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Existence results for double-phase problems via Morse theory

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We obtain nontrivial solutions for a class of double-phase problems using Morse theory. In the absence of a direct sum decomposition, we use a cohomological local splitting to get an estimate of the critical groups at zero.

Keywords: Double-phase problems; Musielak spaces; Morse theory; cohomological local splitting.

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1. Introduction

The study of energy functionals of the form

$$u \mapsto \int_\Omega \mathcal{H}(x, |\nabla u(x)|)dx, \quad \mathcal{H}(x,t) = t^p + a(x)t^q, \quad q > p > 1, \ a(\cdot) \geq 0, \tag{1.1}$$

where the integrand $\mathcal{H}$ switches between two different elliptic behaviors has been intensively studied since the late eighties. This class of energies was introduced by Zhikov to provide models of *strongly anisotropic* materials,see e.g., [19–21] or the monograph [22]. Also, the integrals of (1.1) settle in the framework of the so-called functionals with non-standard growth conditions, according to a terminology introduced by Marcellini [9, 14, 15]. In [22], energies of the form (1.1) are used in the context of homogenization and elasticity and $a(\cdot)$ drives the geometry of a composite of two different materials with hardening powers p and q.

Significant progresses were recently achieved by Mingione *et al.* in the framework of regularity theory for minimizers of (1.1), see e.g., [1–3, 6, 7]. More recently, in [5], a complete study on the existence and properties of a sequence of variational eigenvalues related to $\mathcal{H}$ including a Weyl type estimate for their growth has been performed. The purpose of this paper is to investigate the existence of solutions to the double phase problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 2$, $1 < p < q < N$,

$$\frac{q}{p} < 1 + \frac{1}{N}, \quad a : \overline{\Omega} \to [0,\infty) \text{ is Lipschitz continuous,} \tag{1.3}$$

and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the growth condition

$$|f(x,t)| \leq C(|t|^{r-1} + 1) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{1.4}$$

for some $1 < r < p^*$ and $C > 0$, being $p^* = Np/(N-p)$ the critical Sobolev exponent of $W_0^{1,p}(\Omega)$. Assuming that $f(x,0) \equiv 0$, problem (1.2) has the trivial solution $u(x) = 0$ and we study the critical groups of the associated variational functional at 0, obtaining a nontrivial solution using Morse theory. In the absence of a direct sum decomposition, we use a cohomological local splitting to get an estimate of the critical groups.

Our main result is for the q-superlinear case for the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) \\ \qquad = \lambda|u|^{p-2}u + |u|^{r-2}u + h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\lambda \in \mathbb{R}$ is a parameter, $q < r < p^*$ and h is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|h(x,t)| \leq C(|t|^{\rho-1} + |t|^{\sigma-1}), \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{1.5}$$

for some $p < \sigma < \rho < r$ and $C > 0$. The notion of weak solution for problem (P) is formulated in a suitable Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$ that will be introduced in Sec. 2.

Let $(\lambda_k) \subset \mathbb{R}^+$ be the sequence of (variational) eigenvalues of the p-Laplacian operator defined via cohomological index, cf. formula (2.13). Let us set

$$G(x,t) := \frac{|t|^r}{r} + H(x,t), \quad x \in \Omega, \ t \in \mathbb{R},$$

being $H(x,t) = \int_0^t h(x,\tau)d\tau$. The following is the main result of the paper.

Theorem 1.1. *Assume that conditions* (1.3) *and* (1.5) *hold. Then problem* (P) *has a nontrivial weak solution* $u \in W_0^{1,\mathcal{H}}(\Omega)$ *in each of these cases*:

(1) $\lambda \notin \{\lambda_k\}_{k \geq 1}$;
(2) *for some* $\delta > 0$, $G(x,t) \leq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$;
(3) $G(x,t) \geq c|t|^s$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$ *for some* $s \in (p,q)$ *and* $c > 0$.

To our knowledge this is the first existence result for double-phase problems (1.2) in the framework of Morse theory and it is obtained by analyzing the critical groups $H^q(\Phi^0 \cap U, \Phi^0 \cap U \backslash \{0\})$ of the associated energy functional Φ at zero, $q \in \mathbb{N}$.

2. Preliminaries and Proof

2.1. *Variational setting*

The Musielak–Orlicz space $L^{\mathcal{H}}(\Omega)$ associated with the function

$$\mathcal{H} : \Omega \times [0,\infty) \to [0,\infty), \quad (x,t) \mapsto t^p + a(x)t^q$$

consists of all measurable functions $u : \Omega \to \mathbb{R}$ with the $\mathcal{H}$-modular

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)dx < \infty,$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf\left\{\gamma > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\gamma}\right) \leq 1\right\}.$$

The space $L^{\mathcal{H}}(\Omega)$ is a uniformly convex, and hence reflexive, Banach space. Denoting by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ and by $L^q_a(\Omega)$ the space of all measurable functions $u : \Omega \to \mathbb{R}$ with the seminorm

$$\|u\|_{q,a} := \left(\int_\Omega a(x)|u|^q dx\right)^{1/q} < \infty,$$

we have the continuous embeddings

$$L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_a(\Omega),$$

see [5, Proposition 2.15(i), (iv), (v)]. Since $\rho_{\mathcal{H}}(u/\|u\|_{\mathcal{H}}) = 1$ whenever $u \neq 0$, we have

$$\min\{\|u\|^p_{\mathcal{H}}, \|u\|^q_{\mathcal{H}}\} \leq \|u\|^p_p + \|u\|^q_{q,a} \leq \max\{\|u\|^p_{\mathcal{H}}, \|u\|^q_{\mathcal{H}}\}, \quad \forall\, u \in L^{\mathcal{H}}(\Omega). \tag{2.1}$$

The related Sobolev space $W^{1,\mathcal{H}}(\Omega)$ consists of all functions u in $L^{\mathcal{H}}(\Omega)$ with $|\nabla u| \in L^{\mathcal{H}}(\Omega)$, normed by

$$\|u\|_{1,\mathcal{H}} := \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \||\nabla u|\|_{\mathcal{H}}$. The completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$ is denoted by $W_0^{1,\mathcal{H}}(\Omega)$ and it can be equivalently renormed by

$$\|u\| := \|\nabla u\|_{\mathcal{H}}$$

via a Poincaré-type inequality, cf. [5, Proposition 2.18(iv)], under assumption (1.3). The spaces $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are uniformly convex, and hence reflexive, Banach spaces. The Sobolev embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact since $r < p^*$, cf. [5, Proposition 2.15(iii)]. We have

$$\min\{\|u\|^p, \|u\|^q\} \le \|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q \le \max\{\|u\|^p, \|u\|^q\}, \quad \forall\, u \in W_0^{1,\mathcal{H}}(\Omega), \tag{2.2}$$

by virtue of (2.1). A weak solution of problem (1.2) is a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ satisfying

$$\int_\Omega (|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2})\nabla u \cdot \nabla v dx = \int_\Omega f(x,u)v dx, \quad \forall\, v \in W_0^{1,\mathcal{H}}(\Omega).$$

Weak solutions coincide with critical points of the functional

$$\Phi(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q - F(x,u)\right] dx, \quad u \in W_0^{1,\mathcal{H}}(\Omega),$$

where $F(x,t) = \int_0^t f(x,s)ds$, by the following proposition.

Proposition 2.1 (C^1 energy). *Assume that* (1.4) *holds. Then* Φ *is of class* C^1 *with*

$$\langle \Phi'(u), v\rangle = \int_\Omega [(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2})\nabla u \cdot \nabla v - f(x,u)v]dx, \tag{2.3}$$

for every $u, v \in W_0^{1,\mathcal{H}}(\Omega)$.

Proof. In view of the embeddings mentioned above, (2.3) is clear. To see that Φ' is continuous, suppose that $u_j \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. For all $v \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|v\| = 1$, by the Hölder inequality,

$$\begin{aligned} |\langle \Phi'(u_j) - \Phi'(u), v\rangle| \le{} & \||\nabla u_j|^{p-2}\,\nabla u_j - |\nabla u|^{p-2}\nabla u|\|_{p'}\|\nabla v\|_p \\ & + \|a(x)^{1/q'}|\nabla u_j|^{q-2}\nabla u_j - |\nabla u|^{q-2}\nabla u|\|_{q'}\|\nabla v\|_{q,a} \\ & + \|f(x,u_j) - f(x,u)\|_{r'}\|v\|_r, \end{aligned}$$

where $s' = s/(s-1)$ is the Hölder conjugate of s. Since $L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L_a^q(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$, $\nabla u_j \to \nabla u$ in $L^p(\Omega) \cap L_a^q(\Omega)$, $u_j \to u$ in $L^r(\Omega)$, and $\|\nabla v\|_p$, $\|\nabla v\|_{q,a}$ and $\|v\|_r$ are uniformly bounded, the assertion follows from the dominated convergence theorem and (1.4). □

2.2. *Palais–Smale condition*

The operator $A : W_0^{1,\mathcal{H}}(\Omega) \to (W_0^{1,\mathcal{H}}(\Omega))'$ defined by

$$\langle A(u), v\rangle := \int_\Omega [(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2})\nabla u \cdot \nabla v]dx, \quad u, v \in W_0^{1,\mathcal{H}}(\Omega),$$

where $(W_0^{1,\mathcal{H}}(\Omega))'$ is the dual space of $W_0^{1,\mathcal{H}}(\Omega)$, has the following important property.

Proposition 2.2. *If $u_j \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and $A(u_j)(u_j - u) \to 0$, then $u_j \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.*

Proof. Noting that

$$\langle A(u), v\rangle \leq \|\nabla u\|_p^{p-1} \|\nabla v\|_p + \|\nabla u\|_{q,a}^{q-1} \|\nabla v\|_{q,a} \quad \forall\, u, v \in W_0^{1,\mathcal{H}}(\Omega)$$

by the Hölder inequality, and the equality holds when $u = v$, we have

$$\begin{aligned} 0 &\leq (\|\nabla u_j\|_p^{p-1} - \|\nabla u\|_p^{p-1})(\|\nabla u_j\|_p - \|\nabla u\|_p) \\ &\quad + (\|\nabla u_j\|_{q,a}^{q-1} - \|\nabla u\|_{q,a}^{q-1})(\|\nabla u_j\|_{q,a} - \|\nabla u\|_{q,a}) \\ &\leq \langle A(u_j) - A(u), u_j - u\rangle \to 0, \end{aligned}$$

so that $\|\nabla u_j\|_p \to \|\nabla u\|_p$ and $\|\nabla u_j\|_{q,a} \to \|\nabla u\|_{q,a}$. Then $\nabla u_j \to \nabla u$ in $L^p(\Omega) \cap L_a^q(\Omega)$ by uniform convexity, and hence the conclusion follows from (2.2). □

Recall that the functional Φ satisfies the Palais–Smale compactness condition at the level $c \in \mathbb{R}$, or $(\mathrm{PS})_c$ for short, if every sequence $(u_j) \subset W_0^{1,\mathcal{H}}(\Omega)$ such that $\Phi(u_j) \to c$ and $\Phi'(u_j) \to 0$, called a $(\mathrm{PS})_c$ sequence, has a convergent subsequence. We say that Φ satisfies the (PS) condition if it satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$. When verifying these conditions, it suffices to show that (u_j) is bounded by the following proposition.

Proposition 2.3 (Bounded Palais–Smale condition). *Every bounded sequence $(u_j) \subset W_0^{1,\mathcal{H}}(\Omega)$ such that $\Phi'(u_j) \to 0$ has a convergent subsequence.*

Proof. Since (u_j) is bounded, a renamed subsequence converges to some u weakly in $W_0^{1,\mathcal{H}}(\Omega)$ and strongly in $L^r(\Omega)$. Then

$$\langle A(u_j), u_j - u\rangle = \langle \Phi'(u_j), u_j - u\rangle + \int_\Omega f(x, u_j)(u_j - u)dx \to 0$$

since

$$\left| \int_\Omega f(x, u_j)(u_j - u)dx \right| \leq C(\|u_j\|_r^{r-1} + 1)\|u_j - u\|_r$$

by (1.4) and the Hölder inequality, so the conclusion follows from Proposition 2.2. □

2.3. *Regularity estimates*

For $f \in L^m(\Omega)$ with $m > 1$, solutions of

$$\int_\Omega (|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2})\nabla u \cdot \nabla v dx = \int_\Omega f(x) v dx \quad \forall\, v \in W_0^{1,\mathcal{H}}(\Omega) \qquad (2.4)$$

enjoy the natural L^m-estimates given in the following proposition.

Proposition 2.4. *Let $f \in L^m(\Omega), 1 < m \leq \infty$ and let $u \in W_0^{1,\mathcal{H}}(\Omega)$ satisfy* (2.4)*. Then*

$$\|u\|_r \leq C\|f\|_m^{1/(p-1)}, \tag{2.5}$$

where we have set

$$r = \begin{cases} \dfrac{N(p-1)m}{N-pm}, & 1 < m < \dfrac{N}{p} \\[2ex] \infty, & m > \dfrac{N}{p} \end{cases}$$

and $C = C(N, \Omega, p, m) > 0$.

Proof. For $k, \alpha > 0$ and $t \in \mathbb{R}$, set $t_k = \max\{-k, \min\{t, k\}\}$ and consider the nondecreasing function $g(t) = t_k^\alpha$ (with the agreement $a^\alpha := |a|^{\alpha-1}a$, for $a \in \mathbb{R}$). Testing equation (2.4) with the $g(u) \in W_0^{1,\mathcal{H}}(\Omega)$ provides the inequality

$$\|\nabla G(u)\|_p^p \leq \int_\Omega f(x)g(u)dx,$$

where

$$G(t) := \int_0^t g'(s)^{1/p} ds = \frac{\alpha^{1/p} p}{\alpha + p - 1} t_k^{(\alpha+p-1)/p}, \quad t \in \mathbb{R}.$$

Using the Sobolev inequality on the left and the Hölder inequality on the right now gives

$$\|u_k^{(\alpha+p-1)/p}\|_{p^*}^p \leq C\|f\|_m \|u_k^\alpha\|_{m'}. \tag{2.6}$$

If $1 < m < N/p$, take

$$\alpha = \frac{(p-1)p^*}{pm' - p^*} = \frac{N(p-1)(m-1)}{N - pm} > 0,$$

so that

$$\frac{(\alpha + p - 1)p^*}{p} = \alpha m' =: r.$$

Then $r = N(p-1)m/(N - pm)$ and (2.6) gives $\|u_k\|_r^{pr/p^*} \leq C\|f\|_m \|u_k\|_r^{r/m'}$, so

$$\|u_k\|_r \leq C\|f\|_m^{1/(p-1)}.$$

Letting $k \to \infty$ gives (2.5) for this case. If $N/p < m \leq \infty$, arguing as in [5, Sec. 3.2] gives

$$\|u\|_\infty \leq C\|f\|_m^{1/(p-1)}. \tag{2.7}$$

This concludes the proof. □

2.4. *Critical groups at zero*

In this subsection, we consider the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \lambda|u|^{p-2}u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $\lambda \in \mathbb{R}$ is a parameter and g is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|g(x,t)| \le C(|t|^{r-1} + |t|^{\sigma-1}) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{2.9}$$

for some $p < \sigma < r < p^*$ and $C > 0$. Problem (2.8) has the trivial solution $u = 0$, and we study the critical groups at 0 of the associated functional

$$\Phi(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q - \frac{\lambda}{p}|u|^p - G(x,u)\right] dx, \quad u \in W_0^{1,\mathcal{H}}(\Omega),$$

where $G(x,t) = \int_0^t g(x,s)ds$. Let us recall that the critical groups of Φ at 0 are given by

$$C^q(\Phi,0) := H^q(\Phi^0 \cap U, \Phi^0 \cap U \setminus \{0\}), \quad q \in \mathbb{N}, \tag{2.10}$$

where $\Phi^0 = \{u \in W_0^{1,\mathcal{H}}(\Omega) : \Phi(u) \le 0\}$, U is any neighborhood of 0, and H denotes Alexander–Spanier cohomology with $\mathbb{Z}_2$-coefficients. They are independent of U by the excision property of the cohomology groups. They are also invariant under homotopies that preserve the isolatedness of the critical point by the following proposition (see Chang and Ghoussoub [4] or Corvellec and Hantoute [8]).

Proposition 2.5 (Homotopical invariance). *Let $\Phi_\tau, \tau \in [0,1]$ be a family of C^1-functionals on a Banach space W such that 0 is a critical point of each Φ_τ. If there is a closed neighborhood U of 0 such that*

(1) *each Φ_τ satisfies the* (PS) *condition over U,*
(2) *U contains no other critical point of any Φ_τ,*
(3) *the map $[0,1] \to C^1(U,\mathbb{R}), \tau \mapsto \Phi_\tau$ is continuous,*

then $C^q(\Phi_0,0) \approx C^q(\Phi_1,0)$ for all q.

First we show that the critical groups of Φ at 0 depend only on the values of $g(x,t)$ for small $|t|$.

Lemma 2.6. *Let $\delta > 0$ and let $\vartheta : \mathbb{R} \to [-\delta,\delta]$ be a smooth nondecreasing function such that*

$$\vartheta(t) = -\delta \quad \text{for } t \le -\delta, \quad \vartheta(t) = t \quad \text{for } -\delta/2 \le t \le \delta/2, \quad \vartheta(t) = \delta \quad \text{for } t \ge \delta.$$

Let us set

$$\Phi_1(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q - \frac{\lambda}{p}|u|^p - G(x, \vartheta(u))\right] dx, \quad u \in W_0^{1,\mathcal{H}}(\Omega).$$

If 0 *is an isolated critical point of* Φ, *then it is also an isolated critical point of* Φ_1 *and*

$$C^q(\Phi, 0) \approx C^q(\Phi_1, 0), \quad \textit{for all } q.$$

Proof. We apply Proposition 2.5 to the family of functionals, for $u \in W_0^{1,\mathcal{H}}(\Omega)$ and $\tau \in [0,1]$,

$$\Phi_\tau(u) := \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q - \frac{\lambda}{p}\,|u|^p - G(x, (1-\tau)u + \tau\vartheta(u))\right] dx,$$

in a ball $B_\varepsilon(0) = \{u \in W_0^{1,\mathcal{H}}(\Omega) : \|u\| \le \varepsilon\}$ for $\varepsilon > 0$ small, after noting that $\Phi_0 = \Phi$. Proposition 2.3 implies that each Φ_τ satisfies the Palais–Smale condition over the ball $B_\varepsilon(0)$ and it is readily seen that the map $[0,1] \ni \tau \mapsto \Phi_\tau \in C^1(B_\varepsilon(0), \mathbb{R})$ is continuous, so it only remains to show that for sufficiently small $\varepsilon > 0$, $B_\varepsilon(0)$ contains no critical point of any Φ_τ other than 0. Suppose $u_j \to 0$ in $W_0^{1,\mathcal{H}}(\Omega)$, $\Phi'_{\tau_j}(u_j) = 0, \tau_j \in [0,1]$ and $u_j \ne 0$. Then u_j is a weak solution to

$$\begin{cases} -\mathrm{div}(|\nabla u_j|^{p-2}\nabla u_j + a(x)|\nabla u_j|^{q-2}\nabla u_j) = \lambda|u_j|^{p-2}u_j + g_j(x, u_j) & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega, \end{cases}$$

where we have set

$$g_j(x,t) = (1 - \tau_j + \tau_j\vartheta'(t))g(x, (1-\tau_j)t + \tau_j\vartheta(t)).$$

Since $(1-\tau_j)t + \tau_j\vartheta(t) = t$ for $|t| \le \delta/2$ and $|(1-\tau_j)t + \tau_j\vartheta(t)| \le |t| + \delta < 3|t|$ for $|t| > \delta/2$, the growth estimate (2.9) implies that, for some $C > 0$ independent of j,

$$|g_j(x,t)| \le C(|t|^{r-1} + |t|^{\sigma-1}) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Then $u_j \in L^\infty(\Omega)$ (cf. [5, Sec. 3.2]) with L^∞-bound independent of j. Since $u_j \to 0$ in $W_0^{1,p}(\Omega)$, it follows $\|u_j\|_\ell \to 0$ for any $\ell \ge 1$, as $j \to \infty$. By Proposition 2.4, applied with the choice

$$f_j(x) := \lambda|u_j(x)|^{p-2}u_j(x) + g_j(x, u_j(x)), \quad j \in \mathbb{N}, \ x \in \Omega,$$

we get $\|u_j\|_\infty \to 0$ since for a fixed $m_0 > N/p$ we have, for $j \to \infty$,

$$\int_\Omega |f_j|^{m_0} dx \le C\|u_j\|_{m_0(p-1)}^{m_0(p-1)} + C\|u_j\|_{m_0(r-1)}^{m_0(r-1)} + C\|u_j\|_{m_0(\sigma-1)}^{m_0(\sigma-1)} \to 0.$$

For sufficiently large j we thus have $|u_j(x)| \le \delta/2$ for a.e. $x \in \Omega$ and, hence, $\Phi'(u_j) = \Phi'_{\tau_j}(u_j) = 0$, contradicting the assumption that 0 is an *isolated* critical point of Φ. □

In the absence of a direct sum decomposition, the main technical tool to get an estimate of the critical groups is the notion of cohomological local splitting introduced in Perera, Agarwal and O'Regan [18], which is a variant of the homological linking of Perera [17] (see [13]). The following slightly different form of this notion was given in Degiovanni, Lancelotti, and Perera [11].

Definition 2.7. We say that a C^1-functional Φ on a Banach space W has a cohomological local splitting near 0 in dimension $k \geq 1$ if there are symmetric cones $W_\pm \subset W$ with $W_+ \cap W_- = \{0\}$ and $\rho > 0$ such that

$$i(W \backslash W_+) = i(W_- \backslash \{0\}) = k$$

and

$$\Phi(u) \geq \Phi(0) \quad \forall\, u \in B_\rho \cap W_+, \qquad \Phi(u) \leq \Phi(0) \quad \forall\, u \in B_\rho \cap W_-, \tag{2.11}$$

where i denotes the $\mathbb{Z}_2$-cohomological index and $B_\rho = \{u \in W : \|u\| \leq \rho\}$.

We recall the definition of the cohomological index (see Fadell and Rabinowitz [12]). For a symmetric subset M of $W \backslash \{0\}$, let $\overline{M} = M/\mathbb{Z}_2$ be the quotient space of M with each u and $-u$ identified, let $f : \overline{M} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{M}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{M})$ be the induced homomorphism of the Alexander–Spanier cohomology rings. Then the cohomological index of M is defined by

$$i(M) = \begin{cases} \sup\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\}, & M \neq \emptyset, \\ 0, & M = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \leq m-1$, so $i(S^{m-1}) = m$.

Proposition 2.8 ([11, Proposition 2.1]). *Assume that* 0 *is an isolated critical point of* Φ *and that* Φ *has a cohomological local splitting near* 0 *in dimension* k. *Then it holds* $C^k(\Phi, 0) \neq 0$.

In order to give sufficient conditions for Φ to have a cohomological local splitting near 0, and hence a nontrivial critical group by Proposition 2.8, consider the asymptotic eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Let

$$I(u) = \int_\Omega |\nabla u|^p dx, \quad J(u) = \int_\Omega |u|^p dx, \quad u \in W_0^{1,p}(\Omega),$$

and set

$$\Psi(u) = \frac{1}{J(u)}, \quad u \in \mathcal{M} = \{u \in W_0^{1,p}(\Omega) : I(u) = 1\}.$$

Then eigenvalues of problem (2.12) coincide with critical values of Ψ. Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \geq 1. \tag{2.13}$$

Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow +\infty$ is a sequence of eigenvalues of (2.12) and

$$\begin{aligned}\lambda_k < \lambda_{k+1} \Rightarrow i(\{u \in \mathcal{M} : \Psi(u) \leq \lambda_k\}) \\ = i(\{u \in \mathcal{M} : \Psi(u) < \lambda_{k+1}\}) = k\end{aligned} \tag{2.14}$$

(see [18, Propositions 3.52 and 3.53]). The main result of this subsection is the following theorem.

Theorem 2.9 (Critical groups at 0). *Assume that g satisfies* (2.9) *and* 0 *is an isolated critical point of* Φ.

(1) $C^0(\Phi, 0) \approx \mathbb{Z}_2$ *and* $C^q(\Phi, 0) = 0$ *for* $q \geq 1$ *in the following cases*:
- (a) $\lambda < \lambda_1$;
- (b) $\lambda = \lambda_1$ *and, for some* $\delta > 0$, $G(x, t) \leq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$.

(2) $C^k(\Phi, 0) \neq 0$ *in the following cases*:
- (a) $\lambda_k < \lambda < \lambda_{k+1}$;
- (b) $\lambda_k < \lambda = \lambda_{k+1}$ *and, for some* $\delta > 0$, $G(x, t) \leq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$;
- (c) $\lambda_k = \lambda < \lambda_{k+1}$ *and* $G(x, t) \geq c|t|^s$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$ *for some* $s \in (p, q)$ *and* $c > 0$.

Proof. We have

$$\Phi(u) = \frac{1}{p}[I(u) - \lambda J(u)] + \int_\Omega \left[\frac{a(x)}{q}|\nabla u|^q - G(x, u)\right] dx. \tag{2.15}$$

By (2.9) and the Sobolev embedding, we have

$$\int_\Omega G(x, u)dx = \mathrm{o}(\|\nabla u\|_p^p), \quad \text{as } \|\nabla u\|_p \to 0, \tag{2.16}$$

and in view of Lemma 2.6, without loss of generality, we may assume that the sign conditions on G in (1)(b) and (2)(b) hold for every $t \in \mathbb{R}$.

(1) We show that 0 is a local minimizer of Φ. Since $\Psi(u) \geq \lambda_1$ for all $u \in \mathcal{M}$,we have

$$I(u) \geq \lambda_1 J(u), \quad \forall\, u \in W_0^{1,p}(\Omega). \tag{2.17}$$

(a) By (2.15)–(2.17), we get

$$\Phi(u) \geq \frac{1}{p}\left(1 - \frac{\lambda_+}{\lambda_1} + \mathrm{o}(1)\right) \|\nabla u\|_p^p, \quad \text{as } \|\nabla u\|_p \to 0,$$

where $\lambda_+ = \max\{\lambda, 0\}$. So $\Phi(u) \geq 0$ for all $u \in B_\rho$ for sufficiently small $\rho > 0$ by (2.2).

(b) By (2.15) and (2.17),we get

$$\Phi(u) \geq -\int_\Omega G(x,u)dx \geq 0, \quad \forall\, u \in W_0^{1,\mathcal{H}}(\Omega).$$

(2) We show that Φ has a cohomological local splitting near 0 in dimension k and then apply Proposition 2.8. In light of [10, Theorem 2.3], the set $\{u \in W_0^{1,p}(\Omega) : I(u) \leq \lambda_k J(u)\}$ contains a *symmetric cone* W_- with $i(W_-\backslash\{0\}) = k$ and $\{u \in W_- : \|u\|_p = 1\}$ is bounded in $C^1(\Omega)$, so that, in particular, we have the inequality

$$\int_\Omega \frac{a(x)}{q}|\nabla u|^q dx \leq C\|u\|_p^q, \quad \forall\, u \in W_-, \tag{2.18}$$

for some $C > 0$. Since $W_0^{1,\mathcal{H}}(\Omega)$ is embedded in $W_0^{1,p}(\Omega)$ as a *dense linear subspace*, the inclusion

$$\{u \in W_0^{1,\mathcal{H}}(\Omega) : I(u) < \lambda_{k+1}J(u)\} \subset \{u \in W_0^{1,p}(\Omega) : I(u) < \lambda_{k+1}J(u)\}$$

is a *homotopy equivalence* by Palais (cf. [16, Theorem 17]), so

$$\begin{aligned} &i(\{u \in W_0^{1,\mathcal{H}}(\Omega) : I(u) < \lambda_{k+1}J(u)\}) \\ &\quad = i(\{u \in W_0^{1,p}(\Omega) : I(u) < \lambda_{k+1}J(u)\}) = k \end{aligned}$$

by virtue of (2.14). We take now

$$W_+ := \{u \in W_0^{1,\mathcal{H}}(\Omega) : I(u) \geq \lambda_{k+1}J(u)\}.$$

It only remains to show that (2.11) holds for sufficiently small $\rho > 0$.

(a) For $u \in W_+$, we obtain

$$\Phi(u) \geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_{k+1}} + \mathrm{o}(1)\right)\|\nabla u\|_p^p, \quad \text{as } \|\nabla u\|_p \to 0$$

by virtue of (2.15) and (2.16). So $\Phi(u) \geq 0$ for all $u \in B_\rho \cap W_+$ for sufficiently small $\rho > 0$ by (2.2). For $u \in W_-$,

$$\Phi(u) \leq -\frac{1}{p}\left(\frac{\lambda}{\lambda_k} - 1 + \mathrm{o}(1)\right)\|\nabla u\|_p^p \quad \text{as } \|\nabla u\|_p \to 0$$

by (2.15), (2.16), and (2.18) since $q > p$. So $\Phi(u) \leq 0$ for all $u \in B_\rho \cap W_-$ for small $\rho > 0$ by (2.2).

(b) For $u \in W_+$, we have

$$\Phi(u) \geq -\int_\Omega G(x,u)dx \geq 0$$

by (2.15), and $\Phi(u) \leq 0$ for all $u \in B_\rho \cap W_-$ for small $\rho > 0$ as in (a).

(c) We have $\Phi(u) \geq 0$ for all $u \in B_\rho \cap W_+$ for sufficiently small $\rho > 0$ as in (i). For $u \in W_-$,

$$\Phi(u) \leq C\|u\|_p^q - \frac{\|u\|_p^s}{C}$$

for some $C > 0$ by (2.15), (2.18), and since $s > p$. Since $s < q$, then $\Phi(u) \leq 0$ for all $u \in B_\rho \cap W_-$ for sufficiently small $\rho > 0$ by (2.2). ◻

2.5. *Nontrivial solutions*

In this subsection we obtain a nontrivial solution of the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \lambda|u|^{p-2}u + |u|^{r-2}u + h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.19}$$

where $\lambda \in \mathbb{R}$ is a parameter, $r \in (q, p^*)$, and h is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|h(x,t)| \le C(|t|^{\rho-1} + |t|^{\sigma-1}) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{2.20}$$

for some $p < \sigma < \rho < r$ and $C > 0$. First we verify that the associated functional

$$\Phi(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q - \frac{\lambda}{p}|u|^p - \frac{1}{r}|u|^r - H(x,u)\right] dx, \quad u \in W_0^{1,\mathcal{H}}(\Omega),$$

where $H(x,t) = \int_0^t h(x,s)ds$, satisfies the (PS) condition. We note that

$$\begin{aligned} q\Phi(u) - \langle \Phi'(u), u\rangle = \left(\frac{q}{p} - 1\right) \int_\Omega (|\nabla u|^p - \lambda|u|^p)dx + \left(1 - \frac{q}{r}\right) \int_\Omega |u|^r dx \\ + \int_\Omega (h(x,u)u - qH(x,u))dx. \end{aligned} \tag{2.21}$$

Lemma 2.10 (Palais–Smale condition). *Every sequence $(u_j) \subset W_0^{1,\mathcal{H}}(\Omega)$ such that $(\Phi(u_j))$ is bounded and $\Phi'(u_j) \to 0$ has a convergent subsequence.*

Proof. It suffices to show that (u_j) is bounded by Proposition 2.3. Since $p < q < r$ and $\sigma < \rho < r$, it follows from (2.21), (2.20) and the Hölder and Young's inequalities that $\|u_j\|_r^r \le C + \mathrm{o}(\|u_j\|)$ for some $C > 0$. Then

$$\begin{aligned} \int_\Omega \left[\frac{1}{p}|\nabla u_j|^p + \frac{a(x)}{q}|\nabla u_j|^q\right] dx &= \Phi(u_j) + \int_\Omega \left[\frac{\lambda}{p}|u_j|^p + \frac{1}{r}|u_j|^r + H(x,u_j)\right] dx \\ &\le C + \mathrm{o}(\|u_j\|), \end{aligned}$$

which together with (2.2) gives the desired conclusion. $\square$

Next we study the structure of the sublevel sets of Φ at infinity.

Lemma 2.11. *There exists $\alpha < 0$ such that the sublevel set*

$$\Phi^\alpha := \{u \in W_0^{1,\mathcal{H}}(\Omega) : \Phi(u) \le \alpha\}$$

is contractible in itself.

Proof. Since $p < q < r$ and $\sigma < \rho < r$, it follows from (2.21), (2.20), and the Young's inequality that $\langle \Phi'(u), u\rangle - q\Phi(u)$ is bounded from above, so for $\alpha < 0$

with $|\alpha|$ sufficiently large,

$$\langle \Phi'(u), u \rangle < 0, \quad \forall\, u \in \Phi^\alpha. \tag{2.22}$$

For $u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$, taking into account that $\Phi(tu) \to -\infty$ as $t \to +\infty$, set

$$t(u) = \inf \{t \geq 1 : \Phi(tu) \leq \alpha\},$$

and note that the function $u \mapsto t(u)$ is continuous by (2.22) and the implicit function theorem. Then the map $u \mapsto t(u)\, u$ is a *retraction* of $W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ onto Φ^α, and the conclusion follows since the former is contractible in itself. □

2.6. *Proof of Theorem* 1.1

We are now ready to prove the main result. Let (λ_k) be the sequence of eigenvalues of problem (2.12) defined in (2.13). Suppose that 0 is the only critical point of Φ. Taking $U = W_0^{1,\mathcal{H}}(\Omega)$ in (2.10), we have

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}).$$

Let $\alpha < 0$ be as in Lemma 2.11. Since Φ has no other critical points and satisfies the (PS) condition by Lemma 2.10, Φ^0 is a deformation *retract* of $W_0^{1,\mathcal{H}}(\Omega)$ and Φ^α is a deformation *retract* of $\Phi^0 \setminus \{0\}$ by the second deformation lemma. So

$$C^q(\Phi, 0) \approx H^q(W_0^{1,\mathcal{H}}(\Omega), \Phi^\alpha) = 0 \quad \forall\, q \in \mathbb{N},$$

since Φ^α is contractible in itself, contradicting Theorem 2.9 in each of the cases (1)–(3).

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