DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **11**, Number **3**, June **2018**

doi:10.3934/dcdss.2018031

pp. **561–576**

BIFURCATION RESULTS FOR PROBLEMS WITH FRACTIONAL TRUDINGER-MOSER NONLINEARITY

Kanishka Perera

Department of Mathematical Sciences
Florida Institute of Technology
150 W University Blvd, Melbourne, FL 32901, USA

Marco Squassina

Dipartimento di Matematica e Fisica
Università Cattolica del Sacro Cuore
Via dei Musei 41, I-25121 Brescia, Italy

Abstract. By using a suitable topological argument based on cohomological linking and by exploiting a Trudinger–Moser inequality in fractional spaces recently obtained, we prove existence of multiple solutions for a problem involving the nonlinear fractional laplacian and a related critical exponential nonlinearity. This extends the literature for the N-Laplacian operator.

1. **Introduction.**

1.1. **Overview.** Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and with Lipschitz boundary $\partial\Omega$. We denote by ω_{N-1} the measure of the unit sphere in $\mathbb{R}^N$ and $N' = N/(N-1)$. Since the time when the Trudinger-Moser inequality was first proved (cf. [7, 23, 27])

$$\sup_{u \in W_0^{1,N}(\Omega),\ \|\nabla u\|_N \leq 1} \int_\Omega e^{\alpha_N |u|^{N'}} dx < +\infty, \qquad \alpha_N = N\omega_{N-1}^{1/(N-1)},$$

existence and multiplicity of solutions for various nonlinear problems with exponential nonlinearity were investigated. For instance, Adimurthi [1] proved the existence of a positive solution to the quasi-linear elliptic problem

$$\begin{cases} -\Delta_N u = \lambda\, |u|^{N-2}\, u\, e^{\,|u|^{N'}} & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_N u := \mathrm{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator for $0 < \lambda < \lambda_1(N)$, being $\lambda_1(N) > 0$ the first eigenvalue of Δ_N with Dirichlet boundary conditions, see also [10]. The case $N = 2$ was investigated in [8, 9], where the existence of a nontrivial solution was found for $\lambda \geq \lambda_1$. Recently, in [28] it was proved that problem (1) admits a nontrivial weak solution whenever $\lambda > 0$ is *not* an eigenvalue of $-\Delta_N$ in Ω with Dirichlet boundary conditions. In addition in [28] a bifurcation result

2010 *Mathematics Subject Classification.* Primary: 35J92; Secondary: 35P30.
Key words and phrases. Fractional Trudinger-Moser embedding, exponential nonlinearity, existence of solutions.
The second author is member of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM).

for higher (nonlinear) eigenvalues (which are suitably defined via the cohomological index) is also obtained, yielding in turn multiplicity results.

The issue of Trudinger-Moser type embeddings for fractional spaces is rather delicate and only quite recently, Parini and Ruf [25] (see also the refinement obtained in [17]) provided a partial result in the *Sobolev-Slobodeckij* space

$$W_0^{s,N/s}(\Omega), \quad s \in (0,1),\ N \geq 1,$$

defined as the completion of $C_0^\infty(\Omega)$ for the norm

$$\|u\| = [u]_{s,N/s} := \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{N/s}}{|x-y|^{2N}}\, dxdy \Big)^{s/N}.$$

We also refer the reader to [18, 19, 21, 24] for results in a different functional framework, namely the *Bessel potential* spaces $H^{s,p}$. In fact, they proved that the supremum $\alpha_{N,s}(\Omega)$ of $\alpha \geq 0$ with

$$\sup_{u \in W_0^{s,N/s}(\Omega),\ [u]_{s,N/s} \leq 1} \int_\Omega e^{\alpha\, |u|^{N/(N-s)}}\, dx < +\infty, \tag{2}$$

is positive and finite. Furthermore, they proved the existence of $\alpha^*_{N,s}(\Omega) \geq \alpha_{N,s}(\Omega)$ such that the supremum in (2) is $+\infty$ for $\alpha > \alpha^*_{N,s}(\Omega)$. On the other hand it still remains *unknown* whether

$$\alpha_{N,s}(\Omega) = \alpha^*_{N,s}(\Omega).$$

The case $N = 1$ and $s = 1/2$ was earlier considered in [16] (see also [14]), where the authors study the existence of weak solutions to the problem

$$-\frac{C_s}{2} \int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{1+2s}} dy = f(u), \qquad u \in W_0^{1/2,2}(-1,1),$$

where $C_s > 0$ is a suitable normalization constant. We also mention [11, 12] for other investigations in the one dimensional case on the whole space $\mathbb{R}$, facing the problem of the lack of compactness. In particular in [12], the existence of *ground state* solutions for the problem

$$-\frac{C_s}{2} \int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{1+2s}} dy + u = f(u), \qquad u \in W_0^{1/2,2}(\mathbb{R}),$$

was proved, where f is a Trudinger-Moser critical growth nonlinearity.

To the authors' knowledge, in the framework of the *Sobolev-Slobodeckij* spaces $W_0^{s,N/s}(\Omega)$, fractional counterparts of the local quasilinear N-Laplacian problem (1) were not previously tackled in the literature. This is precisely the goal of this manuscript.

1.2. **The main result.** Let $N \geq 1$ and $s \in (0,1)$. In the following, the standard norm for the L^p space will always be denoted by $|\cdot|_p$. For $\lambda > 0$, we consider the quasilinear problem

$$\begin{cases} (-\Delta)^s_{N/s}\, u = \lambda\, |u|^{(N-2s)/s}\, u\, e^{\,|u|^{N/(N-s)}} & \text{in } \Omega \\ \qquad\quad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}$$

where $(-\Delta)^s_{N/s}$ is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)^s_{N/s}\, u(x) := 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{(N-2s)/s}\,(u(x)-u(y))}{|x-y|^{2N}}\, dy, \quad x \in \mathbb{R}^N.$$

We refer the interested reader to [22] and the references therein for an overview on recent progresses on existence, nonexistence and regularity results for equations involving the *fractional p-laplacian* operator $(-\Delta)_p^s$, $p > 1$. The standard sequence of eigenvalues for $(-\Delta)_{N/s}^s$ via the *Krasnoselskii genus* does not furnish enough information on the structure of sublevels and thus the eigenvalues will be introduced via the *cohomological index*. We consider critical values of the functional

$$\Psi(u) := \frac{1}{|u|_{N/s}^{N/s}}, \quad u \in \mathcal{M}, \quad \mathcal{M} := \{u \in W_0^{s,N/s}(\Omega) : \|u\| = 1\}.$$

Let $\mathcal{F}$ be the class of symmetric sets of $\mathcal{M}$, $i(M)$ the $\mathbb{Z}_2$-cohomological index of a $M \subset \mathcal{F}$ and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \geq 1, \qquad (\lambda_k \to +\infty)$$

Consider also the positive constant

$$\mu_{N,s}(\Omega) := \alpha_{N,s}(\Omega)^{(N-s)/N} \left(\frac{N}{s\,\mathcal{L}(\Omega)}\right)^{s/N},$$

being $\mathcal{L}$ the Lebesgue measure in $\mathbb{R}^N$. The following is our main result

Theorem 1.1. *Assume that* $\lambda_k \leq \lambda < \lambda_{k+1} = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$ *for some* $k, m \geq 1$ *and*

$$\lambda + \mu_{N,s}(\Omega)\, \lambda^{(N-s)/N} > \lambda_{k+1},$$

then problem (3) *has* m *distinct pairs of nontrivial solutions* $\pm u_j^\lambda$, $j = 1, \ldots, m$ *such that* $u_j^\lambda \to 0$ *as* $\lambda \nearrow \lambda_{k+1}$. *In particular, if*

$$\lambda_k \leq \lambda < \lambda_{k+1} < \lambda + \mu_{N,s}(\Omega)\, \lambda^{(N-s)/N}$$

for some $k \geq 1$, *then problem* (3) *has a nontrivial solution.*

This result, which follows from the results in Section 5, is nontrivial since the classical linking arguments of [8, 9] cannot be used in the quasi-linear setting. Instead the abstract machinery developed in [28] will be applied. We also would like to stress that, since the Trudinger-Moser embedding (2) still holds with nonoptimal exponent (contrary to the local case), it is not clear how to prove Brezis-Nirenberg type results, namely that problem (3) admits a nontrivial weak solution whenever $\lambda > 0$ is not an eigenvalue of $(-\Delta)_{N/s}^s$.

2. **Preliminaries.** As anticipated in the introduction, we work in the fractional Sobolev space $W_0^{s,N/s}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the Gagliardo seminorm

$$[u]_{s,N/s} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{N/s}}{|x - y|^{2N}}\, dxdy\right)^{s/N}.$$

Furthermore, since $\partial\Omega$ is assumed to be Lipschitz, we have (cf. [5, Proposition B.1])

$$W_0^{s,N/s}(\Omega) = \left\{u \in L^{N/s}(\mathbb{R}^N) : [u]_{s,N/s} < \infty,\ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}.$$

A function $u \in W_0^{s,N/s}(\Omega)$ is a *weak solution* of problem (3) if

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{(N-2s)/s}\,(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{2N}}\,dxdy$$
$$= \lambda \int_\Omega |u|^{(N-2s)/s}\,u\,e^{|u|^{N/(N-s)}}\,v\,dx, \quad \forall v \in W_0^{s,N/s}(\Omega).$$

As proved in [15, Proposition 2.12], a weak solution turns into a *poinwise* solution if $u \in C^{1,\gamma}_{\rm loc}$ for some $\gamma \in (0,1)$ sufficiently close to 1. The integral on the right-hand side is well-defined in view of [25, Proposition 3.2] and the Hölder inequality. Weak solutions coincide with critical points of the C^1 functional

$$\Phi(u) = \frac{s}{N}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{N/s}}{|x-y|^{2N}}\,dxdy - \lambda\int_\Omega F(u)\,dx, \quad u \in W_0^{s,N/s}(\Omega),$$

where $F(t) = \int_0^t f(\tau)\,d\tau$ and $f(t) = |t|^{(N-2s)/s}\,t\,e^{|t|^{N/(N-s)}}$.

We recall that $W_0^{s,N/s}(\Omega)$ is uniformly convex, and hence reflexive. Indeed, for $u \in W_0^{s,N/s}(\Omega)$, let

$$\widetilde{u}(x,y) := \frac{u(x)-u(y)}{|x-y|^{2s}}, \quad (x,y) \in \mathbb{R}^{2N}.$$

Then the mapping $u \mapsto \widetilde{u}$ is a linear isometry from $W_0^{s,N/s}(\Omega)$ to $L^{N/s}(\mathbb{R}^{2N})$, so the uniform convexity of $L^{N/s}(\mathbb{R}^{2N})$ gives the conclusion.

We also have the following Brézis-Lieb lemma in $W_0^{s,N/s}(\Omega)$.

Lemma 2.1. *If (u_j) is bounded in $W_0^{s,N/s}(\Omega)$ and converges to u a.e. in Ω, then*

$$\|u_j\|^{N/s} - \|u_j - u\|^{N/s} \to \|u\|^{N/s}.$$

Proof. Let

$$\widetilde{u}_j(x,y) = \frac{u_j(x)-u_j(y)}{|x-y|^{2s}}, \qquad \widetilde{u}(x,y) = \frac{u(x)-u(y)}{|x-y|^{2s}},$$

and note that $(\widetilde{u}_j)$ is bounded in $L^{N/s}(\mathbb{R}^{2N})$ and converges to $\widetilde{u}$ a.e. in $\mathbb{R}^{2N}$. Hence

$$|\widetilde{u}_j|_{N/s}^{N/s} - |\widetilde{u}_j - \widetilde{u}|_{N/s}^{N/s} \to |\widetilde{u}|_{N/s}^{N/s}$$

by the Brézis-Lieb lemma [6], where $|\cdot|_{N/s}$ denotes the norm in $L^{N/s}(\mathbb{R}^{2N})$, namely the conclusion. □

It was shown [25, Theorem 1.1] that the supremum $\alpha_{N,s}(\Omega)$ of all $\alpha \geq 0$ such that

$$\sup\left\{\int_\Omega e^{\alpha\,|u|^{N/(N-s)}}\,dx : u \in W_0^{s,N/s}(\Omega),\ [u]_{s,N/s} \leq 1\right\} < +\infty$$

satisfies $0 < \alpha_{N,s}(\Omega) < \infty$. The main result of this section is the following theorem, which is due to P.L. Lions [20] in the local case $s = 1$.

Theorem 2.2. *If (u_j) is a sequence in $W_0^{s,N/s}(\Omega)$ with $\|u_j\| = 1$ for all $j \in \mathbb{N}$ and converging a.e. to a nonzero function u, then*

$$\sup_{j\in\mathbb{N}}\int_\Omega e^{\alpha\,|u_j|^{N/(N-s)}}\,dx < +\infty$$

for all $\alpha < \alpha_{N,s}(\Omega)/(1-\|u\|^{N/s})^{s/(N-s)}$.

Proof. We have

$$|u_j|^{N/(N-s)} \leq (|u| + |u_j - u|)^{N/(N-s)} \leq (p\,|u|)^{N/(N-s)} + (q\,|u_j - u|)^{N/(N-s)},$$

where $1/p + 1/q = 1$. Then

$$\int_\Omega e^{\alpha\,|u_j|^{N/(N-s)}}\,dx \leq \left(\int_\Omega e^{\alpha\,\widetilde{p}\,|u|^{N/(N-s)}}\,dx\right)^{1/p} \left(\int_\Omega e^{\alpha\,\widetilde{q}\,|u_j-u|^{N/(N-s)}}\,dx\right)^{1/q}$$

by the Hölder inequality, where $\widetilde{p} = p^{(2N-s)/(N-s)}$ and $\widetilde{q} = q^{(2N-s)/(N-s)}$. The first integral on the right-hand side is finite, and the second integral equals

$$\int_\Omega e^{\alpha\,\widetilde{q}\,\|u_j-u\|^{N/(N-s)}\,|v_j|^{N/(N-s)}}\,dx,$$

where $v_j = (u_j - u)/\,\|u_j - u\|$. By Lemma 2.1, $\|u_j - u\|^{N/(N-s)} \to (1 - \|u\|^{N/s})^{s/(N-s)}$. Taking $q > 1$ sufficiently close to 1, let

$$\alpha\,\widetilde{q}\,(1 - \|u\|^{N/s})^{s/(N-s)} < \beta < \alpha_{N,s}(\Omega).$$

Then $\alpha\,\widetilde{q}\,\|u_j - u\|^{N/(N-s)} \leq \beta$ and hence the last integral is less than or equal to

$$\int_\Omega e^{\beta\,|v_j|^{N/(N-s)}}\,dx,$$

for all sufficiently large j, which is bounded since $\beta < \alpha_{N,s}(\Omega)$ and $\|v_j\| = 1$. $\square$

We close this preliminary section with a technical lemma.

Lemma 2.3. *For all $t \in \mathbb{R}$,*

1. $F(t) \leq \dfrac{N-s}{N}\,\dfrac{t f(t)}{|t|^{N/(N-s)}}$,
2. $F(t) \leq F(1) + \dfrac{s\,(N-s)}{N^2}\,t f(t)$,
3. $\dfrac{s}{N}\,t f(t) - F(t) \geq \dfrac{s^2}{N^2}\,|t|^{N^2/s(N-s)}$, *in particular,* $t f(t) \geq \dfrac{N}{s}\,F(t)$,
4. $F(t) \leq \dfrac{s}{N}\,|t|^{N/s} + |t|^{N^2/s(N-s)}\,e^{|t|^{N/(N-s)}}$,
5. $F(t) \geq \dfrac{s}{N}\,|t|^{N/s} + \dfrac{s\,(N-s)}{N^2}\,|t|^{N^2/s(N-s)}$.

Proof. Since f is odd, and hence F is even,

$$F(t) = \int_0^{|t|} f(\tau)\,d\tau = \int_0^{|t|} \tau^{(N-s)/s}\,e^{\tau^{N/(N-s)}}\,d\tau.$$

1. Integrating by parts,

$$\begin{aligned} &F(t)\\ =&\frac{N-s}{N}\,|t|^{N/s-N/(N-s)}\,e^{|t|^{N/(N-s)}} - \frac{N-2s}{s}\int_0^{|t|} \tau^{N/s-N/(N-s)-1}\,e^{\tau^{N/(N-s)}}\,d\tau\\ \leq&\frac{N-s}{N}\,\frac{|t|^{N/s}\,e^{|t|^{N/(N-s)}}}{|t|^{N/(N-s)}}\\ =&\frac{N-s}{N}\,\frac{t f(t)}{|t|^{N/(N-s)}}. \end{aligned}$$

2. For $|t| \le 1$, $F(t) \le F(1)$. For $|t| > 1$, $F(t) = F(1) + \int_1^{|t|} f(\tau)\, d\tau$. Integrating by parts,

$$\begin{aligned}
&\int_1^{|t|} f(\tau)\, d\tau \\
=&\frac{s}{N}\, |t|^{N/s}\, e^{\,|t|^{N/(N-s)}} - \frac{se}{N} - \frac{s}{N-s} \int_1^{|t|} \tau^{(N-s)/s+N/(N-s)}\, e^{\,\tau^{N/(N-s)}}\, d\tau \\
\le&\frac{s}{N}\, tf(t) - \frac{s}{N-s} \int_1^{|t|} f(\tau)\, d\tau,
\end{aligned}$$

and hence $\int_1^{|t|} f(\tau)\, d\tau \le \dfrac{s\,(N-s)}{N^2}\, tf(t)$.

3. integrating by parts,

$$\begin{aligned}
F(t) &= \frac{s}{N}\, |t|^{N/s}\, e^{\,|t|^{N/(N-s)}} - \frac{s}{N-s} \int_0^{|t|} \tau^{N/s+N/(N-s)-1}\, e^{\,\tau^{N/(N-s)}}\, d\tau \\
&\le \frac{s}{N}\, tf(t) - \frac{s}{N-s} \int_0^{|t|} \tau^{N^2/s(N-s)-1}\, d\tau \\
&= \frac{s}{N}\, tf(t) - \frac{s^2}{N^2}\, |t|^{N^2/s(N-s)}.
\end{aligned}$$

4. Since $e^\tau \le 1 + \tau e^\tau$ for all $\tau \ge 0$,

$$\begin{aligned}
F(t) &\le \int_0^{|t|} \tau^{(N-s)/s} \left(1 + \tau^{N/(N-s)}\, e^{\,\tau^{N/(N-s)}}\right) d\tau \\
&\le \int_0^{|t|} \tau^{(N-s)/s}\, d\tau + \int_0^{|t|} |t|^{(N-s)/s+N/(N-s)}\, e^{\,|t|^{N/(N-s)}}\, d\tau \\
&= \frac{s}{N}\, |t|^{N/s} + |t|^{N^2/s(N-s)}\, e^{\,|t|^{N/(N-s)}}.
\end{aligned}$$

5. Since $e^\tau \ge 1 + \tau$ for all $\tau \ge 0$,

$$\begin{aligned}
F(t) &\ge \int_0^{|t|} \tau^{(N-s)/s} \left(1 + \tau^{N/(N-s)}\right) d\tau \\
&= \frac{s}{N}\, |t|^{N/s} + \frac{s\,(N-s)}{N^2}\, |t|^{N^2/s(N-s)}.
\end{aligned}$$

This concludes the proof. □

3. **Palais-Smale condition.** Recall that Φ satisfies the $(\mathrm{PS})_c$ condition if every sequence (u_j) in $W_0^{s,N/s}(\Omega)$ such that $\Phi(u_j) \to c$ and $\Phi'(u_j) \to 0$, called a $(\mathrm{PS})_c$ sequence, has a convergent subsequence. The main result of this section is the following theorem.

Theorem 3.1. *Φ satisfies the $(\mathrm{PS})_c$ condition for all $c < \dfrac{s}{N}\, \alpha_{N,s}(\Omega)^{(N-s)/s}$.*

First we prove a lemma.

Lemma 3.2. *If u_j converges to u weakly in $W_0^{s,N/s}(\Omega)$ and a.e. in Ω, and*

$$\sup_{j \in \mathbb{N}} \int_\Omega u_j\, f(u_j)\, dx < \infty, \tag{4}$$

then

$$\int_\Omega F(u_j)\,dx \to \int_\Omega F(u)\,dx.$$

Proof. For any $M > 0$, write

$$\int_\Omega F(u_j)\,dx = \int_{\{|u_j|<M\}} F(u_j)\,dx + \int_{\{|u_j|\geq M\}} F(u_j)\,dx.$$

By Lemma 2.3, 1 and (4), we have

$$\int_{\{|u_j|\geq M\}} F(u_j)\,dx \leq \frac{N-s}{NM^{N/(N-s)}} \int_\Omega u_j\, f(u_j)\,dx = \mathcal{O}\bigg(\frac{1}{M^{N/(N-s)}}\bigg), \text{ as } M \to \infty.$$

Hence

$$\int_\Omega F(u_j)\,dx = \int_{\{|u_j|<M\}} F(u_j)\,dx + \mathcal{O}\bigg(\frac{1}{M^{N/(N-s)}}\bigg),$$

and the desired conclusion follows by letting $j \to \infty$ first and then $M \to \infty$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let (u_j) be a $(\mathrm{PS})_c$ sequence. Then

$$\Phi(u_j) = \frac{s}{N}\,\|u_j\|^{N/s} - \lambda \int_\Omega F(u_j)\,dx = c + \mathrm{o}(1) \tag{5}$$

and

$$\Phi'(u_j)\,u_j = \|u_j\|^{N/s} - \lambda \int_\Omega u_j\, f(u_j)\,dx = \mathrm{o}(\|u_j\|). \tag{6}$$

Since $s/N > s\,(N-s)/N^2$, it follows from Lemma 2.3 2, (5) and (6) that (u_j) is bounded in $W^{s,N/s}_0(\Omega)$. Hence a renamed subsequence converges to some u weakly in $W^{s,N/s}_0(\Omega)$, strongly in $L^p(\Omega)$ for all $p \in [1,\infty)$, and a.e. in Ω. Moreover,

$$\sup_{j\in\mathbb{N}} \int_\Omega u_j\, f(u_j)\,dx < \infty$$

by (6), and hence

$$\int_\Omega F(u_j)\,dx \to \int_\Omega F(u)\,dx \tag{7}$$

by virtue of Lemma 3.2. By Lemma 2.3 3, (5), and (6),

$$\frac{\lambda s^2}{N^2} \int_\Omega |u_j|^{N^2/s(N-s)}\,dx \leq \lambda \int_\Omega \bigg[\frac{s}{N}\,u_j\, f(u_j) - F(u_j)\bigg]\,dx = c + \mathrm{o}(1),$$

so

$$c \geq \frac{\lambda s^2}{N^2} \int_\Omega |u|^{N^2/s(N-s)}\,dx \geq 0.$$

If $c = 0$, then $u = 0$ and hence $\displaystyle\int_\Omega F(u_j)\,dx \to 0$ by (7), so $\|u_j\| \to 0$ by (5).

Now suppose that $0 < c < (s/N)\,\alpha_{N,s}(\Omega)^{(N-s)/s}$. We claim that the weak limit u is nonzero. Suppose $u = 0$. Then

$$\int_\Omega F(u_j)\,dx \to 0 \tag{8}$$

by (7) and hence

$$\|u_j\| \to \bigg(\frac{Nc}{s}\bigg)^{s/N} < \alpha_{N,s}(\Omega)^{(N-s)/N}$$

by (5). Let $(Nc/s)^{s/(N-s)} < \alpha < \alpha_{N,s}(\Omega)$. Then $\|u_j\| \le \alpha^{(N-s)/N}$ for all $j \ge j_0$ for some j_0. Let $1 < q < \alpha_{N,s}(\Omega)/\alpha$. By the Hölder inequality,

$$\int_\Omega u_j\, f(u_j)\, dx \le \left(\int_\Omega |u_j|^{Np/s}\, dx\right)^{1/p} \left(\int_\Omega e^{\,q\, |u_j|^{N/(N-s)}}\, dx\right)^{1/q},$$

where $1/p + 1/q = 1$. The first integral on the right-hand side converges to zero since $u = 0$, while the second integral is bounded for $j \ge j_0$ since $q\, |u_j|^{N/(N-s)} = q\, \alpha\, |\widetilde{u}_j|^{N/(N-s)}$ with $q\, \alpha < \alpha_{N,s}(\Omega)$ and $\widetilde{u}_j = u_j/\alpha^{(N-s)/N}$ satisfies $\|\widetilde{u}_j\| \le 1$, so

$$\int_\Omega u_j\, f(u_j)\, dx \to 0.$$

Then $u_j \to 0$ by (6), and hence $c = 0$ by (5) and (8), a contradiction. So u is nonzero.

Since $\Phi'(u_j) \to 0$,

$$\int_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^{(N-2s)/s}\, (u_j(x) - u_j(y))\, (v(x) - v(y))}{|x-y|^{2N}}\, dxdy - \lambda \int_\Omega f(u_j)\, v\, dx \to 0$$

for all $v \in W^{s,N/s}_0(\Omega)$. For $v \in C^\infty_0(\Omega)$, an argument similar to that in the proof of Lemma 3.2 using the estimate

$$\left|\int_{\{|u_j| \ge M\}} f(u_j)\, v\, dx\right| \le \frac{\sup |v|}{M} \int_\Omega u_j\, f(u_j)\, dx = \mathcal{O}\left(\frac{1}{M}\right)$$

shows that $\int_\Omega f(u_j)\, v\, dx \to \int_\Omega f(u)\, v\, dx$, so

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{(N-2s)/s}\, (u(x) - u(y))\, (v(x) - v(y))}{|x-y|^{2N}}\, dxdy = \lambda \int_\Omega f(u)\, v\, dx.$$

Then this holds for all $v \in W^{s,N/s}_0(\Omega)$ by density, and taking $v = u$ gives

$$\|u\|^{N/s} = \lambda \int_\Omega u\, f(u)\, dx. \tag{9}$$

Next we claim that

$$\int_\Omega u_j\, f(u_j)\, dx \to \int_\Omega u\, f(u)\, dx. \tag{10}$$

We have

$$u_j\, f(u_j) = |u_j|^{N/s}\, e^{\,|u_j|^{N/(N-s)}} = |u_j|^{N/s}\, e^{\,\|u_j\|^{N/(N-s)}\, |\widetilde{u}_j|^{N/(N-s)}}, \tag{11}$$

where $\widetilde{u}_j = u_j/\, \|u_j\|$. By (5) and (7),

$$\|u_j\| \to \left[\frac{N}{s}\, (c + \lambda\beta)\right]^{s/N},$$

where $\beta = \int_\Omega F(u)\, dx$, so $\widetilde{u}_j$ converges a.e. to $\widetilde{u} = u/\, [(N/s)\, (c + \lambda\beta)]^{s/N}$. Then

$$\|u_j\|^{N/(N-s)}\, (1 - \|\widetilde{u}\|^{N/s})^{s/(N-s)} \to \left[\frac{N}{s}\, (c + \lambda\beta) - \|u\|^{N/s}\right]^{s/(N-s)} \le \left(\frac{Nc}{s}\right)^{s/(N-s)}$$

since

$$\|u\|^{N/s} \ge \frac{\lambda N}{s} \int_\Omega F(u)\, dx = \frac{\lambda N \beta}{s}$$

by (9) and Lemma 2.3 3. Let

$$\frac{\left(\dfrac{Nc}{s}\right)^{s/(N-s)}}{(1-\|\widetilde{u}\|^{N/s})^{s/(N-s)}} < \alpha - 2\varepsilon < \alpha < \frac{\alpha_{N,s}(\Omega)}{(1-\|\widetilde{u}\|^{N/s})^{s/(N-s)}}.$$

Then $\|u_j\|^{N/(N-s)} \le \alpha - 2\varepsilon$ for all $j \ge j_0$ for some j_0, and

$$\sup_{j\in\mathbb{N}} \int_\Omega e^{\alpha\,|\widetilde{u}_j|^{N/(N-s)}}\,dx < \infty \tag{12}$$

by Theorem 2.2. For $M > 0$ and $j \ge j_0$, (11) then gives

$$\begin{aligned}
&\int_{\{|u_j|\ge M\}} u_j\, f(u_j)\,dx \\
&\le \int_{\{|u_j|\ge M\}} |u_j|^{N/s}\, e^{(\alpha-2\varepsilon)\,|\widetilde{u}_j|^{N/(N-s)}}\,dx \\
&\le \left(\max_{t\ge 0} t^{N/s}\, e^{-\varepsilon\, t^{N/(N-s)}}\right) \|u_j\|^{N/s}\, e^{-\varepsilon\,(M/\|u_j\|)^{N/(N-s)}} \int_\Omega e^{\alpha\,|\widetilde{u}_j|^{N/(N-s)}}\,dx.
\end{aligned}$$

The last expression goes to zero as $M \to \infty$ uniformly in j since $\|u_j\|$ is bounded and (12) holds, so (10) now follows as in the proof of Lemma 3.2. By (6), (10), and (9),

$$\|u_j\|^{N/s} \to \lambda \int_\Omega u\, f(u)\,dx = \|u\|^{N/s}$$

and hence $\|u_j\| \to \|u\|$, so $u_j \to u$ by the uniform convexity of $W_0^{s,N/s}(\Omega)$. $\square$

4. **Eigenvalue problem.** The asymptotic problem associated with (3) as u goes to zero is the eigenvalue problem

$$\begin{cases} (-\Delta)^s_{N/s}\, u = \lambda\,|u|^{(N-2s)/s}\, u & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{13}$$

The weak formulation of this problem can be written as the operator equation

$$A(u) = \lambda\, B(u), \tag{14}$$

where A and B are the nonlinear operators from $W_0^{s,N/s}(\Omega)$ to its dual $W^{-s,N/(N-s)}(\Omega)$ defined by setting

$$\langle A(u), v\rangle := \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{(N-2s)/s}\,(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{2N}}\,dxdy,$$

$$\langle B(u), v\rangle := \int_\Omega |u|^{(N-2s)/s}\, uv\,dx, \quad u, v \in W_0^{s,N/s}(\Omega),$$

respectively. The operators A and B are homogeneous of degree $(N-s)/s$, odd, and satisfy

$$\langle A(u), v\rangle \le \|u\|^{(N-s)/s}\,\|v\|\,, \quad \langle A(u), u\rangle = \|u\|^{N/s}\,,$$
$$\langle B(u), u\rangle = |u|^{N/s}_{N/s}\,, \quad \forall u, v \in W_0^{s,N/s}(\Omega).$$

Since $W_0^{s,N/s}(\Omega)$ is uniformly convex, then A is of type (S), i.e. every sequence (u_j) in $W_0^{s,N/s}(\Omega)$ such that $u_j \rightharpoonup u$ and $\langle A(u_j), u_j - u\rangle \to 0$ as $j \to \infty$ has a subsequence that converges strongly to u (see e.g. [26, Proposition 1.3]). Moreover, B is a compact operator since the embedding

$$W_0^{s,N/s}(\Omega) \hookrightarrow L^{N/s}(\Omega),$$

is compact. Hence, problem (14) falls into the abstract framework considered in [26, Ch. 4] and we can construct an increasing and unbounded sequence of eigenvalues as follows.

Eigenvalues of problem (13) coincide with critical values of the functional

$$\Psi(u) = \frac{1}{|u|_{N/s}^{N/s}}, \quad u \in \mathcal{M} = \big\{u \in W_0^{s,N/s}(\Omega) : \|u\| = 1\big\}.$$

Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, let $i(M)$ denote the $\mathbb{Z}_2$-cohomological index of $M \in \mathcal{F}$ (see Fadell and Rabinowitz [13]), and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \geq 1.$$

Then

$$\lambda_1 = \inf_{u \in \mathcal{M}} \Psi(u) > 0$$

is the smallest eigenvalue and $\lambda_k \nearrow \infty$ is a sequence of eigenvalues (see [26, Proposition 3.52]). Moreover, denoting by

$$\Psi^a := \{u \in \mathcal{M} : \Psi(u) \leq a\}, \qquad \Psi_a := \{u \in \mathcal{M} : \Psi(u) \geq a\}$$

the sub- and superlevel sets of Ψ, respectively, we have

$$i(\Psi^{\lambda_k}) = i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}}) = k \tag{15}$$

whenever $\lambda_k < \lambda_{k+1}$ (see [26, Proposition 3.53]). The main result of this section is the following.

Theorem 4.1. *If $\lambda_k < \lambda_{k+1}$, then the sublevel set Ψ^{λ_k} contains a compact symmetric subset of index k.*

First a couple of lemmas.

Lemma 4.2. *The operator A is strictly monotone, i.e.,*

$$\langle A(u) - A(v), u - v\rangle > 0$$

for all $u \neq v$ in $W_0^{s,N/s}(\Omega)$.

Proof. By [26, Lemma 6.3], it suffices to show that

$$\langle A(u), v\rangle \leq \|u\|^{(N-s)/s} \|v\|, \quad \forall u, v \in W_0^{s,N/s}(\Omega)$$

and the equality holds if and only if $\alpha u = \beta v$ for some $\alpha, \beta \geq 0$, not both zero. We have

$$\langle A(u), v\rangle \leq \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{(N-s)/s} \, |v(x) - v(y)|}{|x - y|^{2N}} \, dxdy \leq \|u\|^{(N-s)/s} \|v\|$$

by the Hölder inequality. Clearly, equality holds throughout if $\alpha u = \beta v$ for some $\alpha, \beta \geq 0$, not both zero. Conversely, if $\langle A(u), v\rangle = \|u\|^{(N-s)/s} \|v\|$, equality holds in both inequalities. The equality in the second inequality gives

$$\alpha \, |u(x) - u(y)| = \beta \, |v(x) - v(y)| \quad \text{a.e. in } \mathbb{R}^{2N}$$

for some $\alpha, \beta \geq 0$, not both zero, and then the equality in the first inequality gives

$$\alpha\,(u(x) - u(y)) = \beta\,(v(x) - v(y)) \quad \text{a.e. in } \mathbb{R}^{2N}.$$

Since u and v vanish a.e. in $\mathbb{R}^N \setminus \Omega$, it follows that $\alpha u = \beta v$ a.e. in Ω. □

Lemma 4.3. *For each $w \in L^{N/s}(\Omega)$, the problem*

$$\begin{cases} (-\Delta)^s_{N/s}\, u = |w|^{(N-2s)/s}\, w & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{16}$$

has a unique weak solution $u \in W^{s,N/s}_0(\Omega)$. Moreover, the map

$$J : L^{N/s}(\Omega) \to W^{s,N/s}_0(\Omega), \quad w \mapsto u$$

is continuous, homogeneous of degree $(N-s)/s$, and satisfies

$$\frac{\|J(w)\|}{|J(w)|_{N/s}} \leq \frac{\|w\|}{|w|_{N/s}} \tag{17}$$

for all $w \neq 0$ in $L^{N/s}(\Omega)$.

Proof. The existence follows from a standard minimization argument and the uniqueness from Lemma 4.2. Clearly, J is homogeneous of degree $(N-s)/s$. To see that it is continuous, let $w_j \to w$ in $L^{N/s}(\Omega)$ and let $u_j = J(w_j)$, so

$$\langle A(u_j), v\rangle = \int_\Omega |w_j|^{(N-2s)/s}\, w_j v\, dx \quad \forall v \in W^{s,N/s}_0(\Omega). \tag{18}$$

Testing with $v = u_j$ gives

$$\|u_j\|^{N/s} = \int_\Omega |w_j|^{(N-2s)/s}\, w_j u_j\, dx \leq |w_j|^{(N-s)/s}_{N/s}\, |u_j|_{N/s}$$

by the Hölder inequality, which together with the imbedding $W^{s,N/s}_0(\Omega) \hookrightarrow L^{N/s}(\Omega)$ shows that (u_j) is bounded. Therefore, a renamed subsequence of (u_j) converges to some u weakly, strongly in $L^{N/s}(\Omega)$ and a.e. in Ω. Then u is a weak solution of problem (16) as in the proof of Theorem 3.1, so $u = J(w)$. Testing (18) with $u_j - u$ gives

$$\langle A(u_j), u_j - u\rangle = \int_\Omega |w_j|^{(N-2s)/s}\, w_j\,(u_j - u)\, dx \to 0,$$

so $u_j \to u$ for a further subsequence since the operator A is of type (S). Finally, testing

$$\langle A(u), v\rangle = \int_\Omega |w|^{(N-2s)/s}\, wv\, dx$$

with $v = u, w$ and using the Hölder inequality gives

$$\|u\|^{N/s} \leq |w|^{(N-s)/s}_{N/s}\, |u|_{N/s}\,, \qquad |w|^{N/s}_{N/s} \leq \|u\|^{(N-s)/s}\, \|w\|\,,$$

from which (17) follows. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let

$$\pi(u) = \frac{u}{\|u\|}, \quad \widetilde{\pi}(u) = \frac{u}{|u|_{N/s}}, \quad u \in W^{s,N/s}_0(\Omega) \setminus \{0\}$$

be the radial projections onto $\mathcal{M}$ and

$$\widetilde{\mathcal{M}} = \big\{u \in W_0^{s,N/s}(\Omega) : |u|_{N/s} = 1\big\},$$

respectively, let i be the imbedding $W_0^{s,N/s}(\Omega) \hookrightarrow L^{N/s}(\Omega)$, let J be the map defined in Lemma 4.3, and let $\varphi : \Psi^{\lambda_k} \to \mathcal{M}$ be the composition of the maps

$$\Psi^{\lambda_k} \xrightarrow{\ \widetilde{\pi}\ } \widetilde{\mathcal{M}} \xrightarrow{\ i\ } L^{N/s}(\Omega) \setminus \{0\} \xrightarrow{\ J\ } W_0^{s,N/s}(\Omega) \setminus \{0\} \xrightarrow{\ \pi\ } \mathcal{M}.$$

Since i is compact,

$$i(\widetilde{\pi}(\Psi^{\lambda_k})) = \big\{u \in \widetilde{\mathcal{M}} : \|u\|^{N/s} \le \lambda_k\big\}$$

is compact in $L^{N/s}(\Omega)$, and hence $K_0 = \varphi(\Psi^{\lambda_k})$ is compact in $W_0^{s,N/s}(\Omega)$. Since φ is an odd continuous map, $i(K_0) \ge i(\Psi^{\lambda_k})$. For $u \in \Psi^{\lambda_k}$, $\varphi(u) = J(u)/\|J(u)\|$ since J is homogeneous, so

$$\Psi(\varphi(u)) = \frac{\|J(u)\|^{N/s}}{|J(u)|_{N/s}^{N/s}} \le \frac{\|u\|^{N/s}}{|u|_{N/s}^{N/s}} = \Psi(u) \le \lambda_k$$

by (17), and hence $K_0 \subset \Psi^{\lambda_k}$. Then $i(K_0) \le i(\Psi^{\lambda_k})$ by the monotonicity of the index, so $i(K_0) = i(\Psi^{\lambda_k}) = k$ by (15). $\square$

5. **Bifurcation and multiplicity.** In this section we prove the following bifurcation and multiplicity results for problem (3), in which the constant

$$\mu_{N,s}(\Omega) = \alpha_{N,s}(\Omega)^{(N-s)/N} \left(\frac{N}{s\,\mathcal{L}(\Omega)}\right)^{s/N}$$

plays an important role, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Theorem 5.1. *If*

$$\lambda < \lambda_1 < \lambda + \mu_{N,s}(\Omega)\,\lambda^{(N-s)/N},$$

then problem (3) *has a pair of nontrivial solutions* $\pm\, u^\lambda$ *such that* $u^\lambda \to 0$ *as* $\lambda \nearrow \lambda_1$.

Theorem 5.2. *If* $\lambda_k \le \lambda < \lambda_{k+1} = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$ *for some* $k, m \ge 1$ *and*

$$\lambda + \mu_{N,s}(\Omega)\,\lambda^{(N-s)/N} > \lambda_{k+1}, \tag{19}$$

then problem (3) *has* m *distinct pairs of nontrivial solutions* $\pm\, u_j^\lambda$, $j = 1, \ldots, m$ *such that* $u_j^\lambda \to 0$ *as* $\lambda \nearrow \lambda_{k+1}$.

In particular, we have the following existence result.

Corollary 1. *If*

$$\lambda_k \le \lambda < \lambda_{k+1} < \lambda + \mu_{N,s}(\Omega)\,\lambda^{(N-s)/N}$$

for some $k \ge 1$, *then problem* (3) *has a nontrivial solution.*

Remark 1. Since $\lambda \ge \lambda_k$ in Theorem 5.2, (19) holds if

$$\lambda > \lambda_{k+1} - \mu_{N,s}(\Omega)\,\lambda_k^{(N-s)/N},$$

or if

$$\lambda > \left(\frac{\lambda_{k+1} - \lambda_k}{\mu_{N,s}(\Omega)}\right)^{N/(N-s)}.$$

We only give the proof of Theorem 5.2. The proof of Theorem 5.1 is similar and simpler. The proof will be based on an abstract critical point theorem proved in Yang and Perera [28] that generalizes Bartolo et al. [3, Theorem 2.4].

Let Φ be an even C^1-functional on a Banach space W. Let $\mathcal{A}^*$ denote the class of symmetric subsets of W, let $r > 0$, let $S_r = \{u \in W : \|u\| = r\}$, let $0 < b \leq \infty$, and let Γ denote the group of odd homeomorphisms of W that are the identity outside $\Phi^{-1}(0, b)$. The pseudo-index of $M \in \mathcal{A}^*$ related to i, S_r, and Γ is defined by

$$i^*(M) = \min_{\gamma \in \Gamma} i(\gamma(M) \cap S_r)$$

(see Benci [4]).

Theorem 5.3 ([28, Theorem 2.4]). *Let K_0 and B_0 be symmetric subsets of $\mathcal{M} = \{u \in W : \|u\| = 1\}$ such that K_0 is compact, B_0 is closed, and*

$$i(K_0) \geq k + m, \qquad i(\mathcal{M} \setminus B_0) \leq k$$

for some $k \geq 0$ and $m \geq 1$. Assume that there exists $R > r$ such that

$$\sup \Phi(K) \leq 0 < \inf \Phi(B), \qquad \sup \Phi(X) < b,$$

where $K = \{Ru : u \in K_0\}$, $B = \{ru : u \in B_0\}$, and $X = \{tu : u \in K,\ 0 \leq t \leq 1\}$. For $j = k+1, \dots, k+m$, let

$$\mathcal{A}_j^* = \{M \in \mathcal{A}^* : M \text{ is compact and } i^*(M) \geq j\}$$

and set

$$c_j^* := \inf_{M \in \mathcal{A}_j^*} \max_{u \in M} \Phi(u).$$

Then

$$\inf \Phi(B) \leq c_{k+1}^* \leq \dots \leq c_{k+m}^* \leq \sup \Phi(X),$$

in particular, $0 < c_j^* < b$. If, in addition, Φ satisfies the $(\mathrm{PS})_c$ condition for all $c \in (0, b)$, then each c_j^* is a critical value of Φ and there are m distinct pairs of associated critical points.

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2. In view of Theorem 3.1, we apply Theorem 5.3 with

$$b := \frac{s}{N}\, \alpha_{N,s}(\Omega)^{(N-s)/s}.$$

By Theorem 4.1, the sublevel set $\Psi^{\lambda_{k+m}}$ has a compact symmetric subset K_0 with

$$i(K_0) = k + m.$$

We take $B_0 := \Psi_{\lambda_{k+1}}$, so that

$$i(\mathcal{M} \setminus B_0) = k$$

by (15). Let $R > r > 0$ and let K, B, and X be as in Theorem 5.3. By Lemma 2.3 4,

$$\Phi(u) \geq \frac{s}{N} \left(\|u\|^{N/s} - \lambda \int_\Omega |u|^{N/s}\, dx \right) - \lambda \int_\Omega |u|^{N^2/s(N-s)}\, e^{\,|u|^{N/(N-s)}}\, dx,$$

so for $u \in \Psi_{\lambda_{k+1}}$,

$$\Phi(ru) \geq \frac{sr^{N/s}}{N}\left(1 - \frac{\lambda}{\Psi(u)}\right) - \lambda r^{N^2/s(N-s)} \int_\Omega |u|^{N^2/s(N-s)}\, e^{\,r^{N/(N-s)}\,|u|^{N/(N-s)}}\, dx$$

$$\geq r^{N/s}\left[\frac{s}{N}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\right.$$

$$\left. - \lambda r^{N/(N-s)}\left(\int_\Omega |u|^{2N^2/s(N-s)}\, dx\right)^{1/2}\left(\int_\Omega e^{\,2r^{N/(N-s)}\,|u|^{N/(N-s)}}\, dx\right)^{1/2}\right].$$

The first integral in the last expression is bounded since $W_0^{s,N/s}(\Omega) \hookrightarrow L^{2N^2/s(N-s)}(\Omega)$, and the second integral is also bounded if $2r^{N/(N-s)} < \alpha_{N,s}(\Omega)$. Since $\lambda < \lambda_{k+1}$, it follows that $\inf \Phi(B) > 0$ if r is sufficiently small. By Lemma 2.3 5 and the Hölder inequality,

$$\Phi(u) \leq \frac{s}{N}\, \|u\|^{N/s} - \frac{\lambda s\,(N-s)}{N^2} \int_\Omega |u|^{N^2/s(N-s)}\, dx$$

$$\leq \frac{s}{N}\, \|u\|^{N/s} - \frac{\lambda s\,(N-s)}{N^2\, \mathcal{L}(\Omega)^{s/(N-s)}}\left(\int_\Omega |u|^{N/s}\, dx\right)^{N/(N-s)},$$

so for $u \in K_0 \subset \Psi^{\lambda_{k+1}}$,

$$\Phi(Ru) \leq \frac{sR^{N/s}}{N} - \frac{\lambda s\,(N-s)\, R^{N^2/s(N-s)}}{N^2\, \mathcal{L}(\Omega)^{s/(N-s)}\, \Psi(u)^{N/(N-s)}}$$

$$\leq -\frac{sR^{N/s}}{N}\left(\frac{\lambda\,(N-s)\, R^{N/(N-s)}}{\lambda_{k+1}^{N/(N-s)}\, N\, \mathcal{L}(\Omega)^{s/(N-s)}} - 1\right).$$

It follows that $\Phi \leq 0$ on K if R is sufficiently large. By Lemma 2.3 5 and the Hölder inequality,

$$\Phi(u) \leq \frac{s}{N}\, \|u\|^{N/s} - \lambda \int_\Omega \left[\frac{s}{N}\, |u|^{N/s} + \frac{s\,(N-s)}{N^2}\, |u|^{N^2/s(N-s)}\right] dx$$

$$\leq \frac{s}{N}\left(\|u\|^{N/s} - \lambda \int_\Omega |u|^{N/s}\, dx\right) - \frac{\lambda s\,(N-s)}{N^2\, \mathcal{L}(\Omega)^{s/(N-s)}}\left(\int_\Omega |u|^{N/s}\, dx\right)^{N/(N-s)},$$

so for $u \in X$,

$$\Phi(u) \leq \frac{(\lambda_{k+1} - \lambda)\, s}{N} \int_\Omega |u|^{N/s}\, dx - \frac{\lambda s\,(N-s)}{N^2\, \mathcal{L}(\Omega)^{s/(N-s)}}\left(\int_\Omega |u|^{N/s}\, dx\right)^{N/(N-s)}$$

$$\leq \sup_{\rho \geq 0}\left[\frac{(\lambda_{k+1} - \lambda)\, s\rho}{N} - \frac{\lambda s\,(N-s)\, \rho^{N/(N-s)}}{N^2\, \mathcal{L}(\Omega)^{s/(N-s)}}\right]$$

$$= \frac{(\lambda_{k+1} - \lambda)^{N/s}\, s^2\, \mathcal{L}(\Omega)}{\lambda^{(N-s)/s}\, N^2}.$$

So

$$\sup \Phi(X) \leq \frac{(\lambda_{k+1} - \lambda)^{N/s}\, s^2\, \mathcal{L}(\Omega)}{\lambda^{(N-s)/s}\, N^2} < \frac{s}{N}\, \alpha_{N,s}(\Omega)^{(N-s)/s}$$

by (19). Thus, problem (3) has m distinct pairs of nontrivial solutions $\pm\, u_j^\lambda$, $j = 1, \dots, m$ such that

$$0 < \Phi(u_j^\lambda) \le \frac{(\lambda_{k+1} - \lambda)^{N/s}\, s^2\, \mathcal{L}(\Omega)}{\lambda^{(N-s)/s}\, N^2} \tag{20}$$

by Theorem 5.3. To prove that $u_j^\lambda \to 0$ as $\lambda \nearrow \lambda_{k+1}$, it suffices to show that for every sequence $\nu_n \nearrow \lambda_{k+1}$, a subsequence of $v_n := u_j^{\nu_n}$ converges to zero. We have

$$\Phi(v_n) = \frac{s}{N}\, \|v_n\|^{N/s} - \nu_n \int_\Omega F(v_n)\, dx \to 0 \tag{21}$$

by (20) and

$$\Phi'(v_n)\, v_n = \|v_n\|^{N/s} - \nu_n \int_\Omega v_n\, f(v_n)\, dx = 0. \tag{22}$$

Since $s/N > s\,(N-s)/N^2$, it follows from Lemma 2.3 2, (21), and (22) that (v_n) is bounded in $W_0^{s,N/s}(\Omega)$. Hence a renamed subsequence converges to some v weakly in $W_0^{s,N/s}(\Omega)$, strongly in $L^p(\Omega)$ for all $p \in [1, \infty)$, and a.e. in Ω. By Lemma 2.3 3, (21), and (22),

$$\frac{s^2}{N^2} \int_\Omega |v_n|^{N^2/s(N-s)}\, dx \le \int_\Omega \left[\frac{s}{N}\, v_n\, f(v_n) - F(v_n)\right] dx = \frac{\Phi(v_n)}{\nu_n} \le \frac{\Phi(v_n)}{\lambda_k} \to 0,$$

so

$$\int_\Omega |v|^{N^2/s(N-s)}\, dx = 0$$

and hence $v = 0$. Since $\int_\Omega v_n\, f(v_n)\, dx$ is bounded by (22), then

$$\int_\Omega F(v_n)\, dx \to 0,$$

by Lemma 3.2, so $\|v_n\| \to 0$ by (21). □

REFERENCES

[1] Adimurthi, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n–Laplacian, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **17** (1990), 393–413.

[2] Adimurthi and S. L. Yadava, Bifurcation results for semilinear elliptic problems with critical exponent in $\mathbb{R}^2$, *Nonlinear Anal.*, **14** (1990), 607–612.

[3] P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity, *Nonlinear Anal.*, **7** (1983), 981–1012.

[4] V. Benci, On critical point theory for indefinite functionals in the presence of symmetries, *Trans. Amer. Math. Soc.*, **274** (1982), 533–572.

[5] L. Brasco, E. Parini and M. Squassina, Stability of variational eigenvalues for the fractional p–Laplacian, *Discrete Contin. Dyn. Syst. A*, **36** (2016), 1813–1845.

[6] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.,* **88** (1983), 486–490.

[7] L. Carleson and A. Chang, On the existence of an extremal function for an inequality of J. Moser, *Bull. Sci. Math.*, **110** (1986), 113–127.

[8] D. G. de Figueiredo, O. H. Miyagaki and B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, *Calc. Var. Partial Differential Equations*, **3** (1995), 139–153.

[9] D. G. de Figueiredo, O. H. Miyagaki and B. Ruf, Corrigendum to Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, *Calc. Var. Partial Differential Equations*, **4** (1996), p203.

[10] J. M. do'O, Semilinear dirichlet problems for the N–Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range, *Differential Integral Equations*, **9** (1996), 967–979.

[11] J. M. do'O, O. H. Miyagaki and M. Squassina, Nonautonomous fractional problems with exponential growth, *NoDEA Nonlinear Differential Equations Applications*, **22** (2015), 1395–1410.

[12] J. M. do'O, O. H. Miyagaki and M. Squassina, Ground states of nonlocal scalar field equations with Trudinger–Moser critical nonlinearity, *Topol. Meth. Nonlinear Anal.*, **48** (2016), 477–492.

[13] E. R. Fadell and P. H. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems, *Invent. Math.*, **45** (1978), 139–174.

[14] J. Giacomoni, P. K. Mishra and K. Sreenadh, Fractional elliptic equations with critical exponential nonlinearity, *Adv. Nonlinear Anal.*, **5** (2016), 57–74.

[15] A. Iannizzotto, S. Mosconi and M. Squassina, Global Holder regularity for the fractional p–Laplacian, *Rev. Mat. Iberoam.*, **32** (2016), 1355–1394.

[16] A. Iannizzotto and M. Squassina, 1/2–Laplacian problems with exponential nonlinearity, *J. Math. Anal. Appl.,* **414** (2014), 372–385.

[17] S. Iula, A note on the Moser–Trudinger inequality in Sobolev-Slobodeckij spaces in dimension one, preprint, 2016.

[18] S. Iula, A. Maalaoui and L. Martinazzi, A fractional Moser–Trudinger type inequality in one dimension and its critical points, *Differential Integral Equations*, **29** (2016), 455–492.

[19] H. Kozono, T. Sato and H. Wadade, Upper bound of the best constant of a Trudinger-Moser inequality and its application to a Gagliardo–Nirenberg inequality, *Indiana Univ. Math. J.,* **55** (2006), 1951–1974.

[20] P.-L. Lions, The concentration–compactness principle in the calculus of variations. The limit case I, *Rev. Mat. Iberoam.*, **1** (1985), 145–201.

[21] L. Martinazzi, Fractional Adams–Moser–Trudinger type inequalities, *Nonlinear Anal.*, **127** (2015), 263–278.

[22] S. Mosconi and M. Squassina, Recent progresses in the theory of nonlinear nonlocal problems, *Bruno Pini Math. Analysis Sem.*, **2016** (2016), 147–164.

[23] J. Moser, A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.,* **20** (1970/71), 1077–1092.

[24] T. Ozawa, On critical cases of Sobolev's inequalitites, *J. Funct. Anal.,* **127** (1995), 259–269.

[25] E. Parini and B. Ruf, On the Moser–Trudinger inequality in fractional Sobolev-Slobodeckij spaces, *J. Anal. Math.*, to appear, `arXiv:1607.07681`.

[26] K. Perera, R. P. Agarwal and D. O'Regan, *Morse Theoretic Aspects of p–Laplacian Type Operators* **161**, Mathematical surveys and monographs, American Mathematical Society, Providence, RI, 2010.

[27] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.,* **17** (1967), 473–483.

[28] Y. Yang and K. Perera, N–Laplacian problems with critical Trudinger-Moser nonlinearities, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **16** (2016), 1123–1138.

Received May 2017; revised August 2017.

E-mail address: kperera@fit.edu
E-mail address: marco.squassina@unicatt.it