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2\alpha}{2}}}\in L^2(\mathbb{R}^N\times\mathbb{R}^N)\right\}.$$

It is known that

$$\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}dxdy=2C(n,\alpha)^{-1}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u|^2dx,$$

where

$$C(n,\alpha)=\left(\int_{\mathbb{R}^N}\frac{1-\cos\zeta_1}{|\zeta|^{N+2\alpha}}d\zeta\right)^{-1}.$$

We endow the space $H^\alpha(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)}:=\left(\int_{\mathbb{R}^N}|u|^2dx+\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u|^2dx\right)^{1/2}.$$

$H^\alpha(\mathbb{R}^N)$ is also the completion of $C_0^\infty(\mathbb{R}^N)$ with $\|\cdot\|_{H^\alpha(\mathbb{R}^N)}$ and it is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2_\alpha^*]$. The homogeneous space $D^{\alpha,2}(\mathbb{R}^N)$ is

$$D^{\alpha,2}(\mathbb{R}^N) := \left\{ u \in L^{2_\alpha^*}(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2\alpha}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

and it is also the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{\alpha,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx \right)^{1/2}.$$

Lemma 2.1. (Norm equivalence) *Assume* (V$_2$)–(V$_3$). *Then there exist $\varepsilon_0 > 0$ and $\omega > 0$ with*

$$\int_{\mathbb{R}^N} \left((a-\varepsilon)|(-\Delta)^{\frac{\alpha}{2}} u|^2 + V(x)u^2 \right) dx \geq \omega \int_{\mathbb{R}^N} |u|^2 dx,$$

for every $u \in H^\alpha(\mathbb{R}^N)$ and all $\varepsilon \in (0, \varepsilon_0)$.

Proof. By contradiction, for $\varepsilon_0 = \omega = 1/n$ there exist $\varepsilon_n \to 0$ and $\{u_n\}_{n\in\mathbb{N}} \subset H^\alpha(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N} \left((a-\varepsilon_n)|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 + V(x)u_n^2 \right) dx \leq \frac{1}{n} \int_{\mathbb{R}^N} |u_n|^2 dx.$$

Then, up to a standard nomalization, we may assume that $\|u_n\|_{H^\alpha(\mathbb{R}^N)} = 1$ and

$$\int_{\mathbb{R}^N} \left((a-\varepsilon_n)|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 + V(x)u_n^2 \right) dx \leq \frac{1}{n}.$$

In view of (V$_3$), we get $\|u_n\|_2 \to 0$, which implies from (V$_2$) and the above inequality that $\{u_n\}_{n\in\mathbb{N}}$ goes to zero in $D^{\alpha,2}(\mathbb{R}^N)$. Therefore $u_n \to 0$ in $H^\alpha(\mathbb{R}^N)$, which contradicts the normalization. □

Let

$$\mathcal{H} := \left\{ u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}$$

be the Hilbert space equipped with the inner product

$$\langle u, v \rangle_{\mathcal{H}} := a \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} v \, dx + \int_{\mathbb{R}^N} V(x)uv \, dx,$$

and the corresponding induced norm

$$\|u\| := \left(\int_{\mathbb{R}^N} a|(-\Delta)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} V(x)u^2 dx \right)^{1/2}.$$

From Lemma 2.1, it easily follows that the above norm is equivalent to $\|\cdot\|_{H^\alpha}$. A function $u \in \mathcal{H}$ is a (weak) solution to problem (K) if, for every $\varphi \in \mathcal{H}$, we have

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx\right) \int_{\mathbb{R}^N} (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} \varphi dx + \int_{\mathbb{R}^N} V(x) u\varphi dx = \int_{\mathbb{R}^N} f(u)\varphi dx.$$

We stress that, under assumptions (V_1)–(V_3) and (f_1)–(f_3), if u is a weak solution to the above problem, then u is globally bounded and Hölder regular allowing the pointwise reppresentation of $(-\Delta)^\alpha u$ by the results of [13]. In particular $u > 0$ a.e. wherever $u \geq 0$. In what follows, we recall a fractional varsion of Lions lemma whose proof can be seen in [32].

Lemma 2.2. (Lions lemma) *Assume that* $\{u_n\}_{n\in\mathbb{N}}$ *is bounded in* $\mathcal{H}$ *and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^2 dx = 0,$$

for some $r > 0$. *Then* $u_n \to 0$ *in* $L^s(\mathbb{R}^N)$ *for all* $s \in (2, 2^*_\alpha)$.

The energy functional associated with (K), $I : \mathcal{H} \to \mathbb{R}$, is defined as

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(u)dx, \quad u \in \mathcal{H},$$

with $F(u) = \int_0^u f(t)dt$. Obviously $I \in C^1(\mathcal{H})$ and its critical points are weak solutions to (K).

3. The perturbed functional

Since we do not impose the well-known *Ambrosetti–Rabinowitz* condition, the boundedness of the Palais–Smale sequence becomes complicated. To overcome this difficulty, we adopt a monotonicity trick due to Jeanjean [18].

Theorem 3.1. (Monotonicity trick [18]) *Let* $(E, \|\cdot\|)$ *be a real Banach space with its dual space* E' *and* $J \in \mathbb{R}^+$ *an interval. Consider the family of* C^1 *functionals on* E

$$I_\lambda = A(u) - \lambda B(u), \quad \forall \lambda \in J,$$

with B *nonnegative and either* $A(u) \to +\infty$ *or* $B(u) \to +\infty$ *as* $\|u\| \to \infty$, *satisfying* $I_\lambda(0) = 0$. *We set*

$$\Gamma_\lambda := \{\gamma \in C([0,1], E) \mid \gamma(0) = 0,\, I_\lambda(\gamma(1)) < 0\}, \quad \text{for all } \lambda \in J.$$

If for every $\lambda \in J$, Γ_λ *is nonempty and*

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{s\in[0,1]} I_\lambda(\gamma(s)) > 0,$$

then for almost any $\lambda \in J$, I_λ *admits a bounded Palais–Smale sequence* $\{u_n\}_{n\in\mathbb{N}} \subset E$, *namely* $\sup_{n\in\mathbb{N}} \|u_n\| < \infty$, $I_\lambda(u_n) \to c_\lambda$ *and* $I'_\lambda(u_n) \to 0$ *in* E'. *Moreover* $\lambda \to c_\lambda$ *is left continuous.*

Set $J := [\frac{1}{2}, 1]$, $E := \mathcal{H}$ and

$$A(u) := \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx\right)^2, \quad B(u) := \int_{\mathbb{R}^N} F(u) dx.$$

We consider the family of functionals $I_\lambda : \mathcal{H} \to \mathbb{R}$ defined by $I_\lambda(u) = A(u) - \lambda B(u)$, that is

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx\right)^2 - \lambda \int_{\mathbb{R}^N} F(u) dx.$$

It is easy to see that $B(u) \geq 0$ for all $u \in \mathcal{H}$ and $A(u) \to +\infty$ as $\|u\| \to \infty$. In the following H denotes a closed half-space of $\mathbb{R}^N$ containing the origin, $0 \in H$. We denote by $\mathscr{H}$ the set of closed half-spaces of $\mathbb{R}^N$ containing the origin. We shall equip $\mathscr{H}$ with a topology ensuring that $H_n \to H$ as $n \to \infty$ if there is a sequence of isometries $i_n : \mathbb{R}^N \to \mathbb{R}^N$ such that $H_n = i_n(H)$ and i_n converges to the identity. Given $x \in \mathbb{R}^N$, the reflected point $\sigma_H(x)$ will also be denoted by x^H. The polarization of a nonnegative function $u : \mathbb{R}^N \to \mathbb{R}_+$ with respect to H is defined as

$$u^H(x) := \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & \text{for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, & \text{for } x \in \mathbb{R}^N \backslash H. \end{cases}$$

Given u, the Schwarz symmetrization u^* of u is the unique function such that u and u^* are equimeasurable and $u^*(x) = h(|x|)$, where $h : (0, \infty) \to \mathbb{R}_+$ is a continuous and decreasing function.
We set $\mathcal{H}^+ := \{u \in \mathcal{H} : u \geq 0\}$. Now we state a symmetric version of Theorem 3.1.

Lemma 3.2. (Symmetric monotonicity trick [35]) *Under the assumptions of Theorem 3.1 for $E = \mathcal{H}$, assume that $I_\lambda(|u|) \leq I_\lambda(u)$ for any $\lambda \in J$ and $u \in \mathcal{H}$ and*

$$I_\lambda(u^H) \leq I_\lambda(u), \quad \text{for any } \lambda \in J,\ u \in \mathcal{H}^+ \text{ and } H \in \mathscr{H}.$$

Then, for any $p \in [2, 2^*_\alpha]$, I_λ has a bounded Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ with $\|u_n - |u_n|^*\|_p \to 0$.

Lemma 3.3. (Uniform Mountain-Pass geometry) *Assume that* (f_1)–(f_3) *and* (V_1)–(V_3) *hold. Furthermore let $N = 2$ with $\alpha \in (\frac{1}{2}, 1)$ or $N = 3$ with $\alpha \in (\frac{3}{4}, 1)$. Then we have:*

(1) $\Gamma_\lambda \neq \emptyset$, *for every $\lambda \in J$;*

(2) *there exist $r, \eta > 0$ independent of λ, such that $\|u\| = r$ implies $I_\lambda(u) \geq \eta$. In particular $c_\lambda \geq \eta$.*

Proof. (1) For every $\varphi \in \mathcal{H}^+\backslash\{0\}$, taking into account of (f_3), we have

$$I_\lambda(\varphi) \leq \frac{1}{2}\|\varphi\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} \varphi|^2 dx\right)^2 - \frac{D}{2q}\int_{\mathbb{R}^N} \varphi^q dx - \frac{1}{22^*_\alpha}\int_{\mathbb{R}^N} \varphi^{2^*_\alpha} dx.$$

Under the assumptions on N and α, it follows that $2^*_\alpha > 4$. Then there exists $t_0 > 0$ sufficiently large, independent of $\lambda \in J$, such that $I_\lambda(t_0\varphi) < 0$. Setting

$w := t_0\varphi \in \mathcal{H}$, we have $I_\lambda(w) < 0$ and we can define the corresponding Γ_λ. Then, setting $\gamma(t) := tw$, we have $\gamma \in \Gamma_\lambda \neq \emptyset$ for every $\lambda \in J$.

(ii) (f_1)-(f_2) imply that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(s)| \leq \varepsilon|s|^2 + C_\varepsilon|s|^{2^*_\alpha}, \quad \text{for all } s \in \mathbb{R}.$$

Then there exist $\sigma_1, \sigma_2 > 0$ such that

$$I_\lambda(\varphi) \geq \sigma_1\|\varphi\|^2 - \sigma_2\|\varphi\|^{2^*_\alpha}, \quad \text{for every } \varphi \in \mathcal{H}.$$

Hence there exist $r, \eta > 0$, independent of λ, such that for $\|u\| = r$, $I_\lambda(u) \geq \eta$ (and $I_\lambda(\varphi) > 0$ as soon as $\|\varphi\| \leq r$ with $\varphi \neq 0$). Now fix $\lambda \in J$ and $\gamma \in \Gamma_\lambda$. Since $\gamma(0) = 0$ and $I_\lambda(\gamma(1)) < 0$, certainly $\|\gamma(1)\| > r$. By continuity, we conclude that there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = r$. Therefore, for every $\lambda \in J$, we conclude $c_\lambda \geq \inf_{\gamma\in\Gamma_\lambda} I_\lambda(\gamma(t_\gamma)) \geq \eta > 0$. □

Lemma 3.4. (I_λ decreases upon polarization) *Assume* (V_4) *holds. Then for any $\lambda \in J$, for all $u \in \mathcal{H}^+$ and $H \in \mathscr{H}$ there holds $I_\lambda(u^H) \leq I_\lambda(u)$.*

Proof. It is known (see [5, Theorem 2]) that

$$\int_{\mathbb{R}^{2N}} \frac{|u^H(x) - u^H(y)|^2}{|x-y|^{N+2\alpha}}dxdy \leq \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}}dxdy, \quad \text{for all } u \in \mathcal{H}^+.$$

Furthermore, we have (see [38])

$$\int_{\mathbb{R}^N} F(u^H)dx = \int_{\mathbb{R}^N} F(u)dx, \quad \text{for all } u \in \mathcal{H}^+,$$

and, by the monotonicity assumptions on V,

$$\int_{\mathbb{R}^N} V(x)(u^H)^2dx \leq \int_{\mathbb{R}^N} V(x)u^2dx, \quad \text{for all } u \in \mathcal{H}^+,$$

which concludes the proof by the definition of I_λ. □

Assume (V_1)–(V_3) and (f_1)–(f_3). As a consequence we now get the following result.

Corollary 3.5. (Bounded Palais–Smale with sign) For almost every $\lambda \in J$, there is a bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}^+$ such that $I_\lambda(u_n) \to c_\lambda$, $I'_\lambda(u_n) \to 0$. Furthermore, $\|u_n - |u_n|^*\|_{2^*_\alpha} \to 0$ if (V_4) is assumed.

Proof. For a.a. $\lambda \in J$, a bounded (PS)-sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ for I_λ is provided by combining Theorem 3.1 with Lemmas 3.3 and 3.4. Furthermore, if (V_4) holds, using Lemma 3.2 in place of Theorem 3.1, we also get $\|u_n - |u_n|^*\|_{2^*_\alpha} \to 0$. Next we show that we can assume that u_n is nonnegative. Indeed, we know that $\langle I'_\lambda(u_n), u_n^-\rangle = \langle \mu_n, u_n^-\rangle$ with $\mu_n \to 0$ in $\mathcal{H}'$ as $n \to \infty$, with $u_n^- = \min\{u_n, 0\}$, namely ($f(s) = 0$ for $s \leq 0$)

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx\right)\int_{\mathbb{R}^N} (-\Delta)^\alpha u_n u_n^- dx + \int_{\mathbb{R}^N} V(x)|u_n^-|^2dx = \langle \mu_n, u_n^-\rangle.$$

As it is readily checked, for all $x, y \in \mathbb{R}^N$, we have

$$(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y)) \geq (u_n^-(x) - u_n^-(y))^2,$$

which yields that

$$\begin{aligned}
&2C(n,\alpha)^{-1}\int_{\mathbb{R}^3}(-\Delta)^{\alpha}u_n u_n^- dx\\
&\quad=\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{N+2\alpha}}dxdy\\
&\quad\geq\int_{\mathbb{R}^{2N}}\frac{(u_n^-(x)-u_n^-(y))^2}{|x-y|^{N+2\alpha}}dxdy=2C(n,\alpha)^{-1}\|u_n^-\|^2_{D^{\alpha,2}}.
\end{aligned}$$

Thus $\|u_n^-\|=o_n(1)$, which also yields that $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded. We can now prove that $I_\lambda(u_n^+)\to c_\lambda$ and $I'_\lambda(u_n^+)\to 0$. Of course $\|u_n\|^2=\|u_n^+\|^2+o_n(1)$ and

$$\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx\right)^2=\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n^+|^2dx\right)^2+o_n(1).$$

Notice that from (f_1)–(f_2), we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^N}F(u_n)dx-\int_{\mathbb{R}^N}F(u_n^+)dx\right|&\leq C\int_{\mathbb{R}^N}(|u_n|+|u_n|^{2^*_\alpha-1})|u_n^-|\leq C\|u_n^-\|_2\\
&\quad+C\|u_n^-\|_{2^*_\alpha}=o_n(1).
\end{aligned}$$

This shows that $I_\lambda(u_n^+)\to c_\lambda$. We claim that $I'_\lambda(u_n^+)\to 0$. Setting $w_n:=I'_\lambda(u_n)-I'_\lambda(u_n^+)$, it is enough to prove that $w_n\to 0$ in $\mathcal{H}'$. For any $\varphi\in\mathcal{H}$ with $\|\varphi\|_{\mathcal{H}}\leq 1$, we have

$$\begin{aligned}
\langle w_n,\varphi\rangle&=\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx\right)\int_{\mathbb{R}^N}(-\Delta)^{\alpha/2}u_n(-\Delta)^{\alpha/2}\varphi dx\\
&\quad-\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n^+|^2dx\right)\int_{\mathbb{R}^N}(-\Delta)^{\alpha/2}u_n^+(-\Delta)^{\alpha/2}\varphi dx\\
&\quad+\int_{\mathbb{R}^N}V(x)u_n^-\varphi dx-\lambda\int_{\mathbb{R}^N}(f(u_n)-f(u_n^+))\varphi dx\\
&=\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n^+|^2dx\right)\int_{\mathbb{R}^N}(-\Delta)^{\alpha/2}u_n^-(-\Delta)^{\alpha/2}\varphi dx\\
&\quad+\int_{\mathbb{R}^N}V(x)u_n^-\varphi dx-\lambda\int_{\mathbb{R}^N}f(u_n^-)\varphi dx+\langle\xi_n,\varphi\rangle,
\end{aligned}$$

for some $\xi_n\to 0$ in $\mathcal{H}'$. Then, by (f_1)-(f_2), $|\langle w_n,\varphi\rangle|\leq C\|u_n^-\|_{\mathcal{H}}+\|\xi_n\|_{\mathcal{H}'}$, proving the claim. Observe now that by the triangular inequality and the contractility property of the Schwarz symmetrization in L^p-spaces (i.e. $\|w^*-z^*\|_p\leq\|w-z\|_p$ for all $w,z\in L^p(\mathbb{R}^N)$ with $w,z\geq 0$), we get

$$\begin{aligned}
&\left|\|u_n^+-(u_n^+)^*\|_{2^*_\alpha}-\|u_n-|u_n|^*\|_{2^*_\alpha}\right|\\
&\quad\leq\|u_n^-+((u_n^+)^*-|u_n|^*)\|_{2^*_\alpha}\leq\|u_n^-\|_{2^*_\alpha}+\|(u_n^+)^*-|u_n|^*\|_{2^*_\alpha}\\
&\quad\leq\|u_n^-\|_{2^*_\alpha}+\|u_n^+-|u_n|\|_{2^*_\alpha}=2\|u_n^-\|_{2^*_\alpha}\leq C\|u_n^-\|_{\mathcal{H}}=o_n(1).
\end{aligned}$$

Since $\|u_n-|u_n|^*\|_{2^*_\alpha}\to 0$, we have $\|u_n^+-(u_n^+)^*\|_{2^*_\alpha}\to 0$ as $n\to\infty$. This ends the proof. □

4. Upper estimate of c_λ and limit problems

In this section, we give an upper estimate of the mountain pass value c_λ. Moreover, the corresponding limit problem is discussed.

4.1. An energy estimate

Next we provide a crucial energy estimate for c_λ.

Lemma 4.1. (Energy estimate) *Suppose that* (f_1)–(f_3) *and* (V_1)–(V_3) *hold. For any* $\lambda \in [\frac{1}{2}, 1]$, *assume that*

$$q \in \left(\frac{4\alpha}{N-2\alpha}, 2_\alpha^*\right) \quad \text{or} \quad q \in \left(2, \frac{4\alpha}{N-2\alpha}\right] \text{ with } D \text{ large enough.}$$

Then we have

$$c_\lambda < c_\lambda^*, \qquad c_\lambda^* := \frac{aS_\alpha}{2}T^{N-2\alpha} + \frac{bS_\alpha^2}{4}T^{2N-4\alpha} - \frac{\lambda}{2_\alpha^*}T^N,$$

where $T = T(\lambda) > 0$ *continuously depends on* λ.

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with support in $B_2(0)$ such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \in [0,1]$ on $B_2(0)$. It is well known that S_α is achieved by

$$\mathcal{T}(x) := \kappa\big(\mu^2 + |x - x_0|^2\big)^{-\frac{N-2\alpha}{2}}$$

for arbitrary $\kappa \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^N$. Then, taking $x_0 = 0$, we can define

$$v_\varepsilon(x) := \eta(x)u_\varepsilon(x), \quad u_\varepsilon(x) = \varepsilon^{-\frac{N-2\alpha}{2}}u^*(x/\varepsilon), \quad u^*(x) := \frac{\mathcal{T}\left(x/S_\alpha^{1/(2\alpha)}\right)}{\|\mathcal{T}\|_{2_\alpha^*}}.$$

Then $(-\Delta)^\alpha u_\varepsilon = |u_\varepsilon|^{2_\alpha^*-2}u_\varepsilon$ and $\|(-\Delta)^{\frac{\alpha}{2}}u_\varepsilon\|_2^2 = \|u_\varepsilon\|_{2_\alpha^*}^{2_\alpha^*} = S_\alpha^{\frac{N}{2\alpha}}$. As in [33], we have

$$A_\varepsilon := \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}v_\varepsilon(x)|^2 dx = S_\alpha^{\frac{N}{2\alpha}} + \mathcal{O}(\varepsilon^{N-2\alpha}). \tag{4.1}$$

On the other hand, for any $q \in [2, 2_\alpha^*)$, we obtain

$$\int_{\mathbb{R}^N} |v_\varepsilon|^q dx \geq \varepsilon^{N-\frac{(N-2\alpha)q}{2}}\kappa^q\|\mathcal{T}\|_{2_\alpha^*}^{-q}|\mathbb{S}_{N-1}|S_\alpha^{N/(2\alpha)}\int_0^{\frac{1}{\varepsilon S_\alpha^{1/(2\alpha)}}} \frac{r^{N-1}}{(\mu^2+r^2)^{\frac{(N-2\alpha)q}{2}}}dr,$$

where $\mathbb{S}_{N-1}$ is the unit sphere in $\mathbb{R}^N$. Observe that, as $\varepsilon \to 0$,

$$\int_0^{\frac{1}{\varepsilon S_\alpha^{1/(2\alpha)}}} \frac{r^{N-1}}{(\mu^2+r^2)^{\frac{(N-2\alpha)q}{2}}}dr \begin{cases} \to c \in (0,\infty), & \text{if } q > \frac{N}{N-2\alpha}, \\ = \mathcal{O}(\log(\frac{1}{\varepsilon})), & \text{if } q = \frac{N}{N-2\alpha}, \\ = \mathcal{O}(\varepsilon^{(N-2\alpha)q-N}), & \text{if } q < \frac{N}{N-2\alpha}. \end{cases}$$

Then

$$C_\varepsilon := \int_{\mathbb{R}^N} |v_\varepsilon|^q dx \geq \begin{cases} \mathcal{O}(\varepsilon^{N-\frac{(N-2\alpha)q}{2}}), & \text{if } q > \frac{N}{N-2\alpha}, \\ \mathcal{O}(\log(\frac{1}{\varepsilon})\varepsilon^{N-\frac{(N-2\alpha)q}{2}}), & \text{if } q = \frac{N}{N-2\alpha}, \\ \mathcal{O}(\varepsilon^{\frac{(N-2\alpha)q}{2}}), & \text{if } q < \frac{N}{N-2\alpha}. \end{cases} \tag{4.2}$$

Since $2 < \frac{N}{N-2\alpha}$,

$$\int_{\mathbb{R}^N} |v_\varepsilon|^2 dx \geq \mathcal{O}(\varepsilon^{N-2\alpha}).$$

Similar as above,

$$\begin{aligned}\int_{\mathbb{R}^N} |v_\varepsilon|^2 dx &\leq \varepsilon^{2\alpha}\kappa^2 \|\mathcal{T}\|_{2^*_\alpha}^{-2} |\mathbb{S}_{N-1}| S_\alpha^{N/(2\alpha)} \int_0^{\frac{2}{\varepsilon S_\alpha^{1/(2\alpha)}}} \frac{r^{N-1}}{(\mu^2+r^2)^{N-2\alpha}} dr \\ &\leq \mathcal{O}(\varepsilon^{N-2\alpha}).\end{aligned}$$

So that we have

$$B_\varepsilon := \int_{\mathbb{R}^N} |v_\varepsilon|^2 dx = \mathcal{O}(\varepsilon^{N-2\alpha}). \tag{4.3}$$

As can be seen in [33], it holds

$$D_\varepsilon := \int_{\mathbb{R}^N} |v_\varepsilon|^{2^*_\alpha} dx = S_\alpha^{\frac{N}{2\alpha}} + \mathcal{O}(\varepsilon^N).$$

Step 1. For any $\varepsilon > 0$ small there exists $t_0 > 0$ such that $I_\lambda(\gamma_\varepsilon(t_0)) < 0$, where $\gamma_\varepsilon(t) := v_\varepsilon(\cdot/t)$. Indeed, by ($\mathrm{V}_2$) and ($\mathrm{f}_3$), for any $t > 0$,

$$\begin{aligned}I_\lambda(\gamma_\varepsilon(t)) &\leq \frac{a}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}\gamma_\varepsilon(t)|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}\gamma_\varepsilon(t)|^2 dx\right)^2 \\ &\quad + \frac{V_\infty}{2}\int_{\mathbb{R}^N} |\gamma_\varepsilon(t)|^2 dx - \lambda \int_{\mathbb{R}^N} \left[\frac{|\gamma_\varepsilon(t)|^{2^*_\alpha}}{2^*_\alpha} + D\frac{|\gamma_\varepsilon(t)|^q}{q}\right] dx \\ &= \frac{aA_\varepsilon}{2} t^{N-2\alpha} + \frac{bA_\varepsilon^2}{4} t^{2N-4\alpha} + \left(\frac{V_\infty B_\varepsilon}{2} - \frac{\lambda D_\varepsilon}{2^*_\alpha} - \frac{\lambda D C_\varepsilon}{q}\right) t^N. \end{aligned} \tag{4.4}$$

Noting that $2\alpha < N < 4\alpha$, we have $0 < 2N - 4\alpha < N$. Then by (4.3),

$$\frac{V_\infty B_\varepsilon}{2} - \frac{\lambda D_\varepsilon}{2^*_\alpha} \to -\frac{\lambda S_\alpha^{\frac{N}{2\alpha}}}{2^*_\alpha}, \text{ as } \varepsilon \to 0.$$

So it follows from (4.1) that for any $\varepsilon > 0$ small enough, $I_\lambda(\gamma_\varepsilon(t) \to -\infty$ as $t \to +\infty$. Then there exists $t_0 > 0$ such that $I_\lambda(\gamma_\varepsilon(t_0)) < 0$.

Step 2. Notice that, as $t \to 0^+$, we have

$$\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{\alpha}{2}}\gamma_\varepsilon(t)|^2 + |\gamma_\varepsilon(t)|^2\right] dx = t^{N-2\alpha}A_\varepsilon + t^N B_\varepsilon \to 0$$

uniformly for $\varepsilon > 0$ small. We set $\gamma_\varepsilon(0) = 0$. Then $\gamma_\varepsilon(t_0\cdot) \in \Gamma_\lambda$, where Γ_λ is as in Theorem 3.1 and

$$c_\lambda \leq \sup_{t \geq 0} I_\lambda(\gamma_\varepsilon(t)).$$

Recalling that $c_\lambda > 0$, by (4.4), there exists $t_\varepsilon > 0$ such that

$$\sup_{t \geq 0} I_\lambda(\gamma_\varepsilon(t)) = I_\lambda(\gamma_\varepsilon(t_\varepsilon)).$$

By (4.1), (4.2) and (4.4), we get $I_\lambda(\gamma_\varepsilon(t)) \to 0^+$ as $t \to 0^+$ and $I_\lambda(\gamma_\varepsilon(t)) \to -\infty$ as $t \to +\infty$ uniformly for $\varepsilon > 0$ small. Then there exist $t_1, t_2 > 0$ (independent of $\varepsilon > 0$) such that $t_1 \le t_\varepsilon \le t_2$. Let

$$J_\varepsilon(t) := \frac{aA_\varepsilon}{2} t^{N-2\alpha} + \frac{bA_\varepsilon^2}{4} t^{2N-4\alpha} - \frac{\lambda D_\varepsilon}{2_\alpha^*} t^N,$$

then

$$c_\lambda \le \sup_{t\ge 0} J_\varepsilon(t) + \left(\frac{V_\infty B_\varepsilon}{2} - \frac{\lambda D C_\varepsilon}{q}\right) t_\varepsilon^N$$

By formula (4.2), for any $q \in (2, 2_\alpha^*)$, we have

$$C_\varepsilon \ge \mathcal{O}(\varepsilon^{N-\frac{(N-2\alpha)q}{2}}).$$

Then by (4.3), we conclude that

$$c_\lambda \le \sup_{t\ge 0} J_\varepsilon(t) + \mathcal{O}(\varepsilon^{N-2\alpha}) - \mathcal{O}(D\varepsilon^{N-\frac{(N-2\alpha)q}{2}}).$$

Noting that $N-2\alpha > 0$ and $N-(N-2\alpha)q/2 > 0$, we have $\sup_{t\ge 0} J_\varepsilon(t) \ge c_\lambda/2$ uniformly for $\varepsilon > 0$ small. As above, there are $t_3, t_4 > 0$ (independent of $\varepsilon > 0$) such that $\sup_{t\ge 0} J_\varepsilon(t) = \sup_{t\in[t_3,t_4]} J_\varepsilon(t)$. By (4.1),

$$c_\lambda \le \sup_{t\ge 0} K\left(S_\alpha^{\frac{1}{2\alpha}} t\right) + \mathcal{O}\left(\varepsilon^{N-2\alpha}\right) - \mathcal{O}\left(D\varepsilon^{N-\frac{(N-2\alpha)q}{2}}\right), \tag{4.5}$$

where

$$K(t) = \frac{aS_\alpha}{2} t^{N-2\alpha} + \frac{bS_\alpha^2}{4} t^{2N-4\alpha} - \frac{\lambda}{2_\alpha^*} t^N.$$

Observe that for $t > 0$,

$$K'(t) = \frac{(N-2\alpha)t^{N-2\alpha-1}}{2} \tilde{K}(t), \text{ where } \tilde{K}(t) := aS_\alpha + bS_\alpha^2 t^{N-2\alpha} - \lambda t^{2\alpha},$$

and $\tilde{K}'(t) = t^{N-2\alpha-1}(bS_\alpha^2(N-2\alpha) - 2\lambda\alpha t^{4\alpha-N})$. Since $4\alpha > N$, there is a unique $T > 0$ such that $\tilde{K}(t) > 0$ if $t \in (0, T)$ and $\tilde{K}(t) < 0$ if $t > T$. Hence, T is the unique maximum point of K. Then by (4.5),

$$c_\lambda \le K(T) + \mathcal{O}(\varepsilon^{N-2\alpha}) - \mathcal{O}\left(D\varepsilon^{N-\frac{(N-2\alpha)q}{2}}\right). \tag{4.6}$$

If $q > 4\alpha/(N-2\alpha)$, then $0 < N-(N-2\alpha)q/2 < N-2\alpha$, which implies by (4.6) that for any fixed $D > 0$, $c_\lambda < K(T)$ for $\varepsilon > 0$ small. If $2 < q \le 4\alpha/(N-2\alpha)$, for $\varepsilon > 0$ small and $D \ge \varepsilon^{(N-2\alpha)q/2-2\alpha-1}$, then also in this case $c_\lambda < K(T)$, which completes the proof. $\square$

4.2. The limit problem

Note that $V(x) \to V_\infty$ as $|x| \to \infty$. For any $\lambda \in [1/2, 1]$, we consider the problem

$$\begin{cases} \left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx\right)(-\Delta)^\alpha u + V_\infty u = \lambda f(u) & \text{in } \mathbb{R}^N, \\ u \in H^\alpha(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

whose energy functional is defined by

$$I_\lambda^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} (a|(-\Delta)^{\frac{\alpha}{2}} u|^2 + V_\infty u^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx\right)^2 - \lambda \int_{\mathbb{R}^N} F(u)dx.$$

We will use of the following Pohožaev type identity, whose proof is similar as in [11].

Lemma 4.2. (Pohožaev identity) *Let u be a critical point of I_λ^∞ in $\mathcal{H}$ for $\lambda \in [\frac{1}{2}, 1]$. Then $P_\lambda(u) = 0$,*

$$P_\lambda(u) := \frac{N-2\alpha}{2}\int_{\mathbb{R}^N} a|(-\Delta)^{\frac{\alpha}{2}} u|^2 dx + \frac{N-2\alpha}{2} b\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx\right)^2 + \frac{N}{2}\int_{\mathbb{R}^N} V_\infty u^2 dx - N\lambda \int_{\mathbb{R}^N} F(u)dx. \tag{4.7}$$

Notice that $P_\lambda(u) = \frac{d}{dt} I_\lambda^\infty(u(\cdot/t))\big|_{t=1}$.

Lemma 4.3. *For $\lambda \in [\frac{1}{2}, 1]$, if $w_\lambda \in \mathcal{H}\backslash\{0\}$ solves $P_\lambda(w_\lambda) = 0$, then there exists $\gamma_\lambda \in C([0,1], \mathcal{H})$ such that $\gamma_\lambda(0) = 0$, $I_\lambda^\infty(\gamma_\lambda(1)) < 0$, $w_\lambda \in \gamma_\lambda([0,1])$, $0 \notin \gamma_\lambda((0,1])$ and*

$$\max_{t\in[0,1]} I_\lambda^\infty(\gamma_\lambda(t)) = I_\lambda^\infty(w_\lambda).$$

Proof. Note that

$$I_\lambda^\infty(w_\lambda(\cdot/t)) = \frac{t^{N-2\alpha}}{2}\int_{\mathbb{R}^N} a|(-\Delta)^{\frac{\alpha}{2}} w_\lambda|^2 dx + \frac{bt^{2N-4\alpha}}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} w_\lambda|^2 dx\right)^2 + \frac{t^N}{2}\int_{\mathbb{R}^N} V_\infty w_\lambda^2 dx - t^N\lambda \int_{\mathbb{R}^N} F(w_\lambda)dx = 0,$$

which, by (4.7), yields

$$\lim_{t\to\infty} I_\lambda^\infty(w_\lambda(\cdot/t)) < 0.$$

Then there is $t_0 > 0$ such that $I_\lambda^\infty(w_\lambda(\cdot/t_0)) < 0$. Let $\gamma_\lambda(t) = w_\lambda(\cdot/tt_0))$ for $0 < t \le 1$ and $\gamma_\lambda(0) = 0$. Then $\gamma_\lambda \in C([0,1], \mathcal{H})$, $w_\lambda \in \gamma_\lambda([0,1])$ and $\max_{t\in[0,1]} I_\lambda^\infty(\gamma_\lambda(t)) = I_\lambda^\infty(w_\lambda)$ as $t = t_0^{-1}$ is the unique maximum point of $t \mapsto I_\lambda^\infty(\gamma_\lambda(t))$ by Lemma 4.2. $\square$

5. Behaviour of Palais–Smale sequences

By Corollary 3.5, for almost every $\lambda \in [1/2, 1]$, there exists a bounded Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ for I_λ at the level c_λ. Then there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that $u_n \rightharpoonup u_0$ in $\mathcal{H}$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Let

$$v_n^1 := u_n - u_0.$$

Then $v_n^1 \rightharpoonup 0$ in $\mathcal{H}$ and $v_n^1 \to 0$ a.e. Notice that, since $u_n \geq 0$, the dominated convergence theorem implies that $(v_n^1)^- \to 0$ in $L^q(\mathbb{R}^N)$ for any $2 \leq q \leq 2_\alpha^*$.

5.1. Splitting lemmas

Let us set

$$g(t) := f(t) - (t^+)^{2_\alpha^*-1}, \qquad G(t) := \int_0^t g(s)ds.$$

In order to get the profile decomposition of $\{u_n\}_{n\in\mathbb{N}}$, we state the following splitting lemmas.

Lemma 5.1. (Splitting lemma I) *We have*

$$\left|\int_{\mathbb{R}^N} (g(u_n) - g(u_0) - g(v_n^1))\varphi dx\right| \leq o_n(1)\|\varphi\|, \tag{5.1}$$

where $o_n(1) \to 0$ as $n \to \infty$, uniformly for any $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Proof. For each $n \geq 1$, there exists $\theta_n \in (0, 1)$ such that

$$|g(u_n) - g(v_n^1)| \leq |g'(v_n^1 + \theta_n u_0)||u_0|. \tag{5.2}$$

In view of (f_1)-(f_3), for any $\varepsilon > 0$, there exists $\bar{D} > 0$ such that

$$|g(t)| \leq \varepsilon |t|^{2_\alpha^*-1}, \quad \text{for } |t| \geq \bar{D}/2. \tag{5.3}$$

Let $\Omega_n(\bar{D}) := \{x \in \mathbb{R}^N : |u_n(x)| \geq \bar{D}\}$ and for $r > 0$, $B_r := \{x \in \mathbb{R}^N : |x| < r\}$, $B_r^c := \mathbb{R}^N \backslash B_r(0)$. Since $u_0 \in \mathcal{H}$, we have $|B_R^c \cap \{|u_0(x)| \geq \bar{D}/2\}| \to 0$ as $R \to \infty$. Then for ε given as above, there exist $R > 0$ and $\Omega_R \subset \mathbb{R}^N$ with $|\Omega_R| \leq \Lambda_\varepsilon$ such that $|u_0(x)| < \bar{D}/2$ for $x \in B_R^c \backslash \Omega_R$, where $\Lambda_\varepsilon > 0$ will be chosen later small enough. Then, by Hölder's inequality, (5.2) and (5.3), we have

$$\begin{aligned}
&\int_{B_R^c \backslash \Omega_R} |g(u_n) - g(v_n^1)||\varphi| dx \\
&\quad \leq \int_{(B_R^c \backslash \Omega_R) \cap \Omega_n(\bar{D})} |g(u_n) - g(v_n^1)||\varphi| dx \\
&\qquad + \int_{(B_R^c \backslash \Omega_R) \cap \Omega_n^c(\bar{D})} |g(u_n) - g(v_n^1)||\varphi| dx \\
&\quad \leq \varepsilon C \left(\|u_n\|_{2_\alpha^*}^{2_\alpha^*-1} + \|v_n^1\|_{2_\alpha^*}^{2_\alpha^*-1}\right)\|\varphi\| + \max_{|t|\leq 2\bar{D}} |g'(t)| \left(\int_{B_R^c} u_0^2(x) dx\right)^{1/2} \|\varphi\|.
\end{aligned} \tag{5.4}$$

It follows from (f_1) and (f_2) that, for $\varepsilon > 0$ given, there exists $C_\varepsilon = C_\varepsilon(f) > 0$ such that

$$\begin{aligned}
&\int_{\Omega_R} |g(u_n) - g(v_n^1)||\varphi|dx \\
&\quad \le \varepsilon \int_{\Omega_R} \left(|u_n|^{2_\alpha^*-1} + |v_n^1|^{2_\alpha^*-1}\right)|\varphi|dx + C_\varepsilon \int_{\Omega_R} \left(|u_n| + |v_n^1|\right)|\varphi|dx \\
&\quad \le \varepsilon C \left(\|u_n\|_{2_\alpha^*}^{2_\alpha^*-1} + \|v_n^1\|_{2_\alpha^*}^{2_\alpha^*-1}\right)\|\varphi\| + C_\varepsilon|\Omega_R|^{\frac{2\alpha}{N}}\left(\|u_n\|_{2_\alpha^*} + \|v_n^1\|_{2_\alpha^*}\right)\|\varphi\|_{2_\alpha^*}.
\end{aligned} \tag{5.5}$$

By (5.4) and (5.5), by choosing Λ_ε such that $C_\varepsilon \Lambda_\varepsilon^{2\alpha/N} \le \varepsilon$, there exists $C > 0$ with

$$\int_{B_R^c} |g(u_n) - g\left(v_n^1\right)||\varphi|dx \le C\varepsilon\|\varphi\|. \tag{5.6}$$

Moreover,

$$\begin{aligned}
\int_{B_R^c} |g(u_0)||\varphi|dx &\le C\int_{B_R^c} |u_0||\varphi|dx + \int_{B_R^c} |u_0|^{2_\alpha^*-1}|\varphi|dx \\
&\le C\left(\int_{B_R^c} |u_0|^2dx\right)^{1/2}\|\varphi\| + C\left(\int_{B_R^c} |u_0|^{2_\alpha^*}dx\right)^{(2_\alpha^*-1)/2_\alpha^*}\|\varphi\|.
\end{aligned} \tag{5.7}$$

It follows from (5.6) and (5.7) that, for $\varepsilon > 0$ above, we choose $R > 0$ above large enough such that

$$\left|\int_{B_R^c} (g(u_n) - g(u_0) - g(v_n^1))\varphi dx\right| \le C\varepsilon\|\varphi\|, \tag{5.8}$$

where C is independent of n, ε and $\varphi \in C_0^\infty(\mathbb{R}^N)$. On the other hand,

$$\begin{aligned}
&\int_{B_R} |g(u_n) - g(u_0)||\varphi|dx \\
&\quad \le \left(\int_{B_R} |g(u_n) - g(u_0)|^{2_\alpha^*/(2_\alpha^*-1)}dx\right)^{(2_\alpha^*-1)/2_\alpha^*}\left(\int_{B_R} |\varphi|^{2_\alpha^*}\right)^{1/2_\alpha^*}.
\end{aligned}$$

Observe that

$$\lim_{t\to+\infty} \frac{g^{2_\alpha^*/(2_\alpha^*-1)}(t)}{t^{2_\alpha^*}} = \lim_{t\to 0^+} \frac{g^{2_\alpha^*/(2_\alpha^*-1)}(t)}{t^{2_\alpha^*/(2_\alpha^*-1)}} = 0.$$

Then $|g(u_n) - g(u_0)|^{2_\alpha^*/(2_\alpha^*-1)} \to 0$ in $L^1(B_R)$. Hence, we deduce

$$\int_{B_R} |g(u_n) - g(u_0)||\varphi|dx \le o_n(1)\|\varphi\|. \tag{5.9}$$

Similarly, we also obtain that

$$\int_{B_R} |g(v_n^1)|\varphi dx \le o_n(1)\|\varphi\|, \tag{5.10}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. It follows from (5.8), (5.9) and (5.10) that (5.1) holds. $\square$

Lemma 5.2. (Splitting lemma II) *We have*

$$\left|\int_{\mathbb{R}^N}\left(|u_n|^{2^*_\alpha-2}u_n-|u_0|^{2^*_\alpha-2}u_0-|v_n^1|^{2^*_\alpha-2}v_n^1\right)\varphi dx\right|\le o_n(1)\|\varphi\|,$$

where $o_n(1)\to 0$ *as* $n\to\infty$*, uniformly for any* $\varphi\in C_0^\infty(\mathbb{R}^N)$*.*

Proof. For any $\varepsilon>0$, there exists $R=R(\varepsilon)>0$ such that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N\setminus B_R(0)}\left(|u_n|^{2^*_\alpha-2}u_n-|u_0|^{2^*_\alpha-2}u_0-|v_n^1|^{2^*_\alpha-2}v_n^1\right)\varphi dx\right|\\ &\quad\le\left|\int_{\mathbb{R}^N\setminus B_R(0)}\left(|u_n|^{2^*_\alpha-2}u_n-|v_n^1|^{2^*_\alpha-2}v_n^1\right)\varphi dx\right|\\ &\qquad+\left|\int_{\mathbb{R}^N\setminus B_R(0)}|u_0|^{2^*_\alpha-2}u_0\varphi dx\right|\\ &\quad\le C\int_{\mathbb{R}^N\setminus B_R(0)}\left(|u_n|^{2^*_\alpha-2}+|v_n^1|^{2^*_\alpha-2}\right)|u_0\varphi|dx\\ &\qquad+\int_{\mathbb{R}^N\setminus B_R(0)}|u_0|^{2^*_\alpha-1}|\varphi|dx\le C\varepsilon\|\varphi\|.\end{aligned}\tag{5.11}$$

On the other hand, for every $r>0$, we have

$$\begin{aligned}&\left|\int_{B_R(0)}\left(|u_n|^{2^*_\alpha-2}u_n-|u_0|^{2^*_\alpha-2}u-|v_n^1|^{2^*_\alpha-2}v_n^1\right)\varphi dx\right|\\ &\le\int_{B_R(0)\cap\{|v_n^1|\le r\}}\left||u_n|^{2^*_\alpha-2}u_n-|u_0|^{2^*_\alpha-2}u_0-|v_n^1|^{2^*_\alpha-2}v_n^1\right|\varphi dx\\ &\quad+\int_{B_R(0)\cap\{|v_n^1|\ge r\}}\left||u_n|^{2^*_\alpha-2}u_n-|u_0|^{2^*_\alpha-2}u_0-|v_n^1|^{2^*_\alpha-2}v_n^1\right|\varphi dx=:I_1+I_2.\end{aligned}$$

Now, there exists $r=r(R)$ such that $r|B_R(0)|^{1/2^*_\alpha}\le\varepsilon$. Therefore, we have

$$\begin{aligned}I_1&\le C\int_{B_R(0)\cap\{|v_n^1|\le r\}}\left(|u_n|^{2^*_\alpha-2}+|u_0|^{2^*_\alpha-2}+|v_n^1|^{2^*_\alpha-2}\right)|v_n^1\varphi|dx\\ &\le Cr|B_R(0)|^{1/2^*_\alpha}\|\varphi\|\le C\varepsilon\|\varphi\|.\end{aligned}\tag{5.12}$$

For such r,R fixed above, u_n converges to u in measure in $B_R(0)$, i.e. $|B_R(0)\cap\{|v_n|\ge r\}|\to 0$ for $n\to\infty$. Therefore, for $n\ge 1$ large,

$$\begin{aligned}I_2&\le C\int_{B_R(0)\cap\{|v_n^1|\ge r\}}\left(|u_n|^{2^*_\alpha-2}+|v_n^1|^{2^*_\alpha-2}\right)|u_0\varphi|dx\\ &\quad+\int_{B_R(0)\cap\{|v_n^1|\ge r\}}|u_0|^{2^*_\alpha-1}|\varphi|dx\le C\varepsilon\|\varphi\|.\end{aligned}\tag{5.13}$$

Then (5.11), (5.12) and (5.13) yield the assertion. □

Lemma 5.3. (Splitting lemma III) *We have*

$$\int_{\mathbb{R}^N}f(u_n)u_ndx=\int_{\mathbb{R}^N}f(v_n^1)v_n^1dx+\int_{\mathbb{R}^N}f(u_0)u_0dx+o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$. Furthermore

$$\int_{\mathbb{R}^N} F(u_n)dx = \int_{\mathbb{R}^N} F(v_n^1)dx + \int_{\mathbb{R}^N} F(u_0)dx + o_n(1).$$

Proof. Since $f(t) = g(t) + t^{2_\alpha^*-1}$ for $t \geq 0$, by the standard Brezis–Lieb lemma, it suffices to prove

$$\int_{\mathbb{R}^N} g(u_n)u_n dx = \int_{\mathbb{R}^N} g(v_n^1)v_n^1 dx + \int_{\mathbb{R}^N} g(u_0)u_0 dx + o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$. Fixed $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(t)| \leq \varepsilon t^{2_\alpha^*-1} + C_\varepsilon t, \quad t \geq 0. \tag{5.14}$$

Then there exists $R = R(\varepsilon) > 0$ large enough such that

$$\begin{aligned} \left|\int_{\mathbb{R}^N} g(u_0)v_n^1 dx\right| &\leq \int_{B_R} |g(u_0)v_n^1|dx + \int_{B_R^c} |g(u_0)v_n^1|dx \\ &\leq \int_{B_R} \left(\varepsilon|u_0|^{2_\alpha^*-1} + C_\varepsilon|u_0|\right)|v_n^1|dx + \varepsilon\left(\|v_n^1\|_2 + \|v_n^1\|_{2_\alpha^*}\right) \\ &\leq C\varepsilon + C_\varepsilon o_n(1). \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} \left|\int_{\mathbb{R}^N} g\left(v_n^1\right)u_0 dx\right| &\leq \int_{B_R} |g\left(v_n^1\right)u_0|dx + \int_{B_R^c} |g\left(v_n^1\right)u_0|dx \\ &\leq \int_{B_R} \left(\varepsilon|v_n^1|^{2_\alpha^*-1} + C_\varepsilon|v_n^1|\right)|u_0|dx \\ &\quad + \int_{B_R^c} \left(\varepsilon|v_n^1|^{2_\alpha^*-1} + C_\varepsilon|v_n^1|\right)|u_0|dx \\ &\leq C\varepsilon + C_\varepsilon o_n(1). \end{aligned} \tag{5.16}$$

It follows from (5.15), (5.16) and Lemma 5.1 that

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} (g(u_n)u_n - g(u_0)u_0 - g(v_n^1)v_n^1)dx\right| \leq \int_{\mathbb{R}^N} |(g(u_n) - g(u_0) - g(v_n^1))u_n|dx \\ &\quad + \int_{\mathbb{R}^N} |g(v_n^1)u_0|dx + \int_{\mathbb{R}^N} |g(u_0)v_n^1|dx \\ &\leq o_n(1)\|u_n\| + C\varepsilon + C_\varepsilon o_n(1). \end{aligned}$$

Letting $n \to \infty$ and $\varepsilon \to 0^+$ completes the proof of the first assertion. The second assertion follows from the standard Brezis–Lieb lemma and

$$\int_{\mathbb{R}^N} G(u_n)dx = \int_{\mathbb{R}^N} G(v_n^1)dx + \int_{\mathbb{R}^N} G(u_0)dx + o_n(1),$$

whose proof is left to the reader. □

5.2. Profile decomposition

In the following, we give the profile decomposition of $\{u_n\}_{n\in\mathbb{N}}$, which plays a crucial role in getting the compactness. Since $c_\lambda > 0$, for some $\bar{B} > 0$ we have

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \to \bar{B}^2, \qquad \text{as } n \to \infty.$$

Now, for any $u \in \mathcal{H}$, let

$$J_\lambda(u) := \frac{a + b\bar{B}^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 dx - \lambda \int_{\mathbb{R}^N} F(u) dx$$

and

$$J_\lambda^\infty(u) := \frac{a + b\bar{B}^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V_\infty |u|^2 dx - \lambda \int_{\mathbb{R}^N} F(u) dx,$$

which are respectively the corresponding functional of the following problems

$$\begin{aligned} &(a + b\bar{B}^2)(-\Delta)^\alpha u + V(x)u = f(u), \\ &(a + b\bar{B}^2)(-\Delta)^\alpha u + V_\infty u = f(u), \ u \in \mathcal{H}. \end{aligned}$$

Here we point out that in contrast with the original problem (K), the problems above are both *non Kirchhoff*. Now we take advantage of this to get the profile decomposition of $\{u_n\}_{n\in\mathbb{N}}$.

Lemma 5.4. (Profile decomposition) *Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *be the sequence mentioned above and assume that conditions* (V_1)-(V_3), (f_1)-(f_3) *hold and* $N < 4\alpha$. *Then* $J'_\lambda(u_0) = 0$ *with* $u_0 \geq 0$, *and there exist a number* $k \in \mathbb{N} \cup \{0\}$, *nontrivial positive critical points* $w^1, \ldots, w^k \in H^\alpha(\mathbb{R}^N)$ *of* J_λ^∞ *which decay polynomially at infinity as* $w^j(x)|x|^{N+2\alpha} = \mathcal{O}(1)$, *such that*

(i) $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ *if* $i \neq j$, $1 \leq i, j \leq k$, $n \to +\infty$,

(ii) $c_\lambda + \frac{b\bar{B}^4}{4} = J_\lambda(u_0) + \sum_{j=1}^k J_\lambda^\infty(w^j)$,

(iii) $\|u_n - u_0 - \sum_{j=1}^k w^j(\cdot - y_n^j)\| \to 0$,

(iv) $\bar{B}^2 = \|(-\Delta)^{\frac{\alpha}{2}} u_0\|_2^2 + \sum_{j=1}^k \|(-\Delta)^{\frac{\alpha}{2}} w^j\|_2^2$.

Moreover, we agree that in the case $k = 0$ *the above holds without* w^j. *In addition, if* (V_4) *holds, then* $k = 0$ *and* $u_0 \in H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$.

Proof. Observe that, from $I_\lambda(u_n) = c_\lambda + o_n(1)$ and $I'_\lambda(u_n) \to 0$ in $\mathcal{H}'$, we obtain

$$J_\lambda(u_n) = c_\lambda + \frac{b\bar{B}^4}{4} + o_n(1), \qquad J'_\lambda(u_n) \to 0 \quad \text{in } \mathcal{H}'.$$

Then, it is standard to get $J'_\lambda(u_0)\varphi = 0$ for all $\varphi \in \mathcal{H}$. From Lemma 5.3, we get

$$\int_{\mathbb{R}^N} F(v_n^1)dx = \int_{\mathbb{R}^N} F(u_n)dx - \int_{\mathbb{R}^N} F(u_0)dx + o_n(1),$$
$$\int_{\mathbb{R}^N} f(v_n^1)v_n^1 dx = \int_{\mathbb{R}^N} f(u_n)u_n dx - \int_{\mathbb{R}^N} f(u_0)u_0 dx + o_n(1).$$

It follows that

$$J_\lambda(u_n) = J_\lambda(v_n^1) + J_\lambda(u_0) + o_n(1), \tag{5.17}$$
$$J'_\lambda(v_n^1)v_n^1 = J'_\lambda(u_n)u_n - J'_\lambda(u_0)u_0 + o_n(1) = o_n(1). \tag{5.18}$$

On the other hand, by a slight variant of [11, Proposition 4.1], u_0 satisfies the Pohǒzaev identity

$$\frac{N-2\alpha}{2}(a+b\bar{B}^2)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u_0|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(x)\cdot x\, u_0^2 dx$$
$$+ \frac{N}{2}\int_{\mathbb{R}^N} V(x)u_0^2 dx - N\lambda \int_{\mathbb{R}^N} F(u_0)dx = 0.$$

Then by (V_1) and $N < 4\alpha$, we have

$$\begin{aligned} NJ_\lambda(u_0) &= \alpha(a+b\bar{B}^2)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u_0|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(x)\cdot x\, u_0^2 dx \\ &\geq \alpha(a+b\bar{B}^2)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u_0|^2 dx - \frac{1}{2}\|W\|_{\frac{N}{2\alpha}}\|u_0\|_{2_\alpha^*}^2 \\ &\geq \alpha(a+b\bar{B}^2)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u_0|^2 dx - a\alpha\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u_0|^2 dx \\ &= \alpha b\bar{B}^2\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u_0|^2 dx > 0, \end{aligned}$$

which implies that

$$J_\lambda(u_0) \geq \frac{b\bar{B}^2}{4}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u_0|^2 dx. \tag{5.19}$$

We claim that one of the following conclusions holds for v_n^1:

(v1) $v_n^1 \to 0$ in $\mathcal{H}$, or

(v2) there exist $r' > 0$, $\sigma > 0$ and a sequence $\{y_n^1\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty}\int_{B_{r'}(y_n^1)} |v_n^1|^2 dx \geq \sigma > 0. \tag{5.20}$$

Indeed, suppose that (v2) does not occur. Then for any $r > 0$, we have

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} |v_n^1|^2 dx = 0.$$

Therefore, it follows from Lemma 2.2 that $v_n^1 \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in (2, 2_\alpha^*)$. It follows from (5.14) that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^N} |g(v_n^1)v_n^1| dx \leq \varepsilon\left(\int_{\mathbb{R}^N} |v_n^1|^2 + |v_n^1|^{2_\alpha^*}\right)dx + C_\varepsilon\int_{\mathbb{R}^N} |v_n^1|^q dx.$$

So from $v_n^1 \to 0$ in $L^q(\mathbb{R}^N)$ and the arbitrariness of ε, we can easily obtain that

$$\int_{\mathbb{R}^N} f(v_n^1)v_n^1 dx = \int_{\mathbb{R}^N} ((v_n^1)^+)^{2_\alpha^*} dx + o_n(1).$$

Furthermore, from $J'_\lambda(v_n^1)v_n^1 = o_n(1)$ in (5.18), we have

$$\|v_n^1\|^2 + b\bar{B}^2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} v_n^1|^2 dx = \lambda \|(v_n^1)^+\|_{2_\alpha^*}^{2_\alpha^*} + o_n(1). \tag{5.21}$$

In view of conditions (V_2)-(V_3), we can check that $V_\infty > 0$. And so we can also get

$$\int_{\mathbb{R}^N} V(x)|v_n^1|^2 dx = \int_{\mathbb{R}^N} V^+(x)|v_n^1|^2 dx + o_n(1),$$

which, together with the definition of S_α and (5.21), implies that

$$aS_\alpha \left(\int_{\mathbb{R}^N} |v_n^1|^{2_\alpha^*} dx\right)^{\frac{2}{2_\alpha^*}} + bS_\alpha^2 \left(\int_{\mathbb{R}^N} |v_n^1|^{2_\alpha^*} dx\right)^{\frac{4}{2_\alpha^*}} \leq \lambda \int_{\mathbb{R}^N} |v_n^1|^{2_\alpha^*} dx + o_n(1). \tag{5.22}$$

Let $\ell \geq 0$ be such that $\int_{\mathbb{R}^N} |v_n^1|^{2_\alpha^*} dx \to \ell^N$. If $\ell > 0$, then it follows from (5.22) that

$$K'(\ell) = \frac{(N-2\alpha)\ell^{-1}}{2}(aS_\alpha \ell^{N-2\alpha} + bS_\alpha^2 \ell^{2N-4\alpha} - \lambda \ell^N) \leq 0,$$

where K has been defined in Lemma 4.1. This also implies that $\ell \geq T$ (T is the unique maximum point of K). On the other hand, by (5.17) and (5.19), we have

$$\begin{aligned}
c_\lambda + \frac{b\bar{B}^4}{4} &= \int_{\mathbb{R}^N} \left(\frac{a+b\bar{B}^2}{2}\left|(-\Delta)^{\frac{\alpha}{2}} v_n^1\right|^2 + \frac{1}{2}V(x)\left|v_n^1\right|^2 - \frac{\lambda}{2_\alpha^*}\left(\left(v_n^1\right)^+\right)^{2_\alpha^*}\right) dx \\
&\quad + J_\lambda(u_0) + o_n(1) \\
&\geq \int_{\mathbb{R}^N} \left(\left(\frac{a}{2} + \frac{b\bar{B}^2}{4}\right)\left|(-\Delta)^{\frac{\alpha}{2}} v_n^1\right|^2 + \frac{1}{2}V(x)\left|v_n^1\right|^2 - \frac{\lambda}{2_\alpha^*}\left(\left(v_n^1\right)^+\right)^{2_\alpha^*}\right) dx \\
&\quad + \frac{b\bar{B}^4}{4} + o_n(1),
\end{aligned}$$

which, together with (5.21) and the definition of S_α, implies that

$$\begin{aligned}
c_\lambda &\geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) a \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} v_n^1\right|^2 dx + \left(\frac{1}{4} - \frac{1}{2_\alpha^*}\right) b \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} v_n^1\right|^2 dx\right)^2 \\
&\quad + o_n(1) \\
&\geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) aS_\alpha \left(\int_{\mathbb{R}^N} \left|v_n^1\right|^{2_\alpha^*} dx\right)^{\frac{2}{2_\alpha^*}} + \left(\frac{1}{4} - \frac{1}{2_\alpha^*}\right) bS_\alpha^2 \left(\int_{\mathbb{R}^N} \left|v_n^1\right|^{2_\alpha^*} dx\right)^{\frac{4}{2_\alpha^*}} \\
&\quad + o_n(1).
\end{aligned}$$

Thus, combining $\int_{\mathbb{R}^N} |v_n^1|^{2^*_\alpha} dx \to \ell^N$ and $\ell \geq T$, $K'(T) = 0$, we have

$$\begin{aligned} c_\lambda &\geq \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right) aS_\alpha \ell^{N-2\alpha} + \left(\frac{1}{4} - \frac{1}{2^*_\alpha}\right) bS_\alpha^2 \ell^{2N-4\alpha} \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right) aS_\alpha T^{N-2\alpha} + \left(\frac{1}{4} - \frac{1}{2^*_\alpha}\right) bS_\alpha^2 T^{2N-4\alpha} \\ &= \frac{1}{2} aS_\alpha T^{N-2\alpha} + \frac{1}{4} bS_\alpha^2 T^{2N-4\alpha} - \frac{\lambda}{2^*_\alpha} T^N = c_\lambda^*, \end{aligned}$$

contradicting $c_\lambda < c_\lambda^*$. Hence, $\ell = 0$. It follows from (5.21) that $\|v_n^1\| \to 0$, that is, $u_n \to u_0$ in $\mathcal{H}$. Then Lemma 5.4 hold with $k = 0$ if (v2) does not occur. In particular, if we assume (V_4) holds, then by Corollary 3.5, $\|u_n - |u_n|^*\|_{2^*_\alpha} \to 0$. Obviously, $\{|u_n|^*\}_{n\in\mathbb{N}} \subset H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$ is bounded and $\|u_n - |u_n|^*\|_q \to 0$ for $q \in (2, 2^*_\alpha)$. Since $\{|u_n|^*\}_{n\in\mathbb{N}}$ has a strongly convergent subsequence in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*_\alpha)$, without loss of generality, we assume that $u_n \to u_0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*_\alpha)$ and $u_0 = u_0^*$. As a consequence, (v2) does not hold and as above, $u_n \to u_0$ in $\mathcal{H}$.

In the following, otherwise, suppose that (v2) holds, that is (5.20) holds. Consider $v_n^1(\cdot + y_n^1)$. The boundedness of $\{v_n^1\}_{n\in\mathbb{N}}$ and (5.20) imply that $v_n^1(\cdot + y_n^1) \rightharpoonup w^1 \neq 0$ in $\mathcal{H}$. Thus, it follows from $v_n^1 \rightharpoonup 0$ in $\mathcal{H}$ that $\{y_n^1\}_{n\in\mathbb{N}}$ is unbounded and, up to a subsequence, $|y_n^1| \to +\infty$. Let us prove that $(J_\lambda^\infty)'(w^1) = 0$. It suffices to show that $(J_\lambda^\infty)'(v_n^1(\cdot + y_n^1))\varphi \to 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Combining Lemmas 5.1 and 5.2, we obtain

$$|J'_\lambda(u_n)\varphi - J'_\lambda(u_0)\varphi - J'_\lambda(v_n^1)\varphi| \leq o_n(1)\|\varphi\|, \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N),$$

which implies that $|J'_\lambda(v_n^1)\varphi| \leq o_n(1)\|\varphi\|$, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, as $n \to \infty$. Notice that

$$\begin{aligned} &J'_\lambda(v_n^1)\varphi(\cdot - y_n^1) \\ &\quad= \frac{C(n,\alpha)}{2}(a + b\bar{B}^2)\int_{\mathbb{R}^{2N}} \frac{(v_n^1(x) - v_n^1(y))(\varphi(x - y_n^1) - \varphi(y - y_n^1))}{|x-y|^{N+2\alpha}} dxdy \\ &\qquad+ \int_{\mathbb{R}^N} V(x) v_n^1(x)\varphi(x - y_n^1)dx - \lambda\int_{\mathbb{R}^N} g(v_n^1(x))\varphi(x - y_n^1)dx \\ &\qquad- \lambda\int_{\mathbb{R}^N} ((v_n^1(x))^+)^{2^*_\alpha - 1}\varphi(x - y_n^1)dx = o_n(1)\|\varphi(\cdot - y_n^1)\| = o_n(1)\|\varphi\|. \end{aligned}$$

Thus, as $n \to \infty$, it follows that

$$\begin{aligned} &\frac{C(n,\alpha)}{2}(a + b\bar{B}^2)\int_{\mathbb{R}^{2N}} \frac{(v_n^1(x + y_n^1) - v_n(y + y_n^1))(\varphi(x) - \varphi(y))}{|x-y|^{N+2\alpha}} dxdy \\ &\qquad+ \int_{\mathbb{R}^N} V(x + y_n^1) v_n^1(x + y_n^1)\varphi(x)dx - \lambda\int_{\mathbb{R}^N} g(v_n^1(x + y_n^1))\varphi(x)dx \\ &\qquad- \int_{\mathbb{R}^N} ((v_n^1(x + y_n^1))^+)^{2^*_\alpha - 1}\varphi(x)dx = o_n(1)\|\varphi\|. \end{aligned} \tag{5.23}$$

Since $|y_n^1| \to \infty$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} (V(x+y_n^1) - V_\infty) v_n^1(x+y_n^1)\varphi(x)dx \to 0. \tag{5.24}$$

Thus, combining (5.23) and (5.24), we have for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
&(J_\lambda^\infty)'(v_n^1(\cdot + y_n^1))\varphi \\
&\quad = \frac{C(n,\alpha)}{2}(a + b\bar{B}^2)\int_{\mathbb{R}^{2N}} \frac{(v_n^1(x+y_n^1) - v_n(y+y_n^1))(\varphi(x)-\varphi(y))}{|x-y|^{N+2\alpha}}dxdy \\
&\qquad + \int_{\mathbb{R}^N} V_\infty v_n^1(x+y_n^1)\varphi(x)dx - \lambda \int_{\mathbb{R}^N} g(v_n^1(x+y_n^1))\varphi(x)dx \\
&\qquad - \lambda \int_{\mathbb{R}^N} ((v_n^1(x+y_n^1))^+)^{2_\alpha^*-1}\varphi(x)dx = o_n(1).
\end{aligned}$$

Then, $(J_\lambda^\infty)'(w^1) = 0$, $w^1 > 0$ and $w^1(x)|x|^{N+2\alpha} = \mathcal{O}(1)$ as $|x| \to \infty$. Finally, let us set

$$v_n^2(x) = v_n^1(x) - w^1(x - y_n^1), \tag{5.25}$$

then $v_n^2 \rightharpoonup 0$ in $\mathcal{H}$. Since $V(x) \to V_\infty$ as $|x| \to \infty$ and $v_n^1 \to 0$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (V(x) - V_\infty)(v_n^1)^2 dx = o_n(1).$$

It follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} V(x)|v_n^2|^2 dx &= \int_{\mathbb{R}^N} V(x)|v_n^1|^2 dx + \int_{\mathbb{R}^N} V(x+y_n^1)|w^1(x)|^2 dx \\
&\quad - 2\int_{\mathbb{R}^N} V(x+y_n^1)v_n^1(x+y_n^1)w^1(x)dx \\
&= \int_{\mathbb{R}^N} V_\infty |u_n|^2 dx - \int_{\mathbb{R}^N} V_\infty |u_0|^2 dx - \int_{\mathbb{R}^N} V_\infty |w^1|^2 dx + o_n(1) \\
&= \int_{\mathbb{R}^N} V(x)|u_n|^2 dx - \int_{\mathbb{R}^N} V(x)|u_0|^2 dx \\
&\quad - \int_{\mathbb{R}^N} V_\infty |w^1|^2 dx + o_n(1),
\end{aligned} \tag{5.26}$$

and (it is easy to see that $\|(v_n^2)^-\|_{2_\alpha^*}) = o_n(1)$) also

$$\begin{cases}
\|(-\Delta)^{\frac{\alpha}{2}} v_n^2\|_2^2 = \|(-\Delta)^{\frac{\alpha}{2}} u_n\|_2^2 - \|(-\Delta)^{\frac{\alpha}{2}} u_0\|_2^2 - \|(-\Delta)^{\frac{\alpha}{2}} w^1\|_2^2 + o_n(1), \\
\|(v_n^2)^+\|_{2_\alpha^*}^{2_\alpha^*} = \|u_n\|_{2_\alpha^*}^{2_\alpha^*} - \|u_0\|_{2_\alpha^*}^{2_\alpha^*} - \|w^1\|_{2_\alpha^*}^{2_\alpha^*} + o_n(1),
\end{cases} \tag{5.27}$$

$$\int_{\mathbb{R}^N} G(v_n^2)dx = \int_{\mathbb{R}^N} G(u_n)dx - \int_{\mathbb{R}^N} G(u_0)dx - \int_{\mathbb{R}^N} G(w^1)dx. \tag{5.28}$$

It is readily checked that we also have

$$\begin{aligned}
\int_{\mathbb{R}^N} g(v_n^2)\varphi dx &= \int_{\mathbb{R}^N} g(u_n)\varphi dx - \int_{\mathbb{R}^N} g(u_0)\varphi dx \\
&\quad - \int_{\mathbb{R}^N} g(w^1(\cdot - y_n^1))\varphi dx + o_n(1)\|\varphi\|,
\end{aligned} \tag{5.29}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Combining (5.26), (5.27), (5.28) and (5.29), we deduce that

$$\begin{aligned}
&(1)\quad J_\lambda(v_n^2) = J_\lambda(u_n) - J_\lambda(u_0) - J_\lambda^\infty(w^1) + o_n(1),\\
&(2)\quad J'_\lambda(v_n^2)\varphi = J'_\lambda(u_n)\varphi - J'_\lambda(u_0)\varphi - (J_\lambda^\infty)'(w^1(\cdot - y_n^1))\varphi + o_n(1)\|\varphi\|\\
&\qquad = o_n(1)\|\varphi\|,\\
&(3)\quad J_\lambda^\infty(v_n^2) = J_\lambda^\infty(v_n^1) - J_\lambda^\infty(w^1) + o_n(1)
\end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Thus $\{v_n^2\}_{n\in\mathbb{N}}$ is a Palais–Smale sequence and we get

$$J_\lambda(v_n^2) = c_\lambda + \frac{b\bar{B}^4}{4} - J_\lambda(u_0) - J_\lambda^\infty(w^1) + o_n(1) < c_\lambda^* + \frac{b\bar{B}^4}{4}.$$

Remark that one of (v1) and (v2) holds for v_n^2. If $v_n^2 \to 0$ in $\mathcal{H}$, then Lemma 5.4 holds with $k = 1$. Otherwise, $\{v_n^2\}$ is non-vanishing, that is, (v2) holds for v_n^2. Similarly, we repeat the arguments. By iterating this procedure we obtain sequences of points $\{y_n^j\} \subset \mathbb{R}^N$ such that $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ if $i \neq j$ as $n \to +\infty$ and $v_n^j = v_n^{j-1} - w^{j-1}(x - y_n^{j-1})$ (like (5.25)) with $j \geq 2$ such that $v_n^j \rightharpoonup 0$ in $\mathcal{H}$, $(J_\lambda^\infty)'(w^j) = 0$. Using the properties of the weak convergence, we have

$$\begin{aligned}
&(a)\|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{k} \|w^j(\cdot - y_n^j)\|^2 = \|u_n - u_0 - \sum_{j=1}^{k} w^j(\cdot - y_n^j)\|^2 + o_n(1),\\
&(b)J_\lambda(u_n) = J_\lambda(u_0) + \sum_{j=1}^{k} J_\lambda^\infty(w^j) + J_\lambda^\infty(v_n^{k+1}) + o_n(1).
\end{aligned} \tag{5.30}$$

Note that there is $\rho > 0$ such that $\|w\| \geq \rho$ for every nontrivial critical point w of J_λ^∞ and $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. By (5.30)(a), the iteration stops at some k. That is, $v_n^{k+1} \to 0$ in $\mathcal{H}$. We stress that the polynomial decay of the limiting profiles w^j can be justified as in [13, Theorem 3.4 and Theorem 1.5]. The proof is now complete. □

6. Proof of the main results

In order to obtain the existence of ground state solutions of problem (K), our strategy is that we firstly obtain the existence nontrivial solutions of the perturbed problem, then as λ goes to 1, we get a nontrivial solution of the original problem. Finally, thanks to the profile decomposition of the (PS)-sequence, we obtain the existence of ground state solutions of problem (K).

6.1. Nontrivial critical points of I_λ

Lemma 6.1. *Assume that* (V_1)-(V_3) *and* (f_1)-(f_3) *hold. For almost every* $\lambda \in [1/2, 1]$, *there exists* $u_\lambda \in \mathcal{H}\backslash\{0\}$ *such that* $I_\lambda(u_\lambda) = c_\lambda$ *and* $I'_\lambda(u_\lambda) = 0$. *In addition, if* ($V_4$) *holds, then* $u_\lambda \in H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$.

Proof. For almost all $\lambda \in [1/2, 1]$, there is a bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $I_\lambda(u_n) \to c_\lambda$, $I'_\lambda(u_n) \to 0$. From Lemma 5.4, up to a subsequence, there exist $u_0 \in \mathcal{H}$ and $\bar{B} > 0$ such that

$$u_n \rightharpoonup u_0 \quad \text{in } \mathcal{H}, \quad \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \to \bar{B}^2, \text{ as } n \to \infty$$

and $J'_\lambda(u_0) = 0$. Furthermore, there exist $k \in \mathbb{N} \cup \{0\}$, nontrivial critical points $w^1, \ldots, w^k$ of J^∞_λ and k sequences of points $\{y^j_n\} \subset \mathbb{R}^N$, $1 \le j \le k$, such that

$$\left\| u_n - u_0 - \sum_{j=1}^{k} w^j(\cdot - y^j_n) \right\| \to 0, \quad c_\lambda + \frac{b\bar{B}^4}{4} = J_\lambda(u_0) + \sum_{j=1}^{k} J^\infty_\lambda(w^j) \quad (6.1)$$

and

$$\bar{B}^2 = \|(-\Delta)^{\frac{\alpha}{2}} u_0\|_2^2 + \sum_{j=1}^{k} \|(-\Delta)^{\frac{\alpha}{2}} w^j\|_2^2. \quad (6.2)$$

Now we claim that if $u_0 \ne 0$, then by $N < 4\alpha$,

$$J_\lambda(u_0) > \frac{b\bar{B}^2}{4} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_0|^2 dx. \quad (6.3)$$

Indeed, since $J'_\lambda(u_0) = 0$, similar as in [11], we get

$$\begin{aligned} \bar{P}_\lambda(u_0) &:= \frac{N-2\alpha}{2}(a + b\bar{B}^2) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_0|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V(x) u_0^2 dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x), x) u_0^2 dx - N\lambda \int_{\mathbb{R}^N} F(u_0) dx = 0. \end{aligned}$$

By hypothesis (V_1) we have

$$\begin{aligned} J_\lambda(u_0) &= \frac{\alpha}{N}(a + b\bar{B}^2) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_0|^2 dx - \frac{1}{2N} \int_{\mathbb{R}^N} \nabla V(x) \cdot x \, u_0^2 dx \\ &> \frac{\alpha}{N} b\bar{B}^2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_0|^2 dx, \end{aligned}$$

which implies that (6.3) holds. For each nontrivial critical point w^j, $(j = 1, ..., k)$ of J^∞_λ,

$$\begin{aligned} &\frac{N-2\alpha}{2}(a + b\bar{B}^2) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} w^j|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V_\infty |w^j|^2 dx \\ &\quad - N\lambda \int_{\mathbb{R}^N} F(w^j) dx = P^\infty_\lambda(w^j) = 0. \end{aligned}$$

Then it follows from (6.2) that

$$\begin{aligned} &\frac{a(N-2\alpha)}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} w^j|^2 dx + \frac{b(N-2\alpha)}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} w^j|^2 dx \right)^2 \\ &\quad + \frac{N}{2} \int_{\mathbb{R}^N} V_\infty |w^j|^2 dx - N\lambda \int_{\mathbb{R}^N} F(w^j) dx \le 0. \end{aligned}$$

Then there exists $t_j \in (0,1]$ such that

$$\begin{aligned} &\frac{at_j^{N-2\alpha}}{2}(N-2\alpha)\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j|^2dx \\ &\quad+\frac{bt_j^{2N-4\alpha}}{2}(N-2\alpha)\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j|^2dx\right)^2 \\ &\quad+\frac{Nt_j^N}{2}\int_{\mathbb{R}^N}V_\infty|w^j|^2dx-Nt_j^N\lambda\int_{\mathbb{R}^N}F(w^j)dx=0. \end{aligned} \tag{6.4}$$

That is, $w^j(\cdot/t_j)$ satisfies the identity $P_\lambda(u)=0$ and it follows from Lemma 4.3 that there exists $\gamma_\lambda \in C([0,1],\mathcal{H})$ such that $\gamma_\lambda(0)=0$, $I_\lambda^\infty(\gamma_\lambda(1))<0$, $w^j \in \gamma_\lambda([0,1])$ and

$$I_\lambda^\infty(w^j(\cdot/t_j))=\max_{t\in[0,1]}I_\lambda^\infty(\gamma_\lambda(t)).$$

By hypothesis (V_2), we have $\max_{t\in[0,1]}I_\lambda^\infty(\gamma_\lambda(t))\geq\max_{t\in[0,1]}I_\lambda(\gamma_\lambda(t))$, which, by the definition of c_λ, implies that $I_\lambda^\infty(w^j(\frac{\cdot}{t_j}))\geq c_\lambda$. In particular, if $V(x)\not\equiv V_\infty$, then

$$I_\lambda^\infty(w^j(\cdot/t_j))>c_\lambda. \tag{6.5}$$

So by (6.4) we have

$$\begin{aligned} J_\lambda^\infty(w^j)&=J_\lambda^\infty(w^j)-\frac{1}{N}P_\lambda^\infty(w^j)=(a+b\bar{B}^2)\left(\frac12-\frac{1}{2_\alpha^*}\right)\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j|^2dx \\ &\geq\left(\frac12-\frac{1}{2_\alpha^*}\right)a\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j(\frac{x}{t_j})|^2dx \\ &\quad+\left(\frac14-\frac{1}{2_\alpha^*}\right)b\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j(\frac{x}{t_j})|^2dx\right)^2+\frac{b\bar{B}^2}{4}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j|^2dx \\ &=I_\lambda^\infty\left(w^j\left(\frac{\cdot}{t_j}\right)\right)-\frac1N P_\lambda(w^j(\frac{\cdot}{t_j}))+\frac{b\bar{B}^2}{4}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j|^2dx \\ &=I_\lambda^\infty\left(w^j\left(\frac{\cdot}{t_j}\right)\right)+\frac{b\bar{B}^2}{4}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j|^2dx \end{aligned} \tag{6.6}$$

and then we conclude that

$$J_\lambda^\infty(w^j)\geq c_\lambda+\frac{b\bar{B}^2}{4}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}w^j|^2dx,$$

where the inequality is strict if $V(x)\not\equiv V_\infty$. Then by formulas (6.2)-(6.3),

$$c_\lambda+\frac{b\bar{B}^4}{4}=J_\lambda(u_0)+\sum_{j=1}^k J_\lambda^\infty(w^j)\geq kc_\lambda+\frac{b\bar{B}^4}{4}, \tag{6.7}$$

with *strict* inequality if $V(x)\not\equiv V_\infty$ or $u_0\neq 0$.

- If $k=0$, we are done. If condition (V_4) holds, then $k=0$ and u_0 is radial. Then it follows that $I_\lambda(u_0)=J_\lambda(u_0)-b\bar{B}^4/4=c_\lambda$ and $I'_\lambda(u_0)=J'_\lambda(u_0)=0$.
- If $k=1$ and $V(x)\not\equiv V_\infty$ or $u_0\neq 0$, then (6.7) yields a contradiction

• If $k=1$ and $V(x)\equiv V_\infty$ and $u_0=0$, then $\bar{B}^2=\|(-\Delta)^{\frac{\alpha}{2}}w^1\|_2^2$ and it follows from (6.1) that

$$c_\lambda = J_\lambda^\infty(w^1)-\frac{b\bar{B}^4}{4}=J_\lambda^\infty(w^1)-\frac{b}{4}\|(-\Delta)^{\frac{\alpha}{2}}w^1\|_2^4=I_\lambda(w^1),\qquad I_\lambda'(w^1)=0,$$

as desired. Hence, in any case, the assertion follows. □

6.2. Completion of the proof

Choosing a sequence $\{\lambda_n\}_{n\in\mathbb{N}}\subset[\frac{1}{2},1]$ satisfying $\lambda_n\to 1$, we find a sequence of nontrivial critical points $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ (still denoted by $\{u_n\}_{n\in\mathbb{N}}$) of I_{λ_n} and $I_{\lambda_n}(u_n)=c_{\lambda_n}$. In particular, if (V$_4$) holds, then $\{u_n\}_{n\in\mathbb{N}}\subset H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$. Now we show that $\{u_n\}$ is bounded in $\mathcal{H}$. Remark that u_n satisfies the Pohožaev identity as follows

$$\frac{N-2\alpha}{2}\int_{\mathbb{R}^N}a|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx+\frac{N-2\alpha}{2}b\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx\right)^2 \\ +\frac{N}{2}\int_{\mathbb{R}^N}V(x)u_n^2dx+\frac{1}{2}\int_{\mathbb{R}^N}\nabla V(x)\cdot x\,u_n^2dx-N\lambda\int_{\mathbb{R}^N}F(u_n)dx=0.$$

It follows that

$$NI_{\lambda_n}(u_n)=\alpha\int_{\mathbb{R}^N}a|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx+\left(\alpha-\frac{N}{4}\right)b\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx\right)^2 \\ -\frac{1}{2}\int_{\mathbb{R}^N}\nabla V(x)\cdot x\,u_n^2dx.$$

Since c_λ^* is continuous on λ, $I_{\lambda_n}(u_n)=c_{\lambda_n}+o_n(1)<c^*_{\lambda_n}$. It follows from (V$_1$) that there is a positive number $\kappa\in(0,2a\alpha)$ such that $\|W\|_{\frac{N}{2\alpha}}\le\kappa S_\alpha$. Hence,

$$\left(a\alpha-\frac{\kappa}{2}\right)\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx\le NI_{\lambda_n}(u_n),$$

which implies that $\int_{\mathbb{R}^N}a|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx$ is bounded from above. By (V$_3$), (f$_1$)-(f$_2$) and $I'_{\lambda_n}(u_n)u_n=0$, there is $\nu>0$ such that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ with

$$\nu\int_{\mathbb{R}^N}u_n^2dx\le\int_{\mathbb{R}^N}a|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx+\int_{\mathbb{R}^N}V(x)u_n^2dx\le\varepsilon\int_{\mathbb{R}^N}u_n^2dx \\ +C_\varepsilon\int_{\mathbb{R}^N}u_n^{2^*_\alpha}dx,$$

which yields that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. By Theorem 3.1,

$$\lim_{n\to\infty}I(u_n)=\lim_{n\to\infty}\left(I_{\lambda_n}(u_n)+(\lambda_n-1)\int_{\mathbb{R}^N}F(u_n)dx\right)=\lim_{n\to\infty}c_{\lambda_n}=c_1$$

and for any $\varphi\in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty}I'(u_n)\varphi=\lim_{n\to\infty}\left(I'_{\lambda_n}(u_n)\varphi+(\lambda_n-1)\int_{\mathbb{R}^N}f(u_n)\varphi dx\right)=0.$$

That is, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded Palais–Smale sequence for I at level c_1. Then by Lemma 6.1, there is a nontrivial critical point $u_0 \in \mathcal{H}$ (radial, if (V_4) holds) for I and $I(u_0) = c_1$. Set

$$\nu = \inf\{I(u) : u \in \mathcal{H}\backslash\{0\},\, I'(u) = 0\}.$$

Of course $0 < \nu \le I(u_0) = c_1 < \infty$. By the definition of ν, there is $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ with $I(u_n) \to \nu$ and $I'(u_n) = 0$. We deduce that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Up to a sequence, for some $\bar{B} > 0$,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \to \bar{B}^2.$$

Let us set $J(u) := J_1(u)$ and $J^\infty(u) := J_1^\infty(u)$, for any $u \in \mathcal{H}$. From Lemma 5.4 there exists $u_0 \in \mathcal{H}$ such that $u_n \rightharpoonup u_0$ in $\mathcal{H}$ and $J'(u_0) = 0$. Furthermore, there exist $k \in \mathbb{N} \cup \{0\}$, nontrivial critical points $w^1, \ldots, w^k$ of J^∞ and k sequences of points $\{y_n^j\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$, $1 \le j \le k$, such that

$$\left\| u_n - u_0 - \sum_{j=1}^{k} w^j(\cdot - y_n^j) \right\| \to 0, \quad \nu + \frac{b\bar{B}^4}{4} = J(u_0) + \sum_{j=1}^{k} J^\infty(w^j) \tag{6.8}$$

and

$$\bar{B}^2 = \|(-\Delta)^{\frac{\alpha}{2}} u_0\|_2^2 + \sum_{j=1}^{k} \|(-\Delta)^{\frac{\alpha}{2}} w^j\|_2^2.$$

If $k = 0$, we are done. If $k \ge 1$, assume by contradiction that $u_0 \ne 0$. Then, as in Lemma 6.1,

$$J(u_0) > \frac{b\bar{B}^2}{4} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_0|^2 dx, \tag{6.9}$$

for each j there is $t_j \in (0, 1]$ such that $I^\infty(w^j(\cdot/t_j)) \ge c_1$, which is strict if $V(x) \not\equiv V_\infty$, and

$$J^\infty(w^j) \ge c_1 + \frac{b\bar{B}^2}{4} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} w^j|^2 dx,$$

where the inequality is strict if $V(x) \not\equiv V_\infty$. Then by formulas (6.8)-(6.9) and $\nu \le c_1$, we get

$$c_1 + \frac{b\bar{B}^4}{4} \ge \nu + \frac{b\bar{B}^4}{4} = J(u_0) + \sum_{j=1}^{k} J^\infty(w^j) > kc_1 + \frac{b\bar{B}^4}{4},$$

a contradiction. Hence $u_0 = 0$ and $k = 1$, in which case a contradiction follows as in the proof of Lemma 6.1. The proof is complete. □

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References

[1] Alves, C.O., Corrêa, F.: On existence of solutions for a class of problem involving a nonlinear operator. Appl. Nonlinear Anal. **8**, 43–56 (2001)

[2] Ambrosio, V., Isernia, T.: A multiplicity result for a fractional Kirchho equation in $\mathbb{R}^N$ with a general nonlinearity. Commun. Contemp. Math. 1750054 (2017). doi:10.1142/S0219199717500547. (**Published online**)

[3] Autuori, G., Fiscella, A., Pucci, P.: Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity. Nonlinear Anal. **125**, 699–714 (2015)

[4] Azzollini, A.: The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity. Differ. Integral Equ. **25**, 543–554 (2012)

[5] Baernstein, A.: A unified approach to symmetrization. In: Partial Differential Equations of Elliptic Type (Cortona, 1992) Symposia Mathematica, vol. 35, pp. 47–91. Cambridge University Press, Cambridge (1994)

[6] Berestycki, H., Lions, P.: Nonlinear scalar field equations. I. Existence of a ground state. Arch. Ration. Mech. Anal. **82**, 313–345 (1983)

[7] Brezis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic problems involving critical Sobolev exponent. Commun. Pure Appl. Math. **36**, 437–477 (1983)

[8] Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Commun. PDE **32**, 1245–1260 (2007)

[9] Caponi, M., Pucci, P.: Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations. Ann. Mat. Pura Appl. **195**(4), 2099–2129 (2016)

[10] Chipot, M., Lovat, B.: Some remarks on nonlocal elliptic and parabolic problems. Nonlinear Anal. **30**, 4619–4627 (1997)

[11] Chang, X., Wang, Z.: Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity. Nonlinearity **26**, 479–494 (2013)

[12] Dipierro, S., Palatucci, G., Valdinoci, E.: Existence and symmetry results for a Schrödinger type problem involving the fractional laplacian. Le Matematiche **68**(1), 201–216 (2013)

[13] Felmer, P., Quaas, A., Tan, J.: Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian. Proc. R. Soc. Edinb. Sect. A **142**, 1237–1262 (2012)

[14] Fiscella, A., Valdinoci, E.: A critical Kirchhoff type problem involving a nonlocal operator. Nonlinear Anal. **94**, 156–170 (2014)

[15] Fiscella, A., Pucci, P.: p-fractional Kirchhoff equations involving critical nonlinearities. Nonlinear Anal. RWA **35**, 350–378 (2017)

[16] Frank, R., Lenzmann, E.: Uniqueness and non degeneracy of ground states for $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ in $\mathbb{R}$. Acta Math. **210**, 261–318 (2013)

[17] He, X., Zou, W.: Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. J. Differ. Equ. **252**, 1813–1834 (2012)

[18] Jeanjean, L.: On the existence of bounded Palais–Smale sequence and application to a Landesman–Lazer type problem set on $\mathbb{R}^N$. Proc. R. Soc. Edinb. Sect. A **129**, 787–809 (1999)

[19] Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)

[20] Laskin, N.: Fractional Schrödinger equation. Phys. Rev. E **66**, 05618 (2002)

[21] Li, G., Ye, H.: Existence of positive solutions for nonlinear Kirchhoff type problems in $\mathbb{R}^3$ with critical Sobolev exponent and sign-changing nonlinearities. Math. Methods Appl. Sci. **37**(16), 2570–2584 (2014)

[22] Li, G., He, Y.: Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$. J. Differ. Equ. **257**, 566–600 (2014)

[23] Lions, J.: On some questions in boundary value problems of mathematical physics. In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations. Proceedings of the International Symposium Inst. Mat. Univ. Fed. Rio de Janeiro, (1997) In: North-Holland Mathematics Studies, vol. 30, pp. 284–346 (1978)

[24] Liu, Z., Guo, S.: Existence and concentration of positive ground states for a Kirchhoff equation involving critical Sobolev exponent. Z. Angew. Math. Phys. **66**, 747–769 (2015)

[25] Liu, Z., Guo, S.: Existence of positive ground state solutions for Kirchhoff type problems. Nonlinear Anal. **120**, 1–13 (2015)

[26] Ma, T., Rivera, J.: Positive solutions for a nonlinear nonlocal elliptic transmission problem. Appl. Math. Lett. **16**, 243–248 (2003)

[27] Nyamoradi, N.: Existence of three solutions for Kirchhoff nonlocal operators of elliptic type. Math. Commun. **18**, 489–502 (2013)

[28] Perera, K., Zhang, Z.: Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differ. Equ. **221**, 246–255 (2006)

[29] Pucci, P., Xiang, M., Zhang, B.: Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$. Calc. Var. Part. Differ. Equ. **54**, 2785–2806 (2015)

[30] Pucci, P., Saldi, S.: Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators. Rev. Mat. Iberoam. **32**, 1–22 (2016)

[31] Pucci, P., Xiang, M., Zhang, B.: Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations. Adv. Nonlinear Anal. **5**, 27–55 (2016)

[32] Secchi, S.: Ground state solutions for nonlinear fractional Schrodinger equations in R-N. J. Math. Phys. **54**, 031501 (2013). doi:10.1063/1.4793990

[33] Servadei, R., Valdinoci, E.: The Brezis–Nirenberg result for the fractional Laplacian. Trans. Am. Math. Soc. **367**, 67–102 (2015)

[34] Silvestre, L.: Hölder estimates for solutions of integro-differential equations like the fractional Laplace. Indiana Univ. J. Math. **55**, 1155–1174 (2006)

[35] Squassina, M.: On the Struwe–Jeanjean–Toland monotonicity trick. Proc. R. Soc. Edinb. Sect. A **142**, 155–169 (2012)

[36] Tao, F., Wu, X.: Existence and multiplicity of positive solutions for fractional Schrödinger equations with critical growth. Nonlinear Anal. RWA **35**, 158–174 (2017)

[37] Teng, K.: Existence of ground state solutions for the nonlinear fractional Schrödinger–Poisson system with critical Sobolev exponent. J. Differ. Equ. **261**, 3061–3106 (2016)

[38] Van Schaftingen, J.: Symmetrization and minimax principles. Commun. Contemp. Math. **7**, 463–481 (2005)

[39] Xiang, M., Zhang, B., Guo, X.: Infinitely many solutions for a fractional Kirchhoff type problem via Fountain Theorem. Nonlinear Anal. **120**, 299–313 (2015)

[40] Zhang, J.J., Zou, W.M.: A Berestycki–Lions theorem revisted. Commun. Contemp. Math. **14**, 14 (2012)

[41] Zhang, J., Zou, W.M.: The critical case for a Berestycki–Lions theorem. Sci. China Math. **14**, 541–554 (2014)

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