Research Article

Hoai-Minh Nguyen, Andrea Pinamonti, Marco Squassina* and Eugenio Vecchi

New characterizations of magnetic Sobolev spaces

https://doi.org/10.1515/anona-2017-0239
Received October 24, 2017; accepted October 24, 2017

Abstract: We establish two new characterizations of magnetic Sobolev spaces for Lipschitz magnetic fields in terms of nonlocal functionals. The first one is related to the BBM formula, due to Bourgain, Brezis and Mironescu. The second one is related to the work of the first author on the classical Sobolev spaces. We also study the convergence almost everywhere and the convergence in L^1 appearing naturally in these contexts.

Keywords: Magnetic Sobolev spaces, new characterization, nonlocal functionals

MSC 2010: 49A50, 26A33, 82D99

1 Introduction

In electromagnetism, a relevant role in the study of particles which interact with a magnetic field $B = \nabla \times A$, $A : \mathbb{R}^3 \to \mathbb{R}^3$, is played by the magnetic Laplacian $(\nabla - \mathrm{i}A)^2$ (see [2, 16, 27]). This yields to nonlinear Schrödinger equations of the type $-(\nabla - \mathrm{i}A)^2 u + u = f(u)$, which have been studied extensively (see e.g. [1, 13, 15, 17] and the references therein). The linear operator $-(\nabla - \mathrm{i}A)^2 u$ is defined weakly as the differential of the energy functional

$$H^1_A(\mathbb{R}^N) \ni u \mapsto \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2 \, dx$$

over complex-valued functions u on $\mathbb{R}^N$. Here i denotes the imaginary unit and $|\cdot|$ the standard Euclidean norm of $\mathbb{C}^N$. Given a measurable function $A : \mathbb{R}^N \to \mathbb{R}^N$ and given an open subset Ω of $\mathbb{R}^N$, one defines $H^1_A(\Omega)$ as the space of complex-valued functions $u \in L^2(\Omega)$ such that $\|u\|_{H^1_A(\Omega)} < \infty$ for the norm

$$\|u\|_{H^1_A(\Omega)} := \big(\|u\|^2_{L^2(\Omega)} + [u]^2_{H^1_A(\Omega)}\big)^{\frac{1}{2}}, \quad [u]_{H^1_A(\Omega)} := \bigg(\int_\Omega |\nabla u - \mathrm{i}A(x)u|^2 \, dx\bigg)^{\frac{1}{2}}.$$

In [14], some physically motivated nonlocal versions of the local magnetic energy were introduced. In particular, the operator $(-\Delta)^s_A$ is defined as the gradient of the nonlocal energy functional

$$H^s_A(\mathbb{R}^N) \ni u \mapsto (1-s) \iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy,$$

Hoai-Minh Nguyen, Department of Mathematics, Ecole Polytechnique Federale de Lausanne, EPFL SB CAMA, Station 8, CH-1015 Lausanne, Switzerland, e-mail: hoai-minh.nguyen@epfl.ch
Andrea Pinamonti, Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123 Povo (Trento), Italy, e-mail: andrea.pinamonti@unitn.it
***Corresponding author: Marco Squassina,** Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via dei Musei 41, 25121 Brescia, Italy, e-mail: marco.squassina@unicatt.it
Eugenio Vecchi, Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy, e-mail: eugenio.vecchi2@unibo.it

where $s \in (0,1)$. Recently, the existence of ground stated of $(-\Delta)^s_A u + u = f(u)$ was investigated in [11] via Lions concentration compactness arguments. In [28] a connection between the local and nonlocal notions was obtained on bounded domains; precisely, if $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain and $A \in C^2(\mathbb{R}^N)$, then for every $u \in H^1_A(\Omega)$ it holds

$$\lim_{s \nearrow 1}(1-s) \int_\Omega \int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy = Q_N \int_\Omega |\nabla u - \mathrm{i}A(x)u|^2\, dx, \tag{1.1}$$

where

$$Q_N := \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\boldsymbol{\omega} \cdot \sigma|^2\, d\sigma \tag{1.2}$$

being $\mathbb{S}^{N-1}$ the unit sphere in $\mathbb{R}^N$ and $\boldsymbol{\omega}$ an arbitrary unit vector of $\mathbb{R}^N$. See also [23] for the general case of the p-norm with $1 \le p < +\infty$ as well as [24], where the limit as $s \searrow 0$ is covered. This provides a new characterization of the H^1_A norm in terms of nonlocal functionals extending the results by Bourgain, Brezis and Mironescu [3, 4] (see also [12, 25]) to the magnetic setting. Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 1 and less than 1 and set

$$\rho_n(r) := \begin{cases} 2(1-s_n)\mathrm{diam}(\Omega)^{2s_n-2} r^{2-2s_n-N} & \text{for } 0 < r \le \mathrm{diam}(\Omega), \\ 0 & \text{for } r > \mathrm{diam}(\Omega), \end{cases}$$

where $\mathrm{diam}(\Omega)$ denotes the diameter of Ω. We have $\int_0^\infty \rho_n(r) r^{N-1}\, dr = 1$ and, for all $\delta > 0$,

$$\lim_{n\to+\infty} \int_\delta^\infty \rho_n(r) r^{N-1}\, dr = 0.$$

Given $u : \Omega \to \mathbb{C}$ a measurable complex-valued function, we denote

$$\Psi_u(x,y) := e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y), \quad x, y \in \Omega.$$

The function $\Psi_u(\cdot,\cdot)$ also depends on A but for notational ease, we ignore it. Assertion (1.1) can be then written as

$$\lim_{n\to+\infty} \int_\Omega \int_\Omega \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\, dx\, dy = 2Q_N \int_\Omega |\nabla u - \mathrm{i}A(x)u|^2\, dx. \tag{1.3}$$

This paper is concerned with the *whole space* setting. Our first goal is to obtain formula (1.3) for $\Omega = \mathbb{R}^N$ and to provide a characterization of $H^1_A(\mathbb{R}^N)$ in terms of the left-hand side of (1.3) in the spirit of the work of Bourgain, Brezis and Mironescu.

Here and in what follows, a sequence of nonnegative radial functions $\{\rho_n\}_{n\in\mathbb{N}}$ is called a *sequence of mollifiers* if it satisfies the conditions

$$\int_0^\infty \rho_n(r) r^{N-1}\, dr = 1 \quad \text{and} \quad \lim_{n\to+\infty} \int_\delta^\infty \rho_n(r) r^{N-1}\, dr = 0 \quad \text{for all } \delta > 0. \tag{1.4}$$

In this direction, we have the following:

Theorem 1.1. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative radial mollifiers. Then $u \in H^1_A(\mathbb{R}^N)$ if and only if $u \in L^2(\mathbb{R}^N)$ and*

$$\sup_{n\in\mathbb{N}} \iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\, dx\, dy < +\infty. \tag{1.5}$$

Moreover, for $u \in H^1_A(\mathbb{R}^N)$, we have

$$\lim_{n\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\, dx\, dy = 2Q_N \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\, dx, \tag{1.6}$$

and

$$\iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\,dx\,dy \\ \leq 2|\mathbb{S}^{N-1}| \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx + 2|\mathbb{S}^{N-1}|(2 + \|\nabla A\|^2_{L^\infty(\mathbb{R}^N)}) \int_{\mathbb{R}^N} |u|^2\,dx. \tag{1.7}$$

In this paper, $|\mathbb{S}^{N-1}|$ denotes the $(N-1)$-Hausdorff measure of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$.

The proof of Theorem 1.1 is given in Section 2.

Remark 1.1. Similar results as in Theorem 1.1 hold for more general mollifiers $\{\rho_n\}_{n\in\mathbb{N}}$ with slight changes in the constants. See Remark 2.1 for details.

The second goal of this paper is to characterize $H^1_A(\mathbb{R}^N)$ in term of $J_\delta(\cdot)$, where, for $\delta > 0$,

$$J_\delta(u) := \iint_{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta\}} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy \quad \text{for } u \in L^1_{\mathrm{loc}}(\mathbb{R}^N).$$

This is motivated by the characterization of the Sobolev space $H^1(\mathbb{R}^N)$ provided in [5] and [18] (see also [6–10, 19–22]) in terms of the family of nonlocal functionals I_δ which is defined by, for $\delta > 0$,

$$I_\delta(u) := \iint_{\{|u(y)-u(x)|>\delta\}} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy \quad \text{for } u \in L^1_{\mathrm{loc}}(\mathbb{R}^N).$$

It was showed in [5, 18] that if $u \in L^2(\mathbb{R}^N)$, then $u \in H^1(\mathbb{R}^N)$ if and only if $\sup_{0<\delta<1} I_\delta(u) < \infty$; moreover,

$$\lim_{\delta\searrow 0} I_\delta(u) = Q_N \int_\Omega |\nabla u|^2\,dx \quad \text{for } u \in H^1(\mathbb{R}^N).$$

Concerning this direction, we establish the following:

Theorem 1.2. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz. Then $u \in H^1_A(\mathbb{R}^N)$ if and only if $u \in L^2(\mathbb{R}^N)$ and*

$$\sup_{0<\delta<1} J_\delta(u) < +\infty. \tag{1.8}$$

Moreover, we have, for $u \in H^1_A(\mathbb{R}^N)$,

$$\lim_{\delta\searrow 0} J_\delta(u) = Q_N \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx$$

and

$$\sup_{\delta>0} J_\delta(u) \leq C_N \bigg(\int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx + (\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} + 1) \int_{\mathbb{R}^N} |u|^2\,dx \bigg). \tag{1.9}$$

Throughout the paper, we shall denote by C_N a generic positive constant depending only on N and possibly changing from line to line.

The proof of Theorem 1.2 is given in Section 3.

As pointed out in [13], a physically meaning example of magnetic potential in the space is

$$A(x,y,z) = \frac{1}{2}(-y, x, 0), \quad (x,y,z) \in \mathbb{R}^3,$$

which in fact fulfills the requirement of Theorems 1.1 and 1.2 that A is Lipschitz. Furthermore, in the spirit of [10], as a byproduct of Theorems 1.1 and 1.2, for $u \in L^2(\mathbb{R}^N)$, if we have

$$\lim_{n\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\,dx\,dy = 0 \quad \text{or} \quad \lim_{\delta\searrow 0} J_\delta(u) = 0,$$

then

$$\nabla \mathfrak{R} u = -A \mathfrak{I} u, \quad \nabla \mathfrak{I} u = A \mathfrak{R} u,$$

namely the direction of $\nabla \mathfrak{R} u$, $\nabla \mathfrak{I} u$ is that of the magnetic potential A. In the particular case $A = 0$, this implies that u is a constant function.

The L^p versions of the above mentioned results are given in Sections 2 and 3. In addition to these results, we also discuss the convergence almost everywhere and the convergence in L^1 of the quantities appearing in Theorems 1.1 and 1.2 in Section 4.

The paper is organized as follows. The proof of Theorems 1.1 and 1.2 are given in Sections 2 and 3, respectively. The convergence almost everywhere and the convergence in L^1 are investigated in Section 4.

2 Proof of Theorem 1.1 and its L^p version

The proof of Theorem 1.1 can be derived from a few lemmas which we present below. The first one is on (1.7).

Lemma 2.1 (Upper bound). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative radial mollifiers. We have, for all $u \in H^1_A(\mathbb{R}^N)$,*

$$\iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\, dx\, dy \\ \le 2|\mathbb{S}^{N-1}| \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\, dx + 2|\mathbb{S}^{N-1}|(2 + \|\nabla A\|^2_{L^\infty(\mathbb{R}^N)}) \int_{\mathbb{R}^N} |u|^2\, dx.$$

Proof. Since $C^\infty_c(\mathbb{R}^N)$ is dense in $H^1_A(\mathbb{R}^N)$ (cf. [16, Theorem 7.22]), using Fatou's lemma, without loss of generality, one might assume that $u \in C^1_c(\mathbb{R}^N)$. Recall that

$$\int_{\mathbb{R}^N} \rho_n(|z|)\, dz = |\mathbb{S}^{N-1}| \int_0^\infty \rho_n(r) r^{N-1}\, dr = |\mathbb{S}^{N-1}|. \tag{2.1}$$

Since

$$\iint_{\substack{\mathbb{R}^{2N} \\ \{|x-y|\ge 1\}}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\, dx\, dy \\ \le 2 \iint_{\mathbb{R}^{2N}} (|u(y)|^2 + |u(x)|^2)\rho_n(|x-y|)\, dx\, dy \le 4|\mathbb{S}^{N-1}| \int_{\mathbb{R}^N} |u|^2\, dx,$$

it suffices to prove that

$$\iint_{\substack{\mathbb{R}^{2N} \\ \{|x-y|<1\}}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\, dx\, dy \\ \le 2|\mathbb{S}^{N-1}| \Big(\int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\, dx + \|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^2\, dx \Big). \tag{2.2}$$

For a.e. $x, y \in \mathbb{R}^N$, we have

$$\frac{\partial \Psi_u(x,y)}{\partial y} = e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} \nabla u(y) - \mathrm{i}\Big\{ A\Big(\frac{x+y}{2}\Big) + \frac{1}{2}(y-x)\cdot \nabla A\Big(\frac{x+y}{2}\Big)\Big\} e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y).$$

It follows that

$$\Big|\frac{\partial \Psi_u(x,y)}{\partial y}\Big| \le |\nabla u(y) - \mathrm{i}A(y)u(y)| + \Big|A\Big(\frac{x+y}{2}\Big) - A(y)\Big||u(y)| + \frac{1}{2}|y-x|\Big|\nabla A\Big(\frac{x+y}{2}\Big)\Big||u(y)|. \tag{2.3}$$

This implies

$$\Big|\frac{\partial \Psi_u(x,y)}{\partial y}\Big| \le |\nabla u(y) - \mathrm{i}A(y)u(y)| + \|\nabla A\|_{L^\infty(\mathbb{R}^N)}|x-y||u(y)|,$$

which yields, for $x, y \in \mathbb{R}^N$ with $|x-y| < 1$,

$$\frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \le 2 \int_0^1 |\nabla u(ty + (1-t)x) - \mathrm{i}A(ty+(1-t)x)u(ty+(1-t)x)|^2\,dt + 2\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} \int_0^1 |u(ty+(1-t)x)|^2\,dt. \tag{2.4}$$

Since, for $f \in L^2(\mathbb{R}^N)$, in light of (1.4) and (2.1), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_0^1 |f(ty+(1-t)x)|^2 \rho_n(|x-y|)\,dt\,dx\,dy = \int_{\mathbb{R}^N} |f(x)|^2\,dx \int_{\mathbb{R}^N} \rho_n(|z|)\,dz = |\mathbb{S}^{N-1}| \int_{\mathbb{R}^N} |f(x)|^2\,dx,$$

we then derive from (2.4) that

$$\iint_{\substack{\mathbb{R}^{2N}\\ \{|x-y|<1\}}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\,dx\,dy \le 2|\mathbb{S}^{N-1}| \int_{\mathbb{R}^N} |\nabla u(y) - \mathrm{i}A(y)u(y)|^2\,dy + 2|\mathbb{S}^{N-1}|\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u(y)|^2\,dy,$$

which is (2.2). □

We next establish the following result which is used in the proof of (1.6) and in the proof of Theorem 1.2.

Lemma 2.2. *Let $u \in C^2(\mathbb{R}^N)$, $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz, and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative radial mollifiers. Then*

$$\liminf_{n\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|)\,dx\,dy \ge 2Q_N \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx. \tag{2.5}$$

Moreover, for any $(\varepsilon_n) \searrow 0$, there holds

$$\liminf_{n\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^{2+\varepsilon_n}}{|x-y|^{2+\varepsilon_n}} \rho_n(|x-y|)\,dx\,dy \ge 2Q_N \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx. \tag{2.6}$$

Throughout this paper, for $R > 0$, let B_R denote the open ball in $\mathbb{R}^N$ centered at the origin and of radius R.

Proof. Fix $R > 1$ (arbitrary). Using the fact

$$|e^{\mathrm{i}t} - (1+\mathrm{i}t)| \le Ct^2 \quad \text{for } t \in \mathbb{R},$$

we have, for $x, y \in B_R$,

$$\begin{aligned} |\Psi_u(x,y) - (1 + \mathrm{i}(x-y)\cdot A(y))u(y)| &\le \left|\Psi_u(x,y) - \left(1 + \mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)\right)u(y)\right| \\ &\quad + |x-y|\left|A\left(\frac{x+y}{2}\right) - A(y)\right||u(y)| \\ &\le C\|u\|_{C^2(B_R)}(1 + \|A\|_{W^{1,\infty}(B_R)})^2|x-y|^2. \end{aligned}$$

Here and in what follows, C denotes a positive constant. On the other hand, we obtain, for $x, y \in B_R$,

$$|u(x) - u(y) - \nabla u(y)\cdot(x-y)| \le C\|u\|_{C^2(B_R)}|x-y|^2.$$

It follows that

$$|[\Psi_u(x,y) - \Psi_u(x,x)] - (\nabla u(y) - \mathrm{i}A(y)u(y))\cdot(y-x)| \le C\|u\|_{C^2(B_R)}(1 + \|A\|_{W^{1,\infty}(B_R)})^2|x-y|^2. \tag{2.7}$$

Since

$$\lim_{n\to+\infty} \iint_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} |x-y|^2 \rho_n(|x-y|)\,dx\,dy = 0, \tag{2.8}$$

it follows from (2.7) that

$$\begin{aligned}\liminf_{n\to+\infty} \iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} & \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \\ & \geq \liminf_{n\to+\infty} \iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} \frac{|(\nabla u(y)-\mathrm{i}A(y)u(y))\cdot(x-y)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy.\end{aligned}$$

We have, by the definition of Q_N,

$$\liminf_{n\to+\infty} \iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} \frac{|(\nabla u(y)-\mathrm{i}A(y)u(y))\cdot(x-y)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \geq 2Q_N \int\limits_{B_{R-1}} |\nabla u(y)-\mathrm{i}A(y)u(y)|^2\,dy. \tag{2.9}$$

By the arbitrariness of $R > 1$ we get

$$\liminf_{n\to+\infty} \iint\limits_{\substack{\mathbb{R}^{2N}\\ \{|x-y|<1\}}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \geq 2Q_N \int\limits_{\mathbb{R}^N} |\nabla u-\mathrm{i}A(x)u|^2\,dx,$$

which implies (2.5).

Assertion (2.6) can be derived as follows. We have, by Hölder's inequality,

$$\begin{aligned}\iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} & \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \\ & \leq \Bigg(\iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^{2+\varepsilon_n}}{|x-y|^{2+\varepsilon_n}}\rho_n(|x-y|)\,dx\,dy\Bigg)^{\frac{2}{2+\varepsilon_n}} \Bigg(\iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} \rho_n(|x-y|)\,dx\,dy\Bigg)^{\frac{\varepsilon_n}{2+\varepsilon_n}}.\end{aligned}$$

Since, for every $R > 0$, there holds

$$\lim_{n\to+\infty} \Bigg(\iint\limits_{\substack{B_R\times B_R\\ \{|x-y|\leq 1\}}} \rho_n(|x-y|)\,dx\,dy\Bigg)^{\frac{\varepsilon_n}{2+\varepsilon_n}} = 1,$$

we get (2.6) from (2.9) and the arbitrariness of $R > 1$. □

We are ready to prove (1.6).

Lemma 2.3 (Limit formula). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative radial mollifiers. Then, for $u \in H^1_A(\mathbb{R}^N)$,*

$$\lim_{n\to+\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy = 2Q_N \int\limits_{\mathbb{R}^N} |\nabla u-\mathrm{i}A(x)u|^2\,dx.$$

Proof. By Lemma 2.1 and the density of $C_c^\infty(\mathbb{R}^N)$ in $H^1_A(\mathbb{R}^N)$, one might assume that $u \in C_c^2(\mathbb{R}^N)$. From Lemma 2.2, it suffices to prove that, for $u \in C_c^2(\mathbb{R}^N)$,

$$\limsup_{n\to+\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \leq 2Q_N \int\limits_{\mathbb{R}^N} |\nabla u-\mathrm{i}A(x)u|^2\,dx. \tag{2.10}$$

Fix $R > 4$ such that $\operatorname{supp} u \subset B_{R/2}$. Using (2.7) and (2.8), one derives that

$$\begin{aligned}\limsup_{n\to+\infty} \iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} & \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \\ & \leq \limsup_{n\to+\infty} \iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} \frac{|(\nabla u(y)-\mathrm{i}A(y)u(y))\cdot(x-y)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy,\end{aligned}$$

which yields

$$\limsup_{n\to+\infty} \iint\limits_{\substack{B_R\times B_R\\ \{|x-y|<1\}}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \le 2Q_N \int\limits_{\mathbb{R}^N} |\nabla u(y) - \mathrm{i}A(y)u(y)|^2\,dy. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}
&\limsup_{n\to+\infty} \iint\limits_{\substack{\mathbb{R}^{2N}\\ \{|x-y|\ge 1\}}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy\\
&\quad\le \limsup_{n\to+\infty} \iint\limits_{\substack{\mathbb{R}^{2N}\\ \{|x-y|\ge 1\}}} 2(|u(x)|^2+|u(y)|^2)\rho_n(|x-y|)\,dx\,dy = 0,
\end{aligned} \tag{2.12}$$

and the fact that

$$\text{if } (x,y)\notin B_R\times B_R \text{ and } |x-y|<1, \text{ then } |\Psi_u(x,y)-\Psi_u(x,x)| = 0, \tag{2.13}$$

by the choice of R. Combining (2.11), (2.12) and (2.13) yields (2.10). □

The following result is about uniform bounds for the integrals in (1.5).

Lemma 2.4. *Let $A:\mathbb{R}^N\to\mathbb{R}^N$ be Lipschitz and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative radial mollifiers. Then $u\in H^1_A(\mathbb{R}^N)$ if $u\in L^2(\mathbb{R}^N)$ and*

$$\sup_{n\in\mathbb{N}} \iint\limits_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy < +\infty. \tag{2.14}$$

Proof. Let $\{\tau_m\}$ be a sequence of nonnegative mollifiers with $\operatorname{supp}\tau_m\subset B_1$ which is normalized by the condition $\int_{\mathbb{R}^N}\tau_m(x)\,dx = 1$. Set

$$u_m = u * \tau_m.$$

We estimate

$$\iint\limits_{\mathbb{R}^{2N}} \frac{|\Psi_{u_m}(x,y)-\Psi_{u_m}(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy.$$

We have

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^{2N}} \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_m(y)-u_m(x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy\\
&\quad= \iint\limits_{\mathbb{R}^{2N}} \frac{\left|\int_{\mathbb{R}^N}(e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y-z)-u(x-z))\tau_m(z)\,dz\right|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy.
\end{aligned}$$

By the change of variables $y'=y-z$ and $x'=x-z$ and using the inequality $|a+b|^2\le 2(|a|^2+|b|^2)$ for all $a,b\in\mathbb{C}$ and applying Jensen's inequality, we deduce that

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^{2N}} \frac{|\Psi_{u_m}(x,y)-\Psi_{u_m}(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy\\
&\quad\le 2\iint\limits_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy\\
&\qquad+ 2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)}-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}|^2|u(y)|^2}{|x-y|^2}\tau_m(z)\rho_n(|x-y|)\,dz\,dx\,dy.
\end{aligned} \tag{2.15}$$

Since, for $t\in\mathbb{R}$,

$$|e^{\mathrm{i}t}-1|\le C|t|,$$

it follows that, for all $x, y, z \in \mathbb{R}^N$,

$$\big|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)} - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\big| = \big|e^{\mathrm{i}(x-y)\cdot(A(\frac{x+y}{2}+z)-A(\frac{x+y}{2}))} - 1\big| \le C\|\nabla A\|_{L^\infty(\mathbb{R}^N)}|x-y||z| \le C|x-y||z|.$$

Here and in what follows in this proof, C denotes some positive constant independent of m and n. Taking into account the fact that supp $\tau_m \subset B_1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)} - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}|^2|u(y)|^2}{|x-y|^2}\tau_m(z)\rho_n(|x-y|)\,dz\,dx\,dy \\ & \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} C|u(y)|^2\tau_m(z)\rho_n(|x-y|)\,dz\,dx\,dy \le C. \end{aligned} \tag{2.16}$$

Combining (2.14), (2.15) and (2.16) yields

$$\iint_{\mathbb{R}^{2N}} \frac{|\Psi_{u_m}(x,y) - \Psi_{u_m}(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \le C. \tag{2.17}$$

On the other hand, by Lemma 2.2 we have

$$\liminf_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|\Psi_{u_m}(x,y) - \Psi_{u_m}(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \ge 2Q_N \int_{\mathbb{R}^N} |\nabla u_m - \mathrm{i}A(x)u_m|^2\,dx. \tag{2.18}$$

The conclusion now immediately follows from (2.17) and (2.18) after letting $m \to +\infty$. □

We are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Theorem 1.1 is a direct consequence of Lemmas 2.1, 2.3 and 2.4. □

Remark 2.1. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative radial functions such that

$$\int_0^1 \rho_n(r)r^{N-1}\,dr = 1, \qquad \lim_{n\to+\infty}\int_\delta^1 \rho_n(r)r^{N-1}\,dr = 0 \quad \text{for every } \delta > 0,$$

and

$$\lim_{n\to+\infty}\int_1^\infty \rho_n(r)r^{N-3}\,dr = 0.$$

Theorem 1.1 then holds for such a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ provided that the constant 2 in (1.7) is replaced by an appropriate positive constant C independent of u. This follows by taking into account the fact that, for $u \in L^2(\mathbb{R}^N)$,

$$\begin{aligned}\limsup_{n\to+\infty} \iint_{\substack{\mathbb{R}^{2N}\\ \{|x-y|\ge1\}}} & \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|x-y|)\,dx\,dy \\ & \le 2\limsup_{n\to+\infty} \iint_{\substack{\mathbb{R}^{2N}\\ \{|x-y|\ge1\}}} (|u(x)|^2 + |u(y)|^2)\rho_n(|x-y|)|x-y|^{-2}\,dx\,dy = 0.\end{aligned}$$

For example, this applies to the radial sequence

$$\rho_n(r) = 2(1-s_n)r^{2-2s_n-N} \quad \text{for } r > 0,$$

which provides a characterization of $H^1_A(\mathbb{R}^N)$ and yields

$$\lim_{n\to+\infty}(1-s_n)\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s_n}}\,dx\,dy = 2Q_N\int_{\mathbb{R}^N}|\nabla u - \mathrm{i}A(x)u|^2\,dx.$$

Consider now the space $(\mathbb{C}^n, |\cdot|_p)$ $(n \geq 1)$, endowed with the norm

$$|z|_p := (|(\Re z_1, \dots, \Re z_n)|^p + |(\Im z_1, \dots, \Im z_n)|^p)^{\frac{1}{p}},$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^n$ and $\Re a$, $\Im a$ denote the real and imaginary parts of $a \in \mathbb{C}$, respectively. We emphasize that this is not related to the p-norm in $\mathbb{R}^n$. In what follows, we use this notation with $n = N$ and $n = 1$. Notice that $|z|_p = |z|$ whenever $z \in \mathbb{R}^n$, which makes our next statements consistent with the case $A = 0$ and u being a real valued function. Also $|\cdot|_2 = |\cdot|$, consistently with the previous definition. Define, for some $\boldsymbol{\omega} \in \mathbb{S}^{N-1}$,

$$Q_{N,p} := \frac{1}{p} \int_{\mathbb{S}^{N-1}} |\boldsymbol{\omega} \cdot \sigma|_p^p \, d\sigma. \tag{2.19}$$

We have, for $z \in \mathbb{C}^N$, (see [3, 23])

$$\int_{\mathbb{S}^{N-1}} |z \cdot \sigma|_p^p \, d\sigma = \int_{\mathbb{S}^{N-1}} |\Re z \cdot \sigma|^p \, d\sigma + \int_{\mathbb{S}^{N-1}} |\Im z \cdot \sigma|^p \, d\sigma = |\Re z|^p p Q_{N,p} + |\Im z|^p p Q_{N,p} = |z|_p^p p Q_{N,p}. \tag{2.20}$$

Using the same approach and technique, one can prove the following L^p version of Theorem 1.1.

Theorem 2.1. *Let $p \in (1, +\infty)$, $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz, and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative radial mollifiers. Then $u \in W_A^{1,p}(\mathbb{R}^N)$ if and only if $u \in L^p(\mathbb{R}^N)$ and*

$$\sup_{n\in\mathbb{N}} \iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|_p^p}{|x-y|^p} \rho_n(|x-y|) \, dx \, dy < +\infty.$$

Moreover, for $u \in W_A^{1,p}(\mathbb{R}^N)$, we have

$$\lim_{n\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|_p^p}{|x-y|^p} \rho_n(|x-y|) \, dx \, dy = p Q_{N,p} \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|_p^p \, dx$$

and

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|_p^p}{|x-y|^p} \rho_n(|x-y|) \, dx \, dy \\ &\quad \leq C_{N,p} \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|_p^p \, dx + C_{N,p}(2 + \|\nabla A\|_{L^\infty(\mathbb{R}^N)}^p) \int_{\mathbb{R}^N} |u|_p^p \, dx \end{aligned} \tag{2.21}$$

for some positive constant $C_{N,p}$ depending only on N and p.

Remark 2.2. Assume that C is a positive constant such that, for all $a, b \in \mathbb{C}$,

$$|a+b|_p^p \leq C(|a|_p^p + |b|_p^p).$$

Then assertion (2.21) of Theorem 2.1 holds with $C_{N,p} = |\mathbb{S}^{N-1}|C$.

3 Proof of Theorem 1.2 and its L^p version

Let us set, for $\sigma \in \mathbb{S}^{N-1}$,

$$\mathscr{M}_\sigma(g, x) := \sup_{t>0} \frac{1}{t} \int_0^t |g(x + s\sigma)| \, ds.$$

and denote $\mathscr{M}_{e_N}$ by $\mathscr{M}_N$, $e_N := (0, \dots, 0, 1)$. We have the following result which is a direct consequence of the theory of maximal functions, see e.g. [29, Theorem 1, p. 5].

Lemma 3.1 (Maximal function estimate). *There exists a universal constant $C > 0$ such that, for all $\sigma \in \mathbb{S}^{N-1}$,*

$$\int_{\mathbb{R}^N} |\mathscr{M}_\sigma(g,x)|^2\,dx \le C \int_{\mathbb{R}^N} |g|^2\,dx \quad \text{for all } g \in L^2(\mathbb{R}^N).$$

The following lemma yields an upper bound of $J_\delta(u)$ in terms of the norm of u in $H^1_A(\mathbb{R}^N)$.

Lemma 3.2 (Uniform upper bound). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz and $u \in H^1_A(\mathbb{R}^N)$. We have*

$$\sup_{\delta>0} J_\delta(u) \le C_N\Bigg(\int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx + (\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} + 1)\int_{\mathbb{R}^N} |u|^2\,dx\Bigg).$$

Proof. By the density of $C_c^\infty(\mathbb{R}^N)$ in $H^1_A(\mathbb{R}^N)$, using Fatou's lemma, we can assume that $u \in C_c^1(\mathbb{R}^N)$. For each $\delta > 0$, let us define

$$\mathcal{A}_\delta := \{(x,y) \in \mathbb{R}^{2N} : |\Psi_u(x,y) - \Psi_u(x,x)| > \delta,\ |x-y| < 1\}$$

and

$$\mathcal{B}_\delta := \{(x,y) \in \mathbb{R}^{2N} : |\Psi_u(x,y) - \Psi_u(x,x)| > \delta,\ |x-y| \ge 1\}.$$

We have

$$\iint_{\mathbb{R}^{2N}} \frac{\delta^2}{|x-y|^{N+2}}\mathbf{1}_{\mathcal{B}_\delta}\,dx\,dy \le \iint_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^{N+2}}\mathbf{1}_{\{|x-y|\ge 1\}}\,dx\,dy.$$

Since $|\Psi_u(x,y) - \Psi_u(x,x)| \le |u(x)| + |u(y)|$ and

$$\iint_{\substack{\mathbb{R}^{2N}\\ \{|x-y|\ge 1\}}} \frac{|u(x)|^2}{|x-y|^{N+2}}\,dx\,dy \le C_N \int_{\mathbb{R}^N} |u(x)|^2\,dx,$$

it follows that

$$\iint_{\mathbb{R}^{2N}} \frac{\delta^2}{|x-y|^{N+2}}\mathbf{1}_{\mathcal{B}_\delta}\,dx\,dy \le C_N \int_{\mathbb{R}^N} |u(x)|^2\,dx.$$

We are therefore interested in estimating the integral

$$\iint_{\mathcal{A}_\delta} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy.$$

Let us now define

$$\mathcal{X}_\delta := \{(x,h,\sigma) \in \mathbb{R}^N \times (0,1) \times \mathbb{S}^{N-1} : |\Psi_u(x,x+h\sigma) - \Psi_u(x,x)| > \delta\}.$$

Performing the change of variables $y = x + h\sigma$, for $h \in (0,1)$ and $\sigma \in \mathbb{S}^{N-1}$, yields

$$\iint_{\mathcal{A}_\delta} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy = \iiint_{\mathcal{X}_\delta} \frac{\delta^2}{h^3}\,dh\,dx\,d\sigma = \int_{\mathbb{S}^{N-1}} \iint_{\mathcal{C}_\sigma} \frac{\delta^2}{h^3}\,dh\,dx\,d\sigma,$$

where $\mathcal{C}_\sigma$ denotes the set

$$\mathcal{C}_\sigma := \{(x,h) \in \mathbb{R}^N \times (0,1) : |\Psi_u(x,x+h\sigma) - \Psi_u(x,x)| > \delta\}, \quad \sigma \in \mathbb{S}^{N-1}.$$

Without loss of generality it suffices to prove that, for $\sigma = e_N = (0,\dots,0,1) \in \mathbb{S}^{N-1}$,

$$\iint_{\mathcal{C}_{e_N}} \frac{\delta^2}{h^3}\,dh\,dx \le C_N\Bigg(\int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx + \|\nabla A\|^2_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N} |u|^2\,dx\Bigg). \tag{3.1}$$

We have, by virtue of (2.3),

$$|\Psi(x,x+he_N) - \Psi(x,x)| \le h\mathscr{M}_N(|\nabla u - \mathrm{i}Au|, x) + h^2\|\nabla A\|_{L^\infty(\mathbb{R}^N)}\mathscr{M}_N(|u|, x). \tag{3.2}$$

Using the fact that if $a + b > \delta$, then either $a > \frac{\delta}{2}$ or $b > \frac{\delta}{2}$, we derive that

$$\begin{aligned}\iint\limits_{\mathcal{C}_{e_N}} \frac{\delta^2}{h^3}\, dh\, dx &\le \iint\limits_{\{h\mathscr{M}_N(|\nabla u - \mathrm{i}Au|,x) > \frac{\delta}{2}\}} \frac{\delta^2}{h^3}\, dh\, dx + \iint\limits_{\{h^2 \|\nabla A\|_{L^\infty(\mathbb{R}^N)} \mathscr{M}_N(|u|,x) > \frac{\delta}{2}\}} \frac{\delta^2}{h^3}\, dh\, dx \\ &\le \iint\limits_{\{h\mathscr{M}_N(|\nabla u - \mathrm{i}Au|,x) > \frac{\delta}{2}\}} \frac{\delta^2}{h^3}\, dh\, dx + \iint\limits_{\{h \|\nabla A\|_{L^\infty(\mathbb{R}^N)} \mathscr{M}_N(|u|,x) > \frac{\delta}{2}\}} \frac{\delta^2}{h^3}\, dh\, dx,\end{aligned}$$

where the last inequality follows recalling that since $(x, h) \in \mathcal{C}_{e_N}$ then $h \in (0, 1)$. As usual, by using the theory of maximal functions stated in Lemma 3.1, we have

$$\iint\limits_{\{h\mathscr{M}_N(|\nabla u - \mathrm{i}Au|,x) > \frac{\delta}{2}\}} \frac{\delta^2}{h^3}\, dh\, dx \le C_N \int\limits_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\, dx \tag{3.3}$$

and

$$\iint\limits_{\{h \|\nabla A\|_{L^\infty(\mathbb{R}^N)} \mathscr{M}_N(|u|,x) > \frac{\delta}{2}\}} \frac{\delta^2}{h^3}\, dh\, dx \le C_N \|\nabla A\|^2_{L^\infty} \int\limits_{\mathbb{R}^N} |u|^2\, dx. \tag{3.4}$$

Assertion (3.1) follows from (3.3) and (3.4). The proof is complete. □

We next establish the following lemma.

Lemma 3.3 (Limit formula). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz and $u \in H^1_A(\mathbb{R}^N)$. Then*

$$\lim_{\delta \searrow 0} J_\delta(u) = Q_N \int\limits_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\, dx,$$

where Q_N is the constant defined in (1.2).

Proof. By virtue of Lemma 3.2, for every $\delta > 0$ and all $w \in H^1_A(\mathbb{R}^N)$, we have

$$J_\delta(w) \le C_N\Big(\int\limits_{\mathbb{R}^N} |\nabla w - \mathrm{i}A(x)w|^2\, dx + (\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} + 1) \int\limits_{\mathbb{R}^N} |w|^2\, dx \Big). \tag{3.5}$$

Since

$$|\Psi_u(x, y) - \Psi_u(x, x)| \le |\Psi_v(x, y) - \Psi_v(x, x)| + |\Psi_{u-v}(x, y) - \Psi_{u-v}(x, x)|,$$

it follows that, for every $\varepsilon \in (0, 1)$,

$$J_\delta(u) \le \iint\limits_{\{|\Psi_v(x,y) - \Psi_v(x,x)| > (1-\varepsilon)\delta\}} \frac{\delta^2}{|x - y|^{N+2}}\, dx\, dy + \iint\limits_{\{|\Psi_{u-v}(x,y) - \Psi_{u-v}(x,x)| > \varepsilon\delta\}} \frac{\delta^2}{|x - y|^{N+2}}\, dx\, dy.$$

This implies, for $\varepsilon \in (0, 1)$ and $u, v \in H^1_A(\mathbb{R}^N)$,

$$J_\delta(u) \le (1 - \varepsilon)^{-2} J_{(1-\varepsilon)\delta}(v) + \varepsilon^{-2} J_{\varepsilon\delta}(u - v). \tag{3.6}$$

From (3.5) and (3.6), we derive that, for $u, u_n \in H^1_A(\mathbb{R}^N)$ and $\varepsilon \in (0, 1)$,

$$J_\delta(u) - (1-\varepsilon)^{-2} J_{(1-\varepsilon)\delta}(u_n) \le \varepsilon^{-2} C_N\Big(\int\limits_{\mathbb{R}^N} |\nabla(u - u_n) - \mathrm{i}A(x)(u - u_n)|^2\, dx + (\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} + 1) \int\limits_{\mathbb{R}^N} |u - u_n|^2\, dx \Big) \tag{3.7}$$

and

$$(1-\varepsilon)^2 J_{\delta/(1-\varepsilon)}(u_n) - J_\delta(u) \le \varepsilon^{-2} C_N\Big(\int\limits_{\mathbb{R}^N} |\nabla(u - u_n) - \mathrm{i}A(x)(u - u_n)|^2\, dx + (\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} + 1) \int\limits_{\mathbb{R}^N} |u - u_n|^2\, dx \Big). \tag{3.8}$$

Since $C^1_c(\mathbb{R}^N)$ is dense in $H^1_A(\mathbb{R}^N)$, from (3.7) and (3.8), it suffices to prove the assertion for $u \in C^1_c(\mathbb{R}^N)$. This fact is assumed from now on.

Let $R > 0$ be such that $\operatorname{supp} u \subset B_{R/2}$. We claim that, for every $\sigma \in \mathbb{S}^{N-1}$, there holds

$$\lim_{\delta\searrow 0} \iint\limits_{\left\{(x,h)\in B_R\times(0,\infty)\,:\,\left|\frac{\Psi_u(x,x+\delta h\sigma)-\Psi_u(x,x)}{\delta h}\right|h>1\right\}} \frac{1}{h^3}\,dh\,dx = \frac{1}{2}\int\limits_{\mathbb{R}^N} |(\nabla u - \mathrm{i}Au)\cdot\sigma|^2\,dx. \tag{3.9}$$

Without loss of generality, we can assume $\sigma = e_N \in \mathbb{S}^{N-1}$. Then, we aim to prove that

$$\lim_{\delta\searrow 0} \iint\limits_{\left\{(x,h)\in B_R\times(0,\infty)\,:\,\left|\frac{\Psi_u(x,x+\delta he_N)-\Psi_u(x,x)}{\delta h}\right|h>1\right\}} \frac{1}{h^3}\,dh\,dx = \frac{1}{2}\int\limits_{\mathbb{R}^N} \left|\frac{\partial u}{\partial y_N}(x) - \mathrm{i}A_N(x)u(x)\right|^2 dx,$$

where A_N denotes the N-th component of A. To this end, we consider the sets

$$\begin{aligned}
\mathcal{C}_{e_N}(x',\delta) &:= \left\{(x_N,h)\in\mathbb{R}\times(0,\infty)\,:\,\left|\frac{\Psi_u(x,x+\delta he_N)-\Psi_u(x,x)}{\delta h}\right|h>1\right\},\\
\mathcal{E}(x') &:= \left\{(x_N,h)\in\mathbb{R}\times(0,\infty)\,:\,\left|\frac{\partial\Psi_u}{\partial y_N}(x,x)\right|h>1\right\},\\
\mathcal{F}(x') &:= \{(x_N,h)\in\mathbb{R}\times(0,\infty)\,:\,h\mathscr{M}_N(|\nabla u-\mathrm{i}Au|,x)+h^2\|\nabla A\|_{L^\infty(\mathbb{R}^N)}\mathscr{M}_N(|u|,x)>1\}.
\end{aligned}$$

Therefore, we obtain $\chi_{\mathcal{C}_{e_N}(x',\delta)}(x_N,h) \le \chi_{\mathcal{F}(x')}(x_N,h)$ for a.e. $(x,h)\in B_R\times(0,\infty)$ (by (3.2) in the proof of Lemma 3.2) and

$$\int\limits_{B_R}\int\limits_0^\infty \frac{1}{h^3}\chi_{\mathcal{F}(x')}(x_N,h)\,dh\,dx \le \mathcal{I}_1 + \mathcal{I}_2,$$

where we have set

$$\begin{aligned}
\mathcal{I}_1 &:= \iint\limits_{\{(x,h)\in B_R\times(0,\infty)\,:\,\mathscr{M}_N(|\nabla u-\mathrm{i}Au|,x)h>\frac{1}{2}\}} \frac{1}{h^3}\,dh\,dx,\\
\mathcal{I}_2 &:= \iint\limits_{\{(x,h)\in B_R\times(0,\infty)\,:\,h^2\|\nabla A\|_{L^\infty(\mathbb{R}^N)}\mathscr{M}_N(|u|,x)>\frac{1}{2}\}} \frac{1}{h^3}\,dh\,dx,
\end{aligned}$$

and we have denoted χ the characteristic function. We have, by the theory of maximal functions,

$$\mathcal{I}_1 \le C\int\limits_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx,$$

and, by a straightforward computation,

$$\mathcal{I}_2 \le C\|\nabla A\|_{L^\infty(\mathbb{R}^N)}\|u\|_{L^\infty(\mathbb{R}^N)}|B_R|.$$

The validity of claim (3.9) with $\sigma = e_N$ now follows from Dominated Convergence theorem since

$$\lim_{\delta\searrow 0}\chi_{\mathcal{C}_{e_N}(x',\delta)}(x_N,h) = \chi_{\mathcal{E}(x')}(x_N,h) \quad \text{for a.e. } (x,h)\in B_R\times(0,\infty),$$

and, by a direct computation,

$$\int\limits_{B_R}\int\limits_0^\infty \chi_{\mathcal{E}(x')}(x_N,h)\frac{1}{h^3}\,dh\,dx = \frac{1}{2}\int\limits_{B_R}\left|\frac{\partial u}{\partial y_N}(x) - \mathrm{i}A_N(x)u(x)\right|^2 dx.$$

Now, performing a change of variables we get

$$\iint\limits_{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta,\,x\in B_R\}} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy = \int\limits_{B_R}\int\limits_{\mathbb{S}^{N-1}}\int\limits_0^\infty \chi_{\mathcal{C}_\sigma(\delta)}(x,h)\frac{1}{h^3}\,dh\,d\sigma\,dx,$$

where

$$\mathcal{C}_\sigma(\delta) := \left\{(x,h)\in B_R\times(0,\infty)\,:\,\left|\frac{\Psi_u(x,x+\delta h\sigma)-\Psi_u(x,x)}{\delta h}\right|h>1\right\}.$$

Exploiting (3.9), we obtain

$$\lim_{\delta\searrow 0}\iint\limits_{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta,\,x\in B_R\}} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy = \frac{1}{2}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{B_R} |(\nabla u - \mathrm{i}Au)\cdot\sigma|^2\,dx\,d\sigma. \tag{3.10}$$

On the other hand, since $\operatorname{supp} u \subset B_{R/2}$, we have

$$\lim_{\delta\searrow 0} \iint\limits_{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta,\, x\in\mathbb{R}^N\setminus B_R\}} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy = \lim_{\delta\searrow 0} \iint\limits_{\{x\in\mathbb{R}^N\setminus B_R,\, y\in B_{R/2}\}} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy = 0. \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\lim_{\delta\searrow 0} \iint\limits_{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta\}} \frac{\delta^2}{|x-y|^{N+2}}\,dx\,dy = \frac{1}{2}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N} |(\nabla u - \mathrm{i}Au)\cdot\sigma|^2\,dx\,d\sigma.$$

In order to conclude, we notice the following, see (2.20):

$$\int\limits_{\mathbb{S}^{N-1}} |V\cdot\sigma|^2\,d\sigma = 2Q_N|V|^2 \quad \text{for any } V\in\mathbb{C}^N,$$

where Q_N is the constant defined in (1.2). □

We next deal with (1.8).

Lemma 3.4. *Let $u \in L^2(\mathbb{R}^N)$ and let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz. Then $u \in H^1_A(\mathbb{R}^N)$ if*

$$\sup_{\delta\in(0,1)} J_\delta(u) < +\infty. \tag{3.12}$$

Proof. The proof is divided into two steps.

Step 1. We assume that $u \in L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Set

$$L := \sup_{x,y\in\mathbb{R}^N} |\Psi_u(x,y)-\Psi_u(x,x)|.$$

In light of (3.12), we obtain

$$\int_0^L \varepsilon\delta^{\varepsilon-1} J_\delta(u)\,d\delta \le C$$

for some positive constant C independent of $\varepsilon \in (0,1)$. By Fubini's theorem and by the definition of L, we have

$$\int_0^L \varepsilon\delta^{\varepsilon-1} J_\delta(u)\,d\delta = \int\limits_{\mathbb{R}^{2N}} \frac{1}{|x-y|^{N+2}} \int_0^{|\Psi_u(x,y)-\Psi_u(x,x)|} \varepsilon\delta^{\varepsilon+1}\,d\delta\,dx\,dy.$$

It follows that

$$\frac{1}{2+\varepsilon}\iint\limits_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^{2+\varepsilon}}{|x-y|^{2+\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\varepsilon}}\,dx\,dy \le C.$$

By virtue of inequality (2.6) of Lemma 2.4, we have

$$\liminf_{\varepsilon\to 0} \iint\limits_{\mathbb{R}^{2N}} \frac{|\Psi_u(x,y)-\Psi_u(x,x)|^{2+\varepsilon}}{|x-y|^{2+\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\varepsilon}}\,dx\,dy \ge 2Q_N \int\limits_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2\,dx,$$

which implies $u \in H^1_A(\mathbb{R}^N)$.

Step 2. We consider the general case. For $M > 1$, define $\mathcal{T}_M : \mathbb{C}\to\mathbb{C}$ by setting

$$\mathcal{T}_M(z) := \begin{cases} z & \text{if } |z|\le M,\\ \frac{Mz}{|z|} & \text{otherwise,}\end{cases}$$

and denote

$$u_M := \mathcal{T}_M(u).$$

Then we have

$$|\mathcal{T}_M(z_1) - \mathcal{T}_M(z_2)| \le |z_1 - z_2| \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

It follows that

$$|\Psi_{u_M}(x,y) - \Psi_{u_M}(x,x)| \le |\Psi_u(x,y) - \Psi_u(x,x)| \quad \text{for all } x, y \in \mathbb{R}^N.$$

Hence we obtain

$$J_\delta(u_M) \le J_\delta(u). \tag{3.13}$$

Applying the result in Step 1, we have $u_M \in H^1_A(\mathbb{R}^N)$ and hence by Lemma 3.3,

$$\lim_{\delta\to 0} J_\delta(u_M) = 2Q_N \int_{\mathbb{R}^N} |\nabla u_M(x) - \mathrm{i}A(x)u_M(x)|^2\,dx. \tag{3.14}$$

Combining (3.13) and (3.14) and letting $M \to +\infty$, we derive that $u \in H^1_A(\mathbb{R}^N)$. The proof is complete. □

Remark 3.1. Similar approach used for $H^1(\mathbb{R}^N)$ is given in [18].

Proof of Theorem 1.2. The limit formula stated in Theorem 1.2 follows by Lemma 3.3. Now, if $u \in H^1_A(\mathbb{R}^N)$, then (1.9) follows from Lemma 3.2. On the contrary, if $u \in L^2(\mathbb{R}^N)$ and (1.8) holds, it follows from Lemma 3.4 that $u \in H^1_A(\mathbb{R}^N)$. □

Given u a measurable complex-valued function, define, for $1 < p < +\infty$,

$$J_{\delta,p}(u) := \iint_{\{|\Psi_u(x,y)-\Psi_u(x,x)|_p > \delta\}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \quad \text{for } \delta > 0.$$

We have the following L^p-version of Theorem 1.2.

Theorem 3.1. *Let $p \in (1,+\infty)$ and let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz. Then $u \in W^{1,p}_A(\mathbb{R}^N)$ if and only if $u \in L^p(\mathbb{R}^N)$ and*

$$\sup_{0<\delta<1} J_{\delta,p}(u) < \infty.$$

Moreover, we have, for $u \in W^{1,p}_A(\mathbb{R}^N)$,

$$\lim_{\delta\searrow 0} J_{\delta,p}(u) = Q_{N,p} \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|_p^p\,dx$$

and

$$J_{\delta,p}(u) \le C_{N,p}\Big(\int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|_p^p\,dx + (\|\nabla A\|^p_{L^\infty(\mathbb{R}^N)} + 1) \int_{\mathbb{R}^N} |u|_p^p\,dx \Big)$$

for some positive constant $C_{N,p}$ depending only on N and p.

Recall that $Q_{N,p}$ is defined by (2.19).

Proof. We have the maximal function estimates in the form

$$\int_{\mathbb{R}^N} |\mathscr{M}_\sigma(g,x)|_p^p\,dx \le C_p \int_{\mathbb{R}^N} |g|_p^p\,dx \quad \text{for all } g \in L^p(\mathbb{R}^N)$$

for all $\sigma \in \mathbb{S}^{N-1}$ and $g \in L^p(\mathbb{R}^N)$, either complex or real valued. It is readily checked (repeat the proof of [16, Theorem 7.22] with straightforward adaptations) that $C^\infty_c(\mathbb{R}^N)$ is dense in $W^{1,p}_A(\mathbb{R}^N)$. Lemma 3.2 holds in the modified form

$$J_{\delta,p}(u) \le C_{N,p}\Big(\int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|_p^p\,dx + (\|\nabla A\|^p_{L^\infty(\mathbb{R}^N)} + 1) \int_{\mathbb{R}^N} |u|_p^p\,dx \Big)$$

for all $u \in W^{1,p}_A(\mathbb{R}^N)$ and $\delta > 0$. To achieve this conclusion, it is sufficient to observe that, see (3.2),

$$|\Psi(x, x+he_N) - \Psi(x,x)|_p \le h\mathscr{M}_N(|\nabla u - \mathrm{i}Au|_p, x) + h^2 \|\nabla A\|_{L^\infty(\mathbb{R}^N)} \mathscr{M}_N(|u|_p, x).$$

The rest of the proof follows verbatim. Lemma 3.3 holds in the form

$$\lim_{\delta\searrow 0} J_{\delta,p}(u) = Q_{N,p} \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|_p^p \, dx$$

for every $u \in W_A^{1,p}(\mathbb{R}^N)$. In fact, mimicking the proof of Lemma 3.3, one obtains

$$\lim_{\delta\searrow 0} \iint_{\{|\Psi_u(x,y)-\Psi_u(x,x)|_p>\delta\}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy = \frac{1}{p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |(\nabla u - \mathrm{i}Au)\cdot\sigma|^p \, dx \, d\sigma.$$

The final conclusion follows from (2.20). Lemma 3.4 can be modified accordingly with minor modifications, replacing $|\cdot|$ with $|\cdot|_p$. □

4 Convergence almost everywhere and convergence in L^1

Motivated by the work in [8] (see also [26]), we are interested in other modes of convergence in the context of Theorems 1.1 and 1.2. We only consider the case $p = 2$. Similar results hold for $p \in (1, +\infty)$ with similar proofs. We begin with the corresponding results related to Theorem 1.1. For $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, set

$$D_n(u,x) := \int_{\mathbb{R}^N} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} \rho_n(|x-y|) \, dy \quad \text{for } x \in \mathbb{R}^N.$$

Proposition 4.1. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz, $u \in H_A^1(\mathbb{R}^N)$, and let (ρ_n) be a sequence of radial mollifiers such that*

$$\sup_{t>1} \sup_n t^{-2} \rho_n(t) < +\infty.$$

We have

$$\lim_{n\to+\infty} D_n(u,x) = 2Q_N |\nabla u(x) - \mathrm{i}A(x)u(x)|^2 \quad \textit{for a.e. } x \in \mathbb{R}^N,$$

and

$$\lim_{n\to+\infty} D_n(u,\cdot) = 2Q_N |\nabla u(\cdot) - \mathrm{i}A(\cdot)u(\cdot)|^2 \quad \textit{in } L^1(\mathbb{R}^N).$$

Before giving the proof of Proposition 4.1, we recall the following result established in [9, Lemma 1] (see also [8, Lemma 2] for a more general version).

Lemma 4.1. *Let $r > 0$, $x \in \mathbb{R}^N$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. We have*

$$\int_{\mathbb{S}^{N-1}} \int_0^r |f(x+s\sigma)| \, ds \, d\sigma \le C_N r M(f)(x).$$

Here and in what follows, for $x \in \mathbb{R}^N$ and $r > 0$, let $B_x(r)$ denote the open ball in $\mathbb{R}^N$ centered at x and of radius r. Moreover, $M(f)$ denotes the maximal function of f,

$$M(f)(x) := \sup_{r>0} \frac{1}{|B_x(r)|} \int_{B_x(r)} |f(y)| \, dy, \quad x \in \mathbb{R}^N.$$

As a consequence of Lemma 4.1, we have:

Corollary 4.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and let ρ be a nonnegative radial function such that*

$$\int_0^\infty \rho(r) r^{N-1} \, dr = 1. \tag{4.1}$$

Then, for a.e. $x \in \mathbb{R}^N$,

$$\int_{B_x(r)} \int_0^1 |f(t(y-x)+x)| \rho(|y-x|) \, dt \, dy \le C_N M(f)(x).$$

Proof. Using polar coordinates, we have

$$\int_{B_x(r)} \int_0^1 |f(t(y-x)+x)|\rho(|y-x|)\,dt\,dy = \int_0^r \int_{\mathbb{S}^{N-1}} \int_0^1 |f(x+ts\sigma)|s^{N-1}\rho(s)\,dt\,d\sigma\,ds.$$

Applying Lemma 4.1, we obtain, for a.e. $x \in \mathbb{R}^N$,

$$\int_{\mathbb{S}^{N-1}} \int_0^1 |f(x+ts\sigma)|\,dt\,d\sigma \le C_N M(f)(x).$$

It follows from (4.1) that, for a.e. $x \in \mathbb{R}^N$,

$$\int_{B_x(r)} \int_0^1 |f(t(y-x)+x)|\rho(|y-x|)\,dt\,dy \le C_N M(f)(x),$$

which is the conclusion. □

We are ready to give the proof of Proposition 4.1.

Proof of Proposition 4.1. We first establish that, for a.e. $x \in \mathbb{R}^N$,

$$|D_n(u,x)| \le C\big(M(|\nabla u - \mathrm{i}Au|^2)(x) + M(|u|^2)(x)\big) + m \int_{\mathbb{R}^N\setminus B_x(1)} |u(y)|^2\,dy, \tag{4.2}$$

where

$$m := 2 \sup_{t>1} \sup_n t^{-2}\rho_n(t).$$

Here and in what follows in this proof, C denotes a positive constant independent of x. Indeed, we have, as in (2.4), for a.e. $x, y \in \mathbb{R}^N$ with $|y-x| < 1$,

$$\begin{aligned}\frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2} &\le 2\int_0^1 |\nabla u(t(y-x)+x) - \mathrm{i}A(t(y-x)+x)u(t(y-x)+x)|^2\,dt \\ &\quad + 2\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} \int_0^1 |u(t(y-x)+x)|^2\,dt.\end{aligned}$$

This implies, for a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}&\int_{B_x(1)} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|y-x|)\,dy \\ &\le 2\int_{B_x(1)} \int_0^1 |\nabla u(t(y-x)+x) - \mathrm{i}A(t(y-x)+x)u(t(y-x)+x)|^2\rho_n(|y-x|)\,dt\,dy \\ &\quad + 2\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} \int_{B_x(1)} \int_0^1 |u(t(y-x)+x)|^2\rho_n(|y-x|)\,dt\,dy.\end{aligned}$$

Applying Corollary 4.1, we have, for a.e. $x \in \mathbb{R}^N$,

$$\int_{B_x(1)} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|y-x|)\,dy \le CM(|\nabla u - \mathrm{i}Au|^2)(x) + CM(|u|^2)(x). \tag{4.3}$$

On the other hand, we get

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_x(1)} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^2}{|x-y|^2}\rho_n(|y-x|)\,dy &\le 2|u(x)|^2 + 2\int_{\mathbb{R}^N\setminus B_x(1)} |u(y)|^2\rho_n(|y-x|)|x-y|^{-2}\,dy \\ &\le 2|u(x)|^2 + m\int_{\mathbb{R}^N\setminus B_x(1)} |u(y)|^2\,dy.\end{aligned} \tag{4.4}$$

A combination of (4.3) and (4.4) yields (4.2). Set, for $v \in H^1_A(\mathbb{R}^N)$ and $\varepsilon \geq 0$,

$$\Omega_\varepsilon(v) := \{x \in \mathbb{R}^N : \limsup_{n\to+\infty} |D_n(v,x) - 2Q_N|\nabla v(x) - \mathrm{i}A(x)v(x)|^2| > \varepsilon\}.$$

By (2.7), one has, for $v \in C^2_c(\mathbb{R}^N)$ and $\varepsilon \geq 0$,

$$|\Omega_\varepsilon(v)| = 0.$$

Using the theory of maximal functions, see e.g. [29, Theorem 1, p. 5], we derive from (4.2) that, for any $\varepsilon > 0$ and for any $w \in H^1_A(\mathbb{R}^N)$ with $m \int_{\mathbb{R}^N} |w(y)|^2\,dy \leq \frac{\varepsilon}{2}$,

$$|\Omega_\varepsilon(w)| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^N} (|\nabla w(x) - \mathrm{i}A(x)w(x)|^2 + |w(x)|^2)\,dx. \tag{4.5}$$

Fix $\varepsilon > 0$ and let $v \in C^2_c(\mathbb{R}^N)$ with $\max\{1, m\}\|v - u\|_{H^1_A(\mathbb{R}^N)} \leq \frac{\varepsilon}{2}$. We derive from (4.5) that

$$|\Omega_\varepsilon(u)| \leq |\Omega_\varepsilon(u - v)| \leq \frac{C}{\varepsilon}\|v - u\|^2_{H^1_A(\mathbb{R}^N)} \leq C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, one reaches the conclusion that $|\Omega_0(u)| = 0$. The proof is complete. □

We next discuss the corresponding results related to Theorem 1.2. Given $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, set, for $x \in \mathbb{R}^N$,

$$J_\delta(u,x) = \int_{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta\}} \frac{\delta^2}{|x-y|^{N+2}}\,dy.$$

We have:

Proposition 4.2. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz and let $u \in H^1_A(\mathbb{R}^N)$. We have*

$$\lim_{\delta\searrow 0} J_\delta(u,x) = Q_N|\nabla u(x) - \mathrm{i}A(x)u(x)|^2 \quad \textit{for a.e. } x \in \mathbb{R}^N \tag{4.6}$$

and

$$\lim_{\delta\searrow 0} J_\delta(u,\cdot) = Q_N|\nabla u(\cdot) - \mathrm{i}A(\cdot)u(\cdot)|^2 \quad \textit{in } L^1(\mathbb{R}^N). \tag{4.7}$$

Proof. For $v \in H^1_A(\mathbb{R}^N)$, set

$$\mathscr{M}(v,x) = \int_{\mathbb{S}^{N-1}} (|\mathscr{M}_\sigma(|\nabla v - \mathrm{i}Av|, x)|^2 + \|\nabla A\|^2_{L^\infty(\mathbb{R}^N)}|\mathscr{M}_\sigma(|v|,x)|^2)\,d\sigma \quad \text{for } x \in \mathbb{R}^N,$$

and denote

$$\hat{J}_\delta(u,x) = \int_{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta\,:\,|y-x|<1\}} \frac{\delta^2}{|x-y|^{N+2}}\,dy \quad \text{for } x \in \mathbb{R}^N.$$

We first establish a variant of (4.6) and (4.7) in which J_δ is replaced by $\hat{J}_\delta$. Using (3.2), as in the proof of Lemma 3.2, we have, for any $v \in H^1_A(\mathbb{R}^N)$,

$$\hat{J}_\delta(v,x) \leq C_N \mathscr{M}(v,x) \quad \text{for all } \delta > 0.$$

We derive that, for $u, u_n \in H^1_A(\mathbb{R}^N)$, and $\varepsilon \in (0,1)$,

$$\hat{J}_\delta(u,x) - (1-\varepsilon)^{-2}\hat{J}_{(1-\varepsilon)\delta}(u_n,x) \leq \varepsilon^{-2}C_N\mathscr{M}(u - u_n, x) \tag{4.8}$$

and

$$(1-\varepsilon)^2\hat{J}_{\delta/(1-\varepsilon)}(u_n,x) - \hat{J}_\delta(u,x) \leq \varepsilon^{-2}C_N\mathscr{M}(u - u_n, x). \tag{4.9}$$

On the other hand, one can check that, as in the proof of Lemma 3.3, for $u_n \in C^2_c(\mathbb{R}^N)$,

$$\lim_{\delta\searrow 0} \hat{J}_\delta(u_n,x) = Q_N|\nabla u_n(x) - \mathrm{i}A(x)u_n(x)|^2 \quad \text{for } x \in \mathbb{R}^N. \tag{4.10}$$

We derive from (4.8), (4.9) and (4.10) that, for $u \in H^1_A(\mathbb{R}^N)$,

$$\lim_{\delta \searrow 0} \hat{J}_\delta(u, x) = Q_N |\nabla u(x) - \mathrm{i}A(x)u(x)|^2 \quad \text{for a.e. } x \in \mathbb{R}^N, \tag{4.11}$$

and, we hence obtain, by the Dominated Convergence Theorem,

$$\lim_{\delta \searrow 0} \hat{J}_\delta(u, \cdot) = Q_N |\nabla u(\cdot) - \mathrm{i}A(\cdot)u(\cdot)|^2 \quad \text{in } L^1(\mathbb{R}^N), \tag{4.12}$$

since $\mathscr{M}(u, x) \in L^1(\mathbb{R}^N)$. A straightforward computation yields

$$\lim_{\delta \searrow 0} \int\limits_{\{|y-x| \geq 1\}} \frac{\delta^2}{|x-y|^{N+2}}\, dy = 0.$$

It follows that

$$\lim_{\delta \searrow 0} [\hat{J}_\delta(u, x) - J_\delta(u, x)] = 0 \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{4.13}$$

We also have, for $w \in C^2_c(\mathbb{R}^N)$,

$$\lim_{\delta \searrow 0} \iint\limits_{\{|\Psi_w(x,y) - \Psi_w(x,x)| > \delta \,:\, |y-x| \geq 1\}} \frac{\delta^2}{|x-y|^{N+2}}\, dx\, dy \leq \lim_{\delta \searrow 0} \iint\limits_{\{(B_R \times \mathbb{R}^N) \cup (\mathbb{R}^N \times B_R) \,:\, |y-x| \geq 1\}} \frac{\delta^2}{|x-y|^{N+2}}\, dx\, dy = 0,$$

where $R > 0$ is such that $\operatorname{supp} w \subset B_R$. Using Lemma 3.2 and the density of $C^2_c(\mathbb{R}^N)$ in $H^1_A(\mathbb{R}^N)$, we derive that

$$\lim_{\delta \searrow 0} [\hat{J}_\delta(u, \cdot) - J_\delta(u, \cdot)] = 0 \quad \text{in } L^1(\mathbb{R}^N). \tag{4.14}$$

The conclusion now follows from (4.11), (4.12), (4.13) and (4.14). □

Funding: Andrea Pinamonti, Marco Squassina and Eugenio Vecchi are members of *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). Eugenio Vecchi received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA grant agreement No. 607643 (Grant MaNET 'Metric Analysis for Emergent Technologies').

References

[1] G. Arioli and A. Szulkin, A semilinear Schrödinger equation in the presence of a magnetic field, *Arch. Ration. Mech. Anal.* **170** (2003), no. 4, 277–295.
[2] J. Avron, I. Herbst and B. Simon, Schrödinger operators with magnetic fields. I. General interactions, *Duke Math. J.* **45** (1978), no. 4, 847–883.
[3] J. Bourgain, H. Brezis and P. Mironescu, Another look at Sobolev spaces, in: *Optimal Control and Partial Differential Equations*, IOS Press, Amsterdam (2001), 439–455.
[4] J. Bourgain, H. Brezis and P. Mironescu, Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications, *J. Anal. Math.* **87** (2002), 77–101.
[5] J. Bourgain and H.-M. Nguyen, A new characterization of Sobolev spaces, *C. R. Math. Acad. Sci. Paris* **343** (2006), no. 2, 75–80.
[6] H. Brezis, New approximations of the total variation and filters in imaging, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **26** (2015), no. 2, 223–240.
[7] H. Brezis and H.-M. Nguyen, Non-local functionals related to the total variation and connections with image processing, preprint (2016), http://arxiv.org/abs/1608.08204.
[8] H. Brezis and H.-M. Nguyen, The BBM formula revisited, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **27** (2016), no. 4, 515–533.
[9] H. Brezis and H.-M. Nguyen, Two subtle convex nonlocal approximations of the BV-norm, *Nonlinear Anal.* **137** (2016), 222–245.
[10] K. Brezis, How to recognize constant functions. A connection with Sobolev spaces, *Uspekhi Mat. Nauk* **57** (2002), no. 4(346), 59–74.

[11] P. d'Avenia and M. Squassina, Ground states for fractional magnetic operators, *ESAIM Control Optim. Calc. Var.* (2017), DOI 10.1051/cocv/2016071.
[12] J. Dávila, On an open question about functions of bounded variation, *Calc. Var. Partial Differential Equations* **15** (2002), no. 4, 519–527.
[13] M. J. Esteban and P.-L. Lions, Stationary solutions of nonlinear Schrödinger equations with an external magnetic field, in: *Partial Differential Equations and the Calculus of Variations. Vol. I*, Progr. Nonlinear Differential Equations Appl. 1, Birkhäuser, Boston (1989), 401–449.
[14] T. Ichinose, Magnetic relativistic Schrödinger operators and imaginary-time path integrals, in: *Mathematical Physics, Spectral Theory and Stochastic Analysis*, Oper. Theory Adv. Appl. 232, Birkhäuser, Basel (2013), 247–297.
[15] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory. Course of Theoretical Physics. Vol. 3*, Addison-Wesley Ser. Adv. Phys., Addison-Wesley Publishing, Reading, 1958.
[16] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Grad. Stud. Math. 14, American Mathematical Society, Providence, 2001.
[17] D. L. Mills, *Nonlinear Optics*, Springer, Berlin, 1998.
[18] H.-M. Nguyen, Some new characterizations of Sobolev spaces, *J. Funct. Anal.* **237** (2006), no. 2, 689–720.
[19] H.-M. Nguyen, Further characterizations of Sobolev spaces, *J. Eur. Math. Soc. (JEMS)* **10** (2008), no. 1, 191–229.
[20] H.-M. Nguyen, Γ-convergence, Sobolev norms, and BV functions, *Duke Math. J.* **157** (2011), no. 3, 495–533.
[21] H.-M. Nguyen, Some inequalities related to Sobolev norms, *Calc. Var. Partial Differential Equations* **41** (2011), no. 3–4, 483–509.
[22] H.-M. Nguyen, Estimates for the topological degree and related topics, *J. Fixed Point Theory Appl.* **15** (2014), no. 1, 185–215.
[23] A. Pinamonti, M. Squassina and E. Vecchi, Magnetic BV functions and the Bourgain–Brezis–Mironescu formula, preprint (2016), https://arxiv.org/abs/1609.09714; *Adv. Calc. Var.* (2017), DOI 10.1515/acv-2017-0019.
[24] A. Pinamonti, M. Squassina and E. Vecchi, The Maz'ya–Shaposhnikova limit in the magnetic setting, *J. Math. Anal. Appl.* **449** (2017), no. 2, 1152–1159.
[25] A. C. Ponce, A new approach to Sobolev spaces and connections to Γ-convergence, *Calc. Var. Partial Differential Equations* **19** (2004), no. 3, 229–255.
[26] A. C. Ponce and D. Spector, On formulae decoupling the total variation of BV functions, *Nonlinear Anal.* **154** (2017), 241–257.
[27] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I*, 2nd ed., Academic Press, New York, 1980.
[28] M. Squassina and B. Volzone, Bourgain–Brézis–Mironescu formula for magnetic operators, *C. R. Math. Acad. Sci. Paris* **354** (2016), no. 8, 825–831.
[29] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton University Press, Princeton, 1970.