ON FRACTIONAL p-LAPLACIAN PROBLEMS WITH WEIGHT

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ABSTRACT. We investigate the existence of nonnegative solutions for a nonlinear problem involving the fractional p-Laplacian operator. The problem is set on a unbounded domain, and compactness issues have to be handled.

1. INTRODUCTION

The interest for the fractional Laplacian operator $(-\Delta)^s$ and more generally pseudodifferential operators, has constantly increased over the last few years, although such operators have been a classical topic of functional analysis since long ago. Nonlocal operators such as $(-\Delta)^s$ naturally arise in continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastical stabilization of Lévy processes, see e.g. [8, 19, 21]. We refer the reader to [13] and to the references included for a selfcontained overview of the basic properties of fractional Sobolev spaces. If Ω is a smooth bounded domain, for semi-linear problems like

$$\begin{cases} (-\Delta)^s u = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

existence, nonexistence, regularity and maximum principles have been intensively investigated, see e.g. [6, 7, 9, 22–27]. When $\Omega = \mathbb{R}^N$, we refer the reader to [10, 14] where weak solutions in $H^s(\mathbb{R}^N)$ are studied. More recently, for $p > 1$, $s \in (0,1)$ and $N > sp$, motivated by some situations arising in game theory, a nonlinear generalization of this operator has been introduced, see [2, 8]. Precisely, for smooth functions u define

$$(-\Delta)_p^s\, u(x) := 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}\,(u(x) - u(y))}{|x - y|^{N+sp}}\, dy, \quad x \in \mathbb{R}^N.$$

This nonlinear operator is consistent, up to some normalization constant depending upon n and s, with the linear fractional Laplacian $(-\Delta)^s$ in the case $p = 2$. A broad range of existence and multiplicity results for the problem

$$\begin{cases} (-\Delta)_p^s u = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

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has been recently obtained in [17] via tools of Morse theory under different growth assumptions for $f(x,u)$. We refer to [15, 16, 20] for the case $f(x,u) = \lambda|u|^{p-2}u$ and the study of properties of (variational) nonlinear eigenvalues, including their asymptotic behaviour.
In this paper, we are concerned with existence of solutions of

$$\begin{cases} (-\Delta)_p^s u = \varphi(x) f(u), & \text{in } \mathbb{R}^N, \\ u \geqslant 0, \ \ u \neq 0, \end{cases} \tag{1.1}$$

under suitable growth and sign assumptions on the functions φ and f. In the local case, that is formally $s = 1$, necessary and sufficient conditions for the solvability of the problem $-\Delta u = \varphi(x)u^q$ in $\mathbb{R}^N$ with $0 < q < 1$ were investigated in [5], see also [4]. Under some sign condition on φ the problem with $s = 1$ and $p > 1$, which thus involves the p-Laplace operator $\Delta_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ was investigated in [1], see also [11]. If we put

$$F(u) = \int_0^u f(\tau)d\tau, \tag{1.2}$$

the (formal) Pohožaev identity for solutions $u \in W^{s,p}(\mathbb{R}^N)$ of problem (1.1) is

$$\int_{\mathbb{R}^N} \big((N - sp)\varphi(x)f(u)u - pN\varphi(x)F(u) - px \cdot \nabla\varphi(x)F(u)\big) = 0. \tag{1.3}$$

A rigorous justification of (1.3) for $p \neq 2$ is still unavailable due to the lack of suitable regularity results, while in the case $p = 2$, (1.3) has been recently proved in [24], see also [10, 25]. For the case $f(u) = u^q$, the identity yields nonexistence of solutions $u \in W^{s,p}(\mathbb{R}^N)$ provided that

$$x \mapsto (N - sp)\varphi(x) - \frac{pN}{q+1}\varphi(x) - \frac{p}{q+1}x \cdot \nabla\varphi(x) \quad \text{has fixed sign in } \mathbb{R}^N.$$

Then, in particular case when φ is constant $u = 0$ as soon as $q \neq p_s^* - 1$, where we set

$$p_s^* := \frac{Np}{N - sp}.$$

Hence, in general, it is rather natural to impose conditions on φ in order to get the existence of nontrivial solutions to (1.1).
We will assume that $p > 1$, $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $f \in C(\mathbb{R}^+)$ satisfies the following conditions:

- (f_1) $f(\tau) \geqslant 0$, for all $\tau \geqslant 0$;
- (f_2) $\mu\tau^q \leqslant f(\tau) \leqslant c\tau^q$, for all $\tau \geqslant 0$, some $p - 1 < q < p_s^* - 1$ and $c, \mu > 0$;
- (f_3) if F denotes the function in (1.2), there exists $m < p$ such that

$$0 \leqslant (q + 1)F(\tau) - f(\tau)\tau \leqslant C\tau^m, \quad \text{for all } \tau \geqslant 0 \text{ and some } C > 0;$$
$$0 \leqslant f(\tau)\tau - pF(\tau) \leqslant C\tau^{q+1}, \quad \text{for all } \tau \geqslant 0.$$

- (W) $\sup_{\mathbb{R}^N \setminus \Omega} \varphi \leqslant 0 < \inf_{\omega} \varphi$ for some bounded domains $\omega, \Omega \subset \mathbb{R}^N$ with $\omega \subset \Omega$.

In addition to $f(\tau) := \tau^q$ for $\tau \geqslant 0$, another example of nonlinearity satisfying (f_1)-(f_3) is

$$f(s) := \begin{cases} 2\tau^q, & 0 \leqslant \tau \leqslant 1, \\ \tau^q + \tau^{m-1}, & \tau \geqslant 1, \end{cases} \qquad m < p < q + 1.$$

The main result of the paper is the following:

Theorem 1.1. *Assume that* (W) *and* (f_1)-(f_3) *hold. Then problem* (1.1) *has a distributional solution, namely there exists a function* $u \in L^{Np/(N-sp)}(\mathbb{R}^N) \setminus \{0\}$ *with* $u \geqslant 0$,

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} < \infty,$$

and

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}} = \int_{\mathbb{R}^N} \varphi(x) f(u)\psi,$$

for all $\psi \in C_c^\infty(\mathbb{R}^N)$*. The same holds if* (W) *and* (f_1) *hold and* (f_2) *holds with* $0 \leqslant q < p-1$.

We point out that the result is new also for the semi-linear case $p = 2$, $1 < q < 2_s^* - 1$ and $N \geqslant 2$, establishing existence of a nonnegative distributional solution $u \in D^{s,2}(\mathbb{R}^N)$ (see the beginning of Section 2 for its definition) for

$$(-\Delta)^s u = \varphi(x) f(u) \quad \text{in } \mathbb{R}^N.$$

In general, it is not guaranteed that the *distributional* solution u of Theorem 1.1 belongs to the fractional space $W^{s,p}(\mathbb{R}^N)$, that is $u \notin L^p(\mathbb{R}^N)$ might occur. Moreover, if the solution $u \geqslant 0$ of Theorem 1.1 was a *weak* supersolution to $(-\Delta)_p^s u = 0$, by the results of [12] (see also [3, Theorem A.1]), actually $u > 0$. On the other hand, assumption (W) prevents u from being a weak supersolution to $(-\Delta)_p^s u = 0$, since $f(u) \geqslant 0$ and $\varphi(x) \leqslant 0$ for $x \notin \Omega$. That $u > 0$ is expected of course. In fact, if $u \geqslant 0$ was a solution in classical sense and if there exists a point $x_0 \in \mathbb{R}^N$ with $u(x_0) = 0$, then $u(y) \geqslant u(x_0)$ for every $y \in \mathbb{R}^N$, yielding

$$0 > -2\int_{\mathbb{R}^N} \frac{(u(y)-u(x_0))^{p-1}}{|x_0-y|^{N+sp}} dy = \varphi(x_0) f(u(x_0)) = 0,$$

namely a contradiction. The proof of Theorem 1.1 follows the pattern of [1], namely nontrivial nonnegative solutions u_n are constructed for the problem defined on a sequence of balls $B(0, R_n) \subset \mathbb{R}^N$ with $u_n = 0$ on $\mathbb{R}^N \setminus B(0, R_n)$, with $R_n \nearrow \infty$ as $n \to \infty$. Then, relying on uniform estimates, the sequence is shown to converge weakly to a nontrivial distributional solution to (1.1). Both in getting uniform estimates and in proving the nontriviality of the weak limit, the fact that $\varphi(x) \leqslant 0$ outside a bounded domain of $\mathbb{R}^N$ plays a crucial role.

2. Preliminary results

The space $D^{s,p}(\mathbb{R}^N)$ is defined by

$$D^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{\frac{Np}{N-sp}}(\mathbb{R}^N) : \|u\|_{D^{s,p}} < \infty \right\}, \quad \|u\|_{D^{s,p}} := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \right)^{1/p}.$$

Endowed with the norm $\|\cdot\|_{D^{s,p}}$ the space $D^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space. From [13, Theorem 6.5], we know that there exists a positive constant C such that

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C\|u\|_{D^{s,p}}, \qquad \text{for every } u \in D^{s,p}(\mathbb{R}^N), \tag{2.1}$$

and $D^{s,p}(\mathbb{R}^N)$ is embedded into $L^q_{\text{loc}}(\mathbb{R}^N)$, for every $1 \leqslant q \leqslant p_s^*$. We observe that, in general, the integral $\varphi F(u)$ may not belong to $L^1(\mathbb{R}^N)$ for $\varphi \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. Hence, we shall consider a sequence of diverging radii $R_n > 0$ and the spaces

$$X_n := \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus B(0, R_n) \right\}$$

endowed with the norm

$$\|u\|_{X_n} := \|u\|_{D^{s,p}}, \qquad u \in X_n, \tag{2.2}$$

and the functionals $J_n : X_n \to \mathbb{R}$ given by

$$J_n(u) := \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} - \int_{B(0,R_n)} \varphi(x)F(u^+), \quad u \in X_n.$$

We stress that, by means of (2.1) and Hölder inequality, the norm defined in (2.2) is equivalent (with constants depending on the value of n) to the standard norm in $W^{s,p}(\mathbb{R}^N)$, namely $\|u\|_{W^{s,p}} = (\|u\|_p^p + \|u\|_{D^{s,p}}^p)^{1/p}$. We can check that $J_n \in C^1(X_n,\mathbb{R})$ and, for $u, v \in X_n$,

$$J_n'(u)(v) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} - \int_{B(0,R_n)} \varphi(x)f(u^+)v.$$

The truncation with $u^+ := \max\{u,0\}$ in the nonlinearity will allow critical points of J_n be automatically nonnegative, see Lemma 2.2.

Without loss of generality, we may assume that all the balls $B(0,R_n)$ contain the domain Ω for each $n \geqslant 1$ large enough.

Lemma 2.1. *For every $n \geqslant 1$ the functional J_n is weakly lower semi-continuous on X_n.*

Proof. If $(u_j) \subset X_n$ converges weakly to some u in X_n as $j \to \infty$, we have

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \leqslant \liminf_{j\to\infty} \int_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^p}{|x-y|^{N+sp}}.$$

Since (u_j) is bounded in $L^p(B(0,R_n))$ via inequality (2.1), the compact embedding theorem for fractional Sobolev spaces [13, Corollary 7.2] implies that, up to a subsequence, the sequence (u_j) converges strongly to u in $L^r(B(0,R_n))$, for every $1 \leqslant r < p_s^*$ and $u_j(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. In turn, since by condition (f_2) there exists a positive constant $C_n > 0$ with

$$|\varphi(x)F(u_j^+)|\chi_{B(0,R_n)} \leqslant C_n|u_j|^{q+1}, \quad (q+1 < p_s^*),$$

we get by the Dominated Convergence theorem that

$$\lim_{j\to\infty} \int_{B(0,R_n)} \varphi(x)F(u_j^+) = \int_{B(0,R_n)} \varphi(x)F(u^+).$$

This concludes the proof. □

Set $u^\pm = \max\{\pm u, 0\}$. We have the following

Lemma 2.2. *If $J_n'(u) = 0$, for $u \in X_n$. Then $u \geqslant 0$.*

Proof. Observe first that if $u \in X_n$, then $u^\pm \in X_n$. We have

$$\int_{B(0,R_n)} \varphi(x)f(u^+)u^- = 0. \tag{2.3}$$

We recall the elementary inequality

$$|\xi^- - \eta^-|^p \leqslant |\xi-\eta|^{p-2}(\xi-\eta)(\eta^- - \xi^-), \quad \text{for every } \xi, \eta \in \mathbb{R}.$$

Then, recalling (2.3), by testing J_n' with $-u^- \in X_n$ yields

$$\begin{aligned} 0 = J_n'(u)(-u^-) &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(y)-u^-(x))}{|x-y|^{N+sp}} \\ &\geqslant \int_{\mathbb{R}^{2N}} \frac{|u^-(x)-u^-(y)|^p}{|x-y|^{N+sp}}. \end{aligned}$$

This implies that u^- is constant in $\mathbb{R}^N$ and since u^- vanishes outside $B(0,R_n)$, it follows that $u^- = 0$. Hence, $u \geqslant 0$ a.e., concluding the proof. □

In the next two lemmas, we consider the case where (f_2) is satisfied with q small.

Lemma 2.3. *Assume (W), (f_1) and (f_2) with $q+1<p$. Then, for each $n\geqslant 1$, there exists a nonnegative critical point $u_n\in X_n\setminus\{0\}$ of J_n such that*

$$J_n(u_n)=\inf_{X_n} J_n<0.$$

Proof. By virtue of condition (f_2), we have the following inequality

$$\int_{B(0,R_n)}\varphi(x)F(u^+)\leqslant c\int_{B(0,R_n)}\varphi(x)|u^+|^{q+1}.$$

By applying Hölder inequality with $\vartheta:=\frac{p_s^*}{p_s^*-(q+1)}$ and $\alpha:=\frac{p_s^*}{q+1}$, we obtain

$$\begin{aligned}\int_{B(0,R_n)}\varphi(x)|u^+|^{q+1}&\leqslant\|\varphi\|_{L^\vartheta(B(0,R_n))}\|u\|^{q+1}_{L^{p_s^*}(B(0,R_n))}\\&=\|\varphi\|_{L^\vartheta(B(0,R_n))}\|u\|^{q+1}_{L^{p_s^*}(\mathbb{R}^N)}\leqslant C_n\|u\|^{q+1}_{D^{s,p}},\end{aligned}$$

for some $C_n>0$. Then, by using this estimate on J_n, we obtain

$$J_n(u)\geqslant\frac{1}{p}\|u\|^p_{D^{s,p}}-C_n\|u\|^{q+1}_{D^{s,p}}.$$

Since $q+1<p$, and recalling the definition of $\|\cdot\|_{X_n}$, we conclude that $J_n(u)\to+\infty$ when $\|u\|_{X_n}\to\infty$, since $p>q+1$, namely J_n is coercive on X_n. Whence, taking into account Lemma 2.1, by a standard argument of the Calculus of Variations, there exists $u_n\in X_n$ such that $J_n(u_n)=\inf_{X_n}J_n$, which is a critical point of J_n. By Lemma 2.2, we have $u_n\geqslant 0$ a.e. Now, we take $\zeta\in C_c^\infty(\mathbb{R}^N)\setminus\{0\}$ with $\operatorname{supp}(\zeta)\subset\omega$. Then using (f_2) again, we obtain

$$J_n(t\zeta)\leqslant\frac{t^p\|\zeta\|^p_{D^{s,p}}}{p}-\mu t^{q+1}\int_{B(0,R_n)}\varphi(x)|\zeta|^{q+1}=\frac{t^p\|\zeta\|^p_{D^{s,p}}}{p}-\mu t^{q+1}\int_\omega\varphi(x)|\zeta|^{q+1}.$$

Since $\inf_\omega\varphi>0$ we have $\int_\omega\varphi(x)|\zeta|^{q+1}>0$ and we can conclude that there exists $t_n>0$ small enough that $J_n(t_n\zeta)<0$. Since $t_n\zeta\in X_n$, we conclude the proof. $\square$

Lemma 2.4. *Assume (W), (f_1) and (f_2) with $q+1<p$. Let, for each $n\in\mathbb{N}$, $u_n\in X_n\setminus\{0\}$ be the nonnegative critical point of J_n obtained in Lemma 2.3. Then there exist two constants $c<0$ and $M>0$, independent of n, such that:*

(i) $\sup_{n\geqslant 1}J_n(u_n)\leqslant c$.
(ii) $\sup_{n\geqslant 1}\|u_n\|_{X_n}\leqslant M$.

Proof. Taking into account that $u_n\geqslant 0$, that $\omega\subset\Omega\subset B(0,R_n)$ and by assumption (W),

$$\int_{B(0,R_n)}\varphi(x)F(u_n)=\int_\Omega\varphi(x)F(u_n)+\int_{B(0,R_n)\setminus\Omega}\varphi(x)F(u_n)\leqslant\int_\Omega\varphi(x)F(u_n).$$

Hence, in turn, we get

$$J_n(u_n)\geqslant\frac{1}{p}\|u_n\|^p_{D^{s,p}}-\int_\Omega\varphi(x)F(u_n)\geqslant\frac{1}{p}\|u_n\|^p_{D^{s,p}}-C\|u_n\|^{q+1}_{D^{s,p}},\tag{2.4}$$

where Hölder inequality was used as in the proof of Lemma 2.3 but here the positive constant $C:=\delta\|\varphi\|_{L^\vartheta(\Omega)}$, for some $\delta=\delta(\Omega)>0$, is independent of $n\geqslant 1$. We also have, by arguing as in the proof of Lemma 2.3, that for a $\zeta \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ with $\mathrm{supp}(\zeta) \subset \omega$,

$$J_n(\tau\zeta) \leqslant c, \qquad c := \frac{\tau^p \|\zeta\|^p_{D^{s,p}}}{p} - \mu\tau^{q+1}\int_\omega \varphi(x)|\zeta|^{q+1} < 0.$$

for some $\tau > 0$ small enough and independent of $n \geqslant 1$. Thus, we get

$$\sup_{n\geqslant 1} J_n(u_n) = \sup_{n\geqslant 1}\inf_{X_n} J_n \leqslant \sup_{n\geqslant 1} J_n(\tau\zeta) \leqslant c < 0.$$

This proves (i). By means of inequality (2.4), inequality (ii) immediately follows otherwise a contradiction follows by the condition $q+1<p$. □

We now turn to the case $p < q+1$, where φ and f satisfy (W) and (f_i) respectively.

Lemma 2.5. *Assume that (W) and (f_1)-(f_3) hold. Then there exist $\rho, r > 0$ and a function $\psi \in X_n \setminus \{0\}$, independent of $n \geqslant 1$, with $\|\psi\|_{X_n} > \rho$ such that*

(i) $J_n(u) \geqslant r$, for every $u \in X_n$ with $\|u\|_{X_n} = \rho$ and all $n \geqslant 1$;
(ii) $J_n(\psi) \leqslant 0$, for all $n \geqslant 1$.

Proof. We have, arguing as in Lemma 2.4, that for all $u \in X_n$

$$J_n(u) \geqslant \frac{1}{p}\|u\|^p_{D^{s,p}} - C\|u\|^{q+1}_{D^{s,p}},$$

with C independent of $n \geqslant 1$. Take $\rho > 0$ such that $\rho^{q-p+1} < 1/2pC$. Then, if $\|u\|_{D^{s,p}} = \rho$, we obtain $J_n(u) \geqslant r$, with $r := \rho^p/2p > 0$. On the other hand, as in the proof of Lemma 2.4, there exists some $t_0 > 0$ (this time large enough) independent of $n \geqslant 1$ such that $J_n(t_0\zeta) \leqslant 0$ and taking $\psi := t_0\zeta$ we have $J_n(\psi) \leqslant 0$. Up to reducing ρ, we also get $\|\psi\|_{D^{s,p}} = t_0\|\zeta\|_{D^{s,p}} > \rho$. □

By Lemma 2.5, we can define, for each $n \geqslant 1$, the min-max level for J_n:

$$c_n := \inf_{\gamma\in\Gamma_n}\max_{0\leqslant t\leqslant 1} J_n(\gamma(t)), \qquad \Gamma_n := \{\gamma \in C([0,1], X_n); \gamma(0) = 0, \gamma(1) = \psi\}.$$

Using the fact that $X_n \subset X_{n+1}$ we actually have

$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n \geqslant \cdots \geqslant r > 0,$$

so that in particular $c_n \to c$, for some $c \geqslant r > 0$.

Lemma 2.6. *Assume that (W) and (f_1)-(f_3) hold. Then the functional J_n satisfies the $(PS)_c$-condition, for every $c \in \mathbb{R}$ and for all $n \geqslant 1$.*

Proof. Suppose now that $J_n(u_j) \to c$ and $J_n'(u_j) \to 0$ as $j \to \infty$. Then we can write

$$c + o_j(1) = \frac{\|u_j\|^p_{D^{s,p}}}{p} - \int_{B(0,R_n)} \varphi(x)F(u_j^+), \tag{2.5}$$

$$o_j(1)\|u_j\|_{D^{s,p}} = \|u_j\|^p_{D^{s,p}} - \int_{B(0,R_n)} \varphi(x)f(u_j^+)u_j. \tag{2.6}$$

By combining these identities, we obtain

$$\Big(\frac{1}{p} - \frac{1}{q+1}\Big)\|u_j\|^p_{D^{s,p}} - \int_{B(0,R_n)} \varphi(x)\Big(F(u_j^+) - \frac{f(u_j^+)u_j}{q+1}\Big) = c + o_j(1) + o_j(1)\|u_j\|_{D^{s,p}}.$$

In turn, on account of condition (f_3), we have

$$\begin{aligned}\int_{B(0,R_n)} \varphi(x)\Big(F(u_j^+) - \frac{f(u_j^+)u_j}{q+1}\Big) &\leqslant \|\varphi\|_{L^\infty(B(0,R_n))} \int_{B(0,R_n)} \Big|F(u_j^+) - \frac{f(u_j^+)u_j}{q+1}\Big| \\ &\leqslant C_n \int_{B(0,R_n)} |u_j|^m \leqslant C_n \|u_j\|^m_{L^{p_s^*}} \leqslant C_n \|u_j\|^m_{D^{s,p}}.\end{aligned}$$

Therefore, we get

$$c + o_j(1) + o_j(1)\|u_j\|_{D^{s,p}} \geqslant \Big(\frac{1}{p} - \frac{1}{q+1}\Big)\|u_j\|^p_{D^{s,p}} - C_n\|u_j\|^m_{D^{s,p}}.$$

Since $p > m$ and $q+1 > p$, this implies that there exists $C(s,n,p,q,c) > 0$ such that

$$\sup_{j\geqslant 1} \|u_j\|_{D^{s,p}} \leqslant C(s,n,p,q,c),$$

namely the sequence (u_j) is bounded in $D^{s,p}(\mathbb{R}^N)$. In turn, there exists a subsequence, still denoted by (u_j), such that $u_j \rightharpoonup u$ in X_n as $j \to \infty$. We also have that $u_j \to u$ in $L^r(B(0,R_n))$, for any $1 \leqslant r < p_s^*$ by the compact embedding theorem [13, Corollary 7.2] and $u_j(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. For any $\psi \in X_n$, we have

$$\int_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}} = \int_{B(0,R_n)} \varphi(x) f(u_j^+)\psi + \langle J_n'(u_j), \psi\rangle.$$

For each $\psi \in X_n$ fixed, we have by dominated convergence

$$\lim_{j\to\infty} \int_{B(0,R_n)} \varphi(x) f(u_j^+)\psi = \int_{B(0,R_n)} \varphi(x) f(u^+)\psi,$$

since there exists $\eta \in L^{q+1}(\mathbb{R}^N)$ such that $|u_j| \leqslant \eta$ a.e. and, for some $C_n > 0$,

$$|\varphi(x) f(u_j^+)\psi\chi_{B(0,R_n)}| \leqslant C_n |u_j^+|^q |\psi| \leqslant C_n |\eta|^q |\psi| \in L^1(\mathbb{R}^N), \qquad \text{for all } j \geqslant 1.$$

Now, if p' is the conjugate exponent to p, we have

$$\text{the sequence } \left(\frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))}{|x-y|^{(N+sp)/p'}}\right) \text{ is bounded in } L^{p'}(\mathbb{R}^{2n})$$

as well as

$$\frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))}{|x-y|^{(N+sp)/p'}} \to \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{(N+sp)/p'}} \quad \text{a.e. in } \mathbb{R}^{2n}.$$

Also, since $(\psi(x)-\psi(y))/|x-y|^{(N+sp)/p} \in L^p(\mathbb{R}^{2n})$ we have (cf. [18, Lemma 4.8]) that

$$\int_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}$$

converges to

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}.$$

This shows that $u \in X_n$ is a weak solution in $B(0,R_n)$, namely

(2.7)

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}} = \int_{B(0,R_n)} \varphi(x) f(u^+)\psi, \quad \forall \psi \in X_n.$$

Choosing $\psi = u$ in (2.7) and $\psi = u_j$ in the above equation for $J_n'(u_j)$ and since for $C_n > 0$,

$$|\varphi(x) f(u_j^+) u_j \chi_{B(0,R_n)}| \leqslant C_n |u_j|^{q+1} \leqslant C_n |\eta|^{q+1} \in L^1(\mathbb{R}^N), \quad \text{for all } j \geqslant 1,$$

we obtain

$$\begin{aligned}\|u\|^p_{D^{s,p}} &= \int_{B(0,R_n)} \varphi(x) f(u^+) u = \lim_{j\to\infty} \int_{B(0,R_n)} \varphi(x) f(u_j^+) u_j \\ &= \lim_{j\to\infty} \int_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^p}{|x-y|^{N+sp}} = \lim_{j\to\infty} \|u_j\|^p_{D^{s,p}}\end{aligned}$$

Since also $u_j \rightharpoonup u$, we can conclude that $u_j \to u$ in X_n, concluding the proof. $\square$

We can finally state the following

Lemma 2.7. *Assume that* (W) *and* (f_1)-(f_3) *hold. Then, for each* $n \geqslant 1$, *the problem*

$$\begin{cases} (-\Delta)^s_p u = \varphi(x) f(u), & \text{in } B(0,R_n), \\ u = 0, & \text{in } \mathbb{R}^N \setminus B(0,R_n), \end{cases} \tag{2.8}$$

admits a nontrivial nonnegative solution $u_n \in X_n$.

Proof. By Lemmas 2.2, 2.5 and 2.6, the assertion follows by the Mountain Pass Theorem. $\square$

3. Proof of Theorem 1.1

Consider first the case where (W) and (f_1)-(f_3) hold. By virtue of Lemma 2.7, there exists a sequence $(u_n) \subset X_n \subset D^{s,p}(\mathbb{R}^N)$ of nontrivial nonnegative weak solutions to problem (2.8) on the exhausting balls $B(0,R_n)$, namely

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp}} = \int_{B(0,R_n)} \varphi(x) f(u_n) \psi, \tag{3.1}$$

for any $\psi \in D^{s,p}(\mathbb{R}^N)$, with $\psi \equiv 0$ on $\mathbb{R}^N \setminus B(0,R_n)$. We claim that this sequence remains bounded in $D^{s,p}(\mathbb{R}^N)$. In fact, for every $n \geqslant 1$, we can write

$$\frac{\|u_n\|^p_{D^{s,p}}}{p} - \int_{B(0,R_n)} \varphi(x) F(u_n) = c_n, \quad \|u_n\|^p_{D^{s,p}} - \int_{B(0,R_n)} \varphi(x) f(u_n) u_n = 0.$$

By combining these identities, we obtain

$$\Big(\frac{1}{p} - \frac{1}{q+1}\Big) \|u_n\|^p_{D^{s,p}} - \int_{B(0,R_n)} \varphi(x) \Big(F(u_n) - \frac{f(u_n) u_n}{q+1}\Big) = c_n.$$

In turn, on account of conditions (f_3) and (W), we have

$$\begin{aligned}\int_{B(0,R_n)} \varphi(x) \Big(F(u_n) - \frac{f(u_n)u_n}{q+1}\Big) &= \int_{\Omega} \varphi(x) \Big(F(u_n) - \frac{f(u_n)u_n}{q+1}\Big) \\ &+ \int_{B(0,R_n)\setminus\Omega} \varphi(x) \Big(F(u_n) - \frac{f(u_n)u_n}{q+1}\Big) \leqslant \int_{\Omega} \varphi(x) \Big(F(u_n) - \frac{f(u_n)u_n}{q+1}\Big) \\ &\leqslant \|\varphi\|_{L^\infty(\Omega)} \int_{\Omega} \Big|F(u_n) - \frac{f(u_n)u_n}{q+1}\Big| \leqslant C \int_{\Omega} |u_n|^m \leqslant C \|u_n\|^m_{L^{p_s^*}} \leqslant C \|u_n\|^m_{D^{s,p}}.\end{aligned}$$

where $C = C(\Omega)$ is independent of $n \geqslant 1$, Therefore, we can conclude that

$$c_1 \geqslant c_n \geqslant \Big(\frac{1}{p} - \frac{1}{q+1}\Big) \|u_n\|^p_{D^{s,p}} - C \|u_n\|^m_{D^{s,p}}.$$

Since $p > m$ and $q+1 > p$, the claim is proved. Then, there exists a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$ as $n \to \infty$. We also have $u_n \to u$ in $L^r(K)$ for any bounded subset $K \subset \mathbb{R}^N$ and all $1 \leqslant r < p_s^*$ by the compact embedding theorem [13, Corollary 7.2] and $u_n(x) \to u(x)$ for a.e. Arguing as in the proof of Lemma 2.6, it follows that u is a distributional weak solution to problem (1.1). In fact, let $\psi \in C_c^\infty(\mathbb{R}^N)$ and set $K := \operatorname{supt}(\psi)$. Then $\psi \in D^{s,p}(\mathbb{R}^N)$ and $\psi \equiv 0$ on $\mathbb{R}^N \setminus B(0,R_n)$, for $n \geqslant 1$ large enough. The left-hand side of (3.1) converges as in the proof of Lemma 2.6, by means of duality arguments. As far as the right-hand side is concerned, by dominated convergence, we get

$$\begin{aligned}\lim_{n\to\infty}\int_{B(0,R_n)} \varphi(x) f(u_n)\psi &= \lim_{n\to\infty}\int_K \varphi(x) f(u_n)\psi \\ &= \int_K \varphi(x) f(u)\psi = \int_{\mathbb{R}^N} \varphi(x) f(u)\psi,\end{aligned}$$

since there exists $\eta \in L^q(K)$ such that $u_n \leqslant \eta$ a.e. in K for all $n \geqslant 1$ and

$$|\varphi(x) f(u_n)\psi|\chi_K(x) \leqslant C u_n^q \chi_K(x) \leqslant C\eta^q \chi_K(x) \in L^1(K).$$

We will now show that $u \neq 0$. Taking (W) and (f_3) into account, we deduce that

$$\begin{aligned}c_n = \frac{1}{p}\|u_n\|^p_{D^{s,p}} - \int_{B(0,R_n)} \varphi(x) F(u_n) &= \int_{B(0,R_n)} \varphi(x)\Big(\frac{f(u_n)u_n}{p} - F(u_n)\Big) \\ = \int_\Omega \varphi(x)\Big(\frac{f(u_n)u_n}{p} - F(u_n)\Big) &+ \int_{B(0,R_n)\setminus\Omega} \varphi(x)\Big(\frac{f(u_n)u_n}{p} - F(u_n)\Big) \\ \leqslant \int_\Omega \varphi(x)\Big(\frac{f(u_n)u_n}{p} - F(u_n)\Big).\end{aligned}$$

Since (c_n) is monotone and bounded from below by $r > 0$, we have $c_n \to c > 0$ as $n \to \infty$. Whence, assuming by contradiction that $u \equiv 0$, since Ω is bounded and, for all $n \geqslant 1$,

$$0 \leqslant (f(u_n)u_n - pF(u_n))\chi_\Omega \leqslant C u_n^{q+1}\chi_\Omega \leqslant C\eta^{q+1}\chi_\Omega,$$

for some $\eta \in L^{q+1}(\Omega)$, we would obtain by dominated convergence

$$\lim_{n\to\infty}\int_\Omega \varphi(x)\Big(\frac{1}{p}f(u_n)u_n - F(u_n)\Big) = \int_\Omega \varphi(x)\Big(\frac{1}{p}f(u)u - F(u)\Big) = 0,$$

which is a contradiction. Therefore $u \neq 0$ and the proof is complete under the first assumption. If instead condition (f_2) holds for $0 \leqslant q < p-1$, then the proof follows in a similar way, in light of Lemma 2.3 and Lemma 2.4. □

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