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Symmetry Results for Nonvariational Quasi-Linear Elliptic Systems

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Abstract

By virtue of a weak comparison principle in small domains we prove axial symmetry in convex and symmetric smooth bounded domains as well as radial symmetry in balls for regular solutions of a class of quasi-linear elliptic systems in non-variational form. Moreover, in the two dimensional case, we study the system when set in a half-space.

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1 Introduction and main results

The aim of this paper is to get some symmetry and monotonicity results for the solutions $(u,v)\in C^{1,\alpha}(\overline{\Omega})\times C^{1,\alpha}(\overline{\Omega})$ to the following quasi-linear elliptic system

$$\begin{cases} -\Delta_p u = f(u,v) & \text{in } \Omega, \\ -\Delta_m v = g(u,v) & \text{in } \Omega, \\ u>0,\ v>0 & \text{in } \Omega, \\ u=0,\ v=0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N\geq 2$ and $\Delta_p=\operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplacian operator, $|\cdot|$ denoting the standard Euclidean norm in $\mathbb{R}^N$. Furthermore, in the two-dimensional case, we shall also consider the system defined in the half-space. Problem 1.1 is the stationary system corresponding to the parabolic system

$$\begin{cases} u_t-\Delta_p u = f(u,v) & \text{in } \Omega\times(0,\infty), \\ v_t-\Delta_m v = g(u,v) & \text{in } \Omega\times(0,\infty), \end{cases}$$

where the adoption of the p-Laplacian operator inside the diffusion term arises in various applications where the standard linear heat operator $u_t-\Delta$ is replaced by a nonlinear diffusion with gradient dependent diffusivity. The equations in the above system usually arise in the theory of non-Newtonian filtration fluids, in turbulent flows in porous media and in glaciology (cf. [2]).

System (1.1) does not necessarily admit a variational structure and it has been previously studied in the literature both from the point of view of existence and symmetry of smooth solutions. For the existence of a positive radially symmetric C^2 solution in the particular case where $f(u,v)=u^\alpha v^\beta$ and $g(u,v)=u^\gamma v^\delta$ for suitable values of $\alpha,\beta,\gamma,\delta\geq 0$, we refer the reader to [6] and to the reference therein. Concerning the symmetry properties (and a priori estimates) of any smooth solution of (1.1) in the special case $f(u,v)=f(v)$ and $g(u,v)=g(u)$ are positive and nondecreasing functions, we refer to [10] (see also [1]).

In our main results we shall always assume on f,g that

$$f,g\in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2_+) \quad \text{and} \quad f(s,t)>0, \quad g(s,t)>0, \quad \text{for all } s,t>0, \tag{1.2}$$

and that they satisfy the monotonicity (also known as *cooperativity*) conditions

$$\frac{\partial f}{\partial t}(s,t)\geq 0 \quad \text{and} \quad \frac{\partial g}{\partial s}(s,t)\geq 0, \quad \text{for all } s,t>0. \tag{1.3}$$

The sign assumptions (1.2) and (1.3) are natural in the study of this class of problems. Furthermore, it is shown in [16] that conditions (1.3) are, actually, necessary in order to obtain symmetry results for the solutions to (1.1). For useful regularity features of the solutions to (1.1), we refer the reader to [10, Section 2] where the regularity of the quasi-linear equation $-\Delta_p u=h(x)$ is investigated under the assumption that $h\in C^{0,\alpha}\cap W^{1,\sigma}_{\mathrm{loc}}(\Omega)$, where $\sigma\geq\max\{N/2,2\}$. In turn, the regularity properties of (1.1) can be obtained by applying the results of [10] to the choices $h(x) = f(u(x), v(x))$ and $h(x) = g(u(x), v(x))$ where f, g are locally Lipschitz. Under the same cooperativity condition (1.3), for the non-degenerate case $p = 2 = m$, we refer e.g. to [5, 12, 16] and references included.

In the following we present our symmetry results, which complete those of [10], first in the case where system (1.1) is set is a smooth bounded symmetric domain and, then, when it is set in a half-space of $\mathbb{R}^2$.

Our results are based on the use of a refined version of the Moving Plane technique [15] (see also [13]). We will in particular use the moving plane procedure as improved in [4]. In the case of the half-space of $\mathbb{R}^2$, we exploit a geometric idea as in [11], which is more related to the techniques developed in [3].

1.1 System in a smooth bounded domain

In a bounded domain Ω, we consider solutions $u, v \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ to the non-variational quasi-linear system

$$\begin{cases} -\Delta_p u = f(u,v) & \text{in } \Omega, \\ -\Delta_m v = g(u,v) & \text{in } \Omega, \\ u > 0, \ v > 0 & \text{in } \Omega, \\ u = 0, \ v = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.4}$$

Furthermore, we assume that (1.2) and that the cooperativity condition (1.3) is satisfied. Let us set

$$Z_u \equiv \{x \in \Omega : \nabla u(x) = 0\}, \qquad Z_v \equiv \{x \in \Omega : \nabla v(x) = 0\}.$$

The first main result of the paper is the following

Theorem 1.1 *Assume that* (1.2) *and* (1.3) *hold. If* Ω *is convex with respect to the* x_1*-direction, and symmetric with respect to the hyperplane* $T_0 = \{x_1 = 0\}$*, then* u *and* v *are symmetric and nondecreasing in the* x_1*-direction in* $\Omega_0 = \{x_1 < 0\}$*, with*

$$\frac{\partial u}{\partial x_1}(x) > 0 \quad \text{in } \Omega_0 \setminus Z_u, \qquad \frac{\partial v}{\partial x_1}(x) > 0 \quad \text{in } \Omega_0 \setminus Z_v.$$

In particular, if Ω *is a ball, then* u *and* v *are radially symmetric with* $\frac{\partial u}{\partial r}(r) < 0$ *and* $\frac{\partial v}{\partial r}(r) < 0$.

Notice that this result holds true under the same assumptions that were considered in [10] where the particular case $f(u, v) = f(v)$ and $g(u, v) = g(u)$ is considered. More precisely, no monotonicity is requested on the function f (resp. g) with respect to u (resp. v).

The second result is an improvement under some restrictions on the values of p, m, of the previous Theorem 1.1.

Theorem 1.2 *Assume that* (1.2) *and* (1.3) *hold and* $\frac{2N+2}{N+2} < p, m < \infty$. *If* Ω *is convex with respect to the* x_1*-direction and symmetric with respect to the hyperplane* $T_0 = \{x_1 = 0\}$, *then* u *and* v *are symmetric and nondecreasing in the* x_1*-direction in* $\Omega_0 = \{x_1 < 0\}$ *with*

$$\frac{\partial u}{\partial x_1}(x) > 0 \quad \text{in } \Omega_0, \qquad \frac{\partial v}{\partial x_1}(x) > 0 \quad \text{in } \Omega_0.$$

In particular $Z_u \subset T_0$ *and* $Z_v \subset T_0$. *Therefore if for* N *orthogonal directions* e_i *the domain* Ω *is symmetric with respect to any hyperplane* $T_0^{e_i} = \{x \cdot e_i = 0\}$, *then*

$$Z_u = Z_v = \{0\}, \tag{1.5}$$

assuming that 0 *is the center of symmetry.*

1.2 System on a half-space of $\mathbb{R}^2$

Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and consider the system

$$\begin{cases} -\Delta_p u = f(u, v) & \text{in } \mathbb{H}, \\ -\Delta_m v = g(u, v) & \text{in } \mathbb{H}, \\ u > 0, \ v > 0 & \text{in } \mathbb{H}, \\ u = 0, \ v = 0 & \text{on } \partial\mathbb{H}. \end{cases} \tag{1.6}$$

Then we have the following monotonicity result

Theorem 1.3 *Let* (u, v) *be a nontrivial weak* $C^{1,\alpha}_{\text{loc}}(\mathbb{H})$ *solution of* (1.6). *Assume that* (1.2) *and* (1.3) *hold and let* $\frac{3}{2} < p, m < \infty$. *Then*

$$\frac{\partial u}{\partial y}(x, y) > 0 \quad \text{and} \quad \frac{\partial v}{\partial y}(x, y) > 0 \qquad \text{for all } (x, y) \in \overline{\mathbb{H}}.$$

We prove Theorem 1.3 by exploiting a weak comparison principle in small domains (see Proposition 2.1), and some techniques developed in [11], where the monotonicity of the solutions was used to prove some Liouville type theorems for Lane-Emden-Fowler type equations.

Notations.

1. For $n \geq 1$, we denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^n$.
2. $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) is the set of positive (resp. negative) real values.
3. For $p > 1$ we denote by $L^p(\mathbb{R}^n)$ the space of measurable functions u such that $\int_\Omega |u|^p dx < \infty$. The norm $(\int_\Omega |u|^p dx)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|\cdot\|_{L^p(\Omega)}$.

4. For $s \in \mathbb{N}$, we denote by $H^s(\Omega)$ the Sobolev space of functions u in $L^2(\Omega)$ having generalized partial derivatives $\partial_i^k u$ in $L^2(\Omega)$ for all $i = 1, \dots, n$ and any $0 \le k \le s$.

5. The norm $(\int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx)^{1/2}$ in $W_0^{1,p}(\Omega)$ is denoted by $\|\cdot\|_{W_0^{1,p}(\Omega)}$.

6. We denote by $C_0^\infty(\Omega)$ the set of smooth compactly supported functions in Ω.

7. We denote by $B(x_0, R)$ a ball of center x_0 and radius R.

8. We denote by $\mathcal{L}(E)$ the Lebesgue measure of the set $E \subset \mathbb{R}^n$.

2 Proofs of the results

In the next section we shall prove the main results of the paper.

2.1 Proof of Theorem 1.1

First, we have the following weak comparison principle in small sub-domains Ω_0 of Ω.

Proposition 2.1 *Assume that $u, v \in C^1(\overline{\Omega})$ and $\tilde{u}, \tilde{v} \in C^1(\overline{\Omega})$ are solutions to* (1.4). *Let Ω_0 be a bounded smooth domain of $\mathbb{R}^N$ such that $\Omega_0 \subset \Omega$. Then there exists a positive number δ, depending upon f, g, $\|u\|_\infty, \|v\|_\infty, \|\tilde{u}\|_\infty, \|\tilde{v}\|_\infty$, such that if*

$$\mathcal{L}(\Omega_0) \le \delta, \quad u \le \tilde{u} \quad on\ \partial\Omega_0, \quad v \le \tilde{v} \quad on\ \partial\Omega_0,$$

then

$$u \le \tilde{u} \quad on\ \Omega_0, \qquad v \le \tilde{v} \quad on\ \Omega_0.$$

Proof. We consider four different cases:

1. $p > 2$ and $m > 2$;

2. $p \le 2$ and $m > 2$;

3. $p > 2$ and $m \le 2$;

4. $p < 2$ and $m < 2$.

We will show that the result follows in cases (1) and (2), the others cases being similar. We will denote by C a generic positive constant, which may change from line to line throughout the proof.

Case 1. ($p > 2$ and $m > 2$). Let us set

$$U = (u - \tilde{u})^+ \quad \text{and} \quad V = (v - \tilde{v})^+.$$

We will prove the result by showing that, actually, it holds $U \equiv V \equiv 0$. Since both $u \leq \tilde{u}$ on $\partial\Omega_0$ and $v \leq \tilde{v}$ on $\partial\Omega_0$ then the functions U, V belong to $W_0^{1,p}(\Omega_0)$. Therefore, let us consider the variational formulations of the equations of (1.4).

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\varphi)dx = \int_\Omega f(u,v)\varphi dx, \quad \forall\varphi \in C_c^\infty(\Omega), \tag{1.7}$$

$$\int_\Omega |\nabla \tilde{u}|^{p-2}(\nabla \tilde{u}, \nabla\varphi)dx = \int_\Omega f(\tilde{u},\tilde{v})\varphi dx, \quad \forall\varphi \in C_c^\infty(\Omega), \tag{1.8}$$

$$\int_\Omega |\nabla v|^{m-2}(\nabla v, \nabla\varphi)dx = \int_\Omega g(u,v)\varphi dx, \quad \forall\varphi \in C_c^\infty(\Omega), \tag{1.9}$$

$$\int_\Omega |\nabla \tilde{v}|^{m-2}(\nabla \tilde{v}, \nabla\varphi)dx = \int_\Omega g(\tilde{u},\tilde{v})\varphi dx, \quad \forall\varphi \in C_c^\infty(\Omega). \tag{1.10}$$

By a density argument, we can put respectively $\varphi = U$ in equations (1.7) and (1.8) and $\varphi = V$ in equations (1.9) and (1.10). Subtracting, we get

$$\int_{\Omega_0} \big(|\nabla u|^{p-2}\nabla u - |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}, \nabla(u-\tilde{u})^+\big)dx = \int_{\Omega_0} [f(u,v) - f(\tilde{u},\tilde{v})](u-\tilde{u})^+ dx, \tag{1.11}$$

$$\int_{\Omega_0} \big(|\nabla v|^{m-2}\nabla v - |\nabla\tilde{v}|^{m-2}\nabla\tilde{v}, \nabla(v-\tilde{v})^+\big)dx = \int_{\Omega_0} [g(u,v) - g(\tilde{u},\tilde{v})](v-\tilde{v})^+ dx. \tag{1.12}$$

Now we use the following standard estimate

$$(|\eta|^{q-2}\eta - |\eta'|^{q-2}\eta', \eta - \eta') \geq C(|\eta| + |\eta'|)^{q-2}|\eta - \eta'|^2,$$

for all $\eta, \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$ and $q > 1$, from equations (1.11) and (1.12) one has that

$$\int_{\Omega_0} (|\nabla u| + |\nabla\tilde{u}|)^{p-2}|\nabla(u-\tilde{u})^+|^2 dx \leq C\int_{\Omega_0} [f(u,v) - f(\tilde{u},\tilde{v})](u-\tilde{u})^+ dx, \tag{1.13}$$

$$\int_{\Omega_0} (|\nabla v| + |\nabla\tilde{v}|)^{m-2}|\nabla(v-\tilde{v})^+|^2 dx \leq C\int_{\Omega_0} [g(u,v) - g(\tilde{u},\tilde{v})](v-\tilde{v})^+ dx. \tag{1.14}$$

Since f is locally lipschitz continuous and $\{t \mapsto f(s,t)\}$ is nondecreasing, from equation (1.13) it follows

$$\begin{aligned}
\int_{\Omega_0} |\nabla u|^{p-2}|\nabla(u-\tilde{u})^+|^2 dx &\leq C\int_{\Omega_0} \Big[\frac{f(u,v) - f(\tilde{u},v)}{u-\tilde{u}}\Big]((u-\tilde{u})^+)^2 dx \\
&\quad + C\int_{\Omega_0} \Big[\frac{f(\tilde{u},v) - f(\tilde{u},\tilde{v})}{(v-\tilde{v})^+}\Big](u-\tilde{u})^+(v-\tilde{v})^+ dx \\
&\leq C\Big(\int_{\Omega_0} ((u-\tilde{u})^+)^2 dx + \int_{\Omega_0} (u-\tilde{u})^+(v-\tilde{v})^+ dx\Big) \\
&\leq C\Big(\int_{\Omega_0} ((u-\tilde{u})^+)^2 dx + \int_{\Omega_0} ((v-\tilde{v})^+)^2 dx\Big),
\end{aligned} \tag{1.15}$$

where, of course, in the last inequality we have used Young's inequality. Arguing in the same fashion, since g is locally lipschitz continuous and $\{s \mapsto g(s,t)\}$ is nondecreasing, from equation (1.14) one deduces

$$\int_{\Omega_0} |\nabla v|^{m-2} |\nabla (v-\tilde{v})^+|^2 dx \le C\Big(\int_{\Omega_0} ((u-\tilde{u})^+)^2 dx + \int_{\Omega_0} ((v-\tilde{v})^+)^2 dx\Big). \quad (1.16)$$

We know that a weighted Poincaré inequality holds true (cf. [8]), that yields

$$\int_{\Omega_0} ((u-\tilde{u})^+)^2 dx \le C_1(\Omega_0) \int_{\Omega_0} |\nabla u|^{p-2} |\nabla (u-\tilde{u})^+|^2 dx, \quad (1.17)$$

$$\int_{\Omega_0} ((v-\tilde{v})^+)^2 dx \le C_2(\Omega_0) \int_{\Omega_0} |\nabla v|^{m-2} |\nabla (v-\tilde{v})^+|^2 dx, \quad (1.18)$$

where $C_1(\Omega_0) \to 0$, when $\mathcal{L}(\Omega_0) \to 0$, as well as $C_2(\Omega_0) \to 0$, for $\mathcal{L}(\Omega_0) \to 0$. In turn, by combining inequalities (1.15) and (1.16), and setting

$$C_{\Omega_0} = C \max\{C_1(\Omega_0), C_2(\Omega_0)\},$$

we conclude that

$$\begin{aligned}\int_{\Omega_0} |\nabla u|^{p-2} |\nabla (u-\tilde{u})^+|^2 dx &\le C_{\Omega_0}\Big(\int_{\Omega_0} |\nabla u|^{p-2} |\nabla (u-\tilde{u})^+|^2 dx \\ &\quad + \int_{\Omega_0} |\nabla v|^{m-2} |\nabla (v-\tilde{v})^+|^2 dx\Big), \\ \int_{\Omega_0} |\nabla v|^{m-2} |\nabla (v-\tilde{v})^+|^2 dx &\le C_{\Omega_0}\Big(\int_{\Omega_0} |\nabla u|^{p-2} |\nabla (u-\tilde{u})^+|^2 dx \\ &\quad + \int_{\Omega_0} |\nabla v|^{m-2} |\nabla (v-\tilde{v})^+|^2 dx\Big).\end{aligned}$$

By adding these equations, and setting

$$I(\Omega_0) = \int_{\Omega_0} |\nabla u|^{p-2} |\nabla (u-\tilde{u})^+|^2 dx + \int_{\Omega_0} |\nabla v|^{m-2} |\nabla (v-\tilde{v})^+|^2 dx,$$

we obtain

$$I(\Omega_0) \le C_{\Omega_0} I(\Omega_0). \quad (1.19)$$

Now, we choose the value of $\delta > 0$ so small that the condition $\mathcal{L}(\Omega_0) \le \delta$ implies $C_{\Omega_0} < 1$. Therefore, from equation (1.19), we get the desired contradiction. In turn, we get

$$(u-\tilde{u})^+ \equiv 0 \quad \text{and} \quad (v-\tilde{v})^+ \equiv 0,$$

concluding the proof in this case.

Case 2. ($p \le 2$ and $m > 2$). Since $p \le 2$ and $u \in C^{1,\alpha}(\overline{\Omega})$, then equation (1.13) gives

$$\int_{\Omega_0} |\nabla (u-\tilde{u})^+|^2 dx \le C \int_{\Omega_0} [f(u,v) - f(\tilde{u},\tilde{v})](u-\tilde{u})^+ dx. \quad (1.20)$$

Then, arguing as in the previous case, since $f(s,t)$ is locally lipschitz continuous and nondecreasing in t, via the standard Poincaré inequality and the weighted Poincaré inequality (1.18), from inequality (1.20) one has

$$\int_{\Omega_0} |\nabla(u-\tilde{u})^+|^2 dx \leq CC_1(\Omega_0)\Big(\int_{\Omega_0} |\nabla(u-\tilde{u})^+|^2 + \int_{\Omega_0} |\nabla v|^{m-2}|\nabla(v-\tilde{v})^+|^2 dx\Big).$$

In the very same way, one gets

$$\int_{\Omega_0} |\nabla v|^{m-2}|\nabla(v-\tilde{v})^+|^2 dx$$
$$\leq CC_2(\Omega_0)\Big(\int_{\Omega_0} |\nabla(u-\tilde{u})^+|^2 dx + \int_{\Omega_0} |\nabla v|^{m-2}|\nabla(v-\tilde{v})^+|^2 dx\Big).$$

Adding these equations, setting

$$J(\Omega_0) = \int_{\Omega_0} |\nabla(u-\tilde{u})^+|^2 + \int_{\Omega_0} |\nabla v|^{m-2}|\nabla(v-\tilde{v})^+|^2 dx,$$

yields immediately

$$J(\Omega_0) \leq C_{\Omega_0} J(\Omega_0).$$

Arguing as before for the case where $p, m > 2$, by choosing δ sufficiently small that $C_{\Omega_0} < 1$, we get the desired contradiction, concluding the proof.

Let us now recall the fundamental ingredients of the moving plane method. Let Ω be a bounded smooth domain contained in $\mathbb{R}^N$. Let us consider a direction, say x_1 for example. We set

$$T_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}.$$

Given $x \in \mathbb{R}^N$ and $\lambda < 0$ for semplicity, we define

$$x_\lambda := (2\lambda - x_1, x_2, \ldots, x_N), \quad u_\lambda(x) := u(x_\lambda),$$
$$v_\lambda(x) := v(x_\lambda), \quad \Omega_\lambda := \big\{x \in \Omega : x_1 < \lambda\big\}.$$

We also set

$$\Lambda := \sup\big\{\lambda \in \mathbb{R} : x \in \Omega_t \text{ implies } x_\lambda \in \Omega \text{ for all } t \leq \lambda\big\}, \qquad a := \inf_{x\in\Omega} x_1. \quad (1.21)$$

$$Z_{u,\lambda} := \{x \in \Omega_\lambda : \nabla u(x) = \nabla u_\lambda(x) = 0\}, \quad Z_{v,\lambda} := \{x \in \Omega_\lambda : \nabla v(x) = \nabla v_\lambda(x) = 0\}.$$

Proposition 2.2 *Assume that* (1.2) *and* (1.3) *hold, and* $1 < p, m < \infty$. *Let* $(u,v) \in C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega})$ *be a solution to system* (1.4) *and let* Λ *be as in* (1.21). *Then, for any* $a \leq \lambda \leq \Lambda$, *we have*

$$u(x) \leq u_\lambda(x) \quad \textit{and} \quad v(x) \leq v_\lambda(x), \qquad \textit{for all } x \in \Omega_\lambda. \quad (1.22)$$

Moreover, for any λ *such that* $a < \lambda < \Lambda$, *we have*

$$u(x) < u_\lambda(x), \qquad \textit{for all } x \in \Omega_\lambda \setminus Z_{u,\lambda}, \quad (1.23)$$

and

$$v(x) < v_\lambda(x), \qquad \textit{for all } x \in \Omega_\lambda \setminus Z_{v,\lambda}. \tag{1.24}$$

Finally, we have

$$\frac{\partial u}{\partial x_1}(x) \geq 0, \qquad \textit{for all } x \in \Omega_\Lambda, \tag{1.25}$$

where $Z_u = \{x \in \Omega : \nabla u(x) = 0\}$, *and*

$$\frac{\partial v}{\partial x_1}(x) \geq 0, \qquad \textit{for all } x \in \Omega_\Lambda. \tag{1.26}$$

Proof. For $a < \lambda < \Lambda$ and λ sufficiently close to a, we assume that $\mathcal{L}(\Omega_\lambda)$ is as small as we need. In particular, we may assume that Proposition 2.1 works with $\Omega_0 = \Omega_\lambda$. Therefore, we set

$$W_\lambda := u - u_\lambda \quad \text{and} \quad H_\lambda := v - v_\lambda,$$

and we observe that, by construction, we have

$$W_\lambda \leq 0 \quad \text{on } \partial\Omega_\lambda \quad \text{and} \quad H_\lambda \leq 0 \quad \text{on } \partial\Omega_\lambda.$$

In turn, by Proposition 2.1, it follows that

$$W_\lambda \leq 0 \quad \text{in } \Omega_\lambda \quad \text{and} \quad H_\lambda \leq 0 \quad \text{in } \Omega_\lambda.$$

We now define the set

$$\Lambda_0^{u,v} = \big\{\lambda > a : u \leq u_t \text{ and } v \leq v_t \text{ for all } t \in (a, \lambda]\big\}. \tag{1.27}$$

and

$$\lambda_0 = \sup \Lambda_0^{u,v}. \tag{1.28}$$

Note that by continuity, we have $u \leq u_{\lambda_0}$ and $v \leq v_{\lambda_0}$. We have to show that actually $\lambda_0 = \Lambda$. Hence, assume that by contradiction $\lambda_0 < \Lambda$ and argue as follows. Let A be an open set such that

$$Z_u \cap \Omega_{\lambda_0} \subset A \subset \Omega_{\lambda_0},$$

$$Z_v \cap \Omega_{\lambda_0} \subset A \subset \Omega_{\lambda_0}.$$

Note that since $|Z_u| = |Z_v| = 0$ (see [10, Theorem 2.2] and the references therein), we can choose A as small as we like. Notice now that, since f and g are locally Lipschitz continuous, there exists a positive constant Λ such that

$$\frac{\partial f}{\partial s}(s,t) + \Lambda \geq 0, \quad \text{and} \quad \frac{\partial g}{\partial t}(s,t) + \Lambda \geq 0, \qquad \text{for all } s, t > 0. \tag{1.29}$$

Furthermore, $\frac{\partial f}{\partial t}(s,t)$ and $\frac{\partial g}{\partial s}(s,t)$ are non-negative for $s, t > 0$, by assumption. Consequently,

$$-\Delta_p u + \Lambda u = f(u,v) + \Lambda u \leq f(u_\lambda, v_\lambda) + \Lambda u_\lambda = -\Delta_p u_\lambda + \Lambda u_\lambda, \tag{1.30}$$

$$-\Delta_m v + \Lambda v = g(u,v) + \Lambda v \leq g(u_\lambda, v_\lambda) + \Lambda v_\lambda = -\Delta_m v_\lambda + \Lambda v_\lambda, \tag{1.31}$$

for any $a \leq \lambda \leq \lambda_0$. In light of (1.30)-(1.31), we are able to write

$$\begin{cases} -\Delta_p u + \Lambda u \leq -\Delta_p u_\lambda + \Lambda u_\lambda & \text{in } \Omega_\lambda, \\ \quad u \leq u_\lambda & \text{in } \Omega_\lambda, \\ -\Delta_m v \,+\, \Lambda v \leq -\Delta_m v_\lambda \,+\, \Lambda v_\lambda & \text{in } \Omega_\lambda, \\ \quad v \leq v_\lambda & \text{in } \Omega_\lambda. \end{cases} \tag{1.32}$$

Then, by (1.32), and a strong comparison principle [7, Theorem 1.4], we get

$$u < u_{\lambda_0} \qquad \text{or} \qquad u \equiv u_{\lambda_0},$$

in any connected component of $\Omega_{\lambda_0} \setminus Z_u$, and

$$v < v_{\lambda_0} \qquad \text{or} \qquad v \equiv v_{\lambda_0},$$

in any connected component of $\Omega_{\lambda_0} \setminus Z_u$. We claim that

The case $u \equiv u_{\lambda_0}$ in some connected component $\mathcal{C}$ of $\Omega_{\lambda_0} \setminus Z_u$ is not possible.

In fact, by construction, it is $\partial\mathcal{C} \setminus T_{\lambda_0} \subseteq Z_u$. If $u \equiv u_{\lambda_0}$, also the reflection of $\partial\mathcal{C} \setminus T_{\lambda_0}$ with respect to T_{λ_0} in contained in Z_u. Consequently $\Omega \setminus Z_u$ would not be connected, which is a contradiction (see [8, 9]). Consequently

$$u < u_{\lambda_0}, \tag{1.33}$$

in any connected component of $\Omega_{\lambda_0} \setminus Z_u$. In the very same way, we get

$$v < v_{\lambda_0} \tag{1.34}$$

in any connected component of $\Omega_{\lambda_0} \setminus Z_v$. Consider now a compact set K in Ω_{λ_0} such that $\mathcal{L}(\Omega_{\lambda_0} \setminus K)$ is sufficiently small so that Proposition 2.1 can be applied. By what we proved before, $u_{\lambda_0} - u$ and $v_{\lambda_0} - v$ are positive in $K \setminus A$, which is compact. Then, by continuity, we find $\epsilon > 0$ such that, $\lambda_0 + \epsilon < \Lambda$ and for $\lambda < \lambda_0 + \epsilon$ we have that $\mathcal{L}(\Omega_\lambda \setminus (K \setminus A))$ is still sufficiently small as before, and $u_\lambda - u > 0$ in $K \setminus A$, $v_\lambda - v > 0$ in $K \setminus A$. In particular $u_\lambda - u > 0$ and $v_\lambda - v > 0$ on $\partial(K \setminus A)$. Consequently $u \leq u_\lambda$ and $v \leq v_\lambda$ on $\partial(\Omega_\lambda \setminus (K \setminus A))$. By Proposition 2.1 it follows $u \leq u_\lambda$ and $v \leq v_\lambda$ in $\Omega_\lambda \setminus (K \setminus A)$ and, consequently in Ω_λ, which contradicts the assumption $\lambda_0 < \Lambda$. Therefore $\lambda_0 \equiv \Lambda$ and the thesis is proved. The proof of (1.23) and (1.24) follows by the strong comparison theorem exploited as above immediately as above, see (1.33) and (1.34). Finally (1.25) and (1.26) follow by the monotonicity of the solution, which is implicit in the above arguments.

2.2 Proof of Theorem 1.2

First, we give the following definition (cf. [8, 9, 10]).

Definition 2.1 *Let $\rho \in L^1(\Omega)$ and $1 \le q < \infty$. The space $H^{1,q}_\rho(\Omega)$ is defined as the completion of $C^1(\overline{\Omega})$ (or $C^\infty(\overline{\Omega})$) under the norm*

$$\|v\|_{H^{1,q}_\rho} = \|v\|_{L^q(\Omega)} + \|\nabla v\|_{L^q(\Omega,\rho)}, \tag{1.35}$$

where

$$\|\nabla v\|^q_{L^p(\Omega,\rho)} := \int_\Omega |\nabla v(x)|^q \rho(x) dx.$$

We also recall that $H^{1,q}_\rho(\Omega)$ may be equivalently defined as the space of functions having distributional derivatives represented by a function for which the norm defined in (1.35) *is bounded. These two definitions are equivalent if the domain has piecewise regular boundary.*

If $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ is a weak solution of (1.4), then we have

$$L_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \equiv (L^1_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)), L^2_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)),$$

where we have set, for $1 < p, m < \infty$,

$$\begin{aligned} &L^1_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \\ &= \int_\Omega |\nabla u|^{p-2}(\nabla u_{x_i}, \nabla\varphi) + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u, \nabla u_{x_i})(\nabla u, \nabla \varphi) \\ &\quad - \int_\Omega \Big[\frac{\partial f}{\partial s}(u,v)u_{x_i} + \frac{\partial f}{\partial t}(u,v)v_{x_i}\Big]\varphi\, dx, \\ &L^2_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \\ &= \int_\Omega |\nabla v|^{m-2}(\nabla v_{x_i}, \nabla\psi) + (m-2)\int_\Omega |\nabla v|^{m-4}(\nabla v, \nabla v_{x_i})(\nabla v, \nabla \psi) \\ &\quad - \int_\Omega \Big[\frac{\partial g}{\partial s}(u,v)u_{x_i} + \frac{\partial g}{\partial t}(u,v)v_{x_i}\Big]\psi\, dx, \end{aligned}$$

for any $\varphi, \psi \in C^1_0(\Omega)$. Moreover, the following equation holds

$$L_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) = 0, \qquad \text{for all } (\varphi, \psi) \text{ in } H^{1,2}_{0,\rho_u}(\Omega) \times H^{1,2}_{0,\rho_v}(\Omega), \tag{1.36}$$

and all $i, j = 1, \dots, N$, where

$$\rho_u(x) := |\nabla u(x)|^{p-2}, \qquad \rho_v(x) := |\nabla v(x)|^{m-2}.$$

More generally, if $(w, h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega)$, we can define $L_{(u,v)}((w, h), (\varphi, \psi))$ as above.

An immediate consequence is the following

Theorem 2.1 *Assume that* (1.2) *and* (1.3) *hold and that* $\frac{2N+2}{N+2} < p, m < \infty$*. Let*

$$(w, h) \in H^{1,2}_{\rho_u} \cap C(\Omega) \times H^{1,2}_{\rho_v} \cap C(\Omega)$$

be a nonnegative weak solutions of

$$L_{(u,v)}((w, h), (\varphi, \psi)) = 0, \qquad \forall \varphi, \psi \in C^1_0(\Omega).$$

Then, for any domain $\Omega' \subset \Omega$ *with* $w \geq 0$ *in* Ω' *and* $h \geq 0$ *in* Ω'*, one of the following four cases occurs*

(i) $w > 0$ *and* $h \equiv 0$ *in* Ω'*;*

(ii) $w > 0$ *and* $h > 0$ *in* Ω'*;*

(iii) $w \equiv 0$ *and* $h > 0$ *in* Ω'*;*

(iv) $w \equiv 0$ *and* $h \equiv 0$ *in* Ω'*.*

Proof. In light of (1.3), we have $\frac{\partial f}{\partial t}(s, t)$ and $\frac{\partial g}{\partial s}(s, t)$ are non-negative for $s, t > 0$. Then, taking into account (1.29), it follows that w and h are nonnegative functions solving the inequalities

$$\int_\Omega |\nabla u|^{p-2}(\nabla w, \nabla \varphi) + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u, \nabla w)(\nabla u, \nabla \varphi)\, dx + \int_\Omega \Lambda w \varphi\, dx \geq 0,$$

$$\int_\Omega |\nabla v|^{m-2}(\nabla h, \nabla \psi) + (m-2)\int_\Omega |\nabla v|^{m-4}(\nabla v, \nabla h)(\nabla v, \nabla \psi)\, dx + \int_\Omega \Lambda v \psi\, dx \geq 0,$$

for all nonnegative test functions φ and ψ, where Λ is the constant appearing in (1.29). Therefore, we can apply [9, Theorem 1.1] to w and to h separately obtaining that, for every $s > 1$ sufficiently close to 1, there exist positive constants C_1, C_2 such that

$$\|w\|_{L^s(B(x,2\delta))} \leq C_1 \inf_{B(x,\delta)} w \quad \text{and} \quad \|h\|_{L^s(B(x,2\delta))} \leq C_2 \inf_{B(x,\delta)} h. \tag{1.37}$$

Then, in turn, the sets $\{x \in \Omega' : w(x) = 0\}$ and $\{x \in \Omega' : h(x) = 0\}$ are both closed (by continuity) and open (via inequalitites (1.37)) in the domain Ω', yielding the assertion.

We have the following

Proposition 2.3 *Let* $(u, v) \in C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega})$ *be a solution to system* (1.4) *and let* Λ *be as in* (1.21)*. Assume that* (1.2) *and* (1.3) *hold and that* $\frac{2N+2}{N+2} < p, m < \infty$*. Then, for any* $a \leq \lambda \leq \Lambda$*, we have*

$$u(x) < u_\lambda(x) \quad \textit{and} \quad v(x) < v_\lambda(x), \qquad \textit{for all } x \in \Omega_\Lambda. \tag{1.38}$$

$$\frac{\partial u}{\partial x_1}(x) > 0, \qquad \textit{for all } x \in \Omega_\Lambda, \tag{1.39}$$

$$\frac{\partial v}{\partial x_1}(x) > 0, \qquad \textit{for all } x \in \Omega_\Lambda. \tag{1.40}$$

Proof. To prove (1.38) it is sufficient to apply equations (1.30) and (1.32). Instead to get (1.39) and (1.40) we use equations (1.25) and (1.26), together with Theorem 2.1.

2.3 Proof of Theorem 1.3

For any given $x \in \mathbb{R}$, by Hopf boundary Lemma, (see [14]), it follows that

$$u_y(x,0) = \frac{\partial u}{\partial y}(x,0) > 0 \quad \text{and} \quad v_y(x,0) = \frac{\partial v}{\partial y}(x,0) > 0.$$

We can therefore fix x_0 and r such that

$$\frac{\partial u}{\partial y}(x,y) \geq \gamma > 0, \quad \frac{\partial v}{\partial y}(x,y) \geq \gamma > 0 \quad \text{for all } (x,y) \in B_{2r}(x_0) \cap \{y \geq 0\}, \quad (1.41)$$

for some $\gamma > 0$. Now, it follows that, for $\lambda \leq r$ fixed, we have $\frac{\partial u}{\partial y}(x_0, y) > 0$ and $\frac{\partial v}{\partial y}(x_0, y) > 0$, provided $0 \leq y \leq \lambda$ and for every $0 < \lambda' \leq \lambda$ we get $u(x_0, y) < u(x_0, 2\lambda' - y)$ and $v(x_0, y) < v(x_0, 2\lambda' - y)$, provided that $y \in [0, \lambda')$. Therefore we can exploit Theorem 2.2 in the appendix and get that for every $0 < \lambda' \leq \lambda$ we have $u(x_0, y) < u(x_0\,,\, 2\lambda' - y)$ and $v(x_0, y) < v(x_0\,,\, 2\lambda' - y)$ in $\Sigma_{\lambda'} \equiv \{(x,y) : 0 < y < \lambda'\}$. Let us set

$$\Lambda = \{\lambda \in \mathbb{R}^+ : u < u_{\lambda'} \text{ and } v < v_{\lambda'} \text{ in } \Sigma_{\lambda'}, \text{ for all } \lambda' \leq \lambda\},$$

$$\bar{\lambda} = \sup \Lambda.$$

We will prove the theorem, proving that $\bar{\lambda} = \infty$. Note that, by continuity $u \leq u_{\bar{\lambda}}$ and $v \leq v_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$ and also $u < u_{\bar{\lambda}}$ and $v < v_{\bar{\lambda}}$, by the strong comparison principle. Moreover by the above arguments we have $\frac{\partial u}{\partial y}(x,y) \geq 0$ and $\frac{\partial u}{\partial y}(x,y) \geq 0$ in $\Sigma_{\bar{\lambda}}$. Furthermore, by the strong maximum principle for the linearized operator (see Theorem 2.1), it follows that

$$\frac{\partial u}{\partial y}(x,y) > 0 \quad \text{and} \quad \frac{\partial v}{\partial y}(x,y) > 0,$$

in $\Sigma_{\bar{\lambda}}$. To prove that $\bar{\lambda} = \infty$, let us argue by contradiction, and assume $\bar{\lambda} < \infty$. First of all let us show that there exists some $\bar{x} \in \mathbb{R}$ such that

$$\frac{\partial u}{\partial y}(\bar{x}\,,\, \bar{\lambda}) > 0 \quad \text{and} \quad \frac{\partial v}{\partial y}(\bar{x}\,,\, \bar{\lambda}) > 0.$$

Note that by continuity $\frac{\partial u}{\partial y}(x, \bar{\lambda}), \frac{\partial v}{\partial y}(x, \bar{\lambda}) \geq 0$.

Let us first show that there exists a point x_0 where $\frac{\partial u}{\partial y}(x_0\,,\, \bar{\lambda}) > 0$. To prove this we argue by contradiction and assume that

$$\frac{\partial u}{\partial y}(x\,,\, \bar{\lambda}) = 0$$

for every $x \in \mathbb{R}$. Now, consider the function $u^\star(x,y)$ defined in $\Sigma_{2\bar{\lambda}}$ by

$$u_\star(x,y) \equiv \begin{cases} u(x,y) & \text{if } 0 \leq y \leq \bar{\lambda}, \\ u(x, 2\bar{\lambda} - y) & \text{if } \bar{\lambda} \leq y \leq 2\bar{\lambda}, \end{cases}$$

and consider the function $u_\star(x,y)$ defined in $\Sigma_{2\bar\lambda}$ by

$$u^\star(x,y) \equiv \begin{cases} u(x, 2\bar\lambda - y) & \text{if } 0 \le y \le \bar\lambda, \\ u(x,y) & \text{if } \bar\lambda \le y \le 2\bar\lambda. \end{cases}$$

Note that $u_\star$ is the even reflection of $u|_{\Sigma_{\bar\lambda}}$ and $u^\star$ is the even reflection of $u|_{\Sigma_{2\bar\lambda}\setminus\Sigma_{\bar\lambda}}$. Also let $v^\star$ and $v_\star$ defined in a similar fashion.

Since we are assuming that $\frac{\partial u}{\partial y}(x\,,\,\bar\lambda) = 0$ for every $x \in \mathbb{R}$, it follows that $u^\star$ and $u_\star$ are C^1 solutions of $-\Delta_m u^\star = f(u^\star, v^\star)$ and $-\Delta_m u_\star = f(u_\star, v_\star)$ respectively. Since by definition $u < u_{\bar\lambda}$ and $v < v_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$, we have

$$u_\star \le u^\star \quad \text{and} \quad v_\star \le v^\star$$

in $\Sigma_{2\bar\lambda}$. Also $u_\star$ does not coincide with $u^\star$ because of the strict inequality $u < u_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$. Also, arguing as in (1.30) (see also (1.31)), we find $\Lambda > 0$ sufficiently large such that

$$-\Delta_p u_\star + \Lambda u_\star \le -\Delta_p u^\star + \Lambda u^\star.$$

Since $u_\star(x,\bar\lambda) = u^\star(x,\bar\lambda)$ for any $x \in \mathbb{R}$, by the strong comparison principle (see [9, Theorem 1.4]) it would follow that $u_\star \equiv u^\star$ in $\Sigma_{2\bar\lambda}$. This contradiction actually proves that there exists some $x_0 \in \mathbb{R}$ such that $\frac{\partial u}{\partial y}(x_0\,,\,\bar\lambda) > 0$.

Let now $x_0 \in \mathbb{R}$ such that $\frac{\partial u}{\partial y}(x_0\,,\,\bar\lambda) > 0$, and consider an interval $[x_0-\delta\,;\,x_0+\delta]$ where u_y is still strictly positive. We claim that there exists $\bar x \in [x_0-\delta\,;\,x_0+\delta]$ such that $\frac{\partial v}{\partial y}(\bar x\,,\,\bar\lambda) > 0$. To prove this, assume by contradiction that $\frac{\partial v}{\partial y}(x\,,\,\bar\lambda) = 0$ for every $x \in [x_0-\delta\,;\,x_0+\delta]$ and consider $v^\star$ and $v_\star$ as above. Exploiting the strong comparison principle exactly as above in $\{(x,y)\,|\,x \in [x_0-\delta\,;\,x_0+\delta]\}$, we get a contradiction. Therefore we conclude that there exists a $\bar x$ such that $\frac{\partial v}{\partial y}(\bar x\,,\,\bar\lambda) > 0$. For such $\bar x$ we therefore have

$$\frac{\partial u}{\partial y}(\bar x\,,\,\bar\lambda) > 0 \quad \text{and} \quad \frac{\partial v}{\partial y}(\bar x\,,\,\bar\lambda) > 0.$$

Since now we have proved that $\frac{\partial u}{\partial y}(x_0\,,\,y) > 0$ and $\frac{\partial v}{\partial y}(x_0\,,\,y) > 0$ for every $y \in [0,\bar\lambda]$, it follows that we can find $\varepsilon > 0$ such that

a) $\frac{\partial u}{\partial y}(x_0\,,\,y) > 0$ and $\frac{\partial v}{\partial y}(x_0\,,\,y) > 0$ for every $y \in [0,\bar\lambda+\varepsilon]$

b) For every $0 < \lambda' \le \bar\lambda + \varepsilon$ we get $u(x_0,y) < u(x_0\,,\,2\lambda' - y)$ and $v(x_0,y) < v(x_0\,,\,2\lambda' - y)$ provided that $y \in [0\,,\,\lambda')$.

Note that *a*) follows easily by the continuity of the derivatives. The proof of *b*) is standard in the moving plane technique. By Theorem 2.2 we now get that $u < u_{\lambda'}$ and $v < v_{\lambda'}$ for every $0 < \lambda' < \bar\lambda + \varepsilon$ which implies $\sup\Lambda > \bar\lambda$, a contradiction. Therefore $\bar\lambda = \infty$.

Appendix

We state and prove here a theorem which follows some ideas contained in [11]. For the readers convenience we provide a blueprint of the proof, which is also based on Proposition 2.1.

Theorem 2.2 *Assume that* (1.2) *and* (1.3) *hold, and let* (u, v) *be a weak* $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{H}) \times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{H})$ *solution of* (1.6). *Assume that* $\frac{3}{2} < p, m < \infty$. *Let* $x_0 \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}$ *fixed, and assume that*

a) $\frac{\partial u}{\partial y}(x_0\,,\,y) > 0$ *and* $\frac{\partial v}{\partial y}(x_0\,,\,y) > 0$ *for every* $y \in [0, \lambda]$

b) For every $0 < \lambda' \le \lambda$ *we have* $u(x_0, y) < u(x_0\,,\,2\lambda' - y)$ *and* $v(x_0, y) < v(x_0\,,\,2\lambda' - y)$ *(that is* $u < u_{\lambda'}$, $v < v_{\lambda'}$*) provided that* $y \in [0\,,\,\lambda')$.

Then, for every $0 < \lambda' \le \lambda$ *and* $(x, y) \in \Sigma_{\lambda'}$, *it follows that*

$$u(x\,,\,y) < u(x\,,\,2\lambda' - y) \quad \text{and} \quad v(x\,,\,y) < v(x\,,\,2\lambda' - y).$$

Proof. Let L_θ be the vector $(\cos\theta, \sin\theta)$ and V_θ the vector orthogonal to L_θ such that $(V_\theta, e_2) \ge 0$. We define $L_{x_0,s,\theta}$ the line parallel to L_θ passing through (x_0, s). We define $\mathcal{T}_{x_0,s,\theta}$ as the triangle delimited by $L_{x_0,s,\theta}$, $\{y = 0\}$ and $\{x = x_0\}$, and we set $u_{x_0,s,\theta}(x) = u(T_{x_0,s,\theta}(x))$ and $v_{x_0,s,\theta}(x) = v(T_{x_0,s,\theta}(x))$, where $T_{x_0,s,\theta}(x)$ is the point symmetric to x, w.r.t. $L_{x_0,s,\theta}$. It is well known that $u_{x_0,s,\theta}$ and $v_{x_0,s,\theta}$ still are solutions of our system. Also for simplicity we set $u_{x_0,s,0} = u_s$ and $v_{x_0,s,0} = v_s$. Let us now consider $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ fixed as in the statement. We have the following

Claim 1. *There exists* $\delta > 0$ *such that for any* $-\delta \le \theta \le \delta$ *and for any* $0 < \lambda' \le \lambda + \delta$ *we have* $u(x_0, y) < u_{x_0,\lambda',\theta}(x_0, y)$ *and* $v(x_0, y) < v_{x_0,\lambda',\theta}(x_0, y)$ *for every* $0 \le y < \lambda'$.

We argue by contradiction. If the claim were false, we could find a sequence of δ_n converging to 0 and $-\delta_n \le \theta_n \le \delta_n$, $0 < \lambda_n \le \lambda + \delta_n$, $0 \le y_n < \lambda_n$ such that

$$u(x_0, y_n) \ge u_{x_0,\lambda_n,\theta_n}(x_0, y_n) \quad \text{or} \quad v(x_0, y_n) \ge v_{x_0,\lambda_n,\theta_n}(x_0, y_n).$$

For a sequence y_n, eventually considering a subsequence, we may assume that $u(x_0, y_n) \ge u_{x_0,\lambda_n,\theta_n}(x_0, y_n)$ for any $n \in \mathbb{N}$ or $v(x_0, y_n) \ge v_{x_0,\lambda_n,\theta_n}(x_0, y_n)$ for any $n \in \mathbb{N}$. Let us assume that $u(x_0, y_n) \ge u_{x_0,\lambda_n,\theta_n}(x_0, y_n)$ for any $n \in \mathbb{N}$. At the limit, eventually considering subsequences, we may assume that λ_n converges to $\tilde{\lambda} \le \lambda$. In addition y_n converges to $\tilde{y}$ for some $\tilde{y} \le \tilde{\lambda}$. Let us show that $\tilde{y} = \tilde{\lambda}$. If $\tilde{\lambda} = 0$ it also follows $\tilde{y} = \tilde{\lambda} = 0$ since $0 \le y_n < \lambda_n$. If instead $\tilde{\lambda} > 0$, by continuity it follows that $u(x_0, \tilde{y}) \ge u_{\tilde{\lambda}}(x_0, \tilde{y})$. Consequently y_n converges to $\tilde{\lambda} = \tilde{y}$ since we know that $u < u_{\lambda'}$ for all $\lambda' \le \tilde{\lambda}$ in $\Sigma_{\lambda'}$. By the mean value theorem since $u(x_0, y_n) \ge u_{x_0,\lambda_n,\theta_n}(x_0, y_n)$, it follows that $\frac{\partial u}{\partial V_{\theta_n}}(\tilde{x}_n, \tilde{y}_n) \le 0$ at some point $\xi_n \equiv (\tilde{x}_n, \tilde{y}_n)$ lying on the line from (x_0, y_n) to $T_{x_0,\lambda_n,\theta_n}(x_0, y_n)$. We recall that the vector V_{θ_n} is orthogonal to the line $L_{x_0,\lambda_n,\theta_n}$ and V_{θ_n} converges to e_2 since θ_n goes to 0. Passing to the limit it follows that $\frac{\partial u}{\partial y}(x_0, \tilde{\lambda}) \leq 0$ which is impossible by the assumptions, proving the claim.

Let δ be the value provided by Claim 1.

Claim 2. *There is $\rho = \rho(\delta)$ such that, for any $0 < s \leq \rho$, the following inequalities hold: $u < u_{x_0,s,\delta}$ in $\mathcal{T}_{x_0,s,\delta}$ ($u < u_{x_0,s,-\delta}$ in $\mathcal{T}_{x_0,s,-\delta}$) and $v < v_{x_0,s,\delta}$ in $\mathcal{T}_{x_0,s,\delta}$ ($v < v_{x_0,s,-\delta}$ in $\mathcal{T}_{x_0,s,-\delta}$).*

We prove that we can find $\rho = \rho(\delta)$ such that, for every $0 < s \leq \rho$, it follows $u < u_{x_0,s,\delta}$ in $\mathcal{T}_{x_0,s,\delta}$ and $v < v_{x_0,s,\delta}$ in $\mathcal{T}_{x_0,s,\delta}$. If we replace δ by $-\delta$ the proof is exactly the same. To prove this, we can set ρ in such a way that

(i) $\rho < \lambda$, where λ is given in the statement.

(ii) For every $0 < s \leq \rho$ we have $u \leq u_{x_0,s,\delta}$ on $\partial(\mathcal{T}_{x_0,s,\delta})$ and $v \leq v_{x_0,s,\delta}$ on $\partial(\mathcal{T}_{x_0,s,\delta})$.

(iii) For ρ small enough and $0 < s \leq \rho$, $\mathcal{L}(\mathcal{T}_{x_0,s,\delta})$ is so small to exploit Proposition 2.1.

Therefore, given any $0 < s \leq \rho$, if we consider $w_{x_0,s,\delta} = u - u_{x_0,s,\delta}$ and $h_{x_0,s,\delta} = v - v_{x_0,s,\delta}$, we have that $w_{x_0,s,\delta} \leq 0$ and $h_{x_0,s,\delta} \leq 0$ on $\partial\mathcal{T}_{x_0,s,\delta}$ and therefore, by Proposition 2.1, we get $w_{x_0,s,\delta} \leq 0$ and $h_{x_0,s,\delta} \leq 0$ in $\mathcal{T}_{x_0,s,\delta}$. Also, by the strong comparison principle exploited as above (see (1.32) and (1.30)), it follows that the strict inequalities hold. This concludes the proof of Claim 2.

Consider now the values ρ and δ provided by the Claims. Consider $0 < \lambda' \leq \lambda$ and let us fix $0 < \bar{s} < \min\{\rho, \lambda'\}$ so that by Claim 2 we have $w_{x_0,\bar{s},\delta} < 0$ and $h_{x_0,\bar{s},\delta} < 0$ in $\mathcal{T}_{x_0,\bar{s},\delta}$. We now define the continuous function $g(t) = (s(t), \theta(t)) : [0,1] \to \mathbb{R}^2$, by $s(t) = (t\lambda' + (1-t)\bar{s}$ and $\theta(t) = (1-t)\delta$, so that $g(0) = (\bar{s}, \delta)$, $g(1) = (\lambda', 0)$ and $\theta(t) \neq 0$ for every $t \in [0,1)$. Moreover Claim 1 yields $w_{x_0,\bar{s},\delta} \leq 0$ and $h_{x_0,s,\delta} \leq 0$ on $\partial(\mathcal{T}_{x_0,s(t),\theta(t)})$ for every $t \in [0,1)$. Also $w_{x_0,s(t),\theta(t)}$ and $h_{x_0,s,\delta}$ are not identically zero on $\partial(\mathcal{T}_{x_0,s(t),\theta(t)})$, for every $t \in [0,1)$. We now let

$$\overline{T} = \{\tilde{t} \in [0,1] \text{ such that } w_{x_0,\bar{s},\delta}\,;\, h_{x_0,s,\delta} < 0 \text{ in } \mathcal{T}_{x_0,s(t),\theta(t)} \text{ for every } 0 \leq t \leq \tilde{t}\},$$

and $\bar{t} = \sup \overline{T}$, where, possibly, $\bar{t} = 0$. Exploiting the moving-rotating plane technique as in [11] it follows that $\bar{t} = 1$, concluding the proof.

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