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# CONCAVE SOLUTIONS TO FINSLER $p$-LAPLACE TYPE EQUATIONS

Sunra Mosconi[1], Giuseppe Riey[*2] and Marco Squassina[3]

[1]Università di Catania, Viale A. Doria 6, 95125 Catania, Italy

[2]Università della Calabria, Ponte P. Bucci 31B, 87036 Rende, Cosenza, Italy

[3] Università Cattolica del Sacro Cuore, Via della Garzetta 48, 25133 Brescia, Italy

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Abstract. We prove concavity properties for solutions to anisotropic quasilinear equations, extending previous results known in the Euclidean case. We focus the attention on nonsmooth anisotropies and in particular we also allow the functions describing the anisotropies to be not even.

1. **Introduction.** A natural question in the framework of nonlinear elliptic PDEs is whether a solution inherits some qualitative properties from it domain of definition. Starting from [25] extensive research has been developed in order to deduce symmetry of solutions from the symmetry of the domain, via the so called Alexandroff-Serrin moving plane method. But when the symmetry of the domain is dropped, one may wonder if the solutions still exhibit some convexity properties just from the convexity of their domain. As it turns out, classical concavity of, say, positive solutions with zero Dirichlet boundary conditions is often rather demanding: it can be achieved for the torsion problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for example, only for convex $\Omega$ which are suitably small perturbations of ellipsoids [29] (see also [18]) and the first eigenfunctions of the Dirichlet Laplacian are actually never concave, in any convex domain [32, Remark 3.4]. One may instead look for *quasi-concavity* of positive solutions in convex domains, meaning that all their superlevel sets are convex. This is usually accomplished by requiring that for a suitable strictly increasing functions $\varphi$ the composition $\varphi(u)$ is concave, a property called *$\varphi$-concavity* of $u$. Indeed, in the seminal paper [37] it is shown that the solution of the torsion problem (1) for $\Omega$ convex is such that $\sqrt{u}$ is concave and in [12] the authors show that the logarithm of a first positive eigenfunctions of the Dirichlet

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*Corresponding author: Giuseppe Riey.

Laplacian is always concave if the domain is convex. More generally, for the positive solution of

$$\begin{cases} -\Delta u = u^\beta & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a convex $\Omega$, the power $u^{(1-\beta)/2}$ is concave for any $\beta \in [0,1]$, the 0-th power being formally identified with $\log u$, see [33, 34]. These last two investigations, based on the so-called *concavity function method*, gave birth to a rich research field on quasi-convexity properties of solutions to PDEs in the eighties, and we refer to [32] for the relevant bibliography. The concavity function method was also successfully applied to quasilinear equation of $p$-Laplacian type in [40]. Another approach to deduce quasi concavity is to couple the classical continuity argument with the *constant rank method* initiated in [15], ensuring strict convexity of suitable tranformations of $u$. Unfortunately, this technique requires solutions to be at least $C^2$ and is not applicable to problems driven by $p$-Laplacian. Later, a new approach to these problems, the *convex envelope method*, was introduced in [3] in the framework of viscosity solutions to fully nonlinear PDEs.

**Recent contributions**. These three strategies for investigating convexity properties of solutions to PDEs have been revisited, extended and modified in various ways. In the resulting vast literature, representative contributions are for instance [27, 26] for the concavity function method, [35, 8] for the constant rank one and [19, 9, 30] for the convex envelope one.

More recently, a general class of reactions $f$ ensuring quasi-concavity of the positive solutions of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

(and more generally of quasi-linear problems involving the $p$-Laplace operator) has been singled out in [10, 11], providing a precise connection on how the quasi-concavity of the solution is affected by the nonlinear term $f$ through the above mentioned $\varphi$. Indeed, the class of continuous increasing $\varphi$ can be partially ordered in a natural way according to their concavity, and one of the goals of [10] was, given $f$ obeying suitable conditions, to determine a "minimally concave" $\varphi$ ensuring $\varphi$-concavity of the solutions of the corresponding quasilinear problem. Note that, for general positive reactions $f$, quasi-concavity of positive solutions of (2) can fail for some smooth convex $\Omega$, as shown in [28].

In [2] the optimality of the assumptions of [10] was discussed and the results were then extended to cover positive solutions to the quasi-linear problem

$$-\mathrm{div}(\alpha(u)Du) + \frac{1}{2}\alpha'(u)|Du|^2 = f(u)$$

(coupled with zero boundary conditions), related to the so called modified nonlinear Schrödinger equation, under suitable joint hypothesis on $\alpha$ and $f$.

Another direction recently investigated in the literature is the quasi-concavity of the solutions if the nonlinearity is perturbed [14] or if the equation is nonautonomous in the diffusion or on the source term, like in the second order semilinear problem

$$-\mathrm{Tr}\,\left(A(x)\,D^2u\right) = a(x)\,u^\beta,$$

see [1]. Finally, *strict* quasi-concavity of positive solutions to

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has also been investigated, despite the above mentioned failure of the constant rank theorem in the $p$-Laplacian framework. In fact, once concavity of a suitable smooth strictly monotone transformation $\varphi(u)$ is achieved, the constant rank theorem can be applied outside the maximum points of $u$, since the latter is sufficiently regular there. The problem is thus reduced to prove uniqueness of the maximum point of $u$ and we refer to [11] for the relevant literature and open questions in the quasilinear setting.

**Equations considered**. In this paper, given a bounded convex $\Omega \subseteq \mathbb{R}^N$, we investigate the quasi-concavity of positive solutions of

$$\begin{cases} -\text{div}\,(DH(Du)) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3}$$

where $H$ is a continuous convex $p$-homogenous function for some fixed $p > 1$ vanishing only at the origin and $f$ is continuous and fulfils suitable concavity conditions detailed below. We are particularly interested in the case where $H$ is not even and possibly non-smooth, in which case (3) requires a suitable variational formulation due to the possible lack of differentiability of $H$.

When $H$ is even, the operator appearing on the right of (3) is also known as the *Finsler p-Laplacian*, since the corresponding kinetic energy

$$u \mapsto \int_\Omega H(Du)\,dx$$

has density which can be expressed as

$$H(Du) = \Phi(Du)^p$$

for a suitable Finsler norm $\Phi$. Clearly, when the norm is the standard Euclidean one, we are reduced to the usual $p$-Laplacian.

This kind of energies can be used to model several anisotropic phenomena related to physics and biology [4]. In materials science and chemistry a central role is played by *non-smooth* Finsler norms in order to describe the behavior of crystalline microstructures [7, 41]. Non-smooth Finsler norms are also used in control theory to describe the cost functional in some optimization problems [22]. Moreover, in differential geometry it is possible to consider non-even energy densities $H$, as for instance those related to Randers metrics, which have applications also in relativity [39].

Problem (3) and the qualitative properties of its solutions has been thoughtfully investigated in recent years under the assumption that $H$ is smooth and its corresponding Finsler norm has strongly convex unit ball, see for instance [16, 17, 20, 21]. Even under these more stringent assumptions, however, the quasi-concavity of solutions to (3) (in relation with suitable assumptions on the reaction $f$) was not generally known, although the results of [3] and [9] could be, at least formally, applied to get concavity properties of the solutions.

Note that (3) has to be understood, for crystalline non-differentiable energies, in weak sense either as a differential inclusion or as minimisation property of the corresponding energy. This will be made precise in the forthcoming paragraph. The regularity of solutions of (3) is therefore very poor, at best $C^\alpha(\overline{\Omega})$ for some $\alpha \in \,]0, 1[$, and this poses serious issues in the direct applicability of the aforementioned methods. It is worth noting that even when $H$ is smooth away from the origin, as for instance in the model case of the $p$-Laplacian whose corresponding solutions enjoy better regularity, the convex envelope technique of [3] still runs into problems. The very notion of viscosity solution is quite different from the standard one, and the coincidence of weak and viscosity solutions for general quasilinear degenerate/singular equations is object of contemporary research, mainly built around the ideas of [31]. We are not aware of any result of this kind for the general crystalline case we are considering in the present investigation.

**Main result**. In order to state our main result, given a continuous $f \in C^0(\mathbb{R}_+)$, set

$$F(t) = \int_0^t f(s)\, ds$$

and let $\varphi$ be

$$\varphi(t) = \int_1^t F(s)^{-\frac{1}{p}}\, ds\,. \tag{4}$$

which is well defined on $]0, M_f] \cap \mathbb{R}$, for

$$M_f = \inf\{t > 0 : f(t) \leq 0\} \tag{5}$$

(in most instances we will actually have $M_f = +\infty$). Note that in general $\varphi$ may be unbounded near 0 and also possibly near some $\bar{t} > M_f$. In order to deal with possibly non-smooth convex $H$, we restrict to special variational solutions of (3), namely minimisers of the corresponding energy

$$\left\{ w \in W_0^{1,p}(\Omega) : w \geq 0 \right\} \ni w \mapsto J(w) = \int_\Omega \frac{1}{p} H(Dw) - F(w)\, dx. \tag{6}$$

This is not restrictive, since if $H$ is differentiable and $f$ fulfils the additional requirements specified in the statement below, any positive solution of (3) turns out to minimise (6).

**Theorem 1.1.** *Suppose that $H \in C^0(\mathbb{R}^N)$ is convex, positively $p$-homogeneous and vanishes only at the origin, while $f \in C^\alpha(]0, M_f]), \mathbb{R}) \cap C^0([0, \infty[, \mathbb{R})$ fulfils $M_f > 0$. Let $u \in W_0^{1,p}(\Omega)$ be a non-negative, nontrivial minimiser for (6).*

*Assume that*

*1. $F^{\frac{1}{p}}$ is concave and $F/f$ is convex on $]0, M_f[$*

*and one of the following conditions*

*($2_H$) $H$ is strictly convex*

*($2_F$) $t \mapsto F(t^{1/p})$ is strictly concave.*

*Then $u \leq M_f$ and the function $v = \varphi(u)$ is concave in $\Omega$, where $\varphi$ is given in (4).*

**Comments on the statement**.

- We are not assuming any regularity or evenness hypothesis on $H$ in Theorem 1.1. This lack of regularity and symmetry on $H$ forces us to build suitable tools such as comparison principles and Hopf type Lemma without relying on PDE arguments and are therefore, as far as we know, new.
- Due to the assumed $p$-positive homogeneity of $H$, its strict convexity is equivalent to the strict convexity of $\{H \leq 1\}$. Note that in general the *strong* convexity of $\{H \leq 1\}$ (meaning that the principal curvature of its boundary are positively bounded from below) is required to get classical $C^{1,\alpha}$ regularity of the corresponding solution.

- By elementary means, one can show that if $t \mapsto F^{1/p}(t)$ is concave, so is $t \mapsto F(t^{1/p})$, which in turn means that $t \mapsto f(t)/t^{p-1}$ is non-increasing, a common notion in Brezis-Oswald uniqueness type results, cfr. [13, 24].
- If $H \in C^1(\mathbb{R}^N)$ and $t \mapsto F(t^{1/p})$ is concave it will turn out that any solution of (3) is actually a non-negative minimiser for $J$. This fact can be proved (see (29)) in the more general framework of possibly non-differentiable $H$, by using the notion of *energy critical point* developed in Section 3 and inspired by [23].
- Either of condition $(2_H)$ and $(2_F)$ above, coupled with the concavity of $t \mapsto F(t^{1/p})$ implies that the non-negative minimisers of $J$ are essentially unique. More precisely, either $t \mapsto F(t^{1/p})$ is linear and $u$ is a first Dirichlet positive eigenfunction minimising the corresponding Rayleigh quotient (see (7) below), or $u$ is the unique non-negative minimiser of $J$ on $W_0^{1,p}(\Omega)$. This has been proved in [38], see Proposition 3.4 for a precise statement.
- Existence of non-negative minimisers (or, equivalently, of non-negative critical points) for $J$ can be characterised in terms of the asymptotic behaviour at 0 and $+\infty$ of $t \mapsto f(t)/t^{p-1}$, assuming it is non-increasing. This is the content of a Brezis-Oswald type result proved in Proposition 3.3 below, which complement the results of [38].
- Assumption (1) can be weakened to
  (1') $t \mapsto f(t)/t^{p-1}$ *is non increasing and* $t \mapsto e^{(p-1)t}/f(e^t)$ *is convex*
  allowing to establish log-concavity of $u$. This follows coupling the arguments in [10] with tools developed in this manuscript. Note that (1) implies (1′), but in this case if $\varphi$-concavity is strictly stronger that log-concavity.
- The assumption $f \in C^\alpha(]0, M_f])$ is a technical one. Indeed, from the convexity of $F/f$ and the positivity of $f$ in $]0, M_f[$ one infers that $f \in \mathrm{Lip}_{\mathrm{loc}}(]0, M_f[)$ so that we actually require a Hölder control at $M_f$ alone.

**Applications**.

- A natural choice for the energy density is $H(z) = |z|_r^p$ for given $r \in [1, \infty]$ and $p \in ]1, \infty[$, where $|z|_r$ denotes the $\ell^r$ norm on $\mathbb{R}^N$. Note that with this choice $\{H \le 1\}$ is never strongly convex unless $r = 2$, but $H$ is strictly convex for any $r \in ]1, \infty[$, so that assumption (1) on the reaction suffices to get quasi-concavity of the corresponding solutions of (3). If $r = 1$ or $\infty$, $H$ fails to be strictly convex and assumption $(2_F)$ is needed to obtain quasi-concavity of the minimisers.

  In the non-even setting we can choose any convex, bounded, open $K \subseteq \mathbb{R}^N$ containing 0 (but not necessarily symmetric) and consider its Minkowski functional
  $$\Phi(z) = \inf\{t > 0 : z/t \in K\}.$$
  defining the energy density as $H = \Phi^p$. The resulting $H$ fulfils $(2_H)$ as long as $K$ is strictly convex.
- Given $p > 1$, the typically used reaction is $F(t) = c\,t^q$ for $1 \le q < p$, $c > 0$, which then fulfils (1) and $(2_F)$ above. In this case, given any convex, positively $p$-homogeneous $H$ vanishing only at the origin and a convex bounded $\Omega \subseteq \mathbb{R}^N$, a non-negative minimiser $u$ of $J$ has the property that $u^{(p-q)/q}$ is concave. In this case, since $(2_F)$ holds true, $H$ may fail to be strictly convex allowing

for example to establish the power-concavity of the minimisers of the non-homogeneous Rayleigh quotient

$$\inf\left\{\frac{\displaystyle\int_\Omega |Du|_\infty^2\,dx}{\left(\displaystyle\int_\Omega |u|^q\,dx\right)^{2/q}} : u\in W_0^{1,2}(\Omega)\setminus\{0\}\right\}$$

for any $q\in[1,2[$ and convex, bounded $\Omega$.
Other explicit examples where ($2_F$) holds true, thus allowing such a generality for $H$, can be found in [10, Section 2].

- Another application of our main result is when $u$ is a first Dirichlet positive eigenfunction of a convex bounded domain $\Omega$, thus minimising the homogeneous Rayleigh quotient

$$\lambda_{1,H}^+(\Omega)=\inf\left\{\frac{\displaystyle\int_\Omega H(Du)\,dx}{\displaystyle\int_\Omega u^p} : u\in W_0^{1,p}(\Omega)\setminus\{0\}, u\geq 0\right\}. \tag{7}$$

Then $F(t)=\lambda_{1,H}^+(\Omega)\,t^p/p$ and $u$ is log-concave as long as $H$ is strictly convex. Note that $F$ fulfils (1) in this case, but not ($2_F$), and that the requirement $u\geq 0$ in (7) is needed since $H$ may fail to be even.

A sample consequence is the log-concavity of the first positive Dirichlet eigenfunction of the *pseudo* $p$-Laplacian studied in [5], solving

$$-\tilde{\Delta}_p u=-\sum_{i=1}^N |\partial_i u|^{p-2}\partial_i u=\tilde{\lambda}_{1,p}(\Omega)\,u^{p-1}$$

for $p\in\,]1,\infty[$. The energy-density of the corresponding kinetic energy is $H(z)=|z|_p^p$ which, as already noted, is strictly convex but $\{H\leq 1\}$ is not strongly convex.

**Sketch of proof**. The proof of Theorem 1.1 relies on a two-steps approximation tailored on both $H$ and $F$. First we smooth out $H$, by keeping its positive $p$-homogeneity and ensuring a form of strong $p$-ellipticity that the original $H$ may not have. The uniqueness (up to scalar multiples when $t\mapsto F(t^{1/p})$ is linear) of the minimiser proved in Section 3 is key to grant the convergence of the minimisers corresponding to the smoothed functional to the original one. Thus we are reduced to prove $\varphi$-concavity of a minimiser $u$ when $H$ is smooth and $p$-elliptic. Standard regularity theory ensures in this setting up to $C^{1,\alpha}(\overline{\Omega})$ regularity, but since $H$ is not assumed to be even, nor can be its regularisations. An appropriate and apparently new anisotropic version of the Hopf Lemma is proved in subsection 4.2 and turns out to be essential for the second regularisation procedure, since we can infer from it uniform $C^2$ bounds in an inner thin strip arbitrarily near the boundary of $\Omega$. A family of different approximating problem is then built, whose corresponding minimisers are *globally* $C^2$. The form of this approximation (see (47)) has be to chosen carefully, in order to ensure that classical results of Kennington and Korevaar can be applied under conditions involving solely the reaction $f$. Then the strategy of Sakaguchi [40] can be employed, namely to consider separately the concavity function related the corresponding solutions far from $\partial\Omega$ and on the boundary of the aforementioned strip, where uniform $C^2$ bounds are available. By passing to the limit, this allows to conclude the concavity of $\varphi(u)$ on any strongly convex subdomain sufficiently close to $\Omega$, and thus the theorem.

As a final point of interest it may be worth mentioning some cases which, despite natural, are not covered by Theorem 1.1. One may consider the crystalline, 2-homogeneous energy $H(z) = |z|_\infty^2$ and given a convex $\Omega \subseteq \mathbb{R}^2$, look for a positive minimiser $u \in W_0^{1,2}(\Omega)$ of the Rayleigh quotient (7). Note that $H$ is not strictly convex and $F(t^{1/2}) = \lambda_{1,H}(\Omega)\, t/2$ is not strictly concave, thus we are not able to prove through Theorem 1.1 that $\log u$ is concave, as one may naively guess. The reason, as should be clear from the previously described proof of Theorem 1.1, is that we don't known wether the corresponding eigenvalue is simple, a fact that may well be false for some convex $\Omega$.

**Notations**. In the paper $c$ and $C$ (eventually with subscripts) denote constants which are allowed to vary from line to line; their dependance on various parameters will be outlined only when relevant to the proof. For $t \in \mathbb{R}$ we denote $t_+ = \max\{t, 0\}$ and $t_- = \max\{-t, 0\}$.

For $a, b \in \mathbb{R}^N$ we denote by $(a, b)$ the standard Euclidean scalar product, by $|a|$ the Euclidean norm and by $a \otimes b$ the matrix whose entries are $(a \otimes b)_{ij} = a_i b_j$. Recall that, for $v, w \in \mathbb{R}^N$, there holds:

$$\big(a \otimes b\, v, w\big) = \big(b, v\big)\, \big(a, w\big).$$

For a measurable $E \subseteq \mathbb{R}^N$, we let $|E|$ be its $N$-dimensional Lebesgue measure and for $p \geq 1$, the $L^p(E)$ norm of a measurable $u : E \to \mathbb{R}$ will be denoted by $\|u\|_p$ when omitting the domain $E$ of $u$ causes no confusion.

## 2. Preliminary results.

### 2.1. Main assumptions.

Throughout the paper $\Omega$ will be an open subset of $\mathbb{R}^N$ with finite measure, often assumed to be convex and bounded. Recall that a strongly convex set is a smooth convex set such that the principal curvatures of $\partial\Omega$ are positive. Clearly, any strongly convex $\Omega$ is strictly convex, but the opposite may not be true.

Moreover, $H : \mathbb{R}^N \to [0, \infty[$ will denote a continuous convex function, obeying at least the one-sided bound

$$H(z) \geq \frac{1}{C}\, |z|^p. \tag{8}$$

A strengthening of the previous condition will be often assumed, namely

$$\frac{1}{C}\, |z|^p \leq H(z) \leq C\, |z|^p \tag{9}$$

and in many instances $H$ will be additionally required to be *positively $p$-homogeneous* ($p > 1$), meaning

$$H(t\, z) = t^p\, H(z) \qquad \forall t > 0, z \in \mathbb{R}^N.$$

Any such $H$ clearly obeys (9).

The reaction $f$ belongs to $C^0(\mathbb{R})$, is even and satisfies the one-sided growth condition

$$f(t) \leq C\, \big(t^{p-1} + 1\big) \qquad t \geq 0 \tag{10}$$

as well as $M_f > 0$, where $M_f$ is given in (5), (possibly $M_f = +\infty$). Let us remark that the evenness condition on $f$ is assumed only for convenience, since we are interested in non-negative critical points for the corresponding functional. Given such a function, we will set

$$F(t) = \int_0^t f(s)\, ds.$$

and

$$f_+(t) = \max\{0, f(t)\}, \qquad F_+(t) = \int_0^t f_+(s)\, ds.$$

Note that we are making a slight abuse of notation here as $F_+(t) \neq (F(t))_+$. We will often assume (see e. g. the following paragraph) that $F^{1/p}$ is concave on $[0, M[$. Note that from the concavity of $F^{1/p}$ on $[0, \infty[$ we readily infer (10) and, more importantly the following condition

$$\mathbb{R}_+ \ni t \mapsto \frac{f(t)}{t^{p-1}} \quad \text{is non-increasing,} \tag{11}$$

which in turn is equivalent to the concavity of $t \mapsto F(t^{1/p})$ on $\mathbb{R}_+$. Note that the opposite implication is not true, i. e. $t \mapsto F(t^{1/p})$ may be concave but $t \mapsto F^{1/p}(t)$ may fail to be concave, see [10, Remark 3.4].

2.2. **Concavity function.** Given a continuous function $v : \Omega \to \mathbb{R}$ with $\Omega$ convex, its *convexity function* $c : \Omega \times \Omega \times [0, 1] \to \mathbb{R}$ is defined as

$$c_v(x, y, t) = t\, v(x) + (1 - t)\, v(y) - v\big(t\, x + (1 - t)\, y\big).$$

Clearly, $v$ is concave in $\Omega$ if and only if $c_v \leq 0$ in its domain. We recall the following fundamental properties of the concavity function and its relation with solutions of PDE. Recall that a function $g : G \subseteq \mathbb{R}^m \to \mathbb{R}$ with $G$ convex is called *harmonic concave* if for any $x, y \in G$ such that $g(x) + g(y) > 0$ it holds

$$\big(g(x) + g(y)\big)\, g\big(\frac{x + y}{2}\big) \geq 2\, g(x)\, g(y)$$

If $g$ is positive, this is equivalent to the convexity of $1/g$.

**Proposition 2.1** ([27])**.** *Let $\Omega$ be bounded and convex in $\mathbb{R}^N$, $N \geq 2$ and $v \in C^2(\Omega)$ solve*

$$-\mathrm{Tr}\, \big(A(Dv)\, D^2 v\big) = b(x, v, Dv)$$

*where $A \in C^1(\mathbb{R}^N, \mathbb{R}^{N\times N})$ fulfills for some $0 < \lambda \leq \Lambda < \infty$*

$$\lambda\, |\xi|^2 \leq (A(z)\, \xi, \xi) \leq \Lambda\, |\xi|^2 \qquad \textit{for all } z, \xi \in \mathbb{R}^N$$

*while $(x, t, \xi) \mapsto b(x, t, \xi)$ is continuous, differentiable with respect to $x$ and $\xi$ and*

$$\partial_x b, \partial_\xi b \in L^\infty_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^N).$$

*If*

1. *$t \mapsto b(x, t, \xi)$ is non-increasing on $v(\Omega)$ for any $(x, \xi) \in \Omega \times Dv(\Omega)$*
2. *$(x, t) \mapsto b(x, t, \xi)$ is harmonic concave on $\Omega \times v(\Omega)$ for any $\xi \in \mathbb{R}^N$*

*then $c_v$ cannot attain a positive interior maximum in $\Omega \times \Omega \times [0, 1]$.*

*Proof.* This is a rephrasement of [27, Theorem 2.1] applied to $\hat{v} = -v$ with $\hat{A}(\xi) = A(-\xi)$, $\hat{b}(x, t, \xi) = b(x, -t, -\xi)$. The proof in [27] uses the $C^1$ regularity of both $\hat{A}$ and $\hat{b}$ to prove that the *convexity* function

$$\hat{v}\big(\frac{x + y}{2}\big) - \frac{\hat{v}(x) + \hat{v}(y)}{2} \tag{12}$$

satisfies a differential inequality ensuring that it cannot attain a positive interior maximum in $\Omega\times\Omega$. The same proof shows that *for any given* $t\in[0,1]$, the function

$$(x,y)\mapsto \hat v(t\,x+(1-t)\,y)-t\,\hat v(x)-(1-t)\,\hat v(y)$$

cannot attain a positive maximum in $\Omega\times\Omega$. This stronger statement *a fortiori* implies that

$$(x,y,t)\mapsto \hat v(t\,x+(1-t)\,y)-t\,\hat v(x)-(1-t)\,\hat v(y)$$

cannot attain a positive interior maximum in $\Omega\times\Omega\times[0,1]$. We will show how to remove the regularity assumption on $\hat b$ in the proof of [27] for the the convexity function (12). To this end, note that it suffices to study the convexity function near points $(x,y)\in\Omega\times\Omega$ such that

$$\hat v\Big(\frac{x+y}{2}\Big)-\frac{\hat v(x)+\hat v(y)}{2}>0, \tag{13}$$

which form an open set by continuity. The only point where the regularity of $t\mapsto \hat b(x,t,\xi)$ (lacking in our setting) is used is applying Lagrange theorem to deduce the inequality

$$\hat b\big(z,\hat v(z),D\hat v(z)\big)\ge \hat b\Big(z,\frac{\hat v(x)+\hat v(y)}{2},D\hat v(z)\Big)+d(x,y)\left(\hat v(z)-\frac{\hat v(x)+\hat v(y)}{2}\right)$$

for suitable $d(x,y)\ge 0$, where $z=(x+y)/2$. However, this inequality is only needed at points $(x,y)\in\Omega\times\Omega$ fulfilling (13), in which case it is certainly true with $d=0$ regardless of the regularity of $\hat b(z,\cdot,\xi)$, since $\hat b$ is non-decreasing. $\square$

In the following, given $\Omega\subseteq\mathbb{R}^N$ and $\delta>0$, we set

$$\Omega_\delta=\{x\in\Omega:\delta<\operatorname{dist}(x,\partial\Omega)\}. \tag{14}$$

**Proposition 2.2** ([34], Lemma 2.4)**.** *Suppose that $\Omega$ is smooth, bounded and strongly convex and $u\in C^1(\overline{\Omega})\cap C^2(\overline{\Omega}\setminus\Omega_\eta)$ for some $\eta>0$ is such that*

$$u>0\quad in\ \Omega,\qquad u=0\quad on\ \partial\Omega,\qquad \frac{\partial u}{\partial n}>0\quad on\ \partial\Omega. \tag{15}$$

*If $\varphi\in C^2(\mathbb{R}_+;\mathbb{R})$ fulfils*

$$\varphi''<0<\varphi'\ \ near\ 0,\qquad \lim_{t\to0^+}\frac{1}{\varphi'(t)}=\lim_{t\to0^+}\frac{\varphi(t)}{\varphi'(t)}=\lim_{t\to0^+}\frac{\varphi'(t)}{\varphi''(t)}=0, \tag{16}$$

*set $v=\varphi(u)$. Then there exists $\delta\in\,]0,\eta[$ such that*

$$D^2v<0\quad on\ \Omega\setminus\Omega_\delta \tag{17}$$

*and for all $x_0\in\Omega\setminus\Omega_\delta$ and $x\in\Omega\setminus\{x_0\}$ it holds*

$$v(x_0)+(Dv(x_0),x-x_0)>v(x). \tag{18}$$

*Proof.* The proof of (17) is in [34, Lemma 2.4, fact 2]. Let then $\delta_0>0$ be such that $D^2v<0$ in $\Omega\setminus\Omega_{\delta_0}$. We give another proof of (18) since the last part of [34, Lemma 2.4] is a bit obscure. Let

$$A_{x_0}=\{x\in\Omega:v(x_0)+(Dv(x_0),x-x_0)\le v(x)\}\,.$$

Fix $\delta_1=\delta_1(\delta_0,\Omega)\in\,]0,\delta_0[$ such that

$$x_0\in\Omega,\quad \operatorname{dist}(x_0,\partial\Omega)<\delta_1\quad\Longrightarrow\quad B_{\delta_1}(x_0)\cap\Omega\subseteq\Omega\setminus\Omega_{\delta_0}. \tag{19}$$

We then claim that for sufficiently small $\delta<\delta_1$ (depending on $u$ as well) it holds

$$x_0\in\Omega,\quad \operatorname{dist}(x_0,\partial\Omega)<\delta\quad\Longrightarrow\quad A_{x_0}\subseteq B_{\delta_1}(x_0)\cap\Omega. \tag{20}$$

Before proving the claim, let us note that it implies (18) thanks to (19) and the strict concavity of $v$ in the convex open set $B_{\delta_1}(x_0)\cap\Omega$ granted by (17), which force $A_{x_0}=\{x_0\}$.

We prove (20) by contradiction, thus assuming that there exists $\delta_n\downarrow 0$, $x_n\in\Omega\setminus\Omega_{\delta_n}$ and $y_n\in\Omega$ such that

$$|y_n-x_n|\geq\delta_1,\qquad v(x_n)+(Dv(x_n),y_n-x_n)\leq v(y_n).$$

By compactness we can suppose $x_n\to\bar{x}\in\partial\Omega$ and $y_n\to\bar{y}\in\overline{\Omega}$, with $|\bar{y}-\bar{x}|\geq\delta_1$. Denoting by $n(\bar{x})$ the interior normal to $\partial\Omega$ at $\bar{x}$, it holds

$$\lim_n(Du(x_n),y_n-x_n)=(Du(\bar{x}),\bar{y}-\bar{x})=|Du(\bar{x})|\,(n(\bar{x}),\bar{y}-\bar{x})>0$$

by the strict convexity of $\partial\Omega$ and (15). Therefore there exists $\theta>0$ such that for sufficiently large $n$ it holds

$$(Du(x_n),y_n-x_n)>\theta.$$

Recalling the definition of $v$, we thus have

$$\begin{aligned}\varphi'(u(x_n))\left(\frac{\varphi(u(x_n))}{\varphi'(u(x_n))}+(Du(x_n),y_n-x_n)\right)&=v(x_n)+(Dv(x_n),y_n-x_n)\\&\leq v(y_n)=\varphi(u(y_n))\end{aligned}$$

so that for sufficiently large $n$

$$\varphi'(u(x_n))\left(\frac{\varphi(u(x_n))}{\varphi'(u(x_n))}+\theta\right)\leq\varphi(u(y_n)).$$

However, since $u(x_n)\to 0$, the left hand side goes to $+\infty$ by (16) while the right and side is bounded from above (since $\varphi$ is smooth on $]0,+\infty[$ and increasing near 0). This proves claim (20) and then (18). □

**Remark 2.3.** The conditions in (16) can be slightly weakened, see [27, Lemma 3.1 and 3.2].

The next proposition shows the rôle of condition (18) in analysing the boundary behaviour of the convexity function. Notice that we will apply it for convex domains slightly smaller than the domain of definition of the function $v$ defined above, in order to ensure that it is smooth up to the boundary.

**Proposition 2.4** ([34], Lemma 2.1)**.** *Suppose that $\Omega$ is smooth, bounded and strongly convex and $\eta>0$. If $v\in C^1(\overline{\Omega})$ fulfils (18) for all $x_0\in\partial\Omega$ and $x\in\overline{\Omega}$, then $c_v$ cannot attain a positive maximum on $\partial(\Omega\times\Omega)\times[0,1]$.*

Clearly, condition (18) is not stable under $C^2$ convergence, but the conjunction of (17) and (18) is, as has been observed in [10]. We report the argument therein for sake of completeness.

**Proposition 2.5.** *Let $\Omega$ be smooth, bounded and strongly convex, $\eta>0$ and let $v_n\in C^1(\overline{\Omega})\cap C^2(\overline{\Omega}\setminus\Omega_\eta)$ be such that $v_n\to v$ in $C^1(\overline{\Omega})$ and in $C^2(\overline{\Omega}\setminus\Omega_\eta)$. If $v$ fulfils (17) for some $\delta\in]0,\eta]$ and (18) at all points $x_0\in\partial\Omega$ and $x\in\overline{\Omega}\setminus\{x_0\}$ then, for all sufficiently large $n$, $v_n$ fulfils them as well.*

*Proof.* Clearly (17) holds true for sufficiently large $n$, which we'll assume henceforth. If (18) does not hold, for a (not relabelled) subsequence there are points $x_n\in\partial\Omega$ and $y_n\in\overline{\Omega}$ with $y_n\neq x_n$ and

$$v_n(x_n)+(Dv_n(x_n),y_n-x_n)\leq v_n(y_n). \tag{21}$$

By the smoothness and strong convexity of $\partial\Omega$ we can find $c > 0$ such that $B_{c\delta}(\bar{x}) \cap \Omega \subseteq \Omega \setminus \Omega_\delta$. Let $n$ be so large that $B_{c\delta/2}(x_n) \subseteq B_{c\delta}(\bar{x})$. Since $v_n$ is strictly concave on the convex open set $B_{c\delta/2}(x_n) \cap \Omega$, the point $y_n$ cannot lie in $B_{c\delta/2}(x_n)$, and thus

$$|x_n - y_n| \geq c\,\delta/2.$$

By taking a subsequence we can suppose that $x_n \to \bar{x}_0 \in \partial\Omega$, $y_n \to \bar{x} \in \overline{\Omega}$ and from the previous display we get $|\bar{x}_0 - \bar{x}| > 0$. Taking the limit in (21), we reach

$$v(\bar{x})_0 + (Dv(\bar{x}_0), \bar{x} - \bar{x}_0) \leq v(\bar{x}), \qquad \overline{\Omega} \ni \bar{x} \neq \bar{x}_0 \in \partial\Omega,$$

contradicting assumption (18) for $v$. □

The previous Propositions will eventually be applied to $v = \varphi(u)$, with $\varphi$ defined as in (4), in a smaller domain $\Omega' \subset \Omega$. To this end, we recall the following elementary facts from [10].

**Lemma 2.6.** *Let $f \in C^0([0, +\infty[, \mathbb{R})$ fulfill (10). If $F^{1/p}$ is concave and $F/f$ is convex on $]0, M_f] \cap \mathbb{R}$, with $M_f$ as in (5), then*

1. *The function $\varphi$ defined in (4) is invertible and fulfils (16) on $]0, M_f[$.*
2. *If $\psi = \varphi^{-1}$, then $\psi''/\psi'$ is non-increasing and $\psi'/\psi''$ is convex on $]0, \varphi(M_f)[$*

## 3. Critical point theory.

In this section we give a meaning to problem (3), which at the moment is oddly defined as $H$ may fail to be differentiable, by using the notion of *energy critical point*. We then study the existence and uniqueness of the corresponding energy critical points.

### 3.1. Energy critical points.

Given a convex $H \in C^0(\mathbb{R}^N)$ fulfilling (8) and an even $f \in C^0(\mathbb{R})$ obeying (10), the corresponding functional $J$ will be defined as

$$J(v) = \int_\Omega \frac{1}{p}\, H(Dv) - F(v)\, dx, \qquad v \in W_0^{1,p}(\Omega).$$

Note that $J$ is always well defined and $J(u) > -\infty$ for all $u \in W_0^{1,p}(\Omega)$, as $(F(v))_+ \in L^1(\Omega)$ for any $v \in W_0^{1,p}(\Omega)$ (thanks to (10), the finite measure assumption on $\Omega$ and Poincaré inequality) but, as (8) and (10) are only one-sided, the resulting $J$ may assume the value $+\infty$. Note that under assumption (10), $J$ is anyway sequentially l. s. c. in the weak topology of $W_0^{1,p}(\Omega)$, as

$$-\int_\Omega F(v)\, dx = \int_\Omega (F(v))_-\, dx - \int_\Omega (F(v))_+\, dx$$

and the first term is l. s. c. by Fatou lemma while the second is continuous by (10) and the finite measure assumption on $\Omega$. As we are interested in non-negative solutions of (3), we let

$$\left(W_0^{1,p}(\Omega)\right)_+ := \left\{ v \in W_0^{1,p}(\Omega) : v \geq 0 \right\}.$$

and define $u \in \left(W_0^{1,p}(\Omega)\right)_+$ to be a non-negative *energy critical point* for $J$ on $\left(W_0^{1,p}(\Omega)\right)_+$ if $u$ minimises the corresponding semi-linearized convex functional

$$v \mapsto J_u(v) := \int_\Omega \frac{1}{p}\, H(Dv) - f(u)\, v\, dx \tag{22}$$

on $\big(W_0^{1,p}(\Omega)\big)_+$. Alternatively, an energy critical point $u$ for $J$ on the full $W_0^{1,p}(\Omega)$ is a minimiser of (22) over

$$V_u = \Big\{v \in W_0^{1,p}(\Omega) : \big(f(u)\, v\big)_+ \in L^1(\Omega)\Big\}.$$

These different definitions of critical point, taken from [23], deserve some comment. Note that $J_u$, as defined in (22) is always well defined on $\big(W_0^{1,p}(\Omega)\big)_+$ under assumption (10), thanks to the same argument as before and the condition $v \geq 0$. This is not the case for $J_u$ on the whole $W_0^{1,p}(\Omega)$ and in this case the requirement $\big(f(u)\, v\big)_+ \in L^1(\Omega)$ is needed.

Note, moreover, that as before (10) ensures that $J_u(u) > -\infty$ while trivially $J_u(0) = 0$, so that

$$u \in C_J \text{ or } C_J^+ \qquad \Longrightarrow \qquad H(Du) \in L^1(\Omega), \quad f(u)\, u \in L^1(\Omega). \tag{23}$$

The set of all energy critical points for $J$ on $\big(W_0^{1,p}(\Omega)\big)_+$ and $W_0^{1,p}(\Omega)$, respectively, will be denoted by $C_J^+$ and $C_J$, respectively. By using the nonlinearity $f_+$ instead of $f$, we can define the functional

$$J_+(v) = \int_\Omega \frac{1}{p}\, H(Dv) - F_+(v)\, dx$$

and the consider corresponding critical points $C_{J_+}$ and $C_{J_+}^+$.

Under quite general assumptions on $f$, all the previuos notions of critical points coincide.

**Lemma 3.1.** *Let $\Omega \subseteq \mathbb{R}^N$ have finite measure, $H$ be convex and obey (8) and $f \in C^0(\mathbb{R})$ be even, fulfil (10), and*

$$f(t) > 0 \quad \text{for } |t| < M_f, \qquad f(t) \leq 0 \quad \text{for } |t| \geq M_f \tag{24}$$

*for some $M_f \geq 0$, with possibly $M_f = +\infty$. Then*

$$C_J = C_J^+ = C_{J_+}^+ = C_{J_+} \tag{25}$$

*and any energy critical points fulfils $0 \leq u \leq M_f$*

*Proof.* Denote by $C$ any one of the set $C_J, C_{J_+}, C_J^+$ or $C_{J_+}^+$. We can assume that $f$ does not vanishes identically, otherwise the claim is trivial as all the previous sets are $\{0\}$. In this case, $M_f > 0$ in (24). We claim that

$$u \in C \quad \Longrightarrow \quad |u| \leq M_f. \tag{26}$$

Recalling (23), we test the minimality of $u$ with $v = \min\{u, M_f\}$ to get

$$\begin{aligned} \int_\Omega \frac{1}{p}\, H(Du) - f(u)\, u\, dx &\leq \int_\Omega \frac{1}{p}\, H(Dv) - f(u)\, v\, dx \\ &= \int_{\{u \leq M_f\}} \frac{1}{p}\, H(Du) - f(u)\, u\, dx - \int_{\{u > M_f\}} f(u)\, M_f\, dx. \end{aligned} \tag{27}$$

Note from (24) and (23) that it holds

$$0 \leq -\int_{\{u > M_f\}} f(u)\, M_f\, dx \leq -\int_{\{u > M_f\}} f(u)\, u\, dx$$

so that all terms in (27) are finite and rearranging we get

$$\int_{\{u > M_f\}} \frac{1}{p}\, H(Du) - f(u)\,(u - M_f)\, dx \leq 0$$

proving that $u \le M_f$ thanks to (8) and (24). The same conclusion holds if $f$ is replaced by $f_+$, hence (26) is proved if $u \in \big(W_0^{1,p}(\Omega)\big)_+$, so it suffices to consider the case $u \in C = C_J$ or $C_{J_+}$. But in this case, $\hat{u}(x) = -u(-x) \in C$, where $\Omega$ is replaced by $-\Omega$, so that the previous argument ensures $\hat{u} \le M_f$, i. e. $u \ge -M_f$, which concludes the proof of (26) in all cases. Therefore $f(u) = f_+(u) \ge 0$ for any $u \in C$, hence the functionals in (22) are the same for $f$ or $f_+$. It follows that

$$C_{J_+} = C_J, \qquad C^+_{J_+} = C^+_J.$$

Moreover, by [38, Corollary 3.6][1], any $u \in C_{J_+}$ is non-negative in $\Omega$, so that $C_{J_+} \subseteq \big(W_0^{1,p}(\Omega)\big)_+$. Finally, again from $f(u) \ge 0$, we get that minimisation over $\big(W_0^{1,p}(\Omega)\big)_+$ of the functional in (22) is equivalent to minimisation over the whole $W_0^{1,p}(\Omega)$, since

$$\int_\Omega \frac{1}{p}\, H(Dv) - f(u)\, v\, dx \ge \int_\Omega \frac{1}{p}\, H(Dv_+) - f(u)\, v_+\, dx.$$

so that $C^+_{J_+} = C_{J_+}$ and the proof is complete. $\square$

By [38, Theorem 3.2 and Theorem 3.5], if $\Omega$ is bounded, $\partial\Omega$ is Lipschitz and $H$ fulfils the two-sided bound (9), any of the corresponding energy critical point belongs to $C^\alpha(\overline{\Omega})$, with $\alpha \in\, ]0,1[$ and its $C^\alpha(\overline{\Omega})$ norm depends on $\|u\|_p$, $\Omega$ and the structural constants appearing the bounds for $H$ and $f$. Moreover, on each connected component of $\Omega$ either $u$ vanishes identically or it is strictly positive in $\Omega$.

A particular reaction $f$ and the corresponding energy critical points provide the notion of *first positive Dirichlet eigenfunction*. More precisely, the number

$$\lambda^+_{1,H}(\Omega) = \inf\left\{\int_\Omega H(Dv)\, dx : v \ge 0, \|v\|_p = 1\right\}$$

is called the first positive Dirichlet eigenvalue and the corresponding minimisers are the normalised first positive Dirichlet eigenfunctions. The latter are the non-negative energy critical points for $J$ with $f(t) = \lambda^+_{1,H}(\Omega)\, t^{p-1}$.

We finally prove the following form of the weak comparison principle, which holds true under very mild assumptions on $H$ and $\Omega$. It basically ensures that if $u$ solves (3) in $\Omega$ and $\underline{u}$ solves $-\mathrm{div}\,(DH(D\underline{u})) = 0$ in $A \subseteq \Omega$, then $\underline{u} \le u$ in $A$ as long as the inequality holds true on $\partial A$. While usually this is deduced through a strong form of convexity for $H$, here we deal with possibly non-strictly convex $H$.

**Proposition 3.2.** *Let $\Omega$, $H$ and $f$ be as in the previous lemma and $u \in C^+_J$. For $A \subseteq \Omega$ open, let $\underline{u} \in W^{1,p}(A)$ be a minimiser of*

$$\underline{u} + W_0^{1,p}(A) \ni v \mapsto \int_A H(Dv)\, dx$$

*such that $(\underline{u} - u)_+ \in W_0^{1,p}(A)$. Then $u \ge \underline{u}$ in $A$.*

*Proof.* Fix a representative of $u$ and $\underline{u}$, noting that using the assumption $(\underline{u}-u)_+ \in W_0^{1,p}(A)$ this can be done in such a way that $\{\underline{u} > u\} \subseteq A$. By the previous Lemma we can use $f_+$ instead of $f$ in the definition of $J$, hence we can assume $f \ge 0$. By the previous Lemma it holds $0 \le u \le M_f$, with $M_f$ given in (24), possibly infinite. If $M_f$ is finite from $(\underline{u} - M_f)_+ \le (\underline{u} - u)_+ \in W_0^{1,p}(A)$ we infer that

[1] Note that only (8) is used in the proof of point 1 therein.

$\min\{\underline{u}, M_f\} = \underline{u} - (\underline{u} - M_f)_+ \in \underline{u} + W_0^{1,p}(A)$, hence by the minimality of $\underline{u}$ againts the competitor $\min\{\underline{u}, M_f\}$ we find

$$\int_A H(D\underline{u})\, dx \le \int_{\{\underline{u}<M_f\}} H(D\underline{u})\, dx$$

therefore

$$\int_{\{\underline{u}\ge M_f\}} H(D\underline{u})\, dx \le 0$$

and thus $\underline{u} \le M_f$ in $A$. Since $W_0^{1,p}(A) \subseteq W_0^{1,p}(\Omega)$ by extending each element of the former as 0 outside $A$, we can test the minimality of $u$ against $\max\{u, \underline{u}\} = u + (\underline{u} - u)_+ \in W_0^{1,p}(\Omega)$ for the functional in (22). This gives

$$\int_\Omega \frac{1}{p} H(Du) - f(u)u\, dx \le \int_{\{\underline{u}>u\}} \frac{1}{p} H(D\underline{u}) - f(u)\underline{u}\, dx + \int_{\{\underline{u}\le u\}} \frac{1}{p} H(Du) - f(u)u\, dx$$

so that, recalling (23),

$$\int_{\{\underline{u}>u\}} \frac{1}{p} H(Du) - f(u)\, u\, dx \le \int_{\{\underline{u}>u\}} \frac{1}{p} H(D\underline{u}) - f(u)\, \underline{u}\, dx. \tag{28}$$

On the other hand, by the minimality of $\underline{u}$ against the competitor $\min\{u, \underline{u}\} = \underline{u} - (\underline{u} - u)_+ \in \underline{u} + W_0^{1,p}(A)$, we have

$$\int_A H(D\underline{u}) \le \int_{\{\underline{u}\le u\}} H(D\underline{u})\, dx + \int_{\{\underline{u}>u\}} H(Du)\, dx$$

so that

$$\int_{\{\underline{u}>u\}} H(D\underline{u})\, dx \le \int_{\{\underline{u}>u\}} H(Du)\, dx.$$

Inserting the latter into (28) and rearranging we obtain

$$\int_{\{\underline{u}>u\}} f(u)\, (\underline{u} - u)\, dx \le 0.$$

Since $0 \le u \le M_f$ and $f(t) > 0$ for $t \in\, ]0, M_f[$, the previous inequality forces $f(u)\, (\underline{u} - u) = 0$ a. e. on $\{\underline{u} > u\}$ and that equality holds in (28). In particular a. e. point in $\{\underline{u} > u\}$ belongs to either $\{u = 0\}$ or $\{u = M_f\}$. Being $\underline{u} \le M_f$ a. e., the second case cannot occur in a set of positive measure and thus $\{\underline{u} > u\} \subseteq \{u = 0\} \cap A$ a. e..

The equality in (28) then reads

$$\int_{\{\underline{u}>u\}} H(Du)\, dx = \int_{\{\underline{u}>u\}} H(D\underline{u})\, dx$$

and the left hand side vanishes since $Du = 0$ a. e. on $\{u = 0\}$. Moreover, still from $\{\underline{u} > u\} \subseteq \{u = 0\} \cap A$, we see that $\underline{u} = (\underline{u} - u)_+$ a. e. on $A$, hence

$$0 = \int_{\{\underline{u}>u\}} H(D\underline{u})\, dx = \int_A H(D(\underline{u} - u)_+)\, dx$$

which implies that $\underline{u} \le u$ a. e. in $A$ by (8). $\square$

3.2. **Existence and uniqueness.** The existence of non-negative energy critical points can be variationally characterised for $H$ being positively $p$-homogeneous and $f$ fulfilling (11). Indeed, in this case [38, Theorem 5.3, point 1] provides

$$C_J^+ = \operatorname{Argmin} J \tag{29}$$

where, by (25), the minimisation problem on the right can be equivalently settled on either $\big(W_0^{1,p}(\Omega)\big)_+$ or $W_0^{1,p}(\Omega)$.

Necessary and sufficient conditions on $f$ for the existence of positive critical points under assumption (11) have been derived for $\partial\Omega$ of class $C^2$ and $H(z) = |z|^p$ (see [10, Proposition 3.8]) through the Hopf Lemma. Here we consider the case of possibly non-smooth $H$ and $\Omega$, so that the Hopf lemma does not hold.

**Proposition 3.3.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open, connected set with finite measure, $H \in C^0(\mathbb{R}^N)$ be convex, positively $p$-homogeneous and vanishing only at the origin and $f \in C^0(\mathbb{R})$ be even and such that (11) holds true. Then either $C_J^+ \setminus \{0\}$ consists of first positive Dirichlet eigenfunctions or*

$$C_J^+ \setminus \{0\} \neq \emptyset \quad\iff\quad \lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} < \lambda_{1,H}^+(\Omega) < \lim_{t\to 0^+} \frac{f(t)}{t^{p-1}}. \tag{30}$$

*Proof.* Suppose that

$$\mu_\infty := \lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} < \lambda_{1,H}^+(\Omega) < \lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} =: \mu_0$$

holds true. Then for a fixed $\varepsilon \in \,]0, \lambda_{1,H}^+(\Omega) - \mu_\infty[$ there exists $t_0 > 0$ such that

$$F(t) \le (\lambda_{1,H}^+(\Omega) - \varepsilon)\, \frac{t^p}{p} \qquad \text{for } t > t_0$$

hence for any $v \in \big(W_0^{1,p}(\Omega)\big)_+$ it holds

$$J(v) \ge \int_\Omega \frac{1}{p} H(Dv)\, dx - \int_{v\le t_0} F(v)\, dx - (\lambda_{1,H}^+(\Omega) - \varepsilon) \int_\Omega \frac{v^p}{p}\, dx$$

so that by the definition of $\lambda_{1,H}^+(\Omega)$

$$J(v) \ge \frac{\varepsilon}{p\,\lambda_{1,H}^+(\Omega)} \int_\Omega H(Dv)\, dx - \sup_{[0,t_0]} F\, |\Omega|$$

which implies coercivity of $J$ on $\big(W_0^{1,p}(\Omega)\big)_+$ thanks to (9) (which still holds true under the present, weaker assumption on $H$). Therefore $J$ admits a minimiser $u \in C_J^+(\Omega)$. To prove that $u \neq 0$, note that from $\mu_0 > \lambda_{1,H}^+(\Omega)$ we infer that for some positive $\varepsilon$ and $t_1$

$$F(t) \ge (\lambda_{1,H}^+(\Omega) + \varepsilon)\, \frac{t^p}{p} \qquad \text{for } t \in [0, t_1].$$

Therefore, if $w$ is a first positive normalised Dirichlet eigenfunction (which is bounded), for sufficiently small $t > 0$ we have

$$J(t\,w) \le \frac{t^p}{p} \int_\Omega H(Dw) - (\lambda_{1,H}^+(\Omega) + \varepsilon)\, w^p\, dx = -\varepsilon\, \frac{t^p}{p} < 0.$$

Thus $J(u) < 0$ and $u \in C_J^+ \setminus \{0\}$.

Vice-versa, suppose that $u \in C_J^+ \setminus \{0\}$ and that $u$ is not a first positive Dirichlet eigenfunction. By [38, Corollary 2.7] it holds

$$\int_\Omega H(Du)\, dx = \int_\Omega f(u)\, u\, dx$$

so by the definition of $\lambda^+_{1,H}(\Omega)$ and (11)

$$\lambda^+_{1,H}(\Omega) \int_\Omega u^p\, dx \le \int_\Omega H(Du)\, dx = \int_\Omega f(u)\, u\, dx = \int_\Omega \frac{f(u)}{u^{p-1}}\, u^p\, dx \le \mu_0 \int_\Omega u^p\, dx,$$

proving that $\mu_0 \ge \lambda^+_{1,H}(\Omega)$ and furthermore that $\mu_0 > \lambda^+_{1,H}(\Omega)$, since otherwise the previous inequalities are all equalities, forcing

$$f(t) = \lambda^+_{1,H}(\Omega)\, t^p \qquad \text{on } u(\Omega),$$

i. e. that $u$ is a first positive Dirichlet eigenfunction. To prove that $\mu_\infty < \lambda^+_{1,H}(\Omega)$, recall that for any bounded $v \in \big(W^{1,p}_0(\Omega)\big)_+$ it holds (see [38, eq. (5.7)])

$$\int_\Omega H(v)\, dx \ge \int_\Omega \frac{f(u)}{u^{p-1}}\, v^p\, dx$$

and in particular the integrand on the right is in $L^1(\Omega)$. Choosing $v$ to be a first positive Dirichlet eigenfunction in the previous inequality, we obtain by (11)

$$\lambda^+_{1,H}(\Omega) \int_\Omega v^p\, dx = \int_\Omega H(v)\, dx \ge \int_\Omega \frac{f(u)}{u^{p-1}}\, v^p\, dx \ge \mu_\infty \int_\Omega v^p\, dx$$

so that $\mu_\infty \le \lambda^+_{1,H}(\Omega)$ and, as before, equality holds if and only if $u$ is a first positive Dirichlet eigenfunction. □

Regarding uniqueness, we recall the following result from [38, Theorem 5.3].

**Proposition 3.4.** *Under the assumptions of the previous proposition, let $u \in C_J^+ \setminus \{0\}$. Then either $u$ is a first positive Dirichlet eigenfunction or $u$ is the unique positive energy critical point for $J$ in the following cases:*

1. *$H$ is strictly convex*
2. *$t \mapsto f(t)/t^{p-1}$ is strictly decreasing on $u(\Omega)$.*

*Moreover, if $H$ is strictly convex the first positive Dirichlet eigenvalue is simple, meaning that any other first positive Dirichlet eigenfunction is a positive scalar multiple of $u$.*

## 4. **Regularised problems.**

In this section we construct regular problems approximating (3) and derive the relevant regularity properties of the solutions, together with their boundary behaviour.

### 4.1. **Approximation scheme.**

**Lemma 4.1.** *Let $G \in C^\infty(\mathbb{R}^N)$ be such that*

$$G(0) = 0, \qquad D^2G(z) \ge \lambda\, \mathrm{Id} \quad \forall z \in \mathbb{R}^N,$$

*for fixed $\lambda > 0$ and set*

$$\Phi(z) = \inf\{t > 0 : G(z/t) \le 1\}.$$

*Then $\Phi \in C^\infty(\mathbb{R}^N \setminus \{0\})$, is positively 1-homogeneous and*

$$\big(D^2\Phi(z)\, v, v\big) \ge \widehat{\lambda}\, |v|^2, \qquad \forall z, v \text{ such that } \Phi(z) = 1,\ (D\Phi(z), v) = 0 \tag{31}$$

*for $\widehat{\lambda} > 0$ depending on $G$ and $\lambda$.*

*Proof.* The assumptions on $G$ ensure that the latter is strongly convex, so that $\Phi$, being the Minkowski functional of $\{G \le 1\}$, is automatically $C^\infty(\mathbb{R}^N \setminus \{0\})$ and positively 1-homogeneous. By construction it holds $\{G = 1\} = \{\Phi = 1\}$ and

$$G\left(z/\Phi(z)\right) = 1 \qquad \forall z \neq 0$$

which, differentiated, gives

$$\frac{DG(z/\Phi(z))}{\Phi(z)} - \frac{D\Phi(z)\ (DG(z/\Phi(z)), z)}{\Phi^2(z)} = 0,$$

or

$$\Phi(z)\, DG(z/\Phi(z)) = D\Phi(z)\, (DG(z/\Phi(z)), z)\,. \tag{32}$$

Differentiating once more, we obtain for $\Phi(z) = 1$

$$\begin{aligned} &D\Phi(z) \otimes DG(z) + D^2G(z)\ (\mathrm{Id} - z \otimes D\Phi(z)) \\ &= (DG(z), z)\ D^2\Phi(z) + D\Phi(z) \otimes \big(DG(z) + z\, D^2G(z)\, (\mathrm{Id} - z \otimes D\Phi(z))\big)\,. \end{aligned}$$

For such $z$'s, if $v$ obeys $(D\Phi(z), v) = 0$, we have

$$D^2G(z)\, v = (DG(z), z)\ D^2\Phi(z)\, v + D\Phi(z)\ \big(z\, D^2G(z), v\big)$$

and taking the scalar product with $v$ provides by $(D\Phi(z), v) = 0$

$$\big(D^2G(z)\, v, v\big) = (DG(z), z)\ \big(D^2\Phi(z)\, v, v\big)\,.$$

The claim is thus proved with

$$\widehat{\lambda} = \frac{\lambda}{\sup_{\{G=1\}} (DG(z), z)}.$$

□

**Proposition 4.2.** *Let $\Omega \subseteq \mathbb{R}^N$ be open, connected and with finite measure. Suppose that*

1. *$H : \mathbb{R}^N \to [0, +\infty[$ is convex, positively $p$-homogeneous and vanishes only at the origin*
2. *$f \in C^0(\mathbb{R})$ is even and fulfils (11)*

*and that either the convexity of $H$ or the monotonicity of $\mathbb{R}_+ \ni t \mapsto f(t)/t^{p-1}$ are strict. If $u \in C^+_J \setminus \{0\}$ is not a first positive Dirichlet eigenfunction, there exists a sequence of convex, positively $p$-homogeneous $H_n \in C^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$ such that*

$$\lambda_n\, |v|^2\, |z|^{p-2} \le \big(D^2H_n(z)\, v, v\big) \le \Lambda_n\, |v|^2\, |z|^{p-2} \tag{33}$$

*for $0 < \lambda_n \le \Lambda_n$ and corresponding $u_n \in C^+_{J_n}$ with*

$$J_n(v) = \int_\Omega \frac{1}{p}\, H_n(Dv) - F(v)\, dx$$

*such that $u_n \rightharpoonup u$ in $W^{1,p}_0(\Omega)$.*

*Proof.* Fix an even $\varphi \in C^\infty_c(B_1; [0, \infty[)$ such that $\|\varphi\|_1 = 1$ and set, for a sequence $\varepsilon_n \downarrow 0$, $\varphi_n(z) = \varepsilon_n^{-N} \varphi(z/\varepsilon_n)$. By Jensen inequality it holds $\varphi_n * H \ge H$ and for each $n$ the function $\varphi_n * H$ is smooth and convex. We can then set

$$G_n(z) = \varphi_n * H(z) + \varepsilon_n\, \frac{|z|^2}{2}$$

so that each $G_n$ is convex as well and it holds

$$G_n \ge H, \qquad G_n \to H \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^N). \tag{34}$$

Define the convex sets

$$K_n = \{z : G_n(z) \le 1\} \subseteq \{H \le 1\} = K.$$

Thanks to (34) and (9) there exists $\delta > 0$ such that for sufficiently large $n$ it holds

$$B_\delta(0) \subseteq K_n \tag{35}$$

so that for such $n$'s we can then let $\Phi_n$ be the Minkowski functional of $K_n$ and similarly define $\Phi$ as the Minkowski functional of $K := \{H \le 1\}$ (notice that then $H = \Phi^p$). Being $K_n \subseteq K$ it holds $\Phi_n \ge \Phi$ and from (35) we infer

$$\Phi_n(z) \le |z|/\delta \tag{36}$$

for all sufficiently large $n$. Given $z \ne 0$, from

$$1 = G_n(z/\Phi_n(z))$$

and the local uniform convergence of $G_n$ to $H$, we infer that any limit point $\mu \in \mathbb{R}$ of the bounded sequence $(\Phi_n(z))$ fulfils $H(z/\mu) = 1$, i. e. $\mu = \Phi(z)$. Therefore, a sub-subsequence argument ensures that $\Phi_n \to \Phi$ point-wise and thus, being each $\Phi_n$, as well as $\Phi$, convex and finite, locally uniformly on $\mathbb{R}^N$. Finally set

$$H_n(z) = \Phi_n^p(z),$$

which, by (36), satisfies

$$H_n(z) \le \frac{|z|^p}{\delta^p} \qquad \forall z \in \mathbb{R}^N \tag{37}$$

for any sufficiently large $n$. By construction, each $H_n$ is smooth on $\mathbb{R}^N \setminus \{0\}$, strictly convex and positively $p$-homogeneous. Since $D^2 G_n \ge \varepsilon_n \,\mathrm{Id}$, by Lemma 4.1 $\Phi_n$ has strongly convex unit ball in the sense that (31) holds true, so that (33) follows from [21, Proposition 3.1]. Moreover, by the previous analysis of the sequence $\Phi_n$, it holds

$$H_n \ge H, \qquad H_n \to H \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^N)$$

so that in particular $J_n \ge J$. By Proposition 3.3, (30) holds true, hence $J_n$ has a nontrivial non-negative energy critical point $u_n$. By the variational characterisation (29) and Proposition 3.4, $u_n$ is the unique minimiser for $J_n$ on $\big(W^{1,p}_0(\Omega)\big)_+$, hence in particular $J_n(u_n) \le 0$. By (30), for a suitable $\theta \in \,]0,1[$ there exists $L > 0$ such that

$$f(t) \le \theta\, \lambda^+_{1,H}(\Omega) t^{p-1} \qquad \text{for } t > L,$$

so that for $t > 0$ it holds

$$F(t) \le \theta\, \lambda^+_{1,H}(\Omega) \frac{t^p}{p} + L \sup_{[0,L]} f.$$

By $H_n \ge H$, the definition of $\lambda^+_{1,H}(\Omega)$ and (9) we thus infer

$$\begin{aligned} 0 \ge J_n(u_n) \ge J(u_n) &\ge \frac{1}{p} \int_\Omega H(Du_n)\, dx - \frac{\theta\, \lambda^+_{1,H}(\Omega)}{p} \int_\Omega u_n^p\, dx - C\,|\Omega| \\ &\ge \frac{1-\theta}{p} \int_\Omega H(Du_n)\, dx - C\,|\Omega| \ge \frac{1-\theta}{C\,p} \int_\Omega |Du_n|^p\, dx - C\,|\Omega|. \end{aligned}$$

Therefore $(u_n)$ is bounded in $W^{1,p}_0(\Omega)$. Suppose, up to subsequences, that $u_n \rightharpoonup \bar u \in \big(W^{1,p}_0(\Omega)\big)_+$. From the lower semicontinuity of $J$, $J_n \ge J$ and the minimality of $u_n$ we get

$$J(\bar u) \le \varliminf_n J(u_n) \le \varliminf_n J_n(u_n) \le \varliminf_n J_n(u). \tag{38}$$

Finally, by (37) and dominated convergence

$$\lim_n \int_\Omega H_n(Du)\,dx = \int_\Omega H(Du)\,dx$$

so that

$$J(\bar u) \le \varliminf_n J_n(u) = J(u).$$

Since by (29) $u$ is a minimiser for $J$, so is $\bar u$, and using again Proposition 3.4 grants $\bar u = u$. ☐

We also provide a similar approximation scheme for first Dirichlet positive eigenfunctions.

**Proposition 4.3.** *Let $\Omega \subseteq \mathbb{R}^N$ be open, connected and with finite measure and $H : \mathbb{R}^N \to [0, +\infty[$ be strictly convex and positively $p$-homogeneous. If $u$ is a first positive Dirichlet eigenfunction for $\lambda^+_{1,H}(\Omega)$, there exists a sequence of convex, positively $p$-homogeneous $H_n \in C^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$ obeying (33) and corresponding first positive Dirichlet eigenfunctions $u_n$ such that $u_n \rightharpoonup u$ in $W^{1,p}_0(\Omega)$.*

*Proof.* By Proposition 3.4 the strict convexity of $H$ ensures that the eigenvalue $\lambda^+_{1,H}(\Omega)$ is simple, so we can normalise and suppose that $\|u\|_p = 1$. For $H_n : \mathbb{R}^N \to \mathbb{R}$ defined as in the previous proof, consider the first positive Dirichlet eigenfunctions $u_n$ associated to $H_n$, normalised with unitary $L^p$ norm. Fix $v \in C^\infty_c(\Omega)$ such that $\|v\|_p = 1$ and recall that $H_n \to H$ in $C^0_{\rm loc}(\mathbb{R}^N)$, hence

$$\lim_n \int_\Omega H_n(Dv)\,dx = \int_\Omega H(Dv)\,dx$$

Since by the definition of $u_n$ it holds

$$\int_\Omega H_n(Dv)\,dx \ge \int_\Omega H_n(Du_n)\,dx \ge \frac{1}{C}\int_\Omega |Du_n|^p\,dx$$

hence $(u_n)$ is bounded in $W^{1,p}_0(\Omega)$ and we can suppose that $u_n \rightharpoonup \bar u$ and $\|\bar u\|_p = 1$ by the compactness of $W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)$. The chain of inequalities (38) still holds true, proving that $\bar u$ is a first positive Dirichlet eigenfunction and then that $\bar u = u$ by the simplicity of $\lambda^+_{1,H}(\Omega)$ stated in Proposition 3.4. ☐

In [16] is proved an Hopf Lemma for a class of anisotropic operators, where the function giving the anisotropy is even. In the sequel we extend the result to the case of a not even anisotropy.

4.2. **Hopf Lemma.** Let $\Phi \in C^\infty(\mathbb{R}^N \setminus \{0\})$ be convex, positively 1-homogeneous and such that

$$\{\Phi \le 1\} \qquad \text{is bounded and strongly convex.} \tag{39}$$

The *polar* of $\Phi$ is

$$\Phi^\circ(z) = \sup\{(\xi, z) : \Phi(\xi) \le 1\}$$

and $\Phi^\circ$ has the same regularity of $\Phi$ and, for any $x \ne 0$, it holds [6]:

$$\Phi(D\Phi^\circ(x)) \equiv 1 \equiv \Phi^\circ(D\Phi(x))\,, \tag{40}$$

$$\Phi(x)\,D\Phi^\circ(D\Phi(x)) = x = \Phi^\circ(x)\,D\Phi(D\Phi^\circ(x))\,. \tag{41}$$

**Lemma 4.4.** *For $\Phi$ as given and $r > 0$ set*

$$A_r = \{x \in \mathbb{R}^N : r > \check{\Phi}^\circ(x) > r/2\}, \tag{42}$$

*where*

$$\check{\Phi}^\circ(x) = \Phi^\circ(-x).$$

*For any $m > 0$ there exists $\underline{u} \in C^2(\overline{A_r})$ such that*

$$\begin{cases} \mathrm{div}\,(\Phi^{p-1}(D\underline{u})\, D\Phi(D\underline{u})) = 0 & \text{in } A_r \\ \underline{u} = m & \text{on } \{\check{\Phi}^\circ = r/2\} \\ \underline{u} = 0, \quad \partial_n \underline{u} > 0 & \text{on } \{\check{\Phi}^\circ = r\} \end{cases}$$

*where $n$ is the inner normal to $\{\check{\Phi}^\circ \le r\}$.*

*Proof.* We choose $\underline{u}(x) = w(\check{\Phi}^\circ(x))$ for a suitable, decreasing $w \in C^2([r/2, r])$. Since $D\check{\Phi}^\circ(x) = -D\Phi^\circ(-x)$, we have, for $x \neq 0$ (which can be assumed henceforth as $0 \notin A_r$)

$$D\underline{u}(x) = w'(\check{\Phi}^\circ(x))\, D\check{\Phi}^\circ(x) = -w'(\check{\Phi}^\circ(x))\, D\Phi^\circ(-x). \tag{43}$$

Therefore, as we are assuming $w' < 0$, (40) and (41) and the 0-positive homogeneity of $D\check{\Phi}^\circ$ give

$$\Phi(D\underline{u}(x)) = -w'(\check{\Phi}^\circ(x))\, \Phi(D\Phi^\circ(-x)) = -w'(\check{\Phi}^\circ(x)).$$

$$D\Phi(D\underline{u}(x)) = D\Phi(D\Phi^\circ(-x)) = \frac{-x}{\Phi^\circ(-x)} = -\frac{x}{\check{\Phi}^\circ(x)}$$

so that

$$\begin{aligned} -\mathrm{div}\,\big(\Phi^{p-1}(D\underline{u}(x))\, D\Phi(D\underline{u}(x))\big) &= \mathrm{div}\,\left(x\, \frac{(-w'(\check{\Phi}^\circ(x))^{p-1}}{\check{\Phi}^\circ(x)}\right) \\ &= \left(x, D\frac{(-w'(\check{\Phi}^\circ(x))^{p-1}}{\check{\Phi}^\circ(x)}\right) + N\frac{(-w'(\check{\Phi}^\circ(x))^{p-1}}{\check{\Phi}^\circ(x)} \end{aligned}$$

We compute

$$\begin{aligned} D\frac{(-w'(\check{\Phi}^\circ(x))^{p-1}}{\check{\Phi}^\circ(x)} &= -\frac{(p-1)\,(-w'(\check{\Phi}^\circ(x))^{p-2}\, w''(\check{\Phi}^\circ(x))\, D\check{\Phi}^\circ(x)}{\check{\Phi}^\circ(x)} \\ &\quad - \frac{(-w'(\check{\Phi}^\circ(x))^{p-1}\, D\check{\Phi}^\circ(x)}{(\check{\Phi}^\circ(x))^2} \end{aligned}$$

and note that, by the 1-positive homogeneity of $\check{\Phi}^\circ$

$$\big(x, D\check{\Phi}^\circ(x)\big) = \check{\Phi}^\circ(x),$$

hence

$$\mathrm{div}\,\big(\Phi^{p-1}(D\underline{u})\, D\Phi(D\underline{u})\big) = (p-1)\,(-w'(\check{\Phi}^\circ)^{p-2}\, w''(\check{\Phi}^\circ) + (N-1)\, \frac{(-w'(\check{\Phi}^\circ)^{p-1}}{\check{\Phi}^\circ}.$$

It thus suffices to choose a decreasing $w$ so that

$$(p-1)\,(-w'(t))^{p-2}\, w''(t) + (N-1)\, \frac{(-w'(t)^{p-1}}{t} = 0$$

which is equivalent to

$$\big((-w'(t))^{p-1}\, t^{N-1}\big)' = 0.$$

All the strictly decreasing solutions on $\mathbb{R}_+$ of the latter ODE are given by

$$w(t) = \begin{cases} \frac{A}{p-N}\, t^{(p-N)/(p-1)} + B & \text{if } p \neq N \\ A\, \log t + B & \text{if } p = N \end{cases}$$

for arbitrary $A < 0, B \in \mathbb{R}$, and if $m > 0$ it is readily verified that we can indeed choose $A < 0$, $B \in \mathbb{R}$ in such a way that

$$w(r/2) = m, \qquad w(r) = 0$$

and therefore also $w' < 0$. For such a choice, the corresponding $\underline{u} = w(\check{\Phi}^\circ)$ fulfils all the requirements, as the interior normal to $\{\check{\Phi}^\circ \leq r\}$ is $-D\check{\Phi}^\circ/|D\check{\Phi}^\circ|$ and by (43)

$$\partial_n \underline{u} = -\left(\frac{D\check{\Phi}^\circ}{|D\check{\Phi}^\circ|}, w'(\check{\Phi}^\circ)\, D\check{\Phi}^\circ\right) = -w'(\check{\Phi}^\circ)\, |D\check{\Phi}^\circ| > 0$$

□

**Proposition 4.5.** *Suppose $\Omega$ is bounded and connected with $C^2$ boundary, $H \in C^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$ is positively $p$-homogeneous and fulfils (33), and $f \in C^0(\mathbb{R}, [0, +\infty[)$ obeys (10). Then any critical point $u$ for $J$ is $C^{1,\alpha}(\overline{\Omega})$, $u > 0$ in $\Omega$ and*

$$\frac{\partial u}{\partial n} > 0 \qquad \text{on } \partial\Omega \tag{44}$$

*where $n$ is the interior normal to $\partial\Omega$.*

*Proof.* Any critical point for $J$ (which, under the stated assumption is a $C^1$ functional) is a weak solution of (3) with non-negative and subcritical left hand side. The boundedness and positivity of $u$ has already been discussed and its regularity up to the boundary follows from (33) and [36]. Let $\Phi$ be the Minkowski functional of $\{H \leq 1\}$, so that $H = \Phi^p$, and let $A_r$ be given in (42). Since $\partial\Omega$ is $C^2$ and $\check{\Phi}^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, for any $x_0 \in \partial\Omega$ there exists $x_1 \in \Omega$ and $r > 0$ such that

$$x_1 + A_r \subseteq \Omega, \qquad \left(x_1 + \overline{A_r}\right) \cap \partial\Omega = \{x_0\}.$$

Let

$$m = \inf\{u(x) : \check{\Phi}^\circ(x - x_1) = r/2\} > 0$$

and choose the corresponding $\underline{u}$ given in the previous Lemma. The weak comparison principle in Proposition 3.2 ensures that $u \geq \underline{u}$, which in turn implies (44). □

## 5. **Proof of the main result.**

**• Step 1**

Given $u \in C_J^+$, by Lemma 3.1 we can assume that $F > 0$ on $]0, +\infty[$. Consider the sequence $(u_n)$ given in Propositions 4.2 and 4.3, solving the corresponding regularised problems. Suppose we can prove that $c_{\varphi(u_n)} \leq 0$. Since $u_n \to u$ almost everywhere in $\Omega$ , so does $\varphi(u_n)$ to $\varphi(u)$, hence being $u$ (and thus $\varphi(u)$) continuous in $\Omega$, we will have $c_{\varphi(u)} \leq 0$.

Therefore we can assume that $H \in C^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$ fulfils (55) for some given $0 < \lambda \leq \Lambda$ (and is therefore strictly convex). Moreover, proceeding as in [10, Section 4.1] (with obvious modifications in the case $u$ is a first positive Dirichlet eigenfunction), we can assume that $\Omega$ is strongly convex with smooth boundary. In particular, the strong minimum principle and the Hopf Lemma apply, so we can choose $\eta > 0$, $\beta \in\, ]0, 1[$ such that $u \in C^{1,\beta}(\overline{\Omega})$ and furthermore

$$\inf_{\Omega_\eta} u > 0, \qquad \frac{\partial u}{\partial n} > 0 \text{ on } \partial\Omega, \qquad \inf_{\Omega \setminus \Omega_\eta} |Du| > 0. \tag{45}$$

We aim at proving that for any given $\delta \in ]0,\eta[$ (which we'll assume henceforth),

$$c_{\varphi(u)} \leq 0 \qquad \text{in } \Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1], \tag{46}$$

(see (14) for $\Omega_\delta$), which will prove the theorem. In doing so, we can also assume that $\delta$ is so small that $\partial\Omega_{\delta/2}$ is strongly convex and smooth.

• **Step 2**

Denoting $a^{2/p} = (a^2)^{1/p}$ for any $a \in \mathbb{R}$, we define the family of integrands (recall that $F$ is oddly extended to $\mathbb{R}$)

$$G_\varepsilon(t,z) = \frac{1}{p}\left[\varepsilon\, F(t)^{\frac{2}{p}} + H^{\frac{2}{p}}(z)\right]^{\frac{p}{2}} - F(t)$$

and corresponding auxiliary functionals

$$I_\varepsilon(u) = \frac{1}{p}\int_\Omega \left[\varepsilon\, F(u)^{\frac{2}{p}} + H^{\frac{2}{p}}(Du)\right]^{\frac{p}{2}} dx - \int_\Omega F(u)dx \tag{47}$$

If $u$ is not a first positive Dirichlet eigenfunction, by [10, Lemma 4.1], for any sufficiently small $\varepsilon$ problem

$$\inf\left\{I_\varepsilon(u) : u \in W_0^{1,p}(\Omega)\right\}$$

admits a minimiser $u_\varepsilon$ with the property that

$$u_\varepsilon \to u \qquad \text{in } C^\beta(\overline{\Omega}). \tag{48}$$

Indeed, the proof of [10, Lemma 4.1] can be repeated *verbatim* when $u$ is not a first positive Dirichlet eigenfunction, since only the uniqueness of the minimiser $u$ of $J$ is used therein and the latter is granted by the strict convexity of $H$ (due to the previous point) and Lemma 3.1, (29) and Proposition 3.4. When $u$ is a normalised first positive Dirichlet eigenfunction we instead consider

$$\inf\left\{I_\varepsilon(u) : u \in W_0^{1,p}(\Omega), u \geq 0, \|u\|_p = 1\right\}$$

and proceed in the same way, using again Proposition 3.4 to ensure weak convergence of $u_\varepsilon$ to $u$, as well as (48) by uniform $C^\beta(\overline{\Omega})$ bounds.

• **Step 3**

We now improve the convergence of $u_\varepsilon \to u$ beyond the $C^\beta$ level. Any $u_\varepsilon$ defined above satisfies weakly the Euler-Lagrange equation for $I_\varepsilon$,

$$-\mathrm{div}\left(D_z G_\varepsilon(u_\varepsilon, Du_\varepsilon)\right) + \partial_u G_\varepsilon(u_\varepsilon, Du_\varepsilon) = 0$$

which is more explicitly computed as

$$\begin{aligned} -\,\mathrm{div}\left(\left(\varepsilon F(u_\varepsilon)^{\frac{2}{p}} + H^{\frac{2}{p}}(Du_\varepsilon)\right)^{\frac{p-2}{2}} D\frac{H^{\frac{2}{p}}}{2}(Du_\varepsilon)\right)& \\ = f(u_\varepsilon)\left[1 - \frac{\varepsilon}{p}\left(\varepsilon\, F(u_\varepsilon)^{\frac{2}{p}} + H^{\frac{2}{p}}(Du_\varepsilon)\right)^{\frac{p-2}{2}} F(u_\varepsilon)^{\frac{2-p}{p}}\right]&. \end{aligned} \tag{49}$$

Note that (48) and the positivity of $u$ ensure that given any $\delta \in ]0,\eta[$, there exists $C > 0$ such that for a sufficiently small $\varepsilon$ (which will be assumed henceforth) it holds

$$\frac{1}{C} \leq u_\varepsilon \leq C \qquad \text{in } \Omega_{\delta/4} \tag{50}$$

and therefore, $F(u_\varepsilon)$ is uniformly bounded from above and below by a positive constant. Thanks to Lemma A.1, point 2, Tolskdorff local regularity theory [42] applies , ensuring that

$$\|u_\varepsilon\|_{C^{1,\beta}(\overline{\Omega_{\delta/3}})} \le C \tag{51}$$

for a $\beta \in ]0,1[$ (possibly different from the one in (48)) depending on $\Omega, H, N, p$ and $C > 0$ with the same dependencies and additionally on $\delta$ and $\|u\|_{L^\infty(\Omega)}$, but none of them depending on $\varepsilon$. We'll assume henceforth that $\beta < \alpha$, the Hölder continuity coefficient of $f$. In particular It follows that $u_\varepsilon \to u$ in $C^1(\overline{\Omega_{\delta/3}})$ by Ascoli-Arzelá. Further regularity can be obtained noting that by (50) and the boundedness of $H(Du_\varepsilon)$, the matrix $D^2_z G_\varepsilon$ is strongly elliptic in $\Omega_{\delta/4}$ thanks to Lemma A.1, point 2. The difference quotient method yields $u_\varepsilon \in W^{2,2}_{\rm loc}(\Omega_{\delta/4})$, with any partial derivative $w = \partial_i u_\varepsilon$, $i = 1, \dots, N$ obeying

$$-\mathrm{div}\,\left(D^2_z G_\varepsilon(u_\varepsilon, Du_\varepsilon)\, Dw\right) = \mathrm{div}\,\left(\partial_u D_z G_\varepsilon(u_\varepsilon, Du_\varepsilon)\, w + e_i\, \partial_u G_\varepsilon(u_\varepsilon, Du_\varepsilon)\right)$$

weakly in $\Omega_{\delta/4}$. Using the $C^{1,\beta}$ regularity of $u_\varepsilon$, we see that

$$D^2_z G_\varepsilon(u_\varepsilon, Du_\varepsilon) \qquad \text{and} \qquad \partial_u D_z G_\varepsilon(u_\varepsilon, Du_\varepsilon)\, w + e_i\, \partial_u G_\varepsilon(u_\varepsilon, Du_\varepsilon)$$

are $\beta$-Hölder continuous in $\Omega_{\delta/3}$, so that local linear regularity theory ensures $w \in C^{1,\beta}(\Omega_{\delta/3})$, i.e. $u_\varepsilon \in C^{2,\beta}(\Omega_{\delta/3})$. As $\varepsilon \to 0$, the $C^{2,\beta}(\Omega_{\delta/3})$ norm of $u_\varepsilon$ may blow up, but by (45), for sufficiently small $\varepsilon$ it holds

$$\inf_{\Omega \setminus \Omega_\eta} |Du_\varepsilon| > 0,$$

therefore, by looking at (57), we see that $D^2_z G_\varepsilon$ is strongly elliptic in $\Omega \setminus \Omega_\eta$, with ellipticity constants uniformly bounded from below and above as $\varepsilon \to 0$. We conclude by local elliptic regularity theory that, given $\delta \in ]0, \eta[$, for any sufficiently small $\varepsilon$ it holds

$$\|u_\varepsilon\|_{C^{2,\beta}(\overline{\Omega_{\delta/2}} \setminus \Omega_\delta)} \le C \tag{52}$$

with $C$ depending only on $\Omega, H, N, p, \delta, \eta$ and $\|u\|_\infty$, but not on $\varepsilon$.

• **Step 4**

Let $v_\varepsilon = \varphi(u_\varepsilon)$. We claim that, for any sufficiently small $\delta \in ]0, \eta[$ and, correspondingly, sufficiently small $\varepsilon$, $c_{v_\varepsilon}$ cannot assume a positive maximum on $\partial(\Omega_{\delta/2} \times \Omega_{\delta/2}) \times [0,1]$. Indeed, Proposition 2.2 applies to $u$, so that for any sufficiently sufficiently small $\delta$, $v = \varphi(u)$ fulfils (17) in $\Omega \setminus \Omega_\delta$ and (18) for all $x_0 \in \Omega \setminus \Omega_\delta$, $x \in \Omega \setminus \{x_0\}$. From the uniform bounds (51) and (52) proved in the previous step, we infer by Ascoli-Arzelá that

$$u_\varepsilon \to u \qquad \text{in } C^1(\overline{\Omega_{\delta/2}}) \cap C^2(\overline{\Omega_{\delta/2}} \setminus \Omega_\delta).$$

Since $u_\varepsilon$ fulfils (50) for sufficiently small $\varepsilon$ and since $\varphi \in C^{2,\alpha}_{\rm loc}(\mathbb{R}_+)$, the previous convergence holds true for $v_\varepsilon$ as well. Note that $\partial\Omega_{\delta/2} \subseteq \Omega \setminus \Omega_\delta$, hence $v$ fulfils (18) for all $x_0 \in \partial\Omega_{\delta/2}$ and $x \in \overline{\Omega_{\delta/2}} \setminus \{x_0\}$. Then Proposition 2.5, applied to the family $v_\varepsilon$ and $v$ on the strictly convex set $\Omega_{\delta/2}$, ensures that for any sufficiently small $\varepsilon > 0$ (18) holds true for $v_\varepsilon$ at all points $x_0 \in \partial\Omega_{\delta/2}$, $x \in \overline{\Omega_{\delta/2}} \setminus \{x_0\}$. Finally, Proposition 2.4, applied to such functions $v_\varepsilon$ on $\Omega_{\delta/2}$, proves the claim.

• **Step 5**

For $\varepsilon$ and $v_\varepsilon$ as above and $M_f$ as in (5), we look at the equation fulfilled by $v_\varepsilon$. The proof of the last statement of Lemma 3.1 still holds for the functional $I_\varepsilon$ and its minimiser $u_\varepsilon$, showing that

$$M_\varepsilon = \sup_\Omega u_\varepsilon \le M_f.$$

Let $\psi = \varphi^{-1}$, so that for $t \in \,]0, M_\varepsilon]$ it holds

$$\varphi'(t) = F^{-\frac{1}{p}}(t), \qquad \psi'(s) = F^{\frac{1}{p}}(\psi(s)).$$

We thus have $F(u_\varepsilon) = F(\psi(v_\varepsilon))$ and

$$H(Du_\varepsilon) = H(\psi'(v_\varepsilon)\, Dv_\varepsilon) = F(\psi(v_\varepsilon))\, H(Dv_\varepsilon) \tag{53}$$

where we used the positive $p$-homogeneity of $H$ in the last step. Similarly, by the positive 1-homogeneity of $DH^{2/p}$

$$(DH^{\frac{2}{p}})(Du_\varepsilon) = \psi'(v_\varepsilon)\,(DH^{\frac{2}{p}})(Dv_\varepsilon) = F^{\frac{1}{p}}(\psi(v_\varepsilon))\,(DH^{\frac{2}{p}})(Dv_\varepsilon).$$

We look at equation (49) for $v_\varepsilon$. For the left hand side we compute

$$\begin{aligned}
&\operatorname{div}\left(\left(\varepsilon F(u_\varepsilon)^{\frac{2}{p}} + H^{\frac{2}{p}}(Du_\varepsilon)\right)^{\frac{p-2}{2}} D\frac{H^{\frac{2}{p}}}{2}(Du_\varepsilon)\right)\\
&= \operatorname{div}\left(F^{1-\frac{1}{p}}(\psi(v_\varepsilon))\left(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\right)^{\frac{p-2}{2}} D\frac{H^{\frac{2}{p}}}{2}(Dv_\varepsilon)\right)\\
&= \left(1-\frac{1}{p}\right) F^{-\frac{1}{p}}(\psi(v_\varepsilon))\, f(\psi(v_\varepsilon))\, \psi'(v_\varepsilon)\big(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\big)^{\frac{p-2}{2}} \left(D\frac{H^{\frac{2}{p}}}{2}(Dv_\varepsilon), Dv_\varepsilon\right)\\
&\quad + F^{1-\frac{1}{p}}(\psi(v_\varepsilon))\operatorname{div}\left(\left(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\right)^{\frac{p-2}{2}} D\frac{H^{\frac{2}{p}}}{2}(Dv_\varepsilon)\right)\\
&= \left(1-\frac{1}{p}\right) f(\psi(v_\varepsilon))\left(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\right)^{\frac{p-2}{2}} H^{\frac{2}{p}}(Dv_\varepsilon)\\
&\quad + F^{1-\frac{1}{p}}(\psi(v_\varepsilon))\operatorname{div}\left(\left(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\right)^{\frac{p-2}{2}} D\frac{H^{\frac{2}{p}}}{2}(Dv_\varepsilon)\right)
\end{aligned}$$

where we used $\psi'(s) = F^{1/p}(\psi(s))$ and that, being $H^{2/p}$ positively 2-homogeneous,

$$\left(DH^{\frac{2}{p}}(Dv_\varepsilon), Dv_\varepsilon\right) = 2\, H^{\frac{2}{p}}(Dv_\varepsilon).$$

The right-hand side of (49) is, again by (53),

$$\begin{aligned}
&f(u_\varepsilon)\left[1 - \frac{\varepsilon}{p}\left(\varepsilon\, F(u_\varepsilon)^{\frac{2}{p}} + H^{\frac{2}{p}}(Du_\varepsilon)\right)^{\frac{p-2}{2}} F(u_\varepsilon)^{\frac{2-p}{p}}\right]\\
&= f(\psi(v_\varepsilon))\left[1 - \frac{\varepsilon}{p}\left(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\right)^{\frac{p-2}{2}}\right].
\end{aligned}$$

Hence $v_\varepsilon$ satisfies

$$\begin{aligned}
&- F^{1-\frac{1}{p}}(\psi(v_\varepsilon))\operatorname{div}\left(\left(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\right)^{\frac{p-2}{2}} D\frac{H^{\frac{2}{p}}}{2}(Dv_\varepsilon)\right) =\\
&f(\psi(v_\varepsilon))\left[1 - \frac{\varepsilon}{p}\left(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\right)^{\frac{p-2}{2}} + \left(1-\frac{1}{p}\right)\left(\varepsilon + H^{\frac{2}{p}}(Dv_\varepsilon)\right)^{\frac{p-2}{2}} H^{\frac{2}{p}}(Dv_\varepsilon)\right]
\end{aligned}$$

which rewrites as

$$-\operatorname{div}\,(DH_\varepsilon(Dv_\varepsilon)) = \frac{f(\psi(v_\varepsilon))}{F^{1-\frac{1}{p}}(\psi(v_\varepsilon))} b_\varepsilon(Dv_\varepsilon) \tag{54}$$

where

$$H_\varepsilon(z) = \big(\varepsilon + H^{\frac{2}{p}}(z)\big)^{\frac{p}{2}}$$
$$b_\varepsilon(z) = p + \big((p-1)\, H^{\frac{2}{p}}(z) - \varepsilon\big)\big(\varepsilon + H^{\frac{2}{p}}(z)\big)^{\frac{p-2}{2}}.$$

• **Step 6**

We finally exclude that, for $\varepsilon$ and $v_\varepsilon$ as above, $c_{v_\varepsilon}$ attains a positive maximum on $\Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1]$. Note that

$$b_\varepsilon(z) \geq b(0) = p - \varepsilon^{\frac{p}{2}}$$

which is positive for sufficiently small $\varepsilon$. On the other hand, for $s \in v_\varepsilon(\Omega_{\delta/2}) \subseteq ]0, \varphi(M_\varepsilon)]$ it holds

$$\frac{f(\psi(s))}{F^{1-\frac{1}{p}}(\psi(s))} = \big(F^{\frac{1}{p}}\big)'(\psi(s))$$

which is non-increasing since $\psi$ is non-decreasing and $F^{1/p}$ is concave, while

$$\psi''(s) = \big(F^{\frac{1}{p}}(\psi(s))\big)' = \frac{1}{p}\, F^{1-\frac{1}{p}}(\psi(s))\, f(\psi(s))\, \psi'(s) = \frac{F(\psi(s))}{f(\psi(s))}$$

so that

$$\frac{F^{1-\frac{1}{p}}(\psi(s))}{f(\psi(s))} = \frac{\psi''(s)}{\psi'(s)}$$

which is convex by Lemma 2.6, point 2. Finally, since $v_\varepsilon \in C^2(\Omega_{\delta/2})$, (54) rewrites as

$$-\mathrm{Tr}\,\big(D^2 H_\varepsilon(Dv_\varepsilon)\, D^2 v_\varepsilon\big) = \frac{f(\psi(v_\varepsilon))}{F^{1-\frac{1}{p}}(\psi(v_\varepsilon))} b_\varepsilon(Dv_\varepsilon)$$

and since $H(Dv_\varepsilon)$ is bounded in $\Omega_{\delta/2}$, Lemma A.1, point 2, grants the strong ellipticity of $D^2 H_\varepsilon(z)$ for $z \in Dv_\varepsilon(\Omega_{\delta/2})$. Therefore Proposition 2.1 ensures that $c_{v_\varepsilon}$ cannot attain a positive maximum on $\Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1]$. By step 4 we conclude that, given a sufficiently small $\delta$, for any sufficiently small $\varepsilon > 0$, $c_{v_\varepsilon} \leq 0$ on $\Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1]$ and taking the limit for $\varepsilon \downarrow 0$, we infer (46), proving the theorem.

**Appendix** A. **Ellipticity estimates.** In this appendix we prove strong ellipticity estimates for the auxiliary integrands constructed during the proof of Theorem 1.1, starting from a smooth, positively $p$-homogeneous $H$ obeying

$$\lambda\, |z|^{p-2}\, |v|^2 \leq \big(D^2 H(z)\, v, v\big) \leq \Lambda\, |z|^{p-2}\, |v|^2 \qquad \forall z \in \mathbb{R}^N \setminus \{0\}, v \in \mathbb{R}^N. \tag{55}$$

Their proofs are variants of [20, Appendix A], where (55) is shown to be a consequence of the strong convexity of $\{H \leq 1\}$.

**Lemma A.1.** *Suppose $H \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ is positively $p$-homogeneous and fulfils for $0 < \lambda \leq \Lambda$ the ellipticity estimate (55).*

*Then*

1. *$H^{2/p}$ is strongly elliptic in the sense that there exists positive $\widehat{\lambda}$, $\widehat{\Lambda}$ depending only $H$ and $p$ such that for any $z, v \in \mathbb{R}^N$ it holds*

$$\widehat{\lambda}\, |v|^2 \leq \Big(D^2 H^{2/p}(z)\, v, v\Big) \leq \widehat{\Lambda}\, |v|^2 \tag{56}$$

2. *For any $\theta \geq 0$ the function*

$$H_\theta(z) = \left(\theta + H^{\frac{2}{p}}(z)\right)^{\frac{p}{2}}$$

*fulfils*

$$\widetilde{\lambda}\,\left(\theta + H^{\frac{2}{p}}(z)\right)^{\frac{p-2}{2}}\,|v|^2 \leq \left(D^2 H_\theta(z)\,v, v\right) \leq \widetilde{\Lambda}\,\left(\theta + H^{\frac{2}{p}}(z)\right)^{\frac{p-2}{2}}\,|v|^2 \tag{57}$$

*for all $z, v \in \mathbb{R}^N$, where $\widetilde{\lambda}, \widetilde{\Lambda}$ are positive numbers depending on $H$ and $p$ but not on $\theta$.*

*Proof.* From the positive $p$-homogeneity of $H$ we get

$$p\,H(z) = (DH(z), z)\,, \qquad p\,(p-1)H(z) = \left(D^2 H(z)\,z, z\right) \tag{58}$$

as well as

$$D^2 H(z)\,z = (p-1)DH(z). \tag{59}$$

Let

$$n_z = \frac{DH(z)}{|DH(z)|}$$

be the exterior normal to the level sets of $H$. Then by (58) for any $z \neq 0$

$$(z, n_z) = p\,\frac{H(z)}{|DH(z)|} \geq c\,|z|$$

where

$$c = c(H,p) = \inf_{z\neq 0} p\,\frac{H(z)}{|DH(z)|\,|z|} = \inf_{|z|=1} p\,\frac{H(z)}{|DH(z)|}$$

(where we used the $p$-positive homogeneity of $H$ and of $|DH(z)|\,|z|$), which is finite and positive. In particular any $v \in \mathbb{R}^N$ can be uniquely written as

$$v = k\,z + t \qquad \text{with } k \in \mathbb{R},\ t \in \mathbb{R}^N,\ (t, n_z) = 0. \tag{60}$$

We clearly have

$$|v|^2 \leq 2\left(|t|^2 + k^2\,|z|^2\right).$$

On the other hand, by Schwartz inequality

$$|v| \geq |(v, n_z)| = |k|\,(z, n_z) \geq c\,|k|\,|z|,$$

while by triangle inequality and the latter estimate

$$|t| \leq |v| + |k|\,|z| \leq \left(1 + \frac{1}{c}\right)|v|.$$

All in all, we have found a constant $C = C(H,p)$ such that for any $z \neq 0$ and all $v \in \mathbb{R}^N$ decomposed as in (60), it holds

$$\frac{1}{C}\,(k^2\,|z|^2 + |t|^2) \leq |v|^2 \leq C\,(k^2\,|z|^2 + |t|^2). \tag{61}$$

Decomposition (60) allows the following computations for $z \neq 0$. Thanks to (58) and $(DH(z), t) = 0$, we have

$$(DH(z)\,v, v)^2 = k^2 (DH(z), z)^2 = p^2\,H^2(z)\,k^2. \tag{62}$$

On the other hand,

$$\begin{aligned}(D^2 H(z)\,v, v) &= k^2\,(D^2 H(z)\,z, z) + 2\,k\,(D^2 H(z)\,z, t) + (D^2 H(z)\,t, t)\\ &= p\,(p-1)\,H(z)\,k^2 + (D^2 H(z)\,t, t)\end{aligned} \tag{63}$$

thanks to (58), (59) and again $(DH(z), t) = 0$.

With these tools at hand, let us prove assertion (1) of the Lemma. Being $H^{2/p}$ a positively 2-homogeneous function, $D^2 H^{2/p}$ is 0-homogeneous, hence it suffices to consider the case $z \in \{H = 1\}$. We compute

$$D^2 H^{\frac{2}{p}}(z) = \frac{2}{p}\, H^{\frac{2-p}{p}}(z) \left[\frac{2-p}{p}\, H^{-1}(z)\, DH(z) \otimes DH(z) + D^2 H(z)\right]$$

so that for a given $z \in \{H = 1\}$

$$\left(D^2 H^{\frac{2}{p}}(z)\, v, v\right) = \frac{2}{p}\left[\frac{2-p}{p}\,(DH(z)\, v, v)^2 + (D^2 H(z)\, v, v)\right]. \tag{64}$$

Inserting (62) and (63) in (64) gives, for any $z \in \{H = 1\}$,

$$\left(D^2 H^{\frac{2}{p}}(z)\, v, v\right) = 2\, k^2 + \frac{2}{p}\,(D^2 H(z)\, t, t)$$

and using (55) we obtain

$$2\, k^2 + \frac{2\,\lambda}{p}\,|z|^{p-2}\,|t|^2 \le \left(D^2 H^{\frac{2}{p}}(z)\, v, v\right) \le 2\, k^2 + \frac{2\,\Lambda}{p}\,|z|^{p-2}\,|t|^2.$$

Since $|z|$ is uniformly bounded from above and below on $\{H = 1\}$, the two-sided estimate (61) provides (56) for $z \in \{H = 1\}$, and thus for all $z \in \mathbb{R}^N$ by 0-homogeneity.

To prove assertion (2), we set $\widehat{H} = H^{2/p}$, which is 2 homogeneous and satisfies (56), so that

$$H_\theta(z) = \left(\theta + \widehat{H}(z)\right)^{\frac{p}{2}}.$$

A standard computation gives

$$D^2 H_\theta(z) = \frac{p}{2}\left(\theta + \widehat{H}(z)\right)^{\frac{p-2}{2}}\left[\frac{p-2}{2\,(\theta + \widehat{H}(z))} D\widehat{H}(z) \otimes D\widehat{H}(z) + D^2 \widehat{H}(z)\right].$$

For $z \neq 0$ and $v \in \mathbb{R}^N$, we consider again the decomposition (60), so that (62) and (63) applied to $\widehat{H}$ (so that $p = 2$ therein), give

$$\begin{aligned}
&\big(D^2 H_\theta(z)\, v, v\big)\\
&= \frac{p}{2}\left(\theta + \widehat{H}(z)\right)^{\frac{p-2}{2}}\left[\frac{p-2}{2\,(\theta + \widehat{H}(z))}\, 4\,\widehat{H}^2(z)\, k^2 + 2\,\widehat{H}(z)\, k^2 + \big(D^2 H(z)\, t, t\big)\right]\\
&= p\left(\theta + \widehat{H}(z)\right)^{\frac{p-2}{2}}\left[\frac{\theta + (p-1)\,\widehat{H}(z)}{\theta + \widehat{H}(z)}\,\widehat{H}(z)\, k^2 + \frac{1}{2}\left(D^2 \widehat{H}(z)\, t, t\right)\right].
\end{aligned}$$

Next note that

$$a_p := \min\{1, p-1\} = \inf_{t \ge 0} \frac{\theta + (p-1)\, t}{\theta + t}, \quad b_p := \max\{1, p-1\} = \sup_{t \ge 0} \frac{\theta + (p-1)\, t}{\theta + t}$$

which are positive and independent of $\theta$, hence (56) provides

$$\left[a_p\,\widehat{H}(z)\, k^2 + \frac{\widehat{\lambda}}{2}\,|t|^2\right] \le \frac{(D^2 H_\theta(z)\, v, v)}{p\left(\theta + \widehat{H}(z)\right)^{\frac{p-2}{2}}} \le \left[b_p\,\widehat{H}(z)\, k^2 + \frac{\widehat{\Lambda}}{2}\,|t|^2\right] \tag{65}$$

where $\widehat{\lambda}$ and $\widehat{\Lambda}$ are given in (56), hence are independent of $\theta$. Since $\widehat{H}$ is positively 2-homogeneous and vanishes only at the origin, we readily have

$$\frac{1}{C}\,|z|^2 \leq \widehat{H}(z) \leq C\,|z|^2$$

for a constant $C = C(H, p)$, and using these inequalities in (65) together with (61) provides (57) with the stated dependencies. $\square$

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