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# On the symmetry of minimizers in constrained quasi-linear problems

Marco Squassina

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**Abstract.** We provide a simple proof of the radial symmetry of any nonnegative minimizer for a general class of quasi-linear minimization problems.

**Keywords.** Symmetrization, quasi-linear elliptic equations, non-smooth analysis.

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## 1 Introduction and main result

Let $\Omega$ be either $\mathbb{R}^N$ or a ball $B_R(0)$ centered at the origin in $\mathbb{R}^N$, and define the functional $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, $1 < p < N$, by setting

$$\mathcal{E}(u) = \int_\Omega j(u, |Du|) - \int_\Omega F(|x|, u).$$

Moreover, let $\mathcal{C} \subset W_0^{1,p}(\Omega)$ be the set given by

$$\mathcal{C} = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega G(u) = 1 \right\}. \tag{1.1}$$

Let us consider the following minimization problem

$$m = \inf_{v \in \mathcal{C}} \mathcal{E}(v), \quad -\infty < m < +\infty. \tag{1.2}$$

A classical problem in the Calculus of Variations is to establish the existence of a solution to problem (1.2) and, in addition, to detect further qualitative properties of the solutions such as their radial symmetry and monotonicity [4]. The existence of solutions was extensively investigated, starting from the seminal contributions of Lions [23, 24]. The main strategies followed to achieve the latter goal are, on one hand, the moving plane method by Gidas, Ni and Nirenberg [15] and, on the

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other, the symmetrization techniques, initiated by Steiner and Schwarz for sets, for which we refer the reader to the monographs [2, 21, 27] and the classic [28]. For the semi-linear case $p = 2$, $j(s,t) = |t|^2$ and $F = 0$, a pioneering study was performed by Berestycki and Lions in the celebrated paper [3]. General radial symmetry results for $j(s,t) = |t|^2$ have been obtained by Lopes in [25] via a reflection argument and a unique continuation principle. For $j(s,t) = |t|^p$, interesting results have been achieved by Brock in [5] by exploiting rearrangements and strong maximum principle. For further relevant generalizations of these contributions, we refer to the recent work of Mariş [26]. The works [5,25,26] include the case of systems as well and [5,26] also allow multiple constraints (very general in [26]). The existence of a Schwarz symmetric solution of problem (1.2) under general assumptions on $F$ and $j(u,|Du|)$, allowing growth conditions such as

$$\alpha_0|Du|^p \le j(u,|Du|) \le \alpha(|u|)|Du|^p, \quad \alpha_0 > 0,\ \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \text{ continuous},$$

has been recently established [16, 18]. In this paper, focusing on the highly quasilinear character of our minimization problem, we want to provide, under rather weak assumptions, a quite simple proof that *any* given nonnegative minimum $v$ of (1.2) is radially symmetric and decreasing, after a translation, if the set of critical points of $v^*$ has null Lebesgue measure. In general, assuming for instance that $j$ is convex in the gradient and $F$ behaves smoothly, $\mathcal{E}$ is non-smooth unless $j_u = 0$ and, depending upon the growth estimates on $j$, it can be either continuous (if $\alpha$ is bounded from above) or lower semi-continuous. In turn, quite often, techniques of non-smooth analysis are employed.

Given a nonnegative solution $v$ to (1.2), the idea is to construct a related sequence $(v_n)$ (built up by repeatedly polarizing $v$) which is weakly convergent to the Schwarz symmetrization $v^*$ of $v$ in $W_0^{1,p}(\Omega)$. Then, since $(v_n)$ are also solutions to (1.2) they satisfy an Euler–Lagrange equation in a suitable generalized sense (see Section 2.2 and, in particular, Proposition 2.9) obtained by tools of subdifferential calculus for nonsmooth functionals developed by Campa and Degiovanni in [8]. This allows, in turn, to prove the almost everywhere convergence of the gradients $Dv_n$ to $Dv^*$ by applying a powerful result due to Dal Maso and Murat [9] to a suitable sequence of Leray–Lions type operators associated with $j(v_n, |Dv_n|)$. Finally, this leads to the identity $\|Dv\|_{L^p(\Omega)} = \|Dv^*\|_{L^p(\Omega)}$ which provides the desired conclusion that $v$ is nothing but a translation of $v^*$. We stress that, in proving the main result, we never use any form of the strong maximum principle or unique continuation principle. Identity cases for the $p$-Laplacian have been deeply studied since the first pioneering contributions due to Friedman and McLeod [14] and to Brothers and Ziemer [7]. For some recent developments, extensions and new simplified proofs, we refer the reader to the works of Ferone and Volpicelli (see [12, 13] covering both the case of $\mathbb{R}^N$ and of a bounded domain).

Beyond the study of minima, for an investigation of radial symmetry of *minimax* critical points for a class of quasi-linear problems on the ball associated with lower semi-continuous functionals involving $j(u, |Du|)$, we refer to [30] (see also [31] for the case of $C^1$ functionals). We also refer to the monograph [29] and to the references therein for a wide range of results on quasi-linear problems obtained via non-smooth critical point theory.

Throughout the paper, the spaces $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$, for every $p, q \geq 1$, will be endowed, respectively, both for $\Omega = B_R(0)$ or $\Omega = \mathbb{R}^N$, with the usual norms

$$\|u\|_{L^q(\Omega)} = \left(\int_\Omega |u|^q\right)^{1/q}, \quad \|u\|_{W_0^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{j=1}^N \|D_j u\|^p_{L^p(\Omega)}\right)^{1/p}.$$

Next we formulate the assumptions under which our main result will hold.

### 1.1 Assumptions on $j$

For every $s$ in $\mathbb{R}$, the function ($t \in \mathbb{R}^+$)

$$\{t \mapsto j(s,t)\} \text{ is strictly convex and increasing.} \tag{1.3}$$

The functions $j_s$ and $j_t$ and $j_{st}$ denote the derivatives of $j(s,t)$ with respect to the variables $s$ and $t$ and the mixed derivative respectively, which exist continuous. We assume that there exist a positive constant $\alpha_0$ and increasing functions $\alpha$, $\beta$, $\gamma \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\alpha_0 |\xi|^p \leq j(s, |\xi|) \leq \alpha(|s|)|\xi|^p, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and } \xi \in \mathbb{R}^N, \tag{1.4}$$

$$|j_s(s, |\xi|)| \leq \beta(|s|)|\xi|^p, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and } \xi \in \mathbb{R}^N, \tag{1.5}$$

$$|j_t(s, |\xi|)| \leq \gamma(|s|)|\xi|^{p-1}, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and } \xi \in \mathbb{R}^N. \tag{1.6}$$

### 1.2 Assumptions on $F$

$F(|x|, s)$ is the primitive with respect to $s$ of a Carathéodory function $f(|x|, s)$ with $F(|x|, 0) = 0$. Denoting $p^* = Np/(N-p)$, we assume that there exist a positive constant $C$ and a radial function $a \in L^{Np/(N(p-1)+p)}(\Omega)$ such that

$$|f(|x|, s)| \leq a(|x|) + C|s|^{p-1} + C|s|^{p^*-1}, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and } x \in \Omega, \tag{1.7}$$

$$f(|x|, s) \geq f(|y|, s), \quad \text{for every } s \in \mathbb{R}^+ \text{ and } x, y \in \Omega \text{ with } |x| \leq |y|. \tag{1.8}$$

### 1.3 Assumptions on $G$

$G(s)$ is the primitive with respect to $s$ of a continuous function $g$ with $G(0) = 0$. Moreover, there exists a positive constant $C$ such that

$$|g(s)| \leq C|s|^{p-1} + C|s|^{p^*-1}, \quad \text{for every } s \text{ in } \mathbb{R}, \tag{1.9}$$

$$g \text{ is not identically equal to 0 in a right neighborhood of 0.} \tag{1.10}$$

For a given positive $u \in W_0^{1,p}(\Omega)$, we consider the set

$$C^* := \{x \in \Omega : |Du^*(x)| = 0\} \cap (u^*)^{-1}(0, \operatorname{ess\,sup} u).$$

Under assumptions (1.3)–(1.9), the main result of the paper is the following

**Theorem 1.1.** *Assume that $\Omega$ is either $\mathbb{R}^N$ or a ball $B_R(0) \subset \mathbb{R}^N$ and let $u \in \mathcal{C}$ be any nonnegative solution to* (1.2) *such that $\mathcal{L}^N(C^*) = 0$. Then $u = u^*$.*

If the problem is not set in a ball or on the whole space, in general minima could fail to be radially symmetric, even though the domain is invariant under rotations. For instance, Esteban [11] showed that, if $2 < m < 2^*$ and $B$ is a closed ball in $\mathbb{R}^N$, then the problem

$$\min_{u \in H^1(\mathbb{R}^N \setminus B)} \left\{ \int_{\mathbb{R}^N \setminus B} (|Du|^2 + |u|^2) : \int_{\mathbb{R}^N \setminus B} |u|^m = 1 \right\}$$

admits a solution but *no* solution is radially symmetric. See also the discussion by Kawohl in [22, Example 6 and related references] for similar situations of non-symmetric solutions when the problem is defined on an annulus.

Also, as pointed out by Brothers and Ziemer [7, see Section 4] with a counterexample, the condition $\mathcal{L}^N(C^*) = 0$ is necessary in order to ensure that $\|Dv\|_{L^p(\Omega)} = \|Dv^*\|_{L^p(\Omega)}$ implies that $v$ is a translation of $v^*$.

In the particular case where $j(s,t) = |t|^p$, the conclusion of Theorem 1.1 easily follows directly from identity cases for the $p$-Laplacian operator. In fact, if $u \in \mathcal{C}$ is a nonnegative solution to the minimization problem and $u^*$ is the Schwarz symmetrization of $u$, then of course $u^*$ belongs to $\mathcal{C}$ too (Cavalieri's principle). Moreover, in light of the classical Pólya–Szegő inequality and (2.5) of Proposition 2.3, we have

$$\int_\Omega |Du^*|^p \leq \int_\Omega |Du|^p, \quad \int_\Omega F(|x|, u) \leq \int_\Omega F(|x|, u^*). \tag{1.11}$$

Hence,

$$m \le \mathcal{E}(u^*) = \int_\Omega |Du^*|^p - \int_\Omega F(|x|, u^*) \le \int_\Omega |Du|^p - \int_\Omega F(|x|, u) = m.$$

In turn, by (1.11), we have both $\|Du^*\|_{L^p(\Omega)} = \|Du\|_{L^p(\Omega)}$ and $\int_\Omega F(|x|, u) = \int_\Omega F(|x|, u^*)$. Then, by Proposition 2.4, if $\mathcal{L}^N(C^*) = 0$, there is a translate of $u^*$ which is equal to $u$. In the full quasi-linear case, the Pólya-Szegő inequality (cf. Proposition 2.3)

$$\int_\Omega j(u^*, |Du^*|) \le \int_\Omega j(u, |Du|)$$

holds as well when $j(u, |Du|) \in L^1(\Omega)$, and the above argument would lead to the identity

$$\int_\Omega j(u^*, |Du^*|) = \int_\Omega j(u, |Du|). \tag{1.12}$$

It is not clear (we set it as an open problem) if (1.12) plus $\mathcal{L}^N(C^*) = 0$, could yield *directly* the conclusion that there is a translate of $u^*$ which is almost everywhere equal to $u$. It is readily seen that this would hold true knowing in advance that $(u_n) \subset W_0^{1,p}(\Omega)$, $u_n \rightharpoonup v$ weakly and $\int_\Omega j(u_n, |Du_n|)$ convergent to $\int_\Omega j(v, |Dv|)$ imply $\|Du_n\|_{L^p(\Omega)} \to \|Dv\|_{L^p(\Omega)}$, as $n \to \infty$ (or at least $Du_n(x) \to Du(x)$, as $n \to \infty$, for a.e. $x \in \Omega$). This is known to be the case for strictly convex and coercive integrands $j$ which are merely dependent on the gradient, say $j(s,t) = j_0(t)$, see [33]. In this paper we shall solve the problem indirectly, for minima, by reducing to identity cases for the $p$-Laplacian operator. Of course one could derive the radial symmetry information focusing on identity cases of the nonlinear term, namely from $\int_\Omega F(|x|, u) = \int_\Omega F(|x|, u^*)$. For results in this direction, under strict monotonicity assumptions of $f$ such as

$$f(|x|, s) > f(|y|, s), \quad \text{for all } s \in \mathbb{R}^+ \text{ and } x, y \in \Omega \text{ with } |x| < |y|,$$

we refer the reader to [17, Section 6] (see also [5]).

On the basis of the above discussion, the aim of the paper is to focus the attention of the quasi-linear term in the functional $\mathcal{E}$ (we believe this is somehow more natural since the strict convexity of $j(s, \cdot)$ is a very common requirement) and show that, for minima, identity (1.12) implies, as desired, that $u$ corresponds to a translate of $u^*$.

**Remark 1.2.** In light of conditions (1.7) and (1.9), we also have

$$|F(|x|, s)| \le a(|x|)|s| + C|s|^p + C|s|^{p^*}, \quad \text{for every } s \in \mathbb{R} \text{ and } x \in \Omega, \tag{1.13}$$

$$|G(s)| \le C|s|^p + C|s|^{p^*}, \quad \text{for every } s \in \mathbb{R}. \tag{1.14}$$

As a possible variant of the growth condition (1.7) one could as well assume that $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ with $f(|x|, s) \geq 0$ for every $s \in \mathbb{R}^+$ and $x \in \Omega$ and

$$|f(|x|, s)| \leq C|s|^{p-1} + C|s|^{p^*-1}, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and } x \in \Omega, \tag{1.15}$$

yielding, in turn,

$$|F(|x|, s)| \leq C|s|^p + C|s|^{p^*}, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and } x \in \Omega. \tag{1.16}$$

In this case the symmetrization inequality (1.11) for $F$ holds as well (use [19, Corollary 5.2] in place of [19, Corollary 5.5] in the case $\Omega = \mathbb{R}^N$ and [19, Theorem 6.3] in place of [19, Theorem 6.4] in the case $\Omega = B_R(0)$). It is often the case that $F$ satisfies growth conditions which are more restrictive than (1.13) or (1.16) in order to have $m > -\infty$. For instance, assume that $G(s) = |s|^p$, $j(s, t) = |t|^p$ and $F(|x|, s) = |s|^\sigma$. Then, as a simple scaling argument shows, to guarantee that the minimization problem is well defined it is necessary to assume that

$$p < \sigma < p + p^2/N.$$

For $p = 2$, the value $2 + 4/N$ is precisely the well-known threshold for orbital stability of ground states solutions for the nonlinear Schrödinger equation.

**Remark 1.3.** If $\Omega = \mathbb{R}^N$ and we assume, for instance, that $G(s_0) > 0$ at some point $s_0 > 0$, then one can write down a function $\psi_{s_0} \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_c(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} G(\psi_{s_0}) = 1$ (see [3, p.325]). Moreover notice that, in light of the growth condition (1.4), it holds

$$\int_{\mathbb{R}^N} j(\psi_{s_0}, |D\psi_{s_0}|) \leq \alpha(\|\psi_{s_0}\|_{L^\infty}) \int_{\mathbb{R}^N} |D\psi_{s_0}|^p < +\infty.$$

Hence $\psi_{s_0} \in \mathcal{C}$ as well as $\mathcal{E}(\psi_{s_0}) < +\infty$, which guarantees $m < +\infty$.

It would be interesting to extend Theorem 1.1, in a suitable sense, to allow the case of possibly sign-changing solutions, systems and multiple constraints. The main ingredients of the argument are the facts that the functional decreases under both polarization and symmetrization, while the constraint remains invariant to them. This can be achieved for some classes of vectorial problems putting cooperativity conditions on the nonlinear term $F$ and considering $G$ and $j$ involving a combinations of functions depending only on one single variable, in order to exploit Cavalieri's principle and Pólya–Szegő type inequalities. Notice also that the almost everywhere convergence of the gradients due to Dal Maso and Murat [9] is valid for systems of PDEs as well. We leave this issues to further future investigations.

## 2 Preliminary facts

In the section we include some preparatory results.

### 2.1 Polarization and Schwarz symmetrization

For the notions of this section, we refer, for instance, to [6]. A subset $H$ of $\mathbb{R}^N$ is called a polarizer if it is a closed affine half-space of $\mathbb{R}^N$. Given $x \in \mathbb{R}^N$ and a polarizer $H$, the reflection of $x$ with respect to the boundary of $H$ is denoted by $x_H$. The polarization of a function $u : \mathbb{R}^N \to \mathbb{R}^+$ by a polarizer $H$ is the function $u^H : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$u^H(x) := \begin{cases} \max\{u(x), u(x_H)\}, & \text{if } x \in H \\ \min\{u(x), u(x_H)\}, & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases} \tag{2.1}$$

The polarization $\Omega^H \subset \mathbb{R}^N$ of a set $\Omega \subset \mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{\Omega^H} = (\chi_\Omega)^H$, where $\chi$ denotes the characteristic function. The polarization $u^H$ of a nonnegative function $u$ defined on $\Omega \subset \mathbb{R}^N$ is the restriction to $\Omega^H$ of the polarization of the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ of $u$ by zero outside $\Omega$. The Schwarz symmetrization of a set $\Omega \subset \mathbb{R}^N$ is the unique open ball centered at the origin $\Omega^*$ such that $\mathcal{L}^N(\Omega^*) = \mathcal{L}^N(\Omega)$, being $\mathcal{L}^N$ the $N$-dimensional outer Lebesgue measure. If the measure of $\Omega$ is zero we set $\Omega^* = \emptyset$, while if the measure of $\Omega$ is not finite we put $\Omega^* = \mathbb{R}^N$. A measurable function $u$ is admissible for the Schwarz symmetrization if it is nonnegative and, for every $\varepsilon > 0$, the Lebesgue measure of $\{u > \varepsilon\}$ is finite. The Schwarz symmetrization of an admissible function $u : \Omega \to \mathbb{R}^+$ is the unique function $u^* : \Omega^* \to \mathbb{R}^+$ such that, for all $t \in \mathbb{R}$, it holds $\{u^* > t\} = \{u > t\}^*$. Considering the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ of $u$ by zero outside $\Omega$, $u^* = (\tilde{u})^*|_{\Omega^*}$ and $(\tilde{u})^*|_{\mathbb{R}^N \setminus \Omega^*} = 0$.

We shall denote by $\mathcal{H}_*$ the set of all half-spaces corresponding to $(n-1)$-dimensional Euclidean hyperplanes, containing the origin in the interior. As known, for a domain $\Omega$, it holds $\Omega^* = \Omega$ if and only if $\Omega^H = \Omega$, for all $H \in \mathcal{H}_*$ (cf. [6, Lemma 6.3]). We now recall a very useful convergence result (cf. e.g. [32]).

**Proposition 2.1.** *Assume that $\Omega$ is either $\mathbb{R}^N$ or a ball $B_R(0) \subset \mathbb{R}^N$. There exists a sequence of polarizers $(H_m) \subset \mathcal{H}_*$ such that, for any $1 \le p < \infty$ and all $u \in L^p(\Omega)$, the sequence $u_m = u^{H_1 \cdots H_m}$ converges to $u^*$ strongly in $L^p(\Omega)$, namely $\|u_m - u^*\|_{L^p(\Omega)} \to 0$ as $m \to \infty$.*

The next identities hold by direct computation. For the inequality, see [31, Proposition 2.3].

**Proposition 2.2.** *Let $\Omega$ be either $\mathbb{R}^N$ or a ball $B_R(0)$, $u \in W_0^{1,p}(\Omega, \mathbb{R}^+)$ and $H \in \mathcal{H}_*$. Then $u^H \in W_0^{1,p}(\Omega, \mathbb{R}^+)$ and, if $j(u, |Du|) \in L^1(\Omega)$, then $j(u^H, |Du^H|) \in L^1(\Omega)$ and*

$$\int_\Omega j(u, |Du|) = \int_\Omega j(u^H, |Du^H|). \tag{2.2}$$

*In particular,*

$$\int_\Omega |Du|^p = \int_\Omega |Du^H|^p. \tag{2.3}$$

*Furthermore, $F(|x|, u), F(|x|, u^H), G(u), G(u^H) \in L^1(\Omega)$ and*

$$\int_\Omega F(|x|, u^H) \geq \int_\Omega F(|x|, u), \quad \int_\Omega G(u^H) = \int_\Omega G(u),$$

*provided that conditions* (1.7)*,* (1.8) *and* (1.9) *hold.*

We also have the following

**Proposition 2.3.** *Let $\Omega$ be either $\mathbb{R}^N$ or a ball $B_R(0)$ and let $u \in W_0^{1,p}(\Omega, \mathbb{R}^+)$. Then $u^* \in W_0^{1,p}(\Omega, \mathbb{R}^+)$ and, if $j(u, |Du|) \in L^1(\Omega)$, then $j(u^*, |Du^*|) \in L^1(\Omega)$ and*

$$\int_\Omega j(u^*, |Du^*|) \leq \int_\Omega j(u, |Du|). \tag{2.4}$$

*In particular,*

$$\int_\Omega |Du^*|^p \leq \int_\Omega |Du|^p.$$

*Furthermore, $F(|x|, u), F(|x|, u^*), G(u), G(u^*) \in L^1(\Omega)$ and*

$$\int_\Omega F(|x|, u^*) \geq \int_\Omega F(|x|, u), \quad \int_\Omega G(u^*) = \int_\Omega G(u), \tag{2.5}$$

*provided that conditions* (1.7)*,* (1.8) *and* (1.9) *hold.*

Concerning the symmetrization inequality for $F$, it follows by [19, Corollary 5.5] for the case $\Omega = \mathbb{R}^N$ and [19, Theorem 6.4] for the case $\Omega = B_R(0)$. Concerning Cavalieri's principle for $G$, it follows from [19, Theorem 4.4] in the case $\Omega = \mathbb{R}^N$ and from [19, Theorem 6.2] in the case $\Omega = B_R(0)$. Concerning (2.4), it follows by weakly lower semicontinuity by combining (2.2) with the result of approximation (in the $L^p(\Omega)$ norm) of symmetrizations via polarization ( [32]), arguing as in the proof of [6, Theorem 8.2]

The next result is a formulation of the celebrated *Brothers–Ziemer theorem* [7], taken from [12] for the case of $\Omega$ bounded and from [13] for the case $\Omega = \mathbb{R}^N$.

**Proposition 2.4.** *Assume that $\Omega$ is an open, bounded subset of $\mathbb{R}^N$ and let $u \in W_0^{1,p}(\Omega)$ be a nonnegative function, $1 < p < \infty$, such that*

$$\mathcal{L}^N(C^*) = 0, \quad C^* := \{x \in \Omega^* : |Du^*(x)| = 0\} \cap (u^*)^{-1}(0, \operatorname{ess\,sup} u).$$

*Then, if*

$$\|Du^*\|_{L^p(\Omega^*)} = \|Du\|_{L^p(\Omega)},$$

*the domain $\Omega$ is equivalent to a ball and $u = u^*$ a.e. in $\Omega$, up to a translation. Moreover, the same conclusion holds for $\Omega = \mathbb{R}^N$.*

## 2.2 Solutions to the Euler–Lagrange equation

For any $u \in W_0^{1,p}(\Omega)$, define

$$V_u = \{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : u \in L^\infty(\{x \in \Omega : v(x) \neq 0\})\}. \tag{2.6}$$

The vector space $V_u$ was firstly introduced by Degiovanni and Zani in [10] in the case $p = 2$. In [10] it is also proved that $V_u$ with $p = 2$ is dense in $W_0^{1,2}(\Omega)$. This fact extends with the same proof to the general case of any $p \neq 2$. Let $J : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be the functional

$$J(u) = \int_\Omega j(u, |Du|).$$

The following fact can be easily checked. It shows that $V_u$ is a good test space to differentiate non-smooth functionals of calculus of variations satisfying suitable growth conditions.

**Proposition 2.5.** *Assume conditions* (1.4), (1.5) *and* (1.6). *Then, for every $u \in W_0^{1,p}(\Omega)$ with $J(u) < +\infty$ and every $v \in V_u$ we have*

$$j_s(u, |Du|)v \in L^1(\Omega), \quad j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv \in L^1(\Omega),$$

*with the agreement that $j_t(u, |Du|)\frac{Du}{|Du|} = 0$ when $|Du| = 0$ (in view of* (1.6)*). Moreover, the function $\{t \mapsto J(u + tv)\}$ is of class $C^1$ and*

$$J'(u)(v) = \int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|)v.$$

We recall Definitions 4.3 and 5.5 from [8], respectively, adapted to our concrete framework.

**Definition 2.6.** Let $u \in W_0^{1,p}(\Omega)$ with $J(u) < +\infty$. For every $v \in W_0^{1,p}(\Omega)$ and $\varepsilon > 0$ we define $J_\varepsilon^0(u; v)$ to be the infimum of the $r$'s in $\mathbb{R}$ such that there exist $\delta > 0$ and a continuous function

$$\mathcal{V} : B_\delta(u, J(u)) \cap \mathrm{epi}(J) \times ]0, \delta] \to B_\varepsilon(v),$$

which satisfies

$$J(\xi + t\mathcal{V}((\xi, \mu), t)) \leq \mu + rt,$$

whenever $(\xi, \mu) \in B_\delta(u, J(u)) \cap \mathrm{epi}(J)$ and $t \in ]0, \delta]$. Finally, we set

$$J^0(u; v) := \sup_{\varepsilon>0} J_\varepsilon^0(u; v).$$

**Definition 2.7.** Let $u \in W_0^{1,p}(\Omega)$ with $J(u) < +\infty$. For every $v \in W_0^{1,p}(\Omega)$ and $\varepsilon > 0$ we define $\bar{J}_\varepsilon^0(u; v)$ to be the infimum of the $r$'s in $\mathbb{R}$ such that there exist $\delta > 0$ and a continuous function

$$\mathcal{H} : B_\delta(u, J(u)) \cap \mathrm{epi}(J) \times [0, \delta] \to W_0^{1,p}(\Omega),$$

which satisfies $\mathcal{H}((\xi, \mu), 0) = \xi$,

$$\left\| \frac{\mathcal{H}((\xi, \mu), t_1) - \mathcal{H}((\xi, \mu), t_2)}{t_1 - t_2} - v \right\|_{W^{1,p}(\Omega)} < \varepsilon, \tag{2.7}$$

and $J(\mathcal{H}((\xi, \mu), t)) \leq \mu + rt$, whenever $(\xi, \mu) \in B_\delta(u, J(u)) \cap \mathrm{epi}(J)$ and $t, t_1, t_2 \in [0, \delta]$ with $t_1 \neq t_2$. Finally, we set

$$\bar{J}^0(u; v) := \sup_{\varepsilon>0} \bar{J}_\varepsilon^0(u; v).$$

As remarked in [8, cf. p.1037] it always holds $J^0(u; v) \leq \bar{J}^0(u; v)$. Recalling that $\partial J(u)$ is the subdifferential introduced in [8, Definition 4.1], we have the following

**Lemma 2.8.** *Assume conditions* (1.4), (1.5) *and* (1.6). *Let $u \in W_0^{1,p}(\Omega)$ with $J(u) < +\infty$. Then, the following facts hold:*

(i) *for every $v \in V_u$, we have*

$$J^0(u; v) \leq \bar{J}^0(u; v) \leq \int_\Omega j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|) v.$$

(ii) *if* $\partial J(u) \neq \emptyset$, *then* $\partial J(u) = \{\alpha\}$ *with* $\alpha \in W^{-1,p'}(\Omega)$ *and*

$$\int_\Omega j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|) v = \langle \alpha, v \rangle,$$

*for all* $v \in V_u$.

*Proof.* Let $\eta > 0$ with $J(u) < \eta$. Moreover, let $v \in V_u$ and $\varepsilon > 0$. Take now $r \in \mathbb{R}$ with

$$\int_\Omega j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|) v < r. \tag{2.8}$$

Let $\mathcal{T}$ be a $C^\infty(\mathbb{R})$ function such that

$$\mathcal{T}(s) = 1 \text{ on } [-1, 1], \quad \mathcal{T}(s) = 0 \text{ outside } [-2, 2], \quad |\mathcal{T}'(s)| \leq 2 \text{ on } \mathbb{R}. \tag{2.9}$$

Then, there exists $k_0 \geq 1$ such that

$$\left\| \mathcal{T}\left(\frac{u}{k_0}\right) v - v \right\|_{W^{1,p}(\Omega)} < \varepsilon, \tag{2.10}$$

and

$$\int_\Omega j_t(u, |Du|) \frac{Du}{|Du|} \cdot D\left(\mathcal{T}\left(\frac{u}{k_0}\right) v\right) + \int_\Omega j_s(u, |Du|) \mathcal{T}(\frac{u}{k_0}) v < r. \tag{2.11}$$

In fact, setting $v_k = \mathcal{T}(u/k)v$, we have $v_k \in V_u$ for every $k \geq 1$ and $v_k$ converges to $v$ in $W_0^{1,p}(\Omega)$, yielding inequality (2.10), for $k$ large enough. By Proposition 2.5, we can consider $J'(u)(v_k)$ for all $k \geq 1$ and, as $k$ goes to infinity, for a.e. $x \in \Omega$, we have

$$j_s(u(x), |Du(x)|) v_k(x) \to j_s(u(x), |Du(x)|) v(x),$$
$$j_t(u(x), |Du(x)|) \frac{Du(x)}{|Du(x)|} \cdot Dv_k(x) \to j_t(u(x), |Du(x)|) \frac{Du(x)}{|Du(x)|} \cdot Dv(x),$$

as well as

$$|j_s(u, |Du|) v_k| \leq |j_s(u, |Du|)|\,|v|,$$
$$\left| j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv_k \right| \leq |j_t(u, |Du|)|\,|Dv| + 2|v|\,|j_t(u, |Du|)|\,|Du|.$$

Since $v \in V_u$ and by the growth estimates (1.5)–(1.6), by dominated convergence we get

$$\lim_{k\to\infty}\int_\Omega j_s(u,|Du|)v_k = \int_\Omega j_s(u,|Du|)v,$$
$$\lim_{k\to\infty}\int_\Omega j_t(u,|Du|)\frac{Du}{|Du|}\cdot Dv_k = \int_\Omega j_t(u,|Du|)\frac{Du}{|Du|}\cdot Dv,$$

which, together with (2.8), yields (2.11). Let us now prove that there exists $\delta_1 > 0$ such that

$$\left\| \mathcal{T}\left(\frac{z}{k_0}\right)v - v\right\|_{W^{1,p}(\Omega)} < \varepsilon, \tag{2.12}$$

as well as

$$\int_\Omega j_t\left(z+\vartheta\mathcal{T}\left(\frac{z}{k_0}\right)v, \left|Dz+\vartheta D\left(\mathcal{T}(\frac{z}{k_0})v\right)\right|\right)\frac{Dz+\vartheta D(\mathcal{T}(\frac{z}{k_0})v)}{|Dz+\vartheta D(\mathcal{T}(\frac{z}{k_0})v)|} \cdot D\left(\mathcal{T}\left(\frac{z}{k_0}\right)v\right) \tag{2.13}$$
$$+\int_\Omega j_s\left(z+\vartheta\mathcal{T}\left(\frac{z}{k_0}\right)v, \left|Dz+\vartheta D\left(\mathcal{T}(\frac{z}{k_0})v\right)\right|\right)\mathcal{T}\left(\frac{z}{k_0}\right)v < r,$$

for all $z \in B(u,\delta_1)\cap J^\eta$ and $\vartheta \in [0,\delta_1)$. Indeed, take $u_n \in J^\eta$ such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$, $\vartheta_n \to 0$ as $n\to\infty$ and consider $v_n = \mathcal{T}(u_n/k_0)v \in V_{u_n}$. It follows that $v_n$ converges to $\mathcal{T}(u/k_0)v$ strongly in $W_0^{1,p}(\Omega)$, so that (2.12) follows by (2.10). Now, for a.e. $x\in\Omega$, we have

$$j_s(u_n(x)+\vartheta_n v_n(x), |Du_n(x)+\vartheta_n Dv_n(x)|)v_n(x)$$
$$\to j_s(u(x),|Du(x)|)\mathcal{T}\left(\frac{u(x)}{k_0}\right)v(x),$$

and

$$j_t(u_n(x)+\vartheta_n v_n(x), |Du_n(x)+\vartheta_n Dv_n(x)|)\frac{Du_n(x)+\vartheta_n Dv_n(x)}{|Du_n(x)+\vartheta_n Dv_n(x)|}\cdot Dv_n(x)$$
$$\to j_t(u(x),|Du(x)|)\frac{Du(x)}{|Du(x)|}\cdot D\left(\mathcal{T}\left(\frac{u}{k_0}\right)v\right)(x).$$

Moreover, we have

$$
\begin{aligned}
&|j_s(u_n+\vartheta_n v_n,|Du_n+\vartheta_n Dv_n|)v_n| \\
&\qquad\qquad \le 2^{p-1}\beta(2k_0+\|v\|_{L^\infty(\Omega)})\|v\|_{L^\infty(\Omega)}(|Du_n|^p+|Dv_n|^p), \\
&\left| j_t(u_n+\vartheta_n v_n,|Du_n+\vartheta_n Dv_n|)\frac{Du_n+\vartheta_n Dv_n}{|Du_n+\vartheta_n Dv_n|}\cdot Dv_n\right| \\
&\qquad\qquad \le 2^{p-1}\gamma(2k_0+\|v\|_{L^\infty(\Omega)})\big(|Du_n|^{p-1}|Dv_n|+|Dv_n|^p\big).
\end{aligned}
$$

Then, by dominated convergence we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\Omega j_s(u_n+\vartheta_n v_n,|Du_n+\vartheta_n Dv_n|)v_n \\
&\quad=\int_\Omega j_s(u,|Du|)\mathcal{T}\le\left(\frac{u}{k_0}\le\right)v, \\
&\lim_{n\to\infty}\int_\Omega j_t(u_n+\vartheta_n v_n,|Du_n+\vartheta_n Dv_n|)\frac{Du_n+\vartheta_n Dv_n}{|Du_n+\vartheta_n Dv_n|}\cdot Dv_n \\
&\quad=\int_\Omega j_t(u,|Du|)\frac{Du}{|Du|}\cdot D\left(\mathcal{T}\left(\frac{u}{k_0}\right)v\right),
\end{aligned}
$$

which, in light of (2.11), proves (2.13). Then, taking into account that $\{t\mapsto J(z+t\mathcal{T}(\frac{z}{k_0})v)\}$ is of class $C^1$, Lagrange theorem and (2.13) yield, for some $\vartheta\in[0,t]$,

$$
\begin{aligned}
&J\left(z+t\mathcal{T}\left(\frac{z}{k_0}\right)v\right)-J(z)=tJ'\left(z+\vartheta\mathcal{T}\left(\frac{z}{k_0}\right)v\right)\left(\mathcal{T}\left(\frac{z}{k_0}\right)v\right) \\
&=t\int_\Omega\left[j_t\left(z+\vartheta\mathcal{T}\left(\frac{z}{k_0}\right)v,\left|Dz+\vartheta D\left(\mathcal{T}\left(\frac{z}{k_0}\right)v\right)\right|\right)\frac{Dz+\vartheta D(\mathcal{T}(\frac{z}{k_0})v)}{|Dz+\vartheta D(\mathcal{T}(\frac{z}{k_0})v)|}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot D\left(\mathcal{T}\left(\frac{z}{k_0}\right)v\right) \\
&\quad\left.+\,j_s\left(z+\vartheta\mathcal{T}\left(\frac{z}{k_0}\right)v,\left|Dz+\vartheta D\left(\mathcal{T}\left(\frac{z}{k_0}\right)v\right)\right|\right)\mathcal{T}\left(\frac{z}{k_0}\right)v\right]dx\le rt,
\end{aligned}
$$

for all $z\in B(u,\delta_1)\cap J^\eta$ and $t\in[0,\delta_1)$. Let now $\delta\in(0,\delta_1]$ with $J(u)+\delta<\eta$, and define the continuous function $\mathcal{H}:B_\delta(u,J(u))\cap\operatorname{epi}(J)\times[0,\delta]\to W_0^{1,p}(\Omega)$ by setting

$$
\mathcal{H}((z,\mu),t)=z+t\mathcal{T}\left(\frac{z}{k_0}\right)v.
$$

Then, by direct computation, condition (2.7) in Definition 2.7 is satisfied by (2.12). Notice that, for all $((z,\mu),t) \in B_\delta(u, J(u)) \cap \mathrm{epi}(J) \times [0,\delta]$, we have $z \in B(u,\delta_1) \cap J^\eta$ and $t \in [0,\delta_1)$. Hence, by the above inequality, we have

$$J(\mathcal{H}((z,\mu),t)) \le J(z) + rt \le \mu + rt.$$

whenever $(z,\mu) \in B_\delta(u, J(u)) \cap \mathrm{epi}(J)$ and $t \in [0,\delta]$. Then, according to Definition 2.7, we can conclude that $\bar{J}^0_\varepsilon(u;v) \le r$. By the arbitrariness of $r$, it follows that

$$\bar{J}^0_\varepsilon(u;v) \le \int_\Omega j_t(u,|Du|)\frac{Du}{|Du|}\cdot Dv + \int_\Omega j_s(u,|Du|)v.$$

Hence, by the arbitrariness of $\varepsilon$, we get

$$\bar{J}^0(u;v) \le \int_\Omega j_t(u,|Du|)\frac{Du}{|Du|}\cdot Dv + \int_\Omega j_s(u,|Du|)v, \tag{2.14}$$

for all $v \in V_u$, concluding the proof of assertion (i). Concerning (ii), if $\alpha \in \partial J(u) \subset W^{-1,p'}(\Omega)$, by (i) it follows (recall [8, Corollary 4.7(i)] for the first inequality below)

$$\langle \alpha, v\rangle \le J^0(u;v) \le \int_\Omega j_t(u,|Du|)\frac{Du}{|Du|}\cdot Dv + \int_\Omega j_s(u,|Du|)v,$$

for all $v \in V_u$. Since we can exchange $v$ with $-v$ we get

$$\langle \alpha, v\rangle = \int_\Omega j_t(u,|Du|)\frac{Du}{|Du|}\cdot Dv + \int_\Omega j_s(u,|Du|)v,$$

for all $v \in V_u$. By density of $V_u$ in $W^{1,p}_0(\Omega)$, $\partial J(u) = \{\alpha\}$. This concludes the proof. □

Finally, we have the following

**Proposition 2.9.** *Assume* (1.4), (1.5)*,* (1.6)*,* (1.7) *and* (1.9)*. Let* $\mathcal{C}$ *be the set introduced in* (1.1) *and let* $\mathcal{E} : W^{1,p}_0(\Omega) \to \mathbb{R} \cup \{+\infty\}$ *be the functional*

$$\mathcal{E}(u) = \int_\Omega j(u,|Du|) - \int_\Omega F(|x|,u).$$

*Then the functional* $\mathcal{E}^* : W^{1,p}_0(\Omega) \to \mathbb{R} \cup \{+\infty\}$ *defined by*

$$\mathcal{E}^*(u) = \begin{cases} \mathcal{E}(u) & \text{for } u \in \mathcal{C}, \\ +\infty & \text{for } u \notin \mathcal{C}, \end{cases}$$

*is lower semi-continuous on* $W_0^{1,p}(\Omega)$*. Moreover, for every solution* $u \in \mathcal{C}$ *to problem* (1.2) *there exists a Lagrange multiplier* $\lambda \in \mathbb{R}$ *such that*

$$\int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot D\varphi + \int_\Omega j_s(u, |Du|)\varphi - \int_\Omega f(|x|, u)\varphi = \lambda \int_\Omega g(u)\varphi,$$

*for all* $\varphi \in V_u$.

*Proof.* It is readily seen that $\mathcal{E}^*$ is lower semi-continuous, by conditions (1.13) and (1.14). Let $u \in \mathcal{C}$ be any solution to problem (1.2) (it is $J(u) < +\infty$, since $\mathcal{E}(u) = m < +\infty$). Notice that $\mathcal{E}^* = \mathcal{E} + \mathrm{I}_{\mathcal{C}}$, being $\mathrm{I}_{\mathcal{C}}$ the indicator function of $\mathcal{C}$, $\mathrm{I}_{\mathcal{C}}(v) = 0$ if $v \in \mathcal{C}$ and $\mathrm{I}_{\mathcal{C}}(v) = +\infty$ if $v \notin \mathcal{C}$. If $|d\mathcal{E}^*|(u)$ denotes the weak slope of $\mathcal{E}^*$ (see [8, Definition 2.1]), it follows $|d\mathcal{E}^*|(u) = 0$, and then $0 \in \partial\mathcal{E}^*(u)$, by [8, Theorem 4.13(iii)]. Setting

$$V(u) = -\int_\Omega F(|x|, u)dx, \quad W(u) = \int_\Omega G(u)dx,$$

$V, W$ are $C^1$ functionals, in light of (1.7) and (1.9), and $\partial V(u) = \{-f(|x|, u)\}$, $\partial W(u) = \{g(u)\}$. Moreover, by (1.10), we can find $\hat{v} \in W_0^{1,p}(\Omega)$ such that $\int_\Omega g(u)\hat{v} > 0$ and, by density of $V_u$ in $W_0^{1,p}(\Omega)$, $v_+ \in V_u$ with $\int_\Omega g(u)v_+ > 0$. Taking into account Proposition 2.5 and conclusion (i) of Lemma 2.8, $\bar{J}^0(u; v_+) < +\infty$ and $\bar{J}^0(u; -v_+) < +\infty$. Therefore, we are allowed to apply [8, Corollary 5.9(ii)], yielding

$$0 \in \partial\mathcal{E}(u) + \mathbb{R}\partial W(u) = \partial J(u) + \partial V(u) + \mathbb{R}\partial W(u),$$

where the equality is justified by [8, Corollary 5.3(ii)], since $V$ is a $C^1$ functional. Finally, since $\partial J(u) \neq \emptyset$, assertion (ii) of Lemma 2.8 allows to conclude the proof. □

## 3 Proof of Theorem 1.1

Let $u \in W_0^{1,p}(\Omega)$ be a given nonnegative solution to the minimum problem (1.2), namely

$$j(u, |Du|) \in L^1(\Omega), \quad \int_\Omega G(u) = 1, \quad m = \int_\Omega j(u, |Du|) - \int_\Omega F(|x|, u).$$

We shall divide the proof into four steps.

**Step I (existence of approximating minimizers).** In light of Proposition 2.1, we can find a sequence $(u_n) \subset W_0^{1,p}(\Omega)$ of polarizations of $u$, namely $u_n = u^{H_1 \cdots H_n}$, such that $u_n \to u^*$ strongly in $L^p(\Omega)$, as $n \to \infty$. Furthermore, we learn from (2.3) of Proposition 2.2 that

$$\|Du_n\|_{L^p(\Omega)} = \|Du\|_{L^p(\Omega)}, \quad \text{for all } n \geq 1. \tag{3.1}$$

In turn, the sequence $(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ and, up to a subsequence, it converges weakly to $u^*$ in $W_0^{1,p}(\Omega)$ (since $u_n \to u^*$ in $L^p(\Omega)$). Notice also that, again by virtue of Proposition 2.2, it follows that $j(u_n, |Du_n|) \in L^1(\Omega)$ for all $n \geq 1$ and

$$\int_\Omega j(u_n, |Du_n|) = \int_\Omega j(u, |Du|), \quad \text{for all } n \geq 1, \tag{3.2}$$

as well as

$$\int_\Omega F(|x|, u_n) \geq \int_\Omega F(|x|, u), \quad \int_\Omega G(u_n) = \int_\Omega G(u) = 1, \quad \text{for all } n \geq 1.$$

In particular $(u_n)$ is a sequence of minimizers for problem (1.2), since $(u_n) \subset \mathcal{C}$ and

$$m \leq \int_\Omega j(u_n, |Du_n|) - \int_\Omega F(|x|, u_n) \leq \int_\Omega j(u, |Du|) - \int_\Omega F(|x|, u) = m.$$

Furthermore, in light of Proposition 2.3, we have $\int_\Omega G(u^*) = \int_\Omega G(u) = 1$ and

$$\int_\Omega F(|x|, u^*) \geq \int_\Omega F(|x|, u), \quad \int_\Omega j(u^*, |Du^*|) \leq \int_\Omega j(u, |Du|),$$

so that we obtain

$$m \leq \int_\Omega j(u^*, |Du^*|) - \int_\Omega F(|x|, u^*) \leq \int_\Omega j(u, |Du|) - \int_\Omega F(|x|, u) = m,$$

yielding that $u^*$ is a minimizer for (1.2) too, and

$$\int_\Omega j(u^*, |Du^*|) = \int_\Omega j(u, |Du|).$$

In conclusion, by (3.2), we get

$$\int_\Omega j(u_n, |Du_n|) = \int_\Omega j(u^*, |Du^*|), \quad \text{for all } n \geq 1. \tag{3.3}$$

By Proposition 2.9, there exists a sequence $(\lambda_n) \subset \mathbb{R}$ of Lagrange multipliers such that

$$\int_\Omega j_t(u_n, |Du_n|)\frac{Du_n}{|Du_n|} \cdot D\varphi + \int_\Omega j_s(u_n, |Du_n|)\varphi - \int_\Omega f(|x|, u_n)\varphi = \lambda_n \int_\Omega g(u_n)\varphi, \tag{3.4}$$

for all $n \geq 1$ and any $\varphi \in V_{u_n}$.

**Step II (boundedness of $\boldsymbol{\lambda_n}$).** We claim that $(\lambda_n)$ is bounded in $\mathbb{R}$. To prove this, observe first that there exist $\hat{v} \in C_c^\infty(\Omega)$ and $\hat{h} \geq 1$ such that

$$\lim_n \int_\Omega g(u_n)\mathcal{T}\left(\frac{u_n}{\hat{h}}\right)\hat{v} \neq 0, \tag{3.5}$$

being $\mathcal{T}$ the cut-off function defined in (2.9). If this was not the case, for all $v \in C_c^\infty(\Omega)$ and any $h \geq 1$, we would find

$$\int_\Omega g(u^*)\mathcal{T}\left(\frac{u^*}{h}\right)v = \lim_n \int_\Omega g(u_n)\mathcal{T}\left(\frac{u_n}{h}\right)v = 0,$$

by dominated convergence. By the arbitrariness of $h \geq 1$ and dominated convergence, we get

$$\int_\Omega g(u^*)v = 0, \quad \text{for all } v \in C_c^\infty(\Omega),$$

so that $g(u^*) = 0$ a.e. in $\Omega$. This is a contradiction, as $\int_\Omega G(u^*) = \int_\Omega G(u) = 1$ implies that $u^* \not\equiv 0$ which, by assumption (1.10), yields $g(u^*) \not\equiv 0$. Observe now that, for every $v \in C_c^\infty(\Omega)$ and any $h \geq 1$, the function $\mathcal{T}(\frac{u_n}{h})v$ belongs to $V_{u_n}$ and thus it is an admissible test function for (3.4). Therefore, if $\hat{v} \in C_c^\infty(\Omega)$ and $\hat{h} \geq 1$ are as in formula (3.5), inserting $\varphi = \mathcal{T}(\frac{u_n}{\hat{h}})\hat{v}$ into (3.4), we reach the identity

$$\lambda_n \int_\Omega g(u_n)\mathcal{T}\left(\frac{u_n}{\hat{h}}\right)\hat{v} = \sum_{i=1}^3 I_i^n, \tag{3.6}$$

where, denoted by $K$ the support of $\hat{v}$, we have set

$$I_1^n := \int_K \mathcal{T}\left(\frac{u_n}{\hat{h}}\right) j_t(u_n, |Du_n|)\frac{Du_n}{|Du_n|} \cdot D\hat{v},$$

$$I_2^n := \int_K \left[\mathcal{T}\left(\frac{u_n}{\hat{h}}\right) j_s(u_n, |Du_n|) + \mathcal{T}'\left(\frac{u_n}{\hat{h}}\right) j_t(u_n, |Du_n|)\frac{|Du_n|}{\hat{h}}\right]\hat{v},$$

$$I_3^n := -\int_K f(|x|, u_n)\mathcal{T}\left(\frac{u_n}{\hat{h}}\right)\hat{v}.$$

In turn, taking into account the growths (1.5), (1.6) and (1.7), it follows

$$|I_1^n| \le C \int_K |Du_n|^{p-1} |D\hat{v}| \le C \Big( \int_\Omega |Du_n|^p \Big)^{\frac{p-1}{p}} \le C,$$

$$|I_2^n| \le C \int_K |Du_n|^p |\hat{v}| \le C \int_\Omega |Du_n|^p \le C,$$

$$|I_3^n| \le \int_K (a(|x|) + C)|\hat{v}| \le C,$$

for some constant $C = C(\hat{h})$, changing from one line to the next and independent of $n$. Then the claim follows by combining (3.5) and (3.6) and $(\lambda_n)$ admits a convergent subsequence.

**Step III (pointwise convergence).** In this step we prove that, up to a subsequence,

$$Du_n(x) \to Du^*(x), \quad \text{for a.e. } x \in \Omega. \tag{3.7}$$

Let $\Omega_0$ be a fixed bounded subdomain of $\Omega$ (let $\Omega_0 = \Omega$ if $\Omega$ is a ball). We already know that

$$u_n \rightharpoonup u^* \quad \text{weakly in } W^{1,p}(\Omega_0). \tag{3.8}$$

As we have already noticed, for all $h \ge 1$ and $v \in C_c^\infty(\Omega_0)$, the function $\mathcal{T}(\frac{u_n}{h})v$ belongs to the space $V_{u_n}$. Therefore, inserting it into (3.4), we reach

$$\begin{aligned} &\int_{\Omega_0} \mathcal{T}\Big(\frac{u_n}{h}\Big) j_t(u_n, |Du_n|) \frac{Du_n}{|Du_n|} \cdot Dv \\ &\quad = - \int_{\Omega_0} \Big[ \mathcal{T}\Big(\frac{u_n}{h}\Big) j_s(u_n, |Du_n|) + \mathcal{T}'\Big(\frac{u_n}{h}\Big) j_t(u_n, |Du_n|) \frac{|Du_n|}{h} \Big] v \\ &\qquad + \int_{\Omega_0} f(|x|, u_n) \mathcal{T}\Big(\frac{u_n}{h}\Big) v + \lambda_n \int_{\Omega_0} g(u_n) \mathcal{T}\Big(\frac{u_n}{h}\Big) v, \end{aligned}$$

for all $v \in C_c^\infty(\Omega_0)$. This equality can be read as

$$\int_{\Omega_0} b_n(x, Du_n) \cdot Dv = \langle \Phi_n, v \rangle + \langle \mu_n, v \rangle, \quad \forall v \in C_c^\infty(\Omega_0),$$

where we have set

$$
\begin{aligned}
b_n(x,\xi) &:= \mathcal{T}\left(\frac{u_n(x)}{h}\right) j_t(u_n(x),|\xi|)\frac{\xi}{|\xi|}, \quad \text{for a.e. } x\in\Omega_0 \text{ and all } \xi\in\mathbb{R}^N,\\
\langle \mu_n, v\rangle &:= -\int_{\Omega_0}\left[\mathcal{T}'\left(\frac{u_n}{h}\right) j_t(u_n,|Du_n|)\frac{|Du_n|}{h} + \mathcal{T}\left(\frac{u_n}{h}\right) j_s(u_n,|Du_n|)\right] v,\\
\langle \Phi_n, v\rangle &:= \int_{\Omega_0} f(|x|,u_n)\mathcal{T}\left(\frac{u_n}{h}\right) v + \lambda_n \int_{\Omega_0} g(u_n)\mathcal{T}\left(\frac{u_n}{h}\right) v,
\end{aligned}
\tag{3.9}
$$

for every $v\in C_c^\infty(\Omega_0)$. Set also

$$
\begin{aligned}
b(x,\xi) &:= \mathcal{T}\left(\frac{u^*(x)}{h}\right) j_t(u^*(x),|\xi|)\frac{\xi}{|\xi|}, \quad \text{for a.e. } x\in\Omega_0 \text{ and all } \xi\in\mathbb{R}^N,\\
\langle \Phi, v\rangle &:= \int_{\Omega_0} f(|x|,u^*)\mathcal{T}\left(\frac{u^*}{h}\right) v + \lambda \int_{\Omega_0} g(u^*)\mathcal{T}\left(\frac{u^*}{h}\right) v, \quad \forall v\in C_c^\infty(\Omega_0),
\end{aligned}
$$

where $\lambda$ denotes the limit of $(\lambda_n)$, according to Step II. Notice that $(\Phi_n)\subset W^{-1,p'}(\Omega_0)$ and $(\mu_n)$ defines a sequence of Radon measures on $\Omega_0$. Taking into account the strict convexity and monotonicity of $\{t\mapsto j(s,t)\}$ and the growth conditions (1.5)-(1.6), we claim that the operators $b, b_n$ satisfy the following properties:

$$(b_n(x,\xi)-b_n(x,\xi'))\cdot(\xi-\xi')\geq 0, \quad \text{a.e. } x\in\Omega_0, \text{ for all } \xi,\xi'\in\mathbb{R}^N, \tag{3.10}$$

$$(b(x,\xi)-b(x,\xi'))\cdot(\xi-\xi') > 0, \quad \text{a.e. } x\in\Omega_0 \text{ s.t. } u^*(x)\leq h, \xi\neq\xi', \tag{3.11}$$

$$b_n(x,\cdot)\to b(x,\cdot) \text{ as } n\to\infty, \quad \text{a.e. } x\in\Omega_0, \text{ uniformly over compacts}, \tag{3.12}$$

$$b_n(x,Du_n) \quad \text{is bounded in } L^{p'}(\Omega_0,\mathbb{R}^N), \tag{3.13}$$

$$b_n(x,Du^*)\to b(x,Du^*) \text{ as } n\to\infty, \quad \text{strongly in } L^{p'}(\Omega_0,\mathbb{R}^N), \tag{3.14}$$

$$\mu_n \rightharpoonup \mu \text{ as } n\to\infty, \quad \text{weakly* in measure, for some Radon measure } \mu, \tag{3.15}$$

$$\Phi_n\to\Phi \text{ as } n\to\infty, \quad \text{strongly in } W^{-1,p'}(\Omega_0). \tag{3.16}$$

Properties (3.10) and (3.11) follow from the strict convexity of the map $\{\xi\mapsto j(s,|\xi|)\}$ and the definition of $H$. Concerning (3.12), given $x\in\Omega_0$ and a compact $K\subset\mathbb{R}^N$, again by the definition of $H$ and the continuity of $j_t, j_{st}$, for all $\xi\in K$

and all $n \geq 1$ large,

$$
\begin{aligned}
|b_n(x,\xi) - b(x,\xi)| &\leq \left|\mathcal{T}\left(\frac{u_n(x)}{h}\right) - \mathcal{T}\left(\frac{u^*(x)}{h}\right)\right| |j_t(u_n(x),|\xi|)| \\
&\quad + \mathcal{T}\left(\frac{u^*(x)}{h}\right) |j_t(u_n(x),|\xi|) - j_t(u^*(x),|\xi|)| \\
&\leq \sup_{s\in[0,2h+2],\xi\in K} |j_t(s,|\xi|)| \left|\mathcal{T}\left(\frac{u_n(x)}{h}\right) - \mathcal{T}\left(\frac{u^*(x)}{h}\right)\right| \\
&\quad + \mathcal{T}\left(\frac{u^*(x)}{h}\right) |j_{st}(\tau u_n(x) + (1-\tau)u^*(x),|\xi|)||u_n(x) - u^*(x)| \\
&\leq \sup_{s\in[0,2h+2],\xi\in K} |j_t(s,|\xi|)| \times \left|\mathcal{T}\left(\frac{u_n(x)}{h}\right) - \mathcal{T}\left(\frac{u^*(x)}{h}\right)\right| \\
&\quad + \sup_{s\in[0,2h+1],\xi\in K} |j_{st}(s,|\xi|)| \times |u_n(x) - u^*(x)| \\
&\leq C_{h,K}\left\{\left|\mathcal{T}\left(\frac{u_n(x)}{h}\right) - \mathcal{T}\left(\frac{u^*(x)}{h}\right)\right| + |u_n(x) - u^*(x)|\right\},
\end{aligned}
$$

which yields the assertion. Conclusion (3.13) follows from the inequality

$$
|b_n(x,Du_n)|^{p'} = \left|\mathcal{T}\left(\frac{u_n(x)}{h}\right) j_t(u_n(x),|Du_n|)\right|^{p'} \leq (\gamma(2h))^{p'}|Du_n|^p,
$$

for a.e. $x \in \Omega_0$. Moreover, since $b_n(x,Du^*(x)) \to b(x,Du^*(x))$ a.e. in $\Omega_0$ and

$$
|b_n(x,Du^*)|^{p'} = \left|\mathcal{T}\left(\frac{u_n(x)}{h}\right) j_t(u_n(x),|Du^*|)\right|^{p'} \leq (\gamma(2h))^{p'}|Du^*|^p,
$$

for a.e. $x \in \Omega_0$, (3.14) holds as well by dominated convergence. Concerning (3.15), it can be easily verified that the square bracket in (3.9) is a bounded sequence in $L^1(\Omega_0)$ (just argue as in the estimation of the $I_i^n$s), so that, up to a subsequence, the property holds. Let us now prove that (3.16) holds. In fact, for all $v \in W_0^{1,p}(\Omega_0)$ such that $\|v\|_{W^{1,p}(\Omega_0)} \leq 1$, we have

$$
\begin{aligned}
|\langle \Phi_n - \Phi, v\rangle| \leq C &\left\| f(|x|,u_n)\mathcal{T}\left(\frac{u_n}{h}\right) - f(|x|,u^*)\mathcal{T}\left(\frac{u^*}{h}\right)\right\|_{L^{\frac{p^*}{p^*-1}}(\Omega_0)} \\
&+ C\lambda_n \left\| g(u_n)\mathcal{T}\left(\frac{u_n}{h}\right) - g(u^*)\mathcal{T}\left(\frac{u^*}{h}\right)\right\|_{L^{\frac{p^*}{p^*-1}}(\Omega_0)} \\
&+ C|\lambda_n - \lambda| \left\| g(u^*)\mathcal{T}\left(\frac{u^*}{h}\right)\right\|_{L^{\frac{p^*}{p^*-1}}(\Omega_0)}.
\end{aligned}
$$

Notice that, for a.e. in $\Omega_0$,

$$f(|x|,u_n)\mathcal{T}\left(\frac{u_n}{h}\right)-f(|x|,u^*)\mathcal{T}\left(\frac{u^*}{h}\right)\to 0,\quad \text{as } n\to\infty,$$

$$g(u_n)\mathcal{T}\left(\frac{u_n}{h}\right)-g(u^*)\mathcal{T}\left(\frac{u^*}{h}\right)\to 0,\quad \text{as } n\to\infty,$$

$\lambda_n \to \lambda$ as $n \to \infty$ and, by the growth assumptions (1.7) and (1.9),

$$\left|f(|x|,u_n)\mathcal{T}\left(\frac{u_n}{h}\right)-f(|x|,u^*)\mathcal{T}\left(\frac{u^*}{h}\right)\right|^{\frac{p^*}{p^*-1}} \le C_h\, a^{\frac{p^*}{p^*-1}}(|x|)+C_h',$$

$$\left|g(u_n)\mathcal{T}\left(\frac{u_n}{h}\right)-g(u^*)\mathcal{T}\left(\frac{u^*}{h}\right)\right|^{\frac{p^*}{p^*-1}} \le C_h'',$$

for some $C_h, C_h', C_h'' > 0$ and a.e. in $\Omega_0$. Hence, we obtain the property by taking the supremum on $v$ and using the dominated convergence. Therefore, in light of (3.8) and (3.10)–(3.16), we can apply [9, Theorem 5], yielding the almost everywhere convergence of the gradients $Du_n$ to $Du^*$ on the set

$$E_{h,\Omega_0}=\{x\in\Omega_0 : u^*(x)\le h\}.$$

We deduce the desired pointwise convergence (3.7) by the arbitrariness of $h \ge 1$ and $\Omega_0 \subset \Omega$.

**Step IV (proof of the theorem concluded).** In view of (3.7) and (1.4), we have

$$j(u_n,|Du_n|)-\alpha_0|Du_n|^p \ge 0,\quad \text{for all } n\ge 1 \text{ and a.e. in } \Omega,$$

$$j(u_n,|Du_n|)-\alpha_0|Du_n|^p \to j(u^*,|Du^*|)-\alpha_0|Du^*|^p,\quad \text{as } n\to\infty, \text{ a.e. in } \Omega.$$

Taking into account (3.3), by Fatou's lemma, we get

$$\limsup_n \int_\Omega |Du_n|^p \le \int_\Omega |Du^*|^p.$$

Since $(u_n)$ converges to $u^*$ strongly in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$, we can conclude that $u_n \to u^*$ strongly in $W_0^{1,p}(\Omega)$, as $n \to \infty$. Taking the limit into (3.1) we reach

$$\|Du\|_{L^p(\Omega)} = \|Du^*\|_{L^p(\Omega)}.$$

Then, by Proposition 2.4, it follows that $u$ is almost everywhere equal to $u^*$. □

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## Author information

Marco Squassina, Dipartimento di Informatica, Università degli Studi di Verona
Cá Vignal 2, Strada Le Grazie 15, 37134 Verona, Italy.
E-mail: marco.squassina@univr.it