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Symmetry results for the $p(x)$-Laplacian equation

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Abstract. We consider the equation $-\mathrm{div}(|Du|^{p(x)-2}Du) = f(x,u)$ and the related Dirichlet problem. For axially symmetric domains we prove that, under suitable assumptions, there exist mountain-pass solutions which exhibit partial symmetry. Furthermore, we show that semi-stable or non-degenerate smooth solutions need to be radially symmetric in the ball.

Keywords. Quasi-linear elliptic equations, $p(x)$-Laplacian operator, symmetrization, partial symmetry.

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1 Introduction and results

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $p : \overline{\Omega} \to \mathbb{R}$ be a continuous function with

$$1 < p_- := \inf_{\Omega} p \le \sup_{\Omega} p =: p_+ < \infty. \tag{1.1}$$

In the last few years, the interest towards nonlinear elliptic problems of the type

$$-\mathrm{div}(|Du|^{p(x)-2}Du) = f(x,u), \quad \text{in } \Omega, \tag{1.2}$$

has considerably increased and various results appeared in the literature about existence and regularity of weak solutions, see e.g. [7, Chapter 13] and the references therein. The main goal of our paper is to establish some symmetry results for positive solutions, provided that the domain Ω and both functions $p(x)$ and $x \mapsto f(x,s)$ admit some partial or full symmetry in Ω. We shall obtain two type of symmetry results by exploiting two completely different techniques. A first class of results is obtained through suitable versions of the Mountain-Pass Theorem which incorporates symmetry features provided that the functional naturally associated with the problem does increase under polarization [17–20]. In this case we obtain the existence of nontrivial *mountain-pass solutions* with some partial symmetry information if the domain is axially symmetric with respect to a fixed half space

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H with $0 \in \partial H$ or if it is invariant under reflection with respect to any half space H with $0 \in \partial H$. A second class of results is obtained when Ω is a ball in $\mathbb{R}^N$ by exploiting fine regularity estimates for the $C^{1,\alpha}$ solutions, allowing to obtain a meaningful definition for the first eigenvalue of the linearized operator associated with (1.2), see [3,6,9,13]. In this case we obtain that any *semi-stable solution*, namely the first eigenvalue of a suitably defined linearized operator is nonnegative, is radially symmetric when $f(x,s) = f_0(|x|,s)$ and $p(x) = p_0(|x|)$. Whence, in some sense, solutions with some minimality property such as being of mountain-pass type or semi-stable inherit some symmetry from the data of the problem. We now come to the statement of the main results. In the following we denote by $H \subset \mathbb{R}^N$ a closed affine half space of $\mathbb{R}^N$, by $\sigma_H(x)$ the reflected of a point $x \in \mathbb{R}^N$ with respect to ∂H and by $\mathcal{H}_0$ the set of all half spaces $H \subset \mathbb{R}^N$ such that $0 \in \partial H$. The polarization of u by a half space H is denoted by u^H and $\sigma_H(\Omega)$ denotes the set of all reflected points of Ω.

Theorem 1.1. *Assume that $\sigma_H(\Omega) = \Omega$ for some $H \in \mathcal{H}_0$ and, for all $x \in \Omega$*

$$\begin{aligned} p(\sigma_H(x)) &= p(x), \quad q(\sigma_H(x)) = q(x), \\ V(\sigma_H(x)) &= V(x), \quad K(\sigma_H(x)) = K(x). \end{aligned} \tag{1.3}$$

Also, assume that p, q are logarithmic Hölder continuous and $q : \overline{\Omega} \to \mathbb{R}$ is a continuous function with

$$\begin{aligned} &\inf_{x\in\Omega} (q(x) - p(x) + 1) > 0, \\ &\inf_{x\in\Omega} (p^*(x) - q(x) - 1) > 0, \quad p^*(x) = \frac{p(x)N}{N - p(x)}, \end{aligned} \tag{1.4}$$

$V, K \in C(\overline{\Omega})$ with $V(x) \geq V_0 > 0$ for all $x \in \Omega$. Then there exists a nontrivial solution $u \in W_0^{1,p(x)}(\Omega)$ of

$$\begin{cases} -\mathrm{div}(|Du|^{p(x)-2}Du) + V(x)u^{p(x)-1} = K(x)u^{q(x)} & \text{for } x \in \Omega, \\ u \geq 0 & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.5}$$

at the mountain-pass level such that u^H is also a solution of (1.5) *at the same energy level.*

In [1, Lemma 2.5], for the semi-linear case $p(x) = 2$ for every $x \in \Omega$, the authors introduce a new ingredient, namely that if u, u^H are both classical solution of $-\Delta w = f(x,w)$ and f satisfies the invariance

$$f(\sigma_H(x), s) = f(x,s), \quad \text{for all } x \in \Omega \text{ and } s \in \mathbb{R}, \tag{1.6}$$

with respect to some $H \in \mathcal{H}_0$, then either $u(x) > u(\sigma_H(x))$ for all $x \in \mathrm{Int}(H \cap \Omega)$ (resp. $u(x) < u(\sigma_H(x))$ for all $x \in \mathrm{Int}(H \cap \Omega)$) or $u(x) = u(\sigma_H(x))$ for all $x \in \Omega$. On account of Theorem 1.1, it would be interesting to extend these type of results to more general framework. This is to our knowledge an interesting open problem. In the framework of Theorem 1.1, we also have the following

Theorem 1.2. *Assume that* $\sigma_H(\Omega) = \Omega$ *for all* $H \in \mathcal{H}_0$, *and that* (1.3)–(1.4) *hold. Then there exists a nontrivial solution* $u \in W_0^{1,p(x)}(\Omega)$ *of* (1.5) *at the mountain-pass level such that* $u(x) = \psi(|x|, \xi \cdot x)$ *for some unit vector* $\xi \in \mathbb{R}^N$ *and some* $\psi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *with* $\psi(r, \cdot)$ *nondecreasing for all* $r \geq 0$.

The statement of Theorem 1.2 could be easily extended, via minor modifications, to cover the case where the domain is invariant under spherical cap symmetrization [19], $\Omega^* = \Omega$, which is equivalent to $\Omega^H = \Omega$ for every $H \in \mathcal{H}_0$, in place of the more stringent assumption $\sigma_H(\Omega) = \Omega$, for all $H \in \mathcal{H}_0$. It is readily seen that Theorems 1.1 and 1.2 can be extended to cover a more general class of nonlinearities $f(x,s)$ in place of $K(x)s^{q(x)}$ for $s \geq 0$. It is sufficient to assume (1.6) a growth condition such as $|f(x,s)| \leq C + C|s|^{q(x)}$ for all $x \in \Omega$ and $s \in \mathbb{R}$, $f(x,s) = 0$ for $s \leq 0$ (in order to guarantee that the solutions are non-negative), $f(x,s) = o(|s|^{p(x)-1})$ as $s \to 0$ and an Ambrosetti–Rabinowitz type condition: there exists $\mu > 0$ with $\inf\{\mu - p(x) : x \in \Omega\} > 0$ and $R > 0$ such that

$$\mu F(x,s) \leq f(x,s)s \quad \text{for all } x \in \Omega \text{ and } s \geq R,$$

where

$$F(x,s) = \int_0^s f(x,\tau)\,d\tau.$$

We refer the reader to [5], where the mountain-pass geometry and the Palais–Smale condition of

$$\varphi(u) = \int_\Omega \frac{|Du|^{p(x)}}{p(x)} + \int_\Omega \frac{V(x)}{p(x)}|u|^{p(x)} - \int_\Omega F(x,u), \quad u \in W_0^{1,p(x)}(\Omega),$$

are handled in this framework. In the second part of the paper we study the radial symmetry of solutions to (1.2), considering the problem

$$\begin{cases} -\mathrm{div}(|Du|^{p(|x|)-2}Du) = f(|x|,u) & \text{in } \Omega, \\ \qquad\qquad u > 0 & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with $f(t,s)$ locally Lipschitz continuous in $[0,\infty)\times[0,\infty)$ and positive in $[0,\infty)\times$

$(0, \infty)$. Let us recall that the corresponding linearized operator is given by

$$\begin{aligned} L_u(v, \varphi) := \int_\Omega |Du|^{p(x)-2}(Dv, D\varphi) \\ + \int_\Omega (p(x)-2)|Du|^{p(x)-4}(Du, Dv)(Du, D\varphi) \\ - \int_\Omega \partial_s f(|x|, u) v\varphi, \end{aligned}$$

for any $v, \varphi \in H^{1,2}_{0,\rho}$, where the weighted Sobolev space $H^{1,2}_{0,\rho}$ will be suitably defined in Section 5.2. We will prove some summability properties of $|Du|^{-1}$ that will allow us to get a weighted Sobolev type inequality (see Theorem 5.2). This is the key to recover a complete spectral theory for the linearized operator, carried out in Section 5.3. Consequently we can give the following

Definition 1.3. We say that a solution u is semi-stable if

$$\mu_1(L_u, \Omega) \geq 0$$

being $\mu_1(L_u, \Omega)$ the first eigenvalue of the linearized operator L_u in Ω. Furthermore, the solution u is said to be non-degenerate if 0 is not an eigenvalue of the linearized operator L_u in Ω.

Note that, by the variational characterization of the first eigenvalue, it follows that equivalently u is semi-stable if and only if $L_u(\varphi, \varphi) \geq 0$ for any $\varphi \in H^{1,2}_{0,\rho}$. Since the linearized operator arises as second derivative of the energy functional, it follows that the minima of the energy functional are semi-stable solutions. Also, if $f(t, s)$ is decreasing with respect to the s-variable, then it follows that any solution is semi-stable. Moreover in many cases, depending on $p(\cdot)$, it is possible to show that monotone solutions are stable (namely $\mu_1(L_u, \Omega) > 0$) solutions, see e.g. [10]. On the other hand mountain-pass solutions (as the ones previously obtained) generally have Morse index equal to one. That is, the first eigenvalue of the linearized operator is negative, and the second one is non-negative. This is well known in the semi-linear case and we refer to [4] for some remarks regarding the quasi-linear case. We have the following

Theorem 1.4. *Let Ω be a ball or an annulus in $\mathbb{R}^N$ and u be any $C^{1,\alpha}(\overline{\Omega})$ solution to* (1.7)*, with $f(t, s)$ locally Lipschitz continuous in $[0, \infty) \times [0, \infty)$ and positive in $[0, \infty) \times (0, \infty)$. Assume that u is semi-stable. Then u is radially symmetric provided that $p \in C^1(\Omega)$ with $p(|x|) \geq 2$. The same conclusion follows assuming that the solution u in non-degenerate.*

The symmetry result obtained in Theorem 1.4 holds under very general assumptions on the nonlinearity f, assuming that the solution is Semi-stable or non-degenerate. In the semi-linear case $p(x) = 2$, or more generally in the quasi-linear case $p(x) = p$, in the case of a convex domain (not the annulus), it is possible to get similar results exploiting the moving plane technique [15] (see also [11]), without any stability assumption. We refer to [6] and the references therein for a description of the moving planes procedure in the quasi-linear case. Let us mention here that this technique in general cannot be exploited in our case. In fact the moving plane technique is based on the invariance of the equation under reflections with respect to hyperplanes, which is not true in general in the case of $p(x)$-Laplace equations. Let us also point out that our result holds in the case of solutions which are minima of the associated energy functional (and consequently semi-stable). We refer to [8] (see Section 3) for previous results in this setting.

2 Recalls on variable exponent Sobolev spaces

We recall here some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. We set

$$C_+(\overline{\Omega}) = \Big\{h \in C(\overline{\Omega}) : \min_{\overline{\Omega}} h > 1\Big\},$$

and, for $h \in C(\overline{\Omega})$, we denote

$$h_- := \min_{\overline{\Omega}} h \quad \text{and} \quad h_+ := \max_{\overline{\Omega}} h.$$

For $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} : \ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < +\infty\Big\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\Big\},$$

which is a separable and reflexive Banach space. If $u \in L^{p(\cdot)}(\Omega)$, the term

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx$$

is called $p(\cdot)$-modular of u. We summarize here a few basic properties of these spaces, the details being found in [7]. If $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous. For any $u \in L^{p(\cdot)}(\Omega)$

and $v \in L^{p'(\cdot)}(\Omega)$, the following Hölder type inequality holds:

$$\left| \int_\Omega uv\,dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

The norm and $p(\cdot)$-modular of every $u \in L^{p(\cdot)}(\Omega)$ have the following relation:

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \leq \rho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\}.$$

For $p \in C_+(\overline{\Omega})$, the variable exponent Sobolev space is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : D_i u \in L^{p(\cdot)}(\Omega) \text{ for } i = 1, \ldots, N\right\},$$

endowed with the norm

$$\|u\| = \|u\|_{p(\cdot)} + \|Du\|_{p(\cdot)},$$

which is a separable and reflexive Banach space. It is important to note that, unlike the constant exponent case, the smooth functions are in general not dense in $W^{1,p(\cdot)}(\Omega)$. However, as shown in [7], if the exponent variable $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, see [7], then the smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|$. Moreover, the $p(\cdot)$-Poincaré inequality $\|u\|_{p(\cdot)} \leq C\|Du\|_{p(\cdot)}$ holds for all $u \in W_0^{1,p(\cdot)}(\Omega)$, where C depends on p, $|\Omega|$, $\mathrm{diam}(\Omega)$ and N, see [7, Theorem 4.3]. Therefore,

$$\|u\|_{1,p(\cdot)} = \|Du\|_{p(\cdot)}$$

is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. Finally, note that when $s \in C_+(\overline{\Omega})$ and $\inf_\Omega(p^*(x) - s(x)) > 0$, where $p^*(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \geq N$, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact.

Notation. Generic fixed numerical constants will be denoted by C (with subscript in some case), and will be allowed to vary within a single line or formula.

3 Proof of Theorem 1.1

Problem (1.5) is naturally associated with the functional $\varphi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$

$$\varphi(u) = \int_\Omega \frac{|Du|^{p(x)}}{p(x)} + \int_\Omega \frac{V(x)}{p(x)}|u|^{p(x)} - \int_\Omega \frac{K(x)}{q(x)+1}|u^+|^{q(x)+1}. \tag{3.1}$$

It is readily seen that φ is of class C^1 and its critical points correspond to nonnegative weak solutions to (1.5), namely we have

$$\int_\Omega |Du|^{p(x)-2} Du \cdot D\zeta + \int_\Omega V(x)u^{p(x)-1}\zeta = \int_\Omega K(x)u^{q(x)}\zeta,$$

for all $\zeta \in W_0^{1,p(x)}(\Omega)$. For the reader's convenience, we recall that the polarization of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ by a polarizer H is the function $u^H : \mathbb{R}^N \to \mathbb{R}$ defined by

$$u^H(x) := \begin{cases} \max\{u(x), u(\sigma_H(x))\} & \text{if } x \in H, \\ \min\{u(x), u(\sigma_H(x))\} & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases}$$

The polarization $\Omega^H \subset \mathbb{R}^N$ of a set $\Omega \subset \mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{\Omega^H} = (\chi_\Omega)^H$, where χ denotes the characteristic function. The polarization u^H of a function u defined on $\Omega \subset \mathbb{R}^N$ is the restriction to Ω^H of the polarization of the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}$ of u by zero outside Ω. For a domain Ω, the set $\sigma_H(\Omega)$ denotes the set of all reflected points of Ω. In particular, if $H \in \mathcal{H}_0$ and Ω is invariant under reflection with respect to ∂H, namely $\sigma_H(\Omega) = \Omega$, then $u^H : \Omega \to \mathbb{R}$ writes down as

$$u^H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\} & \text{if } x \in H \cap \Omega, \\ \min\{u(x), u(\sigma_H(x))\} & \text{if } x \in (\mathbb{R}^N \setminus H) \cap \Omega. \end{cases} \tag{3.2}$$

3.1 Some preliminary results

In [18], Squassina and Van Schaftingen recently proved the following

Lemma 3.1. *Let* $(X, \|\cdot\|)$ *be a Banach space,* M *be a metric space and* $M_0 \subset M$. *Let us also consider* $\Gamma_0 \subset C(M_0, X)$ *and define the set*

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X, \mathbb{R})$ *satisfies*

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in M} \varphi(\gamma(t)) > \sup_{\gamma_0 \in \Gamma_0} \sup_{t \in M_0} \varphi(\gamma_0(t)) = a,$$

$\Psi \in C(X, X)$ *and*

$$\varphi \circ \Psi \leq \varphi, \quad \Psi(\Gamma) \subset \Gamma,$$

then for every $\epsilon \in]0, \frac{c-a}{2}[$, $\delta > 0$ *and* $\gamma \in \Gamma$ *such that*

$$\sup_M \varphi \circ \gamma \leq c + \epsilon,$$

there exist elements $u, v, w \in X$ such that

(a.1) $c - 2\epsilon \le \varphi(u) \le c + 2\epsilon$,

(a.2) $c - 2\epsilon \le \varphi(v) \le c + 2\epsilon$,

(b.1) $\|u - w\| \le 3\delta$,

(b.2) $\mathrm{dist}_X(w, \gamma(M)) \le \delta$,

(b.3) $\|v - \Psi(w)\| \le 2\delta$,

(c.1) $\|\varphi'(u)\| < 8\epsilon/\delta$,

(c.1) $\|\varphi'(v)\| < 8\epsilon/\delta$.

We now prove the following

Lemma 3.2. *Assume that $\sigma_H(\Omega) = \Omega$ with respect to some half space $H \in \mathcal{H}_0$ and that $p : \overline{\Omega} \to (1, +\infty)$ and $\mu : \overline{\Omega} \to \mathbb{R}^+$ are continuous functions such that*

$$p(\sigma_H(x)) = p(x), \quad \mu(\sigma_H(x)) = \mu(x), \quad \textit{for all } x \in \Omega. \tag{3.3}$$

Then

$$\int_\Omega \mu(x)|Du^H|^{p(x)} = \int_\Omega \mu(x)|Du|^{p(x)}, \quad \textit{for all } u \in W_0^{1,p(x)}(\Omega).$$

Similarly

$$\int_\Omega \mu(x)|u^H|^{p(x)} = \int_\Omega \mu(x)|u|^{p(x)}, \quad \textit{for all } u \in L^{p(x)}(\Omega).$$

Proof. If $u \in W_0^{1,p(x)}(\Omega)$ and $H \in \mathcal{H}_0$, it follows that $u^H \in W_0^{1,p(x)}(\Omega)$. To prove this, it is sufficient to argue as in the beginning of the proof of [16, Proposition 2.3] for the case $\Omega = \mathbb{R}^N$ and then recall that by definition $u^H = (\tilde{u})^H|_\Omega$ and $(\tilde{u})^H|_{\mathbb{R}^N \setminus \Omega} = 0$, being $\sigma_H(\Omega) = \Omega$. Setting

$$v(x) := u(\sigma_H(x)) \quad \text{and} \quad w(x) := u^H(\sigma_H(x)),$$

it follows that v, w belong to $W_0^{1,p(x)}(\Omega)$ and

$$\begin{aligned} Du^H(x) &= \begin{cases} Du(x) & \text{if } x \in \{u > v\} \cap H \cap \Omega, \\ Dv(x) & \text{if } x \in \{u \le v\} \cap H \cap \Omega, \end{cases} \\ Dw(x) &= \begin{cases} Dv(x) & \text{if } x \in \{u > v\} \cap H \cap \Omega, \\ Du(x) & \text{if } x \in \{u \le v\} \cap H \cap \Omega, \end{cases} \end{aligned} \tag{3.4}$$

and, for $x \in H \cap \Omega$, we have

$$u^H(x) = v(x) + (u(x) - v(x))^+ \quad \text{and} \quad w(x) = u(x) - (u(x) - v(x))^+.$$

Writing down σ_H as $\sigma_H(x) = x_0 + Rx$, where R is an orthogonal linear transformation (symmetric, as reflection), taking into account that $|\det R| = 1$ and the formula $|Dv(x)| = |D(u(\sigma_H(x)))| = |R(Du(\sigma_H(x)))| = |(Du)(\sigma_H(x))|$ (and the analogous formula for $|Dw(x)| = |(Du^H)(\sigma_H(x))|$) recalling (3.3), (3.4) and that

$$\Omega \cap (\mathbb{R}^N \setminus H) = \sigma_H(\Omega \cap H),$$

we have

$$\begin{aligned}
\int_\Omega \mu(x)|Du|^{p(x)} &= \int_{H\cap\Omega} \mu(x)|Du|^{p(x)} + \int_{H\cap\Omega} \mu(x)|(Du)(\sigma_H(x))|^{p(x)} \\
&= \int_{H\cap\Omega} \mu(x)|Du|^{p(x)} + \int_{H\cap\Omega} \mu(x)|Dv|^{p(x)} \\
&= \int_{\{u>v\}\cap H\cap\Omega} \mu(x)|Du|^{p(x)} + \int_{\{u>v\}\cap H\cap\Omega} \mu(x)|Dv|^{p(x)} \\
&\qquad + \int_{\{u\le v\}\cap H\cap\Omega} \mu(x)|Dv|^{p(x)} \\
&\qquad + \int_{\{u\le v\}\cap H\cap\Omega} \mu(x)|Du|^{p(x)} \\
&= \int_{H\cap\Omega} \mu(x)|Du^H|^{p(x)} + \int_{H\cap\Omega} \mu(x)|Dw|^{p(x)} \\
&= \int_\Omega \mu(x)|Du^H|^{p(x)}.
\end{aligned}$$

This concludes the proof. □

We can now prove the following

Lemma 3.3. *Assume that $\sigma_H(\Omega) = \Omega$ with respect to some $H \in \mathcal{H}_0$ and that $p : \overline{\Omega} \to (1, +\infty)$ is a continuous functions such that*

$$p(\sigma_H(x)) = p(x), \quad \text{for all } x \in \Omega. \tag{3.5}$$

Then the map

$$\Psi : W_0^{1,p(x)}(\Omega) \to W_0^{1,p(x)}(\Omega), \quad u \mapsto u^H,$$

is well defined and continuous.

Proof. Let $(u_j) \subset W_0^{1,p(x)}(\Omega)$ be a sequence which strongly converges to some $u_0 \in W_0^{1,p(x)}(\Omega)$. Observe that, for every fixed $\lambda > 0$, by applying Lemma 3.2

with $\mu(x) := \lambda^{-p(x)}$ we have

$$\int_\Omega \left(\frac{|Du_j^H|}{\lambda}\right)^{p(x)} = \int_\Omega \left(\frac{|Du_j|}{\lambda}\right)^{p(x)}, \quad \text{for all } j \geq 1. \tag{3.6}$$

Then, by the arbitrariness of λ and the definition on $\|\cdot\|_{L^{p(x)}}$, there holds

$$\sup_{j\geq 1} \|Du_j^H\|_{L^{p(x)}} = \sup_{j\geq 1} \|Du_j\|_{L^{p(x)}} < +\infty.$$

Since (u_j^H) is bounded in the reflexive space $W_0^{1,p(x)}(\Omega)$, up to a subsequence, there exists $w \in W_0^{1,p(x)}(\Omega)$ such that (u_j^H) converges weakly to w as $j \to \infty$. Observe now that, since the polarization is contractive for $L^m(\Omega)$-spaces (precisely, see [19, Proposition 2.3], case of totally invariant domains) and since the injection $i : L^{p(x)}(\Omega) \to L^{p-}(\Omega)$ is continuous, for all $j \geq 1$

$$\begin{aligned}\|u_j^H - u_0^H\|_{L^{p-}(\Omega)} &\leq \|u_j - u_0\|_{L^{p-}(\Omega)} \\ &\leq C\|u_j - u_0\|_{L^{p(x)}(\Omega)} \leq C\|u_j - u_0\|_{W_0^{1,p(x)}(\Omega)},\end{aligned}$$

where in the last inequality we used Poincaré inequality. Hence u_j^H converges to u_0^H strongly in $L^{p-}(\Omega)$. Hence $w = u_0^H$. In conclusion

$$u_j^H \rightharpoonup u_0^H \quad \text{in } W_0^{1,p(x)}(\Omega) \text{ as } j \to \infty,$$

and

$$\lim_{j\to\infty} \|Du_j^H\|_{L^{p(x)}(\Omega)} = \|Du_0^H\|_{L^{p(x)}(\Omega)}.$$

As $W_0^{1,p(x)}(\Omega)$ is uniformly convex (see, for instance, [7, Theorem 8.1.6, p. 243]), we can finally conclude that $u_j^H \to u_0^H$ as $j \to \infty$ in $W_0^{1,p(x)}(\Omega)$. □

3.2 Proof of Theorem 1.1 concluded

With the above results, apply Lemma 3.1 by taking

$$X := W_0^{1,p(x)}(\Omega), \quad M := [0,1], \quad M_0 := \{0,1\}, \quad \Gamma_0 = \{0,\xi\} \tag{3.7}$$

with $\xi \geq 0$ a fixed function with $\xi^H = \xi$ and $\varphi(\xi) < 0$ (for an explicit construction of a function ξ satisfying these conditions, see [5, bottom of p. 613]) and hence

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,p(x)}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = \xi\}.$$

It is readily seen that the functional φ introduced in (3.1) is C^1 smooth. Furthermore,

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t)) > 0 = \max\{\varphi(0), \varphi(\xi)\} = \sup_{\gamma_0\in\{0,\xi\}} \sup_{t\in\{0,1\}} \varphi(\gamma_0(t)) = a,$$

where the first inequality (namely the mountain-pass geometry of φ) can be proved by arguing exactly as in [5, pp. 612–613]. In light of Lemma 3.3 the polarization map is continuous. Also by using Lemma 3.2 with the choices $\mu(x) = p(x)^{-1}$, $\mu(x) = \frac{V(x)}{p(x)}$ and $\mu(x) = \frac{V(x)}{q(x)+1}$ respectively (notice that, on account of (1.3) any of these choices of μ remain invariant under reflection with respect to ∂H), we have

$$\begin{aligned}\varphi(u^H) &= \int_\Omega \frac{|Du^H|^{p(x)}}{p(x)} + \int_\Omega \frac{V(x)}{p(x)}|u^H|^{p(x)} - \int_\Omega \frac{K(x)}{q(x)+1}|(u^+)^H|^{q(x)+1}\\ &= \int_\Omega \frac{|Du|^{p(x)}}{p(x)} + \int_\Omega \frac{V(x)}{p(x)}|u|^{p(x)} - \int_\Omega \frac{K(x)}{q(x)+1}|(u^+)|^{q(x)+1} = \varphi(u)\end{aligned}$$

for every $u \in W_0^{1,p(x)}(\Omega)$. Finally, $\Psi(\Gamma) \subset \Gamma$ since for every $\gamma \in \Gamma$ it follows, again in view of Lemma 3.3, that $\gamma^H \in C([0,1], W_0^{1,p(x)}(\Omega))$ and

$$\gamma^H(0) = (\gamma(0))^H = 0^H = 0 \quad \text{and} \quad \gamma^H(1) = (\gamma(1))^H = \xi^H = \xi.$$

By the definition of c we can find a sequence of curves $(\gamma_j) \subset \Gamma$ such that

$$\sup_{t\in[0,1]} \varphi(\gamma_j([0,1])) \leq c + 1/j^2.$$

Apply now Lemma 3.1 with $\delta_j = 1/j$, $\varepsilon_j = 1/j^2$ and and obtain three sequences (u_j), (v_j) and (w_j) in $W_0^{1,p(x)}(\Omega)$ with

$$\lim_j \varphi(u_j) = \lim_j \varphi(v_j) = c, \quad \lim_j \varphi'(u_j) = \lim_j \varphi'(v_j) = 0$$

and

$$\lim_j \|u_j - w_j\|_{W_0^{1,p(x)}(\Omega)} = 0, \quad \lim_j \|v_j - w_j^H\|_{W_0^{1,p(x)}(\Omega)} = 0.$$

Since φ satisfies the Palais–Smale condition (to this regard, we refer the reader to [5, pp. 614–615], our functional is included in the framework covered therein), up to a subsequence, (u_j) converges to some $u \in W_0^{1,p(x)}(\Omega)$. Hence, the sequence (w_j) also converges to u. By continuity of the polarization, (v_j) converges to u^H. The conclusion follows since φ is of class C^1. □

4 Proof of Theorem 1.2

We recall a definition from [19]. Let X and V be two Banach spaces and let S be a subset of X. We consider two maps $* : S \to V$, $u \mapsto u^*$ (*symmetrization map*) and $h : S \times \mathcal{H}_0 \to S$, $(u, H) \mapsto u^H$ (*polarization map*), where $\mathcal{H}_0$ is a path-connected topological space. We assume:

(i) X is continuously embedded in V,

(ii) h is a continuous mapping,

(iii) for each $u \in S$ and $H \in \mathcal{H}_0$ it holds $(u^*)^H = (u^H)^* = u^*$ and $u^{HH} = u^H$,

(iv) there exists a sequence (H_m) in $\mathcal{H}_0$ such that, for $u \in S$, $u^{H_1 \cdots H_m}$ converges to u^* in V,

(v) for every $u, v \in S$ and $H \in \mathcal{H}_0$ it holds $\|u^H - v^H\|_V \le \|u - v\|_V$.

We recall the main result of [19].

Lemma 4.1. *Let X and V be two Banach spaces, $S \subset X$, $*$ and $\mathcal{H}_0$ satisfying the requirements of the abstract symmetrization framework. Let $\varphi : X \to \mathbb{R}$ be a C^1 functional. Let M be a metric space and let M_0 be a closed subset of M and $\Gamma_0 \subset C(M_0, X)$. Let us define*

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

Assume that

$$+\infty > c = \inf_{\gamma \in \Gamma} \sup_{\tau \in M} \varphi(\gamma(\tau)) > \sup_{\gamma_0 \in \Gamma_0} \sup_{\tau \in M_0} \varphi(\gamma_0(\tau)) = a,$$

and that

$$\forall H \in \mathcal{H}_0,\ \forall u \in S: \quad \varphi(u^H) \le \varphi(u).$$

Then, for every $\varepsilon \in (0, (c-a)/2)$, every $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{\tau \in M} \varphi(\gamma(\tau)) \le c + \varepsilon, \quad \gamma(M) \subset S, \quad \gamma|_{M_0}^{H_0} \in \Gamma_0 \text{ for some } H_0 \in \mathcal{H}_0,$$

there exists $u \in X$ such that

$$c - 2\varepsilon \le \varphi(u) \le c + 2\varepsilon, \quad \|d\varphi(u)\| \le 8\varepsilon/\delta, \quad \|u - u^*\|_V \le K\delta,$$

being K a constant depending upon the embedding $i : X \to V$.

Lemma 4.2. *Assume that $\sigma_H(\Omega) = \Omega$ for all $H \in \mathcal{H}_0$ and that* (1.3) *holds for any $H \in \mathcal{H}_0$. Then the choice $X := S = W_0^{1,p(x)}(\Omega)$ and $V := L^{p-}(\Omega)$ endowed with the natural norms is compatible with abstract symmetrization framework.*

Proof. Since Ω is invariant under reflection with respect to all $H \in \mathcal{H}_0$, it follows that Ω is invariant under cap symmetrization [19]. Of course X is continuously embedded into V. Let us now prove that $h(u, H) := u^H$ is a continuous mapping from $X \times \mathcal{H}_0$ to X. Here $\mathcal{H}_0$ is meant to be endowed with the metric d introduced in [20, Definition 2.35], which makes $\mathcal{H}_0$ a separable metric space. Let (u_j, H_j) be a sequence in $X \times \mathcal{H}_0$ which converges to (u_0, H_0). As for identity (3.6), for every $\lambda > 0$

$$\int_\Omega \left(\frac{|Du_j^{H_j}|}{\lambda}\right)^{p(x)} = \int_\Omega \left(\frac{|Du_j|}{\lambda}\right)^{p(x)}, \quad \text{for all } j \geq 1.$$

Then, it follows that $(u_j^{H_j})$ remains bounded in X and, up to a subsequence, it converges to some function w weakly in X (and strongly in V by the compact embedding theorem). In particular, $(u_j^{H_j})$ converges to w in $L^{p-}(\Omega)$. On the other hand, if $(\vartheta_m) \subset C_c^\infty(\Omega)$ is a sequence converging to u_0 strongly in $L^{p-}(\Omega)$ as $m \to \infty$, for every $j, m \geq 1$, we have

$$\begin{aligned}
\|u_j^{H_j} - u_0^{H_0}\|_{L^{p-}(\Omega)} &\leq \|u_j^{H_j} - u_0^{H_j}\|_{L^{p-}(\Omega)} + \|u_0^{H_j} - u_0^{H_0}\|_{L^{p-}(\Omega)} \\
&\leq \|u_j - u_0\|_{L^{p-}(\Omega)} + \|u_0^{H_j} - \vartheta_m^{H_j}\|_{L^{p-}(\Omega)} \\
&\qquad + \|\vartheta_m^{H_j} - \vartheta_m^{H_0}\|_{L^{p-}(\Omega)} + \|\vartheta_m^{H_0} - u_0^{H_0}\|_{L^{p-}(\Omega)} \\
&\leq C\|u_j - u_0\|_{L^{p(x)}(\Omega)} + 2\|\vartheta_m - u_0\|_{L^{p-}(\Omega)} \\
&\qquad + \|\vartheta_m^{H_j} - \vartheta_m^{H_0}\|_{L^{p-}(\Omega)}.
\end{aligned}$$

Letting $j \to \infty$ at m fixed first and then finally $m \to \infty$, it follows that $(u_j^{H_j})$ converges to $u_0^{H_0}$ in $L^{p-}(\Omega)$. We also used the fact that for a fixed compactly supported function ϑ, it holds that ϑ^{H_j} converges to ϑ^{H_0} uniformly on Ω for $j \to \infty$. By uniqueness, $w = u_0^{H_0}$. In conclusion

$$u_j^{H_j} \rightharpoonup u_0^{H_0}, \quad \text{as } j \to \infty, \quad \text{and} \quad \lim_{j\to\infty} \|Du_j^{H_j}\|_{L^{p(x)}(\Omega)} = \|Du_0^{H_0}\|_{L^{p(x)}(\Omega)}.$$

Then, since as already remarked $W_0^{1,p(x)}(\Omega)$ is uniformly convex, we can conclude that

$$u_j^{H_j} \to u_0^{H_0} \quad \text{as } j \to \infty,$$

concluding the proof of the continuity of h. Also, for all $u \in X$, u belongs to $L^{p-}(\Omega)$ and, in light of [19, Theorem 2.1], there exists a sequence $(H_j) \subset \mathcal{H}_0$ such that, for all $u \in L^\rho(\Omega)$, $\|u^{H_1 \cdots H_j} - u^*\|_{L^{p-}} \to 0$. The contractivity of u^H is the space $L^{p-}(\Omega)$ is a standard fact. □

4.1 Proof of Theorem 1.2 concluded

On account of Lemma 4.2, it is sufficient to argue as for the proof of Theorem 1.1. Applying Lemma 4.1 with the choices (3.7), and with $\delta_j = 1/j$ and $\varepsilon_j = 1/j^2$, we find $(u_j) \subset W_0^{1,p(x)}(\Omega)$ such that $\varphi(u_j) \to c$ and $\varphi'(u_j) \to 0$ as $j \to \infty$ and $\|u_j - u_j^*\|_{L^{p-}(\Omega)} \to 0$ as $j \to \infty$. Since, as already pointed out in the proof of Theorem 1.1, φ satisfies the Palais–Smale condition, it follows that, up to a subsequence, (u_j) converges to some $u \in W_0^{1,p(x)}(\Omega)$. Hence $\varphi(u) = c$ and $\varphi'(u) = 0$. Finally, since

$$\begin{aligned}\|u - u^*\|_{L^{p-}(\Omega)} &\le \|u - u_j\|_{L^{p-}(\Omega)} + \|u_j - u_j^*\|_{L^{p-}(\Omega)} + \|u^* - u_j^*\|_{L^{p-}(\Omega)} \\ &\le 2C\,\|u - u_j\|_{L^{p(x)}(\Omega)} + \|u_j - u_j^*\|_{L^{p-}(\Omega)},\end{aligned}$$

taking into account Poincaré inequality, letting $j \to \infty$, yields $u = u^*$. This concludes the proof. □

5 Proof of Theorem 1.4

We consider $C^{1,\alpha}$ solutions to problem (1.7). Obviously problem (1.7) has to be understood in weak sense, that is $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution to (1.7) if

$$\int_\Omega |Du|^{p(x)-2}(Du, D\varphi) = \int_\Omega f(|x|, u)\varphi, \quad \text{for all } \varphi \in C_c^1(\Omega). \tag{5.1}$$

Throughout this section we shall always assume the assumptions of Theorem 1.4.

5.1 A summability result

We have the following

Lemma 5.1. *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a positive solution to* (1.7). *Then*

$$\int_\Omega \frac{1}{|Du|^{(p(x)-1)r}|x-y|^\gamma} \le C,$$

where C is a positive constant independent of y, $0 \le r < 1$, $\gamma < N-2$ if $N \ge 3$ and $\gamma = 0$ if $N = 2$. In particular it follows that the critical set

$$Z_u = \{x \in \Omega : |Du(x)| = 0\}$$

has zero Lebesgue measure.

Proof. We consider, for $y \in \mathbb{R}^N$, the test function

$$\psi_\varepsilon(x) = (\varepsilon + |Du|^{(p(x)-1)r})^{-1}\eta(\varepsilon + |x-y|)^{-\gamma},$$

where η is a positive smooth cut-off function with $\mathrm{supt}(\eta) = \Omega_0$ such that $\eta = 1$ on $\tilde{\Omega}_0 \subset \Omega_0$ and $\tilde{\Omega}_0 \subset\subset \Omega$ is such that $(\Omega \setminus \tilde{\Omega}_0) \cap Z_u = \emptyset$. In fact, we recall that, in light of the Hopf Boundary Lemma of [21], we have $Z_u \cap \partial\Omega = \emptyset$. Note that ψ_ε is a good test function since it belongs to $W^{1,2}(\Omega)$ by the summability properties of the solutions proved in [2] and thus it can be plugged into (5.1) by density arguments.

Again by the Hopf Boundary Lemma, to achieve the conclusion, it is enough to show that

$$\int_{\tilde{\Omega}_0} \frac{1}{|Du|^{(p(x)-1)r}|x-y|^\gamma} \leq C\,, \tag{5.2}$$

for $\tilde{\Omega}_0 \subset\subset \Omega$. Moreover, without loss of generality, we can reduce to consider the case

$$\max_{x \in \bar{\Omega}_0} \frac{p(x)-2}{p(x)-1} \leq r < 1. \tag{5.3}$$

In fact, once (5.2) holds for $C^{1,\alpha}$ solutions, the same estimation easily follows for $r' < r$. We put ψ_ε as test function in (1.7) and since $f(|x|, u) \geq \sigma$ for some $\sigma > 0$ in the support of ψ_ε, we get

$$\begin{aligned}
&\sigma \int_{\Omega_0} \frac{\eta}{(\varepsilon + |Du|^{(p(x)-1)r})\,(\varepsilon + |x-y|)^\gamma} \\
&\leq \int_{\Omega_0} f(|x|, u)\psi_\varepsilon \\
&\leq \int_{\Omega_0} |Du|^{p(x)-2}|(Du, D\psi_\varepsilon)| \\
&\leq \int_{\Omega_0} (p(x)-1)r \frac{|Du|^{p(x)-2}}{(\varepsilon + |Du|^{(p(x)-1)r})^2} |Du|^{(p(x)-1)r} \frac{1}{(\varepsilon + |x-y|)^\gamma} \eta \|D^2 u\| \\
&\qquad + \int_{\Omega_0} r|\log|Du|| \frac{|Du|^{p(x)-2}}{(\varepsilon + |Du|^{(p(x)-1)r})^2} \\
&\qquad\qquad \times |Du|^{(p(x)-1)r+1} \frac{1}{(\varepsilon + |x-y|)^\gamma} \eta |Dp| \\
&\qquad + \int_{\Omega_0} \frac{|Du|^{p(x)-2}}{(\varepsilon + |Du|^{(p(x)-1)r})} \frac{|Du|\,|D\eta|}{(\varepsilon + |x-y|)^\gamma} \\
&\qquad + \int_{\Omega_0} \gamma \frac{|Du|^{p(x)-2}}{(\varepsilon + |Du|^{(p(x)-1)r})} \frac{\eta |Du|}{(\varepsilon + |x-y|)^{(\gamma+1)}}.
\end{aligned}$$

Since the critical set Z_u is the zero level set of $|Du|^{(p(x)-1)r}$, by Stampacchia's theorem the gradient of $|Du|^{(p(x)-1)r}$ vanishes a.e. in Z_u. In the above calculations we consequently agree that the term $\log|Du|$ make sense outside Z_u, while in Z_u the distributional derivatives of $|Du|^{(p(x)-1)r}$ are zero.

Taking into account that

$$|\log t| \leq C_\delta + t^\delta + t^{-\delta}, \quad t > 0,$$

for all $\delta > 0$ and some $C_\delta > 0$, we have

$$\begin{aligned}
\sigma \int_{\Omega_0} & \frac{\eta}{(\varepsilon + |Du|^{(p(x)-1)r})(\varepsilon + |x-y|)^\gamma} \\
\leq & \int_{\Omega_0} (p(x)-1)r \frac{|Du|^{p(x)-2}}{(\varepsilon + |Du|^{(p(x)-1)r})^2} |Du|^{(p(x)-1)r} \frac{1}{(\varepsilon + |x-y|)^\gamma} \eta \|D^2 u\| \\
& + C \int_{\Omega_0} \frac{|Du|^{p(x)-1}}{(\varepsilon + |Du|^{(p(x)-1)r})} \frac{1}{(\varepsilon + |x-y|)^\gamma} \\
& + C \int_{\Omega_0} \frac{|Du|^{p(x)-1+\delta}}{(\varepsilon + |Du|^{(p(x)-1)r})} \frac{1}{(\varepsilon + |x-y|)^\gamma} \\
& + C \int_{\Omega_0} \frac{|Du|^{p(x)-1-\delta}}{(\varepsilon + |Du|^{(p(x)-1)r})} \frac{1}{(\varepsilon + |x-y|)^\gamma} \\
& + C \int_{\Omega_0} \frac{|Du|^{p(x)-1}}{(\varepsilon + |Du|^{(p(x)-1)r})} \frac{1}{(\varepsilon + |x-y|)^{\gamma+1}} + C,
\end{aligned} \tag{5.4}$$

where δ was fixed small depending on the size of p_-. Since $u \in C^{1,\alpha}$ and $\gamma < N-2$, from (5.4) we get

$$\begin{aligned}
\sigma \int_{\Omega_0} & \frac{\eta}{(\varepsilon + |Du|^{(p(x)-1)r})(\varepsilon + |x-y|)^\gamma} \\
& \leq C \int_{\Omega_0} \frac{|Du|^{(p(x)-2)+(p(x)-1)r}}{(\varepsilon + |Du|^{(p(x)-1)r})^2} \frac{\eta \|D^2 u\|}{(\varepsilon + |x-y|)^\gamma} + C.
\end{aligned}$$

If $\beta \in C(\bar{\Omega}_0)$ is such that

$$\beta(x) = 1 - (p(x)-1)(1-r),$$

with $0 \leq \beta(x) < 1$ by virtue of (5.3), by writing

$$\begin{aligned}
& \frac{|Du|^{(p(x)-2)+(p(x)-1)r}}{(\varepsilon + |Du|^{(p(x)-1)r})^2} \frac{\eta \|D^2 u\|}{(\varepsilon + |x-y|)^\gamma} \\
& \quad = \left[\frac{|Du|^{\frac{3r(p(x)-1)}{2}}}{(\varepsilon + |Du|^{(p(x)-1)r})^2} \frac{\eta^{1/2}}{(\varepsilon + |x-y|)^{\gamma/2}} \right] \left[\frac{\eta^{1/2} |Du|^{\frac{p(x)-2-\beta(x)}{2}} \|D^2 u\|}{(\varepsilon + |x-y|)^{\gamma/2}} \right]
\end{aligned}$$

and using a weighted Young inequality, we finally obtain

$$\begin{aligned}\sigma \int_{\Omega_0} & \frac{\eta}{(\varepsilon + |Du|^{(p(x)-1)r})(\varepsilon + |x-y|)^\gamma} \\ & \leq \delta' \int_{\Omega_0} \frac{\eta}{(\varepsilon + |Du|^{(p(x)-1)r})(\varepsilon + |x-y|)^\gamma} \\ & \quad + \frac{C}{\delta'} \int_{\Omega_0} |Du|^{p(x)-2-\beta(x)} \|D^2 u\|^2 \frac{1}{(\varepsilon + |x-y|)^\gamma} + C.\end{aligned}$$

Repeating step by step the proof of [2, Lemma 3.1] (adding $|x-y|^{-\gamma}$ in all formulas), we end-up with

$$\int_{\Omega_0} \frac{|Du|^{p(x)-2-\beta(x)} \|D^2 u\|^2}{|x-y|^\gamma} \leq C.$$

Hence, choosing $\delta' < \sigma$ we have the desired conclusion letting $\varepsilon \to 0^+$ and recalling that $\eta = 1$ on $\tilde{\Omega}_0$. □

5.2 A weighted Sobolev inequality

Given a solution u to problem (1.7), for $p(x) \geq 2$ we set

$$\rho(x) = |Du(x)|^{p(x)-2}, \quad x \in \Omega,$$

and define the Hilbert space $H_\rho^{1,2}(\Omega)$ as the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|v\|^2_{H_\rho^{1,2}} = \int_\Omega v^2 + \int_\Omega \rho(x)|Dv|^2.$$

Since the domain Ω is smooth, equivalently, $H_\rho^{1,2}$ is composed by the functions v which have distributional derivative with finite norm. The space $H_{0,\rho}^{1,2}$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H_\rho^{1,2}}$ and it is a reflexive Hilbert space.

Moreover let $\mu \in C(\overline{\Omega})$ be such that $0 < \mu_- \leq \mu_+ \leq 1$ and let us define the function

$$V_\mu[g, U](x) := \int_U \frac{g(y)}{|x-y|^{N(1-\mu(x))}} dy. \tag{5.5}$$

By [14, Theorem 3.1] it follows that, for any $1 \leq q(x) \leq \infty$, with

$$\frac{1}{m(x)} - \frac{1}{q(x)} \leq \mu(x)$$

it follows

$$\|V_\mu[g, \Omega](x)\|_{q(\cdot)} \leq \Theta \|g\|_{m(\cdot)}, \tag{5.6}$$

for some positive constant Θ and for any $g \in L^{m(\cdot)}(\Omega)$.

We can now prove the following

Theorem 5.2. *Let $p(x) \geq 2$ for all $x \in \Omega$ and set*

$$\bar{t} := \inf_{x \in \overline{\Omega}} \frac{p(x)-1}{p(x)-2} r,$$

where $r > 0$ is such that

$$\begin{aligned} &\int_\Omega \frac{1}{\rho^{t(x)}|x-y|^\gamma} \leq C(\gamma), \\ &\max_{x \in \overline{\Omega}} \frac{p(x)-2}{p(x)-1} \leq r < 1, \quad t(x) := \frac{p(x)-1}{p(x)-2} r, \end{aligned} \tag{5.7}$$

with $N - 2\bar{t} < \gamma < N-2$ if $N \geq 3$ and $\gamma = 0$ if $N = 2$. Then, for any function $w \in H^{1,2}_{0,\rho}(\Omega)$, we have

$$\|w\|_{q(\cdot)} \leq C \left(\int_\Omega \rho |Dw|^2 \right)^{\frac{1}{2}}, \tag{5.8}$$

for some positive constant C and any $1 \leq q(\cdot) < 2^*(\bar{t})$, where

$$\frac{1}{2^*(\bar{t})} = \frac{1}{2} - \frac{1}{N} + \frac{1}{\bar{t}} \left(\frac{1}{2} - \frac{\gamma}{2N} \right). \tag{5.9}$$

Furthermore the embedding of $H^{1,2}_{0,\rho}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ is compact.

Proof. We can assume that $w \in C^1_c(\Omega)$. Hence standard potential estimates (see [12, Lemma 7.14]) give

$$|w(x)| \leq C \int_\Omega \frac{|Dw(y)|}{|x-y|^{N-1}} dy,$$

where C is a constant depending on the dimension N. Then

$$\begin{aligned} |w(x)| &\leq C \int_\Omega \frac{|Dw(y)|}{|x-y|^{N-1}} dy \\ &\leq C \int_\Omega \frac{1}{\rho^{\frac{1}{2}}|x-y|^{\frac{\gamma}{2\bar{t}}}} \frac{|Dw(y)|\rho^{\frac{1}{2}}}{|x-y|^{N-1-\frac{\gamma}{2\bar{t}}}} dy \\ &\leq C \left(\int_\Omega \frac{1}{\rho^{\bar{t}}|x-y|^\gamma} dy \right)^{\frac{1}{2\bar{t}}} \left(\int_\Omega \frac{\left(|Dw(y)|\rho^{\frac{1}{2}}\right)^{(2\bar{t})'}}{|x-y|^{(N-1-\frac{\gamma}{2\bar{t}})(2\bar{t})'}} dy \right)^{\frac{1}{(2\bar{t})'}}, \end{aligned}$$

where in the last inequality we used Hölder's inequality with $\frac{1}{2\bar{t}} + \frac{1}{(2\bar{t})'} = 1$. Note that, by the definition of $\bar{t}$ and by (5.7), it follows that

$$\int_\Omega \frac{1}{\rho^{\bar{t}}|x-y|^\gamma} \leq C.$$

Hence

$$|w(x)| \leq C\left(\int_\Omega \frac{\left(|Dw(y)|\rho^{\frac{1}{2}}\right)^{(2\bar{t})'}}{|x-y|^{(N-1-\frac{\gamma}{2\bar{t}})(2\bar{t})'}}dy\right)^{\frac{1}{(2\bar{t})'}}. \tag{5.10}$$

We point out that

$$\left(|Dw|\rho^{\frac{1}{2}}\right)^{(2\bar{t})'} \in L^{\frac{2}{(2\bar{t})'}}(\Omega). \tag{5.11}$$

From (5.10), by using equation (5.5) with $\mu = 1 - \frac{1}{N}(N-1-\frac{\gamma}{2\bar{t}})(2\bar{t})'$, we obtain

$$|w(x)| \leq C\left(V_\mu\left[\left(|Dw(y)|\rho^{\frac{1}{2}}\right)^{(2\bar{t})'}, \Omega\right](x)\right)^{\frac{1}{(2\bar{t})'}}.$$

Since $\gamma > N - 2\bar{t}$, we also have $N\bar{t} - 2N + 2\bar{t} + \gamma > 0$ and $\mu > 0$. We shall use now estimate (5.6) (see [14, Theorem 3.1]) with $\frac{1}{m} = (2\bar{t})'/2$, see (5.11). Let us now fix an arbitrary $\tilde{q}(\cdot) > 1$ such that $1/m - 1/\tilde{q}(\cdot) \leq \mu$, which is possible since $1/m - \mu < 1$, as follows by $N\bar{t} - 2N + 2\bar{t} + \gamma > 0$. Therefore, we have

$$\begin{aligned}\|w(x)\|_{\tilde{q}(\cdot)(2\bar{t})'} &\leq C\left\|\left(V_\mu\left[\left(|Dw(y)|\rho^{\frac{1}{2}}\right)^{(2\bar{t})'}, \Omega\right](x)\right)^{\frac{1}{(2\bar{t})'}}\right\|_{\tilde{q}(\cdot)(2\bar{t})'} \\ &\leq C\left\|V_\mu\left[\left(|Dw(y)|\rho^{\frac{1}{2}}\right)^{(2\bar{t})'}, \Omega\right](x)\right\|_{\tilde{q}(\cdot)}^{\frac{1}{(2\bar{t})'}}. \end{aligned} \tag{5.12}$$

From (5.12), by (5.6) we get

$$\|w\|_{\tilde{q}(\cdot)(2\bar{t})'} \leq C\left(\int_\Omega \rho|Dw|^2\right)^{\frac{1}{2}},$$

that gives (5.8) and (5.9) with $q(x) = \tilde{q}(x)(2\bar{t})'$, and consequently for any $q(\cdot)$ as in the statement of the theorem.

Finally the compactness of the embedding follows arguing exactly as in [3]. □

5.3 The eigenvalue problem

Let us consider the linearized operator

$$\begin{aligned} L_u(v,\varphi) := \int_\Omega |Du|^{p(x)-2}(Dv, D\varphi) \\ + \int_\Omega (p(x)-2)|Du|^{p(x)-4}(Du, Dv)(Du, D\varphi) \\ - \int_\Omega \partial_s f(|x|, u)v\varphi, \end{aligned}$$

for any $v, \varphi \in H^{1,2}_{0,\rho}$. We also define $\|\cdot\|_{A_u}$ to be the norm arising from the scalar product

$$\langle v, \varphi \rangle := \int_\Omega |Du|^{p(x)-2}(Dv, D\varphi) \\ + \int_\Omega (p(x)-2)|Du|^{p(x)-4}(Du, Dv)(Du, D\varphi),$$

that is a norm equivalent to $\|v\|_{H^{1,2}_{0,\rho}} = \left(\int_\Omega \rho |Dv|^2\right)^{\frac{1}{2}}$.

Since $\partial_s f(|x|, u) \in L^\infty(\Omega)$, the first eigenvalue $\mu_1(u)$ of the linearized operator is well defined by

$$\mu_1(u) = \inf_{\phi \in H^{1,2}_{0,\rho} \setminus \{0\}} R_u(\phi), \quad R_u(\phi) = \frac{\|\phi\|^2_{A_u} - \int_\Omega \partial_s f(|x|, u)\phi^2}{\int_\Omega \phi^2}.$$

Consider now a minimizing sequence $\phi_n \in H^{1,2}_{0,\rho}$, $\int_\Omega \phi_n^2 = 1$, with $R_u(\phi_n)$ converging to $\mu_1(u)$ as $n \to \infty$. Since $\partial_s f(|x|, u) \in L^\infty(\Omega)$, we have that the sequence $(\|\phi_n\|_{A_u})$ remains bounded. Therefore, up to a subsequence, we get that $\phi_n \rightharpoonup \phi_1$ weakly in $H^{1,2}_{0,\rho}$ and therefore $\phi_n \to \phi_1$ strongly in $L^2(\Omega)$ (by combining Lemma 5.1 and Theorem 5.2). Now, the term $\int_\Omega \partial_s f(|x|, u)\phi^2$ is continuous in $L^2(\Omega)$ and $\|\cdot\|_{A_u}$ is weakly lower semi-continuous in $H^{1,2}_{0,\rho}$. Therefore, $\phi_1 \in H^{1,2}_{0,\rho}$ is such that $\int_\Omega \phi_1^2 = 1$ and $R_u(\phi_1) \le \mu_1(u)$. Hence, $\mu_1(u)$ is attained at ϕ_1. It is now standard to show that ϕ_1 solves $L_u(\phi_1, \varphi) = \int_\Omega \mu_1(u)\phi_1\varphi$ for any $\varphi \in H^{1,2}_{0,\rho}$. Arguing now exactly as in [3, p. 299], we get that every minimizer is of fixed sign and the first eigenspace is one-dimensional.

Remark 5.3. Following [4] it is now possible to develop a complete spectral theory for the linearized operator, showing that it has an increasing discrete sequence of eigenvalues with finite dimensional eigenspaces.

5.4 Proof of Theorem 1.4 completed

Let us write the solution $u = u(r, \theta)$ in polar coordinates, where $r = |x|$ and $\theta = (\theta_1, \ldots, \theta_{n-1})$ are the $n-1$ angular variables. Assume first that u is semi-stable according to Definition 1.3. If u was not radial, then $u_{\theta_i} \neq 0$ and u_{θ_i} changes sign, for some $i \in \{1, \ldots, n-1\}$. Notice now that, since we are considering C^1 solutions, it is clear from the proof that [2, Lemma 3.1] can be stated with $\beta \equiv 0$ and $\varepsilon = 0$. In particular, we get

$$\int_\Omega |Du|^{p(x)-2}|Du_{\theta_i}|^2 \le C \int_\Omega |Du|^{p(x)-2}\|D^2 u\|^2 \le C,$$

and by the boundary conditions we obtain $u_{\theta_i} \in H^{1,2}_{0,\rho}$. It is now easy to see that, since $p(x)$ is radially symmetric, it follows, differentiating the equation in (1.7) with respect to θ_i, that

$$L_u(u_{\theta_i}, \varphi) = 0, \quad \text{for all } \varphi \in H^{1,2}_{0,\rho}. \tag{5.13}$$

In particular, u_{θ_i} is an eigenfunction of the linearized operator corresponding to the 0 eigenvalue. By the semi-stability assumption on u, this implies that u_{θ_i} is the first eigenfunction of L_u and consequently (see Section 5.3) it should have constant sign in Ω. This contradiction shows that u is radially symmetric. If else we assume that u is non-degenerate, the conclusion follows in the same way, noticing that 0 is not an eigenvalue and therefore (5.13) implies that $u_{\theta_i} = 0$. □

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Bibliography

[1] T. Bartsch, T. Weth and M. Willem, Partial symmetry of least energy nodal solutions to some variational problems, *J. Anal. Math.* **96** (2005), 1–18.

[2] A. Canino, P. Le and B. Sciunzi, Local W^{2m}_{loc} regularity for $p(x)$-Laplace equations, *Manuscripta Math.*, in press.

[3] D. Castorina, P. Esposito and B. Sciunzi, Degenerate elliptic equations with singular nonlinearities, *Calc. Var. Partial Differential Equations* **34** (2009), 279–306.

[4] D. Castorina, P. Esposito and B. Sciunzi, Spectral theory for linearized p-Laplace equations, *Nonlinear Anal.* **74** (2011), 3606–3613.

[5] J. Chabrowski and Y. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, *J. Math. Anal. Appl.* **306** (2005), 604–618.

[6] L. Damascelli and B. Sciunzi, Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations, *J. Differential Equations* **206** (2004), 483–515.

[7] L. Diening, P. Harjulehto, P. Hästö and M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics 2017, Springer-Verlag, Heidelberg, 2011.

[8] X. Fan, Existence and uniqueness for the $p(x)$-Laplacian Dirichlet problems, *Math. Nachr.* **284** (2011), 1435–1445.

[9] A. Farina, B. Montoro and B. Sciunzi, Monotonicity of solutions of quasilinear degenerate elliptic equation in half-spaces, preprint.

[10] A. Farina, E. Valdinoci and B. Sciunzi, Bernstein and De Giorgi type problems: New results via a geometric approach, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **7** (2008), 741–791.

[11] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[13] L. Montoro, B. Sciunzi and M. Squassina, Asymptotic symmetry for a class of quasilinear parabolic problems, *Adv. Nonlinear Stud.* **10** (2010), 789–818.

[14] N. G. Samko, S. G. Samko and B. G. Vakulov, Weighted Sobolev theorem in Lebesgue spaces with variable exponent, *J. Math. Anal. Appl.* **335** (2007), 560–583.

[15] J. Serrin, Weighted Sobolev theorem in Lebesgue spaces with variable exponent, *Arch. Ration. Mech. Anal* **43** (1971), 304–318.

[16] D. Smets and M. Willem, Partial symmetry and asymptotic behavior for some elliptic variational problems, *Calc. Var. Partial Differential Equations* **18** (2003), 57–75.

[17] M. Squassina, Radial symmetry of minimax critical points for nonsmooth functionals, *Commun. Contemp. Math.* **13** (2011), 487–508.

[18] M. Squassina and J. Van Schaftingen, Finding critical points whose polarization is a critical point, *Topol. Methods Nonlinear Anal.*, to appear.

[19] J. Van Schaftingen, Symmetrization and minimax principles, *Commun. Contemp. Math.* **7** (2005), 463–481.

[20] J. Van Schaftingen, Approximation of symmetrizations and symmetry of critical points, *Topol. Methods Nonlinear Anal.* **28** (2006), 61–85.

[21] Q. Zhang, A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions, *J. Math. Anal. Appl.* **312** (2005), 24–32.

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