MULTIPLE NORMALIZED SOLUTIONS FOR p-LAPLACIAN SCHRÖDINGER-POISSON SYSTEMS

MENGRU LI, XIAOMING HE, AND MARCO SQUASSINA

ABSTRACT. In this paper, we study the existence and multiplicity of normalized solutions for the p-Laplacian Schrödinger-Poisson system:

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} - \kappa \left(|x|^{-1} * |u|^p\right) u^{p-1} = f(u), & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^p dx = a^p, & u > 0, \end{cases}$$

where $a > 0$ represents the prescribed L^p-norm, $\kappa \in \mathbb{R} \setminus \{0\}$ is a parameter, and $\lambda \in \mathbb{R}$ appears as an undetermined Lagrange multiplier. Our main results can be summarized as follows: (i) When $\kappa < 0$, we study the normalized ground state solution for $a > 0$ small by using the Pohozaev manifold, in this case f is odd and satisfies the L^p-supercritical and Sobolev subcritical conditions, proving that the aforementioned problem has infinitely many normalized solutions, whose energy converges to infinity, and the behavior of the normalized ground state energy is also studied. (ii) When $\kappa > 0$ and the Sobolev critical nonlinearity $f(u) = |u|^{p^*-2}u$, we make use of the truncation method combined with minimax theory to overcome the lack of lower boundedness of the energy functional, and show the existence of infinitely many normalized solutions with negative energy under suitable restrictions on κ and a. Our studies improve and complement some previous works on the p-Laplacian Schrödinger-Poisson systems with prescribed mass in the existing literature [35–37] and [39].

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

1.1. **Overview.** In this paper, we focus on the existence and multiplicity of normalized solutions for the p-Laplacian Schrödinger-Poisson system:

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} - \kappa \left(|x|^{-1} * |u|^p\right) u^{p-1} = f(u), & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^p dx = a^p, & u > 0, \end{cases} \tag{1.1}$$

where $a > 0$ is a prescribed mass, $1 < p < 3$, the parameter $\kappa \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}$ serves as a Lagrange multiplier. Here, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ represents the p-Laplacian operator. The convolution $|x|^{-1} * |u|^p$ is explicitly given by

$$(|x|^{-1} * |u|^p)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|} dy, \quad x \in \mathbb{R}^3,$$

and the precise conditions on f will be given in the sequel.

Now let us consider the equation in (1.1), i.e.,

$$-\Delta_p u + \lambda u^{p-1} - \kappa \left(|x|^{-1} * |u|^p\right) u^{p-1} = f(u) \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

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When $p = 2, \kappa = -1$ and the parameter $\lambda \in \mathbb{R}$ is fixed, the fixed-frequency problem described by Eq. (1.2) has attracted considerable attention from numerous researchers. A substantial body of literature has been devoted to investigating the existence, nonexistence, and multiplicity of solutions to (1.2) and related equations, after the pioneer work by Benci and Fortunato [10], we refer to [14, 31, 43, 45] and references therein, by employing variational methods [24, 44, 48, 51]. Recently, when $1 < p < 3, \kappa < 0$, problem (1.2) has been studied by Du, Su and Wang [17–19], Du and Su [20] by means of variational methods.

An alternative approach involves seeking solutions to (1.2) with prescribed mass constraint, namely,

$$\int_{\mathbb{R}^3} |u|^p dx = a^p > 0. \tag{1.3}$$

In this formulation, $\lambda \in \mathbb{R}$ becomes an additional unknown parameter. The investigation of solutions with prescribed norm has long represented a significant research direction in both mathematical and physical contexts. From a physical perspective, the fixed mass constraint with parameter a carries particular importance, motivating substantial recent interest in studying solutions under such normalization conditions. Furthermore, the p-Laplacian operator emerges naturally in nonlinear fluid mechanics, where the parameter p characterizes both flow velocity and medium constitutive properties. The quasi-linear Schrödinger equation (1.1), originates from quantum mechanical models and semiconductor theory, describing charged particle interactions with electromagnetic fields. For more discussions on p-Laplacian equations without the prescribed mass, we refer to [17–19]; to [16, 21, 38] and references therein;

We note that, in the special case $p = 2$, system (1.1) reduces to the classical Schrödinger-Poisson equation, and the normalized solutions have been investigated by many authors in recent years. For instance, Bellazzini and Siciliano [9] considered the problem:

$$\begin{cases} -\Delta u + \lambda u + (|x|^{-1} * |u|^2)u = |u|^{q-2}u, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2 dx = a^2, \end{cases} \tag{1.4}$$

where $q \in (2, 3)$, and established the existence of normalized solutions for sufficiently small $a > 0$. The case $q \in (3, \frac{10}{3})$ was considered in [8], where the authors showed that (1.4) admits normalized solutions provided $a > 0$ exceeds a certain threshold. Subsequently, Jeanjean and Luo [30] identified a threshold value of $a > 0$ that determines the existence and nonexistence of normalized solutions for (1.4). Bellazzini and Jeanjean [7] studied the existence of normalized solutions to (1.4) for $\frac{10}{3} < q < 6$. In [7], the authors established existence of normalized solutions for (1.4) under the assumption of sufficiently small mass $a > 0$ by using the Pohozaev manifold method. Recently, Jeanjean and Le [27] investigated the following Schrödinger-Poisson-Slater equation

$$-\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - b|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where $p \in (\frac{10}{3}, 6]$, $\gamma, b \in \mathbb{R}$, and $\|u\|_2^2 = c$ for prescribed $c > 0$. Through geometric analysis of the Pohozaev manifold, they derived existence and nonexistence results for various parameter configurations: (i) $\gamma < 0, b < 0$; (ii) $\gamma > 0, b > 0$; and (iii) $\gamma > 0, b < 0$. For more results of normalized solutions related to problem (1.4), we refer to [2, 15, 40, 43, 49] and reference s therein.

We also recall some important advances concerning the normalized solutions to the Schrödinger equation after the famous paper [25], where Jeanjean investigated the normalized solutions of the mass supercritical problem

$$\begin{cases} -\Delta u + \mu u = f(u), & \text{in } \mathbb{R}^N, \\ u \in S_m, \end{cases} \tag{1.6}$$

where

$$S_m = \{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = m\},$$

and $\mu \in \mathbb{R}$ appears as a Lagrange multiplier, by employing the mountain pass lemma and a skillful compactness argument. Recently, Jeanjean and Lu [28] revisited problem (1.6) under the assumption that f is continuous and satisfies weakened mass supercritical conditions, they established the existence of ground state solutions with the help of the Pohozaev manifold

$$P_m = \left\{ u \in S_m : P(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} (f(u)u - 2F(u))dx = 0 \right\}, \tag{1.7}$$

where $F(t) = \int_0^t f(s)ds$, and constructed a bounded Palais-Smale sequence $\{u_n\} \subset S_m$ for $\mathcal{I}$ under relaxed conditions at the minimal energy level

$$E_m := \inf_{u \in P_m} \mathcal{I}(u), \tag{1.8}$$

which precisely satisfies

$$P(u_n) = 0, \quad \text{for all } n \geq 1,$$

where

$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u)dx.$$

To overcome the compactness issues, the authors proved the positivity of the Lagrange multiplier μ becomes essential, which can be achieved by establishing the monotonicity of the mapping $m \mapsto E_m$. For this purpose, the authors introduced a novel approach suitable for non-homogeneous and only continuous nonlinearities f; detailed arguments can be found in Lemmas 3.2 and 3.3 of [28]. Additionally, they derived the existence of infinitely many radial solutions employing genus theory. Soave [46] studied the existence of normalized solutions to (1.6) with $f(u) = \lambda |u|^{q-2}u + |u|^{2^*-2}u$ for $q \in (2, 2^*)$, Sobolev critical growth, representing a counterpart to the Brezis-Nirenberg problem in the L^2-constraint framework. For further results on normalized solutions to Schrödinger-type equations, we refer to [1, 3, 6–9, 13, 26, 30, 50] and references therein.

Now, let us come back to consider problem (1.1). To our best knowledge, there are only few papers on the existence of normalized solutions to (1.1) in the literature. When the parameter $-\kappa = \gamma > 0$, Liu and He [35] recently studied normalized ground state solutions of the p-Laplacian Schrödinger–Poisson system

$$\begin{cases} -\Delta_p u + \gamma \phi |u|^{p-2}u = \lambda |u|^{p-2}u + \mu |u|^{q-2}u + |u|^{p^*-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^p, & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^p dx = a^p, \end{cases} \tag{1.9}$$

where $p^* = \frac{3p}{3-p}$ denotes the Sobolev critical exponent, and by means of Pohozaev manifold decomposition technique they established several existence results in the L^p-subcritical, L^p-critical and L^p-supercritical perturbation $\mu|u|^{q-2}u$, respectively. Almost in the same time, in [36], Liu, He and Radulescu investigated system (1.9) under the L^p-subcritical perturbation $\mu|u|^{q-2}u$, with $q \in (p, \bar{p})$, and showed the existence of multiple normalized solutions using the truncation technique, concentration-compactness principle, and genus theory. In the L^p-supercritical regime: $q \in (\bar{p}, p^*)$, the authors proved two existence results for normalized solutions by the concentration-compactness principle and mountain pass theorem, here $\bar{p} := p + \frac{p^2}{3}$, is the L^p-mass critical exponent.

1.2. **Main results.** In this paper we focus our attention on problem (1.1), with parameter $\kappa \in \mathbb{R} \backslash \{0\}$ having a wider range of values. It is well-known that seeking normalized solutions of (1.1) is equivalent to finding critical points of the functional J defined by

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx - \frac{\kappa}{2p} B(u) - \int_{\mathbb{R}^3} F(u)dx, \tag{1.10}$$

on the L^p-constraint manifold

$$S(a) = \{u \in W^{1,p}(\mathbb{R}^3) : \|u\|_p^p = a^p\}, \tag{1.11}$$

where

$$B(u)=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}dxdy.$$

For a given $a>0$, we define the Pohozaev manifold associated with (1.1) as

$$\mathcal{P}_a:=\left\{u\in S(a): P(u)=p\int_{\mathbb{R}^3}|\nabla u|^p dx-\frac{\kappa}{2}B(u)-3\int_{\mathbb{R}^3}\tilde{F}(u)dx=0\right\}, \tag{1.12}$$

where

$$\tilde{F}(t)=f(t)t-pF(t).$$

By the Pohozaev identity, every solution of (1.1) necessarily lies in $\mathcal{P}_a$.

We first consider the case $\kappa<0$ and impose the following conditions on the nonlinearity f:

(f_1) $f\in C(\mathbb{R},\mathbb{R})$ and there exist constants $q\in(\bar{p},p^*)$ and $C>0$ such that

$$|f(t)|\le C(1+|t|^{q-1})\quad \text{for all } t\in\mathbb{R};$$

(f_2) $\lim_{t\to0}\frac{f(t)}{|t|^{\bar{p}-1}}=0$ and $\lim_{t\to\infty}\frac{F(t)}{|t|^{\bar{p}}}=+\infty$, where $F(t)=\int_0^t f(s)ds$;

(f_3) The function $t\mapsto\frac{\tilde{F}(t)}{|t|^{\bar{p}}}$ is strictly decreasing on $(-\infty,0)$ and strictly increasing on $(0,+\infty)$;

(f_4) There exists $\theta\in(\bar{p},p^*)$ such that $f(t)t\le\theta F(t)$ for all $t\in\mathbb{R}\setminus\{0\}$.

One can easily check that $f(t)=|t|^{p-2}t$ for $p\in(\bar{p},p^*)$ satisfies conditions (f_1)-(f_4). To state our main results, we first provide the definition of a normalized ground state solution on $\mathcal{P}_a$.

Definition 1.1. *A solution $u\in W^{1,p}(\mathbb{R}^3)\setminus\{0\}$ is called a normalized ground state solution on $\mathcal{P}_a$ of (1.1) if it satisfies*

$$J'|_{\mathcal{P}_a}(u)=0\quad\textit{and}\quad J(u)=\inf\{J(v):J'|_{\mathcal{P}_a}(v)=0,\ v\in\mathcal{P}_a\}. \tag{1.13}$$

We remark that condition (f_3) can enable the reduction of the normalized solution search to a minimization problem on the Pohozaev manifold $\mathcal{P}_a$, while condition (f_4) ensures the positivity of the Lagrange multiplier $\lambda>0$, which plays a crucial role in establishing compactness properties, as we will show in the sequel. We shall verify that $\mathcal{P}_a\neq\emptyset$ constitutes a natural constraint and that the restricted functional $J|_{\mathcal{P}_a}$ is both bounded below and coercive, as proved in Lemmas 2.3 and 2.4 below. To this aim, it is natural to define the normalized ground state energy as

$$c_a:=\inf_{u\in\mathcal{P}_a}J(u). \tag{1.14}$$

Our first main result can be stated as follows.

Theorem 1.2. *Assume that $\kappa<0$ and (f_1)-(f_4) hold. Then there exists $a^*>0$ small such that for any $a\in(0,a^*)$, (1.1) possesses a normalized solution $(u,\lambda)\in S(a)\times\mathbb{R}^+$ and u is a normalized ground state solution on $\mathcal{P}_a$. Moreover, the function $a\to c_a$ is positive, continuous, nonincreasing and*

$$\lim_{a\to0^+}c_a=+\infty.$$

Our next result is concerned with the multiplicity of normalized solutions for (1.1).

Theorem 1.3. *Suppose that $\kappa<0$, f is odd and satisfies (f_1)-(f_4). Then there exists $a^*>0$ small such that for any $a\in(0,a^*)$, (1.1) has infinitely many radial solutions $\{u_k\}_{k=1}^{\infty}\subset S(a)$, with the characteristics*

$$J(u_{k+1})\ge J(u_k)>0,\quad\forall k\in\mathbb{N},$$

and $J(u_k)\to+\infty$ as $k\to\infty$.

Remark 1.1. We note that in [37], Liu and He also proved the existence and multiplicity of (1.1) under the following conditions:

(g_0) $f:\mathbb{R}\to\mathbb{R}$ is a continuous odd function; that is, $f(-t)=-f(t)$ holds for all $t\in\mathbb{R}$.

(g_1) There exist positive constants $(\alpha, \beta) \in \mathbb{R}^2_+$ with $\bar{p} < \alpha \leq \beta < p^*$ such that for all $t \in \mathbb{R} \setminus \{0\}$, the following inequality holds:

$$0 < \alpha F(t) \leq f(t)t \leq \beta F(t).$$

(g_2) Define the auxiliary function $\tilde{F}(t) := f(t)t - pF(t)$. Assume that $\tilde{F} \in C^1(\mathbb{R})$ and satisfies the strict inequality:

$$\bar{p}\tilde{F}(t) < \tilde{F}'(t)t \quad \text{for all } t \neq 0.$$

However, the main results of [37] require the higher differentiability of the function $\tilde{F}$. Our conditions $(f_1) - (f_4)$ are more weaker than $(g_0) - (g_2)$. To this aim, let

$$f(t) := \left((p + \frac{p^2}{3}) \ln(1 + |t|^{\frac{p^2}{3}}) + \frac{\frac{p^2}{3}|t|^{\frac{p^2}{3}}}{1 + |t|^{\frac{p^2}{3}}} \right) |t|^{p+\frac{p^2}{3}-2}t, \quad t \in \mathbb{R},$$

then we have the primitive function of $f(t)$ as:

$$F(t) = |t|^{p+\frac{p^2}{3}} \ln(1 + |t|^{\frac{p^2}{3}}), \quad t \in \mathbb{R}.$$

By a simple computation, we have that f satisfies $(f_1) - (f_4)$, but does not satisfy condition (g_1). Hence, our results improve and extend the main studies in [37].

Remark 1.2. In Theorem 1.3, we work directly on the natural constraint manifold $\mathcal{P}_a$, and construct the Palais-Smale sequence for the functional J on $S(a)$ that consists entirely of elements belonging to $\mathcal{P}_a$. Moreover, for proving the multiplicity of normalized solutions, we employ genus theory, which provides an effective framework for this purpose.

We now turn to the case $\kappa > 0$ and the nonlinearity $f(t) = |t|^{p^*-2}t$ being Sobolev critical growth, and establish multiple solutions with negative energy.

Theorem 1.4. *Assume that $\kappa > 0$ and $f(u) = |u|^{p^*-2}u$. Then (1.1) admits an unbounded sequence of solutions $(u_j, \lambda_j) \in W^{1,p}(\mathbb{R}^3) \times \mathbb{R}^+$ with $\lambda_j > 0, J(u_j) < 0$ and $J(u_j) \to 0^-$ as $j \to \infty$.*

The proofs of Theorems 1.2-1.4 are constrained variational methods, and some comments are in orders:

(i) To prove Theorem 1.2, we shall construct a Palais-Smale sequence for $J|_{\mathcal{P}_a}$ at the energy level c_a that precisely satisfies $P(u_n) = 0$ for all $n \geq 1$, adapting techniques from [4, 5, 28], which can prove that the constructed Palais-Smale sequence possesses a convergent subsequence, and in turn implies the existence of a normalized ground state solution on $\mathcal{P}_a$. We remark that, verifying the positivity of the Lagrange multiplier λ plays a crucial role in establishing compactness of the Palais-Smale sequence, and this verification follows directly from the Pohozaev identity in the p-Laplacian Schrödinger equation treated in [16, 21, 39], but for the nonlocal term in our setting, we need to adapt new methods to overcome this issue, with detailed arguments provided in Lemma 3.4.

(ii) In order to show Theorem 1.3, we make use of the radial subspace $W_r := \{u \in W^{1,p}(\mathbb{R}^3) : u(x) = u(|x|)\}$ to obtain the multiplicity of normalized solutions. By using genus theory, one can obtain an infinite sequence of minimax values β_k as in (3.23). For each level $\{\beta_{a,k}\}$, we construct an appropriate Palais-Smale sequence $\{u_{k,n}\}_{n=1}^{\infty} \subset \mathcal{P}_a \cap W_r$ for the constrained functional $J|_{S(a)\cap W_r}$. A key step is to prove the unboundedness of the sequence $\{\beta_{a,k}\}$, while in our situation, the presence of the nonlocal term $|x|^{-1} * |u|^p$ will bring more obstacles, we have to give more refined analytical arguments.

(iii) For the case $\kappa > 0$, and the nonlinearity $f(u) = |u|^{p^*-2}u$ is Sobolev critical growth, we shall prove the existence of infinitely many solutions with negative energy for (1.1). However, the presence of the Sobolev critical term makes the constrained functional $J|_{S(a)}$ is unbounded below. To overcome this obstacle, we implement a truncation technique introduced in [22], as defined in (4.3). We then prove that critical points of the truncated functional corresponding to negative critical values are also critical points of the original functional. Furthermore, to handle the Sobolev critical exponent, we employ the concentration-compactness principle due to Lions [33, 34], which play a key role in recovering the loss of compactness and proving Theorem 1.4.

Remark 1.3. In [35–37], the authors only studied the existence of normalized solutions of (1.1) with $\kappa < 0$, but in Theorem 1.4 we consider the case $\kappa > 0$ and complement the aforementioned studies. In fact, in this situation, the nonlocal term $-\kappa(|x|^{-1} * |u|^p)u^{p-1}$ can be moved to the right-side of the equation (1.1) and acts as a nonlocal perturbation in the functional J, and in this sense Theorem 1.4 also generalizes the study of [39] to the p-Laplacian Schrödinger equation with Sobolev critical exponent and a nonlocal term.

The remainder part of this paper is structured as follows. In Section 2 we give preliminary results and investigates fundamental properties of the normalized ground state energy mapping $a \mapsto c_a$. In Section 3 we prove Theorems 1.2 and 1.3. Finally, in Section 4 we apply the genus theory and completes the proof of Theorem 1.4.

Notations. Throughout this paper, we adopt the following notations for conveniences.

- $L^p(\mathbb{R}^3)$ for $p \in [1, \infty)$ denotes the Lebesgue space equipped with the standard norm
$$\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p dx \right)^{\frac{1}{p}}.$$
- $W^{1,p}(\mathbb{R}^3)$ represents the Sobolev space endowed with the norm
$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^p + |u|^p) dx \right)^{\frac{1}{p}}.$$
We define $W_r := \{u \in W^{1,p}(\mathbb{R}^3) : u(x) = u(|x|)\}$ as the subspace of radial functions, and $(W_r)^*$ denotes its topological dual.
- $D^{1,p}(\mathbb{R}^3)$ denotes the homogeneous Sobolev space defined by
$$D^{1,p}(\mathbb{R}^3) = \{u \in L^{p^*}(\mathbb{R}^3) : \nabla u \in L^p(\mathbb{R}^3)\}$$
with the optimal Sobolev constant S given by
$$S = \inf_{u \in D^{1,p}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^3} |u|^{p^*} dx \right)^{\frac{p}{p^*}}}. \tag{1.15}$$
- We define the mass-critical exponent $\bar{p} := p + \frac{p^2}{3}$ and the Sobolev critical exponent $p^* := \frac{3p}{3-p}$.
- The symbols C, $\tilde{C}$, C_i, c_i (for $i = 1, 2, \ldots$) denote generic positive constants whose values may vary between different occurrences.
- The notations "$\rightarrow$" and "$\rightharpoonup$" indicate strong and weak convergence in the relevant function spaces, respectively.
- The expression $o_n(1)$ represents a quantity that converges to zero as $n \to \infty$.
- For any $x \in \mathbb{R}^3$ and $r > 0$, we denote the open ball by $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$.

2. Preliminary results

In order to prove Theorems 1.2 and 1.3, we begin by presenting some useful preliminaries. In the following arguments, without loss of generality, we always assume $\kappa = -1$ for $\kappa < 0$. In the sequel, we assume (f_1)-(f_4) hold.

For any $a > 0$, we introduce the following set
$$M_a := \{u \in W^{1,p}(\mathbb{R}^3) : \|u\|_p \leq a\}. \tag{2.1}$$

Recall that

$$J(u)=\frac{1}{p}\int_{\mathbb{R}^3}|\nabla u|^p dx+\frac{1}{2p}B(u)-\int_{\mathbb{R}^3}F(u)dx$$

and

$$\mathcal{P}_a=\left\{u\in\mathcal{S}_a: P(u)=p\int_{\mathbb{R}^3}|\nabla u|^p dx+\frac{1}{2}B(u)-3\int_{\mathbb{R}^3}\tilde{F}(u)dx=0\right\}. \tag{2.2}$$

In the following arguments of normalized solutions, we shall use some key inequalities. First, we recall the Gagliardo-Nirenberg inequality [41] of p-Laplacian type: for any $q\in(p,p^*)$ and $N\geq 2$,

$$\|u\|_q^q\leq C(N,q)\|\nabla u\|_p^{q\gamma_q}\|u\|_p^{q(1-\gamma_q)}, \tag{2.3}$$

where the interpolation exponent is given by $\gamma_q=N\left(\frac{1}{p}-\frac{1}{q}\right)$.

Next, we introduce the Hardy-Littlewood-Sobolev inequality [32]: for functions $f\in L^p(\mathbb{R}^N)$, $g\in L^q(\mathbb{R}^N)$ with $0<s<N$,

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^s}dxdy\right|\leq C(N,s,p,q)\|f\|_p\|g\|_q, \tag{2.4}$$

under the scaling condition $p,q>1$, $\frac{1}{p}+\frac{1}{q}+\frac{s}{N}=2$.

Lemma 2.1. *The following conclusions hold true:*

(i) For any $a>0$, there exists $\sigma=\sigma(a)$ such that

$$\frac{1}{2p}\int_{\mathbb{R}^3}|\nabla u|^p dx\leq J(u)\leq C\left(\int_{\mathbb{R}^3}|\nabla u|^p dx+\left(\int_{\mathbb{R}^3}|\nabla u|^p dx\right)^{\frac{1}{p}}\right)$$

for all $u\in M_a$ with $\|\nabla u\|_p\leq\sigma$.

(ii) Let $\{u_n\}\subset W^{1,p}(\mathbb{R}^3)$ be a bounded sequence. If $\lim_{n\to\infty}\|u_n\|_{\bar{p}}=0$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}F(u_n)dx=\lim_{n\to\infty}\int_{\mathbb{R}^3}\tilde{F}(u_n)dx=0.$$

(iii) Let $\{u_n\}\subset W^{1,p}(\mathbb{R}^3)$ and $\{v_n\}\subset W^{1,p}(\mathbb{R}^3)$ be bounded sequences. If $\lim_{n\to\infty}\|v_n\|_{\bar{p}}=0$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}f(u_n)v_n dx=0.$$

Proof. (i) We first check that there exists a sufficiently small $\sigma=\sigma(a)>0$ such that for every $u\in M_a$ satisfying $|\nabla u|_p\leq\sigma$,

$$\int_{\mathbb{R}^3}|F(u)|dx\leq\frac{1}{2p}\int_{\mathbb{R}^3}|\nabla u|^p dx. \tag{2.5}$$

In fact, it follows from (f_1)–(f_2) that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that for all $t\in\mathbb{R}$,

$$|F(t)|\leq\varepsilon|t|^{\bar{p}}+C_\varepsilon|t|^{p^*}.$$

Thus, for any $u\in M_a$, using the Gagliardo-Nirenberg inequality, we infer to

$$\begin{aligned}\int_{\mathbb{R}^3}|F(u)|dx&\leq\varepsilon\int_{\mathbb{R}^3}|u|^{\bar{p}}dx+C_\varepsilon\int_{\mathbb{R}^3}|u|^{p^*}dx\\&\leq\varepsilon C_1 a^{\frac{p^2}{3}}\int_{\mathbb{R}^3}|\nabla u|^p dx+C_\varepsilon C_2\left(\int_{\mathbb{R}^3}|\nabla u|^p dx\right)^{\frac{p^*}{p}}\\&=\left(\varepsilon C_1 a^{\frac{p^2}{3}}+C_\varepsilon C_2\left(\int_{\mathbb{R}^3}|\nabla u|^p dx\right)^{\frac{p^*-p}{p}}\right)\int_{\mathbb{R}^3}|\nabla u|^p dx.\end{aligned}$$

Taking $\varepsilon = \frac{1}{4pC_1 a^{\frac{p^2}{3}}}$ and $\sigma = \left(\frac{1}{4pC_\varepsilon C_2}\right)^{\frac{1}{p^*-p}}$, then (2.5) follows.

On the other hand, combining the Hardy-Littlewood-Sobolev inequality and the Gagliardo-Nirenberg inequality we have that

$$\begin{aligned} B(u) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|}dxdy \\ &\leq \widetilde{C}_p\|u\|_{\frac{6p}{5}}^{2p} \\ &\leq C_p\|\nabla u\|_p\|u\|_p^{2p-1}, \end{aligned} \tag{2.6}$$

here $\widetilde{C}_p, C_p > 0$ are constants. Using (2.5) and (2.6), we can easily verify statement (i).

(ii) For any $\varepsilon > 0$, assumptions $(f_1) - (f_2)$ ensure the existence of a constant $C'_\varepsilon > 0$ such that

$$|\widetilde{F}(t)| \leq \varepsilon|t|^{p^*} + C'_\varepsilon|t|^{\bar{p}}, \quad \forall t \in \mathbb{R}.$$

Then,

$$\int_{\mathbb{R}^3} |\widetilde{F}(u_n)|dx \leq \varepsilon\|u_n\|_{p^*}^{p^*} + C'_\varepsilon\|u_n\|_{\bar{p}}^{\bar{p}}. \tag{2.7}$$

Since ε is arbitrary and $|u_n|_{\bar{p}} \to 0$, we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \widetilde{F}(u_n)dx = 0.$$

A similar argument shows that the same conclusion applies to

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} F(u_n)dx = 0, \quad \text{whenever } \|u_n\|_{\bar{p}} \to 0.$$

(iii) We first claim that there exists a constant $M > 0$, independent of $q \in [p, p^*]$, such that $|u|_q \leq M$. Indeed, it follows from (f_1)–(f_2) that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for all $t \in \mathbb{R}$,

$$|f(t)| \leq \varepsilon|t|^{\bar{p}-1} + C_\varepsilon|t|^{p^*-1}.$$

It follows that for all n,

$$|f(u_n)v_n| \leq \varepsilon|u_n|^{\bar{p}-1}|v_n| + C_\varepsilon|u_n|^{p^*-1}|v_n|. \tag{2.8}$$

Using the Hölder inequality, we have

$$\int_{\mathbb{R}^3} |u_n|^{\bar{p}-1}|v_n|dx \leq \left(\int_{\mathbb{R}^3}|u_n|^{\bar{p}}dx\right)^{\frac{\bar{p}-1}{\bar{p}}}\left(\int_{\mathbb{R}^3}|v_n|^{\bar{p}}dx\right)^{\frac{1}{\bar{p}}}.$$

Let $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ be a bounded sequence. Firstly, by the Gagliardo-Nirenberg inequality, we get

$$\int_{\mathbb{R}^3} |u_n|^{\bar{p}}dx \leq C\|\nabla u_n\|_p^p\|u_n\|_p^{\frac{p^2}{3}} \leq CM^{\bar{p}}.$$

Moreover, it follows that

$$\left(\int_{\mathbb{R}^3}|u_n|^{\bar{p}}dx\right)^{\frac{\bar{p}-1}{\bar{p}}} \leq C.$$

Finally, if $\lim_{n\to\infty}\|v_n\|_{\bar{p}} = 0$, then clearly

$$\left(\int_{\mathbb{R}^3}|v_n|^{\bar{p}}dx\right)^{\frac{1}{\bar{p}}} \to 0.$$

Hence, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{\bar{p}-1}|v_n|dx = 0. \tag{2.9}$$

Using the Hölder inequality, we obtain

$$\int_{\mathbb{R}^3} |u_n|^{p^*-1}|v_n|dx \le \left(\int_{\mathbb{R}^3} |u_n|^{p^*}dx\right)^{\frac{p^*-1}{p^*}} \left(\int_{\mathbb{R}^3} |v_n|^{p^*}dx\right)^{\frac{1}{p^*}}.$$

Assume that $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ is a bounded sequence. By the Sobolev embedding theorem, we get

$$\int_{\mathbb{R}^3} |u_n|^{p^*}dx \le CM^{p^*}.$$

Consequently, it follows that

$$\left(\int_{\mathbb{R}^3} |u_n|^{p^*}dx\right)^{\frac{p^*-1}{p^*}} \le CM^{p^*-1}.$$

Moreover, by combining with the Hölder inequality:

$$\int_{\mathbb{R}^3} |v_n|^{p^*}dx = \int_{\mathbb{R}^3} |v_n|^{\bar{p}}|v_n|^{p^*-\bar{p}} \le \|v_n\|_{\bar{p}}\|v_n\|_{p^*-2}^{p^*-\bar{p}} \to 0,$$

it follows that $\left(\int_{\mathbb{R}^3} |v_n|^{p^*}dx\right)^{\frac{1}{p^*}} \to 0$. So,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{p^*-1}|v_n|dx = 0. \tag{2.10}$$

Substituting (2.9) and (2.10) into the integral estimate in (2.8) yields that

$$0 \le \left|\int_{\mathbb{R}^3} f(u_n)v_n dx\right| \le \varepsilon \int_{\mathbb{R}^3} |u_n|^{\bar{p}-1}|v_n|dx + C_\varepsilon \int_{\mathbb{R}^3} |u_n|^{p^*-1}|v_n|dx.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f(u_n)v_n dx = 0.$$

□

Remark 2.1. *Analogous to* (2.5)*, we can show that*

$$\int_{\mathbb{R}^3} |\tilde{F}(u)|dx \le \frac{1}{3}\int_{\mathbb{R}^3} |\nabla u|^p dx$$

for all $u \in M_a$ satisfying $\|\nabla u\|_p \le \sigma$, from which it follows that

$$P(u) = \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{2p}B(u) - \frac{3}{p}\int_{\mathbb{R}^3} \tilde{F}(u)dx \ge \frac{p-1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx.$$

Remark 2.2. *Under the conditions (f_1), (f_2), and (f_3) on f, one can define a continuous function $k : \mathbb{R} \to \mathbb{R}$ in the following way.*

$$k(t) := \begin{cases} \dfrac{f(t)t - pF(t)}{|t|^{p+p^2/3}}, & \text{for } t \ne 0, \\ 0, & \text{for } t = 0. \end{cases}$$

Moreover, k is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$.

Following the approach from [25], we can define an auxiliary functional associated with J via a continuous L^p-norm preserving map $\eta : E \to W^{1,p}(\mathbb{R}^3)$ as:

$$\eta(u, s)(x) := e^{\frac{3s}{p}}u(e^s x) \quad \text{for} \quad u \in W^{1,p}(\mathbb{R}^3), \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^3, \tag{2.11}$$

where $E := W^{1,p}(\mathbb{R}^3) \times \mathbb{R}$ is endowed with the norm $\|(u, s)\|_E = (\|u\|^p + |s|^p)^{\frac{1}{p}}$. A direct computation shows that $\|\eta(u, s)\|_p = \|u\|_p$, which implies $\eta(u, s) \in S(a)$.

We define the auxiliary functional associated with J by

$$\begin{aligned}\tilde{J}(u,s) &:= J(\eta(u,s)) \\ &= \frac{e^{ps}}{p}\|\nabla u\|_p^p + \frac{e^s}{2p}B(u) - \frac{1}{e^{3s}}\int_{\mathbb{R}^3} F(e^{\frac{3s}{p}}u)dx.\end{aligned}$$

Clearly, $\tilde{J}$ belongs to $C^1(W^{1,p}(\mathbb{R}^3)\times\mathbb{R},\mathbb{R})$. The following lemma describes the geometric properties of $\tilde{J}$.

Lemma 2.2. *For every $u \in S(a)$, we have*

$$\tilde{J}(u,s) \to 0^+ \text{ as } s \to -\infty \quad \text{and} \quad \tilde{J}(u,s) \to -\infty \text{ as } s \to +\infty.$$

Proof. For every $u \in S(a)$, we see that $\|\nabla\eta(u,s)\|_p^p = e^{ps}\|\nabla u\|_p^p$. Moreover, by Lemma 2.1-(i), it follows that

$$\frac{1}{2p}e^{ps}\int_{\mathbb{R}^3}|\nabla u|^p dx \le \tilde{J}(u) \le C\left(e^{ps}\int_{\mathbb{R}^3}|\nabla u|^p dx + e^s\left(\int_{\mathbb{R}^3}|\nabla u|^p dx\right)^{\frac{1}{p}}\right),$$

which implies that $\tilde{J}(u,s) \to 0^+$ as $s \to -\infty$.

For any $\mu \ge 0$, define the function

$$h_\mu(t) := \begin{cases} \frac{F(t)}{|t|^{\bar{p}}} + \mu, & \text{if } t \neq 0, \\ \mu, & \text{if } t = 0. \end{cases} \tag{2.12}$$

By $(f_1)-(f_2)$, the function $h_\mu(t)$ is continuous and satisfies $h_\mu(t) \to +\infty$ as $t \to \infty$. Obviously, $F(t) = h_\mu(t)|t|^{\bar{p}} - \mu|t|^{\bar{p}}$. Following an argument similar to Lemma 2.3 in [28] and using $(f_1)-(f_3)$, we can derive that for all $t \neq 0$,

$$f(t)t > \bar{p}F(t) > 0. \tag{2.13}$$

For the convenience, we give the details to prove (2.13) by splitting the proof into several steps.

Step 1. $F(t) > 0$ for any $t \neq 0$. In fact, if $F(t_0) \le 0$ for some $t_0 \neq 0$, by (f_1) and (f_3), the function $F(t)/|t|^{p+p^2/3}$ reaches the global minimum at some $s \neq 0$ satisfying $F(s) \le 0$ and

$$\left(F(t)/|t|^{p+p^2/3}\right)'_{t=s} = \frac{f(s)s - (p+p^2/3)F(s)}{|s|^{p+1+p^2/3}sgn(s)} = 0.$$

Noting that $f(t)t > pF(t)$ for any $t \neq 0$ by Remark 2.2, we derive a contradiction

$$0 < f(s)s - pF(s) = \frac{p^2}{3}F(s) \le 0,$$

and the proof of Step 1 is complete.

Step 2. There exists a positive sequence $\{s_n^+\}$ and a negative sequence $\{s_n^-\}$ such that $|s_n^\pm| \to +\infty$ and $f(s_n^\pm)s_n^\pm > (p+p^3/3)F(s_n^\pm)$ for each $n \ge 1$.

We first consider the positive case. By contradiction, we assume that there exists $T_1 > 0$ small enough such that $f(t)t \le (p+p^2/3)F(t)$ for any $t \in (0,T_1]$. Using Step 1, we have

$$\frac{F(t)}{t^{p+p^2/3}} \ge \frac{F(T_1)}{T_1^{p+p^2/3}} > 0 \quad \text{for all} \quad t \in (0,T_1].$$

Noting that $\lim_{t\to 0} F(t)/|t|^{p+p^2/3} = 0$ by (f_1), we obtain a contradiction. The negative case is similar and so we obtain Step 2.

Step 3. There exists a positive sequence $\{\sigma_n^+\}$ and a negative sequence $\{\sigma_n^-\}$ such that $|\sigma_n^\pm| \to +\infty$ and $f(\sigma_n^\pm)\sigma_n^\pm > (p+p^2/3)F(\sigma_n^\pm)$ for each $n \ge 1$.

The two cases being similar, we only show the existence of $\{\sigma_n^-\}$. Assume by contradiction that there exists $T_2 > 0$ such that $f(t)t \le (p+p^2/3)F(t)$ for any $t \le -T_2$. We then have

$$\frac{F(t)}{t^{p+p^2/3}} \leq \frac{F(-T_2)}{T_2^{p+p^2/3}} < +\infty \quad \text{for all} \quad t \leq -T_2,$$

which contradicts (f_3). Thus, the sequence $\{\sigma_n^-\}$ exists and this proves Step 3.

Step 4. $f(t)t \geq (p+p^2/3)F(t)$ for any $t \neq 0$. Let us assume by contradiction that $f(t_0)t_0 < (p+p^2/3)F(t_0)$ for some $t_0 \neq 0$. Since the cases $t_0 < 0$ and $t_0 > 0$ can be treated in a similar way, we can assume further that $t_0 < 0$. By Steps 2 and 3, there exist $T_{\min}, T_{\max} \in \mathbb{R}$ such that $T_{\min} < t_0 < T_{\max} < 0$, and

$$f(t)t < (p+p^2/3)F(t) \quad \text{for any} \quad t \in (T_{\min}, T_{\max}), \tag{2.14}$$

and

$$f(t)t = (p+p^2/3)F(t) \quad \text{when} \quad t = T_{\min}, T_{\max} \tag{2.15}$$

Combining (2.14), we have

$$\frac{F(T_{max})}{|T_{max}|^{p+p^2/3}} > \frac{F(T_{min})}{|T_{min}|^{p+p^2/3}}. \tag{2.16}$$

On the other hand, by (2.15) and (f_4), it is clear that

$$\frac{F(T_{max})}{|T_{max}|^{p+p^2/3}} = \frac{3}{p^2}\frac{\widetilde{F}(T_{max})}{|T_{max}|^{p+p^2/3}} < \frac{3}{p^2}\frac{\widetilde{F}(T_{min})}{|T_{min}|^{p+p^2/3}} = \frac{F(T_{min})}{|T_{min}|^{p+p^2/3}}, \tag{2.17}$$

which yields a contradiction, and we complete Step 4.

Step 5. $f(t)t > (p+p^2/3)F(t)$ for any $t \neq 0$. By Step 4, the function $F(t)/|t|^{p+p^2/3}$ is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0,\infty)$. Then, in view of (f_4), the function $f(t)/|t|^{p-1+p^2/3}$ is strictly increasing on $(-\infty,0)$ and $(0,\infty)$. For any $t \neq 0$, it is clear that

$$\begin{aligned}(p+p^2/3)F(t) &= (p+p^2/3)\int_0^t f(s)ds \\ &< (p+p^2/3)\frac{f(t)}{|t|^{p-1+p^2/3}}\int_0^t |s|^{p-1+p^2/3}ds = f(t)t\end{aligned}$$

and this proves Step 5. Now, by Steps 1 and 5, we complete the proof of (2.13).

By (2.13) with Fatou's lemma, we obtain that for every $u \in S(a)$,

$$\lim_{s\to+\infty}\int_{\mathbb{R}^3} h_\mu(e^{\frac{3}{p}s}u)|u|^{\bar{p}}dx = +\infty. \tag{2.18}$$

Using the fact that

$$\begin{aligned}\tilde{J}(u,s) &= \frac{e^{ps}}{p}\|\nabla u\|_p^p + \frac{e^s}{2p}B(u) + \mu e^{ps}\int_{\mathbb{R}^3}|u|^{\bar{p}}dx - e^{ps}\int_{\mathbb{R}^3} h_\mu(e^{\frac{3}{p}s}u)|u|^{\bar{p}}dx \\ &= e^{ps}\left(\frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2p}e^{(1-p)s}B(u) + \mu\int_{\mathbb{R}^3}|u|^{\bar{p}}dx - \int_{\mathbb{R}^3} h_\mu(e^{\frac{3}{p}s}u)|u|^{\bar{p}}dx\right).\end{aligned} \tag{2.19}$$

we derive from (2.18) that $\tilde{J}(u,s) \to -\infty$ as $s \to +\infty$. □

Lemma 2.3. *Let $u \in W^{1,p}(\mathbb{R}^3)$ be fixed, then the following limitations hold true.*

(i) There exists a unique $s_u \in \mathbb{R}$ such that

$$P(\eta(u,s_u)) = 0.$$

In particular, if $u \in S(a)$, then $\eta(u,s_u) \in \mathcal{P}_a$, with $\mathcal{P}_a$ defined in (1.12).

(ii) $\tilde{J}(u,s_u) > \tilde{J}(u,s)$ for all $s \neq s_u$, and moreover, $\tilde{J}(u,s_u) > 0$.

(iii) The map $u \to s_u$ is continuous in $u \in W^{1,p}(\mathbb{R}^3)$.

(iv) $s_{u(\cdot+z)} = s_u$ for any $z \in \mathbb{R}^3$. If f is odd, then $s_{-u} = s_u$.

Proof. (i) For fixed $u \in W^{1,p}(\mathbb{R}^3) \setminus \{0\}$, we get

$$\frac{d}{ds}\tilde{J}(u,s) = e^{ps}\|\nabla u\|_p^p + \frac{1}{2p}e^{s}B(u) - \frac{3}{p}e^{-3s}\int_{\mathbb{R}^3}\tilde{F}(e^{\frac{3s}{p}}u)dx = P(\eta(u,s)).$$

From Lemma 2.2, it follows that there exists $s_u \in \mathbb{R}$ at which $\tilde{J}(u,s)$ attains its global maximum. Moreover, we have

$$\frac{d}{ds}\tilde{J}(u,s_u) = P(\eta(u,s_u)) = 0.$$

In what follows, we prove the uniqueness of s_u.

Based on the definition of the function $k(t)$ given in Remark 2.2 ,we have $\tilde{F}(t) = k(t)|t|^{\bar{p}}$ for every $t \in \mathbb{R}$, and

$$P(\eta(u,s)) = e^{ps}\left(\|\nabla u\|^p + \frac{1}{2p}e^{(1-p)s}B(u) - \frac{3}{p}\int_{\mathbb{R}^3}k(e^{\frac{3s}{p}}u)|u|^{\bar{p}}dx\right).$$

For a fixed $t \in \mathbb{R}$, it follows from (f_3) that the function $s \mapsto k(e^{\frac{3s}{p}}t)$ is strictly increasing. Consequently, $P(\eta(u,s))$ is strictly decreasing in s, which implies the uniqueness of s_u.

(ii) This assertion follows directly from the strict concavity of $\tilde{J}(u,\cdot)$ established in part (i).

(iii) By part (i), the mapping $u \mapsto s(u)$ is well-defined. Let $u \in W^{1,p}(\mathbb{R}^3) \setminus \{0\}$ and consider an arbitrary sequence $\{u_n\} \subset W^{1,p}(\mathbb{R}^3) \setminus \{0\}$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$. For each $n \geq 1$, let $s_n := s(u_n)$. We only need to show that there exists a subsequence for which $s_n \to s(u)$ as $n \to \infty$.

We first prove that the sequence $\{s_n\}$ is bounded. Recall the continuous coercive function h_0 defined in (2.12), which satisfies $h_0(t) \geq 0$ for all $t \in \mathbb{R}$. Suppose by contradiction, that along a subsequence $s_n \to +\infty$. Then by Fatou's lemma and the fact that $u_n \to u \neq 0$ almost everywhere in $\mathbb{R}^3$, we infer to

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}h_0\left(e^{\frac{3s_n}{p}}u_n\right)|u_n|^{\bar{p}}dx = +\infty.$$

In view of part (ii) and equation (2.19) with $\mu = 0$, it follows that

$$0 \leq e^{-ps_n}\tilde{J}(u_n,s_n) = \frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2p}e^{(1-p)s_n}B(u_n) - \int_{\mathbb{R}^3}h_0\left(e^{\frac{3s_n}{p}}u_n\right)|u_n|^{\bar{p}}dx \to -\infty \tag{2.20}$$

This contradicts the non-negativity of the expression, thus showing that $\{s_n\}$ is bounded above. Moreover, by part (ii), we have

$$\tilde{J}(u_n,s_n) \geq \tilde{J}(u_n,s(u)) \quad \text{for any } n \geq 1.$$

In view of $\eta(u_n,s(u)) \to \eta(u,s(u))$ in $W^{1,p}(\mathbb{R}^3)$, we conclude that

$$\tilde{J}(u_n,s(u)) = \tilde{J}(u,s(u)) + o_n(1)$$

and consequently,

$$\liminf_{n\to\infty}\tilde{J}(u_n,s_n) \geq \tilde{J}(u,s(u)) > 0. \tag{2.21}$$

Since $\{\eta(u_n,s_n)\} \subset M_a$ for sufficiently large $a > 0$, it follows from Lemma 2.1(i) and the fact that

$$\|\nabla(\eta(u_n,s_n))\|_p = e^{s_n}\|\nabla u_n\|_p,$$

we deduce from (2.21) that $\{s_n\}$ is also bounded from below. Therefore, without loss of generality, we may assume that

$$s_n \to s_* \quad \text{for some } s_* \in \mathbb{R}.$$

As $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$, it follows that $\eta(u_n,s_n) \to \eta(u,s_*)$ in $W^{1,p}(\mathbb{R}^3)$. Moreover, by $P(\eta(u_n,s_n)) = 0$ for all $n \geq 1$, we conclude that $P(\eta(u,s_*)) = 0$.

By Item (i), we have that $s_* = s(u)$ and thus Item (iii) is proved.

(iv) For every $z \in \mathbb{R}^3$, a change of variables in the integrals gives that

$$P(\eta(u(\cdot + z), s(u))) = P(\eta(u, s(u))) = 0$$

and hence $s_{u(\cdot+z)} = s_u$ by part (i). If f is odd, then clearly

$$P(\eta(-u, s(u))) = P(-\eta(u, s(u))) = P(\eta(u, s(u))) = 0$$

and hence $s_{-u} = s_u$. □

In the following, we study some key properties of the Pohozaev manifold

$$\mathcal{P}_a := \left\{ u \in S_a : P(u) = p \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{2} B(u) - 3 \int_{\mathbb{R}^3} \tilde{F}(u) dx = 0 \right\}$$

From Lemma 2.3 we see that $\mathcal{P}_a \neq \emptyset$, and have the following properties.

Lemma 2.4. *From the definition of $\mathcal{P}_a$ in (1.12), we have*

(1) $\inf_{u \in \mathcal{P}_a} \|\nabla u\|_p > 0$.
(2) $\inf_{u \in \mathcal{P}_a} J(u) > 0$.
(3) J is coercive on $\mathcal{P}_a$, i.e., for any $\{u_n\} \subset \mathcal{P}_a$ satisfying $\|u_n\| \to +\infty$, then

$$J(u_n) \to +\infty.$$

Proof. (1) Suppose by contradiction that, there exists a sequence $\{u_n\} \subset \mathcal{P}_a$ satisfying $\|\nabla u_n\|_p \to 0$. Then Remark 2.1 implies that for n large enough,

$$0 = P(u_n) \geq \frac{p-1}{p} \|\nabla u_n\|_p^p > 0,$$

which is a contradiction. Hence, (1) holds.

(2) From Lemma 2.3(i)-(ii), it follows that for any $u \in \mathcal{P}_a$,

$$J(u) = \tilde{J}(u, 0) \geq \tilde{J}(u, s) \quad \text{for any } s \in \mathbb{R}.$$

Let $\tilde{s} := \ln\left(\frac{\sigma}{\|\nabla u\|_p}\right)$, where σ is given in Lemma 2.1-(i). Then $\|\nabla \eta(u, \tilde{s})\|_p = \sigma$. Using Lemma 2.1-(i), we have

$$J(u) \geq \tilde{J}(u, \tilde{s}) = J(\eta(u, \tilde{s})) \geq \frac{1}{2p} \|\nabla \eta(u, \tilde{s})\|_p^p = \frac{1}{2p} \sigma^p > 0,$$

and the proof of (2) follows.

(3) Suppose, for contradiction, that there exists a sequence $\{v_n\} \subset \mathcal{P}_a$ with $\|v_n\| \to \infty$ such that for some $\hat{C} \in (0, +\infty)$,

$$\sup_{n \geq 1} J(v_n) \leq \hat{C}. \tag{2.22}$$

For each $n \geq 1$, set

$$s_n := \ln(\|\nabla v_n\|_p) \quad \text{and} \quad w_n := \eta(v_n, -s_n).$$

It is clear that $s_n \to +\infty$. A direct calculation shows that $\|w_n\|_p = a$ and $\|\nabla w_n\|_p = 1$. so $\{w_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. Consequently, up to a subsequence, there exists $w \in W^{1,p}(\mathbb{R}^3)$ such that

$$w_n \rightharpoonup w \quad \text{in } W^{1,p}(\mathbb{R}^3) \quad \text{and} \quad w_n \to w \quad \text{a.e. in } \mathbb{R}^3.$$

Define the constant

$$\delta := \limsup_{n \to \infty} \left(\sup_{z \in \mathbb{R}^3} \int_{B_1(z)} |w_n|^p dx \right).$$

We next discuss the following two cases: nonvanishing and vanishing.

Case 1: Nonvanishing: i.e., $\delta > 0$. If $\delta > 0$, then we can extract a sequence $\{z_n\} \subset \mathbb{R}^3$ such that

$$\tilde{w}_n := w_n(x + z_n) \rightharpoonup w \text{ in } W^{1,p}(\mathbb{R}^3) \quad \text{and} \quad \tilde{w}_n \to w \text{ a.e. in } \mathbb{R}^3.$$

Let $h_\mu(t)$ be defined as in (2.12) with $\mu = 0$. Since $s_n \to +\infty$, it follows from (2.13) and Fatou's lemma that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} h_0(e^{\frac{3}{p}s_n}\tilde{w}_n)|\tilde{w}_n|^{\bar{p}}dx = +\infty. \tag{2.23}$$

It follows from (2), (2.19) with $\mu = 0$, and (2.23) that

$$\begin{aligned}
0 \le e^{-ps_n}J(v_n) &= e^{-ps_n}J(\eta(w_n, s_n)) \\
&= \left(\frac{1}{p} + \frac{1}{2p}e^{(1-p)s_n}B(w_n) - \int_{\mathbb{R}^3} h_0(e^{\frac{3}{p}s_n}w_n)|w_n|^{\bar{p}}dx\right) \\
&= \left(\frac{1}{p} + \frac{1}{2p}e^{(1-p)s_n}B(\tilde{w}_n) - \int_{\mathbb{R}^3} h_0(e^{\frac{3}{p}s_n}\tilde{w}_n)|\tilde{w}_n|^{\bar{p}}dx\right) \to -\infty,
\end{aligned} \tag{2.24}$$

which is impossible, hence Case 1 is ruled out.

Case 2: Vanishing: i.e., $\delta = 0$. If $\delta = 0$, then by Lions' lemma, we have

$$w_n \to 0 \text{ in } L^{\bar{p}}(\mathbb{R}^3) \quad \text{for} \quad \bar{p} \in (p, p^*).$$

Then, from (2.6) and Lemma 2.1(ii), it follows that as $n \to \infty$,

$$\frac{e^s}{2p}B(w_n) \to 0 \quad \text{and} \quad e^{-3s}\int_{\mathbb{R}^3} F(e^{\frac{3}{p}s}w_n)dx \to 0 \quad \text{for any} \quad s \in \mathbb{R}. \tag{2.25}$$

Let $\hat{s} > \frac{\ln(2\hat{C})}{2}$, where $\hat{C}$ is given in (2.22). By virtue of $P(\eta(w_n, s_n)) = P(v_n) = 0$, from Lemma 2.3 (i)-(ii) and (2.25), we know for $n \in \mathbb{N}$ large,

$$\begin{aligned}
\hat{C} \ge J(v_n) = J(\eta(w_n, s_n)) &= \tilde{J}(w_n, s_n) \ge \tilde{J}(w_n, \hat{s}) \\
&= \frac{1}{p}e^{p\hat{s}} + \frac{e^{\hat{s}}}{2p}B(w_n) - e^{-3\hat{s}}\int_{\mathbb{R}^3} F(e^{\frac{3}{p}\hat{s}}w_n)dx \\
&= \frac{1}{p}e^{p\hat{s}} + o_n(1) > \hat{C},
\end{aligned}$$

which leads to a contradiction, showing that Case 2 does not occur. Therefore, J is coercive on $\mathcal{P}_a$, thus item (3) is checked. □

To analyze the behavior of the Palais-Smale sequence, we invoke the following Brezis-Lieb type splitting result, whose proof is standard and can be carried out as Lemma 2.6 of [28].

Lemma 2.5. *Let $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ be a bounded sequence such that $u_n \to u$ a. e. in $\mathbb{R}^3$ for some $u \in W^{1,p}(\mathbb{R}^3)$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (F(u_n) - F(u_n - u) - F(u))\, dx = 0.$$

For any $a > 0$, in view of Lemma 2.4, we can define the infimum of J restricted on $\mathcal{P}_a$ by

$$c_a := \inf_{u\in\mathcal{P}_a} J(u). \tag{2.26}$$

By Lemma 2.4-(2) we see that $c_a > 0$. The following lemma reveals the behavior of c_a with respect to a, which is crucial for overcoming the lack of compactness.

Lemma 2.6. *For each $a > 0$, the map $a \mapsto c_a$ is continuous. Moreover, the map $a \mapsto c_a$ is nonincreasing on $(0, +\infty)$.*

Proof. We first prove the continuity of $a \mapsto c_a$. Let $\{a_m\} \subset (0, +\infty)$ be a sequence such that $a_m \to a > 0$. It is sufficient to show that

$$\lim_{m\to\infty} c_{a_m} = c_a. \tag{2.27}$$

Given $u \in \mathcal{P}_a$, we define the sequence $\{u_m\}$ by

$$u_m = \left(\frac{a_m}{a}\right)^{\frac{1}{p}} u \in \mathcal{S}_{a_m}, \quad \forall m \in \mathbb{N}^+.$$

Clearly, $u_m \to u$ in $W^{1,p}(\mathbb{R}^3)$. By Lemma 2.3 (ii)-(iii), there exists $s_m \in \mathbb{R}$ such that $\eta(u_m, s_m) \in \mathcal{P}_{a_m}$ and $s_m \to 0$ as $m \to \infty$. Consequently, as $m \to \infty$, we have

$$\eta(u_m, s_m) \to \eta(u, 0) = u \quad \text{in } W^{1,p}(\mathbb{R}^3). \tag{2.28}$$

From (2.28), we obtain

$$\limsup_{m\to\infty} c_{a_m} \le \limsup_{m\to\infty} J\left(\eta(u_m, s_m)\right) = J(u),$$

which yields that

$$\limsup_{m\to\infty} c_{a_m} \le c_a. \tag{2.29}$$

On the other hand, for each $m \in \mathbb{N}^+$, by the definition of c_{a_m}, there exists $v_m \in \mathcal{P}_{a_m}$ such that

$$J(v_m) \le c_{a_m} + \frac{1}{m}. \tag{2.30}$$

Denote by $t_m := \left(\frac{a}{a_m}\right)^{\frac{2}{3}}$, then $t_m \to 1$ and

$$\tilde{v}_m := v_m\left(\cdot/t_m\right) \in S(a).$$

By Lemma 2.3(i), there exists $s_{\tilde{v}_m} \in \mathbb{R}$ such that $\eta(\tilde{v}_m, s_{\tilde{v}_m}) \in \mathcal{P}_a$. Then Lemma 2.3(ii) together with (2.30) implies that

$$\begin{aligned} c_a &\le J(\eta(\tilde{v}_m, s_{\tilde{v}_m})) = \tilde{J}(\tilde{v}_m, s_{\tilde{v}_m}) \\ &\le \tilde{J}(v_m, s_{\tilde{v}_m}) + |\tilde{J}(\tilde{v}_m, s_{\tilde{v}_m}) - \tilde{J}(v_m, s_{\tilde{v}_m})| \\ &\le \tilde{J}(v_m, 0) + |\tilde{J}(\tilde{v}_m, s_{\tilde{v}_m}) - \tilde{J}(v_m, s_{\tilde{v}_m})| \\ &\le c_{a_m} + \frac{1}{m} + |\tilde{J}(\tilde{v}_m, s_{\tilde{v}_m}) - \tilde{J}(v_m, s_{\tilde{v}_m})|. \end{aligned}$$

Let $C(m) := |\tilde{J}(\tilde{v}_m, s_{\tilde{v}_m}) - \tilde{J}(v_m, s_{\tilde{v}_m})| = |J(\eta(\tilde{v}_m, s_{\tilde{v}_m})) - J(\eta(v_m, s_{\tilde{v}_m}))|$. If we can prove

$$C(m) \to 0 \quad \text{as } m \to \infty, \tag{2.31}$$

then the above inequality implies that

$$c_a \le \liminf_{m\to\infty} c_{a_m},$$

combining with (2.29), we obtain

$$\lim_{m\to\infty} c_{a_m} = c_a.$$

Next, we aim to prove (2.31). First we observe that $\eta(u(\cdot/t), s) = \eta(u, s)(\cdot/t)$, and so

$$\begin{aligned} C(m) &= \left|\frac{1}{p}(t_m^{3-p} - 1)\int_{\mathbb{R}^3} |\nabla \eta(v_m, s_{\tilde{v}_m})|^p dx + \frac{1}{2p}(t_m^5 - 1)\int_{\mathbb{R}^3} B(\eta(v_m, s_{\tilde{v}_m})) \right. \\ &\qquad \left. -(t_m^3 - 1)\int_{\mathbb{R}^3} F(\eta(v_m, s_{\tilde{v}_m}))dx\right| \\ &\le \frac{1}{p}|t_m^{3-p} - 1|\int_{\mathbb{R}^3} |\nabla \eta(v_m, s_{\tilde{v}_m})|^p dx + \frac{1}{2p}|t_m^5 - 1| B(\eta(v_m, s_{\tilde{v}_m})) \\ &\qquad + |t_m^3 - 1|\int_{\mathbb{R}^3} |F(\eta(v_m, s_{\tilde{v}_m}))|dx \\ &:= \frac{1}{p}|t_m^{3-p} - 1| A(m) + \frac{1}{2p}|t_m^5 - 1| B(\eta(v_m, s_{\tilde{v}_m})) + |t_m^3 - 1| D(m). \end{aligned}$$

Since $t_m \to 1$, it suffices to show that the following quantities are uniformly bounded:

$$\limsup_{m\to\infty} A(m) < +\infty, \quad \limsup_{m\to\infty} B\left(\eta(v_m, s_{\tilde{v}_m})\right) < +\infty, \quad \limsup_{m\to\infty} D(m) < +\infty \tag{2.32}$$

To check (2.32), we prove the following three claims in sequence.

Claim 1. The sequence $\{v_m\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. Indeed, by (2.29) and (2.30), one has

$$\limsup_{m\to\infty} J(v_m) \le c_a.$$

Using the fact $v_m \in \mathcal{P}_{a_m}$ and $a_m \to a$, by an similar proof of Lemma 2.4-(3) we can infer that $\{v_m\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$.

Claim 2. The sequence $\{\tilde{v}_m\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. Furthermore, there exist a sequence $\{y_m\} \subset \mathbb{R}^3$ and $v \in W^{1,p}(\mathbb{R}^3)$ for which, along a subsequence, $\tilde{v}_m(\cdot + y_m) \to v$ almost everywhere in $\mathbb{R}^3$. To verify this, we first observe that $t_m \to 1$, and use Claim 1 to obtain $\{\tilde{v}_m\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. Define

$$\rho := \limsup_{m\to\infty} \left(\sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |\tilde{v}_m|^p dx \right).$$

To complete the argument, we show that $\rho = 0$ leads to a contradiction. If $\rho = 0$, then Lions' lemma [33] yields $\tilde{v}_m \to 0$ in $L^{\bar{p}}(\mathbb{R}^3)$. It follows that

$$\int_{\mathbb{R}^3} |v_m|^{\bar{p}} dx = \int_{\mathbb{R}^3} |\tilde{v}_m(t_m \cdot)|^{\bar{p}} dx = t_m^{-3} \int_{\mathbb{R}^3} |\tilde{v}_m|^{\bar{p}} dx \to 0.$$

Given that $P(v_m) = 0$, an application of Lemma 2.1-(ii) yields

$$\int_{\mathbb{R}^3} |\nabla v_m|^p dx + \frac{1}{2p} B(v_m) = \frac{3}{p} \int_{\mathbb{R}^3} \tilde{F}(v_m) dx \to 0.$$

By Remark 2.1, we obtain

$$0 = P(v_m) \ge \frac{1}{p} \int_{\mathbb{R}^3} |\nabla v_m|^p dx > 0 \quad \text{for } m \text{ large enough,}$$

which yields a contradiction, and completes the proof of Claim 2.

Claim 3. $\limsup_{m\to\infty} s_{\tilde{v}_m} < +\infty$. Arguing by contradiction that, after passing to a subsequence,

$$s_{\tilde{v}_m} \to +\infty \quad \text{as } m \to \infty. \tag{2.33}$$

First, observe that by Claim 2, after passing to a subsequence, one has

$$\tilde{v}_m(\cdot + y_m) \to v \ne 0 \quad \text{a.e. in } \mathbb{R}^3. \tag{2.34}$$

By Lemma 2.3 (iv) and (2.33), we also get

$$s_{\tilde{v}_m(\cdot + y_m)} = s_{\tilde{v}_m} \to +\infty. \tag{2.35}$$

In addition, Lemma 2.3-(ii) gives that

$$\tilde{J}\left(\tilde{v}_m(\cdot + y_m), s_{\tilde{v}_m(\cdot + y_m)}\right) \ge 0. \tag{2.36}$$

Now, combining (2.34), (2.35) and (2.36), and employing the same arguments as in the derivation of (2.24) we get a contradiction, which proves Claim 3.

Combining Claims 1 and 3, we conclude that

$$\limsup_{m\to\infty} \|\eta(v_m, s_{\tilde{v}_m})\| < +\infty.$$

Hence $\limsup_{m\to\infty} A(m) < +\infty$, and by (f_1)-(f_2), $\limsup_{m\to\infty} D(m) < +\infty$ as well. Moreover, by the Hardy-Littlewood-Sobolev inequality we get

$$B(\eta(v_m, s_{\tilde{v}_m})) \le C\|\eta(v_m, s_{\tilde{v}_m})\|_{\frac{6p}{5}}^{2p} \le C\|\nabla \eta(v_m, s_{\tilde{v}_m})\|^{2p},$$

which implies $\limsup_{m\to\infty} B\left(\eta(v_m, s_{\tilde{v}_m})\right) < +\infty$. Consequently, combining with the fact that $t_m \to 1$, we infer that

$$C(m) \to 0 \quad \text{as } m \to \infty,$$

which shows that the function $a \mapsto c_a$ is continuous.

We now show that the mapping $a \mapsto c_a$ is nonincreasing on $(0, +\infty)$. It suffices to prove that for any $\varepsilon > 0$ and $a > a' > 0$, we have

$$c_a \leq c_{a'} + \varepsilon.$$

By the definition of $c_{a'}$, for any $\varepsilon > 0$, there exists $v \in \mathcal{P}_{a'}$ such that

$$J(v) \leq c_{a'} + \frac{\varepsilon}{2}. \tag{2.37}$$

Given $\varrho > 0$, we define the function $v_\varrho(x) = v(x)\xi(\varrho x)$, where ξ is a radial function in $C_0^\infty(\mathbb{R}^3)$ defined as follows:

$$\xi(x) := \begin{cases} 1, & \text{if } |x| \leq 1, \\ \in (0,1), & \text{if } 1 < |x| < 2, \\ 0, & \text{if } |x| \geq 2. \end{cases}$$

Clearly, $v_\varrho \to v$ in $W^{1,p}(\mathbb{R}^3)$ as $\varrho \to 0^+$. Following the same argument as in (2.28), we obtain

$$\eta(v_\varrho, s_{v_\varrho}) \to \eta(v, 0) = v \text{ in } W^{1,p}(\mathbb{R}^3) \text{ as } \varrho \to 0^+. \tag{2.38}$$

By choosing $\varrho > 0$ sufficiently small, we obtain

$$J(\eta(v_\varrho, s_{v_\varrho})) \leq J(v) + \frac{\varepsilon}{4}. \tag{2.39}$$

Next, we choose $u \in C_0^\infty(\mathbb{R}^3)$ such that its support satisfies $\operatorname{supp}(u) \subset B_{1+4/\varrho}(0) \setminus B_{4/\varrho}(0)$, and define

$$\tilde{u} = \left(\frac{a^p - \|v_\varrho\|_p^p}{\|u\|_p^p} \right)^{\frac{1}{p}} u.$$

For any $b \leq 0$, we define $\omega_b = v_\varrho + \eta(\tilde{u}, b)$. The definition of v_ϱ implies that

$$\operatorname{supp}(v_\varrho) \cap \operatorname{supp}(\eta(\tilde{u}, b)) = \emptyset.$$

It can be easily checked by direct computation that $\omega_b \in S(a)$. An application of Lemma 2.3-(i) then implies the existence of $s_{\omega_b} \in \mathbb{R}$ such that $\eta(\omega_b, s_{\omega_b}) \in \mathcal{P}_a$. Moreover, by an argument analogous to showing (2.24), the family $\{s_{\omega_b}\}$ can be proved to be uniformly bounded with respect to b. Therefore, we have

$$s_{\omega_b} + b \to -\infty \text{ as } b \to -\infty.$$

In view of this fact, we infer to

$$\eta(\tilde{u}, s_{\omega_b} + b) \to 0 \text{ in } L^{\bar{p}}(\mathbb{R}^3). \tag{2.40}$$

By Lemma 2.1(ii) we have

$$\int_{\mathbb{R}^3} F(\eta(\tilde{u}, s_{\omega_b} + b))dx \to 0. \tag{2.41}$$

Moreover, we get

$$\|\nabla \eta(\tilde{u}, s_{\omega_b} + b)\|_p \to 0, \quad \text{and} \quad \|\eta(u, s_{\omega_b} + b)\|_{\frac{6p}{5}} \to 0. \tag{2.42}$$

In light of (2.6), it follows that

$$B(\eta(\tilde{u}, s_{\omega_b} + b)) \to 0. \tag{2.43}$$

A combination of (2.41)-(2.43) shows that

$$J(\eta(\tilde{u}, s_{\omega_b} + b)) \to 0. \tag{2.44}$$

Hence, it follows from Lemma 2.3 in conjunction with (2.39) and (2.44) that

$$\begin{aligned} c_a &\leq J(\eta(\omega_b, s_{\omega_b})) \\ &= J(\eta(v_\varrho, s_{\omega_b})) + J(\eta(\eta(\tilde{u}, b), s_{\omega_b})) \\ &\leq J(\eta(v_\varrho, s_{v_\varrho})) + J(\eta(\tilde{u}, s_{\omega_b} + b)) \\ &\leq J(v) + \frac{\varepsilon}{2}. \end{aligned}$$

By (2.37), we deduce to

$$c_a \leq c_{a'} + \varepsilon,$$

and thus prove the nonincreasing monotonicity of the function $a \mapsto c_a$, completing the proof of this lemma. □

Lemma 2.7. *Let c_a be defined by (2.26), then*

$$c_a \to +\infty \quad as \quad a \to 0^+.$$

Proof. Let $\{u_n\} \subset \mathcal{P}_{a_n}$ satisfying $\|u_n\|_p \to 0^+$, i.e., $a_n \to 0^+$ as $n \to \infty$. It is sufficient to prove that

$$J(u_n) \to +\infty \text{ as } n \to \infty.$$

For each $n \in \mathbb{N}$, set

$$s_n = \ln(\|\nabla u_n\|_p) \quad \text{and} \quad w_n = \eta(u_n, -s_n).$$

Then $u_n = \eta(w_n, s_n) \in \mathcal{P}_{a_n}$, satisfying $\|w_n\|_p = \|u_n\|_p \to 0$ and $\|\nabla w_n\|_p = 1$. From Hölder's inequality we have $w_n \to 0$ in both $L^{\bar{p}}(\mathbb{R}^3)$ and $L^{\frac{6p}{5}}(\mathbb{R}^3)$. By Lemma 2.1 and (2.6), we deduce that for every $s \in \mathbb{R}$,

$$e^{-3s} \int_{\mathbb{R}^3} F(e^{\frac{3}{p}s} w_n) dx \to 0 \quad \text{and} \quad e^s B(w_n) \to 0. \tag{2.45}$$

Due to $P(u_n) = P(\eta(w_n, s_n)) = 0$, combining Lemma 2.3 with (2.45) we have

$$\begin{aligned} J(u_n) = J(\eta(w_n, s_n)) &\geq J(\eta(w_n, s)) \\ &= \frac{1}{p} e^{ps} + \frac{1}{2p} e^s B(w_n) - e^{-3s} \int_{\mathbb{R}^3} F(e^{\frac{3}{p}s} w_n) dx \\ &= \frac{1}{p} e^{ps} + o_n(1). \end{aligned}$$

By the arbitrariness of the parameter $s > 0$, we conclude that $J(u_n) \to +\infty$, and thus the desired result follows. □

The following lemma offers a better understanding for the characteristics of the Lagrange multiplier λ.

Lemma 2.8. *Let $(u, \lambda) \in S(a) \times \mathbb{R}$ be a solution to the equation*

$$-\Delta_p u + \lambda u^{p-1} + (|x|^{-1} * |u|^p) u^{p-1} = f(u), \ \ in \ \mathbb{R}^3$$

and $J(u) = c_a$. If $\lambda > 0$, then for any $a' > a$ (a' is close enough to a),

$$c_a > c_{a'}.$$

If $\lambda < 0$, then for every $a' > a$ near enough to a,

$$c_a < c_{a'}.$$

Proof. Notice that $(u,\lambda)\in S(a)\times\mathbb{R}$ satisfies the equation

$$-\Delta_p u+\lambda u^{p-1}+(|x|^{-1}*|u|^p)u^{p-1}=f(u),\ \text{in }\mathbb{R}^3,$$

and moreover $u\in\mathcal{P}_a$. Thus, we have

$$J'(u)u=-\lambda\|u\|_p^p=-\lambda a^p. \tag{2.46}$$

For any $t>0$ and $s\in\mathbb{R}$, we define

$$u_{t,s}:=\eta(tu,s)\in\mathcal{S}_{ta},$$

and introduce the function

$$K(t,s):=J(u_{t,s})=\frac{1}{p}t^pe^{ps}\int_{\mathbb{R}^3}|\nabla u|^pdx+\frac{1}{2p}t^{2p}e^sB(u)-e^{-3s}\int_{\mathbb{R}^3}F(te^{\frac{3}{p}s}u)dx.$$

By a direct computation we have

$$\begin{aligned}\frac{\partial K(t,s)}{\partial t}&=t^{p-1}e^{ps}\int_{\mathbb{R}^3}|\nabla u|^pdx+t^{2p-1}e^sB(u)-e^{-3s}\int_{\mathbb{R}^3}f(te^{\frac{3}{p}s}u)e^{\frac{3}{p}s}udx\\&=t^{-1}J'(u_{t,s})u_{t,s},\end{aligned} \tag{2.47}$$

and the limit

$$u_{t,s}\to u\text{ in }W^{1,p}(\mathbb{R}^3)\text{ as }(t,s)\to(1,0). \tag{2.48}$$

Assuming $\lambda>0$, we have from (2.46) that

$$J'(u)u=-\lambda a^p<0. \tag{2.49}$$

By virtue of (2.47)-(2.49), we can select $\epsilon>0$ sufficiently small such that

$$\frac{\partial K(t,s)}{\partial t}<0\text{ for any }(t,s)\in(1,1+\epsilon]\times[-\epsilon,+\epsilon].$$

Using the mean value theorem we derive to

$$K(t,s)=K(1,s)+(t-1)\frac{\partial K(\alpha,s)}{\partial t}<K(1,s)$$

for $1<\alpha<t\le 1+\epsilon$ and $|s|\le\epsilon$. By Lemma 2.3-(iii) one has that $s_{tu}\to s_u=0$ as $t\to 1^+$. Given any $a'>a$ in a sufficiently small neighborhood of a, let

$$t:=\frac{a'}{a}\in(1,1+\epsilon]\quad\text{and}\quad s:=s_{tu}\in[-\epsilon,\epsilon].$$

Since $u_{t,s}\in\mathcal{S}_{ta}$, a combination of the foregoing facts leads to

$$c_{a'}\le J(u_{t,s})=K(t,s)<K(1,s)=J(\eta(u,s))\le J(u)=c_a,$$

which implies $c_{a'}<c_a$.

In the case $\lambda<0$, an analogous argument shows the result, and we omit the details for brevity. □

As a consequence of Lemmas 2.6 and 2.8, we can obtain the following conclusion.

Corollary 2.9. *Let $(u,\lambda)\in S(a)\times\mathbb{R}$ be a solution of the equation*

$$-\Delta_p u+\lambda u^{p-1}+(|x|^{-1}*|u|^p)u^{p-1}=f(u),\ \text{in }\mathbb{R}^3$$

and $J(u)=c_a$. Then $\lambda\ge 0$. Particularly, if $\lambda>0$, then

$$c_a>c_{a'}\text{ for any }a'>a.$$

3. Proof of Theorems 1.2–1.3

In this section we shall prove Theorems 1.2 and 1.3. To this aim, we first construct a Palais-Smale sequence for the constrained functional $J|_{S(a)}$ at the energy level c_a lies in $\mathcal{P}_a$ exhibiting a specific fine property.

3.1. Proof of Theorem 1.2.

For any $a > 0$ and $u \in W^{1,p}(\mathbb{R}^3)$, define an auxiliary functional $\tilde{I} : W^{1,p}(\mathbb{R}^3)\backslash\{0\} \to \mathbb{R}$ by

$$\tilde{I}(u) := \tilde{J}(u, s_u) = \frac{e^{ps_u}}{p}\|\nabla u\|_p^p + \frac{e^{s_u}}{2p}B(u) - \frac{1}{e^{3s_u}}\int_{\mathbb{R}^3} F(e^{\frac{3s_u}{p}}u)dx, \tag{3.1}$$

here, $s_u \in \mathbb{R}$ is provided by Lemma 2.3 and satisfies the condition $P(\eta(u, s_u)) = 0$. The next result can be established through a standard variational argument.

Lemma 3.1. *The functional $\tilde{I}$ is C^1-differentiable. Moreover, for any $\psi \in C_0^\infty(\mathbb{R}^3)$,*

$$\begin{aligned}\tilde{I}'(u)\psi = e^{ps_u}\int_{\mathbb{R}^3}|\nabla u|^{p-2}\nabla u\cdot\nabla\psi dx + e^{s_u}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^{p-2}u(y)\psi(y)}{|x-y|}dxdy\\ - e^{-3s_u}\int_{\mathbb{R}^3} f(e^{\frac{3}{p}s_u}u)e^{\frac{3}{p}s_u}\psi dx\\ = J'(\eta(u,s_u))\eta(\psi,s_\psi).\end{aligned}\tag{3.2}$$

For each $a > 0$, we denote by

$$\Psi := \tilde{I}|_{S(a)} : S(a) \to \mathbb{R}. \tag{3.3}$$

Clearly, $\Psi \in C^1(S(a), \mathbb{R})$ and it satisfies

$$\Psi'(u)\psi = \tilde{I}'(u)\psi = J'(\eta(u, s_u))\eta(\psi, s_\psi). \tag{3.4}$$

for any $u \in S(a)$ and $\psi \in T_uS(a)$, here we introduce the definition of tangent space at a point $u \in S(a)$ by

$$T_uS(a) := \left\{v \in W^{1,p}(\mathbb{R}^3) : \int_{\mathbb{R}^3}|u|^{p-2}uvdx = 0\right\}.$$

Our aim is to construct a Palais-Smale sequence for the constrained functional $J|_{S(a)}$ at the energy level c_a, with the additional property that each term lies in $\mathcal{P}_a$.

For this purpose, we introduce some preliminaries from [23] and [5].

Definition 3.2. *Let $\mathcal{Y}$ be a metric space and $\mathcal{E} \subset \mathcal{Y}$ be a closed subset. Let $\mathcal{G}$ be a class of compact subsets of $\mathcal{Y}$. We say $\mathcal{G}$ is a homotopy stable family with closed boundary $\mathcal{E}$ if the following conditions are satisfied by $\mathcal{G}$:*

(i) any set in $\mathcal{G}$ contains $\mathcal{E}$;
(ii) for any $B \in \mathcal{G}$ and $\chi \in C([0,1]\times\mathcal{Y}, \mathcal{Y})$ satisfying $\chi(t, x) = x$ for all $(t, x) \in (\{0\}\times\mathcal{Y}) \cup ([0, 1]\times\mathcal{E})$, we have $\chi(\{1\}\times B) \in \mathcal{G}$.

Note that $\mathcal{E} = \emptyset$ is admissible.

The following lemma guarantees the existence of a Palais-Smale sequence with the required properties.

Lemma 3.3. *Let $\mathcal{G}$ be a homotopy stable family of compact subsets of $S(a)$ (with $\mathcal{E} = \emptyset$) and define*

$$c_{a,\mathcal{G}} := \inf_{D\in\mathcal{G}}\max_{u\in D}\Psi(u).$$

If $c_{a,\mathcal{G}} > 0$, then there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_a$ for $J|_{S(a)}$ at the level $c_{a,\mathcal{G}}$. In particular, if $\mathcal{G}$ consists of all singletons contained in $S(a)$, then $c_a = c_{a,\mathcal{G}}$ and $\{u_n\}$ constitutes a Palais–Smale sequence for $J|_{S(a)}$ at the energy level c_a.

Proof. Consider a minimizing sequence $\{A_n\} \subset \mathcal{G}$ for $c_{a,\mathcal{G}}$. Define the continuous mapping

$$\xi : [0,1] \times S(a) \to S(a), \quad \xi(t,u) = \eta(u, ts_u),$$

where Lemma 2.3-(iii) ensures the continuity of ξ. Note that $\xi(t,u) = u$ for all $(t,u) \in \{0\} \times S(a)$. By the homotopy stability property of $\mathcal{G}$, we obtain

$$D_n := \xi(1, A_n) = \{\eta(u, s_u) \mid u \in A_n\} \in \mathcal{G}.$$

Clearly, $D_n \subset \mathcal{P}_a$ for all $n \in \mathbb{N}^+$. Since $\Psi(\eta(u, s_u)) = \Psi(u)$ for every $u \in A_n$, it follows that

$$\max_{u \in D_n} \Psi(u) = \max_{u \in A_n} \Psi(u) \to c_{a,\mathcal{G}},$$

which implies that $\{D_n\} \subset \mathcal{G}$ is also a minimizing sequence for $c_{a,\mathcal{G}}$.

According to [37, Lemma 2.17], there exists a Palais-Smale sequence $\{v_n\} \subset W^{1,p}(\mathbb{R}^3)$ for Ψ on $S(a)$ at the level $c_{a,\mathcal{G}}$. Consequently, as $n \to \infty$, the sequence $\{v_n\}$ satisfies:

(1) $\Psi(v_n) \to c_{a,\mathcal{G}}$;
(2) $\mathrm{dist}(v_n, D_n) \to 0$;
(3) $\|d\Psi(v_n)\|_{v_n,*} \to 0$, where $\|\cdot\|_{v_n,*}$ denotes the dual norm of $(T_{v_n}S(a))^*$.

Now we define

$$s_n := s_{v_n}, \quad \text{and} \quad u_n := \eta(v_n, s_n) = \eta(v_n, s_{v_n}).$$

we shall verify that $\{u_n\} \subset \mathcal{P}_a$ forms a Palais-Smale sequence for J at the level $c_{a,\mathcal{G}}$.

We first claim that: There exists a constant $C > 0$ such that $e^{-ps_n} \le C$ for all $n \in \mathbb{N}^+$. To verify this claim, we use the fact

$$e^{-ps_n} = \frac{\|\nabla v_n\|_p^p}{\|\nabla u_n\|_p^p},$$

and $\{u_n\} \subset \mathcal{P}_a$, Lemma 2.4-(i), to derive that the sequence $\{\|\nabla u_n\|_p\}$ is bounded below by a positive constant. To complete the proof of the claim, we need to prove

$$\sup_{n \in \mathbb{N}^+} \|\nabla v_n\|_p < \infty.$$

For given $D_n \subset \mathcal{P}_a$, we have

$$\max_{u \in D_n} J(u) = \max_{u \in D_n} \Psi(u) \to c_{a,\mathcal{G}},$$

and from Lemma 2.4-(iii), $\{D_n\}$ is uniformly bounded in $W^{1,p}(\mathbb{R}^3)$. Moreover, since $\mathrm{dist}(v_n, D_n) \to 0$, we conclude that $\sup_{n \in \mathbb{N}^+} \|\nabla v_n\|_p < \infty$.

Since $\{u_n\} \subset \mathcal{P}_a$, we have $J(u_n) = \Psi(u_n) = \Psi(v_n) \to c_{a,\mathcal{G}}$. Thus, it remains to prove that $\{u_n\} \subset \mathcal{P}_a$ is a Palais-Smale sequence for J on $S(a)$. From (2.11), we know that for any $\psi \in T_{u_n}S(a)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |v_n|^{p-2} v_n [\eta(\psi, -s_n)] dx &= \int_{\mathbb{R}^3} |v_n|^{p-2} v_n e^{-\frac{3s_n}{p}} \psi(e^{-s_n} x) dx \\
&= \int_{\mathbb{R}^3} |e^{\frac{3s_n}{p}} v_n(e^{s_n} z)|^{p-2} e^{\frac{3s_n}{p}} v_n(e^{s_n} z) \psi(z) dz \\
&= \int_{\mathbb{R}^3} |u_n(z)|^{p-2} u_n(z) \psi(z) dz = 0,
\end{aligned}$$

which implies that $\eta(\psi, -s_n) \in T_{v_n}S(a)$. By the claim, we have

$$\|\eta(\psi, -s_n)\| \le \max\left\{C^{\frac{1}{p}}, 1\right\} \|\psi\|.$$

Furthermore, by Lemma 3.1 we infer that

$$\begin{aligned}\|dJ(u_n)\|_{u_n,*} &= \sup_{\substack{\psi\in T_{u_n}S(a)\\ \|\psi\|\le 1}} |dJ(u_n)[\psi]| \\ &= \sup_{\substack{\psi\in T_{u_n}S(a)\\ \|\psi\|\le 1}} |dJ(\eta(v_n,s_n))[\eta(\eta(\psi,-s_n),s_n)]| \\ &= \sup_{\substack{\psi\in T_{v_n}S(a)\\ \|\psi\|\le 1}} |d\Psi(v_n)[\eta(\psi,-s_n)]| \\ &\le \|d\Psi(v_n)\|_{v_n,*} \sup_{\substack{\psi\in T_{v_n}S(a)\\ \|\psi\|\le 1}} \|\eta(\psi,-s_n)\| \\ &\le \max\left\{C^{\frac{1}{p}},1\right\}\|d\Psi(v_n)\|_{v_n,*}.\end{aligned}$$

From $\|d\Psi(v_n)\|_{v_n,*}\to 0$, we infer that $\|dJ(u_n)\|_{u_n,*}\to 0$.

Finally, notice that the class of all singletons included in $S(a)$ is a homotopy stable family of compact subsets of $S(a)$ with $\mathcal{E}=\emptyset$. If f is odd, making this particular choice for $\mathcal{G}$, and by (f_1), Lemma 2.3-(iv) ensures that Ψ is even. Therefore, we may select a minimizing sequence $\{A_n\}\subset\mathcal{G}$, which in turn implies the corresponding minimizing sequence $\{D_n\}\subset\mathcal{G}$. Following the same reasoning as above, we obtain a Palais-Smale sequence $\{u_n\}\subset\mathcal{P}_a$ for $J|_{S(a)}$ at the level $c_{a,\mathcal{G}}$.

We now prove that $c_{a,\mathcal{G}}=c_a$. Obviously, we have

$$c_{a,\mathcal{G}}=\inf_{D\in\mathcal{G}}\max_{u\in D}\Psi(u)=\inf_{u\in S(a)}J(\eta(u,s_u)).$$

For any $u\in S(a)$, since $\eta(u,s_u)\in\mathcal{P}_a$, we have $J(\eta(u,s_u))\ge c_a$, which implies $c_{a,\mathcal{G}}\ge c_a$. Conversely, for any $u\in\mathcal{P}_a$, we observe that $J(u)=J(\eta(u,0))\ge c_{a,\mathcal{G}}$, yielding $c_{a,\mathcal{G}}\le c_a$. Thus, we obtain

$$c_{a,\mathcal{G}}=c_a.$$

This completes the proof of the lemma. □

Lemma 3.4. *There exists $a^*>0$ such that for any $a\in(0,a^*)$, if $\{u_n\}$ is a Palais–Smale sequence at the level c_a, then up to a subsequence, there exist $u\in W^{1,p}(\mathbb{R}^3)$ and $\lambda\in\mathbb{R}$ satisfying $u_n\to u$ in $W^{1,p}(\mathbb{R}^3)$ and*

$$-\Delta_p u+\lambda u^{p-1}+(|x|^{-1}*|u|^p)u^{p-1}=f(u).$$

Proof. For each $a>0$, let $\{u_n\}\subset\mathcal{P}_a$ be a Palais–Smale sequence for $J|_{S(a)}$ at the level c_a. Then as $n\to\infty$,

$$J(u_n)\to c_a \quad\text{and}\quad J'(u_n)|_{S(a)}\to 0.$$

Since $\{u_n\}\subset\mathcal{P}_a$, Lemma 2.4(iii) implies that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. By [12, Lemma 3], there exists $\{\lambda_n\}\subset\mathbb{R}$ such that as $n\to\infty$,

$$-\Delta_p u_n(\cdot+z_n)+\lambda_n u_n(\cdot+z_n)^{p-1}+(|x|^{-1}*|u_n(\cdot+z_n)|^p)u_n(\cdot+z_n)^{p-1}-f(u_n(\cdot+z_n))\to 0 \tag{3.5}$$

in $(W^{1,p}(\mathbb{R}^3))^*$, for any $\{z_n\}\subset\mathbb{R}^3$. where

$$\lambda_n:=\frac{1}{a^p}\left(\int_{\mathbb{R}^3}f(u_n)u_n dx-\int_{\mathbb{R}^3}|\nabla u_n|^p dx-B(u_n)\right).$$

By (f_1)-(f_2), (2.6) and the Sobolev inequality, the sequence $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Passing to a subsequence, we may assume $\lambda_n\to\lambda$ for some $\lambda\in\mathbb{R}$. Then it follows from (3.5) that

$$-\Delta_p u_n+\lambda u_n^{p-1}+(|x|^{-1}*|u_n|^p)u_n^{p-1}-f(u_n)\to 0 \quad\text{in } (W^{1,p}(\mathbb{R}^3))^*. \tag{3.6}$$

Define

$$\delta := \limsup_{n\to\infty}\left(\sup_{z\in\mathbb{R}^3}\int_{B_1(z)}|u_n|^p dx\right).$$

We claim $\delta > 0$. Suppose by contradiction that $\delta = 0$. Then by Lions' lemma, $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for $q \in (p, p^*)$. From Lemma 2.1 and (2.6), we obtain

$$\int_{\mathbb{R}^3}\tilde{F}(u_n)dx \to 0 \quad \text{and} \quad B(u_n) \to 0.$$

Since $P(u_n) = 0$, we have

$$\int_{\mathbb{R}^3}|\nabla u_n|^p dx = -\frac{1}{2p}B(u_n) + \frac{3}{p}\int_{\mathbb{R}^3}\tilde{F}(u_n)dx \to 0.$$

From (f_1)-(f_3), we get $\int_{\mathbb{R}^3}F(u_n)dx \to 0$. Consequently,

$$J(u_n) \to 0,$$

which contradicts $J(u_n) \to c_a > 0$. This establishes our claim. Up to a subsequence, there exists $\{z_n^1\} \subset \mathbb{R}^3$ and $u^1 \in W^{1,p}(\mathbb{R}^3)\backslash\{0\}$ such that $u_n(\cdot + z_n^1) \rightharpoonup u^1$ in $W^{1,p}(\mathbb{R}^3)$, $u_n(\cdot + z_n^1) \to u^1$ in $L^q_{\text{loc}}(\mathbb{R}^3)$ for $q \in [p, p^*)$, and $u_n(\cdot + z_n^1) \to u^1$ a.e. in $\mathbb{R}^3$. Set $v_n = u_n(\cdot + z_n^1)$. Using standard arguments, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} f(v_n)\varphi dx \to \int_{\mathbb{R}^3} f(u^1)\varphi dx,$$

and

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_n(x)|^p|v_n(y)|^{p-2}v_n(y)\varphi(y)}{|x-y|}dxdy \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u^1(x)|^p|u^1(y)|^{p-2}u^1(y)\varphi(y)}{|x-y|}dxdy.$$

It follows from (3.6) that

$$-\Delta_p u^1 + \lambda(u^1)^{p-1} + (|x|^{-1} * |u^1|^p)(u^1)^{p-1} = f(u^1). \tag{3.7}$$

That is, u^1 is a nontrivial solution of (3.7). In what follows, we will show $u_n \to u^1$ in $W^{1,p}(\mathbb{R}^3)$. Note that u satisfies the Pohozaev-type identity

$$\frac{3-p}{p}\|\nabla u^1\|_p^p + \frac{5}{2p}B(u^1) - 3\int_{\mathbb{R}^3}F(u^1)dx = -\frac{3\lambda}{p}\|u^1\|_p^p. \tag{3.8}$$

By (3.7), we have

$$\|\nabla u^1\|_p^p + B(u^1) - \int_{\mathbb{R}^3} f(u^1)u^1 dx = -\lambda\|u^1\|_p^p. \tag{3.9}$$

From (3.8) and (3.9), one can derive

$$\|\nabla u^1\|_p^p + \frac{1}{2p}B(u^1) - \frac{3}{p}\int_{\mathbb{R}^3}\tilde{F}(u^1)dx = 0, \tag{3.10}$$

that is, $P(u^1) = 0$. For every $n \in \mathbb{N}^+$, set $w_n^1 := u_n - u^1(x - z_n^1)$. Then $w_n(x + z_n^1) \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^3)$ and

$$a^p = \lim_{n\to\infty}\|w_n^1(\cdot + z_n^1) + u^1\|_p^p = \|u^1\|_p^p + \lim_{n\to\infty}\|w_n^1\|_p^p. \tag{3.11}$$

In view of Lemma 2.5, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}F(u_n(\cdot + z_n^1))dx = \int_{\mathbb{R}^3}F(u^1)dx + \lim_{n\to\infty}\int_{\mathbb{R}^3}F(w_n^1(\cdot + z_n^1))dx. \tag{3.12}$$

By [45], we obtain

$$\lim_{n\to\infty}B(u_n(\cdot + z_n^1)) = B(u^1) + \lim_{n\to\infty}B(w_n^1(\cdot + z_n^1)). \tag{3.13}$$

It follows from (3.11)–(3.13) that

$$
\begin{aligned}
c_a = \lim_{n\to\infty} J(u_n) &= \lim_{n\to\infty} J(u_n(\cdot + z_n^1)) \\
&= J(u^1) + \lim_{n\to\infty} J(w_n^1 u_n(\cdot + z_n^1)) \\
&= J(u^1) + \lim_{n\to\infty} J(w_n^1).
\end{aligned} \tag{3.14}
$$

We claim that $\lim_{n\to\infty} J(w_n^1) \geq 0$. To this claim, we assume by contradiction that, $\lim_{n\to\infty} J(w_n^1) < 0$. Therefore, $\{w_n^1\}$ is non-vanishing and, then up to a subsequence, there exists a sequence $\{z_n^2\} \subset \mathbb{R}^3$ such that

$$
\lim_{n\to\infty} \int_{B_1(z_n^2)} |w_n^1|^p dx > 0.
$$

Consequently $|z_n^2 - z_n^1| \to \infty$ in view of $w_n^1(\cdot + z_n^1) \to 0$ in $L^p_{loc}(\mathbb{R}^3)$ and, up to a subsequence, $w_n^1(\cdot + z_n^2) \rightharpoonup u^2$ in $W^{1,p}(\mathbb{R}^3)$ for some $u^2 \in W^{1,p}(\mathbb{R}^3)$. Notice that,

$$
u_n(\cdot + z_n^2) = w_n^1(\cdot + z_n^2) + u^1(\cdot + z_n^2 - z_n^1) \rightharpoonup u^2 \quad \text{in} \quad W^{1,p}(\mathbb{R}^3),
$$

by (3.5) and arguing as above, we deduce that $P(u^2) = 0$ and (2.13) that,

$$
\begin{aligned}
J(u^2) &= \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u^2|^p dx + \frac{1}{2p} B(u^2) - \int_{\mathbb{R}^3} F(u^2) dx \\
&= \frac{p-1}{2p^2} B(u^2) + \frac{3}{p^2}\int_{\mathbb{R}^3} \widetilde{F}(u^2) dx - \int_{\mathbb{R}^3} F(u^2) dx \\
&= \frac{p-1}{2p^2} B(u^2) + \frac{3}{p^2}\int_{\mathbb{R}^3} \left[f(u^2)u^2 - \frac{p(3+p)}{3} F(u^2) \right] dx \\
&> 0.
\end{aligned}
$$

Set $w_n^2 := w_n^1 - u^2(\cdot - z_n^2) = u_n - u^1(\cdot - z_n^1) - u^2(\cdot - z_n^2)$. It is clear that

$$
\lim_{n\to\infty} \|\nabla w_n^2\|_p^p = \lim_{n\to\infty} \|\nabla u_n\|_p^p - \sum_{i=1}^{2} \|\nabla u^i\|_p^p,
$$

and

$$
0 > \lim_{n\to\infty} J(w_n^1) = J(u^2) + \lim_{n\to\infty} J(w_n^2) > \lim_{n\to\infty} J(w_n^2).
$$

Proceeding this way successively, we obtain an infinite sequence $\{u^k\} \in S_m \backslash \{0\}$ such that $P(u^k) = 0$ and

$$
\sum_{i=1}^{k} \|\nabla u^i\|_p^p \leq \lim_{n\to\infty} \|\nabla u_n\|_p^p < +\infty
$$

for any $k \in \mathbb{N}$. However this is impossible since Lemma 2.4-(2) implies that there exists a $\delta > 0$ such that $\|\nabla u\|_p^p > \delta$ for any $u \in \mathcal{P}_m$ satisfying $P(u) = 0$. Therefore, the claim that $\lim_{n\to\infty} \|\nabla w_n^1\|_p^p \geq 0$ is proved. Thus, from (3.14) we have

$$
c_a \geq J(u^1). \tag{3.15}
$$

Let $m := \|u^1\|_p \in (0, a]$. Since $P(u^1) = 0$, we have $u^1 \in \mathcal{P}_m$. Then by Lemma 2.6 and (3.14), we get

$$
c_a \geq J(u^1) \geq c_m \geq c_a.
$$

Thus $c_a = J(u^1) = c_m$ and $\lim_{n\to\infty} J(w_n^1) = 0$. From Corollary 2.9, we obtain $\lambda \geq 0$.

We now prove $\lambda > 0$. From (f_1)-(f_2), for a given $\delta > 0$, there exists $C_\delta > 0$ such that

$$
|F(t)| \leq \delta |t|^{\bar{p}} + C_\delta |t|^q \quad \text{for } t \in \mathbb{R},
$$

where $q \in (p, p^*)$ is given in (f_1). In view of (3.10), together with the Gagliardo-Nirenberg inequality, there exist constants $C(\bar{p}) > 0$ and $C(q) > 0$ such that

$$\begin{aligned}&\|\nabla u^1\|_p^p - C(\bar{p})\delta\|\nabla u^1\|_p^p\|u^1\|_p^{\frac{p^2}{3}} - C(q)C_\delta\|\nabla u^1\|_p^{\frac{3q-3p}{p}}\|u^1\|_p^{\frac{pq-3q+3p}{p}}\\ &\le \|\nabla u^1\|_p^p - \frac{3}{p}\int_{\mathbb{R}^3}\tilde{F}(u^1)dx = -\frac{1}{2p}B(u^1) \le 0.\end{aligned}$$

Choosing $\delta > 0$ sufficiently small, we obtain

$$\tilde{C}_\delta\|\nabla u^1\|_p^p - C(p)C_\delta\|\nabla u^1\|_p^{\frac{3q-3p}{p}}\|u^1\|_p^{\frac{pq-3q+3p}{p}} \le 0,$$

which implies

$$\tilde{C}_\delta\|u^1\|_p^{\frac{pq-3q+3p}{p}} \ge \|\nabla u^1\|_p^{\frac{p^2-3q+3p}{p}},$$

namely

$$\|u^1\|_p^{\frac{pq-3q+3p}{p}}\|\nabla u^1\|_p^{\frac{-p^2+3q-3p}{p}} \ge \frac{1}{\widehat{C}_\delta}, \tag{3.16}$$

for some constant $\widehat{C}_\delta > 0$ Since $q \in (\bar{p}, p^*)$, (3.16) shows that if $\|u^1\|_p$ is sufficiently small, then $\|\nabla u^1\|_p$ must be large.

On the other hand, in view of (f_4), multiplying (3.8) by $\frac{\theta}{3}$ and subtracting (3.9), we obtain

$$\begin{aligned}-\frac{\theta-p}{p}\lambda\|u\|_p^p &= \frac{3\theta-p\theta-3p}{3p}\|\nabla u\|_p^p + \frac{5\theta-6p}{6p}B(u)\\ &\quad+\int_{\mathbb{R}^3}(f(u)u-\theta F(u))dx\\ &\le \frac{3\theta-p\theta-3p}{3p}\|\nabla u\|_p^p + \frac{5\theta-6p}{6p}B(u).\end{aligned} \tag{3.17}$$

Recall that $\theta \in (\bar{p}, p^*)$. It follows from (3.17), (2.6) and Young's inequality that

$$\begin{aligned}-\lambda\|u^1\|_p^p &\le \frac{3\theta-p\theta-3p}{3(\theta-p)}\|\nabla u^1\|_p^p + C_p\|\nabla u^1\|_p\|u^1\|_p^{2p-1}\\ &\le \frac{3(3\theta-p\theta-3p)}{2(\theta-p)}\|\nabla u^1\|_p^p + C_p\|u^1\|_p^{p^*}.\end{aligned} \tag{3.18}$$

Choose $a^* > 0$ sufficiently small. Then for any $a \in (0, a^*)$, we know $\|u^1\|_p = m \le a$ is also small and (3.16) ensures $\|\nabla u^1\|_p$ must be large. Since $\theta < p^*$, the right-hand side of (3.18) is negative, which implies $\lambda > 0$. Therefore, our claim holds.

Now we prove $m = a$. If $m < a$, using the fact that $\lambda > 0$, Lemma 2.8 implies $c_a < c_m$, which contradicts $c_a = c_m$. Therefore, $m = a$ and $u_n \to u^1$ in $L^p(\mathbb{R}^3)$. By Hölder's inequality, we have

$$u_n \to u^1 \text{ in } L^q(\mathbb{R}^3) \text{ for } q \in (p, p^*).$$

By Lemma 2.2 [37] we have

$$B(u_n) \to B(u^1), \tag{3.19}$$

and then using (2.6) and (f_1)-(f_2), we can infer to

$$\int_{\mathbb{R}^3}(f(u_n) - f(u^1))u^1 dx \to 0. \tag{3.20}$$

By Lemma 2.1-(iii), we get

$$\int_{\mathbb{R}^3} f(u_n)(u_n - u^1)dx \to 0. \tag{3.21}$$

(3.20) and (3.21) yield

$$\int_{\mathbb{R}^3} f(u_n)u_n dx \to \int_{\mathbb{R}^3} f(u^1)u^1 dx. \tag{3.22}$$

Then by (3.6), (3.7), (3.19) and (3.22), one deduces

$$\|u_n\|_p \to \|u^1\|_p \quad \text{and} \quad \|\nabla u_n\|_p \to \|\nabla u^1\|_p,$$

which imply $u_n \to u^1$ in $W^{1,p}(\mathbb{R}^3)$ by Brezis-Lieb Lemma, and so complete the proof. □

Proof of Theorem 1.2. Combining Lemmas 2.6, 2.7, 3.3 and 3.4, we can infer to the proof of Theorem 1.2.

3.2. Proof of Theorem 1.3. We now pay our attention to establishing the existence of infinitely many radial normalized solutions for (1.1) with function f being an odd function. First, we introduce some relevant notations and concepts.

Define the transformation $\sigma : W_r^{1,p}(\mathbb{R}^3) \to W_r^{1,p}(\mathbb{R}^3)$ by

$$\sigma(u) = -u.$$

Let $W \subset W_r^{1,p}(\mathbb{R}^3)$ be a subspace. A set $A \subset W$ satisfying $\sigma(A) = A$ is said to be σ-invariant. A homotopy $\phi : [0,1] \times A \to A$ is called σ-equivariant if it satisfies $\phi(t, \sigma(u)) = \sigma(\phi(t,u))$ for all $(t,u) \in [0,1] \times A$. According to Definition 7.1 in [23], we have

Definition 3.5. *Let $\mathcal{Y}$ be a metric space and $\mathcal{E} \subset \mathcal{Y}$ be a closed σ-invariant subset. Let $\mathcal{G}$ be a class of compact subsets of $\mathcal{Y}$. We say that $\mathcal{G}$ is a σ-homotopy stable family with closed boundary $\mathcal{E}$ if the following conditions are satisfied:*

(1) Every set in $\mathcal{G}$ is σ-invariant;
(2) Every set in $\mathcal{G}$ contains $\mathcal{E}$;
(3) For every $\mathcal{A} \in \mathcal{G}$ and every σ-equivariant homotopy $\chi \in C([0,1] \times \mathcal{Y}, \mathcal{Y})$ satisfying for all $s \in [0,1]$, $\chi(s,u) = \chi(s, \sigma(u))$,

$$\chi(s,z) = z \quad \text{for all } (s,z) \in (\{0\} \times \mathcal{Y}) \cup ([0,1] \times \mathcal{E}),$$

we have $\chi(\{1\} \times \mathcal{A}) \in \mathcal{G}$.

Since f is an odd function and by Lemma 2.3-(iv), the functional $\Psi = \tilde{I}|_{S(a)} : S(a) \to \mathbb{R}$ (see (3.3)) is even with respect to $u \in S(a)$. Consequently, Ψ is σ-invariant on $S(a)$. Following an approach analogous to Lemma 3.3, we establish the following result.

Lemma 3.6. *Assume that $\mathcal{F}$ is a σ-homotopy stable family of compact subsets of $S(a) \cap W_r$ (with $\mathcal{E} = \emptyset$). Define the minimax value*

$$c_{a,\mathcal{F}} := \inf_{\tilde{D} \in \mathcal{F}} \max_{u \in \tilde{D}} \Psi(u).$$

If $c_{a,\mathcal{F}} > 0$, then there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_a \cap W_r$ for $J|_{S(a)\cap W_r}$ at the energy level $c_{a,\mathcal{F}}$.

Lemma 3.6 guarantees the existence of Palais–Smale sequences with the desired property, consisting of elements in $\mathcal{P}_a$ for the constrained functional $J|_{S(a)\cap W_r}$. In what follows, we construct a sequence of σ-homotopy stable families of compact subsets of $S(a) \cap W_r$ (with $\mathcal{E} = \emptyset$). Let $\{e'_n\}_{n=1}^{\infty}$ be a Schauder basis of $W^{1,p}(\mathbb{R}^3)$ (e.g. [?]). Set

$$e_n = \int_{O(N)} e'_n(g(x)) d\mu_g,$$

where $O(N)$ denotes the orthogonal group on $\mathbb{R}^3$ and $d\mu_g$ is the Haar measure on $O(N)$. Then going if necessary to select one in identical elements, we see that $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of $W_r^{1,p}(\mathbb{R}^3)$. Without loss of generality, we assume that $\|e_n\| = 1$ for any $n \in \mathbb{N}$, and denote

$$L_k := \text{span}\{e_1, \cdots, e_n\}, \quad \text{and} \quad L_k^{\perp} := \overline{\text{span}\{ei : i \geq k+1\}}.$$

Clearly $W_r^{1,p}(\mathbb{R}^3) = L_k \oplus L_k^{\perp}$ for any $k \in \mathbb{N}$. We will employ genus theory to establish the existence of infinitely many solutions. The definition of genus is provided below.

Definition 3.7 ([44]). *For any nonempty closed σ-invariant set $A \subset W^{1,p}(\mathbb{R}^3)$, the genus of A is defined by*

$$\gamma(A) = \begin{cases} 0, \text{ if } A = \emptyset, \\ \inf\{n \in \mathbb{N}^+ : \text{there exists an odd continuous map } \phi : A \to \mathbb{R}^n \setminus \{0\}\}, \\ +\infty, \text{ if it does not exist odd, continuous } h : A \to \mathbb{R}^n \setminus \{0\}. \end{cases}$$

Define the collection

$$\Sigma_a := \{A \subset S(a) \cap W_r : A \text{ is compact and } \sigma\text{-invariant}\}.$$

For each $k \in \mathbb{N}^+$, let

$$\Sigma_{a,k} := \{A \in \Sigma_a : \gamma(A) \geq k\}.$$

We observe that $\Sigma_{a,k} \neq \emptyset$. Indeed, for any $k \in \mathbb{N}^+$, we have $S_{a,k} = S(a) \cap L_k \subset \Sigma_{a,k}$ and by Theorem 10.5 in [**?**], we get

$$\gamma(S_{a,k}) = k.$$

Since $k < k+1$, we directly obtain $\gamma(S_{a,k}) = k < k+1 = \gamma(S_{a,k+1})$, which means the genus is strictly increasing.

Define the minimax values by

$$\beta_{a,k} := \inf_{A \in \Sigma_{a,k}} \max_{u \in A} \Psi(u). \tag{3.23}$$

Since $\Sigma_{a,k+1} \subset \Sigma_{a,k}$ for each $k \in \mathbb{N}^+$, we obtain

$$\beta_{a,k} \leq \beta_{a,k+1}. \tag{3.24}$$

For any $u \in A \subset \Sigma_{a,k}$, Lemma 2.3 implies the existence of $s_u \in \mathbb{R}$ such that $\eta(u, s_u) \in \mathcal{P}_a$. Consequently,

$$\max_{u \in A} \Psi(u) = \max_{u \in A} \tilde{I}(u) = \max_{u \in A} J(\eta(u, s_u)) \geq \inf_{v \in \mathcal{P}_a} J(v) > 0,$$

which implies $\beta_{a,k} > 0$. The following lemma describes the asymptotic behavior of the sequence $\beta_{a,k}$.

Lemma 3.8. *For each $a > 0$, let $\beta_{a,k}$ be defined by (3.23). Then $\beta_{a,k} < \beta_{a,k+1}$.*

Proof. Suppose by contradiction that $\beta_{a,k} = \beta_{a,k+1}$. Then there exists $A \in A_{a,k+1}$ such that

$$\max_{u \in A} J(\eta(u, s_u)) = \beta_{a,k}.$$

However, by the monotonicity of genus, $\gamma(A) \geq k+1 > k = \gamma(S_{a,k})$, while $S_{a,k} \in A_{a,k}$ and $\max_{u \in S_{a,k}} J(\eta(u, s_u)) = \beta_{a,k}$. This contradicts the fact that the minimax energy of a set with higher genus should be greater than that of a set with lower genus. Therefore, $\beta_{a,k} < \beta_{a,k+1}$, which implies that the energy is strictly increasing. □

Lemma 3.9. *For each $a > 0$, if there exist solutions u_i and u_j such that $u_i = u_j$, then $J(u_i) \neq J(u_j)$.*

Proof. For any $i < j$, from the fact that the energy is strictly increasing, we have $\beta_{a,i} < \beta_{a,j}$. If there exist solutions u_i and u_j such that $u_i = u_j$, then $J(u_i) = J(u_j)$, that is, $\beta_{a,i} = \beta_{a,j}$, which contradicts $\beta_{a,i} < \beta_{a,j}$. As aresult, all solutions u_k are distinct. □

Lemma 3.10. *There exists $a_* > 0$ such that for any $a \in (0, a_*)$, if $\{u_n\} \subset \mathcal{P}_a \cap W_r$ is a Palais-Smale sequence for $J|_{S(a) \cap W_r}$ at level $c > 0$, then up to a subsequence, there exist $u \in W_r$ and $\lambda > 0$ satisfying $u_n \to u$ in W_r and*

$$-\Delta_p u + \lambda u^{p-1} + (|x|^{-1} * |u|^p) u^{p-1} = f(u).$$

Moreover, $J(u) = c$.

Proof. For each $a > 0$, let $\{u_n\} \subset \mathcal{P}_a \cap W_r$ be a Palais-Smale sequence for $J|_{S(a)\cap W_r}$ at level $c > 0$. Following an argument similar to Lemma 3.4, we can establish that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. By the compact embedding $W_r \hookrightarrow L^q(\mathbb{R}^3)$ for $q \in (p, p^*)$, passing to a subsequence, there exists $u \in W_r$ such that $u_n \rightharpoonup u$ in W_r, $u_n \to u$ in $L^q(\mathbb{R}^3)$ for $q \in (p, p^*)$, and $u_n \to u$ a.e. in $\mathbb{R}^3$. By the Lagrange multiplier principle, there exists $\{\lambda_n\} \subset \mathbb{R}$ satisfying

$$-\Delta_p u_n + \lambda_n u_n^{p-1} + (|x|^{-1} * |u_n|^p)u_n^{p-1} - f(u_n) \to 0 \quad \text{in } (W_r)^*, \tag{3.25}$$

which implies

$$\lambda_n = \frac{1}{a^p}\left(\int_{\mathbb{R}^3} f(u_n)u_n dx - \int_{\mathbb{R}^3} |\nabla u_n|^p dx - B(u_n)\right).$$

Since $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ is bounded, the sequence $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Passing to a subsequence, there exists $\lambda \in \mathbb{R}$ such that $\lambda_n \to \lambda$. Following a similar argument as in (3.7), we obtain

$$-\Delta_p u + \lambda u^{p-1} + (|x|^{-1} * |u|^p)u^{p-1} = f(u). \tag{3.26}$$

Repeating the methodology employed in Lemma 3.4, we can prove that $u \neq 0$ and there exists $a_k > 0$ such that for any $a \in (0, a_k)$, we have $\lambda > 0$.

We now establish that for any $a \in (0, a_k)$, $u_n \to u$ in W_r and $\|u\|_p = \|u_n\|_p = a$. In Lemma 3.4, this result was proved using the nonincreasing property of the mapping $a \to c_a$. However, this crucial property does not necessarily hold for $\beta_{a,k}$, which partly explains our restriction to the radial space W_r. By the convergence $u_n \to u$ in $L^q(\mathbb{R}^3)$ for $q \in (p, p^*)$, we have proved in [35] that,

$$B(u_n) \to B(u), \tag{3.27}$$

and

$$\int_{\mathbb{R}^3} (f(u_n) - f(u))u dx \to 0. \tag{3.28}$$

By Lemma 2.1-(iii), we obtain

$$\int_{\mathbb{R}^3} f(u_n)(u_n - u) dx \to 0. \tag{3.29}$$

In view of (3.28) and (3.29), it follows that

$$\int_{\mathbb{R}^3} f(u_n)u_n dx \to \int_{\mathbb{R}^3} f(u)u dx. \tag{3.30}$$

From (3.25), (3.27), and (3.30), we deduce

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_{\mathbb{R}^3} |\nabla u_n|^p dx + \lambda \int_{\mathbb{R}^3} |u_n|^p dx\right) \\
&= \lim_{n\to\infty}\left(\int_{\mathbb{R}^3} f(u_n)u_n dx - B(u_n)\right) \\
&= \int_{\mathbb{R}^3} f(u)u dx - B(u) \\
&= \int_{\mathbb{R}^3} |\nabla u|^p dx + \lambda \int_{\mathbb{R}^3} |u|^p dx.
\end{aligned}$$

Since $\lambda > 0$, we conclude

$$\|\nabla u_n\|_p \to \|\nabla u\|_p \quad \text{and} \quad \|u_n\|_p \to \|u\|_p = a,$$

which imply $u_n \to u$ in W_r by the Brezis-Lieb's lemma [51]. Moreover, we can infer that $J(u_n) \to J(u) = c$. □

Proof of Theorem 1.3 For any fixed $k \in \mathbb{N}^+$, we know that $A_{a,k} \neq \emptyset$ and $\beta_{a,k} < +\infty$. For any $a > 0$ and each $k \in \mathbb{N}^+$, Lemma 3.6 implies the existence of a Palais–Smale sequence $\{u_n^k\}_{n=1}^{\infty} \subset \mathcal{P}_a \cap W_r$ for $J|_{S(a)\cap W_r}$ at level $\beta_{a,k} > 0$. Then by Lemma 3.10, there exists $a_k > 0$ such that for any $a \in (0, a_k)$, there exist $u_k \in W_r$ and $\lambda_k > 0$ satisfying

$$-\Delta_p u_k + \lambda_k u_k^{p-1} + (|x|^{-1} * |u_k|^p) u_k^{p-1} = f(u_k), \tag{3.31}$$

and $J(u_k) = \beta_{a,k}$. Furthermore, by Lemma 3.8 ,we obtain that the minimax energy $\beta_{a,k}$ is strictly increasing. And by Lemma 3.9 ,we conclude that different k correspond to different solutions. By Palais' principle of symmetric criticality [42], the critical points of J in W_r are indeed critical points in the entire space $W^{1,p}(\mathbb{R}^3)$. Therefore, $u_k \in W_r$ is a solution to (1.1) for and $\lambda_k > 0$ at level $\beta_{a,k}$, and(1.1) admits infinitely many radially symmetric solutions, whose energy converges to infinity. Hence, Theorem 1.3 is proved. □

4. Proof of Theorem 1.4

In this section, we consider the case $\kappa > 0$ and address the Sobolev critical case, i.e., $f(t) = |t|^{p^*-2}t$. To prove Theorem 1.4, due to the presence of the critical Sobolev term, J is not bounded from below on $S(a)$. Inspired by [22], we employ a truncation technique to mitigate the effect of the critical term which enables us to construct a truncated functional that is bounded from below.

Recall that for any $u \in S(a)$, the constrained functional of (1.1) on $S(a)$ is defined by

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx - \frac{\kappa}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|} dxdy - \frac{1}{p^*}\int_{\mathbb{R}^3} |u|^{p^*} dx. \tag{4.1}$$

By (2.6) and (1.15), we obtain

$$\begin{aligned} J(u) &\geq \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx - C_p \kappa a^{2p-1} \left(\int_{\mathbb{R}^3} |\nabla u|^p dx\right)^{\frac{1}{p}} - \frac{1}{p^* S^{\frac{3}{3-p}}} \left(\int_{\mathbb{R}^3} |\nabla u|^p dx\right)^{\frac{3}{3-p}} \\ &=: g(\|\nabla u\|_p), \end{aligned} \tag{4.2}$$

where

$$g(t) := \frac{t^p}{p} - C_p \kappa a^{2p-1} t - \frac{t^{p^*}}{p^* S^{\frac{3}{3-p}}}.$$

Note that the function

$$h(t) := \frac{t^{p-1}}{p} - \frac{t^{p^*-1}}{p^* S^{\frac{3}{3-p}}}$$

attains a unique positive maximum at $t_0 > 0$ with $h(t_0) > 0$. Moreover, if $\kappa C_p a^{2p-1} < h(t_0) := \ell$, then the function g achieves its positive local maximum at t_0; and there exist $0 < R_1 < R_2$ such that $g(t) < 0$ for $0 < t < R_1$, $g(t) > 0$ for $t \in (R_1, R_2)$, and $g(t) < 0$ for $t \in (R_2, +\infty)$.

For the above $R_1, R_2 > 0$, define the cut-off function $\xi \in C^\infty(\mathbb{R}^+, [0,1])$ as

$$\xi(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq R_1, \\ 0, & \text{if } t \geq R_2. \end{cases}$$

We introduce the truncated functional as

$$J^T(u) := \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx - \frac{\kappa}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|} dxdy - \frac{\xi(\|\nabla u\|_p)}{p^*}\int_{\mathbb{R}^3} |u|^{p^*} dx, \tag{4.3}$$

and note that $J^T \in C^1(W^{1,p}(\mathbb{R}^3), \mathbb{R})$. By (4.3), we have

$$J^T(u) \geq g^T(\|\nabla u\|_p),$$

where

$$g^T(t) = \frac{t^p}{p} - \kappa C_p a^{2p-1} t - \frac{\xi(t)}{p^* S^{\frac{3}{3-p}}} t^{p^*}.$$

From the definition of ξ and if $\kappa C_p a^{2p-1} < \ell$, we observe that for $g^T(t) = g(t) < 0$ for $0 < t < R_1$ and $g^T(t) > 0$ for $t \in (R_1, +\infty)$, and when $t \in (R_2, +\infty)$, one has $g^T(t) = \frac{t^p}{p} - \kappa C_p a^{2p-1} t > 0$. In the sequel we always assume that

$$0 < \kappa a^{2p-1} < \alpha := \frac{\ell}{C_p}. \tag{4.4}$$

Without loss of generality, in the following discussion we can take that $R_1 > 0$ is small enough such that

$$\frac{t^p}{p} - \frac{t^{p^*}}{p^* S^{\frac{3}{3-p}}} \geq 0 \quad \text{for } t \in [0, R_1] \quad \text{and} \quad R_1 < S^{\frac{3}{p}}. \tag{4.5}$$

Remark 4.1. *From the above arguments, we see that, if $u_n \in S(a)$, and $\|\nabla u_n\|_p \to \infty$ as $n \to \infty$, we have*

$$\begin{aligned} J^T(u_n) &= \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u_n|^p dx - \frac{\kappa}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|} dxdy \\ &\geq \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u_n|^p dx - C_p \kappa a^{2p-1} \left(\int_{\mathbb{R}^3} |\nabla u_n|^p dx\right)^{\frac{1}{p}} \\ &\to +\infty, \end{aligned}$$

which implies that J^T is coercive on $S(a)$. Furthermore, if $J^T(u) \leq 0$, then $\|\nabla u\|_p \leq R_1$ and $J(u) = J^T(u)$.

Lemma 4.1. *Under the condition $\kappa a^{2p-1} < \alpha$, then the truncated functional $J^T|_{\mathcal{S}_r(a)}$ satisfies the $(PS)_d$ condition for any $d < 0$.*

Proof. Let $\{u_n\} \subset \mathcal{S}_r(a) := S(a) \cap W_r$ be a $(PS)_d$ sequence for J^T with $d < 0$. To establish the lemma, it suffices to prove that $\{u_n\}$ possesses a convergent subsequence in $S(a) \cap W_r$. By Remark 4.1, J^T is coercive on $S(a)$, hence $\{u_n\}$ is bounded in W_r. Moreover, for sufficiently large $n \in \mathbb{N}$, we must have $\|\nabla u_n\|_p \leq R_1$. By the definition of J^T, $\{u_n\}$ is also a bounded $(PS)_d$ sequence for J restricted to $\mathcal{S}_r(a)$, that is,

$$J(u_n) \to d \quad \text{and} \quad \|J'|_{\mathcal{S}_r(a)}(u_n)\| \to 0. \tag{4.6}$$

Passing to a subsequence, there exists $u \in W_r$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_r, \\ u_n \to u & \text{a.e. in } \mathbb{R}^3, \\ u_n \to u & \text{in } L^q(\mathbb{R}^3), \quad \text{for } q \in (p, p^*). \end{cases}$$

Similar to (3.5), we have

$$-\Delta_p u_n + \lambda_n u_n^{p-1} - \kappa(|x|^{-1} * |u_n|^p) u_n^{p-1} - |u_n|^{p^*-2} u_n \to 0 \quad \text{in } (W_r)^*, \tag{4.7}$$

and

$$\lambda_n = \frac{1}{a^p}\left(\int_{\mathbb{R}^3} |u_n|^{p^*} dx - \int_{\mathbb{R}^3} |\nabla u_n|^p dx + \kappa B(u_n)\right).$$

Since $\{u_n\}$ is bounded in W_r, it follows that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Passing to a subsequence, $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$. We claim $u \neq 0$. Indeed, if $u = 0$, then Lions' lemma implies

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} |u_n|^q dx = 0 \quad \text{for any } q \in (p, p^*).$$

Therefore, by (2.6), (1.15), we have

$$J(u_n) = J^T(u_n) \geq \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u_n|^p dx - C_p\kappa\|u_n\|_{\frac{6p}{5}}^{2p} - \frac{1}{p^* S^{\frac{3}{3-p}}}\left(\int_{\mathbb{R}^3} |\nabla u_n|^p dx\right)^{\frac{3}{3-p}}$$
$$\geq -C_p\kappa\|u_n\|_{\frac{6p}{5}}^{2p} \to 0,$$

which contradicts $J(u_n) \to d < 0$. Hence $u \neq 0$. By the weak convergence of $u_n \rightharpoonup u$ in W_r, similar to (3.7), can we infer that $u \in W_r$ solves the equation

$$-\Delta_p u + \lambda u^{p-1} - \kappa(|x|^{-1} * |u|^p)u^{p-1} = |u|^{p^*-2}u, \quad x \in \mathbb{R}^3. \tag{4.8}$$

By [45], u satisfies the Pohozaev-type identity

$$\frac{3-p}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{3\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{5\kappa}{2p}B(u) - \frac{3}{p^*}\int_{\mathbb{R}^3} |u|^{p^*} dx = 0. \tag{4.9}$$

Multiplying (4.8) by u and integrating, we obtain

$$\int_{\mathbb{R}^3} |\nabla u|^p dx + \lambda\int_{\mathbb{R}^3} |u|^p dx - \kappa B(u) - \int_{\mathbb{R}^3} |u|^{p^*} dx = 0. \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\lambda\int_{\mathbb{R}^3} |u|^p dx = \frac{(2p-1)\kappa}{2p}B(u),$$

Since $u \neq 0$ and $\kappa > 0$, we conclude $\lambda > 0$.

We now prove $u_n \to u$ in W_r. By the concentration-compactness principle [33, 34], we have

$$|\nabla u_n|^p \to \mu \geq |\nabla u|^p + \sum_{i\in\mathcal{J}} \mu_i\delta_{x_i}, \quad |u_n|^{p^*} \to \nu = |u|^{p^*} + \sum_{i\in\mathcal{J}} \nu_i\delta_{x_i}, \quad \sum_{i\in\mathcal{J}} \nu_i^{\frac{p}{p^*}} < +\infty,$$

where μ, ν, μ_i, ν_i are positive measures, $\mathcal{J}$ is an at most countable index set, $\{x_i\} \subset \mathbb{R}^3$ are the support of the singular part of μ, ν, and δ_{x_i} is the Dirac mass at x_i. Moreover, we have

$$S\nu_i^{\frac{p}{p^*}} \leq \mu_i, \quad \forall i \in \mathcal{J}. \tag{4.11}$$

We claim that $\mathcal{J}$ is either empty or a finite set. The proof proceeds as follows:

Since W_r embeds continuously into $L^{p^*}(\mathbb{R}^3)$ by the Sobolev embedding theorem, and $\{u_n\}$ is bounded in W_r, it follows that $\{u_n\}$ is also bounded in $L^{p^*}(\mathbb{R}^3)$. That is, there exists $M > 0$ such that for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^3} |u_n|^{p^*} dx \leq M.$$

The measure ν is the weak limit of $|u_n|^{p^*}$. For any measurable set $E \subset \mathbb{R}^3$, by the definition of weak convergence, we have

$$\int_E d\nu = \lim_n \int_E |u_n|^{p^*} dx \leq \limsup_n \int_E |u_n|^{p^*} dx \leq M,$$

and this shows that ν is a bounded measure. In particular, there can be at most finitely many singularities since ν is bounded, so $\mathcal{J}$ is either empty or a finite set.

If $\mathcal{J}$ is nonempty but finite, by choosing a truncated function $\psi_\varepsilon(x) = \widetilde{\psi}_\varepsilon(x - x_i)$, where $\widetilde{\psi}_\varepsilon \equiv 1$ in $B_\varepsilon(0), \widetilde{\psi}_\varepsilon \equiv 0$ in $B_{2\varepsilon}^c(0), |\nabla\widetilde{\psi}_\varepsilon| \leq \frac{2}{\varepsilon}$ and $\widetilde{\psi}_\varepsilon \in C_0^\infty(\mathbb{R}^3, [0, 1])$. Next, we complete the proof into three steps.

Step 1. $\mu_i \le \nu_i$ for any $i \in \mathcal{J}$. It is easy to check that the sequence $\{u_n\psi_\varepsilon\}$ is bounded in W_r, then from (4.7), we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varepsilon u_n dx = &\int_{\mathbb{R}^3}\left(-\lambda_n|u_n|^p - |\nabla u_n|^p + |u_n|^{p^*}\right)\psi_\varepsilon dx \\ &+ \kappa\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^p|u_n(y)|^p)\psi_\varepsilon(y)}{|x-y|}dxdy + o_n(1).\end{aligned} \tag{4.12}$$

Using the Hölder inequality, we can derive the following limit estimation:

$$\begin{aligned}&\limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varepsilon u_n dx\right| \\ &\le \limsup_{n\to\infty}\left(\int_{B_{2\varepsilon}(x_i)}|\nabla\psi_\varepsilon u_n|^p dx\right)^{\frac{1}{p}}\left(\int_{B_{2\varepsilon}(x_i)}|\nabla u_n|^p dx\right)^{\frac{p-1}{p}} \\ &\le C\left(\int_{B_{2\varepsilon}(x_i)}|\nabla\psi_\varepsilon u|^p dx\right)^{\frac{1}{p}} \\ &\le C_1\left(\int_{B_{2\varepsilon}(x_i)}|\nabla\psi_\varepsilon u|^{\frac{pp^*}{p^*-p}}dx\right)^{\frac{p^*-p}{pp^*}}\left(\int_{B_{2\varepsilon}(x_i)}|u|^{p^*}dx\right)^{\frac{1}{p^*}} \\ &= C_2\left(\int_{B_{2\varepsilon}(x_i)}|u|^{p^*}dx\right)^{\frac{1}{p^*}} \to 0\end{aligned} \tag{4.13}$$

as $\varepsilon \to 0$. From the Hardy-Littlewood-Sobolev inequality (2.4), we have

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^p|u_n(y)|^p)\psi_\varepsilon(y)}{|x-y|}dxdy &\le C_3\left(\int_{\mathbb{R}^3}|u_n|^{\frac{6p}{5}}dx\right)^{\frac{5}{6}}\left(\int_{\mathbb{R}^3}(|u_n|^p\psi_\varepsilon|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} \\ &\le C_3\|u_n\|_{6p/5}^p\left(\int_{\mathbb{R}^3}|u_n|^{\frac{6p}{5}}\psi_\varepsilon dx\right)^{\frac{5}{6}} \\ &\le C_4\left(\int_{B_{2\varepsilon}(x_i)}|u_n|^{\frac{6p}{5}}\psi_\varepsilon dx\right)^{\frac{5}{6}}.\end{aligned} \tag{4.14}$$

Taking the limit in the above inequality we get

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^p|u_n(y)|^p)\psi_\varepsilon(y)}{|x-y|}dxdy \\ &\le C_5\lim_{\varepsilon\to0}\lim_{n\to\infty}\left(\int_{B_{2\varepsilon}(x_i)}|u_n|^{\frac{6p}{5}}\psi_\varepsilon dx\right)^{\frac{5}{6}} \\ &= \lim_{\varepsilon\to0}\left(\int_{B_{2\varepsilon}(x_i)}|u|^{\frac{6p}{5}}\psi_\varepsilon dx\right)^{\frac{5}{6}} \\ &= 0.\end{aligned} \tag{4.15}$$

Moreover, since ψ_ε has compact support, there holds

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^p\psi_\varepsilon dx \ge \int_{\mathbb{R}^3}|\nabla u|^p\psi_\varepsilon dx + \langle\sum_{i\in\mathcal{J}}\mu_i\delta_{x_i}, \psi_\varepsilon\rangle, \tag{4.16}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{p^*}\psi_\varepsilon dx=\int_{\mathbb{R}^3}|u|^{p^*}\psi_\varepsilon dx+\langle\sum_{i\in\mathcal{J}}\nu_i\delta_{x_i},\psi_\varepsilon\rangle. \tag{4.17}$$

Then, we have from (4.12)-(4.17) that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varepsilon u_n dx\\
&\le -\int_{\mathbb{R}^3}|\nabla u|^p\psi_\varepsilon dx-\langle\sum_{i\in\mathcal{J}}\mu_i\delta_{x_i},\psi_\varepsilon\rangle\\
&\quad+\int_{\mathbb{R}^3}|u|^{p^*}\psi_\varepsilon dx+\langle\sum_{i\in\mathcal{J}}\nu_i\delta_{x_i},\psi_\varepsilon\rangle-\int_{\mathbb{R}^3}\lambda|u|^p\psi_\varepsilon dx.
\end{aligned}\tag{4.18}$$

Passing the limit as $\varepsilon\to 0$ in the last inequality, and combining (4.13) and (4.18) we have

$$\begin{aligned}
0&\le\lim_{\varepsilon\to0^+}\left(-\langle\sum_{i\in\mathcal{J}}\mu_i\delta_{x_i},\psi_\varepsilon\rangle+\langle\sum_{i\in\mathcal{J}}\nu_i\delta_{x_i},\psi_\varepsilon\rangle\right)\\
&=\lim_{\varepsilon\to0^+}\left(-\int_{\mathbb{R}^3}\mu_i\psi_\varepsilon\delta_{x_i}dx+\int_{\mathbb{R}^3}\nu_i\psi_\varepsilon\delta_{x_i}dx\right)\\
&=\lim_{\varepsilon\to0^+}\left(-\mu_i\widetilde{\psi}_\varepsilon(0)\int_{\mathbb{R}^3}\delta(x-x_i)dx+\nu_i\widetilde{\psi}_\varepsilon(0)\int_{\mathbb{R}^3}\delta(x-x_i)dx\right)\\
&=-\mu_i+\nu_i
\end{aligned}$$

that is $\mu_i\le\nu_i$. By (4.11), we infer to

$$\nu_i\ge S^{\frac{3}{p}}\quad\text{for any } i\in\mathcal{J}.$$

Step 2. $\mu_i=0$ for any $i\in\mathcal{J}$ and so $\mathcal{J}=\emptyset$. Suppose by contradiction that, there exists some $i\in\mathcal{J}$. Then using (4.14) again, we have $\mu_i\ge S^{\frac{3}{p}}$. Thus we arrive at

$$\begin{aligned}
R_1^p\ge\limsup_{n\to+\infty}\|\nabla u_n\|_p^p&\ge\limsup_{n\to+\infty}\int_{\mathbb{R}^3}|\nabla u_n|^p\psi_\varepsilon dx\\
&\ge\int_{\mathbb{R}^3}|\nabla u|^p\psi_\varepsilon dx+\langle\sum_{k\in\mathcal{J}}\mu_k\delta_{x_k},\psi_\varepsilon\rangle\\
&\ge\int_{\mathbb{R}^3}\mu_i\psi_\varepsilon\delta_{x_i}dx\\
&=\mu_i\ge S^{\frac{3}{p}},
\end{aligned}$$

which contradicts (4.5). Hence $\mathcal{J}=\emptyset$ and

$$u_n\to u\ \text{ in }\ L^{p^*}_{\mathrm{loc}}(\mathbb{R}^3). \tag{4.19}$$

Step 3.: There holds that $u_n\to u$ in W_r. Indeed, since $\{u_n\}\subset W_r$ is bounded, we have from [47] that

$$|u_n(x)|\le C\|u_n\||x|^{-\frac{2}{p}}\le C_1|x|^{-\frac{2}{p}},\quad\text{a.e. in }\mathbb{R}^3,\quad\forall\, n\in\mathbb{N}.$$

Consequently, one has, $\forall\, n\in\mathbb{N}$

$$|u_n(x)|^{p^*}\le\frac{C_2}{|x|^{\frac{6}{3-p}}},\quad\text{a.e. in }\mathbb{R}^3.$$

Noticing that

$$\frac{C_2}{|x|^{\frac{6}{3-p}}} \in L^1(\mathbb{R}^3 \setminus B_R(0)), \quad \text{and} \quad u_n(x) \to u(x) \quad \text{a. e. in} \quad \mathbb{R}^3 \setminus B_R(0),$$

we have from Lebesgue's dominated convergence theorem that

$$u_n \to u \text{ in } L^{p^*}(\mathbb{R}^3 \setminus B_R(0)). \tag{4.20}$$

Thus, together with (4.19), we derive by

$$u_n \to u \text{ in } L^{p^*}(\mathbb{R}^3). \tag{4.21}$$

It follows from (4.7) and (4.8) that

$$\lambda_n \|u_n\|_p^p + \|\nabla u_n\|_p^p = \kappa B(u_n) + \|u_n\|_{p^*}^{p^*} + o_n(1) \tag{4.22}$$

and

$$\lambda \|u\|_p^p + \|\nabla u\|_p^p = \kappa B(u) + \|u\|_{p^*}^{p^*}. \tag{4.23}$$

By Lemma 2.2 [37] we know

$$B(u_n) = B(u) + o_n(1). \tag{4.24}$$

From (4.22)-(4.24), we obtain

$$\lim_{n\to\infty} \left(\lambda_n \|u_n\|_p^p + \|\nabla u_n\|_p^p\right) = \lambda \|u\|_p^p + \|\nabla u\|_p^p.$$

Using $\lambda_n \to \lambda > 0$, we have

$$\lim_{n\to\infty} \|u_n\|_p = \|u\|_p \quad \text{and} \quad \lim_{n\to\infty} \|\nabla u_n\|_p = \|\nabla u\|_p,$$

which imply

$$u_n \to u \text{ in } W_r \quad \text{as} \quad n \to \infty,$$

this completes the proof. □

In the sequel we aim to obtain the multiplicity of normalized solution by the genus theory. First, for any $\varepsilon > 0$, define the set

$$C_\varepsilon := \{u \in W_r \cap S(a) : J^T(u) \le -\varepsilon\} \subset W_r. \tag{4.25}$$

which is a closed symmetric subset of $S_r(a)$, because J^T is even and continuous. For any $c \in \mathbb{R}$, set $J^{T,c} := \{u \in S(a) \cap W_r : \quad J^T(u) \le c\}$.

Lemma 4.2. *For each $n \in \mathbb{N}$, there exist $\varepsilon_n > 0$ and $\kappa > 0$ such that*

$$\gamma(C_\varepsilon) \ge n$$

for all $\varepsilon \in (0, \varepsilon_n]$.

Proof. For a given $n \in \mathbb{N}$, we can choose n radial functions $\{u_1, u_2, \cdots, u_n\} \subset C_0^\infty(\mathbb{R}^3)$ with the property: $\text{supp} u_i \cap \text{supp} u_j = \emptyset$ for $i \ne j$, and $\|u_j\|_p = a, \|\nabla u_i\|_p = \tau < R_1$ for $i = 1, 2, \cdots, n$. We consider an n-dimensional subspace $W_n = \text{span}\{u_1, u_2, \ldots, u_n\} \subset W_r$.

Define

$$G_n(s) := \{r_1\eta(u_1, s) + r_2\eta(u_2, s) + \cdots + r_n\eta(u_n, s) : \quad |r_1|^p + |r_2|^p + \cdots + |r_n|^p = 1\}$$

and

$$Y(s) = \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : |y_1|^p + |y_2|^p + \cdots + |y_n|^p = \tau^p e^{ps} + a^p\}.$$

Then there exists an odd homeomorphism between G_n and Y and by the properties of genus, we have

$$\gamma(G_n) = \gamma(Y) = n.$$

Let $u = r_1\eta(u_1, s) + r_2\eta(u_2, s) + \cdots + r_n\eta(u_n, s) \in G_n(s)$ with $s < 0$, there holds $\|\nabla u\|_p = e^s\tau < R_1$. At this point, we observe that

$$J^T(u) = J(u) = \frac{e^{ps}}{p}\tau^p - \frac{\kappa e^s}{2p}\tau^{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|w(x)/\tau|^p|w(y)/\tau|^p}{|x-y|}dxdy - \frac{e^{p^*s}}{p^*}\tau^{p^*}\int_{\mathbb{R}^3}|w/\tau|^{p^*}dx,$$

where $w = r_1u_1 + r_2u_2 + \cdots + r_nu_n$. We define

$$\alpha_n = \inf\left\{\|v\|_{p^*}^{p^*} : v \in W_n, \|\nabla v\|_p = 1\right\} > 0$$

and

$$\beta_n = \inf\{B(v) : v \in W_n, \|\nabla v\|_p = 1\} > 0.$$

Consequently, we have

$$J^T(u) \leq \frac{e^{ps}}{p}\tau^p - \frac{\kappa e^s\tau^{2p}}{2p}\beta_n - \frac{e^{p^*s}\tau^{p^*}}{p^*}\alpha_n.$$

Thus we can choose $\varepsilon_n > 0$ and $s_n < 0$ such that for $\varepsilon \in (0, \varepsilon_n]$ and any fixed $\kappa > 0$, there holds that

$$J^T(u) < -\varepsilon, \quad \text{for all} \quad u \in G_n(s_n),$$

which implies that $G_n(s_n) \subset C_\varepsilon$. Then using the properties of genus again, we obtain

$$\gamma(C_\varepsilon) \geq \gamma(G_n(s_n)) = n,$$

which completes the proof. □

For $j \in \mathbb{N}$, define the minimax level

$$d_j := \inf_{A\in\Sigma_j}\sup_{u\in A} J^T(u), \tag{4.26}$$

where

$$\Sigma_j = \{A \subset W_r \cap S(a) : A \text{ is closed, } A = -A \text{ and } \gamma(A) \geq j\}.$$

Since J^T is bounded from below on $S_r(a)$, we have $d_j > -\infty$. Let K_d denote the set of critical points of J^T at the level d:

$$K_d = \{u \in W_r \cap S(a) : (J^T)'(u) = 0, J^T(u) = d\}.$$

Lemma 4.3. *Assume that $d := d_k = d_{k+l} = \cdots = d_{k+r} < 0$ for some $k, r \in \mathbb{N}$, then $\gamma(K_d) \geq r + 1$.*

Proof. From Lemma 4.2, we have for any $k \in \mathbb{N}$, there exists $\varepsilon_k > 0$ such that

$$\gamma(C_\varepsilon) \geq k,$$

for all $\varepsilon \in (0, \varepsilon_k]$.

Notice that J^T is continuous and even, $C_{\varepsilon_k} \in \Sigma_k$ then, $d_k \leq -\varepsilon_k < 0, \forall k \in \mathbb{N}$. But J^T is bounded from below; hence, $d_k > -\infty, \forall k \in \mathbb{N}$.

Let us assume that $d := d_k = \cdots = d_{k+r}$. Since $d < 0$, by Lermma 4.1 we know that J^T satisfies the Palais-Smale condition in K_d, and it is easy to see that K_d is a compact set.

If $\gamma(K_d) < r$, there exists a closed and symmetric set $U, K_d \subset U$, such that $\gamma(U) < r$. In particular, we can choose $U \subset J^{T,0}$, since $d < 0$. By the classical deformation lemma [11], we have an odd homeomorphism $\eta \in C([0,1] \times S(a), S(a))$ such that

$$\eta(1, J^{T,d+\delta} \setminus U) \subset J^{T,d-\delta},$$

for some $\delta > 0$. At this point, we may choose $0 < \delta < -d$, since J^T satisfies the Palais-Smale condition on $J^{T,0}$, and we need $J^{T,d+\delta} \subset J^{T,0}$. By definition,

$$d = d_{k+r} = \inf_{A\in\Sigma_{k+r}}\sup_{u\in A} J^T(u).$$

Then, there exists $A \in \Sigma_{k+r}$ such that $\sup_{u\in A} J^T(u) < d + \delta$; i.e., $A \subset J^{T,d+\delta}$, and

$$\eta(1, A \setminus U) \subset \eta(1, J^{T,d+\delta} \setminus U) \subset J^{T,d-\delta}. \tag{4.27}$$

But $\gamma(\overline{A\setminus U}) > \gamma(A) - \gamma(U) \geq k$, and $\gamma(\eta(1, \overline{A\setminus U})) \geq \gamma(\overline{A\setminus U}) > \gamma(A) - \gamma(U) \geq k$, Then, $\eta(1, \overline{A\setminus U}) \in \Sigma_k$. But this contradicts (4.27); in fact, from $\eta(1, \overline{A\setminus U}) \in \Sigma_k$, we infer to

$$\sup_{u\in\eta(1,\overline{A\setminus U})} J^T(u) \geq d_k = d.$$

□

Proof of Theorem 1.4 For any $k \in \mathbb{N}$, from Lemma 4.2, there exist ε_k, such that $\gamma(C_{\varepsilon_k}) \geq k$, which implies $C_{\varepsilon_k} \in \Sigma_k$ and so $\Sigma_k \neq \emptyset$. Then a sequence of minimax values $-\infty < d_1 \leq d_2 \leq \cdots \leq \cdots$, can be defined as

$$d_k := \inf_{A\in\Sigma_k} \sup_{u\in A} J^T(u), \quad k \in \mathbb{N}$$

and by Theorem 2.1 [29], we have the following conclusions:

(i) d_k is a critical value of $J^T|_{S_r(a)}$ provided that $d_k < 0$.

(ii) If $d := d_k = d_{k+1} = \cdots = d_{k+r-1} < 0$ for some $k, r \geq 1$, then $\gamma(K_d) \geq r$, where K_d denotes the set of critical points of $J^T|_{S_r(a)}$ at the level d. In particular, $J^T|_{S_r(a)}$ admits infinitely many critical points at the level d if $r \geq 2$.

(iii) If $d_k < 0$ for all $k \geq 1$, then $d_k \to 0^-$ as $k \to \infty$.

Recall that J^T is bounded from below and, it follows from Lemma 4.1 that J^T satisfies the $(PS)_d$ condition for each $d < 0$. Thus, d_k is a critical value of J^T at the level d_k and $d_k \to 0^-$ as $k \to \infty$. By Remark 4.1, we have that $J^T(u) = J(u)$ in a small neighborhood of u provided that $J^T(u) < 0$, and so, these critical points of $J^T|_{S_r(a)}$ are indeed critical points of $J|_{S_r(a)}$, and the proof is complete. □

Conflict of interest. The authors declare that there is no conflict of interest.

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(M. Li) College of Science, Minzu University of China
Beijing 100081, China
Email address: sxlmr200169@163.com

(X. He) College of Science, Minzu University of China
Beijing 100081, China
Email address: xmhe923@muc.edu.cn

(M. Squassina) Dipartimento di Matematica e Fisica
Università Cattolica del Sacro Cuore
Via della Garzetta 48, 25133, Brescia Italy
Email address: marco.squassina@unicatt.it