MULTIPLE NORMALIZED SOLUTIONS FOR p-LAPLACIAN SCHRÖDINGER-POISSON SYSTEMS

MENGRU LI, XIAOMING HE, AND MARCO SQUASSINA

Abstract. In this paper, we investigates the existence and multiplicity of normalized solutions for the p-Laplacian Schrödinger-Poisson system:

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} - \kappa\left(|x|^{-1} * |u|^p\right) u^{p-1} = f(u) & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^p dx = a^p, \qquad u > 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $a > 0$ represents the prescribed L^p-norm, $\kappa \in \mathbb{R} \setminus \{0\}$ is a parameter, and $\lambda \in \mathbb{R}$ appears as an undetermined Lagrange multiplier. Our principal findings are summarized as follows: (i) For $\kappa < 0$, under the conditions that f is odd and satisfies L^p-supercritical yet Sobolev subcritical growth, we establish the existence of a normalized ground state solution for sufficiently small $a > 0$ by employing the Pohozaev manifold method combined with genus theory. In this setting, we prove that the problem admits infinitely many normalized solutions whose energies tend to infinity. The asymptotic behavior of the normalized ground state energy is also analyzed. (ii) For $\kappa > 0$ and the Sobolev critical nonlinearity $f(u) = |u|^{p^*-2}u$, we use a truncation technique together with the concentration-compactness principle to address the lack of lower boundedness of the energy functional. Under appropriate constraints on κ and a, we demonstrate the existence of infinitely many normalized solutions possessing negative energy. These results extend earlier work by some of the authors on p-Laplacian Schrödinger-Poisson systems.

Contents

1. Introduction and main results

1.1. **Overview.** In this paper, we focus on the existence and multiplicity of normalized solutions for the p-Laplacian Schrödinger-Poisson system:

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} - \kappa\left(|x|^{-1} * |u|^p\right) u^{p-1} = f(u) & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^p dx = a^p, \quad u > 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $a > 0$ is a prescribed mass, $1 < p < 3$, the parameter $\kappa \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}$ serves as a Lagrange multiplier. Here, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ represents the p-Laplacian operator. The convolution $|x|^{-1} * |u|^p$ is explicitly given by

$$(|x|^{-1} * |u|^p)(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|} dy, \quad x \in \mathbb{R}^3,$$

and the precise conditions on f will be given in the sequel.

Now let us consider the equation in (1.1), i.e.,

$$-\Delta_p u + \lambda u^{p-1} - \kappa\left(|x|^{-1} * |u|^p\right) u^{p-1} = f(u) \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

2020 *Mathematics Subject Classification.* 35A15, 35B33, 35J20, 35J60.
Key words and phrases. Schrödinger-Poisson systems; Normalized solutions; Genus theory; Pohozaev manifold.

When $p = 2$ and $\kappa = -1$ with the parameter $\lambda \in \mathbb{R}$ fixed, the fixed-frequency problem corresponding to Eq. (1.2) has attracted considerable attention. Following the pioneering work of Benci and Fortunato [8], a substantial body of literature has been devoted to studying the existence, nonexistence, and multiplicity of solutions to (1.2) and related equations—see, for example, [12, 36, 38] and the references therein—primarily by employing variational methods [37, 44]. More recently, for the case $1 < p < 3$ and $\kappa < 0$, problem (1.2) has been investigated by Du, Su, and Wang [15, 16] using variational approaches.

An alternative approach is to search for solutions to (1.2) under the prescribed mass constraint

$$\int_{\mathbb{R}^3} |u|^p dx = a^p > 0. \tag{1.3}$$

In this formulation, $\lambda \in \mathbb{R}$ emerges as an additional unknown parameter. The study of solutions with prescribed mass has long constituted a major direction of research in mathematical and physical contexts. From a physical standpoint, the fixed mass constraint characterized by parameter a holds particular significance, which has stimulated considerable recent interest in investigating solutions under such normalized conditions. Moreover, the p-Laplacian operator arises naturally in the context of nonlinear fluid mechanics, where the exponent p characterizes both the flow velocity and constitutive properties of the medium. The quasi-linear Schrödinger equation (1.1) originates from quantum mechanical models and semiconductor theory, describing the interaction of charged particles with electromagnetic fields. For further discussions on p-Laplacian equations without prescribed mass constraints, we refer to [14, 17, 31] and the references therein.

We note that, in the special case $p = 2$, system (1.1) reduces to the classical Schrödinger-Poisson equation, and the normalized solutions have been investigated by many authors in recent years. For instance, Bellazzini and Siciliano [7] considered the problem:

$$\begin{cases} -\Delta u + \lambda u + (|x|^{-1} * |u|^2)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2 dx = a^2, \end{cases} \tag{1.4}$$

where $q \in (2, 3)$, and established the existence of normalized solutions for sufficiently small $a > 0$. The case $q \in (3, \frac{10}{3})$ was considered in [6], where the authors showed that (1.4) admits normalized solutions provided $a > 0$ exceeds a certain threshold. Subsequently, Jeanjean and Luo [24] identified a threshold value of $a > 0$ that determines the existence and nonexistence of normalized solutions for (1.4). Bellazzini and Jeanjean [5] studied the existence of normalized solutions to (1.4) for $\frac{10}{3} < q < 6$, the authors established existence of normalized solutions for (1.4) under the assumption of sufficiently small mass $a > 0$ by using the Pohozaev manifold method. Recently, Jeanjean and Le [21] investigated the following Schrödinger-Poisson-Slater equation

$$-\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - b|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where $p \in (\frac{10}{3}, 6]$, $\gamma, b \in \mathbb{R}$, and $\|u\|_2^2 = c$ for prescribed $c > 0$. Through geometric analysis of the Pohozaev manifold, they derived existence and nonexistence results for various parameter configurations: (i) $\gamma < 0, b < 0$; (ii) $\gamma > 0, b > 0$; and (iii) $\gamma > 0, b < 0$. For more results of normalized solutions related to problem (1.4), we refer to [13, 33, 36, 43] and references therein.

We also recall some important advances concerning the normalized solutions to the Schrödinger equation after the famous paper [20], where Jeanjean investigated the normalized solutions of the mass supercritical problem

$$\begin{cases} -\Delta u + \mu u = f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^3} u^2 dx = a^2, \end{cases} \tag{1.6}$$

where $\mu \in \mathbb{R}$ appears as a Lagrange multiplier, by employing the mountain pass lemma and a skillful compactness argument. Recently, Jeanjean and Lu [22] revisited problem (1.6) under the assumption that f is continuous and satisfies weakened mass supercritical conditions, they established the existence of ground state solutions with the help of the Pohozaev manifold. Soave [40] studied the existence of normalized solutions to (1.6) with $f(u) = \lambda|u|^{q-2}u + |u|^{2^*-2}u$ for $q \in (2, 2^*)$, Sobolev critical growth, representing a counterpart to the Brezis-Nirenberg problem in the L^2-constraint framework. For further results on normalized solutions to Schrödinger-type equations, we refer to [1, 5–7, 11, 24, 39] and references therein.

Now, let us come back to consider problem (1.1). As far as we know, there are only few papers on the existence of normalized solutions to (1.1) in the literature. When the parameter $\kappa < 0$, Liu and He [28] recently studied normalized ground state solutions of (1.1) with the nonlinearity $f(u) = \mu|u|^{q-2}u, +|u|^{p^*-2}u, q \in (p, p^*)$, that is the following problem:

$$\begin{cases} -\Delta_p u - \kappa\phi|u|^{p-2}u = \lambda|u|^{p-2}u + \mu|u|^{q-2}u + |u|^{p^*-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^p, & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^p dx = a^p, \end{cases} \tag{1.7}$$

where $p^* = \frac{3p}{3-p}$ denotes the Sobolev critical exponent, and by means of Pohozaev manifold decomposition technique they established several existence results in the L^p-subcritical, L^p-critical and L^p-supercritical perturbation $\mu|u|^{q-2}u$, respectively. In [29], the authors investigated the existence and multiple solutions of system problem (1.7) when the parameters $\kappa < 0$ and $\mu > 0$ is large enough, applying the concentration-compactness principle and mountain pass theorem. In [30], the authors considered problem (1.7) under the *Sobolev subcritical* nonlinearity: $af(u)$, and derived several existence and non-existence results by distinguish the positive and negative signs of parameters $\gamma, a \in \mathbb{R}$, through variational methods.

1.2. **Main results.** In this paper we focus our attention on problem (1.1), with parameter $\kappa \in \mathbb{R}\backslash\{0\}$ having a wider range of values. It is well-known that seeking normalized solutions of (1.1) is equivalent to finding critical points of the functional I defined by

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx - \frac{\kappa}{2p}B(u) - \int_{\mathbb{R}^3} F(u)dx, \tag{1.8}$$

on the L^p-constraint manifold

$$S(a) = \{u \in W^{1,p}(\mathbb{R}^3) : \|u\|_p^p = a^p\}, \tag{1.9}$$

where

$$B(u) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|}dxdy.$$

For a given $a > 0$, we define the Pohozaev manifold associated with (1.1) as

$$\mathcal{P}_a := \left\{u \in S(a) \middle| P(u) = p\int_{\mathbb{R}^3} |\nabla u|^p dx - \frac{\kappa}{2}B(u) - 3\int_{\mathbb{R}^3} \tilde{F}(u)dx = 0\right\}, \tag{1.10}$$

where

$$\tilde{F}(t) = f(t)t - pF(t).$$

By the Pohozaev identity, every solution of (1.1) necessarily lies in $\mathcal{P}_a$.

We first consider the case $\kappa < 0$ and impose the following conditions on the nonlinearity f:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist constants $q \in (\bar{p}, p^*)$ and $C > 0$ such that

$$|f(t)| \leq C(1 + |t|^{q-1}) \quad \text{for all } t \in \mathbb{R};$$

(f_2) $\lim\limits_{t\to 0} \dfrac{f(t)}{|t|^{\bar{p}-1}} = 0$ and $\lim\limits_{t\to\infty} \dfrac{F(t)}{|t|^{\bar{p}}} = +\infty$, where $F(t) = \int_0^t f(s)ds$;

(f_3) The function $t \mapsto \dfrac{\tilde{F}(t)}{|t|^{\bar{p}}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$;

(f_4) There exists $\theta \in (\bar{p}, p^*)$ such that $f(t)t \leq \theta F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$.

One can easily check that $f(t) = |t|^{p-2}t$ for $p \in (\bar{p}, p^*)$ satisfies conditions (f_1)-(f_4). To state our main results, we first provide the definition of a normalized ground state solution on $\mathcal{P}_a$.

Definition 1.1. *A solution $u \in W^{1,p}(\mathbb{R}^3) \setminus \{0\}$ is said to be a normalized ground state solution on $\mathcal{P}_a$ of (1.1) if it verifies*

$$I'|_{\mathcal{P}_a}(u) = 0 \quad \text{and} \quad I(u) = \inf\{I(v) : I'|_{\mathcal{P}_a}(v) = 0,\, v \in \mathcal{P}_a\}. \tag{1.11}$$

We remark that condition (f_3) can enable the reduction of the normalized solution search to a minimization problem on the Pohozaev manifold $\mathcal{P}_a$, while condition (f_4) ensures the positivity of the Lagrange multiplier $\lambda > 0$, which plays a crucial role in establishing compactness properties, as we will show in the sequel. We shall verify that $\mathcal{P}_a \neq \emptyset$ constitutes a natural constraint and that the restricted functional $I|_{\mathcal{P}_a}$ is both bounded below and coercive, as proved in Lemmas 2.3 and 2.4 below. To this aim, it is natural to define the normalized ground state energy as

$$c_a := \inf_{u \in \mathcal{P}_a} I(u). \tag{1.12}$$

Our first main result concerning the monotonicity of ground state energy with respect to mass, which can be stated as follows.

Theorem 1.2. *Suppose that $\kappa < 0$ and (f_1)-(f_4) hold. Then there exists $a_k > 0$ small such that for any $a \in (0, a_k)$, (1.1) has a normalized solution $(u, \lambda) \in S(a) \times \mathbb{R}^+$ and u is a normalized ground state solution on $\mathcal{P}_a$. Moreover, the function $a \to c_a$ is positive, continuous, nonincreasing and*

$$\lim_{a \to 0^+} c_a = +\infty.$$

Our next result is concerned with the multiplicity of normalized solutions for (1.1).

Theorem 1.3. *Suppose that $\kappa < 0$, f is odd and satisfies (f_1)-(f_4). Then there exists $a^* > 0$ small such that for any $a \in (0, a^*)$, (1.1) has infinitely many radial solutions $\{u_k\}_{k=1}^{\infty} \subset S(a)$, with the characteristics*

$$I(u_{k+1}) \geq I(u_k) > 0, \quad \forall k \in \mathbb{N},$$

and $I(u_k) \to +\infty$ when $k \to \infty$.

Remark 1.1. We note that in [30], Liu and He proved the existence and multiplicity of (1.1) under the following conditions:

- (g_0) $f : \mathbb{R} \to \mathbb{R}$ is a continuous odd function; that is, $f(-t) = -f(t)$ holds for all $t \in \mathbb{R}$.
- (g_1) There exist positive constants $(\alpha, \beta) \in \mathbb{R}^2_+$ with $\bar{p} < \alpha \leq \beta < p^*$ such that for all $t \in \mathbb{R} \setminus \{0\}$, the following inequality holds:
$$0 < \alpha F(t) \leq f(t)t \leq \beta F(t).$$
- (g_2) Define the auxiliary function $\tilde{F}(t) := f(t)t - pF(t)$. Assume that $\tilde{F} \in C^1(\mathbb{R})$ and satisfies the strict inequality:
$$\bar{p}\tilde{F}(t) < \tilde{F}'(t)t \quad \text{for all } t \neq 0.$$

However, the main results of [30] require the higher differentiability of the function $\tilde{F}$. Our conditions $(f_1) - (f_4)$ are more weaker than $(g_0) - (g_2)$. To see this, we give the following:

Example 1.1. Let us consider the function:

$$f(t) := \left((p + \frac{p^2}{3}) \ln(1 + |t|^{\frac{p^2}{3}}) + \frac{\frac{p^2}{3}|t|^{\frac{p^2}{3}}}{1 + |t|^{\frac{p^2}{3}}} \right) |t|^{p + \frac{p^2}{3} - 2} t, \quad t \in \mathbb{R},$$

then we have the primitive function of $f(t)$ as:

$$F(t) = |t|^{p + \frac{p^2}{3}} \ln(1 + |t|^{\frac{p^2}{3}}), \quad t \in \mathbb{R}.$$

By a simple calculation, we have that f satisfies $(f_1) - (f_4)$, but does not satisfy the well-known Jeanjean's L^2-mass supercritical growth condition (g_1). Hence, our results improve the main results in [30].

We now turn to the case $\kappa > 0$ and the nonlinearity $f(t) = |t|^{p^*-2}t$ being Sobolev critical growth, and establish multiple solutions with negative energy.

Theorem 1.4. *Suppose that $\kappa > 0$ and $f(u) = |u|^{p^*-2}u$. Then (1.1) has an unbounded sequence of solutions $(u_j, \lambda_j) \in W^{1,p}(\mathbb{R}^3) \times \mathbb{R}^+$ with $\lambda_j > 0, I(u_j) < 0$ and $I(u_j) \to 0^-$ when $j \to \infty$.*

The proofs of Theorems 1.2-1.4 are constrained variational methods, and some comments are in orders:

(i) To prove Theorem 1.2, we shall construct a Palais-Smale sequence for $I|_{\mathcal{P}_a}$ at the energy level c_a that precisely satisfies $P(u_n) = 0$ for all $n \geq 1$, adapting techniques from [3, 4, 22], which can prove that the constructed Palais-Smale sequence admits a convergent subsequence, and in turn implies the existence of a normalized ground state solution on $\mathcal{P}_a$. We remark that, verifying the positivity of the Lagrange multiplier λ plays a crucial role in establishing compactness of the Palais-Smale sequence, and this verification follows directly from the Pohozaev identity in the p-Laplacian Schrödinger equation treated in [14, 17, 32], but for the nonlocal term in our setting, we need to adapt new methods to overcome this issue, with detailed arguments provided in Lemma 3.4.

(ii) In order to show Theorem 1.3, we make use of the radial subspace $W_r := \{u \in W^{1,p}(\mathbb{R}^3) : u(x) = u(|x|)\}$ to obtain the multiplicity of normalized solutions. By using genus theory, one can obtain an infinite sequence of minimax values β_k as in (3.23). For each level $\{\beta_{a,k}\}$, we construct an appropriate Palais-Smale sequence $\{u_{k,n}\}_{n=1}^{\infty} \subset \mathcal{P}_a \cap W_r$ for the constrained functional $I|_{S(a)\cap W_r}$. A key step is to prove the unboundedness of the sequence $\{\beta_{a,k}\}$, while in our situation, the presence of the nonlocal term $|x|^{-1} * |u|^p$ will bring more obstacles, we have to give more refined analytical arguments.

(iii) For the case $\kappa > 0$, and the nonlinearity $f(u) = |u|^{p^*-2}u$ is Sobolev critical growth, we shall prove the existence of infinitely many solutions with negative energy for (1.1). However, the presence of the Sobolev critical term makes the constrained functional $I|_{S(a)}$ is unbounded below. To overcome this obstacle, we implement a truncation technique introduced in [18], as defined in (4.3). We then prove that critical points of the truncated functional corresponding to negative critical values are also critical points of the original functional. Furthermore, to handle the Sobolev critical exponent, we employ the concentration-compactness principle due to Lions [26, 27], which play a key role in recovering the loss of compactness and proving Theorem 1.4.

Remark 1.2. In [28–30], the authors only studied the existence of normalized solutions of (1.1) with $\kappa < 0$, but in Theorem 1.4 we consider the case $\kappa > 0$ and complement the aforementioned studies. Theorem 1.4 also extends the study of [32] to the p-Laplacian Schrödinger equation with Sobolev critical exponent and a nonlocal perturbation term $\kappa\left(|x|^{-1} * |u|^p\right)u^{p-1}$. Our main results also extend the related studies in [5–7, 21, 24] to the more general p-Laplacian cases.

The remainder part of this paper is structured as follows. In Section 2 we give preliminary results and investigates fundamental properties of the normalized ground state energy mapping $a \mapsto c_a$. In Section 3 we prove Theorems 1.2 and 1.3. Finally, in Section 4 we apply the concentration-compactness principle and genus theory and complete the proof of Theorem 1.4.

Notation. Throughout this paper, we adopt the following conventions.

- For $p \in [1, \infty)$, $L^p(\mathbb{R}^3)$ denotes the usual Lebesgue space with norm
$$\|u\|_p = \Big(\int_{\mathbb{R}^3} |u|^p dx\Big)^{\frac{1}{p}}.$$
- $W^{1,p}(\mathbb{R}^3)$ is the standard Sobolev space equipped with the norm
$$\|u\| := \Big(\int_{\mathbb{R}^3} \big(|\nabla u|^p + |u|^p\big) dx\Big)^{\frac{1}{p}}.$$

- $W_r := \{u \in W^{1,p}(\mathbb{R}^3) : u(x) = u(|x|)\}$ denotes the subspace of radially symmetric functions, and $(W_r)^*$ stands for its topological dual.
- $D^{1,p}(\mathbb{R}^3)$ is the homogeneous Sobolev space

$$D^{1,p}(\mathbb{R}^3) = \{u \in L^{p^*}(\mathbb{R}^3) : \nabla u \in L^p(\mathbb{R}^3)\},$$

and S denotes the best Sobolev constant:

$$S = \inf_{u \in D^{1,p}(\mathbb{R}^3)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^3} |\nabla u|^p dx}{\left(\displaystyle\int_{\mathbb{R}^3} |u|^{p^*} dx\right)^{\frac{p}{p^*}}}. \tag{1.13}$$

- The mass-critical exponent and the Sobolev critical exponent are respectively

$$\bar{p} := p + \frac{p^2}{3}, \qquad p^* := \frac{3p}{3-p}.$$

- The letters $C, \widetilde{C}, C_i, c_i$ $(i = 1, 2, \dots)$ denote positive constants whose values may change from line to line.
- We write $\to$ and $\rightharpoonup$ for strong and weak convergence in the relevant function spaces.
- $o_n(1)$ stands for a quantity that tends to 0 as $n \to \infty$.

2. Preliminary results

In order to prove Theorems 1.2 and 1.3, we begin by presenting some useful preliminaries. In the following arguments, without loss of generality, we always assume $\kappa = -1$ for $\kappa < 0$. In the sequel, we assume (f_1)-(f_4) hold.

For each given $a > 0$, we denote by the set

$$M_a := \{u \in W^{1,p}(\mathbb{R}^3) : \|u\|_p \le a\}. \tag{2.1}$$

In the sequel, we shall search for critical points of the functional

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{2p}B(u) - \int_{\mathbb{R}^3} F(u)dx$$

on the Pohozaev manifold

$$\mathcal{P}_a = \left\{u \in S(a) : P(u) = p\int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{2}B(u) - 3\int_{\mathbb{R}^3} \tilde{F}(u)dx = 0\right\}. \tag{2.2}$$

In the following arguments of normalized solutions, we shall use some key inequalities. First, we recall the Gagliardo-Nirenberg inequality [34] of p-Laplacian type: for any $q \in (p, p^*)$ and $N \ge 2$,

$$\|u\|_q^q \le C(N, q)\|\nabla u\|_p^{q\gamma_q}\|u\|_p^{q(1-\gamma_q)}, \tag{2.3}$$

where the interpolation exponent is given by $\gamma_q = N\left(\frac{1}{p} - \frac{1}{q}\right)$.

Next, we introduce the Hardy-Littlewood-Sobolev inequality [25]: for functions $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$ with $0 < s < N$,

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^s} dxdy\right| \le C(N, s, p, q)\|f\|_p\|g\|_q, \tag{2.4}$$

under the scaling condition $p, q > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{s}{N} = 2$.

Lemma 2.1. *The following conclusions hold true:*

(i) For any $a > 0$, there exists $\sigma = \sigma(a)$ such that

$$\frac{1}{2p}\int_{\mathbb{R}^3} |\nabla u|^p dx \le I(u) \le C\left(\int_{\mathbb{R}^3} |\nabla u|^p dx + \left(\int_{\mathbb{R}^3} |\nabla u|^p dx\right)^{\frac{1}{p}}\right)$$

for all $u \in M_a$ with $\|\nabla u\|_p \le \sigma$.

(ii) If $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ is a bounded sequence, and $\lim_{n\to\infty} \|u_n\|_{\bar{p}} = 0$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} F(u_n)dx = \lim_{n\to\infty}\int_{\mathbb{R}^3} \tilde{F}(u_n)dx = 0.$$

(iii) If $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ and $\{v_n\} \subset W^{1,p}(\mathbb{R}^3)$ are bounded sequences, and $\lim_{n\to\infty} \|v_n\|_{\bar{p}} = 0$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f(u_n)v_n dx = 0.$$

Proof. (i) We first check that there exists a sufficiently small $\sigma = \sigma(a) > 0$ such that for every $u \in M_a$ satisfying $|\nabla u|_p \le \sigma$,

$$\int_{\mathbb{R}^3} |F(u)|dx \le \frac{1}{2p}\int_{\mathbb{R}^3} |\nabla u|^p dx. \tag{2.5}$$

In fact, it follows from (f_1)–(f_2) that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $t \in \mathbb{R}$,

$$|F(t)| \le \varepsilon|t|^{\bar{p}} + C_\varepsilon|t|^{p^*}.$$

Thus, for any $u \in M_a$, using the Gagliardo-Nirenberg inequality (2.3), we infer to

$$\begin{aligned}\int_{\mathbb{R}^3} |F(u)|dx &\le \varepsilon\int_{\mathbb{R}^3} |u|^{\bar{p}}dx + C_\varepsilon\int_{\mathbb{R}^3} |u|^{p^*}dx\\ &\le \varepsilon C_1 a^{\frac{p^2}{3}}\int_{\mathbb{R}^3} |\nabla u|^p dx + C_\varepsilon C_2\left(\int_{\mathbb{R}^3} |\nabla u|^p dx\right)^{\frac{p^*}{p}}\\ &= \left(\varepsilon C_1 a^{\frac{p^2}{3}} + C_\varepsilon C_2\left(\int_{\mathbb{R}^3} |\nabla u|^p dx\right)^{\frac{p^*-p}{p}}\right)\int_{\mathbb{R}^3} |\nabla u|^p dx.\end{aligned}$$

Taking $\varepsilon = \frac{1}{4pC_1 a^{\frac{p^2}{3}}}$ and $\sigma = \left(\frac{1}{4pC_\varepsilon C_2}\right)^{\frac{1}{p^*-p}}$, then (2.5) follows.

Taking into account of the Hardy-Littlewood-Sobolev inequality (2.4) and the Gagliardo-Nirenberg inequality (2.3) we can infer that

$$\begin{aligned}B(u) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|}dxdy\\ &\le \widetilde{C}_p\|u\|^{2p}_{\frac{6p}{5}}\\ &\le C_p\|\nabla u\|_p\|u\|_p^{2p-1},\end{aligned} \tag{2.6}$$

here $\widetilde{C}_p, C_p > 0$ are constants. Using (2.5) and (2.6), we can easily verify statement (i).

(ii) For any $\varepsilon > 0$, assumptions $(f_1) - (f_2)$ ensure the existence of a constant $C'_\varepsilon > 0$ such that

$$|\widetilde{F}(t)| \le \varepsilon|t|^{p^*} + C'_\varepsilon|t|^{\bar{p}}, \quad \forall t \in \mathbb{R}.$$

Then,

$$\int_{\mathbb{R}^3} |\widetilde{F}(u_n)|dx \le \varepsilon\|u_n\|^{p^*}_{p^*} + C'_\varepsilon\|u_n\|^{\bar{p}}_{\bar{p}}. \tag{2.7}$$

Since ε is arbitrary and $|u_n|_{\bar{p}} \to 0$, we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \widetilde{F}(u_n)dx = 0.$$

Similarly, we can conclude that,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} F(u_n)dx = 0, \quad \text{whenever } \|u_n\|_{\bar{p}} \to 0.$$

(iii) We first claim that there exists a constant $M > 0$, independent of $q \in [p, p^*]$, such that $|u|_q \leq M$. Indeed, it follows from (f_1)–(f_2) that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for all $t \in \mathbb{R}$,

$$|f(t)| \leq \varepsilon|t|^{\bar{p}-1} + C_\varepsilon|t|^{p^*-1}.$$

It follows that for all n,

$$|f(u_n)v_n| \leq \varepsilon|u_n|^{\bar{p}-1}|v_n| + C_\varepsilon|u_n|^{p^*-1}|v_n|. \tag{2.8}$$

Using the Hölder inequality, we have

$$\int_{\mathbb{R}^3} |u_n|^{\bar{p}-1}|v_n|dx \leq \left(\int_{\mathbb{R}^3} |u_n|^{\bar{p}}dx\right)^{\frac{\bar{p}-1}{\bar{p}}} \left(\int_{\mathbb{R}^3} |v_n|^{\bar{p}}dx\right)^{\frac{1}{\bar{p}}}.$$

Let $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ be a bounded sequence. Firstly, by the Gagliardo-Nirenberg inequality (2.3), we get

$$\int_{\mathbb{R}^3} |u_n|^{\bar{p}}dx \leq C\|\nabla u_n\|_p^p \|u_n\|_p^{\frac{p^2}{3}} \leq CM^{\bar{p}}.$$

Moreover, it follows that

$$\left(\int_{\mathbb{R}^3} |u_n|^{\bar{p}}dx\right)^{\frac{\bar{p}-1}{\bar{p}}} \leq C.$$

Finally, if $\lim\limits_{n\to\infty} \|v_n\|_{\bar{p}} = 0$, then clearly

$$\left(\int_{\mathbb{R}^3} |v_n|^{\bar{p}}dx\right)^{\frac{1}{\bar{p}}} \to 0.$$

Hence, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n|^{\bar{p}-1}|v_n|dx = 0. \tag{2.9}$$

Using the Hölder inequality, we obtain

$$\int_{\mathbb{R}^3} |u_n|^{p^*-1}|v_n|dx \leq \left(\int_{\mathbb{R}^3} |u_n|^{p^*}dx\right)^{\frac{p^*-1}{p^*}} \left(\int_{\mathbb{R}^3} |v_n|^{p^*}dx\right)^{\frac{1}{p^*}}.$$

Assume that $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ is a bounded sequence. By the Sobolev embedding theorem, we get

$$\int_{\mathbb{R}^3} |u_n|^{p^*}dx \leq CM^{p^*}.$$

Consequently, it follows that

$$\left(\int_{\mathbb{R}^3} |u_n|^{p^*}dx\right)^{\frac{p^*-1}{p^*}} \leq CM^{p^*-1}.$$

Moreover, by combining with the Hölder inequality:

$$\int_{\mathbb{R}^3} |v_n|^{p^*}dx = \int_{\mathbb{R}^3} |v_n|^{\bar{p}}|v_n|^{p^*-\bar{p}} \leq \|v_n\|_{\bar{p}}\|v_n\|_{p^*-2}^{p^*-\bar{p}} \to 0,$$

it follows that $\left(\int_{\mathbb{R}^3}|v_n|^{p^*}dx\right)^{\frac{1}{p^*}} \to 0$. So,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{p^*-1}|v_n|dx = 0. \tag{2.10}$$

Substituting (2.9) and (2.10) into the integral estimate in (2.8) yields that

$$0 \le \left|\int_{\mathbb{R}^3} f(u_n)v_n dx\right| \le \varepsilon\int_{\mathbb{R}^3}|u_n|^{\bar{p}-1}|v_n|dx + C_\varepsilon\int_{\mathbb{R}^3}|u_n|^{p^*-1}|v_n|dx.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f(u_n)v_n dx = 0.$$

□

Remark 2.1. *Analogous to (2.5), we can show that*

$$\int_{\mathbb{R}^3}|\tilde{F}(u)|dx \le \frac{1}{3}\int_{\mathbb{R}^3}|\nabla u|^p dx$$

for all $u \in M_a$ satisfying $\|\nabla u\|_p \le \sigma$, from which it follows that

$$P(u) = \int_{\mathbb{R}^3}|\nabla u|^p dx + \frac{1}{2p}B(u) - \frac{3}{p}\int_{\mathbb{R}^3}\tilde{F}(u)dx \ge \frac{p-1}{p}\int_{\mathbb{R}^3}|\nabla u|^p dx.$$

Remark 2.2. *Under the conditions (f_1), (f_2), and (f_3) on f, if we define a continuous function $k : \mathbb{R} \to \mathbb{R}$ as follows:*

$$k(t) := \begin{cases} \dfrac{f(t)t - pF(t)}{|t|^{p+p^2/3}}, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0. \end{cases}$$

Then, k is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$.

We recall the following conclusion from [28].

Proposition 2.1(Lemma 2.2 [28]) *Let $\{u_n\}$ be a sequence in $W^{1,p}(\mathbb{R}^3)$ with $u_n \rightharpoonup u$ weakly and $u_n \to u$ almost everywhere in $\mathbb{R}^3$. Denote by B' the Fréchet derivative of the functional B. Then, as $n \to \infty$, the following hold:*

(i) $B(u_n - u) = B(u_n) - B(u) + o_n(1)$,

(ii) $B'(u_n - u) = B'(u_n) - B'(u) + o_n(1)$ *in* $(W^{1,p}(\mathbb{R}^3))^*$.

Following the approach from [20], we can define an auxiliary functional associated with I via a continuous L^p-norm preserving map $\eta : E \to W^{1,p}(\mathbb{R}^3)$ as:

$$\eta(u,s)(x) := e^{\frac{3s}{p}}u(e^s x) \quad \text{for} \quad u \in W^{1,p}(\mathbb{R}^3), \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^3, \tag{2.11}$$

where $E := W^{1,p}(\mathbb{R}^3)\times\mathbb{R}$ is endowed with the norm $\|(u,s)\|_E = (\|u\|^p + |s|^p)^{\frac{1}{p}}$. A direct computation shows that $\|\eta(u,s)\|_p = \|u\|_p$, which implies $\eta(u,s) \in S(a)$.

We define the auxiliary functional associated with I by

$$\tilde{I}(u,s) := I(\eta(u,s)) = \frac{e^{ps}}{p}\|\nabla u\|_p^p + \frac{e^s}{2p}B(u) - \frac{1}{e^{3s}}\int_{\mathbb{R}^3}F(e^{\frac{3s}{p}}u)dx.$$

Clearly, $\tilde{I}$ belongs to $C^1(W^{1,p}(\mathbb{R}^3)\times\mathbb{R},\mathbb{R})$. The following lemma describes the geometric properties of $\tilde{I}$.

Lemma 2.2. *For every $u \in S(a)$, we have*

$$\tilde{I}(u,s) \to 0^+ \text{ as } s \to -\infty \quad \text{and} \quad \tilde{I}(u,s) \to -\infty \text{ as } s \to +\infty.$$

Proof. For every $u \in S(a)$, we see that $\|\nabla \eta(u,s)\|_p^p = e^{ps}\|\nabla u\|_p^p$. Moreover, by Lemma 2.1-(i), it follows that

$$\frac{1}{2p}e^{ps}\int_{\mathbb{R}^3}|\nabla u|^p dx \leq \tilde{I}(u) \leq C\left(e^{ps}\int_{\mathbb{R}^3}|\nabla u|^p dx + e^s\left(\int_{\mathbb{R}^3}|\nabla u|^p dx\right)^{\frac{1}{p}}\right),$$

which implies that $\tilde{I}(u,s) \to 0^+$ when $s \to -\infty$.

For each given $\mu \geq 0$, we define the function

$$h_\mu(t) := \begin{cases} \frac{F(t)}{|t|^{\bar{p}}} + \mu, & \text{if } t \neq 0, \\ \mu, & \text{if } t = 0. \end{cases} \tag{2.12}$$

By $(f_1) - (f_2)$, the function $h_\mu(t)$ is continuous and satisfies $h_\mu(t) \to +\infty$ as $t \to \infty$. Obviously, $F(t) = h_\mu(t)|t|^{\bar{p}} - \mu|t|^{\bar{p}}$. By a similar argument to Lemma 2.3 in [22] and using $(f_1) - (f_3)$, we can claim that for all $t \neq 0$,

$$f(t)t > \bar{p}F(t) > 0. \tag{2.13}$$

We prove (2.13) by splitting the proof into several steps.

Step 1. $F(t) > 0$ for any $t \neq 0$. Indeed, if $F(t_0) \leq 0$ for some $t_0 \neq 0$, by (f_1) and (f_3), the function $F(t)/|t|^{p+p^2/3}$ achieves its global minimum at some $s \neq 0$ such that $F(s) \leq 0$ and

$$\left(F(t)/|t|^{p+p^2/3}\right)'_{t=s} = \frac{f(s)s - (p+p^2/3)F(s)}{|s|^{p+1+p^2/3}sgn(s)} = 0.$$

In view of $f(t)t > pF(t)$ for any $t \neq 0$, and by Remark 2.2, we can infer to

$$0 < f(s)s - pF(s) = \frac{p^2}{3}F(s) \leq 0,$$

which leads to contradiction and complete the proof of Step 1.

Step 2. There exists a positive sequence $\{s_n^+\}$ and a negative sequence $\{s_n^-\}$ satisfying $|s_n^\pm| \to +\infty$ and $f(s_n^\pm)s_n^\pm > (p + p^3/3)F(s_n^\pm)$ for each $n \geq 1$.

We only deal with the positive case, since the negative case can be treated similarly. Assume by contradiction that, there exists $T_1 > 0$ small enough such that $f(t)t \leq (p+p^2/3)F(t)$ for any $t \in (0, T_1]$. From Step 1, we have

$$\frac{F(t)}{t^{p+p^2/3}} \geq \frac{F(T_1)}{T_1^{p+p^2/3}} > 0 \quad \text{for all} \quad t \in (0, T_1].$$

By virtue of $\lim_{t\to 0} F(t)/|t|^{p+p^2/3} = 0$ and (f_1), we can infer to a contradiction. So, we finish the proof of Step 2.

Step 3. We construct a positive sequence $\{\beta_n^+\}$ and a negative sequence $\{\beta_n^-\}$ so as to $|\beta_n^\pm| \to +\infty$ and $f(\beta_n^\pm)\beta_n^\pm > (p+p^2/3)F(\beta_n^\pm)$ for each $n \geq 1$.

The two cases can treated similarly, we only construct the existence of $\{\beta_n^-\}$. Suppose by contradiction that there exists $T_2 > 0$ such that $f(t)t \leq (p+p^2/3)F(t)$ for each $t \leq -T_2$. We obtain

$$\frac{F(t)}{t^{p+p^2/3}} \leq \frac{F(-T_2)}{T_2^{p+p^2/3}} < +\infty \quad \text{for all} \quad t \leq -T_2,$$

which yields a contradiction to (f_3). Hence, the sequence $\{\beta_n^-\}$ exists and we prove Step 3.

Step 4. $f(t)t \geq (p+p^2/3)F(t)$ for each $t \neq 0$. If not, then $f(t_0)t_0 < (p+p^2/3)F(t_0)$ for some $t_0 \neq 0$. Because the cases $t_0 < 0$ and $t_0 > 0$ can be treated in a similar manner, and so, we consider here that $t_0 < 0$. From Steps 2 and 3, there exists $T_{\min}, T_{\max} \in \mathbb{R}$ such that $T_{\min} < t_0 < T_{\max} < 0$, and

$$f(t)t < (p+p^2/3)F(t) \quad \text{for any} \quad t \in (T_{\min}, T_{\max}), \tag{2.14}$$

and

$$f(t)t = (p+p^2/3)F(t) \quad \text{when} \quad t = T_{\min}, T_{\max} \tag{2.15}$$

It follows from (2.14) that,

$$\frac{F(T_{max})}{|T_{max}|^{p+p^2/3}} > \frac{F(T_{min})}{|T_{min}|^{p+p^2/3}}. \tag{2.16}$$

But, by (2.15) and (f_4), we can infer to

$$\frac{F(T_{max})}{|T_{max}|^{p+p^2/3}} = \frac{3}{p^2}\frac{\widetilde{F}(T_{max})}{|T_{max}|^{p+p^2/3}} < \frac{3}{p^2}\frac{\widetilde{F}(T_{min})}{|T_{min}|^{p+p^2/3}} = \frac{F(T_{min})}{|T_{min}|^{p+p^2/3}}, \tag{2.17}$$

which yields a contradiction, and we complete the proof of Step 4.

Step 5. $f(t)t > (p + p^2/3)F(t)$ for each $t \neq 0$. From Step 4, the function $F(t)/|t|^{p+p^2/3}$ is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, \infty)$. Notice that, by (f_4) we see that $f(t)/|t|^{p-1+p^2/3}$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$. Then, for each $t \neq 0$, we can obtain

$$\begin{aligned}(p + p^2/3)F(t) &= (p + p^2/3)\int_0^t f(s)ds \\ &< (p + p^2/3)\frac{f(t)}{|t|^{p-1+p^2/3}}\int_0^t |s|^{p-1+p^2/3}ds = f(t)t\end{aligned}$$

and this proves Step 5. Thus, from Steps 1 and 5, the conclusion (2.13) follows.

By (2.13) with Fatou's lemma, we obtain that for every $u \in S(a)$,

$$\lim_{s\to+\infty}\int_{\mathbb{R}^3} h_\mu(e^{\frac{3}{p}s}u)|u|^{\bar{p}}dx = +\infty. \tag{2.18}$$

Now, from

$$\begin{aligned}\tilde{I}(u, s) &= \frac{e^{ps}}{p}\|\nabla u\|_p^p + \frac{e^s}{2p}B(u) + \mu e^{ps}\int_{\mathbb{R}^3}|u|^{\bar{p}}dx - e^{ps}\int_{\mathbb{R}^3} h_\mu(e^{\frac{3}{p}s}u)|u|^{\bar{p}}dx \\ &= e^{ps}\left(\frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2p}e^{(1-p)s}B(u) + \mu\int_{\mathbb{R}^3}|u|^{\bar{p}}dx - \int_{\mathbb{R}^3} h_\mu(e^{\frac{3}{p}s}u)|u|^{\bar{p}}dx\right).\end{aligned} \tag{2.19}$$

we derive from (2.18) that $\tilde{I}(u, s) \to -\infty$ as $s \to +\infty$. □

Lemma 2.3. *Let $u \in W^{1,p}(\mathbb{R}^3)$ be fixed, then the following limitations hold true.*

(i) There exists a unique $s_u \in \mathbb{R}$ such that

$$P(\eta(u, s_u)) = 0.$$

In particular, if $u \in S(a)$, then $\eta(u, s_u) \in \mathcal{P}_a$, with $\mathcal{P}_a$ defined in (1.10).

(ii) $\tilde{I}(u, s_u) > \tilde{I}(u, s)$ for all $s \neq s_u$, and moreover, $\tilde{I}(u, s_u) > 0$.

(iii) The map $u \to s_u$ is continuous in $u \in W^{1,p}(\mathbb{R}^3)$.

(iv) $s_{u(\cdot+z)} = s_u$ for any $z \in \mathbb{R}^3$. If f is odd, then $s_{-u} = s_u$.

Proof. (i) For fixed $u \in W^{1,p}(\mathbb{R}^3) \setminus \{0\}$, we get

$$\frac{d}{ds}\tilde{I}(u, s) = e^{ps}\|\nabla u\|_p^p + \frac{1}{2p}e^s B(u) - \frac{3}{p}e^{-3s}\int_{\mathbb{R}^3}\tilde{F}(e^{\frac{3s}{p}}u)dx = P(\eta(u, s)).$$

From Lemma 2.2, it follows that there exists $s_u \in \mathbb{R}$ at which $\tilde{I}(u, s)$ attains its global maximum. Moreover, we have

$$\frac{d}{ds}\tilde{I}(u, s_u) = P(\eta(u, s_u)) = 0.$$

In what follows, we prove the uniqueness of s_u.

By the definition of the function $k(t)$ given in Remark 2.2, we have $\tilde{F}(t) = k(t)|t|^{\bar{p}}$ for every $t \in \mathbb{R}$, and

$$P(\eta(u,s)) = e^{ps}\left(\|\nabla u\|^p + \frac{1}{2p}e^{(1-p)s}B(u) - \frac{3}{p}\int_{\mathbb{R}^3} k(e^{\frac{3s}{p}}u)|u|^{\bar{p}}dx\right).$$

For a fixed $t \in \mathbb{R}$, it follows from (f_3) that the function $s \mapsto k(e^{\frac{3s}{p}}t)$ is strictly increasing. Consequently, $P(\eta(u,s))$ is strictly decreasing in s, which implies the uniqueness of s_u.

(ii) This assertion follows easily by the strict concavity of $\tilde{I}(u,\cdot)$ established in part (i).

(iii) In view of part (i), the mapping $u \mapsto s(u)$ is well-defined. Let $u \in W^{1,p}(\mathbb{R}^3)\setminus\{0\}$ and consider an arbitrary sequence $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)\setminus\{0\}$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$. For each $n \geq 1$, put $s_n := s(u_n)$. We only need to verify that there exists a subsequence for which $s_n \to s(u)$ as $n \to \infty$.

Now, we show that the sequence $\{s_n\}$ is bounded. From the continuous coercive function h_0 defined in (2.12), we have that $h_0(t) \geq 0$ for all $t \in \mathbb{R}$ by (2.13). Suppose by contradiction, there holds that $s_n \to +\infty$, along a subsequence. and then, by Fatou's lemma and $u_n \to u \neq 0$ a. e. in $\mathbb{R}^3$, we infer as

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} h_0\left(e^{\frac{3s_n}{p}}u_n\right)|u_n|^{\bar{p}}dx = +\infty.$$

From part (ii) and equation (2.19) with $\mu = 0$, it follows that

$$0 \leq e^{-ps_n}\tilde{I}(u_n,s_n) = \frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2p}e^{(1-p)s_n}B(u_n) - \int_{\mathbb{R}^3} h_0\left(e^{\frac{3s_n}{p}}u_n\right)|u_n|^{\bar{p}}dx \to -\infty \tag{2.20}$$

This contradicts the non-negativity of the expression, thus showing that $\{s_n\}$ is bounded above. Moreover, by part (ii), we have

$$\tilde{I}(u_n,s_n) \geq \tilde{I}(u_n,s(u)) \qquad \forall\, n \in \mathbb{N}.$$

In view of $\eta(u_n,s(u)) \to \eta(u,s(u))$ in $W^{1,p}(\mathbb{R}^3)$, we conclude that

$$\tilde{I}(u_n,s(u)) = \tilde{I}(u,s(u)) + o_n(1)$$

and consequently,

$$\liminf_{n\to\infty}\tilde{I}(u_n,s_n) \geq \tilde{I}(u,s(u)) > 0. \tag{2.21}$$

Since $\{\eta(u_n,s_n)\} \subset M_a$ for sufficiently large $a > 0$, it follows from Lemma 2.1(i) and the fact that

$$\|\nabla(\eta(u_n,s_n))\|_p = e^{s_n}\|\nabla u_n\|_p,$$

we deduce from (2.21) that $\{s_n\}$ is also bounded from below. Thus, without loss of generality, we may suppose

$$s_n \to s_* \in \mathbb{R}.$$

As $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$, it follows that $\eta(u_n,s_n) \to \eta(u,s_*)$ in $W^{1,p}(\mathbb{R}^3)$. Moreover, by $P(\eta(u_n,s_n)) = 0$ for all $n \geq 1$, we conclude that $P(\eta(u,s_*)) = 0$. By Item (i), we have that $s_* = s(u)$ and so Item (iii) is verified.

(iv) For every $z \in \mathbb{R}^3$, a change of variables in the integrals gives that

$$P(\eta(u(\cdot+z),s(u))) = P(\eta(u,s(u))) = 0$$

and hence $s_{u(\cdot+z)} = s_u$ by part (i). If f is odd, then clearly

$$P(\eta(-u,s(u))) = P(-\eta(u,s(u))) = P(\eta(u,s(u))) = 0$$

and thus, $s_{-u} = s_u$. □

In what follows, we investigate several essential properties of the Pohožaev manifold

$$\mathcal{P}_a := \left\{u \in S(a) \mid P(u) = p\int_{\mathbb{R}^3}|\nabla u|^p\,dx + \frac{1}{2}B(u) - 3\int_{\mathbb{R}^3}\widetilde{F}(u)\,dx = 0\right\}.$$

Lemma 2.3 guarantees that $\mathcal{P}_a \neq \emptyset$. The next lemma collects its basic features.

Lemma 2.4. *Let $\mathcal{P}_a$ be defined as in (1.10). Then the following hold:*

(1) $\inf_{u\in\mathcal{P}_a} \|\nabla u\|_p > 0$.

(2) $\inf_{u\in\mathcal{P}_a} I(u) > 0$.

(3) I is coercive on $\mathcal{P}_a$: assume that $\{u_n\} \subset \mathcal{P}_a$ satisfies $\|u_n\| \to +\infty$, then $I(u_n) \to +\infty$.

Proof. (1) Suppose by contradiction that, there exists a sequence $\{u_n\} \subset \mathcal{P}_a$ such that $\|\nabla u_n\|_p \to 0$. Then Remark 2.1 implies that for sufficiently large n,

$$0 = P(u_n) \geq \frac{p-1}{p}\|\nabla u_n\|_p^p > 0,$$

which is impossible. Hence $\inf_{u\in\mathcal{P}_a} \|\nabla u\|_p > 0$.

(2) For each $u \in \mathcal{P}_a$, Lemma 2.3-(i),(ii) gives that

$$I(u) = \tilde{I}(u,0) \geq \tilde{I}(u,s) \qquad \text{for all } s \in \mathbb{R}.$$

Let $\tilde{s} := \ln\bigl(\sigma/\|\nabla u\|_p\bigr)$, where σ is the constant provided by Lemma 2.1-(i). Then $\|\nabla \eta(u,\tilde{s})\|_p = \sigma$, and using Lemma 2.1-(i) we obtain

$$I(u) \geq \tilde{I}(u,\tilde{s}) = I\bigl(\eta(u,\tilde{s})\bigr) \geq \frac{1}{2p}\|\nabla \eta(u,\tilde{s})\|_p^p = \frac{1}{2p}\sigma^p > 0.$$

Consequently, $\inf_{u\in\mathcal{P}_a} I(u) > 0$.

(3) Suppose by contradiction that, there exist $\hat{C} > 0$ and a sequence $\{v_n\} \subset \mathcal{P}_a$ with $\|v_n\| \to \infty$ such that

$$\sup_{n\geq 1} I(v_n) \leq \hat{C}. \tag{2.22}$$

For each $n \in \mathbb{N}$, let

$$s_n := \ln\bigl(\|\nabla v_n\|_p\bigr), \qquad w_n := \eta(v_n, -s_n).$$

Obviously, $s_n \to +\infty$. A direct computation shows that $\|w_n\|_p = a$ and $\|\nabla w_n\|_p = 1$. Denote by

$$\delta := \limsup_{n\to\infty}\Bigl(\sup_{z\in\mathbb{R}^3}\int_{B_1(z)} |w_n|^p\,dx\Bigr).$$

To obtain a contradiction, we distinguish the following two cases: non-vanishing and vanishing.

Case 1: Non-vanishing: that is $\delta > 0$. In this case, there exists $\{z_n\} \subset \mathbb{R}^3$ such that, setting $\tilde{w}_n := w_n(\cdot + z_n)$, we have $\tilde{w}_n \rightharpoonup w \neq 0$ in $W^{1,p}(\mathbb{R}^3)$ and $\tilde{w}_n \to w$ a.e. in $\mathbb{R}^3$. Let $h_\mu(t)$ be as in (2.12) with $\mu = 0$. Since $s_n \to +\infty$, (2.13) together with Fatou's lemma yields

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} h_0\bigl(e^{\frac{3}{p}s_n}\tilde{w}_n\bigr)\,|\tilde{w}_n|^{\bar{p}}\,dx = +\infty. \tag{2.23}$$

Using item (2), (2.19) with $\mu = 0$, and (2.23), we obtain

$$\begin{aligned}
0 &\leq e^{-ps_n} I(v_n) = e^{-ps_n} I\bigl(\eta(w_n, s_n)\bigr)\\
&= \frac{1}{p} + \frac{1}{2p}e^{(1-p)s_n}B(w_n) - \int_{\mathbb{R}^3} h_0\bigl(e^{\frac{3}{p}s_n}w_n\bigr)\,|w_n|^{\bar{p}}\,dx\\
&= \frac{1}{p} + \frac{1}{2p}e^{(1-p)s_n}B(\tilde{w}_n) - \int_{\mathbb{R}^3} h_0\bigl(e^{\frac{3}{p}s_n}\tilde{w}_n\bigr)\,|\tilde{w}_n|^{\bar{p}}\,dx \to -\infty,
\end{aligned}$$

a contradiction. Hence Case 1 cannot occur.

Case 2: Vanishing, that is $\delta = 0$. In this case, using Lions's lemma, we conclude by $w_n \to 0$ in $L^{\bar{p}}(\mathbb{R}^3)$ for every $\bar{p} \in (p, p^*)$. Consequently, from (2.6) and Lemma 2.1-(ii) we obtain, for any fixed $s \in \mathbb{R}$, as $n \to \infty$,

$$\frac{e^s}{2p}B(w_n) \to 0, \qquad e^{-3s}\int_{\mathbb{R}^3} F\bigl(e^{\frac{3}{p}s}w_n\bigr)\,dx \to 0. \tag{2.24}$$

Take $\hat{s} > \frac{\ln(2\hat{C})}{2}$ with $\hat{C}$ as in (2.22). Since $P(\eta(w_n, s_n)) = P(v_n) = 0$, Lemma 2.3-(i),(ii) together with (2.24) gives, for large n,

$$\begin{aligned} \hat{C} \geq I(v_n) &= I\big(\eta(w_n, s_n)\big) = \tilde{I}(w_n, s_n) \geq \tilde{I}(w_n, \hat{s}) \\ &= \frac{1}{p}e^{p\hat{s}} + \frac{e^{\hat{s}}}{2p}B(w_n) - e^{-3\hat{s}} \int_{\mathbb{R}^3} F\big(e^{\frac{3}{p}\hat{s}} w_n\big)\, dx \\ &= \frac{1}{p}e^{p\hat{s}} + o_n(1) > \hat{C}, \end{aligned}$$

again a contradiction. Thus Case 2 is also impossible. Therefore I is coercive on $\mathcal{P}_a$, and item (3) is proved. □

To understand the structure of Palais-Smale sequences, we employ a Brezis-Lieb type splitting lemma. The proof is standard and follows the same lines as Lemma 2.6 in [22].

Lemma 2.5. *Assume that $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ is a bounded sequence such that $u_n \to u$ a. e. in $\mathbb{R}^3$ for some $u \in W^{1,p}(\mathbb{R}^3)$, then there holds that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (F(u_n) - F(u_n - u) - F(u))\, dx = 0.$$

For each $a > 0$, we define the infimum of I on the Pohozaev manifold $\mathcal{P}_a$ as

$$c_a := \inf_{u \in \mathcal{P}_a} I(u). \tag{2.25}$$

Lemma 2.4-(2) guarantees that $c_a > 0$. The dependence of c_a on the parameter a is described in the following lemma, a result that will play a key role in dealing with the lack of compactness inherent to the problem.

Lemma 2.6. *For every $a > 0$, let c_a be defined as in (2.25). Then the following properties hold:*

(i) The function $a \mapsto c_a$ is continuous on $(0, \infty)$.
(ii) The function $a \mapsto c_a$ is nonincreasing on $(0, \infty)$.

Proof. We first establish the continuity of the mapping $a \mapsto c_a$. Let $\{a_n\} \subset (0, +\infty)$ satisfy $a_n \to a > 0$. It suffices to prove that

$$\lim_{n\to\infty} c_{a_n} = c_a. \tag{2.26}$$

Fix $u \in \mathcal{P}_a$ and define

$$u_n = \Big(\frac{a_n}{a}\Big)^{\frac{1}{p}} u \in S(a_n), \qquad n \in \mathbb{N}^+.$$

Clearly $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$. By Lemma 2.3-(ii),(iii) there exists $s_n \in \mathbb{R}$ such that $\eta(u_n, s_n) \in \mathcal{P}_{a_n}$ and $s_n \to 0$ as $n \to \infty$. Consequently, when $n \to \infty$, we get

$$\eta(u_n, s_n) \to \eta(u, 0) = u \quad \text{in } W^{1,p}(\mathbb{R}^3). \tag{2.27}$$

From (2.25) we obtain

$$\limsup_{m\to\infty} c_{a_n} \leq \limsup_{n\to\infty} I\big(\eta(u_n, s_n)\big) = I(u),$$

hence

$$\limsup_{n\to\infty} c_{a_n} \leq c_a. \tag{2.28}$$

On the other hand, for each $n \in \mathbb{N}^+$ we can choose $v_n \in \mathcal{P}_{a_n}$ with

$$I(v_n) \leq c_{a_n} + \frac{1}{n}. \tag{2.29}$$

Set $t_n := \big(\frac{a}{a_n}\big)^{\frac{2}{3}}$; then $t_n \to 1$ and

$$\tilde{v}_n := v_n(\cdot/t_n) \in S(a).$$

By Lemma 2.3-(i) there exists $s_{\tilde v_n}\in\mathbb{R}$ such that $\eta(\tilde v_n, s_{\tilde v_n})\in\mathcal{P}_a$. Using Lemma 2.3-(ii) together with (2.29) we obtain

$$\begin{aligned} c_a &\le I\big(\eta(\tilde v_n, s_{\tilde v_n})\big)=\tilde I(\tilde v_n, s_{\tilde v_n})\\ &\le \tilde I(v_n, s_{\tilde v_n})+\big|\tilde I(\tilde v_n, s_{\tilde v_n})-\tilde I(v_n, s_{\tilde v_n})\big|\\ &\le \tilde I(v_n, 0)+\big|\tilde I(\tilde v_n, s_{\tilde v_n})-\tilde I(v_n, s_{\tilde v_n})\big|\\ &\le c_{a_n}+\frac{1}{n}+\big|\tilde I(\tilde v_n, s_{\tilde v_n})-\tilde I(v_n, s_{\tilde v_n})\big|. \end{aligned}$$

Denote by

$$C(n):=\big|I\big(\eta(\tilde v_n, s_{\tilde v_n})\big)-I\big(\eta(v_n, s_{\tilde v_n})\big)\big|.$$

If we show

$$C(n)\to 0 \quad \text{as } n\to\infty, \tag{2.30}$$

then the previous inequality yields

$$c_a\le \liminf_{n\to\infty} c_{a_n},$$

which together with (2.28) gives $\lim_{n\to\infty} c_{a_n}=c_a$.

We now prove (2.30). Observing that $\eta(u(\cdot/t),s)=\eta(u,s)(\cdot/t)$, we have

$$\begin{aligned} C(n)&=\Big|\frac{1}{p}(t_n^{3-p}-1)\int_{\mathbb{R}^3}|\nabla\eta(v_n,s_{\tilde v_n})|^p\,dx+\frac{1}{2p}(t_n^5-1)B\big(\eta(v_n,s_{\tilde v_n})\big)\\ &\qquad-(t_n^3-1)\int_{\mathbb{R}^3}F\big(\eta(v_n,s_{\tilde v_n})\big)\,dx\Big|\\ &\le\frac{1}{p}|t_n^{3-p}-1|\int_{\mathbb{R}^3}|\nabla\eta(v_n,s_{\tilde v_n})|^p\,dx+\frac{1}{2p}|t_n^5-1|B\big(\eta(v_n,s_{\tilde v_n})\big)\\ &\qquad+|t_n^3-1|\int_{\mathbb{R}^3}\big|F\big(\eta(v_n,s_{\tilde v_n})\big)\big|\,dx\\ &:=\frac{1}{p}|t_n^{3-p}-1|A(n)+\frac{1}{2p}|t_n^5-1|B\big(\eta(v_n,s_{\tilde v_n})\big)+|t_n^3-1|D(n). \end{aligned}$$

Since $t_n\to 1$, it is enough to verify that the three quantities

$$\limsup_{n\to\infty}A(n),\ \limsup_{n\to\infty}B\big(\eta(v_n,s_{\tilde v_n})\big),\ \limsup_{n\to\infty}D(n) \tag{2.31}$$

are finite. This will be accomplished through three claims.

Claim 1. The sequence $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. To this aim, by (2.28) and (2.29), we have $\limsup_{n\to\infty} I(v_n)\le c_a$. Because $v_n\in\mathcal{P}_{a_n}$ and $a_n\to a$, an argument similar to that in Lemma 2.4-(3) shows that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$.

Claim 2. The sequence $\{\tilde v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$, and moreover, there exist a sequence $\{y_n\}\subset\mathbb{R}^3$ and $v\in W^{1,p}(\mathbb{R}^3)$ such that, along a subsequence, $\tilde v_n(\cdot+y_n)\to v$ a.e. in $\mathbb{R}^3$.

The boundedness of $\{\tilde v_n\}$ follows from Claim 1 and $t_n\to 1$. Define

$$\rho:=\limsup_{n\to\infty}\Big(\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|\tilde v_n|^p\,dx\Big).$$

If $\rho=0$, Lions' lemma [26] implies that $\tilde v_n\to 0$ in $L^{\bar p}(\mathbb{R}^3)$. Consequently,

$$\int_{\mathbb{R}^3}|v_n|^{\bar p}\,dx=\int_{\mathbb{R}^3}|\tilde v_n(t_n\,\cdot)|^{\bar p}\,dx=t_n^{-3}\int_{\mathbb{R}^3}|\tilde v_n|^{\bar p}\,dx\to 0.$$

Since $P(v_n)=0$, Lemma 2.1-(ii) yields

$$\int_{\mathbb{R}^3}|\nabla v_n|^p\,dx+\frac{1}{2p}B(v_n)=\frac{3}{p}\int_{\mathbb{R}^3}\tilde F(v_n)\,dx\to 0.$$

By Remark 2.1 we then obtain, for large n,

$$0=P(v_n)\geq \frac{1}{p}\int_{\mathbb{R}^3}|\nabla v_n|^p\,dx>0,$$

a contradiction. Hence $\rho>0$ and Claim 2 follows.

Claim 3. $\limsup_{n\to\infty} s_{\tilde v_n}<+\infty$. Assume, after passing to a subsequence, that

$$s_{\tilde v_n}\to+\infty\quad (n\to\infty). \tag{2.32}$$

By Claim 2 we may further assume (subsequence) that

$$\tilde v_n(\cdot+y_n)\to v\neq 0\quad \text{a.e. in } \mathbb{R}^3. \tag{2.33}$$

Lemma 2.3-(iv) together with (2.32) gives that

$$s_{\tilde v_n(\cdot+y_n)}=s_{\tilde v_n}\to+\infty. \tag{2.34}$$

Moreover, Lemma 2.3-(ii) implies

$$\tilde I\big(\tilde v_n(\cdot+y_n),s_{\tilde v_n(\cdot+y_n)}\big)\geq 0. \tag{2.35}$$

Combining (2.33), (2.34) and (2.35) and arguing exactly as in the derivation of (2.19) leads to a contradiction. Thus Claim 3 holds. From Claims 1 and 3 we conclude that

$$\begin{aligned}\limsup_{n\to\infty}\big\|\eta(v_n,s_{\tilde v_n})\big\| &=\limsup_{n\to\infty}\left\|e^{\frac{3s_{\tilde v_n}}{p}}v_n(e^{s_{\tilde v_n}}x)\right\|\\ &=\limsup_{n\to\infty}\left(\int_{\mathbb{R}^3}\left|\nabla e^{\frac{3s_{\tilde v_n}}{p}}v_n(e^{s_{\tilde v_n}}x)\right|^p+\left|e^{\frac{3s_{\tilde v_n}}{p}}v_n(e^{s_{\tilde v_n}}x)\right|^p dx\right)^{\frac1p}\\ &=\limsup_{n\to\infty}\left(\int_{\mathbb{R}^3}e^{ps_{\tilde v_n}}\,|\nabla v_n(x)|^p+|v_n(x)|^p\,dx\right)^{\frac1p}\\ &<+\infty.\end{aligned}$$

Consequently $\limsup_{n\to\infty}A(n)<+\infty$, and by (f_1)–(f_2) also $\limsup_{n\to\infty}D(n)<+\infty$. Using the HLS inequality (2.4) and GN inequality (2.3), we obtain

$$B\big(\eta(v_n,s_{\tilde v_n})\big)\leq C\big\|\eta(v_n,s_{\tilde v_n})\big\|_{\frac{6p}{5}}^{2p}\leq Ca\big\|\nabla\eta(v_n,s_{\tilde v_n})\big\|_p^{2p-1},$$

hence $\limsup_{n\to\infty}B\big(\eta(v_n,s_{\tilde v_n})\big)<+\infty$. Since $t_n\to1$, the three terms in the definition of $C(n)$ tend to zero; therefore $C(n)\to0$ as $n\to\infty$. This proves the continuity of $a\mapsto c_a$.

We now show that $a\mapsto c_a$ is nonincreasing on $(0,+\infty)$. It suffices to prove that for every $\varepsilon>0$ and every pair $a>a'>0$,

$$c_a\leq c_{a'}+\varepsilon.$$

By the definition of $c_{a'}$ there exists $v\in\mathcal{P}_{a'}$ with

$$I(v)\leq c_{a'}+\frac{\varepsilon}{2}. \tag{2.36}$$

Fix $\sigma>0$ and define $v_\sigma(x)=v(x)\,\zeta(\sigma x)$, where ζ is a radial function in $C_0^\infty(\mathbb{R}^3)$ satisfying

$$\zeta(x)=\begin{cases}1, & |x|\leq1,\\ \in(0,1), & 1<|x|<2,\\ 0, & |x|\geq2.\end{cases}$$

Clearly $v_\sigma\to v$ in $W^{1,p}(\mathbb{R}^3)$ as $\sigma\to0^+$. Then by Lemma 2.3-(ii1) we infer to

$$\eta(v_\sigma,s_{v_\sigma})\to\eta(v,0)=v\quad\text{in } W^{1,p}(\mathbb{R}^3). \tag{2.37}$$

Thus, we can can choose $\sigma>0$ sufficiently small, such that

$$I\big(\eta(v_\sigma,s_{v_\sigma})\big)\leq I(v)+\frac{\varepsilon}{4}. \tag{2.38}$$

Next, pick $u \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp}(u) \subset B_{1+4/\sigma}(0) \setminus B_{4/\sigma}(0)$ and set

$$\tilde{u} = \Big(\frac{a^p - \|v_\sigma\|_p^p}{\|u\|_p^p}\Big)^{\frac{1}{p}} u.$$

For any given $b \le 0$, we define $\omega_b = v_\sigma + \eta(\tilde{u}, b)$. By construction, one has

$$\operatorname{supp}(v_\sigma) \cap \operatorname{supp}(\eta(\tilde{u}, b)) = \emptyset,$$

and a direct computation shows $\omega_b \in S(a)$. Lemma 2.3-(i) yields $s_{\omega_b} \in \mathbb{R}$ such that $\eta(\omega_b, s_{\omega_b}) \in \mathcal{P}_a$. Moreover, an argument analogous to the one leading to (2.20) proves that $\{s_{\omega_b}\}$ is uniformly bounded in b. Consequently, as $b \to -\infty$,

$$s_{\omega_b} + b \to -\infty,$$

which implies that,

$$\eta(\tilde{u}, s_{\omega_b} + b) \to 0 \quad \text{in } L^{\bar{p}}(\mathbb{R}^3). \tag{2.39}$$

By Lemma 2.1-(ii) we obtain

$$\int_{\mathbb{R}^3} F\big(\eta(\tilde{u}, s_{\omega_b} + b)\big)\, dx \to 0. \tag{2.40}$$

Furthermore,

$$\|\nabla \eta(\tilde{u}, s_{\omega_b} + b)\|_p \to 0, \qquad \|\eta(\tilde{u}, s_{\omega_b} + b)\|_{\frac{6p}{5}} \to 0. \tag{2.41}$$

In view of (2.6) we also have

$$B\big(\eta(\tilde{u}, s_{\omega_b} + b)\big) \to 0. \tag{2.42}$$

Combining (2.40)–(2.42) yields

$$I\big(\eta(\tilde{u}, s_{\omega_b} + b)\big) \to 0. \tag{2.43}$$

Finally, using Lemma 2.3 together with (2.38) and (2.43) we obtain

$$\begin{aligned} c_a &\le I\big(\eta(\omega_b, s_{\omega_b})\big) \\ &= I\big(\eta(v_\sigma, s_{\omega_b})\big) + I\big(\eta(\eta(\tilde{u}, b), s_{\omega_b})\big) \\ &\le I\big(\eta(v_\sigma, s_{v_\sigma})\big) + I\big(\eta(\tilde{u}, s_{\omega_b} + b)\big) \\ &\le I(v) + \frac{\varepsilon}{2}. \end{aligned}$$

By (2.36) this gives $c_a \le c_{a'} + \varepsilon$, completing the proof of the lemma. □

Lemma 2.7. *Let c_a be defined as in (2.25). Then*

$$c_a \to +\infty \quad \textit{as} \quad a \to 0^+.$$

Proof. Consider a sequence $\{u_n\} \subset \mathcal{P}_{a_n}$ satisfying $\|u_n\|_p \to 0^+$, that is, $a_n \to 0^+$ as $n \to \infty$. It is sufficient to prove that

$$I(u_n) \to +\infty \ \text{ as } \ n \to \infty.$$

For each $n \in \mathbb{N}$, set

$$s_n := \ln(\|\nabla u_n\|_p) \quad \text{and} \quad w_n := \eta(u_n, -s_n).$$

Then $u_n = \eta(w_n, s_n) \in \mathcal{P}_{a_n}$, with $\|w_n\|_p = \|u_n\|_p \to 0$ and $\|\nabla w_n\|_p = 1$.

By Hölder's inequality, $w_n \to 0$ in both $L^{\bar{p}}(\mathbb{R}^3)$ and $L^{\frac{6p}{5}}(\mathbb{R}^3)$. Applying Lemma 2.1 and (2.6), we obtain for every $s \in \mathbb{R}$,

$$e^{-3s} \int_{\mathbb{R}^3} F\big(e^{\frac{3}{p}s} w_n\big)\, dx \to 0 \quad \text{and} \quad e^s B(w_n) \to 0 \quad \text{as } n \to \infty. \tag{2.44}$$

Since $P(u_n) = P(\eta(w_n, s_n)) = 0$, Lemma 2.3 together with (2.44) implies

$$\begin{aligned} I(u_n) &= I(\eta(w_n, s_n)) \geq I(\eta(w_n, s)) \\ &= \frac{1}{p}e^{ps} + \frac{1}{2p}e^s B(w_n) - e^{-3s}\int_{\mathbb{R}^3} F\big(e^{\frac{3}{p}s}w_n\big)\,dx \\ &= \frac{1}{p}e^{ps} + o_n(1). \end{aligned}$$

Since $s > 0$ can be chosen arbitrarily large, we conclude that $I(u_n) \to +\infty$ as $n \to \infty$, which yields the desired result. □

The next lemma provides a refined description of the behavior of the Lagrange multiplier λ and its relationship with the energy level c_a.

Lemma 2.8. *Let $(u, \lambda) \in S(a) \times \mathbb{R}$ be a solution of the problem*

$$-\Delta_p u + \lambda u^{p-1} + \big(|x|^{-1} * |u|^p\big)u^{p-1} = f(u) \quad \textit{in } \mathbb{R}^3,$$

with $I(u) = c_a$.

(i) *If $\lambda > 0$, then there exists $\delta > 0$ such that*

$$c_a > c_{a'} \quad \textit{for all } a' \in (a, a+\delta).$$

(ii) *If $\lambda < 0$, then there exists $\delta > 0$ such that*

$$c_a < c_{a'} \quad \textit{for all } a' \in (a, a+\delta).$$

Proof. Since (u, λ) satisfies the equation and $u \in \mathcal{P}_a$, we have the identity

$$I'(u)u = -\lambda\|u\|_p^p = -\lambda a^p. \tag{2.45}$$

For $t > 0$ and $s \in \mathbb{R}$, define the rescaled function

$$u_{t,s} := \eta(tu, s) \in S(ta),$$

and consider the two-parameter functional

$$K(t,s) := I(u_{t,s}) = \frac{1}{p}t^p e^{ps}\int_{\mathbb{R}^3} |\nabla u|^p\,dx + \frac{1}{2p}t^{2p}e^s B(u) - e^{-3s}\int_{\mathbb{R}^3} F\big(te^{\frac{3}{p}s}u\big)\,dx.$$

A direct computation gives

$$\begin{aligned} \frac{\partial K(t,s)}{\partial t} &= t^{p-1}e^{ps}\int_{\mathbb{R}^3} |\nabla u|^p\,dx + t^{2p-1}e^s B(u) - e^{-3s}\int_{\mathbb{R}^3} f\big(te^{\frac{3}{p}s}u\big)e^{\frac{3}{p}s}u\,dx \\ &= t^{-1}I'(u_{t,s})u_{t,s}. \end{aligned} \tag{2.46}$$

Moreover, we have the convergence

$$u_{t,s} \to u \quad \text{in } W^{1,p}(\mathbb{R}^3) \quad \text{as } (t,s) \to (1,0). \tag{2.47}$$

In the case $\lambda > 0$, by (2.45), we can obtain $I'(u)u = -\lambda a^p < 0$. By (2.46)-(2.47), there exists $\epsilon > 0$ such that

$$\frac{\partial K(t,s)}{\partial t} < 0 \quad \text{for all } (t,s) \in (1, 1+\epsilon] \times [-\epsilon, \epsilon].$$

Applying the mean value theorem, for any $t \in (1, 1+\epsilon]$ and $|s| \leq \epsilon$, there exists $\alpha \in (1, t)$ such that

$$K(t,s) = K(1,s) + (t-1)\frac{\partial K(\alpha, s)}{\partial t} < K(1,s).$$

By Lemma 2.3-(iii), we have $s_{tu} \to s_u = 0$ as $t \to 1^+$. Thus, for $a' > a$ sufficiently close to a, we set

$$t := \frac{a'}{a} \in (1, 1+\epsilon] \quad \text{and} \quad s := s_{tu} \in [-\epsilon, \epsilon].$$

Since $u_{t,s} \in S(ta) = S(a')$, we obtain

$$c_{a'} \le I(u_{t,s}) = K(t,s) < K(1,s) = I(\eta(u,s)) \le I(u) = c_a,$$

which proves $c_{a'} < c_a$.

In the case $\lambda < 0$, the argument is analogous to the case $\lambda > 0$ and we omit it for brevity. □

An immediate consequence of Lemmas 2.6 and 2.8 yields the following result.

Corollary 2.9. *Let $(u,\lambda) \in S(a) \times \mathbb{R}$ be a solution of the problem*

$$-\Delta_p u + \lambda u^{p-1} + \big(|x|^{-1} * |u|^p\big) u^{p-1} = f(u) \quad \text{in } \mathbb{R}^3,$$

and suppose that $I(u) = c_a$. Then $\lambda \ge 0$. Moreover, if $\lambda > 0$, we have

$$c_a > c_{a'} \qquad \text{if } \ a' > a.$$

3. Proof of Theorems 1.2–1.3

This section is devoted to the proof of Theorems 1.2 and 1.3. To achieve this, our first step is to construct a Palais–Smale sequence for the constrained functional $I|_{S(a)}$ at the energy level c_a. This sequence will be constructed to lie entirely within the Pohožaev manifold $\mathcal{P}_a$ and to possess a specific refined property.

3.1. Proof of Theorem 1.2. For any $a > 0$ and $u \in W^{1,p}(\mathbb{R}^3)$, we define an auxiliary functional $\tilde{J} : W^{1,p}(\mathbb{R}^3)\backslash\{0\} \to \mathbb{R}$ by

$$\tilde{J}(u) := \tilde{J}(u, s_u) = \frac{e^{ps_u}}{p}\|\nabla u\|_p^p + \frac{e^{s_u}}{2p}B(u) - \frac{1}{e^{3s_u}}\int_{\mathbb{R}^3} F(e^{\frac{3s_u}{p}}u)dx, \tag{3.1}$$

here, $s_u \in \mathbb{R}$ is provided by Lemma 2.3 and satisfies the condition $P(\eta(u,s_u)) = 0$. The next result can be established through a standard variational argument.

Lemma 3.1. *The functional $\tilde{J}$ is C^1-differentiable. Furthermore, for every $\psi \in C_0^\infty(\mathbb{R}^3)$, we have*

$$\begin{aligned}\tilde{J}'(u)\psi &= e^{ps_u}\int_{\mathbb{R}^3}|\nabla u|^{p-2}\nabla u\cdot\nabla\psi dx + e^{s_u}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^{p-2}u(y)\psi(y)}{|x-y|}dxdy\\ &\quad - e^{-3s_u}\int_{\mathbb{R}^3} f(e^{\frac{3}{p}s_u}u)e^{\frac{3}{p}s_u}\psi dx\\ &= I'(\eta(u,s_u))\eta(\psi,s_\psi).\end{aligned} \tag{3.2}$$

For a fixed $a > 0$, we define the restriction of $\tilde{J}$ to the sphere $S(a)$ by

$$\Psi := \tilde{J}|_{S(a)} : \ S(a) \to \mathbb{R}. \tag{3.3}$$

Clearly, $\Psi \in C^1(S(a),\mathbb{R})$ and it satisfies

$$\Psi'(u)\psi = \tilde{J}'(u)\psi = I'(\eta(u,s_u))\eta(\psi,s_\psi). \tag{3.4}$$

for any $u \in S(a)$ and $\psi \in T_uS(a)$, here we introduce the definition of tangent space at a point $u \in S(a)$ by

$$T_uS(a) := \left\{ v \in W^{1,p}(\mathbb{R}^3) : \int_{\mathbb{R}^3}|u|^{p-2}uvdx = 0\right\}.$$

Our aim is to construct a Palais-Smale sequence for the constrained functional $I|_{S(a)}$ at the energy level c_a, with the additional property that each term lies in $\mathcal{P}_a$.

For this purpose, we introduce some basic concepts and tools from [19] and [4] regarding constrained critical point theory and the construction of Palais–Smale sequences. These preliminaries will provide the necessary theoretical framework for our subsequent proofs.

Definition 3.2. *Let $\mathbb{M}$ be a metric space and let $\mathfrak{B} \subset \mathbb{M}$ be a closed subset. Let $\mathfrak{F}$ be a class of compact subsets of $\mathbb{M}$. We say that $\mathfrak{F}$ is a* homotopy stable family with closed boundary $\mathfrak{B}$ *if the following two conditions hold:*

(i) every set in $\mathfrak{F}$ contains $\mathfrak{B}$;
(ii) for every $B \in \mathfrak{F}$ and every continuous map $\varphi \in C([0,1] \times \mathbb{M}, \mathbb{M})$ satisfying

$$\varphi(t,x) = x \qquad \text{for all } (t,x) \in (\{0\} \times \mathbb{M}) \cup ([0,1] \times \mathfrak{B}),$$

one has $\varphi(\{1\} \times B) \in \mathfrak{F}$.

The case $\mathfrak{B} = \emptyset$ is allowed.

The following lemma guarantees the existence of a Palais–Smale sequence with the desired analytical properties, which is essential for applying variational methods in the proof of our main result.

Lemma 3.3. *Let $\mathfrak{F}$ be a homotopy stable family of compact subsets of $S(a)$ (with $\mathfrak{B} = \emptyset$) and define*

$$c_{a,\mathfrak{F}} := \inf_{D \in \mathfrak{F}} \max_{u \in D} \Psi(u).$$

If $c_{a,\mathfrak{F}} > 0$, then there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_a$ for $I|_{S(a)}$ at the level $c_{a,\mathfrak{F}}$. In particular, if $\mathfrak{F}$ consists of all singletons contained in $S(a)$, then $c_a = c_{a,\mathfrak{F}}$ and $\{u_n\}$ constitutes a Palais–Smale sequence for $I|_{S(a)}$ at the energy level c_a.

Proof. Let $\{A_n\} \subset \mathfrak{F}$ be a minimizing sequence for $I|_{S(a)}$ at level $c_{a,\mathfrak{F}}$. Define the continuous mapping

$$\mathcal{H} : [0,1] \times S(a) \longrightarrow S(a), \qquad \mathcal{H}(t,u) = \eta(u, ts_u),$$

whose continuity is guaranteed by Lemma 2.3-(iii). Observe that $\mathcal{H}(t,u) = u$ for every $(t,u) \in \{0\} \times S(a)$. By the homotopy-stability property of $\mathfrak{F}$ we obtain

$$D_n := \mathcal{H}(1, A_n) = \{\, \eta(u, s_u) \mid u \in A_n \,\} \in \mathfrak{F}.$$

Clearly $D_n \subset \mathcal{P}_a$ for all $n \in \mathbb{N}^+$. Since $\Psi(\eta(u,s_u)) = \Psi(u)$ for each $u \in A_n$, it follows that

$$\max_{u \in D_n} \Psi(u) = \max_{u \in A_n} \Psi(u) \longrightarrow c_{a,\mathfrak{F}},$$

so $\{D_n\} \subset \mathfrak{F}$ is also a minimizing sequence for $c_{a,\mathfrak{F}}$.

According to [30, Lemma 2.17], there exists a Palais–Smale sequence $\{v_n\} \subset W^{1,p}(\mathbb{R}^3)$ for Ψ on $S(a)$ at the level $c_{a,\mathfrak{F}}$. Consequently, as $n \to \infty$,

(i) $\Psi(v_n) \to c_{a,\mathfrak{F}}$;
(ii) $\operatorname{dist}(v_n, D_n) \to 0$;
(iii) $\|d\Psi(v_n)\|_{v_n,*} \to 0$, where $\|\cdot\|_{v_n,*}$ denotes the dual norm of $(T_{v_n}S(a))^*$.

We denote by

$$s_n := s_{v_n}, \qquad u_n := \eta(v_n, s_n) = \eta(v_n, s_{v_n}).$$

We shall verify that $\{u_n\} \subset \mathcal{P}_a$ is a Palais–Smale sequence for I at the same level $c_{a,\mathfrak{F}}$.

Claim 1. There exists a constant $C > 0$ such that $e^{-ps_n} \leq C$ for all $n \in \mathbb{N}^+$. Indeed, from the definition of s_n,

$$e^{-ps_n} = \frac{\|\nabla v_n\|_p^p}{\|\nabla u_n\|_p^p}.$$

Because $\{u_n\} \subset \mathcal{P}_a$, Lemma 2.4-(i) implies that $\{\|\nabla u_n\|_p\}$ is bounded below by a positive constant. Hence it suffices to show that $\sup_n \|\nabla v_n\|_p < \infty$.

For each $n \in \mathbb{N}$, $D_n \subset \mathcal{P}_a$, we have that

$$\max_{u \in D_n} I(u) = \max_{u \in D_n} \Psi(u) \to c_{a,\mathfrak{F}}.$$

Lemma 2.4-(iii) implies the uniform boundedness of $\{D_n\}$ in $W^{1,p}(\mathbb{R}^3)$. Since $\operatorname{dist}(v_n, D_n) \to 0$, we obtain $\sup_n \|\nabla v_n\|_p < \infty$, which proves the claim.

Claim 2. $\{u_n\}$ is a Palais-Smale sequence for I on $S(a)$. Since $\{u_n\} \subset \mathcal{P}_a$, we have $I(u_n) = \Psi(u_n) = \Psi(v_n) \to c_{a,\mathfrak{F}}$. It remains to estimate the constrained gradient of I at u_n.

For any $\psi \in T_{u_n}S(a)$, it follows from (2.11) that

$$\begin{aligned}\int_{\mathbb{R}^3} |v_n|^{p-2} v_n\, \eta(\psi, -s_n)\, dx &= \int_{\mathbb{R}^3} |v_n|^{p-2} v_n\, e^{-\frac{3s_n}{p}}\, \psi(e^{-s_n}x)\, dx \\ &= \int_{\mathbb{R}^3} \left|e^{\frac{3s_n}{p}}\, v_n(e^{s_n}z)\right|^{p-2} e^{\frac{3s_n}{p}}\, v_n(e^{s_n}z)\, \psi(z)\, dz \\ &= \int_{\mathbb{R}^3} |u_n(z)|^{p-2} u_n(z) \psi(z)\, dz = 0,\end{aligned}$$

whence $\eta(\psi, -s_n) \in T_{v_n}S(a)$. By Claim 1,

$$\|\eta(\psi, -s_n)\| \le \max\big\{C^{1/p}, 1\big\} \|\psi\|.$$

Using Lemma 3.1, we obtain

$$\begin{aligned}\|dI(u_n)\|_{u_n,*} &= \sup_{\substack{\psi \in T_{u_n}S(a) \\ \|\psi\| \le 1}} |dI(u_n)[\psi]| \\ &= \sup_{\substack{\psi \in T_{u_n}S(a) \\ \|\psi\| \le 1}} |dI(\eta(v_n, s_n))[\eta(\eta(\psi, -s_n), s_n)]| \\ &= \sup_{\substack{\psi \in T_{v_n}S(a) \\ \|\psi\| \le 1}} |d\Psi(v_n)[\eta(\psi, -s_n)]| \\ &\le \|d\Psi(v_n)\|_{v_n,*} \sup_{\substack{\psi \in T_{v_n}S(a) \\ \|\psi\| \le 1}} \|\eta(\psi, -s_n)\| \\ &\le \max\big\{C^{1/p}, 1\big\} \|d\Psi(v_n)\|_{v_n,*}.\end{aligned}$$

Since $\|d\Psi(v_n)\|_{v_n,*} \to 0$, we conclude $\|dI(u_n)\|_{u_n,*} \to 0$, proving Claim 2.

Conclusion. The collection of all singletons contained in $S(a)$ is a homotopy-stable family with empty boundary $\mathfrak{B} = \emptyset$. When f is odd, choosing $\mathfrak{F}$ to be this family, condition (f_1) together with Lemma 2.3-(iv) ensures that Ψ is even. We may then select a minimizing sequence $\{A_n\} \subset \mathfrak{F}$, which yields a corresponding minimizing sequence $\{D_n\} \subset \mathfrak{F}$. Repeating the argument above produces a Palais-Smale sequence $\{u_n\} \subset \mathcal{P}_a$ for $I|_{S(a)}$ at the level $c_{a,\mathfrak{F}}$.

Finally, we verify that $c_{a,\mathfrak{F}} = c_a$. By definition,

$$c_{a,\mathfrak{F}} = \inf_{D \in \mathfrak{F}} \max_{u \in D} \Psi(u) = \inf_{u \in S(a)} I(\eta(u, s_u)).$$

For any $u \in S(a)$, by $\eta(u, s_u) \in \mathcal{P}_a$, we have $I(\eta(u, s_u)) \ge c_a$, hence $c_{a,\mathfrak{F}} \ge c_a$. Conversely, for any $u \in \mathcal{P}_a$, $I(u) = I(\eta(u, 0)) \ge c_{a,\mathfrak{F}}$, so $c_{a,\mathfrak{F}} \le c_a$. Therefore $c_{a,\mathfrak{F}} = c_a$, which completes the proof of the lemma. □

Lemma 3.4. *There exists $a^* > 0$ such that for any $a \in (0, a^*)$, if $\{u_n\}$ is a Palais–Smale sequence at the level c_a, then up to a subsequence, there exist $u \in W^{1,p}(\mathbb{R}^3)$ and $\lambda \in \mathbb{R}$ satisfying $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$ and*

$$-\Delta_p u + \lambda u^{p-1} + (|x|^{-1} * |u|^p) u^{p-1} = f(u).$$

Proof. Let $\{u_n\} \subset \mathcal{P}_a$ be a Palais–Smale sequence for $I|_{S(a)}$ at the level c_a. Then, as $n \to \infty$,

$$I(u_n) \to c_a, \qquad I'(u_n)|_{S(a)} \to 0.$$

Since $\{u_n\} \subset \mathcal{P}_a$, Lemma 2.4-(iii) implies that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. By [10, Lemma 3], there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that, for any $\{z_n\} \subset \mathbb{R}^3$,

$$-\Delta_p u_n(\cdot + z_n) + \lambda_n u_n(\cdot + z_n)^{p-1} + (|x|^{-1} * |u_n(\cdot + z_n)|^p) u_n(\cdot + z_n)^{p-1} - f(u_n(\cdot + z_n)) \to 0 \tag{3.5}$$

in $(W^{1,p}(\mathbb{R}^3))^*$, where

$$\lambda_n := \frac{1}{a^p}\Big(\int_{\mathbb{R}^3} f(u_n)u_n\,dx - \int_{\mathbb{R}^3} |\nabla u_n|^p\,dx - B(u_n)\Big).$$

Owing to (f_1)–(f_2), inequality (2.6) and the Sobolev inequality, the sequence $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Passing to a subsequence, we may assume $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$. From (3.5) it then follows that

$$-\Delta_p u_n + \lambda u_n^{p-1} + (|x|^{-1} * |u_n|^p) u_n^{p-1} - f(u_n) \to 0 \quad \text{in } (W^{1,p}(\mathbb{R}^3))^*. \tag{3.6}$$

Define the limit

$$\delta := \limsup_{n\to\infty}\Big(\sup_{z\in\mathbb{R}^3}\int_{B_1(z)} |u_n|^p\,dx\Big).$$

We claim $\delta > 0$. Otherwise, if $\delta = 0$, then Lions's lemma yields $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for every $q \in (p, p^*)$. Lemma 2.1 together with (2.6) gives

$$\int_{\mathbb{R}^3} \tilde{F}(u_n)\,dx \to 0, \qquad B(u_n) \to 0.$$

Since $P(u_n) = 0$, we infer to

$$\int_{\mathbb{R}^3} |\nabla u_n|^p\,dx = -\frac{1}{2p}B(u_n) + \frac{3}{p}\int_{\mathbb{R}^3} \tilde{F}(u_n)\,dx \to 0.$$

From (f_1)–(f_3) we also obtain $\int_{\mathbb{R}^3} F(u_n)\,dx \to 0$, whence $I(u_n) \to 0$, contradicting $I(u_n) \to c_a > 0$.

Consequently, up to a subsequence there exist $\{z_n^1\} \subset \mathbb{R}^3$ and $u^1 \in W^{1,p}(\mathbb{R}^3) \setminus \{0\}$ such that

$$\begin{cases} u_n(\cdot + z_n^1) \rightharpoonup u^1 & \text{in } W^{1,p}(\mathbb{R}^3), \\ u_n(\cdot + z_n^1) \to u^1 & \text{in } L^q_{\text{loc}}(\mathbb{R}^3), \quad \forall\, q \in [p, p^*), \\ u_n(\cdot + z_n^1) \to u^1 & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Set $v_n := u_n(\cdot + z_n^1)$. Standard arguments show that, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} f(v_n)\varphi\,dx \longrightarrow \int_{\mathbb{R}^3} f(u^1)\varphi\,dx$$

and

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|v_n(x)|^p|v_n(y)|^{p-2}v_n(y)\varphi(y)}{|x-y|}\,dxdy \longrightarrow \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u^1(x)|^p|u^1(y)|^{p-2}u^1(y)\varphi(y)}{|x-y|}\,dxdy.$$

Thereby, by (3.6) we obtain

$$-\Delta_p u^1 + \lambda(u^1)^{p-1} + (|x|^{-1} * |u^1|^p)(u^1)^{p-1} = f(u^1). \tag{3.7}$$

Thus u^1 is a nontrivial solution of (3.7). Moreover, u^1 satisfies the Pohozaev-type identity

$$\frac{3-p}{p}\|\nabla u^1\|_p^p + \frac{5}{2p}B(u^1) - 3\int_{\mathbb{R}^3} F(u^1)\,dx = -\frac{3\lambda}{p}\|u^1\|_p^p. \tag{3.8}$$

Multiplying (3.7) by u^1 and integrating shows that

$$\|\nabla u^1\|_p^p + B(u^1) - \int_{\mathbb{R}^3} f(u^1)u^1\,dx = -\lambda\|u^1\|_p^p. \tag{3.9}$$

From (3.8) and (3.9), we can deduce by

$$\|\nabla u^1\|_p^p + \frac{1}{2p}B(u^1) - \frac{3}{p}\int_{\mathbb{R}^3} \tilde{F}(u^1)\,dx = 0, \tag{3.10}$$

i.e. $P(u^1) = 0$.

For each $n \in \mathbb{N}^+$, set $w_n^1 := u_n - u^1(\cdot - z_n^1)$. Then $w_n^1(\cdot + z_n^1) \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^3)$ and

$$a^p = \lim_{n\to\infty} \|w_n^1(\cdot + z_n^1) + u^1\|_p^p = \|u^1\|_p^p + \lim_{n\to\infty} \|w_n^1\|_p^p. \tag{3.11}$$

From Lemma 2.5 we drive that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n(\cdot + z_n^1))\,dx = \int_{\mathbb{R}^3} F(u^1)\,dx + \lim_{n\to\infty} \int_{\mathbb{R}^3} F(w_n^1(\cdot + z_n^1))\,dx. \tag{3.12}$$

Moreover, by Proposition 2.1, we have

$$\lim_{n\to\infty} B(u_n(\cdot + z_n^1)) = B(u^1) + \lim_{n\to\infty} B(w_n^1(\cdot + z_n^1)). \tag{3.13}$$

Combining (3.11)-(3.13), we obtain

$$\begin{aligned} c_a &= \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n(\cdot + z_n^1)) \\ &= I(u^1) + \lim_{n\to\infty} I(w_n^1(\cdot + z_n^1)) \\ &= I(u^1) + \lim_{n\to\infty} I(w_n^1). \end{aligned} \tag{3.14}$$

Next, we prove $\lim\limits_{n\to\infty} I(w_n^1) \geq 0$. In fact, if $\lim_{n\to\infty} I(w_n^1) < 0$, then $\{w_n^1\}$ is non-vanishing, and passing to a further subsequence, there exists $\{z_n^2\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty} \int_{B_1(z_n^2)} |w_n^1|^p\,dx > 0.$$

Because $w_n^1(\cdot + z_n^1) \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^3)$, we have $|z_n^2 - z_n^1| \to \infty$. Up to another subsequence, $w_n^1(\cdot + z_n^2) \rightharpoonup u^2$ in $W^{1,p}(\mathbb{R}^3)$ for some $u^2 \in W^{1,p}(\mathbb{R}^3)$. Notice that

$$u_n(\cdot + z_n^2) = w_n^1(\cdot + z_n^2) + u^1(\cdot + z_n^2 - z_n^1) \rightharpoonup u^2 \qquad \text{in } W^{1,p}(\mathbb{R}^3).$$

Arguing as before, from (3.5) we infer $P(u^2) = 0$. Moreover, using (2.13), we deduce to

$$\begin{aligned} I(u^2) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u^2|^p\,dx + \frac{1}{2p} B(u^2) - \int_{\mathbb{R}^3} F(u^2)\,dx \\ &= \frac{p-1}{2p^2} B(u^2) + \frac{3}{p^2} \int_{\mathbb{R}^3} \tilde{F}(u^2)\,dx - \int_{\mathbb{R}^3} F(u^2)\,dx \\ &= \frac{p-1}{2p^2} B(u^2) + \frac{3}{p^2} \int_{\mathbb{R}^3} \Big[f(u^2)u^2 - \frac{p(3+p)}{3} F(u^2)\Big]dx \\ &> 0. \end{aligned}$$

Define $w_n^2 := w_n^1 - u^2(\cdot - z_n^2) = u_n - u^1(\cdot - z_n^1) - u^2(\cdot - z_n^2)$. Thus, Brezis-Lieb's lemma and Proposition 2.1 imply that

$$\lim_{n\to\infty} \|\nabla w_n^2\|_p^p = \lim_{n\to\infty} \|\nabla u_n\|_p^p - \sum_{i=1}^{2} \|\nabla u^i\|_p^p,$$

and

$$0 > \lim_{n\to\infty} I(w_n^1) = I(u^2) + \lim_{n\to\infty} I(w_n^2) > \lim_{n\to\infty} I(w_n^2).$$

Proceeding inductively, we can obtain an infinite sequence $\{u^k\} \subset S(m) \setminus \{0\}$ with $P(u^k) = 0$ and

$$\sum_{i=1}^{k} \|\nabla u^i\|_p^p \leq \lim_{n\to\infty} \|\nabla u_n\|_p^p < +\infty \qquad \text{for every } k \in \mathbb{N}.$$

This is impossible because Lemma 2.4(2) implies the existence of a $\delta > 0$ such that $\|\nabla u\|_p^p > \delta$ for all $u \in \mathcal{P}_m$ with $P(u) = 0$. Hence the claim is proved, and (3.14) gives

$$c_a \geq I(u^1). \tag{3.15}$$

Set $m := \|u^1\|_p \in (0,a]$. Since $P(u^1)=0$, we have $u^1 \in \mathcal{P}_m$. Lemma 2.6 together with (3.14) yields

$$c_a \geq I(u^1) \geq c_m \geq c_a,$$

whence $c_a = I(u^1) = c_m$ and $\lim_{n\to\infty} I(w_n^1) = 0$. Corollary 2.9 then gives $\lambda \geq 0$.

In the sequel, we show the positivity of λ. To this aim, we note that, from (f_1)-(f_2), for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|F(t)| \leq \delta |t|^{\bar{p}} + C_\delta |t|^q, \quad \forall t \in \mathbb{R},$$

where $q \in (p, p^*)$ is the exponent appearing in (f_1). Using (3.10) and the Gagliardo–Nirenberg inequality(2.3), we can find constants $C(\bar{p}), C(q) > 0$ for which

$$\begin{aligned}
&\|\nabla u^1\|_p^p - C(\bar{p})\delta \|\nabla u^1\|_p^p \|u^1\|_p^{\frac{p^2}{3}} - C(q)C_\delta \|\nabla u^1\|_p^{\frac{3q-3p}{p}} \|u^1\|_p^{\frac{pq-3q+3p}{p}} \\
&\leq \|\nabla u^1\|_p^p - \frac{3}{p}\int_{\mathbb{R}^3} \tilde{F}(u^1)\,dx = -\frac{1}{2p}B(u^1) \leq 0.
\end{aligned}$$

Choosing $\delta > 0$ sufficiently small we obtain

$$\widetilde{C}_\delta \|\nabla u^1\|_p^p - C(p)C_\delta \|\nabla u^1\|_p^{\frac{3q-3p}{p}} \|u^1\|_p^{\frac{pq-3q+3p}{p}} \leq 0,$$

which implies

$$\widetilde{C}_\delta \|u^1\|_p^{\frac{pq-3q+3p}{p}} \geq \|\nabla u^1\|_p^{\frac{p^2-3q+3p}{p}},$$

i.e.

$$\|u^1\|_p^{\frac{pq-3q+3p}{p}} \|\nabla u^1\|_p^{\frac{-p^2+3q-3p}{p}} \geq \frac{1}{\widehat{C}_\delta} \tag{3.16}$$

for some constant $\widehat{C}_\delta > 0$. Because $q \in (\bar{p}, p^*)$, inequality (3.16) shows that if $\|u^1\|_p$ is small enough, then $\|\nabla u^1\|_p$ must be large.

On the other hand, multiplying (3.8) by $\theta/3$ and subtracting (3.9) gives, in view of (f_4),

$$\begin{aligned}
-\frac{\theta-p}{p}\lambda \|u^1\|_p^p &= \frac{3\theta - p\theta - 3p}{3p}\|\nabla u^1\|_p^p + \frac{5\theta-6p}{6p}B(u^1) \\
&\quad + \int_{\mathbb{R}^3} \big(f(u^1)u^1 - \theta F(u^1)\big)\,dx \\
&\leq \frac{3\theta - p\theta - 3p}{3p}\|\nabla u^1\|_p^p + \frac{5\theta-6p}{6p}B(u^1).
\end{aligned} \tag{3.17}$$

Recall that $\theta \in (\bar{p}, p^*)$. Combining (3.17) with (2.6) and Young's inequality yields

$$\begin{aligned}
-\lambda \|u^1\|_p^p &\leq \frac{3\theta - p\theta - 3p}{3(\theta-p)}\|\nabla u^1\|_p^p + C_1 \|\nabla u^1\|_p \|u^1\|_p^{2p-1} \\
&\leq \frac{\theta(3-p)-3p}{3(\theta-p)}\|\nabla u^1\|_p^p + C_1\left(\frac{3p-\theta(3-p)}{6(\theta-p)C_1}\|\nabla u^1\|_p^p + C_2\|u^1\|_p^{\frac{p(2p-1)}{p-1}}\right) \\
&= \frac{\theta(3-p)-3p}{6(\theta-p)}\|\nabla u^1\|_p^p + C_3 \|u^1\|_p^{\frac{p(2p-1)}{p-1}}.
\end{aligned} \tag{3.18}$$

Choose $a^* > 0$ sufficiently small. For any $a \in (0, a^*)$ we have $m = \|u^1\|_p \leq a$, hence $\|u^1\|_p$ is small; by (3.16) this forces $\|\nabla u^1\|_p$ to be large. Because $\theta < p^*$, the right-hand side of (3.18) is negative, and consequently $\lambda > 0$.

It remains to show that $m = a$. If $m < a$, the fact $\lambda > 0$ together with Lemma 2.8 would give $c_a < c_m$, contradicting $c_a = c_m$. Hence $m = a$, and therefore $u_n \to u^1$ in $L^p(\mathbb{R}^3)$. Using Hölder inequality, we see that $u_n \to u^1$ in $L^q(\mathbb{R}^3)$ for each $q \in (p, p^*)$. By Proposition 2.1 we get

$$B(u_n) \to B(u^1). \tag{3.19}$$

Applying (2.6) and (f_1)-(f_2), we also obtain

$$\int_{\mathbb{R}^3} \big(f(u_n) - f(u^1)\big) u^1 \, dx \to 0. \tag{3.20}$$

Lemma 2.1-(iii) implies that

$$\int_{\mathbb{R}^3} f(u_n)(u_n - u^1) \, dx \to 0. \tag{3.21}$$

From (3.20) and (3.21) we infer to

$$\int_{\mathbb{R}^3} f(u_n) u_n \, dx \longrightarrow \int_{\mathbb{R}^3} f(u^1) u^1 \, dx. \tag{3.22}$$

Finally, combining (3.6), (3.7), (3.19) and (3.22) we can deduce that

$$\|u_n\|_p \to \|u^1\|_p, \qquad \|\nabla u_n\|_p \to \|\nabla u^1\|_p,$$

and by the Brezis-Lieb lemma we have that $u_n \to u^1$ in $W^{1,p}(\mathbb{R}^3)$, completing the proof. □

Proof of Theorem 1.2. Combining Lemmas 2.6, 2.7, 3.3 and 3.4, we complete the proof of Theorem 1.2.

3.2. Proof of Theorem 1.3. We now turn to establishing the existence of infinitely many radial normalized solutions for (1.1) under the assumption that the nonlinearity f is odd. First, we introduce some relevant notation and concepts.

Define the transformation $\sigma : W_r^{1,p}(\mathbb{R}^3) \to W_r^{1,p}(\mathbb{R}^3)$ by

$$\sigma(u) = -u.$$

Given a subspace $W \subset W_r^{1,p}(\mathbb{R}^3)$, a set $A \subset W$ is called σ-invariant if $\sigma(A) = A$. A homotopy $\phi : [0,1] \times A \to A$ is said to be σ-equivariant if

$$\phi(t, \sigma(u)) = \sigma(\phi(t,u)), \quad \forall\, (t,u) \in [0,1] \times A.$$

From [19], we have the following

Definition 3.5. *([19]) Let $\mathbb{M}$ be a metric space and let $\mathfrak{B} \subset \mathbb{M}$ be a closed σ-invariant subset. A class $\mathfrak{F}$ of compact subsets of $\mathbb{M}$ is called a* σ-homotopy stable family with closed boundary $\mathfrak{B}$ *if the following hold:*

(1) Every $A \in \mathfrak{F}$ is σ-invariant;
(2) Every $A \in \mathfrak{F}$ contains $\mathfrak{B}$;
(3) For each $A \in \mathfrak{F}$ and every σ-equivariant homotopy $\varphi \in C([0,1] \times \mathbb{M}, \mathbb{M})$ such that

$$\varphi(s,u) = \varphi(s, \sigma(u)), \quad \forall\, s \in [0,1],\ u \in \mathbb{M},$$

and

$$\varphi(s,z) = z, \quad \forall\, (s,z) \in \big(\{0\} \times \mathbb{M}\big) \cup \big([0,1] \times \mathfrak{B}\big),$$

one has $\varphi\big(\{1\} \times A\big) \in \mathfrak{F}$.

Since f is an odd function and by Lemma 2.3-(iv), the functional $\Psi = \tilde{I}|_{S(a)} : S(a) \to \mathbb{R}$ (see (3.3)) is even with respect to $u \in S(a)$. Consequently, Ψ is σ-invariant on $S(a)$. Following an approach analogous to Lemma 3.3, we establish the following result.

Lemma 3.6. *Assume that $\mathscr{F}$ is a σ-homotopy stable family of compact subsets of $S(a) \cap W_r$ (with $\mathfrak{B} = \emptyset$). Set*

$$c_{a,\mathscr{F}} := \inf_{\tilde{D} \in \mathscr{F}} \max_{u \in \tilde{D}} \Psi(u).$$

If $c_{a,\mathscr{F}} > 0$, then there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_a \cap W_r$ for $I|_{S(a)\cap W_r}$ at the level $c_{a,\mathscr{F}}$.

Lemma 3.6 ensures the existence of Palais–Smale sequences, lying in $\mathcal{P}_a$, for the constrained functional $I|_{S(a)\cap W_r}$. In the sequel we construct a sequence of σ-homotopy stable families of compact subsets of $S(a)\cap W_r$ (with $\mathfrak{B}=\emptyset$).

Let $\{e_n'\}_{n=1}^{\infty}$ be a Schauder basis of $W^{1,p}(\mathbb{R}^3)$ (see e.g. [42]). Set

$$e_n = \int_{O(N)} e_n'\big(g(x)\big)\, d\mu_g,$$

where $O(N)$ is the orthogonal group on $\mathbb{R}^3$ and $d\mu_g$ denotes the Haar measure on $O(N)$. After deleting possible repeated elements, $\{e_n\}_{n=1}^{\infty}$ becomes a Schauder basis of $W_r^{1,p}(\mathbb{R}^3)$. Without loss of generality we may assume $\|e_n\|=1$ for every $n\in\mathbb{N}$, and we write

$$L_k := \operatorname{span}\{e_1,\dots,e_k\}, \qquad L_k^{\perp} := \overline{\operatorname{span}\{e_i : i\geq k+1\}}.$$

Clearly $W_r^{1,p}(\mathbb{R}^3) = L_k \oplus L_k^{\perp}$ for all $k\in\mathbb{N}$. We shall use genus theory to prove the existence of infinitely many solutions; the precise definition of genus is recalled below.

Definition 3.7 ([37]). *For any nonempty closed σ-invariant set $\mathcal{A}\subset W^{1,p}(\mathbb{R}^3)$, the genus of $\mathcal{A}$ is defined by*

$$\gamma(\mathcal{A}) := \begin{cases} 0, & \text{if } \mathcal{A}=\emptyset,\\ \inf\Big\{n\in\mathbb{N}^+ : \exists \text{ an odd continuous map } \phi:\mathcal{A}\to\mathbb{R}^n\setminus\{0\}\Big\}, & \\ +\infty, & \text{if no such map exists.}\end{cases}$$

Define the collection

$$\Sigma_a := \{\mathcal{A}\subset S(a)\cap W_r : \mathcal{A} \text{ is compact and } \sigma\text{-invariant}\}.$$

and for each $k\in\mathbb{N}^+$ set

$$\Sigma_{a,k} := \{\mathcal{A}\in\Sigma_a : \gamma(\mathcal{A})\geq k\}.$$

We observe that $\Sigma_{a,k}\neq\emptyset$. Indeed, for any $k\in\mathbb{N}^+$, we have $S_{a,k}=S(a)\cap L_k\subset\Sigma_{a,k}$ Theorem 10.5 of [2] gives $\gamma(S_{a,k})=k$. Since $k<k+1$, we obtain $\gamma(S_{a,k})=k<k+1=\gamma(S_{a,k+1})$; hence the genus is strictly increasing with k.

Introduce the minimax levels

$$\beta_{a,k} := \inf_{\mathcal{A}\in\Sigma_{a,k}} \max_{u\in\mathcal{A}} \Psi(u). \tag{3.23}$$

Because $\Sigma_{a,k+1}\subset\Sigma_{a,k}$ for every k, we have

$$\beta_{a,k}\leq\beta_{a,k+1}.$$

For any $\mathcal{A}\in\Sigma_{a,k}$ and $u\in\mathcal{A}$, Lemma 2.3 guarantees a number $s_u\in\mathbb{R}$ such that $\eta(u,s_u)\in\mathcal{P}_a$. Consequently,

$$\max_{u\in\mathcal{A}}\Psi(u)=\max_{u\in\mathcal{A}}\widetilde{I}(u)=\max_{u\in\mathcal{A}} I\big(\eta(u,s_u)\big)\geq\inf_{v\in\mathcal{P}_a} I(v)>0,$$

so $\beta_{a,k}>0$. The next lemma describes the asymptotic behaviour of the sequence $\{\beta_{a,k}\}$.

Lemma 3.8. *For each $a>0$, let $\beta_{a,k}$ be defined by (3.23). Then $\beta_{a,k}<\beta_{a,k+1}$.*

Proof. Suppose by contradiction that $\beta_{a,k}=\beta_{a,k+1}$. Then there exists $A\in\Sigma_{a,k+1}$ such that

$$\max_{u\in A} I(\eta(u,s_u))=\beta_{a,k}.$$

However, by the monotonicity of genus, $\gamma(A)\geq k+1>k=\gamma(S_{a,k})$, while $S_{a,k}\in\Sigma_{a,k}$ and $\max_{u\in S_{a,k}} I(\eta(u,s_u))=\beta_{a,k}$. This contradicts the fact that the minimax energy of a set with higher genus should be greater than that of a set with lower genus. Therefore, $\beta_{a,k}<\beta_{a,k+1}$, which implies that the energy is strictly increasing. □

Lemma 3.9. *For each $a > 0$, if there exist solutions u_i and u_j such that $u_i = u_j$, then $I(u_i) \neq I(u_j)$.*

Proof. For any $i < j$, from the fact that the energy is strictly increasing, we have $\beta_{a,i} < \beta_{a,j}$. If there exist solutions u_i and u_j such that $u_i = u_j$, then $I(u_i) = I(u_j)$, that is, $\beta_{a,i} = \beta_{a,j}$, which contradicts $\beta_{a,i} < \beta_{a,j}$. As a result, all solutions u_k are distinct. □

Lemma 3.10. *There exists $a_k > 0$ such that, for every $a \in (0, a_k)$, the following holds: if $\{u_n\} \subset \mathcal{P}_a \cap W_r$ is a Palais–Smale sequence for $I|_{S(a)\cap W_r}$ at a positive level $c > 0$, then a subsequence converges strongly in W_r to a function $u \in W_r$, and there exists $\lambda > 0$ satisfying*

$$-\Delta_p u + \lambda |u|^{p-2} u + \big(|x|^{-1} * |u|^p\big)|u|^{p-2}u = f(u), \quad x \in \mathbb{R}^3.$$

Moreover, $I(u) = c$.

Proof. For each $a > 0$, let $\{u_n\} \subset \mathcal{P}_a \cap W_r$ be a Palais–Smale sequence for the functional I restricted to $S(a) \cap W_r$ at the level $c > 0$. Following the argument of Lemma 3.4, one obtains that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. By the compact embedding $W_r \hookrightarrow L^q(\mathbb{R}^3)$ for $q \in (p, p^*)$, after passing to a subsequence we can find $u \in W_r$ such that $u_n \rightharpoonup u$ in W_r, $u_n \to u$ in $L^q(\mathbb{R}^3), \forall q \in (p, p^*)$, and $u_n \to u$ a.e. in $\mathbb{R}^3$. The Lagrange multiplier principle provides a sequence $\{\lambda_n\} \subset \mathbb{R}$ satisfying

$$-\Delta_p u_n + \lambda_n |u_n|^{p-2} u_n + \big(|x|^{-1} * |u_n|^p\big)|u_n|^{p-2}u_n - f(u_n) \longrightarrow 0 \quad \text{in } (W_r)^*, \tag{3.24}$$

which yields the identity

$$\lambda_n = \frac{1}{a^p}\Big(\int_{\mathbb{R}^3} f(u_n)u_n\,dx - \int_{\mathbb{R}^3} |\nabla u_n|^p\,dx - B(u_n)\Big).$$

Since $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$, the sequence $\{\lambda_n\}$ remains bounded in $\mathbb{R}$. After extracting a subsequence, we may assume $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$. Arguing as in the derivation of (3.7), we conclude that

$$-\Delta_p u + \lambda |u|^{p-2} u + \big(|x|^{-1} * |u|^p\big)|u|^{p-2}u = f(u). \tag{3.25}$$

Following the same line of argument as in Lemma 3.4, we show that $u \neq 0$. Moreover, there exists $a_k > 0$ such that for every $a \in (0, a_k)$, the Lagrange multiplier satisfies $\lambda > 0$.

We next prove that for every $a \in (0, a_k)$, there holds that $u_n \to u$ in W_r, and $\|u\|_p = \|u_n\|_p = a$. In Lemma 3.4 this was obtained from the non-increasing behavior of the map $a \mapsto c_a$, a property that is not guaranteed for $\beta_{a,k}$; this partly explains why we work in the radial subspace W_r.

By Proposition 2.1, we have that

$$B(u_n) \longrightarrow B(u), \tag{3.26}$$

and since $u_n \to u$ in $L^q(\mathbb{R}^3)$ for $q \in (p, p^*)$, arguing as in the proof of (3.19)-(3.22), we can derive as

$$\int_{\mathbb{R}^3} \big(f(u_n) - f(u)\big)u\,dx \longrightarrow 0. \tag{3.27}$$

Lemma 2.1-(iii) gives

$$\int_{\mathbb{R}^3} f(u_n)(u_n - u)\,dx \longrightarrow 0. \tag{3.28}$$

Combining (3.27) and (3.28), we obtain

$$\int_{\mathbb{R}^3} f(u_n)u_n\,dx \longrightarrow \int_{\mathbb{R}^3} f(u)u\,dx. \tag{3.29}$$

From (3.24), (3.26) and (3.29), we derive

$$\begin{aligned}\lim_{n\to\infty}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^p\,dx+\lambda\int_{\mathbb{R}^3}|u_n|^p\,dx\Big)&\\=\lim_{n\to\infty}\Big(\int_{\mathbb{R}^3}f(u_n)u_n\,dx-B(u_n)\Big)&\\=\int_{\mathbb{R}^3}f(u)u\,dx-B(u)&\\=\int_{\mathbb{R}^3}|\nabla u|^p\,dx+\lambda\int_{\mathbb{R}^3}|u|^p\,dx.&\end{aligned}$$

Since $\lambda>0$, the previous equality implies

$$\|\nabla u_n\|_p\to\|\nabla u\|_p,\qquad \|u_n\|_p\to\|u\|_p=a,$$

and by the Brezis-Lieb lemma [44] we conclude that $u_n\to u$ in W_r. Consequently, the convergence $I(u_n)\to I(u)=c$ follows directly. □

Proof of Theorem 1.3. For each fixed $k\in\mathbb{N}^+$ the set $\Sigma_{a,k}$ is non-empty and $\beta_{a,k}<+\infty$. Given $a>0$, Lemma 3.6 yields a Palais–Smale sequence $\{u_n^k\}_{n=1}^\infty\subset\mathcal{P}_a\cap W_r$ for the restricted functional $I|_{S(a)\cap W_r}$ at the level $\beta_{a,k}>0$. Applying Lemma 3.10, we can find numbers $a_k>0$, a function $u_k\in W_r$ and a Lagrange multiplier $\lambda_k>0$ such that, for every $a\in(0,a_k)$,

$$-\Delta_p u_k+\lambda_k|u_k|^{p-2}u_k+\big(|x|^{-1}*|u_k|^p\big)|u_k|^{p-2}u_k=f(u_k),\qquad x\in\mathbb{R}^3,$$

and $I(u_k)=\beta_{a,k}$. Lemma 3.8 tells us that the minimax values $\beta_{a,k}$ are strictly increasing in k, while Lemma 3.9 guarantees that distinct indices k give rise to distinct solutions. By Palais's principle of symmetric criticality [35], every critical point of I in the radial subspace W_r is in fact a critical point in the full space $W^{1,p}(\mathbb{R}^3)$. Consequently, each $u_k\in W_r$ solves (1.1) with $\lambda_k>0$ at the energy level $\beta_{a,k}$, and (1.1) possesses infinitely many radially symmetric solutions whose energies tend to infinity. This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.4

In this section we deal with the Sobolev critical case $f(t)=|t|^{p^*-2}t$ under the assumption $\kappa>0$. To establish Theorem 1.4, we note that the critical Sobolev nonlinearity causes the functional I to be unbounded from below on $S(a)$. Following the idea of [18], we apply a truncation method to control the influence of the critical term. This procedure allows us to define a modified functional that is bounded from below.

Recall that for any $u\in S(a)$, the constrained functional of (1.1) on $S(a)$ is defined by

$$I(u)=\frac{1}{p}\int_{\mathbb{R}^3}|\nabla u|^p dx-\frac{\kappa}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}dxdy-\frac{1}{p^*}\int_{\mathbb{R}^3}|u|^{p^*}dx. \tag{4.1}$$

By (2.3) and (1.13), we obtain

$$\begin{aligned}I(u)\geq{}&\frac{1}{p}\int_{\mathbb{R}^3}|\nabla u|^p dx-C_p\kappa a^{2p-1}\left(\int_{\mathbb{R}^3}|\nabla u|^p dx\right)^{\frac{1}{p}}-\frac{1}{p^*S^{\frac{3}{3-p}}}\left(\int_{\mathbb{R}^3}|\nabla u|^p dx\right)^{\frac{3}{3-p}}\\&=:g(\|\nabla u\|_p),\end{aligned} \tag{4.2}$$

where

$$g(t):=\frac{t^p}{p}-C_p\kappa a^{2p-1}t-\frac{t^{p^*}}{p^*S^{\frac{3}{3-p}}}.$$

Observe that the function

$$h(t):=\frac{t^{p-1}}{p}-\frac{t^{p^*-1}}{p^*S^{\frac{3}{3-p}}}$$

attains a unique positive maximum at a point $t_0 > 0$, and $h(t_0) > 0$. If $\kappa C_p a^{2p-1} < h(t_0) =: \ell$, then g also possesses a positive local maximum at t_0, and there exist numbers $0 < R_1 < R_2$ such that

$$g(t) < 0 \text{ for } 0 < t < R_1, \qquad g(t) > 0 \text{ for } R_1 < t < R_2, \qquad g(t) < 0 \text{ for } t > R_2.$$

Using these radii $R_1, R_2 > 0$, we define a cut-off function $\xi \in C^\infty(\mathbb{R}^+, [0,1])$ by

$$\xi(t) = \begin{cases} 1, & 0 \le t \le R_1, \\ 0, & t \ge R_2. \end{cases}$$

We introduce the truncated functional as

$$I^T(u) := \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx - \frac{\kappa}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|} dxdy - \frac{\xi(\|\nabla u\|_p)}{p^*}\int_{\mathbb{R}^3} |u|^{p^*} dx, \tag{4.3}$$

and note that $I^T \in C^1(W^{1,p}(\mathbb{R}^3), \mathbb{R})$. By (4.3), we have

$$I^T(u) \ge g^T(\|\nabla u\|_p),$$

where

$$g^T(t) = \frac{t^p}{p} - \kappa C_p a^{2p-1} t - \frac{\xi(t)}{p^* S^{\frac{3}{3-p}}} t^{p^*}.$$

From the definition of ξ and if $\kappa C_p a^{2p-1} < \ell$, we observe that for $g^T(t) = g(t) < 0$ for $0 < t < R_1$ and $g^T(t) > 0$ for $t \in (R_1, +\infty)$, and when $t \in (R_2, +\infty)$, one has $g^T(t) = \frac{t^p}{p} - \kappa C_p a^{2p-1} t > 0$. In the sequel we always assume that

$$0 < \kappa a^{2p-1} < \alpha := \frac{\ell}{C_p}. \tag{4.4}$$

Without loss of generality, in the following discussion we can take that $R_1 > 0$ is small enough such that

$$\frac{t^p}{p} - \frac{t^{p^*}}{p^* S^{\frac{3}{3-p}}} \ge 0 \quad \text{for } t \in [0, R_1] \quad \text{and} \quad R_1 < S^{\frac{3}{p}}. \tag{4.5}$$

Remark 4.1. *From the above arguments, we see that, if $u_n \in S(a)$, and $\|\nabla u_n\|_p \to \infty$ as $n \to \infty$, we have*

$$\begin{aligned} I^T(u_n) &= \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u_n|^p dx - \frac{\kappa}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|} dxdy \\ &\ge \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u_n|^p dx - C_p \kappa a^{2p-1} \left(\int_{\mathbb{R}^3} |\nabla u_n|^p dx\right)^{\frac{1}{p}} \\ &\to +\infty, \end{aligned}$$

which implies that I^T is coercive on $S(a)$. Furthermore, if $I^T(u) \le 0$, then $\|\nabla u\|_p \le R_1$ and $I(u) = I^T(u)$.

Lemma 4.1. *Under the condition $\kappa a^{2p-1} < \alpha$, then the truncated functional $I^T|_{\mathcal{S}_r(a)}$ satisfies the $(PS)_d$ condition for any $d < 0$.*

Proof. Let $\{u_n\} \subset \mathcal{S}_r(a) := S(a) \cap W_r$ be a Palais-Smale sequence for I^T at a level $d < 0$. To establish the lemma, it suffices to prove that $\{u_n\}$ possesses a convergent subsequence in $\mathcal{S}_r(a)$. By the coercivity of I^T on $S(a)$ (see Remark 4.1), the sequence $\{u_n\}$ is bounded in W_r. Moreover, for all sufficiently large n, we have $\|\nabla u_n\|_p \le R_1$. Due to the construction of I^T, the same sequence $\{u_n\}$ forms a bounded Palais-Smale sequence for the restriction $I|_{\mathcal{S}_r(a)}$; namely,

$$I(u_n) \to d \quad \text{and} \quad \|I'|_{\mathcal{S}_r(a)}(u_n)\| \to 0. \tag{4.6}$$

Up to a subsequence, there exists $u \in W_r$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_r, \\ u_n \to u & \text{a. e. in } \mathbb{R}^3, \\ u_n \to u & \text{in } L^q(\mathbb{R}^3) \text{ for } q \in (p, p^*). \end{cases}$$

Arguing as in (3.5), we obtain

$$-\Delta_p u_n + \lambda_n u_n^{p-1} - \kappa\big(|x|^{-1} * |u_n|^p\big) u_n^{p-1} - |u_n|^{p^*-2} u_n \to 0 \quad \text{in } (W_r)^*, \tag{4.7}$$

with

$$\lambda_n = \frac{1}{a^p}\Big(\int_{\mathbb{R}^3} |u_n|^{p^*} dx - \int_{\mathbb{R}^3} |\nabla u_n|^p dx + \kappa B(u_n)\Big).$$

The boundedness of $\{u_n\}$ in W_r implies that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Consequently, passing to a further subsequence, λ_n converges to some $\lambda \in \mathbb{R}$. We claim that $u \neq 0$. If, on the contrary, $u = 0$, then Lions' lemma yields

$$\lim_{n \to +\infty} \int_{\mathbb{R}^3} |u_n|^q dx = 0 \quad \text{for all } q \in (p, p^*).$$

Hence, by (2.6) and (4.5),

$$\begin{aligned} I(u_n) = I^T(u_n) &\geq \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u_n|^p dx - C_p \kappa \|u_n\|_{\frac{6p}{5}}^{2p} - \frac{1}{p^* S^{\frac{p}{3-p}}}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^p dx\Big)^{\frac{3}{3-p}} \\ &\geq -C_p \kappa \|u_n\|_{\frac{6p}{5}}^{2p} \to 0, \end{aligned}$$

which contradicts $I(u_n) \to d < 0$. Therefore $u \neq 0$. Using the weak convergence $u_n \rightharpoonup u$ in W_r and arguing similarly to (3.7), we deduce that $u \in W_r$ satisfies

$$-\Delta_p u + \lambda u^{p-1} - \kappa\big(|x|^{-1} * |u|^p\big) u^{p-1} = |u|^{p^*-2} u, \qquad x \in \mathbb{R}^3. \tag{4.8}$$

According to [38], u fulfills the Pohozaev-type identity

$$\frac{3-p}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{3\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{5\kappa}{2p} B(u) - \frac{3}{p^*}\int_{\mathbb{R}^3} |u|^{p^*} dx = 0. \tag{4.9}$$

Multiplying (4.8) by u and integrating gives

$$\int_{\mathbb{R}^3} |\nabla u|^p dx + \lambda \int_{\mathbb{R}^3} |u|^p dx - \kappa B(u) - \int_{\mathbb{R}^3} |u|^{p^*} dx = 0. \tag{4.10}$$

Combining (4.9) and (4.10), we find

$$\lambda \int_{\mathbb{R}^3} |u|^p dx = \frac{(2p-1)\kappa}{2p} B(u).$$

Since $u \neq 0$ and $\kappa > 0$, it follows that $\lambda > 0$.

We now prove the strong convergence $u_n \to u$ in W_r. By the concentration-compactness principle [26, 27], we have

$$|\nabla u_n|^p \rightharpoonup \mu \geq |\nabla u|^p + \sum_{i \in \mathcal{J}} \mu_i \delta_{x_i}, \qquad |u_n|^{p^*} \rightharpoonup \nu = |u|^{p^*} + \sum_{i \in \mathcal{J}} \nu_i \delta_{x_i},$$

with $\sum_{i \in \mathcal{J}} \nu_i^{\frac{p}{p^*}} < +\infty$. Here μ, ν, μ_i, ν_i are positive measures, $\mathcal{J}$ is an at most countable index set, $\{x_i\} \subset \mathbb{R}^3$ are the atoms of the singular parts of μ and ν, and δ_{x_i} denotes the Dirac mass at x_i. Moreover,

$$S \nu_i^{\frac{p}{p^*}} \leq \mu_i \quad \text{for all } i \in \mathcal{J}. \tag{4.11}$$

We claim that $\mathcal{J}$ is either empty or a finite set. The argument is as follows. The continuous Sobolev embedding $W_r \hookrightarrow L^{p^*}(\mathbb{R}^3)$ together with the boundedness of $\{u_n\}$ in W_r implies that $\{u_n\}$ is bounded in $L^{p^*}(\mathbb{R}^3)$. Hence there exists $M > 0$ such that

$$\int_{\mathbb{R}^3} |u_n|^{p^*} dx \le M \quad \text{for all } n \in \mathbb{N}.$$

Since ν is the weak limit of $|u_n|^{p^*}$, for any measurable $E \subset \mathbb{R}^3$,

$$\int_E d\nu = \lim_n \int_E |u_n|^{p^*} dx \le \limsup_n \int_E |u_n|^{p^*} dx \le M,$$

showing that ν is a bounded measure. In particular, ν can carry at most finitely many atoms; consequently $\mathcal{J}$ is either empty or finite.

Assume now that $\mathcal{J}$ is nonempty (hence finite). Choose a cut-off function $\psi_\varepsilon(x) = \widetilde{\psi}_\varepsilon(x - x_i)$, where $\widetilde{\psi}_\varepsilon \equiv 1$ in $B_\varepsilon(0)$, $\widetilde{\psi}_\varepsilon \equiv 0$ in $B^c_{2\varepsilon}(0)$, $|\nabla \widetilde{\psi}_\varepsilon| \le 2/\varepsilon$, and $\widetilde{\psi}_\varepsilon \in C_0^\infty(\mathbb{R}^3, [0,1])$. The remainder of the proof is divided into three steps.

Step 1. $\mu_i \le \nu_i$ for every $i \in \mathcal{J}$. It is straightforward to verify that $\{u_n \psi_\varepsilon\}$ is bounded in W_r. From (4.7), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi_\varepsilon \, u_n \, dx &= \int_{\mathbb{R}^3} \Big(-\lambda_n |u_n|^p - |\nabla u_n|^p + |u_n|^{p^*}\Big)\psi_\varepsilon \, dx \\ &\quad + \kappa \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^p |u_n(y)|^p \psi_\varepsilon(y)}{|x-y|} \, dxdy + o_n(1).\end{aligned} \tag{4.12}$$

Using Hölder's inequality, we infer to

$$\begin{aligned}&\limsup_{n\to\infty} \Big|\int_{\mathbb{R}^3} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi_\varepsilon \, u_n \, dx\Big| \\ &\le \limsup_{n\to\infty} \Big(\int_{B_{2\varepsilon}(x_i)} |\nabla \psi_\varepsilon \, u_n|^p dx\Big)^{\frac{1}{p}} \Big(\int_{B_{2\varepsilon}(x_i)} |\nabla u_n|^p dx\Big)^{\frac{p-1}{p}} \\ &\le C \Big(\int_{B_{2\varepsilon}(x_i)} |\nabla \psi_\varepsilon u|^p dx\Big)^{\frac{1}{p}} \\ &\le C_1 \Big(\int_{B_{2\varepsilon}(x_i)} |\nabla \psi_\varepsilon|^{\frac{pp^*}{p^*-p}} dx\Big)^{\frac{p^*-p}{pp^*}} \Big(\int_{B_{2\varepsilon}(x_i)} |u|^{p^*} dx\Big)^{\frac{1}{p^*}} \\ &= C_2 \Big(\int_{B_{2\varepsilon}(x_i)} |u|^{p^*} dx\Big)^{\frac{1}{p^*}} \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned} \tag{4.13}$$

By the Hardy-Littlewood-Sobolev inequality (2.4), one has

$$\begin{aligned}&\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^p |u_n(y)|^p \psi_\varepsilon(y)}{|x-y|} dxdy \\ &\qquad \le C_3 \Big(\int_{\mathbb{R}^3} |u_n|^{\frac{6p}{5}} dx\Big)^{\frac{5}{6}} \Big(\int_{\mathbb{R}^3} (|u_n|^p \psi_\varepsilon)^{\frac{6}{5}} dx\Big)^{\frac{5}{6}} \\ &\qquad \le C_3 \|u_n\|^p_{6p/5} \Big(\int_{\mathbb{R}^3} |u_n|^{\frac{6p}{5}} \psi_\varepsilon dx\Big)^{\frac{5}{6}} \\ &\qquad \le C_4 \Big(\int_{B_{2\varepsilon}(x_i)} |u_n|^{\frac{6p}{5}} \psi_\varepsilon dx\Big)^{\frac{5}{6}}.\end{aligned} \tag{4.14}$$

Taking limits, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u_n(x)|^p|u_n(y)|^p\psi_\varepsilon(y)}{|x-y|}dxdy\\
&\le C_5\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big(\int_{B_{2\varepsilon}(x_i)}|u_n|^{\frac{6p}{5}}\psi_\varepsilon dx\Big)^{\frac{5}{6}}\\
&=\lim_{\varepsilon\to 0}\Big(\int_{B_{2\varepsilon}(x_i)}|u|^{\frac{6p}{5}}\psi_\varepsilon dx\Big)^{\frac{5}{6}}=0.
\end{aligned}\tag{4.15}
$$

Furthermore, because ψ_ε has compact support,

$$
\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^p\psi_\varepsilon dx\ge\int_{\mathbb{R}^3}|\nabla u|^p\psi_\varepsilon dx+\big\langle\sum_{i\in\mathcal{J}}\mu_i\delta_{x_i},\psi_\varepsilon\big\rangle,\tag{4.16}
$$

$$
\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{p^*}\psi_\varepsilon dx=\int_{\mathbb{R}^3}|u|^{p^*}\psi_\varepsilon dx+\big\langle\sum_{i\in\mathcal{J}}\nu_i\delta_{x_i},\psi_\varepsilon\big\rangle.\tag{4.17}
$$

Combining (4.12)-(4.17), we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\psi_\varepsilon\,u_n\,dx\\
&\le-\int_{\mathbb{R}^3}|\nabla u|^p\psi_\varepsilon dx-\big\langle\sum_{i\in\mathcal{J}}\mu_i\delta_{x_i},\psi_\varepsilon\big\rangle\\
&\quad+\int_{\mathbb{R}^3}|u|^{p^*}\psi_\varepsilon dx+\big\langle\sum_{i\in\mathcal{J}}\nu_i\delta_{x_i},\psi_\varepsilon\big\rangle-\int_{\mathbb{R}^3}\lambda|u|^p\psi_\varepsilon dx.
\end{aligned}\tag{4.18}
$$

Letting $\varepsilon\to 0$ in the last inequality and using (4.13) and (4.18), we get

$$
\begin{aligned}
0&\le\lim_{\varepsilon\to 0^+}\Big(-\big\langle\sum_{i\in\mathcal{J}}\mu_i\delta_{x_i},\psi_\varepsilon\big\rangle+\big\langle\sum_{i\in\mathcal{J}}\nu_i\delta_{x_i},\psi_\varepsilon\big\rangle\Big)\\
&=\lim_{\varepsilon\to 0^+}\Big(-\mu_i\widetilde{\psi}_\varepsilon(0)+\nu_i\widetilde{\psi}_\varepsilon(0)\Big)\\
&=-\mu_i+\nu_i,
\end{aligned}
$$

whence $\mu_i\le\nu_i$. Together with (4.11) we obtain

$$
\nu_i\ge S^{\frac{3}{p}}\quad\text{for all } i\in\mathcal{J}.
$$

Step 2. $\mu_i=0$ for every $i\in\mathcal{J}$, and consequently $\mathcal{J}=\emptyset$. Suppose, by contradiction, that there exists some $i\in\mathcal{J}$. Then from (4.11) we have $\mu_i\ge S^{\frac{3}{p}}$. Thus,

$$
\begin{aligned}
R_1^p\ge\limsup_{n\to+\infty}\|\nabla u_n\|_p^p&\ge\limsup_{n\to+\infty}\int_{\mathbb{R}^3}|\nabla u_n|^p\psi_\varepsilon dx\\
&\ge\int_{\mathbb{R}^3}|\nabla u|^p\psi_\varepsilon dx+\big\langle\sum_{k\in\mathcal{J}}\mu_k\delta_{x_k},\psi_\varepsilon\big\rangle\\
&\ge\mu_i\ge S^{\frac{3}{p}},
\end{aligned}
$$

which contradicts (4.5). Hence $\mathcal{J}=\emptyset$ and

$$
u_n\to u\quad\text{in } L^{p^*}_{\rm loc}(\mathbb{R}^3).\tag{4.19}
$$

Step 3. Strong convergence $u_n\to u$ in W_r. Since $\{u_n\}\subset W_r$ is bounded, it follows from [41] that

$$
|u_n(x)|\le C\|u_n\|\,|x|^{-\frac{2}{p}}\le C_1|x|^{-\frac{2}{p}}\quad\text{a.e. in }\mathbb{R}^3,\ \forall n\in\mathbb{N}.
$$

Consequently, for all $n \in \mathbb{N}$,

$$|u_n(x)|^{p^*} \le \frac{C_2}{|x|^{\frac{6}{3-p}}} \quad \text{a.e. in } \mathbb{R}^3.$$

Noticing that $C_2/|x|^{\frac{6}{3-p}} \in L^1(\mathbb{R}^3 \setminus B_R(0))$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^3 \setminus B_R(0)$, Lebesgue's dominated convergence theorem yields

$$u_n \to u \quad \text{in } L^{p^*}(\mathbb{R}^3 \setminus B_R(0)). \tag{4.20}$$

Together with (4.19) this gives

$$u_n \to u \quad \text{in } L^{p^*}(\mathbb{R}^3). \tag{4.21}$$

From (4.7) and (4.8) we have

$$\lambda_n\|u_n\|_p^p + \|\nabla u_n\|_p^p = \kappa B(u_n) + \|u_n\|_{p^*}^{p^*} + o_n(1) \tag{4.22}$$

and

$$\lambda\|u\|_p^p + \|\nabla u\|_p^p = \kappa B(u) + \|u\|_{p^*}^{p^*}. \tag{4.23}$$

By Proposition 2.1, we have

$$B(u_n) = B(u) + o_n(1). \tag{4.24}$$

Thereby, from (4.22)-(4.24) we deduce

$$\lim_{n\to\infty}\Big(\lambda_n\|u_n\|_p^p + \|\nabla u_n\|_p^p\Big) = \lambda\|u\|_p^p + \|\nabla u\|_p^p.$$

Using $\lambda_n \to \lambda > 0$, we obtain

$$\lim_{n\to\infty}\|u_n\|_p = \|u\|_p \quad \text{and} \quad \lim_{n\to\infty}\|\nabla u_n\|_p = \|\nabla u\|_p,$$

which implies

$$u_n \to u \quad \text{in } W_r \ \text{ as } n \to \infty.$$

This completes the proof. □

In the sequel we aim to obtain the multiplicity of normalized solution by the genus theory. First, for any $\varepsilon > 0$, define the set

$$C_\varepsilon := \{u \in W_r \cap S(a) : I^T(u) \le -\varepsilon\} \subset W_r. \tag{4.25}$$

which is a closed symmetric subset of $S_r(a)$, because I^T is even and continuous. For any $c \in \mathbb{R}$, set $I^{T,c} := \{u \in S(a) \cap W_r : \ \ I^T(u) \le c\}$.

Lemma 4.2. *For each $n \in \mathbb{N}$, there exist $\varepsilon_n > 0$ and $\kappa > 0$ such that*

$$\gamma(C_\varepsilon) \ge n$$

for all $\varepsilon \in (0, \varepsilon_n]$.

Proof. For a given $n \in \mathbb{N}$, we select n radial functions

$$\{u_1, u_2, \dots, u_n\} \subset C_0^\infty(\mathbb{R}^3),$$

satisfying:

$$\operatorname{supp} u_i \cap \operatorname{supp} u_j = \emptyset \quad (i \ne j), \qquad \|u_j\|_p = a, \qquad \|\nabla u_i\|_p = \tau < R_1 \quad (i = 1, \dots, n).$$

We then define the n-dimensional subspace

$$W_n := \operatorname{span}\{u_1, u_2, \dots, u_n\} \subset W_r.$$

Define

$$G_n(s) := \Big\{\sum_{i=1}^n r_i\,\eta(u_i, s) : \sum_{i=1}^n |r_i|^p = 1\Big\},$$

and

$$Y(s) := \Big\{(y_1, \dots, y_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |y_i|^p = \tau^p e^{ps} + a^p\Big\}.$$

There exists an odd homeomorphism between $G_n(s)$ and $Y(s)$; consequently, by the properties of the genus,

$$\gamma\big(G_n(s)\big) = \gamma\big(Y(s)\big) = n.$$

Now, let $u = \sum_{i=1}^n r_i\, \eta(u_i, s) \in G_n(s)$ with $s < 0$. Then

$$\|\nabla u\|_p = e^s \tau < R_1.$$

Observe now that for $u = \sum_{i=1}^n r_i\, \eta(u_i, s) \in G_n(s)$,

$$I^T(u) = I(u) = \frac{e^{ps}}{p}\tau^p - \frac{\kappa e^s}{2p}\tau^{2p} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|w(x)/\tau|^p |w(y)/\tau|^p}{|x-y|}\, dx\, dy - \frac{e^{p^* s}}{p^*}\tau^{p^*} \int_{\mathbb{R}^3} |w/\tau|^{p^*}\, dx,$$

where $w = \sum_{i=1}^n r_i u_i$. We define

$$\alpha_n := \inf\big\{\|v\|_{p^*}^{p^*} : v \in W_n,\ \|\nabla v\|_p = 1\big\} > 0,$$

$$\beta_n := \inf\big\{B(v) : v \in W_n,\ \|\nabla v\|_p = 1\big\} > 0.$$

Thus, we obtain the estimate

$$I^T(u) \le \frac{e^{ps}}{p}\tau^p - \frac{\kappa e^s \tau^{2p}}{2p}\beta_n - \frac{e^{p^* s}\tau^{p^*}}{p^*}\alpha_n.$$

Hence we can select $\varepsilon_n > 0$ and $s_n < 0$ such that, for every $\varepsilon \in (0, \varepsilon_n]$ and any fixed $\kappa > 0$,

$$I^T(u) < -\varepsilon,\ \forall u \in G_n(s_n),$$

which implies $G_n(s_n) \subset C_\varepsilon$. Using once more the monotonicity of the genus, we conclude

$$\gamma(C_\varepsilon) \ge \gamma\big(G_n(s_n)\big) = n,$$

which finishes the proof. □

For each $j \in \mathbb{N}$, we introduce the minimax value

$$d_j := \inf_{\tilde{A} \in \Sigma_j} \sup_{u \in \tilde{A}} I^T(u), \tag{4.26}$$

where

$$\Sigma_j := \big\{\tilde{A} \subset W_r \cap S(a) : \tilde{A} \text{ is closed, symmetric } (\tilde{A} = -\tilde{A}), \text{ and } \gamma(\tilde{A}) \ge j\big\}.$$

Because I^T is bounded from below on $S_r(a)$, it follows that $d_j > -\infty$. For a level $d \in \mathbb{R}$, we denote by $\mathcal{K}_d$ the set of critical points of I^T at that level:

$$\mathcal{K}_d := \big\{u \in W_r \cap S(a) : (I^T)'(u) = 0,\ I^T(u) = d\big\}.$$

Lemma 4.3. *Assume that* $d := d_k = d_{k+1} = \cdots = d_{k+r} < 0$ *for some* $k, r \in \mathbb{N}$, *then* $\gamma(\mathcal{K}_d) \ge r + 1$.

Proof. From Lemma 4.2, for any $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$, such that

$$\gamma(C_\varepsilon) \ge k,\ \forall \varepsilon \in (0, \varepsilon_k].$$

Since I^T is continuous and even, $C_{\varepsilon_k} \in \Sigma_k$; consequently,

$$d_k \le -\varepsilon_k < 0,\ \forall k \in \mathbb{N}.$$

Because I^T is bounded from below, we also have $d_k > -\infty$ for every k.

Now assume $d := d_k = \cdots = d_{k+r}$. As $d < 0$, Lemma 4.1 implies that I^T satisfies the Palais-Smale condition on $\mathcal{K}_d$, and it is straightforward to verify that $\mathcal{K}_d$ is compact.

If $\gamma(\mathcal{K}_d) < r$, then there exists a closed symmetric set U with $\mathcal{K}_d \subset U$ such that $\gamma(U) < r$. In particular, we may choose $U \subset I^{T,0}$ because $d < 0$. By the classical deformation lemma [9], there exists an odd homeomorphism $\eta \in C\big([0,1] \times S(a), S(a)\big)$ satisfying

$$\eta\big(1, I^{T,d+\delta} \setminus U\big) \subset I^{T,d-\delta},$$

for some $\delta > 0$. At this stage we select $0 < \delta < -d$; since I^T satisfies the Palais-Smale condition on $I^{T,0}$, we require $I^{T,d+\delta} \subset I^{T,0}$. By definition, we have

$$d = d_{k+r} = \inf_{\tilde{A} \in \Sigma_{k+r}} \sup_{u \in \tilde{A}} I^T(u).$$

Hence there exists $\tilde{A} \in \Sigma_{k+r}$ with $\sup_{u \in \tilde{A}} I^T(u) < d + \delta$; i.e., $\tilde{A} \subset I^{T,d+\delta}$. Consequently,

$$\eta(1, \tilde{A} \setminus U) \subset \eta\big(1, I^{T,d+\delta} \setminus U\big) \subset I^{T,d-\delta}. \tag{4.27}$$

Observe that

$$\gamma(\overline{\tilde{A} \setminus U}) \geq \gamma(\tilde{A}) - \gamma(U) \geq k,$$

and

$$\gamma\Big(\eta\big(1, \overline{\tilde{A} \setminus U}\big)\Big) \geq \gamma\big(\overline{\tilde{A} \setminus U}\big) \geq \gamma(\tilde{A}) - \gamma(U) \geq k.$$

Thus $\eta\big(1, \overline{\tilde{A} \setminus U}\big) \in \Sigma_k$. This, however, contradicts (4.27). Indeed, from $\eta\big(1, \overline{\tilde{A} \setminus U}\big) \in \Sigma_k$ we obtain

$$\sup_{u \in \eta(1, \overline{\tilde{A} \setminus U})} I^T(u) \geq d_k = d,$$

whereas (4.27) implies the supremum is at most $d - \delta < d$. □

Proof of Theorem 1.4. For each $k \in \mathbb{N}$, Lemma 4.2 yields an ε_k such that $\gamma(C_{\varepsilon_k}) \geq k$. Hence $C_{\varepsilon_k} \in \Sigma_k$ and $\Sigma_k \neq \emptyset$. We can therefore define a non-increasing sequence of minimax values

$$d_k := \inf_{\tilde{A} \in \Sigma_k} \sup_{u \in \tilde{A}} I^T(u), \ \forall k \in \mathbb{N},$$

satisfying $-\infty < d_1 \leq d_2 \leq \cdots$. By Theorem 2.1 of [23] we obtain the following:

(i) If $d_k < 0$, then d_k is a critical value of $I^T|_{S_r(a)}$.
(ii) Assume $d := d_k = d_{k+1} = \cdots = d_{k+r-1} < 0$ for some $k, r \geq 1$, and let $\mathcal{K}_d$ be the set of critical points of $I^T|_{S_r(a)}$ at level d. Then $\gamma(\mathcal{K}_d) \geq r$; in particular, if $r \geq 2$, the functional $I^T|_{S_r(a)}$ possesses infinitely many critical points at level d.
(iii) If $d_k < 0$ for every $k \geq 1$, then $d_k \to 0^-$ as $k \to \infty$.

The functional I^T is bounded from below, and Lemma 4.1 guarantees that it satisfies the $(PS)_d$ condition for all $d < 0$. Consequently, each d_k is indeed a critical value of I^T, and $d_k \to 0^-$ as $k \to \infty$.

According to Remark 4.1, the equality $I^T(u) = I(u)$ holds in a small neighborhood of u whenever $I^T(u) < 0$. Thus the critical points of $I^T|_{S_r(a)}$ obtained above are also critical points of $I|_{S_r(a)}$. This completes the proof. □

Conflict of interest. The authors declare that there is no conflict of interest.

Data availability. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (11771468, 11971027, 12171497). The last author is member of INdAM-GNAMPA.

References

[1] C.O. Alves, C. Ji, O.H. Miyagaki, Normalized solutions for a Schrödinger equation with critical growth in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations* **61**(2022), Paper No. 18, 24 p. 3

[2] A. Ambrosetti, A. Malchiodi, Nonlinear Analysis and Semilinear Elliptic Problems, Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007. 26

[3] T. Bartsch, N. Soave, A natural constraint approach to normalized solutions on nonlinear Schrödinger equations and systems, *J. Funct. Anal.* **272**(2017), 4998-5037. 5

[4] T. Bartsch, N. Soave, Multiple normalized solutions for a competing system of Schrödinger equations, *Calc. Var. Partial Differential Equations* **58**(2019), No. 1, Paper No. 22, 24 p. 5, 19

[5] J. Bellazzini, L. Jeanjean, T. Luo, Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations, *Proc. Lond. Math. Soc.* **107**(2013), 303-339. 2, 3, 5

[6] J. Bellazzini, G. Siciliano, Scaling properties of functionals and existence of constrained minimizers, *J. Funct. Anal.* **261**(2011), 2486-2507. 2

[7] J. Bellazzini, G. Siciliano, Stable standing waves for a class of nonlinear Schrödinger-Poisson equations, *Z. Angew. Math. Phys.* **62**(2011), 267-280. 2, 3, 5

[8] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonl. Anal.* **11** (1998), 283-293. 2

[9] V. Benci, On critical point theory for indefinite functionals in the presence of symmetries, *Trans. Amer. Math. Soc.* **274** (1982), 533-572. 35

[10] H. Berestycki, P.L. Lions, Nonlinear scalar field equations II: existence of infinitely many solutions, *Arch. Ration. Mech. Anal.* **82**(1983), 347-375. 22

[11] B. Bieganowski, J. Mederski, Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth, *J. Funct. Anal.* **280**(2021), 109-135. 3

[12] G. Cerami, G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, *J. Differential Equations* **248** (2010), 521-543. 2

[13] S. Chen, X. Tang, S. Yuan, Normalized solutions for Schrödinger-Poisson equations with general nonlinearities, *J. Math. Anal. Appl.* **481**(2020), No. 1, Article ID 123447, 24 p. 2

[14] S. Deng, Q. Wu, Normalized solutions for p-Laplacian equation with critical Sobolev exponent and mixed nonlinearities, arxiv preprint arxiv:2306.06709, 2023 arxiv.org. 2, 5

[15] Y. Du, J. Su, C. Wang, The Schrödinger-Poisson system with p-Laplacian, *Appl. Math. Lett.* **120**(2021), Article ID 107286, 7 p. 2

[16] Y. Du, J. Su, C. Wang, On the critical Schrödinger-Poisson system with p-Laplacian, *Commun. Pure Appl. Anal.* **21** (2022), 1329-1342. 2

[17] X. Feng, Y. Li, Normalized solutions for some quasilinear elliptic equation with critical Sobolev exponent, arxiv preprint arxiv:2306.10207, 2023 arxiv.org. 2, 5

[18] J. Garcia Azorero, I. Peral Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, *Trans. Amer. Math. Soc.* **323** (1991), 877-895. 5, 28

[19] N. Ghoussoub, Duality and Perturbation Methods in Critical Point Theory, Cambridge University Press, Cambridge, 1993. 19, 25

[20] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal. TMA* **28**(1997), 1633-1659. 2, 9

[21] L. Jeanjean, T. T. Le, Multiple normalized solutions for a Sobolev critical Schrödinger-Poisson-Slater equation, *J. Differential Equations* **303** (2021), 277-325. 2, 5

[22] L. Jeanjean, S.-S. Lu, A mass supercritical problem revisited, *Calc. Var. Partial Differential Equations* **59**(2020), No. 5, Paper No. 174, 42 p. 2, 5, 10, 14

[23] L. Jeanjean, S.-S. Lu, Nonradial normalized solutions for nonlinear scalar field equations, *Nonlinearity* **32** (2019), 4942-4966. 35

[24] L. Jeanjean, T. Luo, Sharp nonexistence results of prescribed L^2-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations, *Z. Angew. Math. Phys.* **64** (2013), 937-954.

2, 3, 5

[25] E.H. Lieb, M. Loss, Analysis, volume 14 of Graduate Studies in Mathematics, 2nd edn. American Mathematical Society, Providence, RI, 2001. 6

[26] P.L. Lions, The concentration compactness principle in the calculus of variations: the locally compact case. Parts 1. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **1** (1984), 109-145. 5, 15, 30

[27] P.L. Lions, The concentration compactness principle in the calculus of variations: the locally compact case, Parts 2. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **1**(1984), 223-283. 5, 30

[28] K. Liu, X. He, Solutions with prescribed mass for the Sobolev critical Schrödinger-Poisson system with p-Laplacian, *Bull. Math. Sci.* **15**(2025), No. 2, Article ID 2550006, 45 p. 3, 5, 9

[29] K. Liu, X. He, V. D. Rădulescu, Solutions with prescribed mass for the p-Laplacian Schrödinger–Poisson system with critical growth, *J. Differential Equations* **444**(2025), 113570. 3

[30] K. Liu, X. He, Normalized solutions to the p-Laplacian Schrödinger–Poisson system with mass supercritical growth, *Bull. Math. Sci.* (2025), 2550021. 3, 4, 5, 20

[31] Y. Liu, R. Zhang, X. Zhang, Normalized solutions to the quasilinear Schrödinger system with p-Laplacian under the L^p-mass supercritical case, *J. Math. Anal. Appl.* **550** (2025), 129594. https://doi.org/10.1016/j.jmaa.2025.129594. 2

[32] Q. Lou, X. Zhang, Z. Zhang, Normalized solutions to p-Laplacian equation: Sobolev critical case, *Topol. Methods Nonlinear Anal.* **64**(2024), 409-439. 5

[33] T. Luo, Multiplicity of normalized solutions for a class of nonlinear Schrödinger-Poisson-Slater equations, *J. Math. Anal. Appl.* **416**(2014), 195-204. 2

[34] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **13**(1959), 115-162. 6

[35] R.S. Palais, The principle of symmetric criticality, *Commun. Math. Phys.* **69** (1979), 19-30. 28

[36] X. Peng, Existence andmultiplicity of solutions for the Schrödinger-Poisson equation with prescribed mass, *Anal. Math. Phys.* **14**(2024), Article number 102, 41 p. 2

[37] P.H. Rabinowitz, Minimax Methods in Critical Points Theory with Application to Differential Equations, In: CBMS Regional Conference Series in Mathematics, vol. 65. American Mathematical Society, Providence, RI, 1986. 2, 26

[38] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.* **237**(2006), 655-674. 2, 30

[39] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities, *J. Differential Equations* **269** (2020), 6941-6987. 3

[40] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case, *J. Funct. Anal.* **279**(2020), No. 6, Article ID 108610, 42 p. 2

[41] J. Su, Z.-Q. Wang, M. Willem, Weighted Sobolev embedding with unbounded and decaying radial potentials, *J. Differential Equations* **238** (2007), 201-219. 32

[42] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, msterdam: North-Holland, 1978. 26

[43] Q. Wang, A. Qian, Normalized solutions to the Schrödinger-Poisson-Slater equation with general nonlinearity: mass supercritical case, *Anal. Math. Phys.* **13**(2023), No. 2, Paper No. 35, 37 p. 2

[44] M. Willem, Minimax theorems. In: Progress in Nonlinear Differential Equations and Their Applications, vol. 24. Birkhauser Boston Inc, Boston, 1996. 2, 28

(M. Li) College of Science, Minzu University of China
Beijing 100081, China
Email address: sxlmr200169@163.com

(X. He) College of Science, Minzu University of China
Beijing 100081, China
Email address: xmhe923@muc.edu.cn

(M. Squassina) Dipartimento di Matematica e Fisica Università Cattolica del Sacro Cuore,
Via dei Musei 41, I-25121 Brescia, Italy
Email address: marco.squassina@unicatt.it