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ENERGY CONVEXITY ESTIMATES FOR NON-DEGENERATE GROUND STATES OF NONLINEAR 1D SCHRÖDINGER SYSTEMS

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Abstract. We study the spectral structure of the complex linearized operator for a class of nonlinear Schrödinger systems, obtaining as byproduct some interesting properties of non-degenerate ground state of the associated elliptic system, such as being isolated and orbitally stable.

1. Introduction and main results.

In the last few years, the interest in the study of Schrödinger systems has considerably increased, in particular, for the following class of two weakly coupled nonlinear Schrödinger equations

$$\begin{cases} \mathrm{i}\partial_t\phi_1 + \dfrac{1}{2}\partial_{xx}\phi_1 + \big(|\phi_1|^{2p} + \beta|\phi_2|^{p+1}|\phi_1|^{p-1}\big)\phi_1 = 0 & \text{in } \mathbb{R}\times\mathbb{R}^+, \\ \mathrm{i}\partial_t\phi_2 + \dfrac{1}{2}\partial_{xx}\phi_2 + \big(|\phi_2|^{2p} + \beta|\phi_1|^{p+1}|\phi_2|^{p-1}\big)\phi_2 = 0 & \text{in } \mathbb{R}\times\mathbb{R}^+, \\ \phi_1(0,x) = \phi_1^0(x), \quad \phi_2(0,x) = \phi_2^0(x) & \text{in } \mathbb{R}, \end{cases} \tag{1.1}$$

where $\Phi = (\phi_1, \phi_2)$ and $\phi_i : [0,\infty)\times\mathbb{R} \to \mathbb{C}$, $\phi_i^0 : \mathbb{R} \to \mathbb{C}$, $0 < p < 2$. Usually the coupling constant $\beta > 0$ models the birefringence effects inside a given anisotropic material (see e.g. [13], [14]). A soliton or standing wave solution is a solution of the form $\Phi(x,t) = (u_1(x)e^{\mathrm{i}t}, u_2(x)e^{\mathrm{i}t})$ where $U(x) = (u_1(x), u_2(x))$ solves the elliptic system

$$\begin{cases} -\dfrac{1}{2}\partial_{xx}r_1 + r_1 = r_1^{2p+1} + \beta r_1^p r_2^{p+1} & \text{in } \mathbb{R}, \\ -\dfrac{1}{2}\partial_{xx}r_2 + r_2 = r_2^{2p+1} + \beta r_2^p r_1^{p+1} & \text{in } \mathbb{R}. \end{cases} \tag{1.2}$$

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Among all the solutions of (1.2) there are the ground states, namely least energy solutions. It is known (see e.g. [11], [19]) that for $p \geq 1$ there exists a ground state $R = (r_1, r_2) \in C^2(\mathbb{R}) \cap W^{2,s}(\mathbb{R})$ for any positive s; Moreover, R has nonnegative components r_i which are even, decreasing on $\mathbb{R}^+$ and exponentially decaying. In [12] it is shown that R can be characterized as a solutions of the following minimization problem

$$\mathcal{E}(R) = \inf_{\mathcal{M}} \mathcal{E}(V) \qquad \text{where} \qquad \mathcal{M} := \left\{ V \in H^1(\mathbb{R}) \times H^1(\mathbb{R}),\, \|V\|_2 = \|R\|_2 \right\}, \tag{1.3}$$

and

$$\mathcal{E}(V) = \mathcal{E}(v_1, v_2) = \frac{1}{2}\,\|\partial_x V\|_2^2 - \frac{1}{p+1} \int \left(|v_1|^{2p+2} + |v_2|^{2p+2} + 2\beta |v_1 v_2|^{p+1}\right), \tag{1.4}$$

when the exponent p satisfies

$$1 \leq p < 2. \tag{1.5}$$

The interest in finding ground states is also motivated by their properties with respect of the analysis of the dynamical system (1.1), such as stability properties. For the single Schrödinger equation many notions of stability have been introduced and proved, among all, we recall [5] and [21, 22]; in the former it is proved that the ground state, which is unique, of the equation

$$-\frac{1}{2}\partial_{xx} z + z = z^{2p+1} \quad \text{in } \mathbb{R}, \tag{1.6}$$

is orbitally stable, that is, roughly speaking, if ϕ^0 is a function close to z with respect to the H^1 norm then the solution of the Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t \phi + \dfrac{1}{2}\partial_{xx}\phi + |\phi|^{2p}\phi = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ \phi(0, x) = \phi^0(x) & \text{in } \mathbb{R}, \end{cases} \tag{1.7}$$

where $\phi : [0, \infty) \times \mathbb{R} \to \mathbb{C}$, $\phi^0 : \mathbb{R} \to \mathbb{C}$ and $1 \leq p < 2$, remains close to z up to phase rotations and translations. This kind of results has been extended to Schrödinger systems in [17] and [16] in the one dimensionale case. In [21, 22] the stability analysis for the single equation becomes deeper assuming that z is non-degenerate, that is the linearized operator for (1.6) has a 1-dimensional kernel which is spanned by $\partial_x z$. More precisely, it is proved that for every $\phi \in H^1(\mathbb{R})$ such that $\|\phi\|_{L^2} = \|z\|_{L^2}$, the following inequality holds

$$\mathcal{E}(\phi) - \mathcal{E}(z) \geq C \inf_{\substack{x_0 \in \mathbb{R} \\ \theta \in [0, 2\pi)}} \|\phi - e^{i\theta} z(\cdot - x_0)\|_{H^1}^2, \tag{1.8}$$

for some positive constant C, provided that the energy $\mathcal{E}(\phi)$ is sufficiently close to $\mathcal{E}(z)$. Here, $\mathcal{E}$ is the energy defined in (1.4) once we consider $V = (z, 0)$. Inequality (1.8) allows to provide not only the same orbital stability result proved in [5], but it also permits to derive explicit differential equation to which the phase and position adjustment have to obey for the ground state to be linearly stable. Moreover, (1.8) tells us that the energy functional can be seen as a Lyapunov functional, as it measures the deviation of the solution of (1.1) from the ground state orbit.

The main goal of this paper is to extend inequality (1.8) to the more general framework of 1D vector Schrödinger problems. In order to do this we are lead to consider non-degenerate ground state for system (1.2). This notion is introduced in the following definition.

Definition 1.1. We will say that a ground state solution $R = (r_1, r_2)$ of system (1.2) is non-degenerate if the set of solutions of the linearized system

$$\begin{cases} -\frac{1}{2}\partial_{xx}\phi + \phi = [(2p+1)r_1^{2p} + \beta p r_1^{p-1} r_2^{p+1}]\phi + \beta(p+1) r_1^p r_2^p \psi & \text{in } \mathbb{R}, \\ -\frac{1}{2}\partial_{xx}\psi + \psi = [(2p+1)r_2^{2p} + \beta p r_1^{p+1} r_2^{p-1}]\psi + \beta(p+1) r_1^p r_2^p \phi & \text{in } \mathbb{R}, \end{cases} \tag{1.9}$$

is an 1-dimensional vector space and any solution (ϕ, ψ) of (1.9) is given by $\theta\partial_x R$, for some $\theta \in \mathbb{R}$.

The main result of the paper is stated in the following

Theorem 1.2. *Let R be non-degenerate and assume (1.5). Then there exists a real constant $C > 0$ such that, for every $\Phi \in H^1 \times H^1$ with*

$$\|\Phi\|_{L^2\times L^2} = \|R\|_{L^2\times L^2},$$

the following inequality holds

$$\begin{aligned} \mathcal{E}(\Phi) - \mathcal{E}(R) \geq & C \inf_{\substack{x\in\mathbb{R} \\ \theta\in[0,2\pi)^2}} \|\Phi - (e^{\mathrm{i}\theta_1} r_1(\cdot - x), e^{\mathrm{i}\theta_2} r_2(\cdot - x))\|^2_{H^1\times H^1} \\ & + o\Big(\inf_{\substack{x\in\mathbb{R} \\ \theta\in[0,2\pi)^2}} \|\Phi - (e^{\mathrm{i}\theta_1} r_1(\cdot - x), e^{\mathrm{i}\theta_2} r_2(\cdot - x))\|^2_{H^1\times H^1} \Big) \end{aligned}$$

where $o(x)$ satisfies $o(x)/x \to 0$ as $x \to 0$.

As interesting consequences, we will obtain the property of being isolated, and of being orbitally stable for a non-degenerate ground state. In [12] it has been recently proved that the set of ground states of (1.2) enjoys the orbital stability property. To this respect, we have to recall that up to now it is not yet been proved a uniqueness result for ground state solutions of the system (1.2). Therefore, a solution of (1.1) which starts near a ground state R, may leave the orbit around R and approach the orbit generated by another ground state. But, this is not the case, once we know that the ground states are isolated. This property is easily obtained as a consequence of Theorem 1.2 as stated in the following corollary.

Corollary 1.3. *Let R be non-degenerate and assume (1.5). Then R is isolated, that is, if there exists a ground state of (1.2) S satisfying $\|R - S\|_{\mathbb{H}^1} < \delta$ for a $\delta > 0$ sufficiently small, then $S = R$ up to a translation and a phase change.*

Then, we can also prove the following

Corollary 1.4. *Let R be non-degenerate and assume (1.5). Then R is orbitally stable.*

We recall that a ground state $R = (r_1, r_2)$ is said to be orbitally stable if for any given $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that

$$\sup_{t\in[0,\infty)} \inf_{\substack{x\in\mathbb{R} \\ \theta\in[0,2\pi)^2}} \|\Psi(t,\cdot) - (e^{\mathrm{i}\theta_1} r_1(\cdot - x), e^{\mathrm{i}\theta_2} r_2(\cdot - x)\|_{H^1\times H^1} < \varepsilon,$$

provided that

$$\inf_{\substack{x\in\mathbb{R} \\ \theta\in[0,2\pi)^2}} \|\Psi^0 - (e^{\mathrm{i}\theta_1} r_1(\cdot - x), e^{\mathrm{i}\theta_2} r_2(\cdot - x)\|_{H^1\times H^1} < \delta,$$

where Ψ is the solution of (1.1) with initial datum Ψ^0.

Theorem 1.2 plays a very important role also in the study of the so-called *soliton dynamics* for Schrödinger. More precisely, when one considers (1.1) when the

Plank's constant $\hbar$ explicitly appears in the equations, and studies the evolution, in the semi-classical limit ($\hbar \to 0$), of the solution of (1.1) starting from a $\hbar$-scaling of a soliton, once the action of external forces appears. We refer the reader to [3, 9, 10] for the scalar case and to [15] for systems, where the authors have recently showed, in semi-classical regime, how the soliton dynamics can be derived from Theorem 1.2.

Finally, we have to point out that some of our results can be proved in general dimension $n \geq 1$ as well, with minor changes. Unfortunately, this is not the case for our main Theorem, since, in order to work on the linearized equation, and to perform Taylor expansion on the energy functional $\mathcal{E}$, we need enough regularity on the nonlinear term and this forces us to restrict the range of p because of the presence of the coupling term. Of course, it is a really interesting open problem, to prove the assertion of Theorem 1.2 for any $n \geq 1$ and any $0 < p < 2/n$.

In Section 2, we will study some delicate spectral properties of the linearized system introduced in Definition 1.1. The proofs of Theorem 1.2 and of Corollaries 1.3 and 1.4 will be carried out in Section 3. Finally, in Section 4, we shall prove that there exists a non-degenerate ground state for system (1.2).

2. Spectral analysis of the linearized operators.

In this section we will prove some important properties concerning the linearized Schrödinger system associated with (1.1). We will make use of the functional spaces $\mathbb{L}^2 = L^2(\mathbb{R}, \mathbb{C}) \times L^2(\mathbb{R}, \mathbb{C})$ and $\mathbb{H}^1 = H^1(\mathbb{R}, \mathbb{C}) \times H^1(\mathbb{R}, \mathbb{C})$. We recall that the inner product between $u, v \in \mathbb{C}$ is given by $u \cdot v = \Re(u\bar{v}) = 1/2(u\bar{v} + v\bar{u})$. It is known (see [4, 20]) that (1.1) is well locally posed in time, for any p, in the space $\mathbb{H}^1$ endowed with the norm $\|\Phi\|^2_{\mathbb{H}^1} = \|\partial_x \Phi\|_2^2 + \|\Phi\|_2^2$ for every $\Phi = (\phi_1, \phi_2) \in \mathbb{H}^1$. Moreover we set the $\mathbb{L}^q$ norm as $\|\Phi\|_q^q = \|\phi_1\|_q^q + \|\phi_2\|_q^q$ for any $q \in [1, \infty)$, we denote by (U, V) the inner scalar product in $\mathbb{L}^2$ and by $(U, V)_{\mathbb{H}^1}$ the inner scalar product in $\mathbb{H}^1$. In [7] it is proved that, for p satisfying $0 < p < 2$ the solution of the Cauchy problem (1.1) exists globally in time and the mass of a solution and its total energy are preserved in time, that is having defined the total energy of system (1.1) as

$$\mathcal{E}(\Phi(t)) = \frac{1}{2} \|\partial_x \Phi(t)\|_2^2 - \int F(\Phi(t)) \tag{2.1}$$

where

$$F(U) = F(u_1, u_2) = \frac{1}{p+1} \Big(|u_1|^{2p+2} + |u_2|^{2p+2} + 2\beta |u_1 u_2|^{p+1} \Big), \tag{2.2}$$

the following conservation laws hold (see [7]):

$$\|\phi_1\|_2^2 = \|\phi_1^0\|_2^2, \quad \|\phi_2\|_2^2 = \|\phi_2^0\|_2^2, \quad \mathcal{E}(\Phi(t)) = \mathcal{E}(0) = \frac{1}{2} \left\|\partial_x \Phi^0\right\|_2^2 - \int F\left(\Phi^0\right).$$

Setting $\phi_i = r_i + \varepsilon w_i$, $i = 1, 2$, the linearized Schrödinger system at r_i in w_i is

$$\begin{cases} i\partial_t w_1 + \dfrac{1}{2}\partial_{xx} w_1 - w_1 + G_1(w_1, w_2) = 0 & \text{in } \mathbb{R}, \\ i\partial_t w_2 + \dfrac{1}{2}\partial_{xx} w_2 - w_2 + G_2(w_1, w_2) = 0 & \text{in } \mathbb{R}, \end{cases} \tag{2.3}$$

where we have set

$$\begin{aligned}G_1(w_1,w_2) &= \left[r_1^{2p}+\beta r_1^{p-1}r_2^{p+1}\right]w_1\\&\quad+\left[2pr_1^{2p}+\beta(p-1)r_1^{p-1}r_2^{p+1}\right]\Re(w_1)+\beta(p+1)r_1^pr_2^p\Re(w_2),\\G_2(w_1,w_2) &= \left[r_2^{2p}+\beta r_1^{p+1}r_2^{p-1}\right]w_2\\&\quad+\left[2pr_2^{2p}+\beta(p-1)r_1^{p+1}r_2^{p-1}\right]\Re(w_2)+\beta(p+1)r_1^pr_2^p\Re(w_1).\end{aligned}$$

System (2.3) can be written down as $\partial_t W = LW$, where $L:\mathbb{L}^2\times\mathbb{L}^2\to\mathbb{L}^2\times\mathbb{L}^2$ is the operator defined by

$$L=\begin{pmatrix}0 & L_-\\ -L_+ & 0\end{pmatrix},\qquad W\in\mathbb{C}^2,\ W=(w_1,w_2)$$

and where the operators $L_-,\ L_+ : L^2(\mathbb{R},\mathbb{R})\times L^2(\mathbb{R},\mathbb{R})\to L^2(\mathbb{R},\mathbb{R})\times L^2(\mathbb{R},\mathbb{R})$ acting respectively on the real and imaginary parts of w_i. are the following

$$L_+=\begin{pmatrix}L_+^{11} & L_+^{12}\\ L_+^{21} & L_+^{22}\end{pmatrix}\qquad L_-=\begin{pmatrix}L_-^{11} & 0\\ 0 & L_-^{22}\end{pmatrix}\tag{2.4}$$

where $L^{ij}_{+,-}:L^2(\mathbb{R},\mathbb{R})\to L^2(\mathbb{R},\mathbb{R})$ are defined by

$$\begin{aligned}L_+^{11} &= -\frac{1}{2}\partial_{xx}+1-H^{11}(R)\qquad L_+^{12}=L_+^{21}=-H^{12}(R)\\L_+^{22} &= -\frac{1}{2}\partial_{xx}+1-H^{22}(R)\\L_-^{11} &= -\frac{1}{2}\partial_{xx}+1-\left[r_1^{2p}+\beta r_1^{p-1}r_2^{p+1}\right]\\L_-^{22} &= -\frac{1}{2}\partial_{xx}+1-\left[r_2^{2p}+\beta r_1^{p+1}r_2^{p-1}\right]\end{aligned}$$

and the Hessian matrix $H_F(U)=(H^{ij}):(\mathbb{R}^+)^2\to M_{2\times 2}(\mathbb{R})$ is given by

$$\begin{aligned}H^{11} &= (2p+1)u_1^{2p}+p\beta u_1^{p-1}u_2^{p+1}\qquad H^{12}=H^{21}=(p+1)\beta u_2^pu_1^p\\H^{22} &= (2p+1)u_2^{2p}+p\beta u_2^{p-1}u_1^{p+1}.\end{aligned}$$

We will study L_+ on $\mathcal{V}$, namely the closed subspace of $\mathbb{H}^1$ defined as

$$\mathcal{V}=\left\{U\in\mathbb{H}^1\,:\,(U,R)=0\right\}.\tag{2.5}$$

The first important property of L_+ on $\mathcal{V}$ is proved in the following proposition.

Proposition 2.1. *Assume* (1.5) *and that R a ground state of* (1.2)*. Then*

$$\inf_{\mathcal{V}}\,(L_+(U),U)=0.$$

Proof. First notice that $U_*=(r_1',r_2')$ belongs to $\mathcal{V}$ and U_* satisfies $(L_+(U_*),U_*)=0$, showing that the infimum is less or equal than zero. On the other hand, since R solves problem (1.3), of course R is also a minimum point of $\mathcal{I}=\mathcal{E}(\Phi)+\|\Phi\|_2^2$ on $\mathcal{M}$. Consequently, for any smooth curve $\varphi:[-1,1]\to\mathcal{M}$ such that $\varphi(0)=R$, it follows

$$\left.\frac{d^2\mathcal{I}(\varphi(s))}{ds^2}\right|_{s=0}\geq 0.$$

Therefore, taking into account that $\mathcal{I}'(R) = 0$, we get

$$\begin{aligned}0 &\le \langle \mathcal{I}''(\varphi(s))\varphi'(s), \varphi'(s)\rangle\Big|_{s=0} + \langle \mathcal{I}'(\varphi(s)), \varphi''(s)\rangle\Big|_{s=0} \\ &= \langle \mathcal{I}''(R)\varphi'(0), \varphi'(0)\rangle + \langle \mathcal{I}'(R), \varphi''(0)\rangle = \langle \mathcal{I}''(R)\varphi'(0), \varphi'(0)\rangle.\end{aligned}$$

Now, taking into account that the map $s \mapsto \|\varphi(s)\|_2$ is constant, it readily follows that $\varphi'(0)$ belongs to $\mathcal{V}$, which yields the assertion by the arbitrariness of φ. $\square$

The above result is the first step to show that L_+ is coercive once we restrict it on a closed subspace of $\mathcal{V}$, as shown in the following proposition.

Proposition 2.2. *Assume* (1.5) *and that R is a ground state of* (1.2) *satisfying Definition 1.1. Then*

$$\inf_{U\in\mathcal{V}_0,} \frac{(L_+(U), U)}{\|U\|_2^2} > 0, \quad \mathcal{V}_0 = \{U \in \mathbb{H}^1 : (U, R) = (U, H_F(R)\partial_x R) = 0\}. \tag{2.6}$$

Proof. Denoting with α the infimum

$$\alpha = \inf_{\|V\|_{L^2}=1,\, V\in\mathcal{V}_0} (L_+(V), V),$$

first notice that Proposition 2.1 implies that α is nonnegative, so that we only have to show that α is not zero. Let us argue by contradiction and suppose that $\alpha = 0$. Taken U_n a minimizing sequence, from the regularity properties of R it follows that U_n is bounded in $\mathbb{H}^1$. These gives us a function $U \in \mathbb{H}^1$, such that $U_n \rightharpoonup U$ weakly (up to a subsequence) in $\mathbb{H}^1$, implying that $U \in \mathcal{V}_0$. From Proposition 2.1 and (2.6), we get

$$\begin{aligned}0 \le (L_+(U), U) &\le \liminf_{n\to\infty}\{\|U_n\|_{\mathbb{H}^1}^2 - (U_n, H_F(R)U_n)\} \\ &= \lim_{n\to\infty} (L_+(U_n), U_n) = 0.\end{aligned}$$

So that U solves $(L_+(U), U) = 0$ and $(L_+(U_n), U_n) \to (L_+(U), U)$. Moreover,

$$\begin{aligned}\|U\|_{\mathbb{H}^1}^2 &\le \liminf_{n\to\infty} \|U_n\|_{\mathbb{H}^1}^2 \le \limsup_{n\to\infty} \|U_n\|_{\mathbb{H}^1}^2 \\ &= \lim_{n\to\infty} \{(L_+(U_n), U_n) + (U_n, H_F(R)U_n)\} \\ &= (L_+(U), U) + (U, H_F(R)U) = \|U\|_{\mathbb{H}^1}^2,\end{aligned}$$

from which $U_n \to U$ strongly in $\mathbb{H}^1$, so that $\|U\|_{\mathbb{L}^2} = 1$ and U solves the constrained minimization problem (2.6). When we derive the functional $(L_+(V), V)/\|V\|_{\mathbb{L}^2}^2$ and use that $(L_+(U), U) = 0$ we obtain that there exists Lagrange multipliers μ, $\gamma \in \mathbb{R}$ such that

$$(L_+U, V) = \mu\,(R, V) + (\gamma \cdot H_F(R)\partial_x R, V)\,, \qquad \text{for every } V \in \mathbb{H}^1. \tag{2.7}$$

Choosing as test function $V = \partial_x R$ and taking into consideration that $(R, \partial_j R) = 0$, gives

$$0 = (L_+(U), \partial_x R) = (\gamma \cdot H_F(R)\partial_x R, \partial_x R) = \gamma(H_F(R)\partial_x R, \partial_x R),$$

where we have taken into account that L_+ is a self-adjoint operator and $\partial_x R = (\partial_x r_1, \partial_x r_2)$ is a solution of $L_+V = 0$. Since R has even components the summands on the right hand side are nonzero, so that $\gamma = 0$. As a consequence, U solves $L_+U = \mu R$. Moreover, we consider the vector $x \cdot \partial_x R$, whose components are $x \cdot$

$\partial_x R = (x\partial_x r_1, x\partial_x r_2)$ and we compute $L_+(x \cdot \partial_x R)$. After some simple calculations, one reaches

$$L_+(x \cdot \partial_x R) = (-\partial_{xx} r_1, -\partial_{xx} r_2),$$
$$L_+(R/p) = -2(r_1^{2p+1} + \beta r_2^{p+1} r_1^p, r_2^{2p+1} + \beta r_1^{p+1} r_2^p).$$

Then, in turn, we get $L_+(R/p + x \cdot \partial_x R) = -2R$, and by linearity

$$L_+\left(-\mu/2(R/p + x \cdot \partial_x R)\right) = \mu R.$$

Then, Definition 1.1 (nondegeneracy) immediately yields

$$U = -\mu/2(R/p + x \cdot \partial_x R) + \theta \cdot \partial_x R \tag{2.8}$$

for some constant $\theta \in \mathbb{R}$. Now we have to show that $\theta = 0$, by using the available constraints. By applying to equation (2.8) the self-adjoint operator $H_F = H_F(R)$, we get

$$H_F U = -\frac{\mu}{2p} H_F R - \frac{\mu}{2} H_F x \cdot \partial_x R + H_F \theta \cdot \partial_x R.$$

As $U \in \mathcal{V}_0$, it results $(H_F U, \partial_x R) = (U, H_F \partial_x R) = 0$. Furthermore, since R is a radial solution of (1.2), we also have that $(H_F R, \partial_x R) = (H_F x \cdot \partial_x R, \partial_x R) = 0$. On the other hand

$$(H_F \theta \cdot \partial_x R, \partial_x R) = \theta (H_F \partial_x R, \partial_x R) = c\theta$$

with $c \neq 0$, so it has to be $\theta = 0$. Then (2.8) reduces to

$$U = -\frac{\mu}{2p} R - \frac{\mu}{2} x \cdot \partial_x R.$$

Computing the L^2-scalar product with R and keeping in mind that $U \in \mathcal{V}_0$ yields

$$0 = (U, R) = -\frac{\mu}{2}\left[\frac{1}{p}\|R\|_2^2 + (x \cdot \partial_x R, R)\right].$$

As far as concern the last term in the previous relation, we integrate by parts and obtain

$$(x \cdot \partial_x R, R) = -\frac{1}{2}\|R\|_2^2.$$

The last two equations and (1.5) give the desired contradiction. $\square$

Remark 2.3. The argument in the proof of the previous Proposition shows that there exists a positive constant α_0 such that

$$(L_+ V, V) \geq \alpha_0 \|V\|_2^2, \qquad \text{for all } V \in \mathcal{V}_0. \tag{2.9}$$

Moreover, if we consider $|||U||| = \sqrt{(L_+ U, U)}$ for every $U \in \mathcal{V}_0$, we obtain that $|||\cdot|||$ satisfies all the required properties of a norm, by (2.9) and by the self-adjointness property of L_+. In addition, every Cauchy sequence $\{U_n\}$ with respect to $|||\cdot|||$ has a strong limit U belonging L^2; moreover U satisfies all the orthogonality relations required in $\mathcal{V}_0$. Besides, computing $(L_+(U_n - U_m), U_n - U_m)$ gives that also $\{\partial_x U_n\}$ is a Cauchy sequence in L^2 then U is necessarily the strong limit of $\{U_n\}$ in $\mathbb{H}^1$. Finally, $|||U_n - U||| \to 0$ by the definition of L_+. As a consequence, $\mathcal{V}_0$ is a Banach space with respect to this norm, and we get the equivalence with the standard $\mathbb{H}^1$ norm, namely there exists $\alpha > 0$ such that

$$(L_+ V, V) \geq \alpha \|V\|_{\mathbb{H}^1}^2, \qquad \text{for all } V \in \mathcal{V}_0.$$

Before stating our next result let us prove the following lemma.

Lemma 2.4. *Let us take $\Phi \in \mathbb{L}^2$ such that $\|\Phi\|_2 = \|R\|_2$ and consider the difference $W = \Phi - R$. Denoting with U and V the real and imaginary part of W, it results*

$$(R, U) = -\frac{1}{2}\left[\|U\|_2^2 + \|V\|_2^2\right] = -\frac{1}{2}\|W\|_2^2 \tag{2.10}$$

Proof. The above identity immediately follows by imposing $\|R + W\|_2^2 = \|R\|_2^2$ and by recalling that R is a real function. □

Proposition 2.5. *Assume (1.5) and that R satisfies Definition 1.1. Moreover, let us take $W = U + iV$ satisfying (2.10) with U verifying*

$$(U, H_F(R)\partial_x R) = 0. \tag{2.11}$$

Then, there exists positive constants D, D_i such that

$$(L_+U, U) \geq D\|U\|_{\mathbb{H}^1}^2 - D_1\|W\|_{\mathbb{H}^1}^4 - D_2\|W\|_{\mathbb{H}^1}^3. \tag{2.12}$$

Proof. Without loss of generality, we can suppose that $\|R\|_2 = 1$; moreover, we decompose U as $U = U_{||} + U_{\perp}$ where $U_{||} = (U, R)\, R$, while $U_{\perp} = U - U_{||}$ is orthogonal to R with respect to the L^2 scalar product. Since L_+ is self-adjoint it results

$$(L_+U, U) = \left(L_+U_{||}, U_{||}\right) + 2\left(L_+U_{\perp}, U_{||}\right) + \left(L_+U_{\perp}, U_{\perp}\right). \tag{2.13}$$

Next, we study separately the summands on the right hand side of this formula. Observe that, taking into account identity (2.10), we have

$$\|\partial_x U_{\perp}\|_2^2 \geq \|\partial_x U\|_2^2 - C\|W\|_2^2\|\partial_x W\|_2, \tag{2.14}$$

for some positive constant C. Since $(U_{||}, H_F(R)\partial_x R) = 0$, condition (2.11) implies that also $U_{\perp}$ has to be orthogonal to $H_F(R)\partial_x R$, hence $U_{\perp}$ is in $\mathcal{V}_0$. Then Remark 2.3, (2.14) and (2.10) give us

$$\begin{aligned}(L_+U_{\perp}, U_{\perp}) &\geq D\|U_{\perp}\|_{\mathbb{H}^1}^2 \geq D\|U\|_{\mathbb{H}^1}^2 - CD\|W\|_2^2\|\partial_x W\|_2 - D\|U_{||}\|_2^2 \\ &= D\|U\|_{\mathbb{H}^1}^2 - d_1\|W\|_2^2\left[\|W\|_2^2 + \|\partial_x W\|_2\right].\end{aligned} \tag{2.15}$$

We also obtain from (2.10) that

$$\begin{aligned}\left(L_+U_{\perp}, U_{||}\right) &= (R, U)\,(L_+U_{\perp}, R) = -\frac{1}{2}\|W\|_2^2\,(L_+U_{\perp}, R) \\ &\geq -d_2\|W\|_2^2(\|\partial_x W\|_2 + \|W\|_2).\end{aligned} \tag{2.16}$$

As far as concern the last term in (2.13), it results

$$\left(L_+U_{||}, U_{||}\right) = (U, R)^2\,(L_+R, R) = \frac{1}{4}\|W\|_2^4\,(L_+R, R) \geq -d_3\|W\|_2^4.$$

This last equation, joint with (2.15) and (2.16) yields the conclusion. □

Proposition 2.6. *It results* $\displaystyle\inf_{V \neq 0,\ (v_i, r_i)_{H^1} = 0} \frac{(L_-(V), V)}{\|V\|_2^2} > 0.$

Proof. Let us first prove that L_- is a positive operator. Denoting with $\sigma_d(L_-)$ the discrete spectrum of the operator L_- it results

$$\sigma_d(L_-) = \sigma_d(L_-^{11}) \cup \sigma_d(L_-^{22}). \tag{2.17}$$

Indeed, if $\lambda \in \sigma_d(L_-^{11})$ we get that $L_-^{11}(u) = \lambda u$, then $\lambda \in \sigma_d(L_-)$ with eigenfunction $U = (u, 0)$, analogous argument holds for $\lambda \in \sigma_d(L_-^{22})$, proving that $\sigma_d(L_-^{11}) \cup \sigma_d(L_-^{22}) \subseteq \sigma_d(L_-)$. On the other hand, if $\lambda \in \sigma_d(L_-)$ there exists $U = (u_1, u_2) \neq (0, 0)$ such that

$$L_-^{11}u_1 = \lambda u_1, \quad L_-^{22}u_2 = \lambda u_2$$

so that, if $u_1 \neq 0$ $\lambda \in \sigma_d(L_-^{11})$, otherwise $u_2 \neq 0$ and $\lambda \in \sigma_d(L_-^{22})$, showing (2.17). Moreover, since $L_- R = 0$, with $R = (r_1, r_2) \neq (0,0)$, $r_i \geq 0$, we get that $\lambda = 0$ is the first eigenvalue of L_-^{11} and L_-^{22} when both r_1, $r_2 \neq 0$. Besides, if for example $r_1 \equiv 0$, $\lambda = 0$ is the first eigenvalue of L_-^{22}, while $L_-^{11} = -\partial_{xx} + 1$ and its discrete spectrum is empty (see e.g. Chapter 3 in [2]), yielding that $\lambda = 0$ is the first eigenvalue of L_-. Then $(L_-(V), V) \geq 0$ for every function $V \in \mathbb{H}^1$, proving that L_- is a positive operator. Arguing now as in the proof of Proposition 2.2, and considering the (nonnegative) infimum

$$\alpha = \inf_{\|V\|_{L^2}=1,\ (V_i, r_i)_{H^1}=0} (L_-(V), V),$$

assuming by contradiction that $\alpha = 0$, we find that there exists a nonzero minimizer U (satisfying the constraints) for the problem such that

$$(L_- U, U) = 0 \tag{2.18}$$

Taking into account that the constraints $(U_i, r_i)_{H^1} = 0$ can be written in the L^2 form

$$(q_-^{11}(R)R_1, U) = 0, \qquad (q_-^{22}(R)R_2, U) = 0, \tag{2.19}$$

where we have set $R_1 = (r_1, 0)$, $R_2 = (0, r_2)$ and

$$q_-^{11}(R) = r_1^{2p} + \beta r_1^{p-1} r_2^{p+1}, \qquad q_-^{22}(R) = r_2^{2p} + \beta r_1^{p+1} r_2^{p-1}.$$

we have three lagrange parameters $\lambda, \gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$(L_- U, V) = \lambda(U, V) + \gamma_1(q_-^{11}(R)R_1, V) + \gamma_2(q_-^{22}(R)R_2, V)$$

for all $V \in \mathbb{H}^1$. Hence, by choosing $V = U$ and taking into account (2.18) and that U satisfies the constraints (2.19), we immediately get $\lambda = 0$. Choosing now $V = R_1$ and $V = R_2$ and taking into account L_- is self-adjoint and that $L_- R_i = 0$ we obtain $\gamma_1 = \gamma_2 = 0$. Therefore, we conclude that

$$L_- U = 0,$$

namely $L_-^{11} u_1 = 0$ and $L_-^{22} u_2 = 0$ where we set $U = (u_1, u_2)$. In turn, u_i is a first eigenfunction of L_-^{ii}, which yields $u_i \in \mathrm{span}(r_i)$ since the first eigenvalue is simple (see e.g. Theorem 3.4 in [2]). This is of course a contradiction with (2.19). Hence $\alpha > 0$ and the proof is complete. □

Remark 2.7. Arguing as in Remark 2.3, it is possible to find a positive constant $\alpha > 0$ such that

$$(L_- V, V) \geq \alpha \|V\|_{\mathbb{H}^1}^2, \qquad \text{for all } V \in \mathbb{H}^1 \text{ with } (v_i, r_i)_{H^1} = 0,\ i = 1, 2.$$

3. **Proofs of the main results.** In order to prove Theorem 1.2, the following characterization will be crucial.

Proposition 3.1. *Let us consider $y_0 \in \mathbb{R}$ and $\Gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ be such that*

$$\min_{\substack{x_0 \in \mathbb{R} \\ \Theta \in \mathbb{R}^2}} \|(\phi_1(\cdot + x_0)e^{\mathrm{i}\theta_1}, \phi_2(\cdot + x_0)e^{\mathrm{i}\theta_2}) - R\|_{\mathbb{H}^1}^2 = \|(\phi_1(\cdot + y_0, t)e^{\mathrm{i}\gamma_1}, \phi_2(\cdot + y_0)e^{\mathrm{i}\gamma_2}) - R\|_{\mathbb{H}^1}^2 \tag{3.1}$$

Then, writing

$$(\phi_1(\cdot + y_0, t)e^{\mathrm{i}\gamma_1}, \phi_2(\cdot + y_0, t)e^{\mathrm{i}\gamma_2}) = R + W,$$

where $W = U + \mathrm{i}V$, the following orthogonality condition are satisfied

$$(U, H_F(R)\partial_x R) = 0, \qquad (v_1, r_1)_{H^1} = (v_2, r_2)_{H^1} = 0. \tag{3.2}$$

Proof. Let us introduce the functions P, $Q : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ defined by

$$P(x_0,\Theta) = P(x_0,\theta_1,\theta_2) = \|(\phi_1(\cdot + x_0)e^{\mathrm{i}\theta_1}, \phi_2(\cdot + x_0)e^{\mathrm{i}\theta_2}) - R\|_2^2$$
$$Q(x_0,\Theta) = Q(x_0,\theta_1,\theta_2) = \|(\partial_x\phi_1(\cdot + x_0)e^{\mathrm{i}\theta_1}, \partial_x\phi_2(\cdot + x_0)e^{\mathrm{i}\theta_2}) - \partial_x R\|_2^2.$$

Writing down the partial derivatives of P and Q and integrating by parts, give us

$$\begin{aligned}
\partial_{x_0}P(x_0,\Theta) &= \sum_{j=1}^{2}\int \left(\phi_j e^{\mathrm{i}\theta_j} - r_j\right) e^{-\mathrm{i}\theta_j}\partial_{x_0}\overline{\phi}_j + \left(\overline{\phi}_j e^{-\mathrm{i}\theta_j} - r_j\right) e^{\mathrm{i}\theta_j}\partial_{x_0}\phi_j \\
&= -2\sum_{j=1}^{2}\int r_j \Re\left(e^{\mathrm{i}\theta_j}\partial_{x_0}\phi_j\right); \\
\partial_{x_0}Q(x_0,\Theta) &= \sum_{j=1}^{2}\int \partial_x\left(\phi_j e^{\mathrm{i}\theta_j} - r_j\right)\partial_x\partial_{x_0}\overline{\phi}_j e^{-\mathrm{i}\theta_j} + \partial_x\left(\overline{\phi}_j e^{-\mathrm{i}\theta_j} - r_j\right)\partial_x\partial_{x_0}\phi_j e^{\mathrm{i}\theta_j} \\
&= -2\sum_{j=1}^{2}\int \partial_x r_j \Re\left(\partial_x\partial_{x_0}\phi_j e^{\mathrm{i}\theta_j}\right); \\
\frac{\partial P}{\partial\theta_j}(x_0,\Theta) &= \mathrm{i}\int \left[-\left(\phi_j e^{\mathrm{i}\theta_j} - r_j\right) e^{-\mathrm{i}\theta_j}\overline{\phi}_j + \left(\overline{\phi}_j e^{-\mathrm{i}\theta_j} - r_j\right) e^{\mathrm{i}\theta_j}\phi_j\right] \\
&= 2\int r_j \Im\left(e^{\mathrm{i}\theta_j}\phi_j\right); \\
\frac{\partial Q}{\partial\theta_j}(x_0,\Theta) &= \mathrm{i}\int \left[-\partial_x\left(\phi_j e^{\mathrm{i}\theta_j} - r_j\right)\partial_x\overline{\phi}_j e^{-\mathrm{i}\theta_j} + \partial_x\left(\overline{\phi}_j e^{-\mathrm{i}\theta_j} - r_j\right)\partial_x\phi_j e^{\mathrm{i}\theta_j}\right] \\
&= 2\int \partial_x r_j \Im\left(\partial_x\phi_j e^{\mathrm{i}\theta_j}\right).
\end{aligned}$$

If $x_0 = y_0$ and $\Gamma = (\gamma_1,\gamma_2)$ realize the minimum in the minimization problem (3.1), then the following equations are satisfied

$$\begin{aligned}
\frac{\partial(P+Q)}{\partial x_0}(x_0,\Theta) &= -2\sum_{j=1}^{2}\int \Big[r_j(x)\Re\Big(e^{\mathrm{i}\gamma_j}\frac{\partial\phi_j}{\partial x_0}(x-y_0)\Big) \\
&\qquad + \partial_x r_j(x)\Re\Big(e^{\mathrm{i}\gamma_j}\partial_x\frac{\partial\phi_j}{\partial x_0}(x-y_0)\Big)\Big] = 0, \\
\frac{\partial(P+Q)}{\partial\theta_j}(x_0,\Theta) &= 2\int \left[r_j(x)\Im\left(e^{\mathrm{i}\gamma_j}\phi_j(x-y_0)\right) + \partial_x r_j(x)\Im\left(e^{\mathrm{i}\gamma_j}\partial_x\phi_j(x-y_0)\right)\right] = 0.
\end{aligned}$$

Denoting with U and V the real and imaginary (respectively) part of $W = \Phi(x - y_0)e^{\mathrm{i}\Gamma} - R(x)$ and taking into account that R is real and does not depend on x_0, it follows

$$\begin{aligned}
\frac{\partial(P+Q)}{\partial x_0}(x_0,\Theta) &= \sum_{j=1}^{2}\int \left[r_j\frac{\partial u_j}{\partial x_0} + \partial_x r_j\partial_x\frac{\partial u_j}{\partial x_0}\right] \\
&= -\sum_{j=1}^{2}\int \left[u_j\frac{\partial r_j}{\partial x_0} + \partial_x u_j\partial_x\frac{\partial r_j}{\partial x_0}\right] = 0, \\
\frac{\partial(P+Q)}{\partial\theta_j}(x_0,\Theta) &= \int \left[r_j v_j + \partial_x r_j\partial_x v_j\right] = 0, \quad j = 1,2.
\end{aligned}$$

The second line of the above equations can be read as the orthogonality conditions on V in (3.2). As far as regards U, we only have to notice that $\partial_x R$ satisfies the linearized system of (1.2) so that all the conditions in (3.2) are proved. $\square$

We are now ready to complete the proof of the main result, Theorem 1.2.

Proof of Theorem 1.2 concluded. Let us consider $\Phi \in \mathbb{H}^1$ with $\|\Phi\|_2 = \|R\|_2$ and $W(x) = \Phi(x-y_0)e^{i\Gamma} - R(x)$, where $y_0 \in \mathbb{R}$ and $\Gamma \in \mathbb{R}^2$ satisfy the minimality conditions (3.1). We want to control the $\mathbb{H}^1$ norm of W in terms of the difference $\mathcal{I}(\Phi) - \mathcal{I}(R)$, being $\mathcal{I}$ is the action functional associated to the system and defined as

$$\mathcal{I}(\Phi) = \mathcal{E}(\Phi) + \|\Phi\|_2^2.$$

To this aim, we first compute the difference $\mathcal{I}(\Phi) - \mathcal{I}(R)$ and we use scale invariance, obtaining $\mathcal{I}(\Phi) - \mathcal{I}(R) = \mathcal{I}(R+W) - \mathcal{I}(R)$. Then, recalling that $\langle \mathcal{I}'(R), W\rangle = 0$, Taylor expansion gives

$$\begin{aligned}\mathcal{I}(\Phi) - \mathcal{I}(R) &= \mathcal{I}(R+W) - \mathcal{I}(R) = \langle \mathcal{I}'(R), W\rangle + \langle \mathcal{I}''(R+\vartheta W)W, W\rangle \\ &= \langle \mathcal{I}''(R)W, W\rangle + \langle \mathcal{I}''(R+\vartheta W)W, W\rangle - \langle \mathcal{I}''(R)W, W\rangle.\end{aligned}$$

In order to evaluate the difference on the right hand side we will use the C^2 regularity of $\mathcal{I}$, at this point it is crucial (1.5). For simplicity, let us consider separately the nonlinear terms in $\mathcal{I}$. The term $G : \mathbb{H}^1 \to \mathbb{R}$ defined by

$$G(U) = G(u_1, u_2) = \|u_1\|_{2p+2}^{2p+2} + \|u_2\|_{2p+2}^{2p+2},$$

is of class C^3, as $p \geq 1$, so that

$$\langle G''(R+\vartheta W)W, W\rangle - \langle G''(R)W, W\rangle \geq -c_1\|W\|_{\mathbb{H}^1}^3. \tag{3.3}$$

As far as concern the coupling term $\Upsilon : \mathbb{H}^1 \to \mathbb{R}$ defined by $\Upsilon(U) = \Upsilon(u_1, u_2) = \|u_1 u_2\|_{p+1}^{p+1}$, it results

$$\begin{aligned}\langle \Upsilon''(U)W, W\rangle =&(p^2-1)\int |u_1|^{p-3}|u_2|^{p-3}\left[|u_2|^4\Re^2(u_1)|w_1|^2 + |u_1|^4\Re^2(u_2)|w_2|^2\right] \\ &+ (p+1)\int |u_1|^{p-1}|u_2|^{p-1}\left[|u_2|^2|w_1|^2 + |u_1|^2|w_2|^2\right] \\ &+ 2(p+1)^2\int |u_1|^{p-1}|u_2|^{p-1}\Re(u_1)\Re(u_2)\Re(w_1\overline{w}_2).\end{aligned}$$

When we write the difference $\langle \Upsilon''(R)W, W\rangle - \langle \Upsilon''(R+\vartheta W)W, W\rangle$ we use that R is a real function and we control the first two terms with the real parts by the modulus; finally we use the inequality

$$\big||r_j + \vartheta w_j|^{p-1} - |r_j|^{p-1}\big| \leq C|w_j|^{p-1},$$

to get

$$\langle \Upsilon''(R)W, W\rangle - \langle \Upsilon''(R+\vartheta W)W, W\rangle \geq -c_1\|W\|_{\mathbb{H}^1}^{2+\mu} \qquad \text{for some } \mu > 0. \tag{3.4}$$

This inequality joint with (3.3) implies that

$$\langle \mathcal{I}''(R+\vartheta W)W, W\rangle - \langle \mathcal{I}''(R)W, W\rangle \geq -C\|W\|_{\mathbb{H}^1}^{2+\mu}. \tag{3.5}$$

Therefore,

$$\mathcal{I}(\Phi) - \mathcal{I}(R) \geq \langle \mathcal{I}''(R)W, W\rangle - C\|W\|_{\mathbb{H}^1}^{2+\mu} = \langle L_- V, V\rangle + \langle L_+ U, U\rangle - C\|W\|_{\mathbb{H}^1}^{2+\mu}.$$

Taking into account the orthogonality conditions of Proposition 3.1, the assertion now follows from Proposition 2.5 and Remark 2.7. $\square$

Proof of Corollary 1.3. Let δ be a positive number to be chosen later. Moreover, let $R = (r_1, r_2) \in \mathbb{H}^1$ and $S = (s_1, s_1) \in \mathbb{H}^1$ be two given non-degenerate ground state solutions to system (1.2) such that

$$\|R - S\|^2_{\mathbb{H}^1} < \delta.$$

Then, taking into account the variational characterization (1.3) for ground states, we learn that

$$\mathcal{E}(R) = \mathcal{E}(S), \qquad \|R\|_{\mathbb{L}^2} = \|S\|_{\mathbb{L}^2}.$$

Notice also that

$$\inf_{\substack{x_0 \in \mathbb{R} \\ \theta \in \mathbb{R}^2}} \|R - (e^{i\theta_1} s_1(\cdot - x_0), e^{i\theta_2} s_2(\cdot - x_0))\|^2_{\mathbb{H}^1} \leq \|R - S\|^2_{\mathbb{H}^1} < \delta.$$

Therefore, by applying Theorem 1.2, if $\delta > 0$ is chosen sufficiently small, we get

$$\inf_{\substack{x_0 \in \mathbb{R} \\ \theta \in \mathbb{R}^2}} \|R - (e^{i\theta_1} s_1(\cdot - x_0), e^{i\theta_2} s_2(\cdot - x_0))\|^2_{\mathbb{H}^1} \leq 0.$$

In turn we conclude that $R = S$, up to a suitable translation and phase change. □

Proof of Corollary 1.4. Let $T > 0$ and let us fix $\varepsilon > 0$ sufficiently small. Consider the solution Ψ of system (1.1) with initial datum Ψ^0. By the conservation laws, we have

$$\|\Psi(t)\|_{\mathbb{L}^2} = \|\Psi^0\|_{\mathbb{L}^2}, \quad \mathcal{E}(\Psi(t)) = \mathcal{E}(\Psi^0), \quad \text{for all } t \in [0, \infty).$$

By the continuity of the energy $\mathcal{E}$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\mathcal{E}(\Psi(t)) - \mathcal{E}(R) = \mathcal{E}(\Psi^0) - \mathcal{E}(R) < \varepsilon, \quad \text{for all } t \in [0, \infty),$$

provided that

$$\inf_{\substack{\theta \in \mathbb{R}^2 \\ x \in \mathbb{R}}} \|\Psi^0(\cdot) - (e^{i\theta_1} r_1(\cdot - x), e^{i\theta_2} r_2(\cdot - x))\|^2_{\mathbb{H}^1} < \delta. \tag{3.6}$$

Then, if we define for any $t > 0$ the positive number

$$\Gamma_{\Psi(t)} = \inf_{\substack{\theta \in \mathbb{R}^2 \\ x \in \mathbb{R}}} \|\Psi(t) - (e^{i\theta_1} r_1(\cdot - x), e^{i\theta_2} r_2(\cdot - x))\|^2_{\mathbb{H}^1},$$

we learn from Theorem 1.2 that there exist two positive constants $\mathcal{A}$ and C such that

$$\Gamma_{\Psi(t)} \leq C(\mathcal{E}(\Psi(t)) - \mathcal{E}(R)), \tag{3.7}$$

provided that $\Gamma_{\Psi(t)} < \mathcal{A}$. Let us define the value

$$T_0 := \sup\big\{t \in [0, T] : \ \Gamma_{\Psi(s)} < \mathcal{A} \text{ for all } s \in [0, t)\big\}.$$

Of course, it holds $T \geq T_0 > 0$ by means of (3.6) (up to reducing the size of δ, if necessary) and the continuity of $\Psi(t)$. Hence, we deduce that

$$\begin{aligned} \sup_{t \in [0, T_0]} \inf_{\substack{\theta \in \mathbb{R}^2 \\ x \in \mathbb{R}}} &\|\Psi(t, \cdot) - (e^{i\theta_1} r_1(\cdot - x), e^{i\theta_2} r_2(\cdot - x))\|^2_{\mathbb{H}^1} \\ &\leq C(\mathcal{E}(\Psi(t)) - \mathcal{E}(R)) = C(\mathcal{E}(\Psi^0) - \mathcal{E}(R)) < C\varepsilon. \end{aligned} \tag{3.8}$$

On the other hand, it is readily seen that, from this inequality, one obtains $T_0 = T$. In fact, assume by contradiction that $T_0 < T$. Then, since by (3.8)

$$\Gamma_{\Psi(T_0)} = \inf_{\substack{\theta \in \mathbb{R}^2 \\ x \in \mathbb{R}}} \|\Psi(T_0, \cdot) - (e^{i\theta_1} r_1(\cdot - x), e^{i\theta_2} r_2(\cdot - x))\|^2_{\mathbb{H}^1} < C\varepsilon,$$

inequality $\Gamma_{\Psi(t)} < \mathcal{A}$ holds true by continuity for any $t \in [T_0, T_0+\rho)$, for some small $\rho > 0$, which is a contradiction by the definition of T_0. Hence $T_0 = T$ and, for any $T > 0$, from (3.8) we get

$$\sup_{t\in[0,T]} \inf_{\substack{\theta\in\mathbb{R}^2 \\ x\in\mathbb{R}}} \|\Psi(t,\cdot) - (e^{i\theta_1} r_1(\cdot - x), e^{i\theta_2} r_2(\cdot - x))\|^2_{\mathbb{H}^1} < C\varepsilon,$$

which is the desired property on $[0,T]$. By the arbitrariness of T the assertion follows. To cover the general case of perturbations Ψ^0 which do not preserve the $\mathbb{L}^2$, namely $\|\Psi_0\|_{\mathbb{L}^2} \neq \|R\|_{\mathbb{L}^2}$, it is sufficient to follow the deformation argument used in the scalar case, see [22, bottom of page 59]. In fact, if R is a non-degenerate ground state solution to (1.2), then $R_\lambda(x) = (r_{1,\lambda}, r_{2,\lambda}) = (\lambda^{1/p} r_1(\lambda x), \lambda^{1/p} r_2(\lambda x))$ is a non-degenerate ground state solution to the system

$$\begin{cases} -\dfrac{1}{2}\partial_{xx} r_{1,\lambda} + \lambda^2 r_{1,\lambda} = r_{1,\lambda}^{2p+1} + \beta r_{1,\lambda}^{p} r_{2,\lambda}^{p+1} & \text{in} \mathbb{R} \\ -\dfrac{1}{2}\partial_{xx} r_{2,\lambda} + \lambda^2 r_{2,\lambda} = r_{2,\lambda}^{2p+1} + \beta r_{2,\lambda}^{p} r_{2,\lambda}^{p+1} & \text{in} \mathbb{R} \end{cases}$$

for all $\lambda > 0$, so that, thanks to assumption (1.5) we can choose $\lambda > 0$ such that $\|R_\lambda\|_{\mathbb{L}^2} = \|\Psi_0\|_{\mathbb{L}^2}$. □

4. **Existence of a non-degenerate ground state.** In the following section we will show that there exists a non-degenerate ground state Z. More precisely, let us consider z be the unique positive radial least energy solution of (1.6) and let a be given by

$$a = (1+\beta)^{-1/2p}. \tag{4.1}$$

We will prove the following result.

Theorem 4.1. *Let a be given in (4.1), then the vector $Z = a(z,z)$ is a non-degenerate ground state of system (1.2) for every $p > 0$, $\beta > 1$ and $p \neq \beta$.*

Remark 4.2. In [11] it is proved that for $\beta \leq 1$ every ground state of (1.2) necessarily has one trivial component, that is the reason of the assumption $\beta > 1$. Moreover, it can been easily seen that for $p = \beta$ the ground state Z is a degenerate solution that is why we assume $p \neq \beta$.

This result will be a consequence of the two following results.

Theorem 4.3. *Let a be given in (4.1), then the vector $Z = a(z,z)$ is a ground state of system (1.2) for every $p > 0$, $\beta > 1$.*

Theorem 4.4. *Let a be given in (4.1), then the vector $Z = a(z,z)$ is a non-degenerate ground state of system (1.2) for every $p > 0$, $\beta > 1$ and $p \neq \beta$.*

Remark 4.5. In [7] it is studied the global existence for the Cauchy problem (1.1) and it is proved that the solution exists for any time if $p < 2/n$, while it can blow up if $p \geq 2/n$. In the critical case $p = 2/n$ it is given a bound on the L^2-norm of the initial data which guarantees the global existence of the solution (see Theorem 2). Since Theorem 4.3 shows that the test functions used in [7] to estimate the blow-up threshold belong to the set of ground state solutions, as a by product, we obtain that the bound given in [7] is the exact threshold value.

Remark 4.6. The above results have been proved for $p = 1$, respectively, in [19] and [6] in any dimension. Actually, the same arguments work for any $p > 0$. In the following we include the details for completeness. Let us notice that the same proof of Theorem 4.3 holds in dimension greater than one; in addition, the arguments used in [6] hold for $p \in (0, 2/n)$ for every $n \geq 1$. Thus, the vector Z is a non-degenerate ground state solution of (1.2) in any dimension $n \geq 1$, our conjecture is that it is the only one if $\beta > 1$. Here our interest, is restricted to the one dimension setting so that we will see the proof of Theorem 4.1 in this case.

4.1. **Proof of Theorem 4.3.** First, we recall this simple facts.

Proposition 4.7. *Let us set*

$$S_1 = \inf_{H^1(\mathbb{R})\setminus\{0\}} \frac{\|u\|_{H^1}^2}{\|u\|_{2p+2}^2}, \qquad T_1 = \inf_{\mathcal{N}_1} \Big\{ \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{2p+2}\|u\|_{2p+2}^{2p+2} \Big\},$$

where

$$\mathcal{N}_1 = \big\{ u \in H^1(\mathbb{R}) : \ u \neq 0, \ \|u\|_{H^1}^2 = \|u\|_{2p+2}^{2p+2} \big\}.$$

Then, the following equality holds

$$T_1 = \frac{1}{2}\frac{p}{p+1}(S_1)^{(p+1)/p}.$$

Proof. As z solves the minimization problems that defines S_1 and T_1, using (1.6) we get

$$S_1 = \frac{\|z\|_{H^1}^2}{\|z\|_{2p+2}^2} = \frac{\|z\|_{H^1}^2}{\|z\|^{2/(p+1)}} = \|z\|_{H^1}^{2p/(p+1)} = \|z\|_{2p+2}^{2p},$$

namely

$$\|z\|_{H^1}^2 = S_1^{(p+1)/p} \quad \text{ and } \quad \|z\|_{2p+2} = S_1^{1/2p}. \tag{4.2}$$

Using these equalities in the definition of T_1 permits to conclude the proof. $\square$

Define now the sets

$$\mathcal{N}_0 = \Big\{ U \in \mathbb{H}^1 : U \neq (0,0), \ \|U\|_{\mathbb{H}^1}^2 = \|U\|_{2p+2}^{2p+2} + 2\beta\|u_1u_2\|_{p+1}^{p+1} \Big\},$$

$$\mathcal{N} = \big\{ U \in \mathbb{H}^1 : u_i \neq 0, \ \|u_i\|_{H^1}^2 = \|u_i\|_{2p+2}^{2p+2} + \beta\|u_1u_2\|_{p+1}^{p+1}, \ i = 1,2 \big\}.$$

Moreover, if $\mathbb{H}_r^1$ is the set of radial function of $\mathbb{H}^1$, we introduce the numbers

$$A_0 = \inf_{U\in\mathcal{N}_0} \mathcal{I}(U), \qquad A = \inf_{U\in\mathcal{N}} \mathcal{I}(U), \qquad A_r = \inf_{U\in\mathcal{N}\cap\mathbb{H}_r^1} \mathcal{I}(U), \tag{4.3}$$

where

$$\mathcal{I}(U) = \frac{1}{2}\|U\|_{\mathbb{H}^1}^2 - \frac{1}{2p+2}\|U\|_{2p+2}^{2p+2} - \frac{1}{p+1}\beta\|u_1u_2\|_{p+1}^{p+1}.$$

Let a be a positive number. Writing down the equations that define $\mathcal{N}$ and recalling that z satisfies (1.6) it is easy to see that $a(z,z) \in \mathcal{N}$ if a satisfies (4.1).

Concerning the infimum problems A_0, A, A_r, in [19] the following result is proved for $p = 1$; actually the same proof holds for any p satisfying (1.5), we include some details.

Proposition 4.8. *Let a satisfies (4.1). Then the following inequalities hold*

$$0 < A_0 \leq A \leq A_r \leq \frac{p}{p+1}a^2 S_1^{(p+1)/p}, \tag{4.4}$$

where the values A_0 and A_r are defined in (4.3).

Proof. First note that, taken any $U = (u_1, u_2) \in \mathcal{N}_0$, the value $\mathcal{I}(U)$ is equal to

$$\mathcal{I}(U) = \frac{1}{2}\Big(\frac{p}{p+1}\Big)\left[\|U\|_{2p+2}^{2p+2} + 2\beta\|u_1u_2\|_{p+1}^{p+1}\right] = \frac{1}{2}\Big(\frac{p}{p+1}\Big)\|U\|_{\mathbb{H}^1}^2. \tag{4.5}$$

Moreover, since $a(z,z) \in \mathcal{N}$ and has radial components, recalling (4.2) we get

$$\begin{aligned} A_r \le \mathcal{I}(az, az) &= \frac{1}{2}\Big(\frac{p}{p+1}\Big)\|(az,az)\|_{H^1}^2 \\ &= \Big(\frac{p}{p+1}\Big)a^2\|z\|_{H^1}^2 = \Big(\frac{p}{p+1}\Big)a^2 S_1^{(p+1)/p}, \end{aligned} \tag{4.6}$$

which is the last inequality on the right-hand side in (4.4). It just remains to show that $A_0 > 0$. To this aim, take $U \in \mathcal{N}_0$ and observe that Hölder and Sobolev inequalities imply that there exist positive constants C_0, C_1 such that

$$\|U\|_{\mathbb{H}^1}^2 = \|U\|_{2p+2}^{2p+2} + 2\beta\|u_1u_2\|_{p+1}^{p+1} \le C_0\|U\|_{2p+2}^{2p+2} \le C_1\|U\|_{\mathbb{H}^1}^{2p+2}$$

so that the norm $\|U\|_{\mathbb{H}^1}$ remains uniformly away from zero. Hence, recalling formula (4.5), we conclude the proof. ☐

We are now ready to complete the proof of Theorem 4.3.

Proof of Theorem 4.3 concluded. We will obtain Theorem 4.3 by showing that the infimum A equals A_r and it is achieved at the couple $a(z,z)$, which is thus a ground state solution of (1.2).

First, let $(U_m) = (u_{m,1}, u_{m,2}) \subset \mathcal{N}$ be a minimizing sequence for A, namely $\mathcal{I}(U_m) = A + o(1)$ as $m \to \infty$. Let us set $y_{m,i} = \|u_{m,i}\|_{2p+2}^2$ for any $m \in \mathbb{N}$ and $i = 1, 2$. Hence, by the definition of S_1 and Hölder inequality, it follows that, for all $m \in \mathbb{N}$,

$$\begin{aligned} S_1 y_{m,1} \le \|u_{m,1}\|_{H^1}^2 &= \|u_{m,1}\|_{2p+2}^{2p+2} + \beta\|u_{m,1}u_{m,2}\|_{p+1}^{p+1} \\ &\le y_{m,1}^{p+1} + \beta y_{m,1}^{(p+1)/2} y_{m,2}^{(p+1)/2}, \end{aligned} \tag{4.7}$$

for all $m \in \mathbb{N}$. Of course, for all $m \in \mathbb{N}$, the analogous inequality holds

$$\begin{aligned} S_1 y_{m,2} \le \|u_{m,2}\|_{H^1}^2 &= \|u_{m,2}\|_{2p+2}^{2p+2} + \beta\|u_{m,1}u_{m,2}\|_{p+1}^{p+1} \\ &\le y_{m,2}^{p+1} + \beta y_{m,1}^{(p+1)/2} y_{m,2}^{(p+1)/2}. \end{aligned} \tag{4.8}$$

Furthermore, taking into account formula (4.5), by addition of the first inequalities in (4.7) and (4.8) one obtains

$$S_1(y_{m,1} + y_{m,2}) \le 2\frac{p+1}{p}\mathcal{I}(U_n) = 2\frac{p+1}{p}A + o(1), \quad \text{as } m \to \infty. \tag{4.9}$$

By combining this inequality with Proposition 4.8 gives

$$S_1(y_{m,1} + y_{m,2}) \le 2a^2 S_1^{(p+1)/p} + o(1), \quad \text{as } m \to \infty.$$

Hence, defining $z_{m,i} = y_{m,i}/S_1^{1/p}$, we derive $z_{m,1} + z_{m,2} \le 2a^2 + o(1)$, as m tends to infinity. Also, by dividing (4.7) by $S_1 y_{m,1}$ and (4.8) by $S_1 y_{m,2}$ and using $S_1 = S_1^{(p-1)/2p} S_1^{(p+1)/2p}$ we obtain that, as $m \to \infty$, $(z_{m,1}, z_{m,2})$ satisfies the following system of inequalities

$$\begin{cases} z_{m,1} + z_{m,2} \le 2a^2 + o(1), \\ z_{m,1}^p + \beta z_{m,1}^{(p-1)/2} z_{m,2}^{(p+1)/2} \ge 1, \\ z_{m,2}^p + \beta z_{m,1}^{(p+1)/2} z_{m,2}^{(p-1)/2} \ge 1. \end{cases}$$

Taking into account (4.1) we are lead to the study of the associated algebraic system of inequalities

$$\begin{cases} x + y \leq 2a^2, \\ x^p + \beta x^{(p-1)/2} y^{(p+1)/2} \geq (1+\beta)a^{2p}, \\ y^p + \beta x^{(p+1)/2} y^{(p-1)/2} \geq (1+\beta)a^{2p}, \end{cases} \tag{4.10}$$

for which we refer to Figure 1.

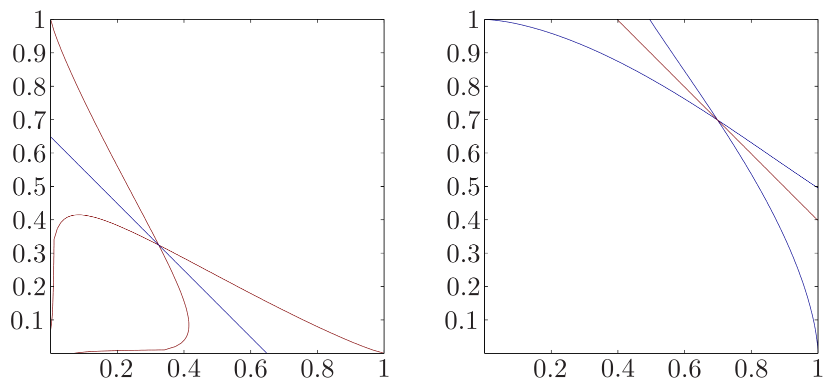

FIGURE 1. Plot of the three curves involved in the algebraic system of inequalities (4.10) in the case $p = 0.7$ (left) and $p = 2.2$ (right). For $p = 1$ the system becomes linear. In all cases the curves intersect to a unique point on the bisecting line. This is also the unique solution to (4.10) (region above both red curves and below the blue line). The value of ω was set to 1.2.

Then, for $\beta > 1$ and any $i = 1, 2$, the sequence $(z_{m,i})$ remains bounded away from zero and it has to be $z_{m,1} \to a^2$ and $z_{m,2} \to a^2$ as $m \to \infty$, so that looking at the first (in)equality of (4.10) with $x = y$ (by figure 1) yields $x = y = a^2$), so that $y_{m,1} \to a^2 S_1^{1/p}$, and $y_{m,2} \to a^2 S_1^{1/p}$, as m diverges. Whence, passing to the limit in formula (4.9), in light of Proposition 4.8 we obtain

$$2S_1^{(p+1)/p} a^2 \leq 2\frac{p+1}{p} A \leq 2a^2 S_1^{(p+1)/p}$$

so that, (4.6), gives

$$A \leq A_r \leq \mathcal{I}(az, az) \leq \Big(\frac{p}{p+1}\Big) a^2 \left(S_1\right)^{(p+1)/p} = A,$$

which gives $A = A_r = \mathcal{I}(az, az)$, concluding the proof. □

4.2. **Proof of Theorem 4.4.** According to Section 4.1, let us consider $Z = a(z, z)$ the particular ground state solution of (1.2), with a given in (4.1); we will now show the non-degeneracy property of Z. First, notice that the linearized system (1.9) can be obtained using the operator L_+ acting on Z, and by the explicit expression of Z we get

$$L_+ = \begin{pmatrix} -\frac{1}{2}\partial_{xx} + 1 & 0 \\ 0 & -\frac{1}{2}\partial_{xx} + 1 \end{pmatrix} - \begin{pmatrix} \frac{p(2+\beta)+1}{1+\beta} z^{2p} & \frac{\beta(p+1)}{1+\beta} z^{2p} \\ \frac{\beta(p+1)}{1+\beta} z^{2p} & \frac{p(2+\beta)+1}{1+\beta} z^{2p} \end{pmatrix}.$$

In accordance with Section 2, we denote with $H_F(Z)$ the second matrix on the right hand side. The quadratic form related to $H_F(Z)$ can be diagonalized by an orthonormal change of coordinates, introducing

$$w_1 = \frac{\sqrt{2}}{2}(\phi_1 + \phi_2), \quad w_2 = \frac{\sqrt{2}}{2}(\phi_1 - \phi_2). \tag{4.11}$$

Since we have

$$\begin{aligned} \mathrm{Tr}(H_F(Z)) &= 2\frac{(2+\beta)p+1}{1+\beta} = (2p+1) + \frac{2p+1-\beta}{1+\beta}, \\ \mathrm{Det}(H_F(Z)) &= \frac{(2p+1)(2p+1-\beta)}{1+\beta}, \end{aligned}$$

it follows that its eigenvalues are

$$\lambda_1 = 2p+1, \qquad \lambda_2 = \frac{2p+1-\beta}{1+\beta} \in (-1, 2p+1) \tag{4.12}$$

so the linear elliptic system $L_+\Phi = 0$ decouples and reduces to

$$\begin{cases} -\frac{1}{2}\partial_{xx}w_1 + w_1 = (2p+1)z^{2p}(x)w_1, & \text{in } \mathbb{R} \\ -\frac{1}{2}\partial_{xx}w_2 + w_2 = \frac{2p+1-\beta}{1+\beta}z^{2p}(x)w_2, & \text{in } \mathbb{R}. \end{cases} \tag{4.13}$$

Taking into account that the weight z is exponentially decaying, the spectrum of the linear self-adjoint operator $-\frac{1}{2}\partial_{xx} + \mathrm{Id} - \mu z^{2p}$ is discrete. Furthermore, from [21, (a) and (b) of Proposition 2.8] with proofs for $n = 1$ in [21, Appendix A], we learn that the eigenvalues of

$$-\frac{1}{2}\partial_{xx}w + w - \mu z^{2p}(x)w = 0 \qquad \text{in } \mathbb{R}, \tag{4.14}$$

are given by $\mu_1 = 1$, $\mu_2 = 2p+1$, $\mu_3 > 2p+1$, and, denoting by V_{μ_i} the eigenspace corresponding to the eigenvalue μ_i, we have $V_{\mu_1} = \mathrm{span}\{z\}$, $V_{\mu_2} = \mathrm{span}\{\partial_x z\}$. Therefore, from the first equation of (4.13) we deduce $w_1 \in \mathrm{span}\{\partial_x z\}$. From (4.12) we also deduce, from the second equation of (4.13), that $w_2 = 0$. In turn, by the orthonormal change of coordinates (4.11) we obtain $\phi_1 = \phi_2 = c\partial_x z$, for some coefficient $c \in \mathbb{R}$. Whence $\mathrm{Ker}(L_+) = \langle \partial_x Z_\beta \rangle$, which concludes the proof. □

References

[1] A. Ambrosetti and E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations,* J. London Math. Soc., **75** (2007), 67–82.

[2] F. A. Berezin and M. A. Shubin, "The Schrdinger Equation," Translated from the 1983 Russian edition by Yu. Rajabov, D. A. Lei tes and N. A. Sakharova and revised by Shubin. With contributions by G. L. Litvinov and Leites. Mathematics and its Applications (Soviet Series), 66. Kluwer Academic Publishers Group, Dordrecht, 1991. xviii+555 pp. ISBN: 0-7923-1218-X.

[3] J. Bronski and R. Jerrard, *Soliton dynamics in a potential,* Math. Res. Letters, **7** (2000), 329–342.

[4] T. Cazenave, "An Introduction to Nonlinear Schrödinger Equations," Textos de Métodos Matemáticos **26**, Universidade Federal do Rio de Janeiro 1996.

[5] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations,* Comm. Math. Phys., **85** (1982), 549–561.

[6] E. N. Dancer and J. Wei, *Spike solutions in coupled nonlinear Schrödinger equations with attractive interaction,* Trans. Amer. Math. Soc., **361** (2009), 1189–1208.

[7] L. Fanelli and E. Montefusco, *On the blow-up threshold for two coupled nonlinear Schrödinger equations,* J. Phys. A: Math. Theor., **40** (2007), 14139–14150.

[8] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I,* J. Funct. Anal., **74** (1987), 160–197.

[9] S. Keraani, *Semiclassical limit of a class of Schrödinger equation with potential,* Comm. Partial Differential Equations, **27** (2002), 693–704.

[10] S. Keraani, *Semiclassical limit for nonlinear Schrödinger equation with potential. II,* Asymptotic Anal., **47** (2006), 171–186.

[11] L. A. Maia, E. Montefusco and B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system,* J. Differential Equations, **229** (2006), 743–767.

[12] L. A. Maia, E. Montefusco and B. Pellacci, *Orbital stability of ground state solutions of coupled nonlinear Schrödinger equations,* Adv. Nonlin. Stud. in press. Preprint. arXiv:0809.3320

[13] S. V. Manakov, *On the theory of two-dimensional stationary self-focusing of electromagnetic waves,* Sov. Phys. JETP, **38** (1974), 248–253.

[14] C. R. Menyuk, *Nonlinear pulse propagation in birefringent optical fibers*, IEEE J. Quantum Electron, **23** (1987), 174–176.

[15] E. Montefusco, B. Pellacci and M. Squassina, *Soliton dynamics for CNLS systems with potentials,* Asympt. Anal., **66** (2010), 61-86.

[16] N. V. Nguyen and Z. Q. Wang, *Orbital stability of solitary waves for a nonlinear Schrödinger system.* Preprint.

[17] M. Ohta, *Stability of solitary waves for coupled nonlinear Schrödinger equations,* Nonlinear Anal., **26** (1995), 933–939.

[18] J. Shatah and W. Strauss, *Instability of nonlinear bound states,* Comm. Math. Phys., **100** (1985), 173–190.

[19] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in* $\mathbb{R}^n$*,* Comm. Math. Phys., **271** (2007), 199–221.

[20] C. Sulem, P. L. Sulem, "The Nonlinear Schrödinger Equation," Self-Focusing and Wave Collapse. Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999. xvi+350 pp. ISBN: 0-387-98611-1.

[21] M. I. Weinstein, *Modulational stability of ground state of nonlinear Schrödinger equations,* SIAM J. Math. Anal., **16** (1985), 472–491.

[22] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations,* Comm. Pure Appl. Math., **39** (1986), 51–67.

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