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Research Article

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Critical Nonlocal Systems with Concave-Convex Powers

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Abstract: By using the fibering method jointly with Nehari manifold techniques, we obtain the existence of multiple solutions to a fractional p-Laplacian system involving critical concave-convex nonlinearities, provided that a suitable smallness condition on the parameters involved is assumed. The result is obtained although there is no general classification for the optimizers of the critical fractional Sobolev embedding.

Keywords: Critical Fractional p-Laplacian System, Concave-Convex Nonlinearities, Nehari Manifold

MSC 2010: 35J20, 35J60, 47G20

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1 Introduction

In this work, we study the multiplicity of solutions to the following fractional elliptic system:

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)_p^s v = \mu |v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded set in $\mathbb{R}^n$, $n > ps$ with $s \in (0,1)$, $\lambda, \mu > 0$ are two parameters, $1 < q < p$ and $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = p_s^*$, where $p_s^* = \frac{np}{n-ps}$ is the fractional critical Sobolev exponent, and $(-\Delta)_p^s$ is the fractional p-Laplacian operator, defined on smooth functions as

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \to 0} \int\limits_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{n+ps}}\, dy, \quad x \in \mathbb{R}^n.$$

This definition is consistent, up to a normalization constant depending on n and s, with the linear fractional Laplacian $(-\Delta)^s$ for the case $p = 2$. If we set $\alpha = \beta$, $\alpha + \beta = r$, $\lambda = \mu$ and $u = v$, then system (1.1) reduces to the following fractional equation with concave-convex nonlinearities:

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{q-2}u + |u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

where $1 < q < p$ and $p < r < p_s^*$. In [14] Goyal and Sreenadh studied the existence and multiplicity of nonnegative solutions to the nonlocal problem (1.2) for subcritical concave-convex nonlinearities. For the fractional

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p-Laplacian, consider the following general problem:

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

So far various results have been obtained for these kind of problems. Lindgren and Lindqvist [19] considered the eigenvalue problem associated with $(-\Delta)_p^s$ and obtained some properties of the first and of higher (variational) eigenvalues. Some results about the existence of solutions have been considered in [13, 18, 21], see also the references therein. Let us also mention [22] where, by using variational methods and topological degree theory, Pucci, Xiang and Zhang proved multiplicity results for fractional p-Kirchhoff equations.

On the other hand, the fractional problems for $p = 2$ have been investigated by many researchers, see, for example, [2, 6, 23] for the critical case and [11] for the fractional Kirchhoff type problem. In particular, Brändle et al. [3] studied the fractional Laplacian equation involving a concave-convex nonlinearity in the subcritical case. The existence and multiplicity of solutions for system (1.1), when $s = 1$, were considered by many authors, see [16, 17, 24] and references therein. In particular, Hsu [16] obtained multiple solutions for the following critical elliptic system:

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta_p v = \mu |v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $q < p$ and $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = \frac{np}{n-p}$. For system (1.1) with $p = 2$, we mention [10, 15]. Moreover, Giacomoni, Mishra and Sreenadh [12] showed the existence of multiple solutions for critical growth fractional elliptic systems with exponential nonlinearity by analyzing the fibering maps.

However, as far as we know, there are a few results on the case $p \neq 2$ with concave-convex critical nonlinearities. Recently, Chen and Deng [7] studied system (1.1) with a *subcritical* concave-convex type nonlinearity, i.e., when $\alpha + \beta < p_s^*$. Motivated by the above results, in the present paper, we are interested in the multiplicity of solutions for the *critical* fractional p-Laplacian system (1.1), i.e., when

$$\alpha + \beta = p_s^*.$$

We denote by $W^{s,p}(\Omega)$ the usual fractional Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \bigg(\int\limits_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \bigg)^{\frac{1}{p}}.$$

Set $Q := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$. We define

$$X := \bigg\{ u\colon \mathbb{R}^n \to \mathbb{R} \text{ measurable, } u|_\Omega \in L^p(\Omega) \text{ and } \int\limits_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy < \infty \bigg\}.$$

The space X is endowed with the norm

$$\|u\|_X := \|u\|_{L^p(\Omega)} + \bigg(\int\limits_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \bigg)^{\frac{1}{p}}.$$

The space X_0 is defined as $X_0 := \{u \in X : u = 0 \text{ on } \mathcal{C}\Omega\}$ or, equivalently, as $\overline{C_0^\infty(\Omega)}^X$ and, for any $p > 1$, it is a uniformly convex Banach space endowed with the norm

$$\|u\|_{X_0} = \bigg(\int\limits_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \bigg)^{\frac{1}{p}}. \tag{1.3}$$

Since $u = 0$ in $\mathbb{R}^n \setminus \Omega$, the integral in (1.3) can be extended to all $\mathbb{R}^n$. The embedding $X_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, p_s^*]$ and compact for $r \in [1, p_s^*)$. We set $E := X_0 \times X_0$, with the norm

$$\|(u,v)\| = (\|u\|_{X_0}^p + \|v\|_{X_0}^p)^{\frac{1}{p}} = \bigg(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy + \int_Q \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \bigg)^{\frac{1}{p}}.$$

For convenience, we define

$$\mathcal{A}(u,\phi) := \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+ps}}\,dx\,dy. \tag{1.4}$$

Definition 1.1. We say that $(u,v) \in E$ is a weak solution of problem (1.1) if

$$\mathcal{A}(u,\phi) + \mathcal{A}(v,\psi) = \int_\Omega (\lambda |u|^{q-2}u\phi + \mu |v|^{q-2}v\psi)\,dx + \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^\beta \phi\,dx + \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^{\beta-2}v\psi\,dx$$

for all $(\phi,\psi) \in E$.

In the sequel we omit the term *weak* when referring to solutions which satisfy Definition 1.1. Let $s \in (0,1)$, $p > 1$ and let Ω be a bounded domain of $\mathbb{R}^n$. The next theorem is our main result.

Theorem 1.2. *Assume that*

$$p^2 s < n < \begin{cases} \infty & \text{if } p \geq 2, \\ \frac{ps}{2-p} & \text{if } p < 2, \end{cases} \qquad \frac{n(p-1)}{n-ps} \leq q < p, \qquad \alpha + \beta = \frac{np}{n-ps}. \tag{1.5}$$

Then there exists a positive constant $\Lambda_* = \Lambda_*(p,q,s,n,|\Omega|)$ such that for λ, μ satisfying

$$0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_*,$$

system (1.1) *admits at least two nontrivial solutions.*

For the critical case, since the embedding $X_0 \hookrightarrow L^{p_s^*}(\mathbb{R}^n)$ fails to be compact, the energy functional does *not* satisfy the Palais–Smale condition globally, but that holds true when the energy level falls inside a suitable range related to the best fractional critical Sobolev constant S, namely,

$$S := \inf_{u \in X_0 \setminus \{0\}} \frac{\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy}{\big(\int_\Omega |u(x)|^{\frac{np}{n-ps}}\,dx\big)^{\frac{n-ps}{n}}}. \tag{1.6}$$

For the critical fractional case with $p \neq 2$, the main difficulty is the lack of an explicit formula for minimizers of S, which is very often a key tool to handle the estimates leading to the compactness range of the functional. It was conjectured that, up to a multiplicative constant, all minimizers are of the form $U(\frac{x-x_0}{\epsilon})$, with

$$U(x) = (1+|x|^{\frac{p}{p-1}})^{-\frac{n-ps}{p}}, \quad x \in \mathbb{R}^n.$$

This conjecture was proved in [8] for $p = 2$, but for $p \neq 2$, it is not even known if these functions are minimizers of S. On the other hand, as in [20], we can overcome this difficulty by the optimal asymptotic behavior of minimizers, which was recently obtained in [4]. This will allow us to prove Lemma 4.10, related to the Palais–Smale condition. That is the only point where the restriction (1.6) on p, q, n comes into play. On the other hand, we point out that, as detected in [20], $n = p^2 s$ corresponds to the critical dimension for the nonlocal Brézis–Nirenberg problem.

This paper is organized as follows. In Section 2, we give some notations and preliminaries for the Nehari manifold and fibering maps. In Section 3, we show that the $(\mathrm{PS})_c$ condition holds for $J_{\lambda,\mu}$, with c in certain interval. In Sections 4 and 5, we complete the proof of Theorem 1.2.

2 The Fibering Properties

In this section, we give some notations and preliminaries for the Nehari manifold and the analysis of the fibering maps. Being a weak solution $(u, v) \in E$ is equivalent to being a critical point of the following C^1 functional on E:

$$J_{\lambda,\mu}(u,v) := \frac{1}{p}\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy + \frac{1}{p}\int_Q \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,dx\,dy - \frac{1}{q}\int_\Omega (\lambda|u|^q+\mu|v|^q)\,dx - \frac{2}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta\,dx.$$

By a direct calculation, we have that $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$ and

$$\begin{aligned}\langle J'_{\lambda,\mu}(u,v),(\phi,\psi)\rangle &= \mathcal{A}(u,\phi)+\mathcal{A}(v,\psi) - \int_\Omega (\lambda|u|^{q-2}u\phi + \mu|v|^{q-2}v\psi)\,dx \\ &\quad - \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^\beta\phi\,dx - \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^{\beta-2}v\psi\,dx\end{aligned}$$

for any $(\phi,\psi) \in E$. We will study critical points of the functional $J_{\lambda,\mu}$ on E. Consider the *Nehari* manifold

$$\mathcal{N}_{\lambda,\mu} = \{(u,v) \in E \setminus \{(0,0)\} : \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\}.$$

Then $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $(u,v) \neq (0,0)$ and

$$\|(u,v)\|^p = \int_\Omega (\lambda|u|^q+\mu|v|^q)\,dx + 2\int_\Omega |u|^\alpha|v|^\beta\,dx.$$

The Nehari manifold $\mathcal{N}_{\lambda,\mu}$ is closely linked to the behavior of a function of the form $\varphi_{u,v}\colon t \mapsto J_{\lambda,\mu}(tu,tv)$ for $t > 0$, defined by

$$\varphi_{u,v}(t) := J_{\lambda,\mu}(tu,tv) = \frac{t^p}{p}\|(u,v)\|^p - \frac{t^q}{q}\int_\Omega (\lambda|u|^q+\mu|v|^q)\,dx - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta\,dx.$$

Such maps are known as *fibering maps* and were introduced by Drabek and Pohozaev in [9].

Lemma 2.1 (Fibering Map). *Let $(u,v) \in E \setminus \{(0,0)\}$. Then $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\varphi'_{u,v}(t) = 0$.*

Proof. The result is a consequence of the fact that $\varphi'_{u,v}(t) = \langle J'_{\lambda,\mu}(tu,tv),(u,v)\rangle$. □

We note that

$$\varphi'_{u,v}(t) = t^{p-1}\|(u,v)\|^p - t^{q-1}\int_\Omega (\lambda|u|^q+\mu|v|^q)\,dx - 2t^{\alpha+\beta-1}\int_\Omega |u|^\alpha|v|^\beta\,dx \tag{2.1}$$

and

$$\varphi''_{u,v}(t) = (p-1)t^{p-2}\|(u,v)\|^p - (q-1)t^{q-2}\int_\Omega (\lambda|u|^q+\mu|v|^q)\,dx - 2(\alpha+\beta-1)t^{\alpha+\beta-2}\int_\Omega |u|^\alpha|v|^\beta\,dx.$$

By Lemma 2.1, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\varphi'_{u,v}(1) = 0$. Hence, for $(u,v) \in \mathcal{N}_{\lambda,\mu}$, (2.1) yields

$$\begin{aligned}\varphi''_{u,v}(1) &= (p-1)\|(u,v)\|^p - (q-1)\int_\Omega (\lambda|u|^q+\mu|v|^q)\,dx - 2(\alpha+\beta-1)\int_\Omega |u|^\alpha|v|^\beta\,dx \\ &= 2(p-(\alpha+\beta))\int_\Omega |u|^\alpha|v|^\beta\,dx + (p-q)\int_\Omega (\lambda|u|^q+\mu|v|^q)\,dx \\ &= (p-q)\|(u,v)\|^p - 2((\alpha+\beta)-q)\int_\Omega |u|^\alpha|v|^\beta\,dx \\ &= (p-(\alpha+\beta))\|(u,v)\|^p + ((\alpha+\beta)-q)\int_\Omega (\lambda|u|^q+\mu|v|^q)\,dx.\end{aligned} \tag{2.2}$$

Thus, it is natural to split $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima and points of inflection of $\varphi_{u,v}$, namely,

$$\mathcal{N}^+_{\lambda,\mu} := \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \varphi''_{u,v}(1) > 0\},$$
$$\mathcal{N}^-_{\lambda,\mu} := \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \varphi''_{u,v}(1) < 0\},$$
$$\mathcal{N}^0_{\lambda,\mu} := \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \varphi''_{u,v}(1) = 0\}.$$

We will prove the existence of solutions of problem (1.1) by investigating the existence of minimizers of the functional $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$. Although $\mathcal{N}_{\lambda,\mu}$ is a subset of E, we can see that the local minimizers on the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ are usually critical points of $J_{\lambda,\mu}$. We have the following lemma.

Lemma 2.2 (Natural Constraint). *Suppose that (u_0, v_0) is a local minimizer of the functional $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and that $(u_0, v_0) \notin \mathcal{N}^0_{\lambda,\mu}$. Then (u_0, v_0) is a critical point of $J_{\lambda,\mu}$.*

Proof. The proof is a standard corollary of the lagrange multiplier rule, where the constraint is

$$Q(u,v) = \|(u,v)\|^p - \int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx - 2\int_\Omega |u|^\alpha |v|^\beta\,dx,$$

after observing that, for $(u,v) \in \mathcal{N}_{\lambda,\mu}$,

$$\begin{aligned}\langle Q'(u,v),(u,v)\rangle &= p\|(u,v)\|^p - q\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx - 2(\alpha+\beta)\int_\Omega |u|^\alpha|v|^\beta\,dx\\ &= (p-1)\|(u,v)\|^p - (q-1)\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx - 2(\alpha+\beta-1)\int_\Omega |u|^\alpha|v|^\beta\,dx\\ &= \varphi''_{u,v}(1) \neq 0,\end{aligned}$$

by the assumption that $(u,v) \notin \mathcal{N}^0_{\lambda,\mu}$. □

In order to understand the Nehari manifold and the fibering maps, we consider $\Psi_{u,v} : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\Psi_{u,v}(t) := t^{p-(\alpha+\beta)}\|(u,v)\|^p - t^{q-(\alpha+\beta)}\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx.$$

By simple computations, we have the following results.

Lemma 2.3 (Properties of $\Psi_{u,v}$). *Let $(u,v) \in E \setminus \{(0,0)\}$. Then $\Psi_{u,v}$ satisfies the following properties:*
(a) *$\Psi_{u,v}(t)$ has a unique critical point at*

$$t_{\max}(u,v) := \left(\frac{(\alpha+\beta-q)\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx}{(\alpha+\beta-p)\|(u,v)\|^p}\right)^{\frac{1}{p-q}} > 0,$$

(b) *$\Psi_{u,v}(t)$ is strictly increasing on $(0, t_{\max}(u,v))$ and strictly decreasing on $(t_{\max}(u,v), +\infty)$,*
(c) *$\lim_{t\to0^+}\Psi_{u,v}(t) = -\infty$ and $\lim_{t\to+\infty}\Psi_{u,v}(t) = 0$.*

Lemma 2.4 (Characterization of $\mathcal{N}^\pm_{\lambda,\mu}$). *We have $(tu, tv) \in \mathcal{N}^\pm_{\lambda,\mu}$ if and only if $\pm\Psi'_{u,v}(t) > 0$.*

Proof. It is clear that for $t > 0$, $(tu, tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\Psi_{u,v}(t) = 2\int_\Omega |u|^\alpha|v|^\beta\,dx. \tag{2.3}$$

Moreover,

$$\Psi'_{u,v}(t) = (p-(\alpha+\beta))t^{p-(\alpha+\beta)-1}\|(u,v)\|^p - (q-(\alpha+\beta))t^{q-(\alpha+\beta)-1}\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx,$$

and if $(tu, tv) \in \mathcal{N}_{\lambda,\mu}$, then

$$t^{\alpha+\beta-1}\Psi'_{u,v}(t) = \varphi''_{u,v}(t) = t^{-2}\varphi''_{tu,tv}(1). \tag{2.4}$$

Hence, $(tu, tv) \in \mathcal{N}^+_{\lambda,\mu}$ (resp. $\mathcal{N}^-_{\lambda,\mu}$) if and only if $\Psi'_{u,v}(t) > 0$ (resp. < 0). □

Lemma 2.5 (Elements of $\mathcal{N}^{\pm}_{\lambda,\mu}$). *Let us set*

$$\Lambda_1 = \Big(\frac{p-q}{2(\alpha+\beta-q)}\Big)^{\frac{p}{\alpha+\beta-p}}\Big(\frac{\alpha+\beta-q}{\alpha+\beta-p}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\Big)^{-\frac{p}{p-q}} S^{\frac{\alpha+\beta}{\alpha+\beta-p}+\frac{q}{p-q}}, \tag{2.5}$$

with S being the best constant for the Sobolev embedding of X_0 into $L^{p_s^*}(\mathbb{R}^n)$. If $(u,v)\in E\setminus\{(0,0)\}$, then for any λ, μ satisfying

$$0<\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}<\Lambda_1,$$

there exist unique $t_1, t_2>0$ such that $t_1<t_{\max}(u,v)<t_2$ and

$$(t_1u,t_1v)\in\mathcal{N}^+_{\lambda,\mu},\quad (t_2u,t_2v)\in\mathcal{N}^-_{\lambda,\mu}.$$

Moreover,

$$J_{\lambda,\mu}(t_1u,t_1v)=\inf_{0\le t\le t_{\max}} J_{\lambda,\mu}(tu,tv),\quad J_{\lambda,\mu}(t_2u,t_2v)=\sup_{t\ge 0} J_{\lambda,\mu}(tu,tv).$$

Proof. As $\int_\Omega |u|^\alpha|v|^\beta\,dx>0$, we know that (2.3) has *no* solution if and only if λ and μ satisfy the condition

$$2\int_\Omega |u|^\alpha|v|^\beta\,dx>\Psi_{u,v}(t_{\max}(u,v)).$$

By Lemma 2.3, we have

$$\begin{aligned}\Psi_{u,v}(t_{\max}(u,v)) &= \Big[\Big(\frac{\alpha+\beta-q}{\alpha+\beta-p}\Big)^{\frac{p-(\alpha+\beta)}{p-q}}-\Big(\frac{\alpha+\beta-q}{\alpha+\beta-p}\Big)^{\frac{q-(\alpha+\beta)}{p-q}}\Big]\frac{\big(\int_\Omega(\lambda|u|^q+\mu|v|^q)\,dx\big)^{\frac{p-(\alpha+\beta)}{p-q}}}{\|(u,v)\|^{\frac{p(q-(\alpha+\beta))}{p-q}}}\\ &= \frac{p-q}{\alpha+\beta-q}\Big(\frac{\alpha+\beta-q}{\alpha+\beta-p}\Big)^{\frac{p-(\alpha+\beta)}{p-q}}\frac{\big(\int_\Omega(\lambda|u|^q+\mu|v|^q)\,dx\big)^{\frac{p-(\alpha+\beta)}{p-q}}}{\|(u,v)\|^{\frac{p(q-(\alpha+\beta))}{p-q}}}.\end{aligned}$$

By Hölder's inequality and the definition of S, we find

$$\int_\Omega(\lambda|u|^q+\mu|v|^q)\,dx\le S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-q}{p}}\|(u,v)\|^q.$$

Then, since $q<p<\alpha+\beta=p_s^*$, we have

$$\begin{aligned}\Psi_{u,v}(t_{\max}(u,v)) &\ge \frac{p-q}{\alpha+\beta-q}\Big(\frac{\alpha+\beta-q}{\alpha+\beta-p}\Big)^{\frac{p-(\alpha+\beta)}{p-q}}\frac{[S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-q}{p}}\|(u,v)\|^q]^{\frac{p-(\alpha+\beta)}{p-q}}}{\|(u,v)\|^{\frac{p(q-(\alpha+\beta))}{p-q}}}\\ &= \frac{p-q}{\alpha+\beta-q}\Big(\frac{\alpha+\beta-q}{\alpha+\beta-p}\Big)^{\frac{p-(\alpha+\beta)}{p-q}}[S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}]^{\frac{p-(\alpha+\beta)}{p-q}}(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-(\alpha+\beta)}{p}}\|(u,v)\|^{\alpha+\beta}.\end{aligned} \tag{2.6}$$

On the other hand, using Young's inequality and the definition of S, we have

$$2\int_\Omega |u|^\alpha|v|^\beta\,dx\le 2\Big(\frac{\alpha}{\alpha+\beta}\int_\Omega|u|^{\alpha+\beta}\,dx+\frac{\beta}{\alpha+\beta}\int_\Omega|v|^{\alpha+\beta}\,dx\Big)\le 2S^{-\frac{\alpha+\beta}{p}}\|(u,v)\|^{\alpha+\beta}.$$

For any λ, μ satisfying $0<\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}<\Lambda_1$, with Λ_1 given in (2.5), we have

$$2S^{-\frac{\alpha+\beta}{p}}\le\frac{p-q}{\alpha+\beta-q}\Big(\frac{\alpha+\beta-q}{\alpha+\beta-p}\Big)^{\frac{p-(\alpha+\beta)}{p-q}}[S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}]^{\frac{p-(\alpha+\beta)}{p-q}}(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-(\alpha+\beta)}{p}}. \tag{2.7}$$

Thus, from (2.6) and (2.7), if λ, μ satisfy $0<\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}<\Lambda_1$, we have

$$\begin{aligned}0<2\int_\Omega|u|^\alpha|v|^\beta\,dx &\le 2S^{-\frac{\alpha+\beta}{p}}\|(u,v)\|^{\alpha+\beta}\\ &\le\frac{p-q}{\alpha+\beta-q}\Big(\frac{\alpha+\beta-q}{\alpha+\beta-p}\Big)^{\frac{p-(\alpha+\beta)}{p-q}}[S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}]^{\frac{p-(\alpha+\beta)}{p-q}}(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-(\alpha+\beta)}{p}}\|(u,v)\|^{\alpha+\beta}\\ &<\Psi_{u,v}(t_{\max}(u,v)).\end{aligned}$$

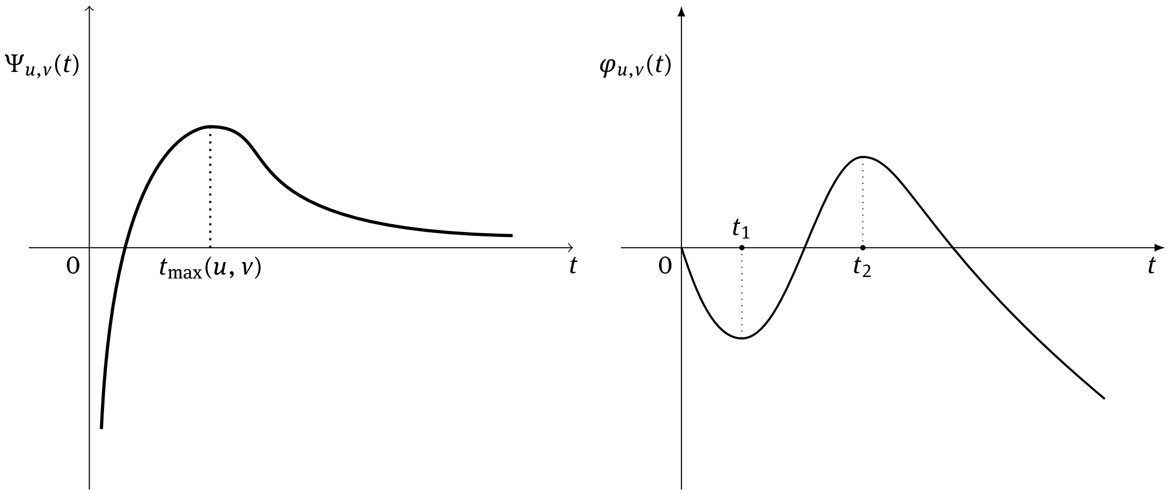

Figure 1. The graphs of $\Psi_{u,v}$ and $\varphi_{u,v}$.

Then, there exist unique $t_1 > 0$ and $t_2 > 0$, with $t_1 < t_{\max}(u,v) < t_2$, such that

$$\Psi_{u,v}(t_1) = \Psi_{u,v}(t_2) = 2\int_\Omega |u|^\alpha |v|^\beta \,dx, \quad \Psi'_{u,v}(t_1) > 0, \quad \Psi'_{u,v}(t_2) < 0.$$

In turn, (2.1) and (2.3) give that $\varphi'_{u,v}(t_1) = \varphi'_{u,v}(t_2) = 0$. By (2.4), we have that $\varphi''_{u,v}(t_1) > 0$ and $\varphi''_{u,v}(t_2) < 0$. These facts imply that $\varphi_{u,v}$ has a local minimum at t_1 and a local maximum at t_2 such that $(t_1u, t_1v) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t_2u, t_2v) \in \mathcal{N}^-_{\lambda,\mu}$. Since $\varphi_{u,v}(t) = J_{\lambda,\mu}(tu, tv)$, we have $J_{\lambda,\mu}(t_2u, t_2v) \geq J_{\lambda,\mu}(tu, tv) \geq J_{\lambda,\mu}(t_1u, t_1v)$ for each $t \in [t_1, t_2]$ and $J_{\lambda,\mu}(t_1u, t_1v) \leq J_{\lambda,\mu}(tu, tv)$ for each $t \in [0, t_1]$. Thus,

$$J_{\lambda,\mu}(t_1u, t_1v) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda,\mu}(tu, tv), \quad J_{\lambda,\mu}(t_2u, t_2v) = \sup_{t \geq 0} J_{\lambda,\mu}(tu, tv).$$

The graphs of $\Psi_{u,v}$ and $\varphi_{u,v}$ can be seen in Figure 1. □

3 The Palais–Smale Condition

In this section, we show that the functional $J_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_c$ condition.

Definition 3.1. Let $c \in \mathbb{R}$, let E be a Banach space and let $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$. We say that $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is a $(\mathrm{PS})_c$ sequence in E for $J_{\lambda,\mu}$ if $J_{\lambda,\mu}(u_k, v_k) = c + o(1)$ and $J'_{\lambda,\mu}(u_k, v_k) = o(1)$ strongly in E^* as $k \to \infty$. We say that $J_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ for $J_{\lambda,\mu}$ in E admits a convergent subsequence.

Lemma 3.2 (Boundedness of $(\mathrm{PS})_c$ Sequences). *If $\{(u_k, v_k)\}_{k\in\mathbb{N}} \subset E$ is a $(\mathrm{PS})_c$ sequence for $J_{\lambda,\mu}$, then it follows that $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is bounded in E.*

Proof. If $\{(u_k, v_k)\} \subset E$ is a $(\mathrm{PS})_c$ sequence for $J_{\lambda,\mu}$, then we have

$$J_{\lambda,\mu}(u_k, v_k) \to c, \quad J'_{\lambda,\mu}(u_k, v_k) \to 0 \quad \text{in } E^* \text{ as } k \to \infty.$$

That is,

$$\frac{1}{p}\|(u_k, v_k)\|^p - \frac{1}{q}\int_\Omega (\lambda|u_k|^q + \mu|v_k|^q)\,dx - \frac{2}{\alpha+\beta}\int_\Omega |u_k|^\alpha |v_k|^\beta \,dx = c + o_k(1), \tag{3.1}$$

$$\|(u_k, v_k)\|^p - \int_\Omega (\lambda|u_k|^q + \mu|v_k|^q)\,dx - 2\int_\Omega |u_k|^\alpha |v_k|^\beta \,dx = o_k(\|(u_k, v_k)\|) \quad \text{as } k \to \infty. \tag{3.2}$$

We shall show that (u_k, v_k) is bounded in E by contradiction. Assume that $\|(u_k, v_k)\| \to \infty$, and set

$$\tilde{u}_k := \frac{u_k}{\|(u_k, v_k)\|}, \quad \tilde{v}_k := \frac{v_k}{\|(u_k, v_k)\|}.$$

Then $\|(\tilde{u}_k, \tilde{v}_k)\| = 1$. There is a subsequence, still denoted by $(\tilde{u}_k, \tilde{v}_k)$, with $(\tilde{u}_k, \tilde{v}_k) \rightharpoonup (\tilde{u}, \tilde{v}) \in E$ and

$$\tilde{u}_k \to \tilde{u}, \quad \tilde{v}_k \to \tilde{v} \quad \text{in } L^r(\mathbb{R}^n), \qquad \tilde{u}_k \to \tilde{u}, \quad \tilde{v}_k \to \tilde{v} \quad \text{a.e. in } \mathbb{R}^n,$$

for any $1 \le r < p_s^* = \frac{np}{n-ps}$. Then, the Dominated Convergence Theorem yields

$$\int_\Omega (\lambda|\tilde{u}_k|^q + \mu|\tilde{v}_k|^q)\,dx \to \int_\Omega (\lambda|\tilde{u}|^q + \mu|\tilde{v}|^q)\,dx \quad \text{as } k \to \infty. \tag{3.3}$$

Moreover, from (3.1) and (3.2), we find that $(\tilde{u}_k, \tilde{v}_k)$ satisfy

$$\frac{1}{p}\|(\tilde{u}_k, \tilde{v}_k)\|^p - \frac{\|(u_k, v_k)\|^{q-p}}{q}\int_\Omega (\lambda|\tilde{u}_k|^q + \mu|\tilde{v}_k|^q)\,dx - \frac{2\|(u_k, v_k)\|^{\alpha+\beta-p}}{\alpha+\beta}\int_\Omega |\tilde{u}_k|^\alpha|\tilde{v}_k|^\beta\,dx = o_k(1),$$
$$\|(\tilde{u}_k, \tilde{v}_k)\|^p - \|(u_k, v_k)\|^{q-p}\int_\Omega (\lambda|\tilde{u}_k|^q + \mu|\tilde{v}_k|^q)\,dx - 2\|(u_k, v_k)\|^{\alpha+\beta-p}\int_\Omega |\tilde{u}_k|^\alpha|\tilde{v}_k|^\beta\,dx = o_k(1).$$

From the above two equalities and (3.3), we obtain

$$\begin{aligned}\|(\tilde{u}_k, \tilde{v}_k)\|^p &= \frac{p(\alpha+\beta-q)}{q(\alpha+\beta-p)}\|(u_k, v_k)\|^{q-p}\int_\Omega (\lambda|\tilde{u}_k|^q + \mu|\tilde{v}_k|^q)\,dx + o_k(1)\\ &= \frac{p(\alpha+\beta-q)}{q(\alpha+\beta-p)}\|(u_k, v_k)\|^{q-p}\int_\Omega (\lambda|\tilde{u}|^q + \mu|\tilde{v}|^q)\,dx + o_k(1).\end{aligned}$$

Since $1 < q < p$ and $\|(u_k, v_k)\| \to \infty$, we get $\|(\tilde{u}_k, \tilde{v}_k)\|^p \to 0$, which contradicts $\|(\tilde{u}_k, \tilde{v}_k)\| = 1$. □

Lemma 3.3 (Uniform Lower Bound). *If $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is a $(\mathrm{PS})_c$ sequence for $J_{\lambda,\mu}$ with $(u_k, v_k) \rightharpoonup (u, v)$ in E, then $J'_{\lambda,\mu}(u, v) = 0$, and there exists a positive constant C_0 depending on p, q, s, n, S and $|\Omega|$ such that*

$$J_{\lambda,\mu}(u, v) \ge -C_0(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}), \tag{3.4}$$

where we have set

$$C_0 := \frac{p-q}{pqp_s^*}\frac{(p_s^*-q)^{\frac{p}{p-q}}}{(p_s^*-p)^{\frac{q}{p-q}}}|\Omega|^{\frac{p(p_s^*-q)}{p_s^*(p-q)}}S^{-\frac{q}{p-q}}, \tag{3.5}$$

with S being the best constant for the Sobolev embedding of X_0 into $L^{p_s^*}(\mathbb{R}^n)$.

Proof. Assume that $\{(u_k, v_k)\} \subset E$ is a $(\mathrm{PS})_c$ sequence for $J_{\lambda,\mu}$ with $(u_k, v_k) \rightharpoonup (u, v)$ in E. That is,

$$J'_{\lambda,\mu}(u_k, v_k) = o(1) \quad \text{strongly in } E^* \text{ as } k \to \infty.$$

Let $(\phi, \psi) \in E$. Then we have

$$\begin{aligned}\langle J'_{\lambda,\mu}(u_k, v_k) - J'_{\lambda,\mu}(u, v), (\phi, \psi)\rangle = {}& \mathcal{A}(u_k, \phi) - \mathcal{A}(u, \phi) + \mathcal{A}(v_k, \psi) - \mathcal{A}(v, \psi)\\ &- \lambda\int_\Omega (|u_k|^{q-2}u_k - |u|^{q-2}u)\phi\,dx - \mu\int_\Omega (|v_k|^{q-2}v_k - |v|^{q-2}v)\psi\,dx\\ &- \frac{2\alpha}{\alpha+\beta}\int_\Omega (|u_k|^{\alpha-2}u_k|v_k|^\beta - |u|^{\alpha-2}u|v|^\beta)\phi\,dx\\ &- \frac{2\beta}{\alpha+\beta}\int_\Omega (|u_k|^\alpha|v_k|^{\beta-2}v_k - |u|^\alpha|v|^{\beta-2}v)\psi\,dx,\end{aligned}$$

where $\mathcal{A}$ is defined in (1.4). We claim that, from $(u_k, v_k) \rightharpoonup (u, v)$ in E, we have

$$\lim_k \mathcal{A}(u_k, \phi) = \mathcal{A}(u, \phi), \quad \lim_k \mathcal{A}(v_k, \psi) = \mathcal{A}(v, \psi) \quad \text{for any } \phi, \psi \in X_0 \text{ as } k \to \infty.$$

In fact, the sequences

$$\left\{\frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))}{|x-y|^{\frac{n+ps}{p'}}}\right\}_{k\in\mathbb{N}}, \quad \left\{\frac{|v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))}{|x-y|^{\frac{n+ps}{p'}}}\right\}_{k\in\mathbb{N}}$$

are bounded in $L^{p'}(\mathbb{R}^n)$ and by the pointwise converge $u_k \to u$ and $v_k \to v$, we have

$$\frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))}{|x-y|^{\frac{n+ps}{p'}}} \xrightarrow{L^{p'}(\mathbb{R}^n)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{\frac{n+ps}{p'}}},$$

and

$$\frac{|v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))}{|x-y|^{\frac{n+ps}{p'}}} \xrightarrow{L^{p'}(\mathbb{R}^n)} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{\frac{n+ps}{p'}}}.$$

Since

$$\frac{\phi(x) - \phi(y)}{|x-y|^{\frac{n+ps}{p}}} \in L^p(\mathbb{R}^n), \quad \frac{\psi(x) - \psi(y)}{|x-y|^{\frac{n+ps}{p}}} \in L^p(\mathbb{R}^n),$$

the claim follows. The sequences u_k and v_k are bounded in X_0, and then in $L^{p_s^*}(\Omega)$. Then $u_k \to u$ and $v_k \to v$ weakly in $L^{p_s^*}(\mathbb{R}^n)$. Furthermore, we obtain

$$|u_k|^{q-2}u_k \xrightarrow{L^{q'}(\Omega)} |u|^{q-2}u, \quad |v_k|^{q-2}v_k \xrightarrow{L^{q'}(\Omega)} |v|^{q-2}v,$$
$$|u_k|^{\alpha-2}u_k|v_k|^\beta \xrightarrow{L^{\frac{\alpha+\beta}{\alpha+\beta-1}}(\Omega)} |u|^{\alpha-2}u|v|^\beta, \quad |u_k|^\alpha|v_k|^{\beta-2}v_k \xrightarrow{L^{\frac{\alpha+\beta}{\alpha+\beta-1}}(\Omega)} |u|^\alpha|v|^{\beta-2}v.$$

Since $\phi, \psi \in X_0 \subset L^q(\Omega) \cap L^{\alpha+\beta}(\Omega)$, it follows that, as $k \to \infty$,

$$\int_\Omega (|u_k|^{q-2}u_k - |u|^{q-2}u)\phi\,dx \to 0, \quad \int_\Omega (|v_k|^{q-2}v_k - |v|^{q-2}v)\psi\,dx \to 0,$$

and

$$\int_\Omega (|u_k|^{\alpha-2}u_k|v_k|^\beta - |u|^{\alpha-2}u|v|^\beta)\phi\,dx \to 0, \quad \int_\Omega (|u_k|^\alpha|v_k|^{\beta-2}v_k - |u|^\alpha|v|^{\beta-2}v)\psi\,dx \to 0.$$

Hence,

$$\langle J'_{\lambda,\mu}(u_k, v_k) - J'_{\lambda,\mu}(u, v), (\phi, \psi)\rangle \to 0 \quad \text{for all } (\phi, \psi) \in E,$$

which yields $J'_{\lambda,\mu}(u, v) = 0$. In particular, we get

$$\langle J'_{\lambda,\mu}(u, v), (u, v)\rangle = 0,$$

i.e.,

$$2\int_\Omega |u|^\alpha|v|^\beta\,dx = \|(u, v)\|^p - \int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx.$$

Then

$$\begin{aligned} J_{\lambda,\mu}(u, v) &= \Big(\frac{1}{p} - \frac{1}{p_s^*}\Big)\|(u, v)\|^p - \Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx \\ &= \frac{s}{n}\|(u, v)\|^p - \Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx. \end{aligned} \tag{3.6}$$

By Hölder's inequality, the Sobolev embedding, (1.6) and Young's inequality, we have

$$\begin{aligned}\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx &\le |\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}(\lambda\|u\|_{X_0}^q + \mu\|v\|_{X_0}^q)\\
&= \Big(\Big[\frac{p}{q}\frac{s}{n}\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)^{-1}\Big]^{\frac{q}{p}}\|u\|_{X_0}^q\Big)\Big(\Big[\frac{p}{q}\frac{s}{n}\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)^{-1}\Big]^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\lambda\Big)\\
&\quad + \Big(\Big[\frac{p}{q}\frac{s}{n}\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)^{-1}\Big]^{\frac{q}{p}}\|v\|_{X_0}^q\Big)\Big(\Big[\frac{p}{q}\frac{s}{n}\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)^{-1}\Big]^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\mu\Big)\\
&\le \frac{s}{n}\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)^{-1}(\|u\|_{X_0}^p + \|v\|_{X_0}^p) + \widehat{C}(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})\\
&= \frac{s}{n}\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)^{-1}\|(u,v)\|^p + \widehat{C}(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}),\end{aligned} \tag{3.7}$$

with

$$\widehat{C} = \frac{p-q}{p}\Big(\Big[\frac{p}{q}\frac{s}{n}\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)^{-1}\Big]^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\Big)^{\frac{p}{p-q}} = \frac{p-q}{p}\Big(\frac{p_s^*-q}{p_s^*-p}\Big)^{\frac{q}{p-q}}|\Omega|^{\frac{p(p_s^*-q)}{p_s^*(p-q)}} S^{-\frac{q}{p-q}}.$$

Then (3.4) follows from (3.6) and (3.7) with $C_0 = (\frac{1}{q} - \frac{1}{p_s^*})\widehat{C}$. □

Let us set

$$S_{\alpha,\beta} := \inf_{(u,v)\in E\setminus\{0\}} \frac{\|(u,v)\|^p}{\big(\int_\Omega |u|^\alpha|v|^\beta\,dx\big)^{\frac{p}{\alpha+\beta}}}. \tag{3.8}$$

We have the following result which provides a connection between $S_{\alpha,\beta}$ and S. The proof essentially follows by the line of arguments used in [1] but, for the sake of self-containedness, we include it.

Lemma 3.4 ($S_{\alpha,\beta}$ versus S). *We have*

$$S_{\alpha,\beta} = \Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{\alpha+\beta}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha+\beta}}\Big]S. \tag{3.9}$$

Proof. Let $\{\omega_n\}_{n\in\mathbb{N}} \subset X_0$ be a minimization sequence for S. Let $s, t > 0$ be chosen later and consider the sequences $u_n := s\omega_n$ and $v_n := t\omega_n$ in X_0. By the definition of $S_{\alpha,\beta}$, we have

$$\frac{s^p + t^p}{(s^\alpha t^\beta)^{\frac{p}{p_s^*}}} \frac{\int_{\mathbb{R}^{2n}} \frac{|\omega_n(x)-\omega_n(y)|^p}{|x-y|^{n+ps}}\,dx\,dy}{\big(\int_\Omega |\omega_n|^{p_s^*}\,dx\big)^{\frac{p}{p_s^*}}} \ge S_{\alpha,\beta}. \tag{3.10}$$

Observe that

$$\frac{s^p + t^p}{(s^\alpha t^\beta)^{\frac{p}{p_s^*}}} = \Big(\frac{s}{t}\Big)^{\frac{p\beta}{p_s^*}} + \Big(\frac{s}{t}\Big)^{-\frac{p\alpha}{p_s^*}}.$$

Let us consider the function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$g(x) := x^{\frac{p\beta}{p_s^*}} + x^{\frac{-p\alpha}{p_s^*}}.$$

Then we have

$$\frac{s^p + t^p}{(s^\alpha t^\beta)^{\frac{p}{p_s^*}}} = g\Big(\frac{s}{t}\Big),$$

and the function g achieves its minimum at point $x_0 = (\frac{\alpha}{\beta})^{\frac{1}{p}}$ with minimum value

$$\min_{x\in\mathbb{R}^+} g(x) = \Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{p_s^*}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{p_s^*}}.$$

Choosing s, t in (3.10) such that $\frac{s}{t} = (\frac{\alpha}{\beta})^{\frac{1}{p}}$ and letting $n \to \infty$ yields

$$\Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{p_s^*}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{p_s^*}}\Big]S \ge S_{\alpha,\beta}. \tag{3.11}$$

On the other hand, let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset E \setminus \{(0,0)\}$ be a minimizing sequence for $S_{\alpha,\beta}$. Set $z_n := s_n v_n$ for $s_n > 0$ with $\int_\Omega |u_n|^{p_s^*}\,dx = \int_\Omega |z_n|^{p_s^*}\,dx$. Then Young's inequality implies

$$\int_\Omega |u_n|^\alpha |z_n|^\beta\,dx \le \frac{\alpha}{\alpha+\beta}\int_\Omega |u_n|^{\alpha+\beta}\,dx + \frac{\beta}{\alpha+\beta}\int_\Omega |z_n|^{\alpha+\beta}\,dx = \int_\Omega |z_n|^{\alpha+\beta}\,dx = \int_\Omega |u_n|^{\alpha+\beta}\,dx.$$

Then we have

$$\begin{aligned}\frac{\int_{\mathbb{R}^{2n}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{n+ps}}\,dx\,dy + \int_{\mathbb{R}^{2n}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{n+ps}}\,dx\,dy}{\big(\int_\Omega |u_n|^\alpha |v_n|^\beta\,dx\big)^{\frac{p}{\alpha+\beta}}} &= \frac{s_n^{\frac{p\beta}{\alpha+\beta}}\big(\int_{\mathbb{R}^{2n}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{n+ps}}\,dx\,dy + \int_{\mathbb{R}^{2n}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\big)}{\big(\int_\Omega |u_n|^\alpha |z_n|^\beta\,dx\big)^{\frac{p}{\alpha+\beta}}}\\ &\ge s_n^{\frac{p\beta}{\alpha+\beta}} \frac{\int_{\mathbb{R}^{2n}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{n+ps}}\,dx\,dy}{\big(\int_\Omega |u_n|^{\alpha+\beta}\,dx\big)^{\frac{p}{\alpha+\beta}}} + s_n^{\frac{p\beta}{\alpha+\beta}} s_n^{-p} \frac{\int_{\mathbb{R}^{2n}} \frac{|z_n(x)-z_n(y)|^p}{|x-y|^{n+ps}}\,dx\,dy}{\big(\int_\Omega |z_n|^{\alpha+\beta}\,dx\big)^{\frac{p}{\alpha+\beta}}}\\ &\ge g(s_n)S\\ &\ge \Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{p_s^*}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{p_s^*}}\Big]S.\end{aligned}$$

In the last inequality, passing to the limit as $n \to \infty$, we obtain

$$\Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{p_s^*}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{p_s^*}}\Big]S \le S_{\alpha,\beta}. \tag{3.12}$$

Thus, (3.9) follows from (3.11) and (3.12). □

Lemma 3.5 (Palais–Smale Range). *The functional $J_{\lambda,\mu}$ satisfies the* (PS)$_c$ *condition with c satisfying*

$$-\infty < c < c_\infty = \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}} - C_0\big(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\big), \tag{3.13}$$

where C_0 is the positive constant defined in (3.5).

Proof. Let $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ be a (PS)$_c$ sequence of $J_{\lambda,\mu}$ in E. Then

$$\frac{1}{p}\|(u_k, v_k)\|^p - \frac{1}{q}\int_\Omega (\lambda|u_k|^q + \mu|v_k|^q)\,dx - \frac{2}{p_s^*}\int_\Omega |u_k|^\alpha |v_k|^\beta\,dx = c + o_k(1), \tag{3.14}$$

$$\|(u_k, v_k)\|^p - \int_\Omega (\lambda|u_k|^q + \mu|v_k|^q)\,dx - 2\int_\Omega |u_k|^\alpha |v_k|^\beta\,dx = o_k(1). \tag{3.15}$$

We know, by Lemma 3.2, that $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is bounded in E. Then, up to a subsequence, $(u_k, v_k) \rightharpoonup (u, v)$ in E and, by Lemma 3.3, we have that (u, v) is a critical point of $J_{\lambda,\mu}$.

Next we show that (u_k, v_k) converges strongly to (u, v) as $k \to \infty$ in E. Since $u_k \to u$ and $v_k \to v$ in $L^r(\mathbb{R}^n)$, we obtain

$$\int_\Omega (\lambda|u_k|^q + \mu|v_k|^q)\,dx \to \int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx \quad \text{as } k \to \infty.$$

Moreover, by variants of the Brezis–Lieb Lemma, we can easily get (cf. [5, Lemma 2.2])

$$\|(u_k, v_k)\|^p = \|(u_k - u, v_k - v)\|^p + \|(u, v)\|^p + o_k(1) \tag{3.16}$$

and

$$\int_\Omega |u_k|^\alpha |v_k|^\beta\,dx = \int_\Omega |u_k - u|^\alpha |v_k - v|^\beta\,dx + \int_\Omega |u|^\alpha |v|^\beta\,dx + o_k(1). \tag{3.17}$$

Taking (3.16) and (3.17) into (3.14) and (3.15), we find that

$$\frac{1}{p}\|(u_k - u, v_k - v)\|^p - \frac{2}{p_s^*}\int_\Omega |u_k - u|^\alpha |v_k - v|^\beta\,dx = c - J_{\lambda,\mu}(u, v) + o_k(1) \tag{3.18}$$

and

$$\|(u_k - u, v_k - v)\|^p = 2\int_\Omega |u_k - u|^\alpha |v_k - v|^\beta \, dx + o_k(1).$$

Hence, we may assume that

$$\|(u_k - u, v_k - v)\|^p \to m, \quad 2\int_\Omega |u_k - u|^\alpha |v_k - v|^\beta \, dx \to m \quad \text{as } k \to \infty. \tag{3.19}$$

If $m = 0$, we are done. Suppose $m > 0$. Then, from (3.19) and the definition of $S_{\alpha,\beta}$ in (3.8), we have

$$S_{\alpha,\beta}\Big(\frac{m}{2}\Big)^{\frac{p}{p_s^*}} = S_{\alpha,\beta} \lim_{k\to\infty}\Big(\int_\Omega |u_k - u|^\alpha |v_k - v|^\beta \, dx\Big)^{\frac{p}{p_s^*}} \le \lim_{k\to\infty} \|(u_k - u, v_k - v)\|^p = m,$$

which yields $m \ge 2(\frac{S_{\alpha,\beta}}{2})^{\frac{n}{ps}}$. From (3.18), we obtain

$$c = \frac{s}{n} m + J_{\lambda,\mu}(u, v).$$

By Lemma 3.3 and for $m \ge 2(\frac{S_{\alpha,\beta}}{2})^{\frac{n}{ps}}$, we find

$$c \ge \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}} - C_0(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}),$$

which is impossible for

$$-\infty < c < \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}} - C_0(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}).$$

□

4 Existence of Solutions

We start with some lemmas.

Lemma 4.1 ($\mathcal{N}^0_{\lambda,\mu}$ is Empty). *Let λ, μ be such that $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$, where Λ_1 is as in (2.5). Then $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.*

Proof. From the proof of Lemma 2.5, we have that there exist *exactly* two numbers $t_2 > t_1 > 0$ such that $\varphi'_{u,v}(t_1) = \varphi'_{u,v}(t_2) = 0$. Furthermore, $\varphi''_{u,v}(t_1) > 0 > \varphi''_{u,v}(t_2)$. If, by contradiction, $(u, v) \in \mathcal{N}^0_{\lambda,\mu}$, then we have that $\varphi'_{u,v}(1) = 0$ with $\varphi''_{u,v}(1) = 0$. Then, either $t_1 = 1$ or $t_2 = 1$. In turn, either $\varphi''_{u,v}(1) > 0$ or $\varphi''_{u,v}(1) < 0$, a contradiction. □

Lemma 4.2 (Coercivity). *The functional $J_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$ for all $\lambda > 0$ and $\mu > 0$.*

Proof. Let $\lambda > 0$ and $\mu > 0$ and pick $(u, v) \in \mathcal{N}_{\lambda,\mu}$. Then, we have

$$\begin{aligned} J_{\lambda,\mu}(u, v) &= \Big(\frac{1}{p} - \frac{1}{\alpha+\beta}\Big)\|(u, v)\|^p - \Big(\frac{1}{q} - \frac{1}{\alpha+\beta}\Big)\int_\Omega (\lambda|u|^q + \mu|v|^q)\, dx \\ &\ge \Big(\frac{1}{p} - \frac{1}{\alpha+\beta}\Big)\|(u, v)\|^p - \Big(\frac{1}{q} - \frac{1}{\alpha+\beta}\Big) S^{-\frac{q}{p}} |\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}} (\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p-q}{p}} \|(u, v)\|^q, \end{aligned}$$

which yields the assertion. □

By Lemmas 4.1 and 4.2, for any λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$, we have

$$\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu},$$

and $J_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$. Therefore, we may define

$$c_{\lambda,\mu} := \inf_{\mathcal{N}_{\lambda,\mu}} J_{\lambda,\mu}, \quad c^\pm_{\lambda,\mu} := \inf_{\mathcal{N}^\pm_{\lambda,\mu}} J_{\lambda,\mu}.$$

Of course, by Lemma 4.2, we have $c_{\lambda,\mu}, c^\pm_{\lambda,\mu} > -\infty$. The following result is valid.

Lemma 4.3 ($c^+_{\lambda,\mu} < 0$ and $c^-_{\lambda,\mu} > 0$). *Let Λ_1 be as in (2.5). Then the following facts hold:*
(i) *if $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$, then $c_{\lambda,\mu} \le c^+_{\lambda,\mu} < 0$,*
(ii) *if $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < (\frac{q}{p})^{\frac{p}{p-q}}\Lambda_1$, then $c^-_{\lambda,\mu} > d_0$ for some $d_0 = d_0(\lambda,\mu,p,q,n,s,|\Omega|) > 0$.*

Proof. Let us prove (i). Let $(u,v) \in \mathcal{N}^+_{\lambda,\mu}$. Then we have $\varphi''_{u,v}(1) > 0$, which combined with (2.2) yields

$$\frac{p-q}{2(\alpha+\beta-q)}\|(u,v)\|^p > \int_\Omega |u|^\alpha |v|^\beta \, dx.$$

Therefore,

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \Big(\frac{1}{p}-\frac{1}{q}\Big)\|(u,v)\|^p + 2\Big(\frac{1}{q}-\frac{1}{\alpha+\beta}\Big)\int_\Omega |u|^\alpha|v|^\beta\,dx \\ &< \Big[\Big(\frac{1}{p}-\frac{1}{q}\Big) + \Big(\frac{1}{q}-\frac{1}{\alpha+\beta}\Big)\frac{p-q}{\alpha+\beta-q}\Big]\|(u,v)\|^p \\ &= -\frac{(p-q)(\alpha+\beta-p)}{pq(\alpha+\beta)}\|(u,v)\|^p < 0. \end{aligned}$$

Therefore, $c_{\lambda,\mu} \le c^+_{\lambda,\mu} < 0$ follows from the definitions of $c_{\lambda,\mu}$ and $c^+_{\lambda,\mu}$.

Let us now come to (ii). Let $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. Then, we have $\varphi''_{u,v}(1) < 0$, which combined with (2.2) yields

$$\frac{p-q}{2(\alpha+\beta-q)}\|(u,v)\|^p < \int_\Omega |u|^\alpha |v|^\beta \, dx.$$

By Young's inequality and the definition of S, we obtain

$$\int_\Omega |u|^\alpha|v|^\beta\,dx \le \frac{\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha+\beta}\,dx + \frac{\beta}{\alpha+\beta}\int_\Omega |v|^{\alpha+\beta}\,dx \le S^{-\frac{\alpha+\beta}{p}}\|(u,v)\|^{\alpha+\beta}.$$

Thus,

$$\|(u,v)\| > \Big(\frac{p-q}{2(\alpha+\beta-q)}\Big)^{\frac{1}{\alpha+\beta-p}} S^{\frac{\alpha+\beta}{p(\alpha+\beta-p)}}.$$

Moreover, by Hölder's inequality and the definition of S, we find

$$\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx \le S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\big(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\big)^{\frac{p-q}{p}}\|(u,v)\|^q.$$

Therefore, if $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < (\frac{q}{p})^{\frac{p}{p-q}}\Lambda_1$, then we have

$$\begin{aligned} J_{\lambda,\mu}(u,v) &\ge \|(u,v)\|^q\Big[\Big(\frac{1}{p}-\frac{1}{\alpha+\beta}\Big)\|(u,v)\|^{p-q} - \Big(\frac{1}{q}-\frac{1}{\alpha+\beta}\Big)S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\big(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}\big)^{\frac{p-q}{p}}\Big] \\ &> \|(u,v)\|^q\Big[\Big(\frac{1}{p}-\frac{1}{\alpha+\beta}\Big)\Big(\frac{p-q}{2(\alpha+\beta-q)}\Big)^{\frac{p-q}{\alpha+\beta-p}} S^{\frac{(\alpha+\beta)(p-q)}{p(\alpha+\beta-p)}} - \Big(\frac{1}{q}-\frac{1}{\alpha+\beta}\Big)S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\big(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}\big)^{\frac{p-q}{p}}\Big] \\ &\ge d_0 > 0. \end{aligned}$$

□

4.1 The First Solution

We now prove the existence of a first solution (u_1, v_1) to (1.1). First, we need some preliminary results.

Lemma 4.4 (Curves into $\mathcal{N}_{\lambda,\mu}$). *Let Λ_1 be as in (2.5) and assume that $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$. Then, for any $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, there exists $\epsilon > 0$ and a differentiable map $\xi : B(0,\epsilon) \subset E \to \mathbb{R}^+$, with $\xi(0) = 1$, such that $\xi(\omega)(z-\omega) \in \mathcal{N}_{\lambda,\mu}$ and*

$$\langle \xi'(0), \omega\rangle = -\frac{p\mathcal{A}(u,\omega_1) + p\mathcal{A}(v,\omega_2) - K_{\lambda,\mu}(z,\omega) - 2\int_\Omega(\alpha|u|^{\alpha-2}u\omega_1|v|^\beta + \beta|u|^\alpha|v|^{\beta-2}v\omega_2)\,dx}{(p-q)\|(u,v)\|^p - 2(\alpha+\beta-q)\int_\Omega |u|^\alpha|u|^\beta\,dx} \tag{4.1}$$

for all $\omega = (\omega_1,\omega_2) \in E$, where

$$K_{\lambda,\mu}(z,\omega) = q\int_\Omega (\lambda|u|^{q-2}u\omega_1 + \mu|v|^{q-2}v\omega_2)\,dx.$$

Proof. For $z = (u, v) \in \mathcal{N}_{\lambda,\mu}$, define a function $F_z\colon \mathbb{R}^+ \times E \to \mathbb{R}$ by

$$\begin{aligned} F_z(\xi, \omega) &:= \langle J'_{\lambda,\mu}(\xi(z-\omega)), \xi(z-\omega)\rangle \\ &= \xi^p(\mathcal{A}(u-\omega_1, u-\omega_1) + \mathcal{A}(v-\omega_2, v-\omega_2)) - \xi^q \int_\Omega (\lambda|u-\omega_1|^q + \mu|v-\omega_2|^q)\,dx \\ &\quad - 2\xi^{\alpha+\beta}\int_\Omega |u-\omega_1|^\alpha|v-\omega_2|^\beta\,dx, \quad \xi \in \mathbb{R}^+,\ \omega \in E. \end{aligned}$$

Then $F_z(1,0) = \langle J'_{\lambda,\mu}(z), z\rangle = 0$ and, by Lemma 4.1, we have

$$\begin{aligned} \frac{d}{d\xi}F_z(1,(0,0)) &= p\|(u,v)\|^p - q\int_\Omega(\lambda|u|^q + \mu|v|^q)\,dx - 2(\alpha+\beta)\int_\Omega |u|^\alpha|v|^\beta\,dx \\ &= (p-q)\|(u,v)\|^p - 2(\alpha+\beta-q)\int_\Omega |u|^\alpha|u|^\beta\,dx \neq 0. \end{aligned}$$

By the Implicit Function Theorem, there exist $\epsilon > 0$ and a C^1 map $\xi\colon B(0,\epsilon) \subset E \to \mathbb{R}^+$, with $\xi(0) = 1$, such that

$$\langle \xi'(0), \omega\rangle = -\frac{p\mathcal{A}(u,\omega_1) + p\mathcal{A}(v,\omega_2) - K_{\lambda,\mu}(z,\omega) - 2\int_\Omega(\alpha|u|^{\alpha-2}u\omega_1|v|^\beta + \beta|u|^\alpha|v|^{\beta-2}v\omega_2)\,dx}{(p-q)\|(u,v)\|^p - 2(\alpha+\beta-q)\int_\Omega |u|^\alpha|u|^\beta\,dx}$$

and $F_z(\xi(\omega),\omega) = 0$ for all $\omega \in B(0,\epsilon)$, which is equivalent to

$$\langle J'_{\lambda,\mu}(\xi(\omega)(z-\omega)), \xi(\omega)(z-\omega)\rangle = 0 \quad \text{for all } \omega \in B(0,\epsilon),$$

i.e., $\xi(\omega)(z-\omega) \in \mathcal{N}_{\lambda,\mu}$. □

Lemma 4.5 (Curves into $\mathcal{N}^-_{\lambda,\mu}$). *Let Λ_1 be as in* (2.5) *and assume $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$. Then, for each $z \in \mathcal{N}^-_{\lambda,\mu}$, there exist $\epsilon > 0$ and a differentiable map $\xi^-\colon B(0,\epsilon) \subset E \to \mathbb{R}^+$, with $\xi^-(0) = 1$, such that $\xi^-(\omega)(z-\omega) \in \mathcal{N}^-_{\lambda,\mu}$ and*

$$\langle (\xi^-)'(0), \omega\rangle = -\frac{p\mathcal{A}(u,\omega_1) + p\mathcal{A}(v,\omega_2) - K_{\lambda,\mu}(z,\omega) - 2\int_\Omega(\alpha|u|^{\alpha-2}u\omega_1|v|^\beta + \beta|u|^\alpha|v|^{\beta-2}v\omega_2)\,dx}{(p-q)\|(u,v)\|^p - 2(\alpha+\beta-q)\int_\Omega |u|^\alpha|u|^\beta\,dx}$$

for every $\omega \in B(0,\epsilon)$.

Proof. Arguing as in the proof of Lemma 4.4, there exist $\epsilon > 0$ and a differentiable map $\xi^-\colon B(0,\epsilon) \subset E \to \mathbb{R}^+$ such that $\xi^-(0) = 1$, $\xi^-(\omega)(z-\omega) \in \mathcal{N}_{\lambda,\mu}$ for all $\omega \in B(0,\epsilon)$ and satisfying (4.1). Since

$$\varphi''_{u,v}(1) = (p-q)\|(u,v)\|^p - 2(\alpha+\beta-q)\int_\Omega |u|^\alpha|v|^\beta\,dx < 0,$$

by continuity, we have

$$\begin{aligned} \varphi''_{\xi^-(\omega)(u-\omega_1),\xi^-(\omega)(v-\omega_2)}(1) &= (p-q)\|(\xi^-(\omega)(u-\omega_1), \xi^-(\omega)(v-\omega_2))\|^p \\ &\quad - 2((\alpha+\beta)-q)\int_\Omega |\xi^-(\omega)(u-\omega_1)|^\alpha|\xi^-(\omega)(v-\omega_2)|^\beta\,dx < 0 \end{aligned}$$

for ϵ sufficiently small, which implies $\xi^-(\omega)(z-\omega) \in \mathcal{N}^-_{\lambda,\mu}$. □

Proposition 4.6 ($(\mathrm{PS})_{c_{\lambda,\mu}}$-Sequences). *The following facts hold:*

(i) *If $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$, then there exists a $(\mathrm{PS})_{c_{\lambda,\mu}}$-sequence $\{(u_k, v_k)\} \subset \mathcal{N}_{\lambda,\mu}$ for $J_{\lambda,\mu}$.*

(ii) *If $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < (\frac{q}{p})^{\frac{p}{p-q}}\Lambda_1$, then there exists a $(\mathrm{PS})_{c^-_{\lambda,\mu}}$-sequence $\{(u_k, v_k)\} \subset \mathcal{N}^-_{\lambda,\mu}$ for $J_{\lambda,\mu}$.*

Proof. (i) By Ekeland's Variational Principle, there exists a minimizing sequence $\{(u_k, v_k)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(u_k, v_k) < c_{\lambda,\mu} + \frac{1}{k}, \quad J_{\lambda,\mu}(u_k, v_k) < J_{\lambda,\mu}(w_1, w_2) + \frac{1}{k}\|(w_1, w_2) - (u_k, v_k)\| \tag{4.2}$$

for each $(w_1, w_2) \in \mathcal{N}_{\lambda,\mu}$. Taking k large and using $c_{\lambda,\mu} < 0$, we have

$$J_{\lambda,\mu}(u_k, v_k) = \Big(\frac{1}{p} - \frac{1}{\alpha+\beta}\Big)\|(u_k, v_k)\|^p - \Big(\frac{1}{q} - \frac{1}{\alpha+\beta}\Big)\int_\Omega (\lambda|u_k|^q + \mu|v_k|^q)\,dx < \frac{c_{\lambda,\mu}}{2}. \tag{4.3}$$

This yields

$$-\frac{q(\alpha+\beta)}{2(\alpha+\beta-q)}c_{\lambda,\mu} < \int_\Omega (\lambda|u_k|^q + \mu|v_k|^q)\,dx \le S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p-q}{p}}\|(u_k, v_k)\|^q. \tag{4.4}$$

Consequently, $(u_k, v_k) \not\equiv 0$ and, by combining it with (4.3) and (4.4), and using Hölder's inequality, we have

$$\|(u_k, v_k)\| > \Big[-\frac{q(\alpha+\beta)}{2(\alpha+\beta-q)}c_{\lambda,\mu}S^{\frac{q}{p}}|\Omega|^{-\frac{\alpha+\beta-q}{\alpha+\beta}}(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{q-p}{p}}\Big]^{\frac{1}{q}},$$

$$\|(u_k, v_k)\| < \Big[\frac{p(\alpha+\beta-q)}{q(\alpha+\beta-p)}S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p-q}{p}}\Big]^{\frac{1}{p-q}}. \tag{4.5}$$

Now we prove that $\|J'_{\lambda,\mu}(u_k, v_k)\|_{E^{-1}} \to 0$ as $k \to \infty$. Fix $k \in \mathbb{N}$. By applying Lemma 4.4 to $z_k = (u_k, v_k)$, we obtain a function $\xi_k : B(0, \epsilon_k) \to \mathbb{R}^+$, for some $\epsilon_k > 0$, such that $\xi_k(h)(z_k - h) \in \mathcal{N}_{\lambda,\mu}$. Take $0 < \rho < \epsilon_k$. Let $w \in E$ with $w \not\equiv 0$ and put $h^* = \frac{\rho w}{\|w\|}$. We set $h_\rho = \xi_k(h^*)(z_k - h^*)$. Then $h_\rho \in \mathcal{N}_{\lambda,\mu}$ and from (4.2) we have

$$J_{\lambda,\mu}(h_\rho) - J_{\lambda,\mu}(z_k) \ge -\frac{1}{k}\|h_\rho - z_k\|.$$

By the Mean Value Theorem, we get

$$\langle J'_{\lambda,\mu}(z_k), h_\rho - z_k\rangle + o(\|h_\rho - z_k\|) \ge -\frac{1}{k}\|h_\rho - z_k\|.$$

Thus, we have

$$\langle J'_{\lambda,\mu}(z_k), -h^*\rangle + (\xi_k(h^*) - 1)\langle J'_{\lambda,\mu}(z_k), z_k - h^*\rangle \ge -\frac{1}{k}\|h_\rho - z_k\| + o(\|h_\rho - z_k\|).$$

Whence, from the fact that $\xi_k(h^*)(z_k - h^*) \in \mathcal{N}_{\lambda,\mu}$, it follows that

$$-\rho\Big\langle J'_{\lambda,\mu}(z_k), \frac{w}{\|w\|}\Big\rangle + (\xi_k(h^*) - 1)\langle J'_{\lambda,\mu}(z_k) - J'_{\lambda,\mu}(h_\rho), z_k - h^*\rangle \ge -\frac{1}{k}\|h_\rho - z_k\| + o(\|h_\rho - z_k\|).$$

Hence, we get

$$\Big\langle J'_{\lambda,\mu}(z_k), \frac{w}{\|w\|}\Big\rangle \le \frac{1}{k\rho}\|h_\rho - z_k\| + \frac{o(\|h_\rho - z_k\|)}{\rho} + \frac{(\xi_k(h^*) - 1)}{\rho}\langle J'_{\lambda,\mu}(z_k) - J'_{\lambda,\mu}(h_\rho), z_k - h^*\rangle. \tag{4.6}$$

Since

$$\|h_\rho - z_k\| \le \rho|\xi_k(h^*)| + |\xi_k(h^*) - 1|\|z_k\| \quad \text{and} \quad \lim_{\rho\to 0}\frac{|\xi_k(h^*) - 1|}{\rho} \le \|\xi'_k(0)\|,$$

for $k \in \mathbb{N}$ fixed, if $\rho \to 0$ in (4.6), then, by virtue of (4.5), we can choose $C > 0$ independent of ρ such that

$$\Big\langle J'_{\lambda,\mu}(z_k), \frac{w}{\|w\|}\Big\rangle \le \frac{C}{k}(1 + \|\xi'_k(0)\|).$$

Thus, we are done if $\sup_{k\in\mathbb{N}}\|\xi'_k(0)\|_{E^*} < \infty$. By (4.1), (4.5) and Hölder's inequality, we have

$$|\langle \xi'_k(0), h\rangle| \le \frac{C_1\|h\|}{|(p-q)\|(u_k, v_k)\|^p - 2(\alpha+\beta-q)\int_\Omega |u_k|^\alpha v_k|^\beta\,dx|}$$

for some $C_1 > 0$. We only need to prove that

$$\Big|(p-q)\|(u_k, v_k)\|^p - 2(\alpha+\beta-q)\int_\Omega |u_k|^\alpha|v_k|^\beta\,dx\Big| \ge C_2$$

for some $C_2 > 0$ and k large. By contradiction, suppose that there exists a subsequence $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ with

$$(p-q)\|(u_k, v_k)\|^p - 2(\alpha+\beta-q)\int_\Omega |u_k|^\alpha|v_k|^\beta\,dx = o_k(1). \tag{4.7}$$

By (4.7) and the fact that $(u_k, v_k) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\|(u_k, v_k)\|^p = \frac{2(\alpha+\beta-q)}{p-q} \int_\Omega |u_k|^\alpha |v_k|^\beta \, dx + o_k(1), \tag{4.8}$$

$$\|(u_k, v_k)\|^p = \frac{\alpha+\beta-q}{\alpha+\beta-p} \int_\Omega (\lambda|u_k|^q + \mu|v_k|^q) \, dx + o_k(1). \tag{4.9}$$

By Young's inequality, it follows that

$$\int_\Omega |u_k|^\alpha |v_k|^\beta \, dx \le S^{-\frac{\alpha+\beta}{p}} \|(u_k, v_k)\|^{\alpha+\beta}.$$

By this and (4.8), we get

$$\|(u_k, v_k)\| \ge \Big(\frac{p-q}{2(\alpha+\beta-q)} S^{\frac{\alpha+\beta}{p}}\Big)^{\frac{1}{\alpha+\beta-p}} + o_k(1). \tag{4.10}$$

Moreover, from (4.9) and by Hölder's inequality, we obtain

$$\|(u_k, v_k)\|^p \le \frac{\alpha+\beta-q}{\alpha+\beta-p} |\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}} S^{-\frac{q}{p}} (\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p-q}{p}} \|(u_k, v_k)\|^q + o_k(1).$$

Thus,

$$\|(u_k, v_k)\| \le \Big(\frac{\alpha+\beta-q}{\alpha+\beta-p} S^{-\frac{q}{p}} |\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\Big)^{\frac{1}{p-q}} (\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{1}{p}} + o_k(1). \tag{4.11}$$

From (4.10) and (4.11), and for k large enough, we get

$$\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} \ge \Big(\frac{p-q}{2(\alpha+\beta-q)}\Big)^{\frac{p}{\alpha+\beta-p}} \Big(\frac{\alpha+\beta-q}{\alpha+\beta-p} |\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\Big)^{-\frac{p}{p-q}} S^{\frac{\alpha+\beta}{\alpha+\beta-p} + \frac{q}{p-q}} = \Lambda_1,$$

which contradicts $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$. Therefore,

$$\langle J'_{\lambda,\mu}(u_k, v_k), \|w\|^{-1} w\rangle \le \frac{C}{k}.$$

This proves (i). By Lemma 4.5, using the same arguments, we can get (ii). □

Here is the main result of the section.

Proposition 4.7 (Existence of the First Solution). *Let Λ_1 be as in (2.5). Assume that $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$. Then there exists $(u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$ with the following properties:*
(i) $J_{\lambda,\mu}(u_1, v_1) = c_{\lambda,\mu} = c^+_{\lambda,\mu} < 0$,
(ii) (u_1, v_1) *is a solution of problem* (1.1).

Proof. By Proposition 4.6 (i), there exists a bounded minimizing sequence $\{(u_k, v_k)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$\lim_{k\to\infty} J_{\lambda,\mu}(u_k, v_k) = c_{\lambda,\mu} \le c^+_{\lambda,\mu} < 0, \qquad J'_{\lambda,\mu}(u_k, v_k) = o_k(1) \quad \text{in } E^*.$$

Then there exists $(u_1, v_1) \in E$ such that, up to a subsequence, $u_k \rightharpoonup u_1$, $v_k \rightharpoonup v_1$ in X_0 as well as $u_k \to u_1$ and $v_k \to v_1$ strongly in $L^r(\Omega)$ for any $1 \le r < p^*$. Then, the Dominated Convergence Theorem yields

$$\int_\Omega (\lambda|u_k|^q + \mu|v_k|^q) \, dx \to \int_\Omega (\lambda|u_1|^q + \mu|v_1|^q) \, dx \quad \text{as } k \to \infty.$$

It is easy to get that (u_1, v_1) is a weak solution of (1.1), cf. Lemma 3.3. Now, since $(u_k, v_k) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\begin{aligned} J_{\lambda,\mu}(u_k, v_k) &= \frac{\alpha+\beta-p}{p(\alpha+\beta)} \|(u_k, v_k)\|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)} \int_\Omega (\lambda|u_k|^q + \mu|v_k|^q) \, dx \\ &\ge -\frac{\alpha+\beta-q}{q(\alpha+\beta)} \int_\Omega (\lambda|u_k|^q + \mu|v_k|^q) \, dx. \end{aligned}$$

Then, from $c_{\lambda,\mu} < 0$, we get

$$\int_\Omega (\lambda|u_1|^q + \mu|v_1|^q)\,dx \geq -\frac{q(\alpha+\beta)}{\alpha+\beta-q} c_{\lambda,\mu} > 0.$$

Therefore, $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of (1.1).

Next, we show that $(u_k, v_k) \to (u_1, v_1)$ strongly in E and $J_{\lambda,\mu}(u_1, v_1) = c^+_{\lambda,\mu}$. In fact, since $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}$, in light of Fatou's lemma, we get

$$\begin{aligned}
c_{\lambda,\mu} &\leq J_{\lambda,\mu}(u_1, v_1) \\
&= \frac{\alpha+\beta-p}{p(\alpha+\beta)}\|(u_1, v_1)\|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega (\lambda|u_1|^q + \mu|v_1|^q)\,dx \\
&\leq \liminf_{k\to\infty}\bigg(\frac{\alpha+\beta-p}{p(\alpha+\beta)}\|(u_k, v_k)\|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega (\lambda|u_k|^q + \mu|v_k|^q)\,dx\bigg) \\
&= \liminf_{k\to\infty} J_{\lambda,\mu}(u_k, v_k) = c_{\lambda,\mu}.
\end{aligned}$$

This implies that $J_{\lambda,\mu}(u_1, v_1) = c_{\lambda,\mu}$ and $\|(u_k, v_k)\|^p \to \|(u_1, v_1)\|^p$. We also have

$$\|(u_k - u_1, v_k - v_1)\|^p = \|(u_k, v_k)\|^p - \|(u_1, v_1)\|^p + o_k(1).$$

Therefore $(u_k, v_k) \to (u_1, v_1)$ strongly in E. We claim that $(u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$, which yields $c_{\lambda,\mu} = c^+_{\lambda,\mu}$. Assume, by contradiction, that $(u_1, v_1) \in \mathcal{N}^-_{\lambda,\mu}$. By Lemma 2.5, there exist unique $t_2 > t_1 > 0$ such that

$$(t_1 u_1, t_1 v_1) \in \mathcal{N}^+_{\lambda,\mu}, \quad (t_2 u_1, t_2 v_1) \in \mathcal{N}^-_{\lambda,\mu}.$$

In particular, we have $t_1 < t_2 = 1$. Since

$$\frac{d}{dt} J_{\lambda,\mu}(t_1 u_1, t_1 v_1) = 0, \quad \frac{d^2}{dt^2} J_{\lambda,\mu}(t_1 u_1, t_1 v_1) > 0,$$

there exists $t^* \in (t_1, 1]$ such that $J_{\lambda,\mu}(t_1 u_1, t_1 v_1) < J_{\lambda,\mu}(t^* u_1, t^* v_1)$. Then

$$c_{\lambda,\mu} \leq J_{\lambda,\mu}(t_1 u_1, t_1 v_1) < J_{\lambda,\mu}(t^* u_1, t^* v_1) \leq J_{\lambda,\mu}(u_1, v_1) = c_{\lambda,\mu},$$

which is a contradiction. Hence, $(u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$. □

4.2 The Second Solution

We next establish the existence of a minimum for $J_{\lambda,\mu}|_{\mathcal{N}^-_{\lambda,\mu}}$. Let S be as in (1.6). From [4], we know that for $1 < p < \infty$, $s \in (0, 1)$, $n > ps$, there exists a minimizer for S, and for every minimizer U, there exist $x_0 \in \mathbb{R}^n$ and a constant sign monotone function $u\colon \mathbb{R} \to \mathbb{R}$ such that $U(x) = u(|x - x_0|)$. In the following, we shall fix a radially symmetric nonnegative decreasing minimizer $U = U(r)$ for S. Multiplying U by a positive constant if necessary, we may assume that

$$(-\Delta)^s_p U = U^{p^*_s - 1} \quad \text{in } \mathbb{R}^n. \tag{4.12}$$

For any $\epsilon > 0$, we note that the function

$$U_\epsilon(x) = \frac{1}{\epsilon^{\frac{n-ps}{p}}} U\Big(\frac{|x|}{\epsilon}\Big)$$

is also a minimizer for S satisfying (4.12). In [4], the following asymptotic estimates for U were provided.

Lemma 4.8 (Optimal Decay). *There exist $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r > 1$,*

$$\frac{c_1}{r^{\frac{n-ps}{p-1}}} \leq U(r) \leq \frac{c_2}{r^{\frac{n-ps}{p-1}}}, \quad \frac{U(\theta r)}{U(r)} \leq \frac{1}{2}.$$

Assume, without loss of generality, that $0 \in \Omega$. For $\epsilon, \delta > 0$, let

$$m_{\epsilon,\delta} = \frac{U_\epsilon(\delta)}{U_\epsilon(\delta) - U_\epsilon(\theta\delta)}, \quad g_{\epsilon,\delta}(t) = \begin{cases} 0 & \text{if } 0 \le t \le U_\epsilon(\theta\delta), \\ m^p_{\epsilon,\delta}(t - U_\epsilon(\theta\delta)) & \text{if } U_\epsilon(\theta\delta) \le t \le U_\epsilon(\delta), \\ t + U_\epsilon(\delta)(m^{p-1}_{\epsilon,\delta} - 1) & \text{if } t \ge U_\epsilon(\delta), \end{cases}$$

and

$$G_{\epsilon,\delta}(t) = \int_0^t g'_{\epsilon,\delta}(\tau)^{\frac{1}{p}}\,d\tau = \begin{cases} 0 & \text{if } 0 \le t \le U_\epsilon(\theta\delta), \\ m_{\epsilon,\delta}(t - U_\epsilon(\theta\delta)) & \text{if } U_\epsilon(\theta\delta) \le t \le U_\epsilon(\delta), \\ t & \text{if } t \ge U_\epsilon(\delta). \end{cases}$$

The functions $g_{\epsilon,\delta}$ and $G_{\epsilon,\delta}$ are nondecreasing and absolutely continuous. Consider the radially symmetric nonincreasing function

$$u_{\epsilon,\delta}(r) = G_{\epsilon,\delta}(U_\epsilon(r)), \tag{4.13}$$

which satisfies

$$u_{\epsilon,\delta}(r) = \begin{cases} U_\epsilon(r) & \text{if } r \le \delta, \\ 0 & \text{if } r \ge \theta\delta. \end{cases}$$

We have the following estimates for $u_{\epsilon,\delta}$, which were proved in [20, Lemma 2.7].

Lemma 4.9 (Norm Estimates). *There exists a constant $C = C(n, p, s) > 0$ such that for any $0 < \epsilon \le \frac{\delta}{2}$, the following estimates hold:*

$$\int_{\mathbb{R}^{2n}} \frac{|u_{\epsilon,\delta}(x) - u_{\epsilon,\delta}(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \le S^{\frac{n}{ps}} + \mathcal{O}\Big(\Big(\frac{\epsilon}{\delta}\Big)^{\frac{n-ps}{p-1}}\Big), \qquad \int_{\mathbb{R}^n} |u_{\epsilon,\delta}(x)|^{p_s^*}\,dx \ge S^{\frac{n}{ps}} - C\Big(\Big(\frac{\epsilon}{\delta}\Big)^{\frac{n}{p-1}}\Big).$$

Next, we prove an important technical lemma. This is the only point where we use conditions (1.5) on p, s, q, n.

Lemma 4.10 ($c^-_{\lambda,\mu} < c_\infty$). *Assume that conditions* (1.5) *hold. Then there exists $\Lambda_2 > 0$ such that, for λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_2$, there exists $(u, v) \in E \setminus \{(0, 0)\}$, with $u \ge 0$, $v \ge 0$, such that*

$$\sup_{t\ge 0} J_{\lambda,\mu}(tu, tv) < c_\infty,$$

where c_∞ is the constant given in (3.13). *In particular, $c^-_{\lambda,\mu} < c_\infty$ for all λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_2$.*

Proof. Write $J_{\lambda,\mu}(u, v) = J(u, v) - K(u, v)$ where the functions $J\colon E \to \mathbb{R}$ and $K\colon E \to \mathbb{R}$ are defined by

$$J(u, v) = \frac{1}{p}\|(u, v)\|^p - \frac{2}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta\,dx, \quad K(u, v) = \frac{1}{q}\int_\Omega (\lambda|u|^q + \mu|v|^q)\,dx.$$

Set $u_0 := \alpha^{\frac{1}{p}} u_{\epsilon,\delta}$, $v_0 := \beta^{\frac{1}{p}} u_{\epsilon,\delta}$, where $u_{\epsilon,\delta}$ is defined by (4.13). The map $h(t) := J(tu_0, tv_0)$ satisfies $h(0) = 0$, $h(t) > 0$ for $t > 0$ small, and $h(t) < 0$ for $t > 0$ large. Moreover, h maximizes at the point

$$t_* := \Big(\frac{\|(u_0, v_0)\|^p}{2\int_\Omega |u_0|^\alpha |v_0|^\beta\,dx}\Big)^{\frac{1}{\alpha+\beta-p}}.$$

Thus, we have

$$\begin{aligned} \sup_{t\ge 0} J(tu_0, tv_0) = h(t_*) &= \frac{t_*^p}{p}\|(u_0, v_0)\|^p - \frac{2t_*^{\alpha+\beta}}{\alpha+\beta}\int_\Omega |u_0|^\alpha |v_0|^\beta\,dx \\ &= \Big(\frac{1}{p} - \frac{1}{\alpha+\beta}\Big)\frac{\|(u_0, v_0)\|^{\frac{p(\alpha+\beta)}{\alpha+\beta-p}}}{(2\int_\Omega |u_0|^\alpha |v_0|^\beta\,dx)^{\frac{p}{\alpha+\beta-p}}} \\ &= \Big(\frac{1}{p} - \frac{1}{\alpha+\beta}\Big)\frac{(\alpha+\beta)^{\frac{\alpha+\beta}{\alpha+\beta-p}}}{2^{\frac{p}{\alpha+\beta-p}}\alpha^{\frac{\alpha}{\alpha+\beta-p}}\beta^{\frac{\beta}{\alpha+\beta-p}}}\frac{\|u_{\epsilon,\delta}\|_{X_0}^{\frac{p(\alpha+\beta)}{\alpha+\beta-p}}}{(\int_\Omega |u_{\epsilon,\delta}|^{\alpha+\beta}\,dx)^{\frac{p}{\alpha+\beta-p}}} \\ &= \frac{s}{n}\frac{1}{2^{\frac{n-ps}{ps}}}\Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{\alpha+\beta}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha+\beta}}\Big]^{\frac{n}{ps}}\Big[\frac{\|u_{\epsilon,\delta}\|^p_{X_0}}{(\int_\Omega |u_{\epsilon,\delta}|^{p_s^*}\,dx)^{\frac{p}{p_s^*}}}\Big]^{\frac{n}{ps}}. \end{aligned}$$

From Lemma 4.9 and (3.9), we have

$$\sup_{t\geq 0} J(tu_0, tv_0) \leq \frac{s}{n}\frac{1}{2^{\frac{n-ps}{ps}}}\Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{\alpha+\beta}}+\Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha+\beta}}\Big]^{\frac{n}{ps}}\Big[\frac{S^{\frac{n}{ps}}+\mathcal{O}((\frac{\epsilon}{\delta})^{\frac{n-ps}{p-1}})}{(S^{\frac{n}{ps}}-C((\frac{\epsilon}{\delta})^{\frac{n}{p-1}}))^{\frac{p}{p_s^*}}}\Big]^{\frac{n}{ps}}$$
$$\leq \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}}+\mathcal{O}\Big(\Big(\frac{\epsilon}{\delta}\Big)^{\frac{n-ps}{p-1}}\Big). \tag{4.14}$$

Let $\delta_1 > 0$ be such that for all λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \delta_1$, the following holds:

$$c_\infty = \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}} - C_0(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}) > 0.$$

We have

$$J_{\lambda,\mu}(tu_0, tv_0) \leq \frac{t^p}{p}\|(u_0, v_0)\|^p \leq Ct^p \quad \text{for } t \geq 0 \text{ and } \lambda, \mu > 0.$$

Thus, there exists $t_0 \in (0, 1)$ such that

$$\sup_{0\leq t\leq t_0} J_{\lambda,\mu}(tu_0, tv_0) < c_\infty$$

for all λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \delta_1$. Since $\alpha, \beta > 1$, from (4.13) and (4.14), it follows that

$$\sup_{t\geq t_0} J_{\lambda,\mu}(tu_0, tv_0) = \sup_{t\geq t_0}[J(tu_0, tv_0) - K(tu_0, tv_0)]$$
$$\leq \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}}+\mathcal{O}\Big(\Big(\frac{\epsilon}{\delta}\Big)^{\frac{n-ps}{p-1}}\Big)-\frac{t_0^q}{q}(\lambda\alpha^{\frac{q}{p}}+\mu\beta^{\frac{q}{p}})\int_{B(0,\delta)}|u_{\epsilon,\delta}|^q\,dx$$
$$\leq \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}}+\mathcal{O}\Big(\Big(\frac{\epsilon}{\delta}\Big)^{\frac{n-ps}{p-1}}\Big)-\frac{t_0^q}{q}(\lambda+\mu)\int_{B(0,\delta)}|u_{\epsilon,\delta}|^q\,dx.$$

Fix now $\delta > 0$ sufficiently small such that $B_{\theta\delta}(0) \Subset \Omega$ (we assume without loss of generality that $0 \in \Omega$), so that $\operatorname{supp}(u_{\epsilon,\delta}) \subset \Omega$, according to formula (4.13). By means of Lemma 4.8, for any $0 < \epsilon \leq \frac{\delta}{2}$, we have

$$\int_{B(0,\delta)}|u_{\epsilon,\delta}(x)|^q\,dx = \int_{B(0,\delta)}|U_\epsilon(x)|^q\,dx = \epsilon^{n-\frac{n-ps}{p}q}\int_{B(0,\frac{\delta}{\epsilon})}|U(x)|^q\,dx$$
$$\geq \epsilon^{n-\frac{n-ps}{p}q}\omega_{n-1}\int_1^{\delta/\epsilon}U(r)^q r^{n-1}\,dr \geq \epsilon^{n-\frac{n-ps}{p}q}\omega_{n-1}c_1^q\int_1^{\delta/\epsilon}r^{n-\frac{n-ps}{p-1}q-1}\,dr$$
$$\simeq C\begin{cases}\epsilon^{n-\frac{n-ps}{p}q} & \text{if } q > \frac{n(p-1)}{n-ps},\\ \epsilon^{n-\frac{n-ps}{p}q}|\log\epsilon| & \text{if } q = \frac{n(p-1)}{n-ps},\\ \epsilon^{\frac{(n-ps)q}{p(p-1)}} & \text{if } q < \frac{n(p-1)}{n-ps}.\end{cases}$$

Therefore, taking into account conditions (1.5), we have

$$\sup_{t\geq t_0} J_{\lambda,\mu}(tu_0, tv_0) \leq \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}} + C(\epsilon^{\frac{n-ps}{p-1}}) - C(\lambda+\mu)\begin{cases}\epsilon^{n-\frac{n-ps}{p}q} & \text{if } q > \frac{n(p-1)}{n-ps},\\ \epsilon^{n-\frac{n-ps}{p}q}|\log\epsilon| & \text{if } q = \frac{n(p-1)}{n-ps}.\end{cases}$$

For $\epsilon = (\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}} \in (0, \frac{\delta}{2})$, we get

$$\sup_{t\geq t_0} J_{\lambda,\mu}(tu_0, tv_0) \leq \frac{2s}{n}\Big(\frac{S_{\alpha,\beta}}{2}\Big)^{\frac{n}{ps}} + C(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})$$
$$- C(\lambda+\mu)\begin{cases}(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}(n-\frac{n-ps}{p}q)} & \text{if } q > \frac{n(p-1)}{n-ps},\\ (\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{n(p-1)}{p(n-ps)}}|\log(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})| & \text{if } q = \frac{n(p-1)}{n-ps}.\end{cases}$$

If $q > \frac{n(p-1)}{n-ps}$, we can choose $\delta_2 > 0$ such that for λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \delta_2$,

$$C(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}) - C(\lambda + \mu)(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}(n-\frac{n-ps}{p}q)} < -C_0(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}), \tag{4.15}$$

where C_0 is the positive constant defined in (3.5). In fact, (4.15) holds if

$$1 + \frac{p}{p-q}\frac{p-1}{n-ps}\Big(n - \frac{n-ps}{p}q\Big) < \frac{p}{p-q} \iff q > \frac{n(p-1)}{n-ps}.$$

If instead $q = \frac{n(p-1)}{n-ps}$, we can choose $\delta_3 > 0$ such that for λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \delta_3$,

$$C(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}) - C(\lambda + \mu)(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{n(p-1)}{p(n-ps)}}|\log(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})| < -C_0(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})$$

as $|\log(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})| \to +\infty$ for $\lambda, \mu \to 0$, and

$$(\lambda + \mu)(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{n(p-1)}{p(n-ps)}} \simeq (\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}).$$

Then, taking

$$\Lambda_2 = \min\Big\{\delta_1, \delta_2, \delta_3, \Big(\frac{\delta}{2}\Big)^{\frac{n-ps}{p-1}}\Big\} > 0,$$

for all λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_2$, we have

$$\sup_{t\geq 0} J_{\lambda,\mu}(tu, tv) < c_\infty. \tag{4.16}$$

Since $(u_0, v_0) \neq (0, 0)$, from Lemma 2.5 and (4.16), there exists $t_2 > 0$ such that $(t_2u_0, t_2v_0) \in \mathcal{N}^-_{\lambda,\mu}$ and

$$c^-_{\lambda,\mu} \leq J_{\lambda,\mu}(t_2u_0, t_2v_0) \leq \sup_{t\geq 0} J_{\lambda,\mu}(tu_0, tv_0) < c_\infty$$

for all λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_2$. This concludes the proof. □

Proposition 4.11 (Existence of the Second Solution). *There exists a positive constant $\Lambda_3 > 0$, such that for λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_3$, the functional $J_{\lambda,\mu}$ has a minimizer (u_2, v_2) in $\mathcal{N}^-_{\lambda,\mu}$ with the following properties:*
(i) $J_{\lambda,\mu}(u_2, v_2) = c^-_{\lambda,\mu}$,
(ii) (u_2, v_2) *is a solution of problem* (1.1).

Proof. Let Λ_2 be as in Lemma 4.10, and set

$$\Lambda_3 := \Big\{\Lambda_2, \Big(\frac{q}{p}\Big)^{\frac{p}{p-q}}\Lambda_1\Big\}.$$

By means of Proposition 4.6 (ii), for all λ, μ satisfying $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_3$, there exists a bounded $(PS)_{c^-_{\lambda,\mu}}$ sequence $(\tilde{u}_k, \tilde{v}_k)\} \subset \mathcal{N}^-_{\lambda,\mu}$ for $J_{\lambda,\mu}$. By the same argument used in the proof of Proposition 4.7, there exists $(u_2, v_2) \in E$ such that, up to a subsequence, $\tilde{u}_k \to u_2, \tilde{v}_k \to v_2$ strongly in E and $J_{\lambda,\mu}(u_2, v_2) = c^-_{\lambda,\mu}$. Moreover, (u_2, v_2) is a solution of problem (1.1).

Next we show that $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$. In fact, since $(\tilde{u}_k, \tilde{v}_k) \in \mathcal{N}^-_{\lambda,\mu}$, we have

$$\varphi''_{\tilde{u}_k,\tilde{v}_k}(1) = (p-q)\|(\tilde{u}_k, \tilde{v}_k)\|^p - 2((\alpha+\beta) - q)\int_\Omega |\tilde{u}_k|^\alpha|\tilde{v}_k|^\beta\,dx < 0.$$

Since $\tilde{u}_k \to u_2, \tilde{v}_k \to v_2$ strongly in E, passing to the limit, we obtain

$$\varphi''_{u_2,v_2}(1) = (p-q)\|(u_2, v_2)\|^p - 2((\alpha+\beta) - q)\int_\Omega |u_2|^\alpha|v_2|^\beta\,dx \leq 0.$$

Since $\mathcal{N}^0_{\lambda,\mu} = \emptyset$, we conclude that $\varphi''_{u_2,v_2}(1) < 0$, i.e., $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$. □

5 Proof of Theorem 1.2

Now we are ready to prove our main result.

Proof of Theorem 1.2. Taking $\Lambda_* = \min\{\Lambda_1, \Lambda_2, \Lambda_3\}$, by Propositions 4.7 and 4.11, we know that for all λ, μ satisfying

$$0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_*,$$

problem (1.1) has two solutions $(u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$ and $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$ in E. Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, these two solutions are distinct.

We next show that (u_1, v_1) and (u_2, v_2) are not semi-trivial. We know that

$$J_{\lambda,\mu}(u_1, v_1) < 0, \quad J_{\lambda,\mu}(u_2, v_2) > 0. \tag{5.1}$$

We note that if $(u, 0)$ (or $(0, v)$) is a semi-trivial solution of problem (1.1), then (1.1) reduces to

$$\begin{cases} (-\Delta)^s_p u = \lambda |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{5.2}$$

Then

$$J_{\lambda,\mu}(u, 0) = \frac{1}{p} \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}}\, dx\, dy - \frac{\lambda}{q} \int_\Omega |u|^q\, dx = -\frac{p-q}{pq} \|u\|^p_{X_0} < 0. \tag{5.3}$$

From (5.1) and (5.3), we get that (u_2, v_2) is *not* semi-trivial. Now we prove that (u_1, v_1) is not semi-trivial. Without loss of generality, we may assume that $v_1 \equiv 0$. Then u_1 is a nontrivial solution of (5.2), and

$$\|(u_1, 0)\|^p = \|u_1\|^p_{X_0} = \lambda \int_\Omega |u_1|^q\, dx > 0.$$

Moreover, we may choose $w \in X_0 \setminus \{0\}$ such that

$$\|(0, w)\|^p = \|w\|^p_{X_0} = \mu \int_\Omega |w|^q\, dx > 0.$$

By Lemma 2.5 there exists a unique $0 < t_1 < t_{\max}(u_1, w)$ such that $(t_1 u_1, t_1 w) \in \mathcal{N}^+_{\lambda,\mu}$, where

$$t_{\max}(u_1, w) = \left(\frac{(\alpha + \beta - q) \int_\Omega (\lambda |u_1|^q + \mu |w|^q)\, dx}{(\alpha + \beta - p) \|(u_1, w)\|^p} \right)^{\frac{1}{p-q}} = \left(\frac{\alpha + \beta - q}{\alpha + \beta - p} \right)^{\frac{1}{p-q}} > 1.$$

Furthermore,

$$J_{\lambda,\mu}(t_1 u_1, t_1 w) = \inf_{0 \le t \le t_{\max}} J_{\lambda,\mu}(t u_1, t w).$$

This together with the fact that $(u_1, 0) \in \mathcal{N}^+_{\lambda,\mu}$ imply that

$$c^+_{\lambda,\mu} \le J_{\lambda,\mu}(t_1 u_1, t_1 w) \le J_{\lambda,\mu}(u_1, w) < J_{\lambda,\mu}(u_1, 0) = c^+_{\lambda,\mu},$$

which is a contradiction. Hence, (u_1, v_1) is *not* semi-trivial too. The proof is now complete. □

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