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RADIAL SYMMETRY OF MINIMAX CRITICAL
POINTS FOR NONSMOOTH FUNCTIONALS

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We obtain the existence of radially symmetric and decreasing solutions to a general
class of quasi-linear elliptic problems by a nonsmooth version of a symmetric minimax
principle recently obtained by Van Schaftingen.

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symmetrization; Schwarz symmetrization; quasi-linear elliptic equations.

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1. Introduction and Main Result

1.1. *Introduction*

The main goal of this paper is to provide, in the framework of nonsmooth critical
point theory (cf. [11, 12, 18] and references therein), a general minimax varia-
tional principle for a class of lower semi-continuous functionals satisfying certain
monotonicity properties under polarization, allowing to detect critical points in the
sense of the weak slope (cf. Definitions 3.1 and 3.4) of minimax type which are
radially symmetric and decreasing. In the case of C^1 smooth functionals these type
of results were studied by Van Schaftingen in [29] (see also [30–32]), where vari-
ous applications to semi-linear elliptic equations of the form $-\Delta u = g(|x|, u)$ in Ω
with $u = 0$ on $\partial\Omega$ were also derived under suitable assumptions on g, when Ω is
either a ball B_1 or an annulus (see also [33]). On the other hand, typically, in the
general context of quasi-linear problems of variational type, the energy functional
$f : H_0^1(B_1) \to \mathbb{R} \cup \{+\infty\}$ is, say,

$$f(u) = \int_{B_1} j(u, |\nabla u|)dx - \int_{B_1} G(|x|, u)dx, \tag{1.1}$$

and under reasonable assumptions f is merely either lower semi-continuous or con-
tinuous on $H_0^1(B_1)$, depending on the growth conditions which are imposed on j

and G (cf. [7, 28, 34]). A class of minimization problems, constrained to the unit sphere of $L^p(\mathbb{R}^N)$, for functionals (1.1) defined on the whole $\mathbb{R}^N$ has been recently investigated in [21] by exploiting the following generalized Polya–Szegö and Hardy–Littlewood type inequalities for Schwarz symmetrization

$$\int_{\mathbb{R}^N} j(u^*, |\nabla u^*|)dx \le \int_{\mathbb{R}^N} j(u, |\nabla u|)dx,$$

$$\int_{\mathbb{R}^N} G(|x|, u)dx \le \int_{\mathbb{R}^N} G(|x|, u^*)dx,$$

the latter holding true under suitable monotonicity conditions on G in the radial argument. These inequalities also hold in the unit ball B_1 and immediately yield $f(u^*) \le f(u)$ for all $u \in H_0^1(B_1)$, namely (1.1) decreases upon Schwarz symmetrization. In turn, the existence of a global minimizer for f on a sphere $\{u \in H_0^1(B_1) : \|u\|_{L^p} = 1\}$, with $p \ge 2$, immediately yields the existence of a radially symmetric and decreasing minimizer. The first of the previous symmetrization inequalities holds under mild assumptions, allowing $j(s, |\xi|)$ to be unbounded with respect to s, say, for instance $j(s, |\xi|) \le \alpha(|s|)|\xi|^2$ where $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function. This constrained minimization problems arise, for instance, in the study of standing wave solutions for semi-linear and quasi-linear Schrödinger equations (see [9] for a recent study). Concerning the study of free critical points for f, in [28] it was obtained existence of infinitely many critical points via $\mathbb{Z}_2$-symmetric nonsmooth mountain pass theorems, under (a subset of) the assumptions listed in Sec. 1.2 (see also [35] and references therein for various applications of nonsmooth critical point theory to quasi-linear elliptic problems). In this paper, we shall prove a general nonsmooth minimax principle (cf. Theorem 3.9) and, in turn, we shall derive the main abstract result of the paper (cf. Theorem 3.10), a symmetric version of Theorem 3.9 working for a large class of lower semi-continuous functionals (in abstract spaces) which are decreasing upon (abstract) polarization. In our main concrete result (cf. Theorem 1.2) we state the existence of a nontrivial radially symmetric and decreasing distributional solution of problem

$$\begin{cases} -\mathrm{div}(j_\xi(u, |\nabla u|)) + j_s(u, |\nabla u|) = g(|x|, u), & \text{in } B_1, \\ u = 0, & \text{on } \partial B_1, \end{cases} \tag{P}$$

corresponding to the *mountain pass critical level* of the functional f. In particular, to mention a case connected with a physically meaningful situation and that can be covered in our framework, we get a radial distributional mountain pass solution to the problem, say in $B_1 \subset \mathbb{R}^3$,

$$\begin{cases} -\Delta u - \dfrac{1}{2} u \Delta u^2 = \gamma(|x|)|u|^{p-2}u, & \text{in } B_1, \\ u = 0, & \text{on } \partial B_1, \end{cases} \tag{1.2}$$

where $4 < p < 6$ and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded decreasing function. This boundary value problem corresponds to (P) with the choice

$$j(s, \xi) = \frac{1}{2}(1 + s^2)|\xi|^2, \quad \text{for } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^3,$$

and all the assumptions of the framework of section (1.2) are fulfilled. Problem (1.2) is related to a quasi-linear Schrödinger equation that arises, for instance, in dissipative quantum mechanics ([22]), plasma physics and fluid mechanics ([17]) and in condensed matter theory ([25]).

The weak slope critical points u of f naturally correspond to generalized solutions (see Definition 4.1) of problem (P), which become in turn distributional by showing that u is bounded. We point out that Theorem 3.10 often provides, in general contexts, also an alternative tool to concentration compactness arguments, see Remark 3.11 for more details. Even in the classical cases such as $j(s, \xi) = |\xi|^2$, if the nonlinearity $g(|x|, s) = D_s G(|x|, s)$ is a merely continuous function, the moving plane argument (cf. [15]) and the homotopy argument due to Brock (cf. [4]) yielding local symmetry of positive solutions cannot be applied. In the general quasi-linear setting, even for functions g of class C^1, to the author's knowledge, no symmetry results based upon moving plane arguments are available in the current literature. On the contrary, for the p-Laplacian operator $j(|\xi|) = |\xi|^p$, there are various results for positive solutions and $\mathrm{Lip}_{\mathrm{loc}}$ and autonomous nonlinearities (cf. [5, 13, 14] for equations and [26] for systems). Finally we notice that, in some cases, the symmetry can be inferred by Palais's *symmetric criticality principle* (cf. [27]) restricting the functional to radial functions. Of course, in this case, one would loose the global mountain pass minimization property.

1.2. *The main concrete result*

Let B_1 be the unit ball in $\mathbb{R}^N$ centered at the origin, $N \geq 3$ and let $f : H_0^1(B_1) \to \mathbb{R} \cup \{+\infty\}$ be the functional defined by

$$f(u) = \int_{B_1} j(u, |\nabla u|)dx - \int_{B_1} G(|x|, u)dx, \tag{1.3}$$

where $j(s, |\xi|) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is of class C^1. We consider the following assumptions.

1.2.1. *Assumptions on j*

We assume that for every s in $\mathbb{R}$

$$\{|\xi| \mapsto j(s, |\xi|)\} \quad \text{is strictly convex and increasing.} \tag{1.4}$$

Moreover, there exist a constant $\alpha_0 > 0$ and a positive increasing function $\alpha \in C(\mathbb{R})$ such that, for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, it holds

$$\alpha_0|\xi|^2 \leq j(s, |\xi|) \leq \alpha(|s|)|\xi|^2. \tag{1.5}$$

The functions $j_s(s, |\xi|)$ and $j_\xi(s, \xi)$ denote the derivatives of $j(s, \xi)$ with respect to the variables s and ξ respectively. Regarding the function $j_s(s, |\xi|)$, we assume that there exist a positive increasing function $\beta \in C(\mathbb{R})$ and a positive constant R such that

$$|j_s(s, |\xi|)| \leq \beta(|s|)|\xi|^2, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and all } \xi \in \mathbb{R}^N, \tag{1.6}$$

$$j_s(s, |\xi|)s \geq 0, \qquad \text{for every } s \text{ in } \mathbb{R} \text{ with } |s| \geq R \text{ and all } \xi \in \mathbb{R}^N. \tag{1.7}$$

Furthermore, we assume that

$$j(-s, |\xi|) \leq j(s, |\xi|), \quad \text{for every } s \text{ in } \mathbb{R}^- \text{ and all } \xi \in \mathbb{R}^N. \tag{1.8}$$

1.2.2. *Assumptions on g, G*

The function $G(|x|, s)$ is the primitive with respect to s of a Carathéodory function $g(|x|, s)$ such that $G(|x|, 0) = 0$. We assume that there exist $p \in (2, 2N/(N-2))$, a positive constant C, $\mu > 2$ and $R' > 0$ such that

$$|g(|x|, s)| \leq C(1 + |s|^{p-1}), \quad \text{for every } s \text{ in } \mathbb{R} \text{ and } x \in B_1, \tag{1.9}$$

$$0 < \mu G(|x|, s)$$
$$\leq g(|x|, s)s, \qquad \text{for every } s \text{ in } \mathbb{R} \text{ with } |s| \geq R' \text{ and } x \in B_1, \tag{1.10}$$

$$\lim_{s \to 0} \frac{g(|x|, s)}{s} = 0, \qquad \text{uniformly in } B_1, \tag{1.11}$$

$$g(|x|, s) \geq g(|y|, s), \qquad \text{for every } s \in \mathbb{R}^+ \text{ and } x, y \in B_1 \text{ with } |x| \leq |y|, \tag{1.12}$$

$$G(|x|, s) \leq G(|x|, -s), \qquad \text{for every } s \in \mathbb{R}^- \text{ and } x \in B_1. \tag{1.13}$$

1.2.3. *Joint assumptions of j and g*

There exist $R'' > 0$ and $\delta > 0$ such that

$$\mu j(s, |\xi|) - j_s(s, |\xi|)s - (1 + \delta)j_\xi(s, |\xi|) \cdot \xi \geq 0, \quad \text{for every } s \in \mathbb{R} \text{ with } |s| \geq R'' \tag{1.14}$$

and all $\xi \in \mathbb{R}^N$. Finally, it holds

$$\lim_{|s| \to \infty} \frac{\alpha(|s|)}{|s|^{\mu-2}} = 0. \tag{1.15}$$

Remark 1.1. The asymptotic sign condition (1.7) is typical for quasi-linear elliptic problems and, in general, plays a role both in the regularity theory (see Frehse's counterexample in [20]) and in the verification of the Palais–Smale condition (see e.g. [28, 34]). Assumption (1.12) is necessary in order to get some inequality for $\int_{B_1} G(|x|, u)$ under polarization, while (1.14) is used to prove that Palais–Smale

sequences are bounded (see [28, 34]). Assumption (1.15) is needed for the functional to satisfy some Mountain Pass geometry (see [28]). Finally, we point out that the growth (1.9) on g could be weakened, allowing that for all $\varepsilon > 0$ there exists $a_\varepsilon \in L^r(B_1)$ with $r > \frac{2N}{N+2}$ such that $|g(|x|, s)| \leq a_\varepsilon(x) + \varepsilon|s|^{(N+2)/(N-2)}$ for all $x \in B_1$ and $s \in \mathbb{R}$. If $r > N/2$ the solutions are bounded (cf. [28, Theorem 7.1(b)]).

1.2.4. *Statement*

The principal result is the following general Ambrosetti-Rabinowitz [2] mountain type theorem which includes the additional information on the radial symmetry of the solution.

Theorem 1.2. *Assume that conditions* (1.4)–(1.15) *hold. Then there exists a nontrivial nonnegative radially symmetric and decreasing mountain pass solution* $u \in H_0^1 \cap L^\infty(B_1)$ *to problem* (P).

Although we state our result in the unit ball, a similar statement could be provided for the functional defined in the unit ball or in the annulus without the monotonicity condition (1.12) on g, yielding a solution u which is invariant under spherical cap symmetrization, namely u depends solely upon $|x|$ and an angular variable. In [29] further applications of the symmetric minimax principle for C^1 smooth functionals are provided, for instance the case of linking geometry. We limit ourself to the statement of Theorem 1.2 although also in the nonsmooth setting some of the applications discussed in [29] could be derived from Theorem 3.10 under appropriate assumptions.

Remark 1.3. Theorem 1.2 covers, in particular, the case where the function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, which appears in growth condition (1.5), is bounded. Under this assumption, the functional f defined in (1.3) is continuous on $H_0^1(B_1)$ (and condition (1.15) is automatically fulfilled). For this framework, we refer the reader to the lectures [8] and to the references therein.

Remark 1.4. We considered the framework of lower semi-continuous functionals defined on the Sobolev space $H_0^1(B_1)$ for simplicity. In fact, a result such as Theorem 1.2 could be stated in the more general setting of functionals over the Sobolev space $W_0^{1,q}(B_1)$, for any $q \geq 2$, as the results from [28] that we used in the proof of Theorem 1.2 can be extended to the case $q \neq 2$. We also refer to [34, 35] and references therein for further results.

2. Tools from Symmetrization Theory

2.1. *Abstract symmetrization*

We recall a definition from [29].

Let X and V be two Banach spaces and $S \subset X$. We consider two maps $* : S \to V$, $u \mapsto u^*$ (*symmetrization map*) and $h : S \times \mathcal{H}_* \to S$, $(u, H) \mapsto u^H$ (*polarization*

map), where $\mathcal{H}_*$ is a path-connected topological space. We assume that the following conditions hold:

(1) X is continuously embedded in V;
(2) h is a continuous mapping;
(3) for each $u \in S$ and $H \in \mathcal{H}_*$ it holds $(u^*)^H = (u^H)^* = u^*$ and $u^{HH} = u^H$;
(4) there exists a sequence (H_m) in $\mathcal{H}_*$ such that, for $u \in S$, $u^{H_1 \cdots H_m}$ converges to u^* in V;
(5) for every $u, v \in S$ and $H \in \mathcal{H}_*$ it holds $\|u^H - v^H\|_V \leq \|u - v\|_V$.

Furthermore $* : S \to V$ can be extended to the whole space X by setting $u^* := (\Theta(u))^*$ for all $u \in X$, where $\Theta : (X, \| \cdot \|_V) \to (S, \| \cdot \|_V)$ is a Lipschitz function such that $\Theta|_S = \mathrm{Id}|_S$.

Remark 2.1. There exists a positive constant C_Θ such that

$$\forall \, u, v \in X : \|u^* - v^*\|_V \leq C_\Theta \|u - v\|_V. \tag{2.1}$$

In fact, given $u, v \in X$, by assumption (5) it holds

$$\|(\Theta(u))^{H_1 \cdots H_m} - (\Theta(v))^{H_1 \cdots H_m}\|_V \leq \|\Theta(u) - \Theta(v)\|_V \leq C_\Theta \|u - v\|_V,$$

where (H_m) is a sequence in $\mathcal{H}_*$ according to assumption (4) and C_Θ is the positive Lipschitz constant of Θ. Then, by the definition of $* : X \to V$, we obtain

$$\|u^* - v^*\|_V \leq \|(\Theta(u))^* - (\Theta(u))^{H_1 \cdots H_m}\|_V + \|(\Theta(u))^{H_1 \cdots H_m} - (\Theta(v))^{H_1 \cdots H_m}\|_V$$

$$+ \|(\Theta(v))^{H_1 \cdots H_m} - (\Theta(v))^*\|_V$$

$$\leq \|(\Theta(u))^* - (\Theta(u))^{H_1 \cdots H_m}\|_V + C_\Theta \|u - v\|_V$$

$$+ \|(\Theta(v))^{H_1 \cdots H_m} - (\Theta(v))^*\|_V,$$

for all $m \geq 1$. Therefore, taking into account the convergence property in assumption (4), letting $m \to \infty$ into the previous inequality immediately yields (2.1).

2.1.1. *Polarization*

A subset H of $\mathbb{R}^N$ is called a polarizer if it is a closed affine half-space of $\mathbb{R}^N$, namely the set of points x which satisfy $\alpha \cdot x \leq \beta$ for some $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ with $|\alpha| = 1$. Given x in $\mathbb{R}^N$ and a polarizer H the reflection of x with respect to the boundary of H is denoted by x_H. The polarization of a function $u : \mathbb{R}^N \to \mathbb{R}^+$ by a polarizer H is the function $u^H : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\}, & \text{if } x \in H \\ \min\{u(x), u(x_H)\}, & \text{if } x \in \mathbb{R}^N \backslash H. \end{cases} \tag{2.2}$$

The polarization $C^H \subset \mathbb{R}^N$ of a set $C \subset \mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{C^H} = (\chi_C)^H$, where χ denotes the characteristic function. The polarization

u^H of a positive function u defined on $C \subset \mathbb{R}^N$ is the restriction to C^H of the polarization of the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ of u by zero outside C. The polarization of a function which may change sign is defined by $u^H := |u|^H$, for any given polarizer H.

2.1.2. *Schwarz symmetrization*

The Schwarz symmetrization of a set $C \subset \mathbb{R}^N$ is the unique open ball centered at the origin C^* such that $\mathcal{L}^N(C^*) = \mathcal{L}^N(C)$, being $\mathcal{L}^N$ the N-dimensional outer Lebesgue measure. If the measure of C is zero we set $C^* = \emptyset$, while if the measure of C is not finite we put $C^* = \mathbb{R}^N$. A measurable function u is admissible for the Schwarz symmetrization if it is nonnegative and, for every $\varepsilon > 0$, the Lebesgue measure of $\{u > \varepsilon\}$ is finite. The Schwarz symmetrization of an admissible function $u : C \to \mathbb{R}^+$ is the unique function $u^* : C^* \to \mathbb{R}^+$ such that, for all $t \in \mathbb{R}$, it holds $\{u^* > t\} = \{u > t\}^*$. Considering the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ of u by zero outside C, then $u^* = (\tilde{u})^*|_{C^*}$ and $(\tilde{u})^*|_{\mathbb{R}^N \setminus C^*} = 0$. The symmetrization for u which are not nonnegative can be the defined by $u^* := |u|^*$.

Remark 2.2. Let us set $\mathcal{H}_* = \{H \in \mathcal{H} : 0 \in H \text{ or } H = H_{+\infty}\}$, $\Omega = B(0,1)$ and either

$$X = W_0^{1,p}(\Omega), \quad S = W_0^{1,p}(\Omega, \mathbb{R}^+), \quad V = L^p \cap L^{p^*}(\Omega),$$

or

$$X = S = W_0^{1,p}(\Omega), \quad V = L^p \cap L^{p^*}(\Omega), \quad u^H := |u|^H, \quad u^* := |u|^*.$$

Then the requirements (1)–(5) in the abstract symmetrization framework are satisfied (cf. [29, Propositions 2.3, 2.5 and Theorem 2.1, see the examples at p. 469]). Furthermore $u = u^*$ if and only if $u = u^H$ for all $H \in \mathcal{H}_*$ and $C = C^*$ if and only if $C = C^H$, for every $H \in \mathcal{H}_*$.

Remark 2.3. Different types of symmetrization can be considered, such as the Steiner symmetrization or the spherical cap symmetrization, for which the abstract framework above is fulfilled and our main result would work. We refer the interested reader to [29] and to the references therein for further details. See also [30, 31].

2.2. *Symmetric approximation of curves*

In general, except in the one-dimensional case (see [10]) the Schwarz symmetric rearrangement is not a continuous function (see [1]). To overcome this problem, we recall a very useful and general approximation tool for continuous curves in Banach spaces provided in Van Schaftingen's paper (see [29, Proposition 3.1]).

Proposition 2.4. *Let X and V be two Banach spaces, $S \subseteq X$, $*$ and $\mathcal{H}_*$ which satisfy the requirements of the abstract symmetrization framework. Let M be a metric space, M_0 and M_1 be disjoint closed sets of M and $\gamma \in C(M, X)$. Let*

$H_0 \in \mathcal{H}_*$ and $\gamma(M) \subset S$. Then, for every $\delta > 0$, there exists a curve $\tilde{\gamma} \in C(M, X)$ such that

$$\|\tilde{\gamma}(\tau) - \gamma(\tau)^*\|_V \leq \delta, \quad \forall\, \tau \in M_1,$$

$\tilde{\gamma}(\tau) = \gamma(\tau)^{H_1 \cdots H_{[\theta]} H_\theta}$ for all $\tau \in M$, with $H_s \in \mathcal{H}_*$ for $s \geq 0$, $\tilde{\gamma}(\tau) = \gamma(\tau)^{H_0}$ for $\tau \in M_0$. Here $[\theta]$ denotes the largest integer less than or equal to θ.

3. Tools from Nonsmooth Critical Point Theory

3.1. *Preliminary notions and results*

In this section we consider abstract notions and results that will be used in the proof of the main results. For the definitions, we refer to [12, 18, 23, 24], where the theory was developed.

We denote with $B(u, \delta)$ the open ball of center u and of radius δ. We recall the definition of the weak slope for a continuous function.

Definition 3.1. Let X be a metric space, $g : X \to \mathbb{R}$ a continuous function, and $u \in X$. We denote by $|dg|(u)$ the supremum of the real numbers σ in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B(u, \delta) \times [0, \delta] \to X$, such that, for every v in $B(u, \delta)$, and for every t in $[0, \delta]$ it results

$$d(\mathcal{H}(v, t), v) \leq t, \quad g(\mathcal{H}(v, t)) \leq g(v) - \sigma t.$$

The extended real number $|dg|(u)$ is called the weak slope of g at u.

The next result establishes the connection between the weak slope of a function g and its differential $dg(u)$, in the case where g is of class C^1. For the proof, see [18, Corollary 2.12].

Proposition 3.2. *If X is an open subset of a normed space E and g is a function of class C^1 on X, then $|dg|(u) = \|dg(u)\|$ for every $u \in X$.*

Let now X be a metric space and let $f : X \to \bar{\mathbb{R}}$ be a function. We set

$$\text{dom}(f) = \{u \in X : f(u) < +\infty\} \quad \text{and} \quad \text{epi}(f) = \{(u, \xi) \in X \times \mathbb{R} : f(u) \leq \xi\},$$

and let us define the function $\mathcal{G}_f : \text{epi}(f) \to \mathbb{R}$ by

$$\mathcal{G}_f(u, \xi) = \xi. \tag{3.1}$$

In the following, epi(f) will be endowed with the metric

$$d((u, \xi), (v, \mu)) = (d(u, v)^2 + (\xi - \mu)^2)^{1/2},$$

so that the function $\mathcal{G}_f$ is Lipschitz continuous of constant 1.

Let us recall [16] a device allowing to reduce the study of continuous or lower semi-continuous functionals to that of Lipschitz functionals. We refer to [18, Proposition 2.3].

Lemma 3.3. *Let $f : X \to \mathbb{R}$ be a continuous function. Then, for every $(u, \xi) \in$ epi(f), we have*

$$|d\mathcal{G}_f|(u, \xi) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1 + |df|(u)^2}} & \text{if } f(u) = \xi \text{ and } |df|(u) < +\infty, \\[3mm] 1 & \text{if } f(u) < \xi \text{ or } |df|(u) = +\infty. \end{cases}$$

On the basis of the previous result, one can define the weak slope of a lower semi-continuous function f by using $|d\mathcal{G}_f|(u, f(u))$. More precisely, we have the following

Definition 3.4. *Let $f : X \to \bar{\mathbb{R}}$ be a lower semi-continuous function. For every $u \in X$ such that $f(u) \in \mathbb{R}$, let*

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}}, & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1, \\[3mm] +\infty, & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

The previous notions allow us to give the following

Definition 3.5. *Let X be a metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function. We say that $u \in \text{dom}(f)$ is a (lower) critical point of f if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (lower) critical value of f if there exists a (lower) critical point $u \in \text{dom}(f)$ of f with $f(u) = c$.*

Definition 3.6. *Let X be a metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function and let $c \in \mathbb{R}$. We say that f satisfies the Palais–Smale condition at level c $((PS)_c$ in short), if every sequence (u_n) in $\text{dom}(f)$ such that $|df|(u_n) \to 0$ and $f(u_n) \to c$ admits a subsequence (u_{n_k}) converging in X.*

In [12, 18] variational methods for lower semi-continuous functionals are developed. Moreover, it is shown that the following condition is fundamental in order to apply the abstract theory to the study of lower semi-continuous functions

$$\forall (u, \xi) \in \text{epi}(f) : f(u) < \xi \quad \Rightarrow \quad |d\mathcal{G}_f|(u, \xi) = 1. \tag{3.2}$$

Let $\rho > 0$ and assume that epi(f) is endowed with the metric

$$d_\rho((u, \xi), (v, \mu)) = (d(u, v)^2 + \rho^2(\xi - \mu)^2)^{1/2}. \tag{3.3}$$

Clearly the metric d_ρ is equivalent to the metric d on epi(f). Moreover, with respect to d_ρ the function $\mathcal{G}_f$ is Lipschitz continuous of constant $1/\rho$.

Proposition 3.7. *Let $f : X \to \bar{\mathbb{R}}$ be a function. Then*

$$|d_\rho \mathcal{G}_f|(u, \xi) = \frac{|d\mathcal{G}_f|(u, \xi)}{\sqrt{1 + (\rho^2 - 1)|d\mathcal{G}_f|(u, \xi)^2}},$$

for every $(u, \xi) \in$ epi(f). In particular, if $|d\mathcal{G}_f|(u, \xi) = 1$, it follows $|d_\rho \mathcal{G}_f|(u, \xi) = \frac{1}{\rho}$.

Proof. The proof follows the lines of the proof of [18, Proposition 2.3]. On the other hand, for the sake of completeness, we sketch the proof. Let us first prove that

$$|d_\rho \mathcal{G}_f|(u,\xi) \geq \frac{|d\mathcal{G}_f|(u,\xi)}{\sqrt{1+(\rho^2-1)|d\mathcal{G}_f|(u,\xi)^2}}.$$

If $|d\mathcal{G}_f|(u,\xi) = 0$, then there is nothing to prove. Otherwise, let $0 < \sigma < |d\mathcal{G}_f|(u,\xi)$ and let $\mathcal{H} : B_\delta(u,\xi) \times [0,\delta] \to \text{epi}(f)$ be a continuous map, according to the definition of weak slope, so that

$$d(\mathcal{H}_1((v,\mu),t),v)^2 + |\mathcal{H}_2((v,\mu),t) - \mu|^2 \leq t^2, \quad \mathcal{H}_2((v,\mu),t) \leq \mu - \sigma t,$$

for all $t \in [0,\delta]$ and every $(v,\mu) \in B_\delta(u,\xi)$. Let now choose $\delta' > 0$ (with $\delta' = \delta$ if $\rho \leq 1$) be such that $\delta' \leq \delta\sqrt{1+(\rho^2-1)\sigma^2}$ and consider the continuous function $\mathcal{K} = (\mathcal{K}_1,\mathcal{K}_2) : B_{\delta'}(u,\xi) \times [0,\delta'] \to \text{epi}(f)$ defined by

$$\mathcal{K}_1((v,\mu),t) = \mathcal{H}_1\left((v,\mu), \frac{t}{\sqrt{1+(\rho^2-1)\sigma^2}}\right),$$

$$\mathcal{K}_2((v,\mu),t) = \mu - \frac{\sigma t}{\sqrt{1+(\rho^2-1)\sigma^2}}.$$

Of course $\mathcal{K}((v,\mu),t) \in \text{epi}(f)$ for all $t \in [0,\delta']$ and every $(v,\mu) \in B_{\delta'}(u,\xi)$. Moreover, we have

$$d_\rho(\mathcal{K}((v,\mu),t),(v,\mu))^2 = d(\mathcal{K}_1((v,\mu),t),v)^2 + \rho^2|\mathcal{K}_2((v,\mu),t) - \mu|^2$$

$$= d\left(\mathcal{H}_1\left((v,\mu), \frac{t}{\sqrt{1+(\rho^2-1)\sigma^2}}\right), v\right)^2$$

$$+ \rho^2 \frac{\sigma^2 t^2}{1+(\rho^2-1)\sigma^2}$$

$$\leq \frac{t^2}{1+(\rho^2-1)\sigma^2} + \frac{(\rho^2-1)\sigma^2 t^2}{1+(\rho^2-1)\sigma^2} = t^2,$$

for all $t \in [0,\delta']$ and every $(v,\mu) \in B_{\delta'}(u,\xi)$. Furthermore, we have

$$\mathcal{G}_f(\mathcal{K}((v,\mu),t)) = \mathcal{K}_2((v,\mu),t) = \mathcal{G}_f(v,\mu) - \frac{\sigma}{\sqrt{1+(\rho^2-1)\sigma^2}}t,$$

for all $t \in [0,\delta']$ and every $(v,\mu) \in B_{\delta'}(u,\xi)$. In turn, we have

$$|d_\rho \mathcal{G}_f|(u,\xi) \geq \frac{\sigma}{\sqrt{1+(\rho^2-1)\sigma^2}},$$

yielding, by the arbitrariness of σ,

$$|d_\rho \mathcal{G}_f|(u,\xi) \geq \frac{|d\mathcal{G}_f|(u,\xi)}{\sqrt{1+(\rho^2-1)|d\mathcal{G}_f|(u,\xi)^2}}.$$

Concerning the proof of the opposite inequality, it is sufficient to show that

$$|d\mathcal{G}_f|(u,\xi) \geq \frac{|d_\rho\mathcal{G}_f|(u,\xi)}{\sqrt{1 - (\rho^2 - 1)|d_\rho\mathcal{G}_f|(u,\xi)^2}}.$$

This can be achieved by arguing as above by considering, in place of $\mathcal{K}$, the continuous function $\hat{\mathcal{K}} = (\hat{\mathcal{K}}_1, \hat{\mathcal{K}}_2) : B_{\delta'}(u,\xi) \times [0,\delta'] \to \mathrm{epi}(f)$ defined by

$$\hat{\mathcal{K}}_1((v,\mu),t) = \mathcal{H}_1\left((v,\mu), \frac{t}{\sqrt{1 - (\rho^2 - 1)\sigma^2}}\right),$$

$$\hat{\mathcal{K}}_2((v,\mu),t) = \mu - \frac{\sigma t}{\sqrt{1 - (\rho^2 - 1)\sigma^2}}.$$

This concludes the proof. $\qquad\qquad\square$

Remark 3.8. The notion of weak slope for a function $f : X \to \overline{\mathbb{R}}$ (not even assumed to be lower semi-continuous) was also provided (see [6, Definition 2.1]) in terms of local deformations, consistently with Definition 3.1 (see [6, Proposition 2.2]). Of course, the extended real number $|df|(u)$ is independent of ρ and, arguing as in [6, Proposition 2.3], it is possible to show that

$$|d_\rho\mathcal{G}_f|(u, f(u)) = \frac{|df|(u)}{\sqrt{1 + \rho^2 |df|(u)^2}}, \quad |d\mathcal{G}_f|(u, f(u)) = \frac{|df|(u)}{\sqrt{1 + |df|(u)^2}},$$

for every u with $|df|(u) < +\infty$. In turn, combining these equalities one immediately obtains the assertion of Proposition 3.7 with $\xi = f(u)$.

3.2. *The non-symmetric minimax theorem*

In the framework of the previous section we have the following nonsmooth minimax principle (for C^1 functionals, see the corresponding version in [36]).

Theorem 3.9. *Let X be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function satisfying* (3.2). *Let $\mathbb{D}$ and $\mathbb{S}$ denote the closed unit ball and the sphere in $\mathbb{R}^N$ respectively and $\Gamma_0 \subset C(\mathbb{S}, X)$. Let us define*

$$\Gamma = \{\gamma \in C(\mathbb{D}, X) : \gamma|_\mathbb{S} \in \Gamma_0\}.$$

Assume that

$$+\infty > c = \inf_{\gamma\in\Gamma} \sup_{\tau\in\mathbb{D}} f(\gamma(\tau)) > \sup_{\gamma_0\in\Gamma_0} \sup_{\tau\in\mathbb{S}} f(\gamma_0(\tau)) = a.$$

Then, for every $\varepsilon \in (0, (c-a)/2)$, every $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{\tau\in\mathbb{D}} f(\gamma(\tau)) \leq c + \varepsilon,$$

there exists $u \in X$ such that

$$c - 2\varepsilon \le f(u) \le c + 2\varepsilon,$$

$$\text{dist}(u, \gamma(\mathbb{D}) \cap f^{-1}([c - 3\varepsilon, c + 3\varepsilon])) \le 3\delta, \quad |df|(u) \le 3\varepsilon/\delta.$$

Proof. We divide the proof into two cases.

Case I. We prove the result for continuous functions $f : X \to \mathbb{R}$. In this case, the assertion follows as a direct application of the quantitative nonsmooth deformation theorem [11] with the stronger conclusion that there exists $u \in X$ with $c - 2\varepsilon \le f(u) \le c + 2\varepsilon$ and

$$\text{dist}(u, \gamma(\mathbb{D}) \cap f^{-1}([c - 2\varepsilon, c + 2\varepsilon])) \le 2\delta, \quad |df|(u) \le \varepsilon/\delta. \tag{3.4}$$

In fact, if this was not the case, by applying [11, Theorem 2.3] with the choice of the closed set $A = \gamma(\mathbb{D}) \cap f^{-1}([c - 2\varepsilon, c + 2\varepsilon]$, one could find a deformation $\eta : X \times [0, 1] \to X$ such that $d(\eta(u, t), u) \le 2\delta t$ for all $u \in X$ and $t \in [0, 1]$, $f(\eta(u, t)) < f(u)$ for all $u \in X$ and $t \in [0, 1]$ with $\eta(u, t) \ne u$ and

$$u \in A, \quad c - \varepsilon \le f(u) \le c + \varepsilon \implies f(\eta(u, 1)) \le c - \varepsilon. \tag{3.5}$$

If $\Xi : X \to [0, 1]$ is a continuous function such that $\Xi(u) = 0$ if $f(u) \le a$ and $\Xi(u) = 1$ if $f(u) \ge c - \varepsilon$, considering $\tilde{\gamma} \in C(\mathbb{D}, X)$ defined by $\tilde{\gamma}(\tau) = \eta(\gamma(\tau), \Xi(\gamma(\tau)))$, it follows that $\tilde{\gamma} \in \Gamma$, since for all $\tau \in \mathbb{S}$ we have $\Xi(\gamma(\tau)) = 0$, due to

$$f(\gamma(\tau)) \le \sup_{\gamma_0 \in \Gamma_0} \sup_{\tau \in \mathbb{S}} f(\gamma_0(\tau)) = a.$$

Given an arbitrary $\tau \in \mathbb{D}$, either $f(\gamma(\tau)) < c - \varepsilon$ and thus $f(\tilde{\gamma}(\tau)) \le f(\gamma(\tau)) < c - \varepsilon$ or $f(\gamma(\tau)) \ge c - \varepsilon$, in which case, by the definition of Ξ and (3.5), we get

$$f(\tilde{\gamma}(\tau)) = f(\eta(\gamma(\tau), \Xi(\gamma(\tau)))) = f(\eta(\gamma(\tau), 1)) \le c - \varepsilon.$$

Hence, we conclude that $f \circ \tilde{\gamma}|_{\mathbb{D}} \le c - \varepsilon$, providing the desired contradiction with the definition of c and concluding the proof for the case of $f : X \to \mathbb{R}$ continuous.

Case II. We cover the general case of lower semi-continuous functions $f : X \to \mathbb{R} \cup \{+\infty\}$. We introduce the sets $\hat{\Gamma}_0 \subset C(\mathbb{S}, \text{epi}(f))$ and $\hat{\Gamma} \subset C(\mathbb{D}, \text{epi}(f))$ by setting

$$\hat{\Gamma}_0 = \{\hat{\gamma} \in C(\mathbb{S}, \text{epi}(f)) : \hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2) \text{ with } \hat{\gamma}_1 \in \Gamma_0 \text{ and } \hat{\gamma}_2(\tau) \le a \text{ for all } \tau \in \mathbb{S}\},$$

$$\hat{\Gamma} = \{\hat{\gamma} \in C(\mathbb{D}, \text{epi}(f)) : \hat{\gamma}|_{\mathbb{S}} \in \hat{\Gamma}_0\}.$$

The space $\text{epi}(f)$ is equipped with the metric d_ρ defined in (3.3), for $\rho > 0$. As we prove below, $\hat{\Gamma} \ne \emptyset$. Of course, by the definition of $\hat{\Gamma}_0$, we have

$$\sup_{\hat{\gamma} \in \hat{\Gamma}_0} \sup_{\tau \in \mathbb{S}} \mathcal{G}_f(\hat{\gamma}(\tau)) = a. \tag{3.6}$$

Let us now prove that

$$\inf_{\hat{\gamma}\in\hat{\Gamma}}\sup_{\tau\in\mathbb{D}}\mathcal{G}_f(\hat{\gamma}(\tau)) = c. \tag{3.7}$$

We first show that

$$\inf_{\hat{\gamma}\in\hat{\Gamma}}\sup_{\tau\in\mathbb{D}}\mathcal{G}_f(\hat{\gamma}(\tau)) \le c. \tag{3.8}$$

In fact, given $b > c$, let $\gamma \in \Gamma$ be such that

$$a < \alpha := \sup_{\tau\in\mathbb{D}} f(\gamma(\tau)) \le b.$$

Consider now the continuous function $\vartheta : \mathbb{D} \to \mathbb{D}$ defined by setting $\vartheta(\tau) = \tau|\tau|^{-1}$ for all $\tau \in \overline{\mathbb{D}\backslash\mathbb{D}/2}$ and $\vartheta(\tau) = 2\tau$ for all $\tau \in \mathbb{D}/2$, and define $\hat{\gamma}_1 : \mathbb{D} \to X$ by setting $\hat{\gamma}_1(\tau) = \gamma(\vartheta(\tau))$ for all $\tau \in \mathbb{D}$. Furthermore, for any $M \ge M_0$ with

$$M_0 := \max_{\tau\in\mathbb{S}} \max_{\tilde{\tau}\in\mathbb{D}/2} \frac{b-a}{|\tilde{\tau}-\tau|} = 2(b-a) > 0,$$

we introduce a continuous function $\hat{\gamma}_2 : \mathbb{D} \to \mathbb{R}$ by setting

$$\hat{\gamma}_2(\tau) := \sup\{f(\hat{\gamma}_1(\tilde{\tau})) - M|\tau - \tilde{\tau}| : \tilde{\tau} \in \mathbb{D}\}.$$

Of course $f(\hat{\gamma}_1(\tau)) \le \hat{\gamma}_2(\tau)$ for $\tau \in \mathbb{D}$ and, by an easy check,

$$\max_{\tau\in\mathbb{D}} \hat{\gamma}_2(\tau) = \alpha.$$

Furthermore, being $M \ge M_0$ and $f(\hat{\gamma}_1)|_{\overline{\mathbb{D}\backslash\mathbb{D}/2}} \le a$, it is readily seen that $\hat{\gamma}_2(\tau) \le a$ for all $\tau \in \mathbb{S}$. Therefore, taking into account that by construction $\hat{\gamma}_1|_{\mathbb{S}} = \gamma \circ \vartheta|_{\mathbb{S}} = \gamma|_{\mathbb{S}} \in \Gamma_0$, it follows that $\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2) \in \hat{\Gamma}$, yielding

$$\inf_{\hat{\gamma}\in\hat{\Gamma}}\sup_{\tau\in\mathbb{D}}\mathcal{G}_f(\hat{\gamma}(\tau)) \le \sup_{\tau\in\mathbb{D}}\mathcal{G}_f(\hat{\gamma}(\tau)) = \alpha = \sup_{\tau\in\mathbb{D}} f(\gamma(\tau)) \le b,$$

which proves (3.8) by the arbitrariness of b. On the contrary, given d with

$$d > \inf_{\hat{\gamma}\in\hat{\Gamma}}\sup_{\tau\in\mathbb{D}}\mathcal{G}_f(\hat{\gamma}(\tau)),$$

we find $\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2) \in \hat{\Gamma}$ with

$$\sup_{\tau\in\mathbb{D}}\mathcal{G}_f(\hat{\gamma}(\tau)) \le d.$$

Then, we have $\hat{\gamma}_1 \in \Gamma$ and $f(\hat{\gamma}_1(\tau)) \le \hat{\gamma}_2(\tau) = \mathcal{G}_f(\hat{\gamma}(\tau)) \le d$, for all $\tau \in \mathbb{D}$. In particular we get $c \le d$, yielding the desired inequality by the arbitrariness of d. This concludes the proof of formula (3.7). At this point, in light of (3.6) and (3.7), given $\varepsilon \in (0, (c-a)/2)$, $\delta > 0$ and $\gamma \in \Gamma$ with $\sup_{\tau\in\mathbb{D}} f(\gamma(\tau)) \le c + \varepsilon$, if $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are defined as before, we have $\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2) \in \hat{\Gamma}$ with $\hat{\gamma}_1(\mathbb{D}) = \gamma(\mathbb{D})$ and

$$\sup_{\tau\in\mathbb{D}}\mathcal{G}_f(\hat{\gamma}(\tau)) \le c + \varepsilon,$$

and we can apply the theorem (cf. (3.4)) to the continuous function $\mathcal{G}_f$, yielding the existence of a pair $(u, \lambda) \in \mathrm{epi}(f)$ such that $c - 2\varepsilon \leq \lambda \leq c + 2\varepsilon$ and

$$\mathrm{dist}_\rho((u, \lambda), \hat{\gamma}(\mathbb{D}) \cap \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon])) \leq 2\delta, \quad |d_\rho \mathcal{G}_f|(u, \lambda) \leq \varepsilon/\delta. \qquad (3.9)$$

Now, by choosing $\rho := \frac{2\sqrt{2}\delta}{3\varepsilon}$ for the metric in $\mathrm{epi}(f)$, we have

$$|d_\rho \mathcal{G}_f|(u, \lambda) \leq \varepsilon/\delta < \frac{1}{\rho}.$$

Therefore, by virtue of Proposition 3.7 and in light of condition (3.2), we deduce that $\lambda = f(u)$, which yields

$$c - 2\varepsilon \leq f(u) \leq c + 2\varepsilon, \quad \mathrm{dist}(u, \gamma(\mathbb{D}) \cap f^{-1}([c - 3\varepsilon, c + 3\varepsilon])) \leq 3\delta. \qquad (3.10)$$

Concerning the second assertion, observe that from the first inequality of (3.9), replacing δ with a slightly larger δ if necessary, there exists $\tau \in \mathbb{D}$ such that

$$d(u, \hat{\gamma}_1(\tau)) \leq 2\delta, \quad c - 2\varepsilon \leq \hat{\gamma}_2(\tau) \leq c + 2\varepsilon.$$

Now, by continuity, there exists $\delta' > 0$ such that

$$\forall \tilde{\tau} \in \mathbb{D} : |\tilde{\tau} - \tau| \leq \delta' \implies d(\hat{\gamma}_1(\tilde{\tau}), \hat{\gamma}_1(\tau)) \leq \delta, \quad c - 3\varepsilon \leq \hat{\gamma}_2(\tilde{\tau}) \leq c + 3\varepsilon.$$

Observe now that, for any given $\mu \in \mathbb{R}$, it follows

$$f(\hat{\gamma}_1)|_{\{\tilde{\tau} \in \mathbb{D} : |\tilde{\tau} - \tau| \leq \delta'\}} \leq \mu \implies \hat{\gamma}_2(\tau) \leq \mu,$$

if $M \geq \max\{M_0, \frac{c + \varepsilon - \mu}{\delta'}\}$ in the definition of $\hat{\gamma}_2$. In fact, it holds

$$\forall \tilde{\tau} \in \mathbb{D} : |\tilde{\tau} - \tau| \leq \delta' \implies f(\hat{\gamma}_1(\tilde{\tau})) - M|\tau - \tilde{\tau}| \leq \mu,$$

$$\forall \tilde{\tau} \in \mathbb{D} : |\tilde{\tau} - \tau| > \delta' \implies f(\hat{\gamma}_1(\tilde{\tau})) - M|\tau - \tilde{\tau}| \leq c + \varepsilon - M\delta' \leq \mu.$$

Hence, since $\hat{\gamma}_2(\tau) > c - 3\varepsilon$, if $M \geq \max\{M_0, \frac{4\varepsilon}{\delta'}\}$ in the definition of $\hat{\gamma}_2$, we have

$$\exists \tilde{\tau} \in \mathbb{D} : |\tilde{\tau} - \tau| \leq \delta' \quad \text{and} \quad c - 3\varepsilon < f(\hat{\gamma}_1(\tilde{\tau})) \leq \hat{\gamma}_2(\tilde{\tau}) \leq c + 3\varepsilon.$$

Since $\hat{\gamma}_1(\tilde{\tau}) \in \gamma(\mathbb{D}) \cap f^{-1}([c - 3\varepsilon, c + 3\varepsilon])$, we obtain

$$\mathrm{dist}(u, \gamma(\mathbb{D}) \cap f^{-1}([c - 3\varepsilon, c + 3\varepsilon]))$$

$$\leq d(u, \hat{\gamma}_1(\tilde{\tau})) \leq d(u, \hat{\gamma}_1(\tau)) + d(\hat{\gamma}_1(\tau), \hat{\gamma}_1(\tilde{\tau})) \leq 3\delta.$$

Finally, by virtue of Proposition 3.7, it holds

$$|df|(u) = \frac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}} = \frac{|d_\rho \mathcal{G}_f|(u, f(u))}{\sqrt{1 - \rho^2 |d_\rho \mathcal{G}_f|(u, f(u))^2}}$$

$$\leq \frac{\varepsilon/\delta}{\sqrt{1 - \rho^2 \varepsilon^2/\delta^2}} = 3\varepsilon/\delta.$$

This concludes the proof. $\qquad \qquad \square$

3.3. *The symmetric minimax theorem*

The main abstract tool of the paper is a symmetric version of Theorem 3.9, namely the following

Theorem 3.10. *Let X and V be two Banach spaces, $S \subset X$, $*$ and $\mathcal{H}_*$ satisfying the requirements of the abstract symmetrization framework. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function satisfying* (3.2). *Let $\mathbb{D}$ and $\mathbb{S}$ denote the closed unit ball and the sphere in $\mathbb{R}^N$ respectively and $\Gamma_0 \subset C(\mathbb{S}, X)$. Let us define*

$$\Gamma = \{\gamma \in C(\mathbb{D}, X) : \gamma|_\mathbb{S} \in \Gamma_0\}.$$

Assume that

$$+\infty > c = \inf_{\gamma \in \Gamma} \sup_{\tau \in \mathbb{D}} f(\gamma(\tau)) > \sup_{\gamma_0 \in \Gamma_0} \sup_{\tau \in \mathbb{S}} f(\gamma_0(\tau)) = a,$$

and that

$$\forall H \in \mathcal{H}_*, \quad \forall u \in S : f(u^H) \leq f(u).$$

Then, for every $\varepsilon \in (0, (c - a)/3)$, every $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{\tau \in \mathbb{D}} f(\gamma(\tau)) \leq c + \varepsilon, \quad \gamma(\mathbb{D}) \subset S, \quad \gamma|_\mathbb{S}^{H_0} \in \Gamma_0 \quad \text{for some } H_0 \in \mathcal{H}_*,$$

there exists $u \in X$ such that

$$c - 2\varepsilon \leq f(u) \leq c + 2\varepsilon, \quad |df|(u) \leq 3\varepsilon/\delta, \quad \|u - u^*\|_V \leq 3((1 + C_\Theta)K + 1)\delta,$$

$$(3.11)$$

being K the norm of the embedding map $i : X \to V$ and C_Θ the Lipschitz constant of Θ.

Proof. Let $\varepsilon \in (0, (c - a)/3)$, $\delta > 0$ and $\gamma \in \Gamma$ satisfying the assumptions. Moreover, let $\vartheta : \mathbb{D} \to \mathbb{D}$ be the continuous function introduced in the proof of Theorem 3.9, and consider the function $\eta : \mathbb{D} \to X$, defined as $\eta(\tau) := \gamma(\vartheta(\tau))$ for all $\tau \in \mathbb{D}$. Then $\eta \in \Gamma$, we have $\eta(\mathbb{D}) = \gamma(\vartheta(\mathbb{D})) = \gamma(\mathbb{D}) \subset S$ and, setting

$$M_1 := \overline{(f \circ \eta)^{-1}([c - 3\varepsilon, c + \varepsilon])},$$

$M_1 \subset \mathbb{D}$ is of course closed and $M_1 \cap \mathbb{S} = \emptyset$. In fact, assume by contradiction that this is not the case and let $\tau \in M_1 \cap \mathbb{S}$. Then

$$\tau \in \mathbb{S}, \quad \tau = \lim_j \tau_j, \quad c - 3\varepsilon \leq f(\gamma(\vartheta(\tau_j))) \leq c + \varepsilon, \quad \text{for all } j \geq 1.$$

In particular, $\tau_j \in \overline{\mathbb{D} \backslash \mathbb{D}/2}$ eventually for $j \geq 1$, so that $\vartheta(\tau_j) \in \mathbb{S}$ eventually for $j \geq 1$. Therefore, for such $j \geq 1$, we obtain

$$c - 3\varepsilon \leq f(\gamma(\vartheta(\tau_j))) \leq \sup_{\tau \in \mathbb{S}} f(\gamma(\tau)) \leq \sup_{\gamma_0 \in \Gamma_0} \sup_{\tau \in \mathbb{S}} f(\gamma_0(\tau)) = a < c - 3\varepsilon$$

yielding the desired contradiction. Now, from Proposition 2.4 (applied with the choice $M = \mathbb{D}$ and $M_0 = \mathbb{S}$) there exists a curve $\tilde{\eta} \in C(\mathbb{D}, X)$ with $\tilde{\eta}|_{\mathbb{S}} = \eta|_{\mathbb{S}}^{H_0} = \gamma|_{\mathbb{S}}^{H_0} \in \Gamma_0$ for the polarizer H_0 (so that $\tilde{\eta} \in \Gamma$) such that $\|\tilde{\eta}(\tau) - \eta(\tau)^*\|_V \leq 3\delta$, for all $\tau \in M_1$. Notice that, by construction, $\tilde{\eta}(\tau)^* = \eta(\tau)^*$ and $f(\tilde{\eta}(\tau)) \leq f(\eta(\tau))$ for every $\tau \in \mathbb{D}$, as $\tilde{\eta}$ is built from η through polarizations. Hence, we obtain

$$\sup_{\tau \in \mathbb{D}} f(\tilde{\eta}(\tau)) \leq \sup_{\tau \in \mathbb{D}} f(\eta(\tau)) = \sup_{\tau \in \mathbb{D}} f(\gamma(\tau)) \leq c + \varepsilon.$$

By applying Theorem 3.9 to $\tilde{\eta}$ and since

$$\tilde{\eta}(\mathbb{D}) \cap f^{-1}([c - 3\varepsilon, c + 3\varepsilon]) \subset \tilde{\eta}(M_1),$$

there exists $u \in X$ such that $\mathrm{dist}(u, \tilde{\eta}(M_1)) \leq 3\delta$ and the first two inequalities in the above conclusion (3.11) hold. According to the abstract symmetrization framework u^* is well defined. Furthermore, in light of inequality (2.1) (cf. Remark 2.1), for all $\tau \in M_1$ we have

$$\|\eta(\tau)^* - u^*\|_V = \|\tilde{\eta}(\tau)^* - u^*\|_V \leq C_\Theta \|u - \tilde{\eta}(\tau)\|_V.$$

Then, the last assertion in (3.11) follows by adding and subtracting $\tilde{\eta}(\tau)$ and $\eta(\tau)^*$ with $\tau \in M_1$, as in the end of [29, proof of Theorem 3.2], namely

$$\|u - u^*\|_V \leq \inf_{\tau \in M_1} [\|u - \tilde{\eta}(\tau)\|_V + \|\tilde{\eta}(\tau) - \eta(\tau)^*\|_V + \|\eta(\tau)^* - u^*\|_V]$$

$$\leq \inf_{\tau \in M_1} [(1 + C_\Theta)\|u - \tilde{\eta}(\tau)\|_V + \|\tilde{\eta}(\tau) - \eta(\tau)^*\|_V]$$

$$\leq 3((1 + C_\Theta)K + 1)\delta.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.11. Let X and V be two Banach spaces such that X is continuously embedded in V and let $S \subset X$. We consider a symmetrization map $* : S \to V$ which satisfies the requirements of the abstract symmetrization framework. Theorem 3.10 provides, in some sense, a useful alternative to concentration compactness. In fact, for a broad range of lower semi-continuous functionals $f : X \to \mathbb{R} \cup \{+\infty\}$ possessing a mountain pass geometry, Theorem 3.10 yields a sequence of functions $(u_h) \subset X$ such that, as $h \to \infty$,

$$f(u_h) \to c, \quad |df|(u_h) \to 0, \quad \|u_h - u_h^*\|_V \to 0. \tag{3.12}$$

It is often the case that the first two limits yield the boundedness of (u_h) in X, so that $u_h \to u$ weakly in X for some $u \in X$ and that the symmetric sequence $(u_h^*) \subset X_r$ converges strongly, up to a subsequence, to some $v \in X_r$ in some

subspace V' with $V \subset V'$ with continuous injection $i : V \to V'$. In particular,

$$u_h^* \to v \quad \text{in } V' \text{ as } h \to \infty, \quad \|u_h - u_h^*\|_{V'} \leq C\|u_h - u_h^*\|_V \to 0,$$

which yields

$$u_h \to u \text{ weakly in } X \quad \text{and} \quad \text{strongly in } V'. \tag{3.13}$$

This conclusion is often sufficient in order to prove, after some work, that the Palais–Smale sequence (u_h) converges to u strongly in X. As a concrete functional framework one can think, for instance, to the case (here Ω can be the whole $\mathbb{R}^N$) where

$$X = W_0^{1,p}(\Omega), \quad V = L^p \cap L^{p^*}(\Omega), \quad V' = L^m(\Omega), \quad p < m < p^*.$$

Therefore, if (u_h) is bounded in $W_0^{1,p}(\Omega)$, the sequence (u_h^*) is bounded in $W_0^{1,p}(\Omega)$ too by the Polya–Szegö inequality and compact in $L^m(\Omega)$ with $p < m < p^*$ in light of [3, Theorem A.I′, p. 341]. Finally, the injection $i : L^p \cap L^{p^*}(\Omega) \to L^m(\Omega)$ is, of course, continuous. For an application of conclusion (3.13) in the case $p = 2$, $\Omega = \mathbb{R}^N$ and $f \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$, see [29, Theorem 4.5].

Remark 3.12. As pointed out in [29] the condition that $\gamma|_{\mathbb{S}}^{H_0} \in \Gamma_0$ for some polarizer $H_0 \in \mathcal{H}_*$ imposes a minimality condition on the energy levels on which one can guarantee the symmetry of critical points.

4. Proof of Theorem 1.2

4.1. *Some preliminary lemmas*

Given a fixed function u in $H_0^1(B_1)$, we define the following subspace of $H_0^1(B_1)$

$$W_u = \{v \in H_0^1(B_1) : j_\xi(u, |\nabla u|) \cdot \nabla v \in L^1(B_1) \text{ and } j_s(u, |\nabla u|)v \in L^1(B_1)\}. \tag{4.1}$$

The space W_u is dense in $H_0^1(B_1)$. It was originally introduced in [19] and subsequently used also throughout [28]. We give the definition of generalized solution.

Definition 4.1. We say that u is a generalized solution to (P) if $u \in H_0^1(B_1)$ and it results $j_\xi(u, |\nabla u|) \cdot \nabla u \in L^1(B_1)$, $j_s(u, |\nabla u|)u \in L^1(B_1)$ and

$$\int_{B_1} j_\xi(u, |\nabla u|) \cdot \nabla v \, dx + \int_{B_1} j_s(u, |\nabla u|)v \, dx = \int_{B_1} g(|x|, u)v \, dx, \quad \forall v \in W_u.$$

We recall some preliminary results.

Lemma 4.2. *Assume that conditions* (1.4)–(1.15) *hold. If $u \in \mathrm{dom}(f)$ is a critical point of f, namely $|df|(u) = 0$, then u is a generalized solution to*

$$\begin{cases} -\mathrm{div}(j_\xi(u, |\nabla u|)) + j_s(u, |\nabla u|) = g(|x|, u), & \text{in } B_1, \\ u = 0, & \text{on } \partial B_1. \end{cases}$$

Furthermore, if $j_\xi(u, |\nabla u|) \cdot \nabla u \in L^1(B_1)$, then u is a distributional solution.

Proof. Combine [28, Proposition 6.4 and Theorem 4.10].   □

Lemma 4.3. *Assume that conditions* (1.4)–(1.15) *hold. Then, for every* $(u, \xi) \in$ epi(J) *with* $f(u) < \xi$, *there holds* $|d\mathcal{G}_f|(u, \xi) = 1$.

Proof. See [28, Theorem 6.1].   □

Lemma 4.4. *Assume that conditions* (1.4)–(1.15) *hold. Then, there exists* $e \in H_0^1(B_1)$ *such that* $f(e) < 0$ *and* $\rho, \sigma > 0$ *such that* $f(u) \geq \sigma$ *for all* $u \in H_0^1(B_1)$ *with* $\|u\|_{H_0^1} = \rho$.

Proof. See [28, the beginning of the proof of Theorem 2.3].   □

Lemma 4.5. *Assume that conditions* (1.4)–(1.15) *hold. Then the functional* f *satisfies the* $(PS)_c$ *condition at every level* $c \in \mathbb{R}$.

Proof. See [28, Theorem 6.9].   □

Lemma 4.6. *Let* $u \in H_0^1(B_1, \mathbb{R}^+)$ *and let* H *be a given half-space in* $\mathcal{H}_*$. *Then*

$$\int_{B_1} j(u, |\nabla u|)dx = \int_{B_1} j(u^H, |\nabla u^H|)dx, \tag{4.2}$$

provided that $0 \in H$ *and that both integrals are finite. Furthermore, under* (1.12),

$$\int_{B_1} G(|x|, u)dx \leq \int_{B_1} G(|x|, u^H)dx.$$

Proof. See e.g. [21, Lemma 2.5] and [29, Proposition 2.3], respectively. The statements therein are provided for functions $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$, for instance (4.2) reads as

$$\int_{\mathbb{R}^N} j(\tilde{u}, |\nabla \tilde{u}|)dx = \int_{\mathbb{R}^N} j(\tilde{u}^H, |\nabla \tilde{u}^H|)dx. \tag{4.3}$$

On the other hand, given a function u in $H_0^1(B_1, \mathbb{R}^+)$, if $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ is the extension of u by zero outside B_1, we have $\tilde{u} \in H_0^1(B_1, \mathbb{R}^+)$ and $\tilde{u}^H|_{\mathbb{R}^N \setminus B_1} = 0$ for all H in $\mathcal{H}_*$. In fact, if $x \in (\mathbb{R}^N \setminus B_1) \cap H$, then $\tilde{u}^H(x) = \max\{\tilde{u}(x), \tilde{u}(x_H)\} = 0$, being $x, x_H \in \mathbb{R}^N \setminus B_1$ (since 0 belongs to H). If, instead, $x \in (\mathbb{R}^N \setminus B_1) \cap (\mathbb{R}^N \setminus H)$, then $\tilde{u}^H(x) = \min\{\tilde{u}(x), \tilde{u}(x_H)\} = \min\{0, \tilde{u}(x_H)\} = 0$, being $\tilde{u} \geq 0$. The, recalling that by definition $u^H := \tilde{u}^H|_{B_1^H} = \tilde{u}^H|_{B_1}$ (since $B^H = B$ for all H in $\mathcal{H}_*$), the desired conclusion (4.2) then follows from (4.3), being $j(s, 0) = 0$.   □

4.2. *Proof of Theorem* 1.2 *concluded*

In view of Lemma 4.6 we have $f(u^H) \leq f(u)$, for every $u \in H_0^1(B_1, \mathbb{R}^+)$ and all polarizer $H \in \mathcal{H}_*$. This holds for all $u \in H_0^1(B_1)$ as well. In fact, notice that, since for sign changing functions $u^H := |u|^H$, taking into account assumptions (1.8)

and (1.13), we obtain that

$$f(u^H) = f(|u|^H) \le f(|u|) \le f(u), \quad \text{for all } u \in H_0^1(B_1) \quad \text{and} \quad H \in \mathcal{H}_*. \quad (4.4)$$

By virtue of Lemma 4.3, we are allowed to apply the abstract symmetric minimax Theorem 3.10 to the lower semi-continuous functional $f : X \to \mathbb{R} \cup \{+\infty\}$ by choosing $X = S = H_0^1(B_1)$, $V = L^2 \cap L^{2^*}(B_1)$, $\mathbb{D} = [0,1]$, $\mathbb{S} = \{0,1\}$,

$$\Gamma = \{\gamma \in C([0,1], H_0^1(B_1)) : \gamma|_{\{0,1\}} \in \Gamma_0\}$$

and $\Gamma_0 = \{0, e \colon e \in H_0^1(B_1) \text{ is such that } f(e) < 0\}$. If follows that $\Gamma \neq \emptyset$ in light of Lemma 4.4, also yielding $c > 0 = a$ by definition of a and c in Theorem 3.10. Of course, it holds $f(0^H) = f(0) = 0$ and $f(e^H) \le f(e) < 0$, for any polarizer H, so that $\gamma(0)^H, \gamma(1)^H \in \Gamma_0$, for any $\gamma \in \Gamma$. Moreover, take $\varepsilon = \varepsilon_h = 1/h^2$, $\delta = \delta_h = 1/h$, $\gamma = \gamma_h \in C([0,1], H_0^1(B_1))$ such that

$$\sup_{\tau \in [0,1]} f(\gamma_h(\tau)) \le c + \frac{1}{h^2}. \quad (4.5)$$

Hence, Theorem 3.10 yields a sequence $(u_h) \subset H_0^1(B_1)$ such that

$$c - \frac{2}{h^2} \le f(u_h) \le c + \frac{2}{h^2}, \quad |df|(u_h) \le \frac{8}{h}, \quad \|u_h - u_h^*\|_{L^2(B_1)} \le \frac{3((1+C_\Theta)K + 1)}{h}.$$

In particular, (u_h) is a Palais–Smale sequence at level c. By means of Lemma 4.5, up to a subsequence, (u_h) strongly converges in $H_0^1(B_1)$ to some $\hat{u} \in H_0^1(B_1)$ with $f(\hat{u}) = c > 0$ (hence $\hat{u}$ is nontrivial) and $|df|(\hat{u}) = 0$. In light of Lemma 4.2, it follows that $\hat{u}$ is a generalized solution of the problem. Taking into account the growth condition (1.9) on g, by virtue of [28, Theorem 7.1(b)] it follows that $\hat{u} \in L^\infty(\Omega)$. Now, by combining assumptions (1.4) and (1.5), it holds $|j_\xi(s, |\xi|)| \le 4\alpha(|s|)|\xi|$, for every $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$ (cf. [28, Remark 4.1]). Then, again by Lemma 4.2, it follows that $\hat{u}$ is a distributional solution, being

$$\int_{B_1} |j_\xi(\hat{u}, |\nabla \hat{u}|) \cdot \nabla \hat{u}| dx \le \int_{B_1} 4\alpha(\hat{u})|\nabla \hat{u}|^2 dx \le 4\alpha(M) \int_{B_1} |\nabla \hat{u}|^2 dx < \infty,$$

where $M = \text{esssup}_{B_1} |u|$. Finally, from $\|u_h - u_h^*\|_{L^2(B_1)} \to 0$ and $u_h \to \hat{u}$ in $L^2(B_1)$ as $h \to \infty$, we get $u_h^* \to \hat{u}^*$ and $u_h^* \to \hat{u}$ in $L^2(B_1)$, so that $\hat{u} = \hat{u}^*$ by uniqueness of the limit. This concludes the proof. $\qquad \square$

Remark 4.7. In the proof of Theorem 1.2, we applied the abstract machinery, Theorem 3.10, with the choice $X = S = H_0^1(B_1)$ by defining $u^H := |u|^H$ and $u^* := |u|^*$, for all $u \in X$ (see also Remark 2.2). In a similar fashion, we could have applied Theorem 3.10 with the choice $X = H_0^1(B_1)$ and $S = H_0^1(B_1, \mathbb{R}^+) \subset X$ by only extending $* : S \to V$ to $* : X \to V$ (according to the abstract symmetrization framework, cf. Sec. 2.1) by setting $u^* = |u|^*$, hence taking $\Theta(u) := |u|$, for all $u \in X$, which is Lipschitz of constant 1 as a map from $(X, \|\cdot\|_V)$ to $(S, \|\cdot\|_V)$. In this case, instead of exploiting assumptions (1.8) and (1.13) (which guarantee

$f(|u|) \leq f(u)$, for all u in X) in the verification of (4.4) we would have used them to find the curves $\gamma_h \in \Gamma$ with $\gamma_h([0,1]) \subset S$ which satisfy inequality (4.5) by just taking $\gamma_h(\tau) := |\tilde{\gamma}_h(\tau)|$ for all $\tau \in [0,1]$, being $\tilde{\gamma}_h \in \Gamma$ some curve which satisfies inequality (4.5) by the definition of the minimax value c. In fact, in light of conditions (1.8) and (1.13), we obtain $f(\gamma_h(\tau)) = f(|\tilde{\gamma}_h(\tau)|) \leq f(\tilde{\gamma}_h(\tau)) \leq c + \frac{1}{h^2}$, for all h and $\tau \in [0,1]$.

Acknowledgments

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