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Soliton dynamics for CNLS systems with potentials

Eugenio Montefusco [a], Benedetta Pellacci [b] and Marco Squassina [c]

[a] *Dipartimento di Matematica, Sapienza Università di Roma, Piazzale A. Moro 5, I-00185 Roma, Italy*
E-mail: montefusco@mat.uniroma1.it
[b] *Dipartimento di Scienze Applicate, Università degli Studi di Napoli "Parthenope", CDN Isola C4, I-80143 Napoli, Italy*
E-mail: benedetta.pellacci@uniparthenope.it
[c] *Dipartimento di Informatica, Università degli Studi di Verona, Cá Vignal 2, Strada Le Grazie 15, I-37134 Verona, Italy*
E-mail: marco.squassina@univr.it

Abstract. The semiclassical limit of a weakly coupled nonlinear focusing Schrödinger system in presence of a nonconstant potential is studied. The initial data is of the form (u_1, u_2) with $u_i = r_i(\frac{x-\tilde{x}}{\varepsilon})\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}}$, where (r_1, r_2) is a real ground state solution, belonging to a suitable class, of an associated autonomous elliptic system. For ε sufficiently small, the solution (ϕ_1, ϕ_2) will been shown to have, locally in time, the form $(r_1(\frac{x-x(t)}{\varepsilon})\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\xi(t)}, r_2(\frac{x-x(t)}{\varepsilon})\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\xi(t)})$, where $(x(t), \xi(t))$ is the solution of the Hamiltonian system $\dot{x}(t) = \xi(t)$, $\dot{\xi}(t) = -\nabla V(x(t))$ with $x(0) = \tilde{x}$ and $\xi(0) = \tilde{\xi}$.

Keywords: weakly coupled nonlinear Schrödinger systems, concentration phenomena, semiclassical limit, orbital stability of ground states, soliton dynamics

1. Introduction and main result

1.1. Introduction

In recent years much interest has been devoted to the study of systems of weakly coupled nonlinear Schrödinger equations. This interest is motivated by many physical experiments especially in nonlinear optics and in the theory of Bose–Einstein condensates (see e.g. [1,17,24,26]). Existence results of *ground and bound states* solutions have been obtained by different authors (see e.g. [3,5,13,21,22,30]). A very interesting aspect regards the dynamics, in the semiclassical limit, of a *general solution*, that is to consider the nonlinear Schrödinger system

$$\begin{cases} \mathrm{i}\varepsilon\partial_t\phi_1 + \dfrac{\varepsilon^2}{2}\Delta\phi_1 - V(x)\phi_1 + \phi_1\big(|\phi_1|^{2p} + \beta|\phi_2|^{p+1}|\phi_1|^{p-1}\big) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \mathrm{i}\varepsilon\partial_t\phi_2 + \dfrac{\varepsilon^2}{2}\Delta\phi_2 - V(x)\phi_2 + \phi_2\big(|\phi_2|^{2p} + \beta|\phi_1|^{p+1}|\phi_2|^{p-1}\big) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \phi_1(0,x) = \phi_1^0(x), \qquad \phi_2(0,x) = \phi_2^0(x), \end{cases} \tag{1.1}$$

with $0 < p < 2/N$, $N \geqslant 1$, and $\beta > 0$ is a constant modeling the birefringence effect of the material. The potential $V(x)$ is a regular function in $\mathbb{R}^N$ modeling the action of external forces (see (1.11)), $\phi_i : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$ are complex valued functions and $\varepsilon > 0$ is a small parameter playing the rôle of Planck's constant. The task to be tackled with respect to this system is to recover the full dynamics of a solution $(\phi_1^\varepsilon, \phi_2^\varepsilon)$ as a point particle subjected to Galileian motion for the parameter ε sufficiently small. Since the famous papers [2,14,16], a large amount of work has been dedicated to this study in the case of a single Schrödinger equation and for a special class of solutions, namely standing wave solutions (see [4] and the references therein). When considering this particular kind of solutions one is naturally lead to study the following elliptic system corresponding to the physically relevant case $p = 1$ (that is Kerr nonlinearities)

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V(x)v = v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

so that the analysis reduces to the study of the asymptotic behavior of solutions of an elliptic system. The concentration of a least energy solution around the local minima (possibly degenerate) of the potential V has been studied in [27], where some sufficient and necessary conditions have been established. To our knowledge the semiclassical dynamics of different kinds of solutions of a single Schrödinger equation has been tackled in the series of papers [7,18,19] (see also [6] for recent developments on the long term soliton dynamics), assuming that the initial datum is of the form $r((x - \tilde{x})/\varepsilon)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}}$, where r is the unique ground state solution of an associated elliptic problem (see Eq. (1.8)) and $\tilde{x}, \tilde{\xi} \in \mathbb{R}^N$. This choice of initial data corresponds to the study of a different situation from the previous one. Indeed, it is taken into consideration the semiclassical dynamics of ground state solutions of the autonomous elliptic equation once the action of external forces occurs. In these papers it is proved that the solution is approximated by the ground state r – up to translations and phase changes – and the translations and phase changes are precisely related with the solution of a Newtonian system in $\mathbb{R}^N$ governed by the gradient of the potential V. Here we want to recover similar results for system (1.1) taking as initial data

$$\phi_1^0(x) = r_1\left(\frac{x - \tilde{x}}{\varepsilon}\right)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}}, \qquad \phi_2^0(x) = r_2\left(\frac{x - \tilde{x}}{\varepsilon}\right)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}}, \tag{1.3}$$

where the vector $R = (r_1, r_2)$ is a suitable ground state (see Definition 1.3) of the associated elliptic system

$$\begin{cases} -\dfrac{1}{2}\Delta r_1 + r_1 = r_1\big(|r_1|^{2p} + \beta|r_2|^{p+1}|r_1|^{p-1}\big) & \text{in } \mathbb{R}^N, \\ -\dfrac{1}{2}\Delta r_2 + r_2 = r_2\big(|r_2|^{2p} + \beta|r_1|^{p+1}|r_2|^{p-1}\big) & \text{in } \mathbb{R}^N. \end{cases} \tag{E}$$

When studying the dynamics of systems some new difficulties can arise. First of all, we have to take into account that, up to now, it is still not known if a uniqueness result (up to translations in $\mathbb{R}^N$) for real ground state solutions of (E) holds. This is expected, at least in the case where $\beta > 1$. Besides, also nondegeneracy properties (in the sense provided in [12,28]) are proved in some particular cases

[12,28]. These obstacles lead us to restrict the set of admissible ground state solutions we will take into consideration (see Definition 1.3) in the study of soliton dynamics.

Our first main result (Theorem 1.5) will give the desired asymptotic behaviour. Indeed, we will show that a solution which starts from (1.3) (for a suitable ground state R) will remain close to the set of ground state solutions, up to translations and phase rotations. Furthermore, in the second result (Theorem 1.9), we will prove that the mass densities associated with the solution ϕ_i converge – in the dual space of $C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ – to the delta measure with mass given by $\|r_i\|_{L^2}$ and concentrated along $x(t)$, solution to the (driving) Newtonian differential equation

$$\ddot{x}(t) = -\nabla V\big(x(t)\big), \qquad x(0) = \tilde{x}, \qquad \dot{x}(0) = \tilde{\xi}, \tag{1.4}$$

where $\tilde{x}$ and $\tilde{\xi}$ are fixed in the initial data of (1.1). A similar result for each single component of the momentum density is lost as a consequence of the birefringence effect. However, we can afford the desired result for a balance on the total momentum density. This shows that – in the semiclassical regime – the solution moves as a point particle under the Galileian law given by the Hamiltonian system (1.4). In the case of V constant our statements are related with the results obtained, by linearization procedure, in [31] for the single equation. Here, by a different approach, we show that (1.4) gives a modulation equation for the solution generated by the initial data (1.3). Although we cannot predict the shape of the solution, we know that the dynamic of the mass center is described by (1.4). The arguments will follow [7,18,19], where the case of a single Schrödinger equations has been considered. The main ingredients are the *conservation laws* of (1.1) and of the Hamiltonian associated with the ODE in (1.4) and a *modulational stability* property for a suitable class of ground state solutions for the associated autonomous elliptic system (E), recently proved in [28] by the authors in the same spirit of the works [31, 32] on scalar Schrödinger equations.

The problem for the single equation has been also studied using the WKB analysis (see, for example, [9] and the references therein), to our knowledge, there are no results for the system using this approach. Some of the arguments and estimates in the paper are strongly based upon those of [19]. On the other hand, for the sake of self-containedness, we prefer to include all the details in the proofs.

1.2. Admissible ground state solutions

Let $\mathbb{H}_\varepsilon$ be the space of the vectors $\Phi = (\phi_1, \phi_2)$ in $\mathbb{H} = H^1(\mathbb{R}^N; \mathbb{C}^2)$ endowed with the rescaled norm

$$\|\Phi\|^2_{\mathbb{H}_\varepsilon} = \frac{1}{\varepsilon^N}\|\Phi\|_2^2 + \frac{1}{\varepsilon^{N-2}}\|\nabla\Phi\|_2^2,$$

where $\|\Phi\|_2^2 = \|(\phi_1,\phi_2)\|_2^2 = \|\phi_1\|_2^2 + \|\phi_2\|_2^2$ and $\|\phi_i\|_2^2 = \|\phi_i\|_{L^2}^2$ is the standard norm in the Lebesgue space L^2 given by $\|\phi_i\|_2^2 = \int \phi_i(x)\bar{\phi}_i(x)\,\mathrm{d}x$.

We aim to study the semiclassical dynamics of a least energy solution of problem (E) once the action of external forces is taken into consideration.

In [3,22,30] it is proved that there exists a least action solution $R = (r_1, r_2) \neq (0,0)$ of (E) which has nonnegative components. Moreover, R is a solution to the following minimization problem (cf. Theorems 3.4 and 3.6 in [23]):

$$\mathcal{E}(R) = \min_{\mathcal{M}} \mathcal{E}, \quad \text{where } \mathcal{M} = \big\{U \in \mathbb{H}\colon\ \|U\|_2 = \|R\|_2\big\}, \tag{1.5}$$

where the functional $\mathcal{E}:\mathbb{H}\to\mathbb{R}$ is defined by

$$\mathcal{E}(U)=\frac{1}{2}\|\nabla U\|_2^2-\int F_\beta(U)\,\mathrm{d}x, \tag{1.6}$$

$$F_\beta(U)=\frac{1}{p+1}\bigl(|u_1|^{2p+2}+|u_2|^{2p+2}+2\beta|u_1|^{p+1}|u_2|^{p+1}\bigr) \tag{1.7}$$

for any $U=(u_1,u_2)\in\mathbb{H}$. We shall denote with $\mathcal{G}$ the set of the (complex) ground state solutions.

Remark 1.1. Any element $V=(v_1,v_2)$ of $\mathcal{G}$ has the form

$$V(x)=\bigl(\mathrm{e}^{\mathrm{i}\theta_1}|v_1(x)|,\mathrm{e}^{\mathrm{i}\theta_2}|v_2(x)|\bigr),\quad x\in\mathbb{R}^N,$$

for some $\theta_1,\theta_2\in S^1$ (so that $(|v_1|,|v_2|)$ is a real, positive, ground state solution). Indeed, if we consider the minimization problems

$$\sigma_{\mathbb{C}}=\inf\bigl\{\mathcal{E}(V)\colon V\in\mathbb{H},\|V\|_{L^2}=\|R\|_{L^2}\bigr\},$$
$$\sigma_{\mathbb{R}}=\inf\bigl\{\mathcal{E}(V)\colon V\in H^1\bigl(\mathbb{R}^N;\mathbb{R}^2\bigr)\|V\|_{L^2}=\|R\|_{L^2}\bigr\}$$

it results that $\sigma_{\mathbb{C}}=\sigma_{\mathbb{R}}$. Trivially one has $\sigma_{\mathbb{C}}\leqslant\sigma_{\mathbb{R}}$. Moreover, if $V=(v_1,v_2)\in\mathbb{H}$, due to the well-known pointwise inequality $|\nabla|v_i(x)||\leqslant|\nabla v_i(x)|$ for a.e. $x\in\mathbb{R}^N$, it holds

$$\int\bigl|\nabla|v_i(x)|\bigr|^2\,\mathrm{d}x\leqslant\int\bigl|\nabla v_i(x)\bigr|^2\,\mathrm{d}x,\quad i=1,2,$$

so that also $\mathcal{E}(|v_1|,|v_2|)\leqslant\mathcal{E}(V)$. In particular, we conclude that $\sigma_{\mathbb{R}}\leqslant\sigma_{\mathbb{C}}$, yielding the desired equality $\sigma_{\mathbb{C}}=\sigma_{\mathbb{R}}$. Let now $V=(v_1,v_2)$ be a *solution* to $\sigma_{\mathbb{C}}$ and assume by contradiction that, for some $i=1,2$,

$$\mathcal{L}^N\bigl(\bigl\{x\in\mathbb{R}^N\colon|\nabla|v_i|(x)|<\bigl|\nabla v_i(x)\bigr|\bigr\}\bigr)>0,$$

where $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$. Then $\|(|v_1|,|v_2|)\|_{L^2}=\|V\|_{L^2}$, and

$$\sigma_{\mathbb{R}}\leqslant\frac{1}{2}\sum_{i=1}^{2}\int\bigl|\nabla|v_i|\bigr|^2\,\mathrm{d}x-\int F_\beta\bigl(|v_1|,|v_2|\bigr)\,\mathrm{d}x<\frac{1}{2}\sum_{i=1}^{2}\int|\nabla v_i|^2\,\mathrm{d}x-\int F_\beta(v_1,v_2)\,\mathrm{d}x=\sigma_{\mathbb{C}},$$

which is a contradiction, being $\sigma_{\mathbb{C}}=\sigma_{\mathbb{R}}$. Hence, we have $|\nabla|v_i(x)||=|\nabla v_i(x)|$ for a.e. $x\in\mathbb{R}^N$ and any $i=1,2$. This is true if and only if $\mathcal{R}e\,v_i\nabla(\mathcal{I}m\,v_i)=\mathcal{I}m\,v_i\nabla(\mathcal{R}e\,v_i)$. In turn, if this last condition holds, we get

$$\bar{v}_i\nabla v_i=\mathcal{R}e\,v_i\nabla(\mathcal{R}e\,v_i)+\mathcal{I}m\,v_i\nabla(\mathcal{I}m\,v_i),\quad\text{a.e. in }\mathbb{R}^N,$$

which implies that $\mathcal{R}e(i\bar{v}_i(x)\nabla v_i(x))=0$ a.e. in $\mathbb{R}^N$. Finally, for any $i=1,2$, from this last identity one immediately finds $\theta_i\in S^1$ with $v_i=\mathrm{e}^{\mathrm{i}\theta_i}|v_i|$, concluding the proof.

In the scalar case, the ground state solution for the equation

$$-\frac{1}{2}\Delta r + r = r^{2p+1} \quad \text{in } \mathbb{R}^N \tag{1.8}$$

is always unique (up to translations) and nondegenerate (see e.g. [20,25,31]). For system (E), in general, the uniqueness and nondegeneracy of ground state solutions is a delicate open question.

The so-called *modulational stability* property of ground states solutions plays an important rôle in soliton dynamics on finite time intervals. More precisely, in the scalar case, some delicate spectral estimates for the seld-adjoint operator $\mathcal{E}''(r)$ were obtained in [31,32], allowing to get the following energy convexity result.

Theorem 1.2. *Let r be a ground state solution of Eq.* (1.8) *with $p < 2/N$. Let $\phi \in H^1(\mathbb{R}^N, \mathbb{C})$ be such that $\|\phi\|_2 = \|r\|_2$ and define the positive number*

$$\Gamma_\phi = \inf_{\substack{y\in\mathbb{R}^N \\ \theta\in[0,2\pi)}} \big\|\phi(\cdot) - \mathrm{e}^{\mathrm{i}\theta} r(\cdot - y)\big\|_{H^1}^2.$$

Then there exist two positive constants $\mathcal{A}$ and C such that

$$\Gamma_\phi \leqslant C\big(\mathcal{E}(\phi) - \mathcal{E}(R)\big),$$

provided that $\mathcal{E}(\phi) - \mathcal{E}(R) < \mathcal{A}$.

For systems, we consider the following definition.

Definition 1.3. We say that a ground state solution $R = (r_1, r_2)$ of system (E) is admissible for the modulational stability property to hold, and we shall write that $R \in \mathcal{R}$, if $r_i \in H^2(\mathbb{R}^N)$ are radial, $|x| r_i \in L^2(\mathbb{R}^N)$, the corresponding solution $\phi_i(t)$ belongs to $H^2(\mathbb{R}^N)$ for all times $t > 0$ and the following property holds: let $\Phi \in \mathbb{H}$ be such that $\|\Phi\|_2 = \|R\|_2$ and define the positive number

$$\Gamma_\Phi := \inf_{\substack{y\in\mathbb{R}^N \\ \theta_1,\theta_2\in[0,2\pi)}} \big\|\Phi(\cdot) - \big(\mathrm{e}^{\mathrm{i}\theta_1} r_1(\cdot - y), \mathrm{e}^{\mathrm{i}\theta_2} r_2(\cdot - y)\big)\big\|_{\mathbb{H}}^2. \tag{1.9}$$

Then there exist a continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\frac{\rho(\xi)}{\xi} \to 0$ as $\xi \to 0^+$ and a positive constant C such that

$$\rho(\Gamma_\Phi) + \Gamma_\Phi \leqslant C\big(\mathcal{E}(\Phi) - \mathcal{E}(R)\big).$$

In particular, there exist two positive constants $\mathcal{A}$ and C' such that

$$\Gamma_\Phi \leqslant C'\big(\mathcal{E}(\Phi) - \mathcal{E}(R)\big), \tag{1.10}$$

provided that $\Gamma_\Phi < \mathcal{A}$.

In the one-dimensional case, for an important physical class, there exists a ground state solution of system (E) which belongs to the class $\mathcal{R}$ (see [28]).

Theorem 1.4. *Assume that $N = 1$, $p \in [1,2)$ and $\beta > 1$. Then there exists a ground state solution $R = (r_1, r_2)$ of system* (E) *which belongs to the class $\mathcal{R}$.*

1.3. Statement of the main results

The action of external forces is represented by a potential $V : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$V \text{ is a } C^3 \text{ function bounded with its derivatives,} \tag{1.11}$$

and we will study the asymptotic behavior (locally in time) as $\varepsilon \to 0$ of the solution of the following Cauchy problem

$$\begin{cases} \mathrm{i}\varepsilon\,\partial_t\phi_1 + \dfrac{\varepsilon^2}{2}\Delta\phi_1 - V(x)\phi_1 + \phi_1\big(|\phi_1|^{2p} + \beta|\phi_2|^{p+1}|\phi_1|^{p-1}\big) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \mathrm{i}\varepsilon\,\partial_t\phi_2 + \dfrac{\varepsilon^2}{2}\Delta\phi_2 - V(x)\phi_2 + \phi_2\big(|\phi_2|^{2p} + \beta|\phi_1|^{p+1}|\phi_2|^{p-1}\big) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \phi_1(x,0) = r_1\Big(\dfrac{x-\tilde{x}}{\varepsilon}\Big)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}}, \qquad \phi_2(x,0) = r_2\Big(\dfrac{x-\tilde{x}}{\varepsilon}\Big)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}}, \end{cases} \tag{S_ε}$$

where $\tilde{x}, \tilde{\xi} \in \mathbb{R}^N$ $N \geqslant 1$, the exponent p is such that

$$0 < p < 2/N. \tag{1.12}$$

It is known (see [15]) that, under these assumptions, and for any initial datum in L^2, there exists a unique solution $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ of the Cauchy problem that exists globally in time. We have chosen as initial data a scaling of a real vector $R = (r_1, r_2)$ belonging to $\mathcal{R}$.

The first main result is the following theorem.

Theorem 1.5. *Let $R = (r_1, r_2)$ be a ground state solution of* (E) *which belongs to the class $\mathcal{R}$. Under assumptions* (1.11), (1.12)*, let $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ be the family of solutions to system* (S_ε)*. Furthermore, let $(x(t), \xi(t))$ be the solution of the Hamiltonian system*

$$\begin{cases} \dot{x}(t) = \xi(t), \\ \dot{\xi}(t) = -\nabla V\big(x(t)\big), \\ x(0) = \tilde{x}, \\ \xi(0) = \tilde{\xi}. \end{cases} \tag{1.13}$$

Then, there exists a locally uniformly bounded family of functions $\theta_i^\varepsilon : \mathbb{R}^+ \to S^1$, $i = 1, 2$, such that, defining the vector $Q_\varepsilon(t) = (q_1^\varepsilon(x,t), q_2^\varepsilon(x,t))$ by

$$q_i^\varepsilon(x,t) = r_i\Big(\frac{x - x(t)}{\varepsilon}\Big)\mathrm{e}^{(\mathrm{i}/\varepsilon)[x\cdot\xi(t) + \theta_i^\varepsilon(t)]},$$

it holds

$$\left\|\Phi^{\varepsilon}(t) - Q_{\varepsilon}(t)\right\|_{\mathbb{H}_{\varepsilon}} \leqslant \mathcal{O}(\varepsilon), \quad \text{as } \varepsilon \to 0 \tag{1.14}$$

locally uniformly in time.

Roughly speaking, the theorem states that, in the semiclassical regime, the modulus of the solution Φ^{ε} is approximated, locally uniformly in time, by the admissible real ground state (r_1, r_2) concentrated in $x(t)$, up to a suitable phase rotation. Theorem 1.5 can also be read as a description of the *slow dynamic* of the system close to the invariant manifold of the standing waves generated by ground state solutions. This topic has been studied, for the single equation, in [29].

Remark 1.6. Suppose that $\tilde{\xi} = 0$ and $\tilde{x}$ is a critical point of the potential V. Then the constant function $(x(t), \xi(t)) = (\tilde{x}, 0)$, for all $t \in \mathbb{R}^+$, is the solution to system (1.13). As a consequence, from Theorem 1.5, the approximated solutions is of the form

$$r_i\left(\frac{x - \tilde{x}}{\varepsilon}\right)\mathrm{e}^{(\mathrm{i}/\varepsilon)\theta_i^{\varepsilon}(t)}, \quad x \in \mathbb{R}^N, t > 0,$$

that is, in the semiclassical regime, the solution concentrates around the critical points of the potential V. This is a remark related to [27] where we have considered as initial data ground states solutions of an associated nonautonomous elliptic problem.

Remark 1.7. As a corollary of Theorem 1.5 we point out that, in the particular case of a constant potential, the approximated solution has components

$$r_i\left(\frac{x - \tilde{x} - \tilde{\xi}t}{\varepsilon}\right)\mathrm{e}^{(\mathrm{i}/\varepsilon)[x\cdot\tilde{\xi} + \theta_i^{\varepsilon}(t)]}, \quad x \in \mathbb{R}^N, t > 0.$$

Hence, the mass center $x(t)$ of $\Phi(t, x)$ moves with constant velocity $\tilde{\xi}$ realizing a uniform motion. This topic has been tackled, for the single equation, in [31].

Remark 1.8. For values of $\beta > 1$ both components of the ground states R are nontrivial and, for $R \in \mathcal{R}$, the solution of the Cauchy problem are approximated by a vector with both nontrivial components. We expect that ground state solutions for $\beta > 1$ are unique (up to translations in $\mathbb{R}^N$) and nondegenerate.

We can also analyze the behavior of total momentum density defined by

$$P^{\varepsilon}(x, t) := p_1^{\varepsilon}(x, t) + p_2^{\varepsilon}(x, t) \quad \text{for } x \in \mathbb{R}^N,\ t > 0, \tag{1.15}$$

where

$$p_i^{\varepsilon}(x, t) := \frac{1}{\varepsilon^{N-1}}\mathcal{I}m\big(\bar{\phi}_i^{\varepsilon}(x, t)\nabla\phi_i^{\varepsilon}(x, t)\big) \quad \text{for } i = 1, 2, x \in \mathbb{R}^N, t > 0. \tag{1.16}$$

Moreover, let $M(t) := (m_1 + m_2)\xi(t)$ be the total momentum of the particle $x(t)$ solution of (1.13), where

$$m_i := \|r_i\|_2^2 \quad \text{for } i = 1, 2. \tag{1.17}$$

The information about the asymptotic behavior of P^ε and of the mass densities $|\phi_i^\varepsilon|^2/\varepsilon^N$ are contained in the following result.

Theorem 1.9. *Under the assumptions of Theorem* 1.5*, there exists* $\varepsilon_0 > 0$ *such that*

$$\begin{aligned} &\big\| \big(|\phi_1^\varepsilon|^2/\varepsilon^N \,\mathrm{d}x, |\phi_2^\varepsilon|^2/\varepsilon^N \,\mathrm{d}x \big) - (m_1, m_2)\delta_{x(t)} \big\|_{(C^2\times C^2)^*} \leqslant \mathcal{O}(\varepsilon^2), \\ &\big\| P^\varepsilon(t,x)\,\mathrm{d}x - M(t)\delta_{x(t)} \big\|_{(C^2)^*} \leqslant \mathcal{O}(\varepsilon^2) \end{aligned}$$

for every $\varepsilon \in (0, \varepsilon_0)$ *and locally uniformly in time.*

Remark 1.10. Essentially, the theorem states that, in the semiclassical regime, the mass densities of the components ϕ_i of the solution Φ^ε behave as a point particle located in $x(t)$ of mass respectively m_i and the total momentum behaves like $M(t)\delta_{x(t)}$. It should be stressed that we can obtain the asymptotic behavior for each single mass density, while we can only afford the same result for the total momentum. The result will follow by a more general technical statement (Theorem 2.4).

Remark 1.11. The hypotheses on the potential V can be slightly weakened. Indeed, we can assume that V is bounded from below and that $\partial^\alpha V$ are bounded only for $|\alpha| = 2$ or $|\alpha| = 3$. This allows to include the important class of harmonic potentials (used e.g. in Bose–Einstein theory), such as

$$V(x) = \frac{1}{2}\sum_{j=1}^{N} \omega_j^2 x_j^2, \quad \omega_j \in \mathbb{R}, j = 1, \dots, N.$$

Hence, Eq. (1.13) reduces to the system of harmonic oscillators

$$\ddot{x}_j(t) + \omega_j^2 x_j(t) = 0, \quad j = 1, \dots, N. \tag{1.18}$$

For instance, in the 2D case, renaming $x_1(t) = x(t)$ and $x_2(t) = y(t)$ the ground states solutions are driven around (and concentrating) along the lines of a *Lissajous curves* having *periodic* or *quasi-periodic* behavior depending on the case when the ratio ω_i/ω_j is, respectively, a *rational* or an *irrational* number. See Figs 1 and 2 for the corresponding phase portrait in some 2D cases, depending on the values of ω_i/ω_j.

The paper is organized as follows.

In Section 2 we set up the main ingredients for the proofs as well as state two technical approximation results (Theorems 2.2 and 2.4) in a general framework. In Section 3 we will collect some preliminary technical facts that will be useful to prove the results. In Section 4 we will include the core computations regarding energy and momentum estimates in the semiclassical regime. Finally, in Section 5, the main results (Theorems 1.5 and 1.9) will be proved.

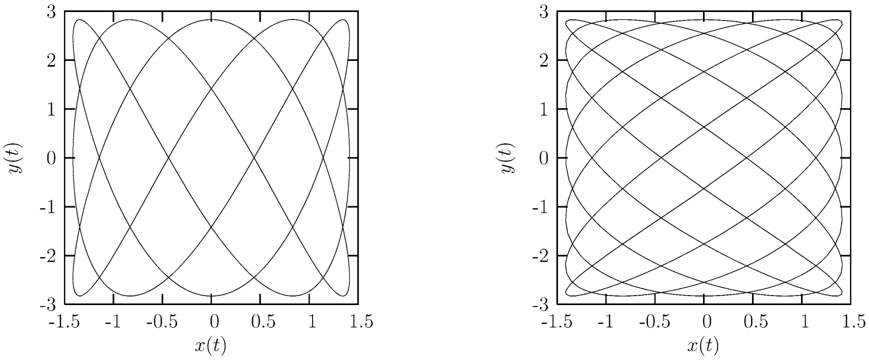

Fig. 1. Phase portrait of system (1.18) in 2D with $\omega_1/\omega_2 = 3/5$ (left) and $\omega_1/\omega_2 = 7/5$ (right). Notice the periodic behaviour.

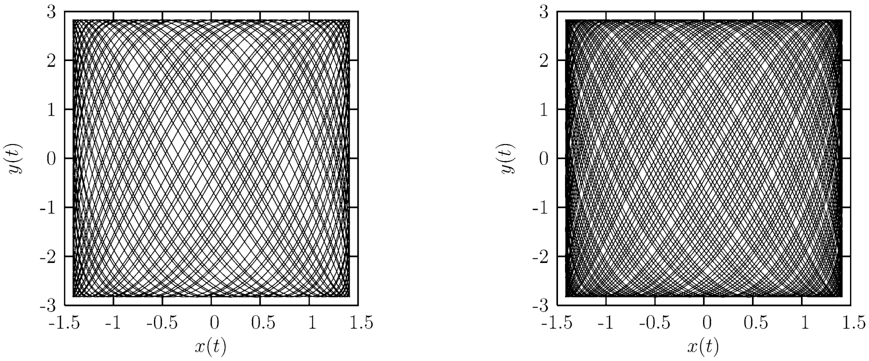

Fig. 2. Phase portrait of system (1.18) in 2D with $\omega_1/\omega_2 = \sqrt{3}/3$ increasing the integration time from $t \in [0, 40\pi]$ (left) to $t \in [0, 60\pi]$ (right). Notice the quasi-periodic behaviour, the plane is filling up.

2. A more general Schrödinger system

In the following sections we will study the behavior, for sufficiently small ε, of a solution $\Phi = (\phi_1, \phi_2)$ of the more general Schrödinger system

$$\begin{cases} \mathrm{i}\varepsilon\,\partial_t\phi_1 + \dfrac{\varepsilon^2}{2}\Delta\phi_1 - V(x)\phi_1 + \phi_1\big(|\phi_1|^{2p} + \beta|\phi_2|^{p+1}|\phi_1|^{p-1}\big) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \mathrm{i}\varepsilon\,\partial_t\phi_2 + \dfrac{\varepsilon^2}{2}\Delta\phi_2 - W(x)\phi_2 + \phi_2\big(|\phi_2|^{2p} + \beta|\phi_1|^{p+1}|\phi_2|^{p-1}\big) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \phi_1(0,x) = r_1\Big(\dfrac{x-\tilde{x}}{\varepsilon}\Big)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}_1}, \qquad \phi_2(0,x) = r_2\Big(\dfrac{x-\tilde{x}}{\varepsilon}\Big)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}_2}, \end{cases} \tag{F$_\varepsilon$}$$

where p verifies (1.12), the potentials V, W both satisfy (1.11) and (r_1, r_2) is a real ground state solution of problem (E). As for the case of a single potential, we get a unique globally defined $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ that depends continuously on the initial data (see, e.g. [15], Theorem 1). Moreover, if the initial data are chosen in $H^2 \times H^2$, then $\Phi^\varepsilon(t)$ enjoys the same regularity property for all positive times $t > 0$ (see e.g. [10]).

Remark 2.1. With no loss of generality, we can assume $V, W \geqslant 0$. Indeed, if ϕ_1, ϕ_2 is a solution to (F$_\varepsilon$), since V, W are bounded from below by (1.11), there exist $\mu > 0$ such that $V(x) + \mu \geqslant 0$ and

$W(x) + \mu \geqslant 0$, for all $x \in \mathbb{R}^N$. Then $\hat{\phi}_1 = \phi_1 \mathrm{e}^{-\mathrm{i}(\mu t/\varepsilon)}$ and $\hat{\phi}_2 = \phi_2 \mathrm{e}^{-\mathrm{i}(\mu t/\varepsilon)}$ is a solution of (F_ε) with $V + \mu$ (resp. $W + \mu$) in place of V (resp. W).

We will show that the dynamics of $(\phi_1^\varepsilon, \phi_2^\varepsilon)$ is governed by the solutions

$$X = (x_1, x_2) : \mathbb{R} \to \mathbb{R}^{2N}, \qquad \Theta = (\xi_1, \xi_2) : \mathbb{R} \to \mathbb{R}^{2N},$$

of the following Hamiltonian systems

$$\begin{cases} \dot{x}_1(t) = \xi_1(t), \\ \dot{\xi}_1(t) = -\nabla V\big(x_1(t)\big), \\ \big(x_1(0), \xi_1(0)\big) = (\tilde{x}, \tilde{\xi}_1), \end{cases} \qquad \begin{cases} \dot{x}_2(t) = \xi_2(t), \\ \dot{\xi}_2(t) = -\nabla W\big(x_2(t)\big), \\ \big(x_2(0), \xi_2(0)\big) = (\tilde{x}, \tilde{\xi}_2). \end{cases} \tag{H}$$

Notice that the Hamiltonians related to these systems are

$$H_1(t) = \frac{1}{2}\big|\xi_1(t)\big|^2 + V\big(x_1(t)\big), \qquad H_2(t) = \frac{1}{2}\big|\xi_2(t)\big|^2 + W\big(x_2(t)\big) \tag{2.1}$$

and are conserved in time. Under assumptions (1.11) it is immediate to check that the Hamiltonian systems (H) have global solutions. With respect to the asymptotic behavior of the solution of (F_ε) we can prove the following results.

2.1. *Two more general results*

We now state two technical theorems that will yield, as a corollary, Theorems 1.5 and 1.9.

Theorem 2.2. *Assume* (1.12) *and that V, W both satisfy* (1.11). *Let $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ be the family of solutions to system* (F_ε). *Then, there exist $\varepsilon_0 > 0$, $T_*^\varepsilon > 0$, a family of continuous functions $\varrho^\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ with $\varrho^\varepsilon(0) = \mathcal{O}(\varepsilon^2)$, locally uniformly bounded sequences of functions $\theta_i^\varepsilon : \mathbb{R}^+ \to S^1$ and a positive constant C, such that, defining the vector $Q_\varepsilon(t) = (q_1^\varepsilon(x,t), q_2^\varepsilon(x,t))$ by*

$$q_i^\varepsilon(x,t) = r_i\left(\frac{x - x_1(t)}{\varepsilon}\right)\mathrm{e}^{(\mathrm{i}/\varepsilon)[x\cdot\xi_i(t) + \theta_i^\varepsilon(t)]}, \quad i = 1, 2,$$

it results

$$\big\|\Phi^\varepsilon(t) - Q_\varepsilon(t)\big\|_{\mathbb{H}_\varepsilon} \leqslant C\sqrt{\varrho^\varepsilon(t) + \left(\frac{\varrho^\varepsilon(t)}{\varepsilon}\right)^2}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and all $t \in [0, T_*^\varepsilon]$, where $x_1(t)$ is the first component of the Hamiltonian system for V in (H).

Remark 2.3. Theorem 2.2 is quite instrumental in the context of our paper, as we cannot guarantee in the general case of different potentials that the function ϱ^ε is small as ε vanishes, locally uniformly in time. Moreover, the time dependent shifting of the components q_i into $x_1(t)$ is quite arbitrary, a similar statement could be written with the component $x_2(t)$ in place of $x_1(t)$, this arbitrariness is a consequence of the same initial data $\tilde{x}$ in (H) for both x_1 and x_2. The task of different initial data in (H) for x_1 and x_2 is to our knowledge an open problem.

In the following, if ξ_i are the second components of the systems in (H), we set

$$M(t) := m_1\xi_1(t) + m_2\xi_2(t), \quad t > 0. \tag{2.2}$$

If $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ is the family of solutions to (F_ε), we have the following theorem.

Theorem 2.4. *There exist $\varepsilon_0 > 0$ and $T_*^\varepsilon > 0$ and a family of continuous functions $\varrho^\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ with $\varrho^\varepsilon(0) = \mathcal{O}(\varepsilon^2)$ such that*

$$\begin{aligned}
&\big\|\big(|\phi_1^\varepsilon|^2/\varepsilon^N \,\mathrm{d}x, |\phi_2^\varepsilon|^2/\varepsilon^N \,\mathrm{d}x\big) - (m_1, m_2)\delta_{x_1(t)}\big\|_{(C^2\times C^2)^*} \leqslant \varrho^\varepsilon(t),\\
&\big\|P^\varepsilon(t,x)\,\mathrm{d}x - M(t)\delta_{x_1(t)}\big\|_{(C^2)^*} \leqslant \varrho^\varepsilon(t)
\end{aligned}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and all $t \in [0, T_*^\varepsilon]$.

3. Some preliminary results

In this section we recall and show some results we will use in proving Theorems 1.5, 1.9, 2.2 and 2.4. First we recall the following conservation laws.

Proposition 3.1. *The mass components of a solution Φ of (F_ε),*

$$\mathcal{N}_i^\varepsilon(t) := \frac{1}{\varepsilon^N}\big\|\phi_i^\varepsilon(t)\big\|_{L^2}^2 \quad \text{for } i = 1,2, t > 0, \tag{3.1}$$

are conserved in time. Moreover, also the total energy defined by

$$E^\varepsilon(t) = E_1^\varepsilon(t) + E_2^\varepsilon(t) \tag{3.2}$$

is conserved as time varies, where

$$\begin{aligned}
E_1^\varepsilon(t) &= \frac{1}{2\varepsilon^{N-2}}\big\|\nabla\phi_1^\varepsilon\big\|_{L^2}^2 + \frac{1}{\varepsilon^N}\int V(x)\big|\phi_1^\varepsilon\big|^2 \,\mathrm{d}x - \frac{1}{2\varepsilon^N}\int F_\beta\big(\Phi^\varepsilon\big)\,\mathrm{d}x,\\
E_2^\varepsilon(t) &= \frac{1}{2\varepsilon^{N-2}}\big\|\nabla\phi_2^\varepsilon\big\|_2^2 + \frac{1}{\varepsilon^N}\int W(x)\big|\phi_2^\varepsilon\big|^2 \,\mathrm{d}x - \frac{1}{2\varepsilon^N}\int F_\beta\big(\Phi^\varepsilon\big)\,\mathrm{d}x.
\end{aligned}$$

Proof. This is a standard fact. For the proof, see e.g. [15]. □

Remark 3.2. From the preceding proposition we obtain that, due to the form of our initial data, the mass components $\mathcal{N}_i^\varepsilon(t)$ do not actually depend on ε. Indeed, for $i = 1,2$,

$$\mathcal{N}_i^\varepsilon(t) = \mathcal{N}_i^\varepsilon(0) = \frac{1}{\varepsilon^N}\int \big|\phi_i^\varepsilon(x,0)\big|^2 \,\mathrm{d}x = \frac{1}{\varepsilon^N}\int \left|r_i\left(\frac{x-\tilde{x}}{\varepsilon}\right)\right|^2 \mathrm{d}x = m_i. \tag{3.3}$$

Thus, the quantities $\phi_i^\varepsilon/\varepsilon^{N/2}$ have constant norm in L^2 equal, respectively, to m_i. In Theorem 2.4 we will show that, for sufficiently small values of ε, the mass densities behave, point-wise with respect to t, as a δ functional concentrated in $x_1(t)$.

In the following we will often make use of the following simple lemma.

Lemma 3.3. *Let $A \in C^2(\mathbb{R}^N)$ be such that $A, D_j A, D^2_{ij} A$ are uniformly bounded and let $R = (r_1, r_2)$ be a ground state solution of problem* (E). *Then, for every $y \in \mathbb{R}^N$ fixed, there exists a positive constant C_0 such that*

$$\left| \int [A(\varepsilon x + y) - A(y)] r_i^2(x)\,\mathrm{d}x \right| \leqslant C_0 \varepsilon^2. \tag{3.4}$$

Proof. By virtue of the regularity properties of the function A and Taylor expansion theorem we get

$$\frac{1}{\varepsilon^2}\left| \int [A(\varepsilon x + y) - A(y)] r_i^2(x)\,\mathrm{d}x \right| \leqslant \frac{1}{\varepsilon}|\nabla A(y)| \left| \int x r_i^2(x)\,\mathrm{d}x \right| + \|\operatorname{Hes}(A)\|_\infty \int |x|^2 r_i^2(x)\,\mathrm{d}x,$$

where $\|\operatorname{Hes}(A)\|_\infty$ denotes the L^∞ norm of the Hessian matrix associated to the function A. The first integral on the right-hand side is zero since each component r_i is radial. The second integral is finite, since $|x| r_i \in L^2(\mathbb{R}^N)$. □

In order to show the desired asymptotic behavior we will use the following property of the functional δ_y on the space $C^2(\mathbb{R}^N)$.

Lemma 3.4. *There exist K_0, K_1, K_2 positive constants, such that, if $\|\delta_y - \delta_z\|_{C^{2*}} \leqslant K_0$ then*

$$K_1|y - z| \leqslant \|\delta_y - \delta_z\|_{C^{2*}} \leqslant K_2|y - z|.$$

Proof. For the proof see [19], Lemmas 3.1 and 3.2. □

The following lemma will be used in proving our main result.

Lemma 3.5. *Let $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ be a solution of* (F_ε) *and consider the vector functions $\alpha_i : \mathbb{R} \to \mathbb{R}^N$ defined by*

$$\alpha_i^\varepsilon(t) = \int p_i^\varepsilon(x,t)\,\mathrm{d}x - m_i \xi_i(t), \quad t > 0, i = 1, 2, \tag{3.5}$$

where the ξ_i's are defined in (H) *and the m_i's are defined in* (1.17)*, for $i = 1, 2$. Then $\{t \mapsto \alpha_i^\varepsilon(t)\}$ is a continuous function and $\alpha_i^\varepsilon(0) = 0$, for $i = 1, 2$.*

Remark 3.6. The integral in (3.5) defines a vector whose components are the integral of $\mathcal{I}m(\bar{\phi}_i^\varepsilon\, \partial\phi_i^\varepsilon / \partial x_j)/\varepsilon^{N-1}$ for $j = 1, \ldots, N$, so that $\alpha_i^\varepsilon : \mathbb{R} \to \mathbb{R}^N$.

Proof of Lemma 3.5. The continuity of α_i immediately follows from the regularity properties of the solution ϕ_i^ε. In order to complete the proof, first note that, for all $x \in \mathbb{R}^N$,

$$\bar{\phi}_i^\varepsilon(x,0) \nabla \phi_i^\varepsilon(x,0) = \frac{\mathrm{i}}{\varepsilon} \tilde{\xi}_i r_i^2\left(\frac{x - \tilde{x}}{\varepsilon}\right) + \frac{1}{\varepsilon} r_i\left(\frac{x - \tilde{x}}{\varepsilon}\right) \nabla r_i\left(\frac{x - \tilde{x}}{\varepsilon}\right),$$

so that, as r_i is a real function, the conclusion follows by a change of variable. □

Lemma 3.7. *Let V and W both satisfying assumptions* (1.11) *and let $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ be a solution of* (F_ε)*. Moreover, let A a positive constant defined by*

$$A = K_1 \sup_{[0,T_0]} \big[|x_1(t)| + |x_2(t)| \big] + K_0, \tag{3.6}$$

where $x_i(t)$ is defined in (H)*, K_0 and K_1 are defined in Lemma* 3.4*, and let χ be a $C^\infty(\mathbb{R}^N)$ function such that $0 \leqslant \chi \leqslant 1$ and*

$$\chi(x) = 1 \quad \textit{if } |x| < A, \qquad \chi(x) = 0 \quad \textit{if } |x| > 2A. \tag{3.7}$$

Then the functions

$$\begin{cases} \eta_1^\varepsilon(t) = m_1 V\big(x_1(t)\big) - \dfrac{1}{\varepsilon^N} \displaystyle\int \chi(x) V(x) \big|\phi_1^\varepsilon(x,t)\big|^2 \,\mathrm{d}x, \\ \eta_2^\varepsilon(t) = m_2 W\big(x_2(t)\big) - \dfrac{1}{\varepsilon^N} \displaystyle\int \chi(x) W(x) \big|\phi_2^\varepsilon(x,t)\big|^2 \,\mathrm{d}x \end{cases} \tag{3.8}$$

are continuous and satisfy $|\eta_i^\varepsilon(0)| = \mathcal{O}(\varepsilon^2)$ for $i = 1, 2$.

Proof. The continuity of η_i^ε immediately follows from the regularity properties of the solution ϕ_i^ε. We will prove the conclusion only for $\eta_1^\varepsilon(0)$, the result for $\eta_2^\varepsilon(0)$ can be showed in an analogous way. We have

$$\begin{aligned} \big|\eta_1^\varepsilon(0)\big| &= \left| m_1 V\big(x_1(0)\big) - \frac{1}{\varepsilon^N} \int \chi(x) V(x) \big|\phi_1^\varepsilon(x,0)\big|^2 \,\mathrm{d}x \right| \\ &\leqslant \left| m_1 V(\tilde{x}) - \frac{1}{\varepsilon^N} \int V(x) r_1^2\left(\frac{x-\tilde{x}}{\varepsilon}\right) \mathrm{d}x \right| + \frac{1}{\varepsilon^N} \int_{|x|>A} \big(1-\chi(x)\big) V(x) r_1^2\left(\frac{x-\tilde{x}}{\varepsilon}\right) \mathrm{d}x. \end{aligned}$$

Then, by Lemma 3.3, and a change of variables imply

$$\big|\eta_1^\varepsilon(0)\big| \leqslant \mathcal{O}\big(\varepsilon^2\big) + \int \big(1 - \chi(\tilde{x} + \varepsilon y)\big) V(\tilde{x} + \varepsilon y) r_1^2(y) \,\mathrm{d}y.$$

The properties of χ and r_1 and assumption (1.11) yield the conclusion. □

We will also use the following identities.

Lemma 3.8. *The following identities holds for $i = 1, 2$.*

$$\frac{1}{\varepsilon^N} \frac{\partial |\phi_i^\varepsilon|^2}{\partial t}(x,t) = -\operatorname{div}_x p_i^\varepsilon(x,t), \quad x \in \mathbb{R}^N, t > 0. \tag{3.9}$$

Moreover, for all $t > 0$, it results

$$\int \frac{\partial P^\varepsilon}{\partial t}(x,t) \,\mathrm{d}x = -\frac{1}{\varepsilon^N} \int \nabla V(x) \big|\phi_1^\varepsilon(x,t)\big|^2 \,\mathrm{d}x - \frac{1}{\varepsilon^N} \int \nabla W(x) \big|\phi_2^\varepsilon(x,t)\big|^2 \,\mathrm{d}x, \tag{3.10}$$

where $P^\varepsilon(x,t)$ is the total momentum density defined in (1.15)*.*

Remark 3.9. It follows from identity (3.10) that for systems with constant potentials the total momentum $\int P^\varepsilon \,\mathrm{d}x$ is a constant of motion.

Remark 3.10. As evident from identity (3.10) as well as physically reasonable, in the case of systems of Schrödinger equations, the balance for the momentum needs to be stated for the sum P^ε instead on the single components p_i^ε. See also identities (3.11) and (3.12) in the proof, where the coupling terms appear.

Proof of Lemma 3.8. In order to prove identity (3.9) note that

$$-\operatorname{div}_x p_i^\varepsilon = -\frac{1}{\varepsilon^{N-1}}\mathcal{I}m(\bar\phi_i^\varepsilon \Delta\phi_i^\varepsilon), \qquad \frac{1}{\varepsilon^N}\frac{\partial|\phi_i^\varepsilon|^2}{\partial t} = \frac{2}{\varepsilon^N}\mathcal{R}e\big((\phi_i^\varepsilon)_t\bar\phi_i^\varepsilon\big).$$

Since ϕ_i^ε solves the corresponding equation in system (F_ε), we can multiply the equation by $\bar\phi_i^\varepsilon$ and add this identity to its conjugate; the conclusion follows from the properties of the nonlinearity. Concerning identity (3.10), observe first that, setting $(p_1^\varepsilon)_j(x,t) = \varepsilon^{1-N}\mathcal{I}m(\bar\phi_1^\varepsilon(x,t)\,\partial_j\phi_1^\varepsilon(x,t))$ for any j and $\partial_j = \partial_{x_j}$, it holds

$$\begin{aligned}
\frac{\partial(p_1^\varepsilon)_j}{\partial t} &= \varepsilon^{1-N}\mathcal{I}m\big(\partial_t\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon\big) + \varepsilon^{1-N}\mathcal{I}m\big(\bar\phi_1^\varepsilon\,\partial_j(\partial_t\phi_1^\varepsilon)\big)\\
&= \varepsilon^{1-N}\mathcal{I}m\big(\partial_t\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon\big) + \varepsilon^{1-N}\mathcal{I}m\big(\partial_j(\bar\phi_1^\varepsilon\,\partial_t\phi_1^\varepsilon)\big) - \varepsilon^{1-N}\mathcal{I}m\big(\partial_j\bar\phi_1^\varepsilon\,\partial_t\phi_1^\varepsilon\big)\\
&= 2\varepsilon^{1-N}\mathcal{I}m\big(\partial_t\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon\big) + \varepsilon^{1-N}\mathcal{I}m\big(\partial_j(\bar\phi_1^\varepsilon\,\partial_t\phi_1^\varepsilon)\big).
\end{aligned}$$

In particular the second term integrates to zero. Concerning the first addendum, take the first equation of system (F_ε), conjugate it and multiply it by $2\varepsilon^{-N}\,\partial_j\phi_1$. It follows

$$\begin{aligned}
2\varepsilon^{1-N}\mathcal{I}m\big(\partial_t\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon\big) &= -\varepsilon^{2-N}\mathcal{R}e\big(\Delta\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon\big) + 2\varepsilon^{-N}V(x)\mathcal{R}e\big(\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon\big)\\
&\quad - 2\varepsilon^{-N}\big|\phi_1^\varepsilon\big|^{2p}\mathcal{R}e\big(\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon\big) - 2\beta\varepsilon^{-N}\big|\phi_2^\varepsilon\big|^{p+1}\big|\phi_1^\varepsilon\big|^{p-1}\mathcal{R}e\big(\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon\big)\\
&= -\varepsilon^{2-N}\mathcal{R}e\big(\partial_i(\partial_i\bar\phi_1^\varepsilon\,\partial_j\phi_1^\varepsilon)\big) + \varepsilon^{2-N}\,\partial_j\left(\frac{|\partial_i\phi_1^\varepsilon|^2}{2}\right)\\
&\quad + \varepsilon^{-N}\,\partial_j\big(V(x)\big|\phi_1^\varepsilon\big|^2\big) - \varepsilon^{-N}\,\partial_jV(x)\big|\phi_1^\varepsilon\big|^2\\
&\quad - \varepsilon^{-N}\,\partial_j\left(\frac{|\phi_1^\varepsilon|^{2p+2}}{p+1}\right) - 2\beta\varepsilon^{-N}|\phi_2^\varepsilon|^{p+1}\,\partial_j\left(\frac{|\phi_1^\varepsilon|^{p+1}}{p+1}\right).
\end{aligned}$$

Of course, one can argue in a similar fashion for the second component ϕ_2. Then, taking into account that all the terms in the previous identity but $\partial_jV(x)|\phi_1^\varepsilon|^2$ and $|\phi_2^\varepsilon|^{p+1}\partial_j|\phi_1^\varepsilon|^{p+1}$ integrate to zero due to the H^2 regularity of ϕ_1, we reach

$$\int\frac{\partial(p_1^\varepsilon)_j}{\partial t}\,\mathrm{d}x = -\frac{1}{\varepsilon^N}\int\frac{\partial V}{\partial x_j}(x)\big|\phi_1^\varepsilon\big|^2\,\mathrm{d}x - \frac{2\beta}{\varepsilon^N}\int\big|\phi_2^\varepsilon\big|^{p+1}\,\partial_j\left(\frac{|\phi_1^\varepsilon|^{p+1}}{p+1}\right)\mathrm{d}x, \tag{3.11}$$

$$\int\frac{\partial(p_2^\varepsilon)_j}{\partial t}\,\mathrm{d}x = -\frac{1}{\varepsilon^N}\int\frac{\partial W}{\partial x_j}(x)\big|\phi_2^\varepsilon\big|^2\,\mathrm{d}x - \frac{2\beta}{\varepsilon^N}\int|\phi_1^\varepsilon|^{p+1}\,\partial_j\left(\frac{|\phi_2^\varepsilon|^{p+1}}{p+1}\right)\mathrm{d}x. \tag{3.12}$$

Adding these identities for any j and taking into account that by the regularity properties of ϕ_i^ε it holds $\int \partial_j(|\phi_1^\varepsilon|^{p+1}|\phi_2^\varepsilon|^{p+1})\,\mathrm{d}x = 0$, formula (3.10) immediately follows. □

4. Energy, mass and momentum estimates

4.1. Energy estimates in the semiclassical regime

In order to obtain the desired asymptotic behavior stated in Theorems 1.5, 1.9, 2.2 and 2.4, we will first prove a key inequality concerning the functional $\mathcal{E}$ defined in (1.6). As pointed out in the Introduction, the main ingredients involved are the conservations laws of the Schrödinger system and of the Hamiltonians functions and a modulational stability property for admissible ground states.

The idea is to evaluate the functional $\mathcal{E}$ on the vector $\Upsilon^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$ whose components are given by

$$v_i^\varepsilon(x,t) = \mathrm{e}^{-(\mathrm{i}/\varepsilon)\xi_i(t)\cdot[\varepsilon x + x_1(t)]}\phi_i^\varepsilon\big(\varepsilon x + x_1(t), t\big), \tag{4.1}$$

where $X = (x_1, x_2), \Theta = (\xi_1, \xi_2)$ are the solution of the system (H). More precisely, we will prove the following result.

Theorem 4.1. *Let $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ be a family of solutions of* (F_ε), *and let Υ^ε be the vector defined in* (4.1). *Then, there exist ε_0 and T_*^ε such that for every $\varepsilon \in (0, \varepsilon_0)$ and for every $t \in [0, T_*^\varepsilon)$, it holds*

$$0 \leqslant \mathcal{E}(\Upsilon^\varepsilon) - \mathcal{E}(R) \leqslant \alpha^\varepsilon + \eta^\varepsilon + \mathcal{O}(\varepsilon^2), \tag{4.2}$$

where we have set

$$\alpha^\varepsilon(t) = \big|\big(\xi_1(t), \xi_2(t)\big) \cdot \big(\alpha_1^\varepsilon(t), \alpha_2^\varepsilon(t)\big)\big|, \qquad \eta^\varepsilon(t) = \big|\eta_1^\varepsilon(t) + \eta_2^\varepsilon(t)\big|, \tag{4.3}$$

α_i, η_i are given in (3.5), (3.8) *and $R = (r_1, r_2)$ is the real ground state belonging to the class $\mathcal{R}$ taken as initial datum in* (F_ε). *Moreover, there exist families of functions θ_i^ε, y_1^ε and a positive constant L such that*

$$\left\|\Phi^\varepsilon - \left(\mathrm{e}^{(\mathrm{i}/\varepsilon)(x\xi_1 + \theta_1^\varepsilon)} r_1\left(\frac{x - y_1^\varepsilon}{\varepsilon}\right), \mathrm{e}^{(\mathrm{i}/\varepsilon)(x\xi_2 + \theta_2^\varepsilon)} r_2\left(\frac{x - y_1^\varepsilon}{\varepsilon}\right)\right)\right\|_{\mathbb{H}_\varepsilon}^2 \leqslant L\big[\alpha^\varepsilon + \eta^\varepsilon + \mathcal{O}(\varepsilon^2)\big] \tag{4.4}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and all $t \in [0, T_*^\varepsilon)$.

Proof. By a change of variable and Proposition 3.1, we get

$$\big\|v_i^\varepsilon(\cdot,t)\big\|_2^2 = \big\|\phi_i^\varepsilon\big(\varepsilon x + x_1(t), t\big)\big\|_2^2 = \frac{1}{\varepsilon^N}\big\|\phi_i^\varepsilon(\cdot,t)\big\|_2^2 = m_i, \quad t > 0, i = 1, 2, \tag{4.5}$$

where m_i are defined in (1.17). Hence the mass of v_i^ε is conserved during the evolution. Moreover, by a change of variable, and recalling definition (1.16) we have

$$\mathcal{E}(\Upsilon^\varepsilon) = \frac{1}{2\varepsilon^{N-2}}\|\nabla\Phi^\varepsilon\|_2^2 + \frac{1}{2}\big(m_1|\xi_1|^2 + m_2|\xi_2|^2\big) - \frac{1}{\varepsilon^N}F_\beta(\Phi^\varepsilon)$$
$$- \int \big(\xi_1(t), \xi_2(t)\big) \cdot \big(p_1^\varepsilon(x,t), p_2^\varepsilon(x,t)\big)\,\mathrm{d}x.$$

Then, taking into account the form of the total energy functional, we obtain

$$\mathcal{E}(\Upsilon^\varepsilon) = E^\varepsilon(t) - \frac{1}{\varepsilon^N}\int \bigl[V(x)\bigl|\phi_1^\varepsilon\bigr|^2 + W(x)\bigl|\phi_2^\varepsilon\bigr|^2\bigr]\,\mathrm{d}x + \frac{1}{2}\bigl(m_1|\xi_1|^2 + m_2|\xi_2|^2\bigr)$$
$$- \int \bigl(\xi_1(t), \xi_2(t)\bigr)\cdot\bigl(p_1^\varepsilon(x,t), p_2^\varepsilon(x,t)\bigr)\,\mathrm{d}x.$$

Moreover, using Proposition 3.1 and performing a change of variable we get

$$\begin{aligned}
E^\varepsilon(t) &= E^\varepsilon(0)\\
&= E^\varepsilon\biggl(r_1\Bigl(\frac{x-\tilde{x}}{\varepsilon}\Bigr)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}_1}, r_2\Bigl(\frac{x-\tilde{x}}{\varepsilon}\Bigr)\mathrm{e}^{(\mathrm{i}/\varepsilon)x\cdot\tilde{\xi}_2}\biggr)\\
&= \mathcal{E}(R) + \frac{1}{2}\bigl(m_1|\tilde{\xi}_1|^2 + m_2|\tilde{\xi}_2|^2\bigr) + \int\bigl[V(\varepsilon x + \tilde{x})|r_1|^2 + W(\varepsilon x + \tilde{x})|r_2|^2\bigr]\,\mathrm{d}x,
\end{aligned}$$

this joint with Lemma 3.3 and the conservation of the Hamiltonians $H_i(t)$ yield

$$\begin{aligned}
\mathcal{E}(\Upsilon^\varepsilon) - \mathcal{E}(R) &= \frac{1}{2}\bigl[m_1\bigl(\bigl|\tilde{\xi}_1(t)\bigr|^2 + \bigl|\xi_1(t)\bigr|^2\bigr) + m_2\bigl(\bigl|\tilde{\xi}_2(t)\bigr|^2 + \bigl|\xi_2(t)\bigr|^2\bigr)\bigr]\\
&\quad - \int\bigl(\xi_1(t), \xi_2(t)\bigr)\cdot\bigl(p_1^\varepsilon(x,t), p_2^\varepsilon(x,t)\bigr)\,\mathrm{d}x\\
&\quad + m_1V(\tilde{x}) + m_2W(\tilde{x}) - \frac{1}{\varepsilon^N}\int\bigl[V(x)\bigl|\phi_1^\varepsilon\bigr|^2 + W(x)\bigl|\phi_2^\varepsilon\bigr|^2\bigr]\,\mathrm{d}x\\
&= m_1\bigl[\bigl|\xi_1(t)\bigr|^2 + V\bigl(x_1(t)\bigr)\bigr] + m_2\bigl[\bigl|\xi_2(t)\bigr|^2 + W\bigl(x_2(t)\bigr)\bigr]\\
&\quad - \int\bigl(\xi_1(t), \xi_2(t)\bigr)\cdot\bigl(p_1^\varepsilon(x,t), p_2^\varepsilon(x,t)\bigr)\,\mathrm{d}x\\
&\quad - \frac{1}{\varepsilon^N}\int\bigl[V(x)\bigl|\phi_1^\varepsilon\bigr|^2 + W(x)\bigl|\phi_2^\varepsilon\bigr|^2\bigr]\,\mathrm{d}x + \mathcal{O}\bigl(\varepsilon^2\bigr).
\end{aligned}$$

Using the definitions of α_i and η_i, we get

$$\begin{aligned}
\mathcal{E}(\Upsilon^\varepsilon) - \mathcal{E}(R) &\leqslant -\bigl(\xi_1(t), \xi_2(t)\bigr)\cdot\bigl(\alpha_1^\varepsilon(t), \alpha_2^\varepsilon(t)\bigr) + \eta^\varepsilon(t)\\
&\quad - \frac{1}{\varepsilon^N}\int\bigl(1-\chi(x)\bigr)\bigl[V(x)\bigl|\phi_1^\varepsilon\bigr|^2 + W(x)\bigl|\phi_2^\varepsilon\bigr|^2\bigr]\,\mathrm{d}x + \mathcal{O}\bigl(\varepsilon^2\bigr).
\end{aligned}$$

Since V and W are nonnegative functions, by (4.3) it follows that

$$\mathcal{E}\bigl(\Upsilon^\varepsilon(t)\bigr) - \mathcal{E}(R) \leqslant \alpha^\varepsilon(t) + \eta^\varepsilon(t) + \mathcal{O}\bigl(\varepsilon^2\bigr).$$

Finally, (1.5) and (4.5) imply the first conclusion of Theorem 4.1, where the positive time T_*^ε is built up as follows. Let $T_0 > 0$ (to be chosen later). In order to conclude the proof of the result, notice that $\alpha_i(t)$ and $\eta_i(t)$ are continuous functions by Lemmas 3.5 and 3.7. Moreover, let $\Gamma_{\Upsilon^\varepsilon(t)}$ be the positive number given in (1.9) for $\Phi = \Upsilon^\varepsilon(t)$. Notice that $\{t \mapsto \Gamma_{\Upsilon^\varepsilon(t)}\}$ is continuous and, in view of the choice of the initial data (1.3), it holds $\Gamma_{\Upsilon^\varepsilon(0)} = 0$. Hence, for every fixed $h_0, h_1 > 0$, we can define the time $T_*^\varepsilon > 0$ by

$$T_*^\varepsilon := \sup\bigl\{t \in [0, T^0]:\ \max\bigl\{\alpha^\varepsilon(s), \eta^\varepsilon(s)\bigr\} \leqslant h_0, \Gamma_{\Upsilon^\varepsilon(s)} \leqslant h_1, \text{ for all } s \in (0,t)\bigr\}. \tag{4.6}$$

Notice that, by (4.2) and choosing ε_0 sufficiently small we derive, for all $t \in [0, T_*^\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$, that $0 \leqslant \mathcal{E}(\Upsilon^\varepsilon(t)) - \mathcal{E}(R) \leqslant 3h_0$. Now we choose h_1 so small that $h_1 < \mathcal{A}$, where $\mathcal{A}$ is the constant appearing in the statement of the admissible ground state (see Definition (1.3)). Therefore, from conclusion (1.10), there exists a positive constant L such that

$$\Gamma_{\Upsilon^\varepsilon(t)} \leqslant L\bigl(\mathcal{E}\bigl(\Upsilon^\varepsilon(t)\bigr) - \mathcal{E}(R)\bigr) \leqslant L\bigl[\alpha^\varepsilon(t) + \eta^\varepsilon(t) + \mathcal{O}\bigl(\varepsilon^2\bigr)\bigr] \tag{4.7}$$

for every $t \in [0, T_*^\varepsilon)$ and all $\varepsilon \in (0, \varepsilon_0)$. In turn, there exist two families of functions $\tilde{y}^\varepsilon(t)$ and $\tilde{\theta}_i^\varepsilon(t)$ $i = 1, 2$ such that

$$\bigl\|\Upsilon^\varepsilon(\cdot, t) - \bigl(\mathrm{e}^{\mathrm{i}\tilde{\theta}_1^\varepsilon(t)} r_1\bigl(\cdot + \tilde{y}^\varepsilon(t)\bigr), \mathrm{e}^{\mathrm{i}\tilde{\theta}_2^\varepsilon(t)} r_2\bigl(\cdot + \tilde{y}^\varepsilon(t)\bigr)\bigr)\bigr\|_{\mathbb{H}}^2 \leqslant L\bigl[\alpha^\varepsilon(t) + \eta^\varepsilon(t) + \mathcal{O}\bigl(\varepsilon^2\bigr)\bigr] \tag{4.8}$$

for every $t \in [0, T_*^\varepsilon)$ and all $\varepsilon \in (0, \varepsilon_0)$. Making a change of variable and using the notation

$$\theta_i^\varepsilon(t) := \varepsilon\tilde{\theta}_i^\varepsilon(t), \qquad y_1^\varepsilon(t) := x_1(t) - \varepsilon\tilde{y}^\varepsilon(t),$$

the assertion follows. □

Remark 4.2. The previous result holds for every $t \in [0, T_*^\varepsilon)$ where T_*^ε is found in (4.6) and $T_*^\varepsilon \leqslant T_0$. But, we have not fixed T_0 yet. This will be done in Lemma 4.6.

4.2. Mass and total momentum estimates

The next lemmas will be used to prove the desired asymptotic behavior. We start with the study of the asymptotic behavior of the mass densities and the total momentum density. From now on we shall set

$$\hat{\alpha}^\varepsilon(t) := \alpha^\varepsilon(t) + \bigl|\alpha_1^\varepsilon(t) + \alpha_2^\varepsilon(t)\bigr|, \quad t > 0.$$

Lemma 4.3. *There exists a positive constant L_1 such that*

$$\begin{aligned}&\bigl\|\bigl(\bigl|\phi_1^\varepsilon\bigr|^2/\varepsilon^N\,\mathrm{d}x, \bigl|\phi_2^\varepsilon\bigr|^2/\varepsilon^N\,\mathrm{d}x\bigr) - (m_1, m_2)\delta_{y_1^\varepsilon(t)}\bigr\|_{(C^2\times C^2)^*} + \bigl\|P^\varepsilon(t,x)\,\mathrm{d}x - M(t)\delta_{y_1^\varepsilon(t)}\bigr\|_{(C^2)^*}\\ &\quad\leqslant L_1\bigl[\hat{\alpha}^\varepsilon(t) + \eta^\varepsilon(t) + \mathcal{O}\bigl(\varepsilon^2\bigr)\bigr]\end{aligned}$$

for every $t \in [0, T_*^\varepsilon]$ and $\varepsilon \in (0, \varepsilon_0)$.

Remark 4.4. This result will immediately imply Theorem 1.5, once we have shown the desired asymptotic behavior of $\hat{\alpha}^\varepsilon(t) + \eta^\varepsilon(t)$ and of the functional $\delta_{y_1^\varepsilon}$.

Proof of Lemma 4.3. For a given $v \in H^1$, a direct calculation yields

$$\big|\nabla|v|\big|^2 = \frac{1}{2}|\nabla v|^2 + \frac{1}{4|v|^2}\sum_{j=1}^{N}\big[(v_j)^2(\bar v)^2 + (v)^2(\bar v_j)^2\big] = |\nabla v|^2 - \frac{|\mathcal{I}m(\bar v \nabla v)|^2}{|v|^2},$$

where $v_j = v_{x_j}$ and, in the last term, it appears the square of the modulus of the vector whose components are $\mathcal{I}m(\bar v v_j)$. Then, we obtain

$$\mathcal{E}(\Upsilon^\varepsilon) = \mathcal{E}\big(|v_1^\varepsilon|, |v_2^\varepsilon|\big) + \int \frac{|\mathcal{I}m(\bar v_1^\varepsilon \nabla v_1^\varepsilon)|^2}{|v_1^\varepsilon|^2}\,\mathrm{d}x + \int \frac{|\mathcal{I}m(\bar v_2^\varepsilon \nabla v_2^\varepsilon)|^2}{|v_2^\varepsilon|^2}\,\mathrm{d}x.$$

In turn, using Theorem 4.1, it follows that, as ε vanishes,

$$\begin{aligned} 0 &\leqslant \mathcal{E}\big(|v_1^\varepsilon|, |v_2^\varepsilon|\big) - \mathcal{E}(R) + \int \frac{|\mathcal{I}m(\bar v_1^\varepsilon \nabla v_1^\varepsilon)|^2}{|v_1^\varepsilon|^2}\,\mathrm{d}x + \int \frac{|\mathcal{I}m(\bar v_2^\varepsilon \nabla v_2^\varepsilon)|^2}{|v_2^\varepsilon|^2}\,\mathrm{d}x \\ &\leqslant \alpha^\varepsilon(t) + \eta^\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big). \end{aligned}$$

Moreover, since $\|(|v_1^\varepsilon|, |v_2^\varepsilon|)\|_2 = \|(v_1^\varepsilon, v_2^\varepsilon)\|_2 = \|R\|_2$, we can conclude that

$$\int \frac{|\mathcal{I}m(\bar v_1^\varepsilon \nabla v_1^\varepsilon)|^2}{|v_1^\varepsilon|^2}\,\mathrm{d}x + \int \frac{|\mathcal{I}m(\bar v_2^\varepsilon \nabla v_2^\varepsilon)|^2}{|v_2^\varepsilon|^2}\,\mathrm{d}x \leqslant \alpha^\varepsilon(t) + \eta^\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big) \tag{4.9}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and for every $t \in [0, T_*^\varepsilon]$. Using (4.1) and (4.5), for any $i = 1, 2$ we get

$$\begin{aligned} \frac{|\mathcal{I}m(\bar v_i^\varepsilon \nabla v_i^\varepsilon)|^2}{|v_i^\varepsilon|^2} &= \frac{|\varepsilon \mathcal{I}m(\bar\phi_i^\varepsilon(\varepsilon x + x_1, t)\nabla\phi_i^\varepsilon(\varepsilon x + x_1, t)) - \xi_i|\phi_i^\varepsilon(\varepsilon x + x_1, t)|^2|^2}{|\phi_i^\varepsilon(\varepsilon x + x_1, t)|^2} \\ &= \varepsilon^2 \frac{|\mathcal{I}m(\bar\phi_i^\varepsilon(\varepsilon x + x_1, t)\nabla\phi_i^\varepsilon(\varepsilon x + x_1(t), t))|^2}{|\phi_i^\varepsilon(\varepsilon x + x_1, t)|^2} + \xi_i^2\big|\phi_i^\varepsilon(\varepsilon x + x_1, t)\big|^2 \\ &\quad - 2\varepsilon\xi_i \mathcal{I}m\big(\bar\phi_i^\varepsilon(\varepsilon x + x_1, t)\nabla\phi_i^\varepsilon\big(\varepsilon x + x_1(t), t\big)\big). \end{aligned}$$

Whence, performing a change of variable and using definition (1.16), we derive

$$\int \frac{|\mathcal{I}m(\bar v_i^\varepsilon \nabla v_i^\varepsilon)|^2}{|v_i^\varepsilon|^2}\,\mathrm{d}x = \varepsilon^N \int \frac{|p_i^\varepsilon(x,t)|^2}{|\phi_i^\varepsilon|^2}\,\mathrm{d}x + m_i\xi_i^2 - 2\xi_i \int p_i^\varepsilon(x,t)\,\mathrm{d}x. \tag{4.10}$$

Notice that

$$\begin{aligned} &\int \left|\varepsilon^{N/2}\frac{p_i^\varepsilon}{|\phi_i^\varepsilon|} - \frac{\int p_i^\varepsilon\,\mathrm{d}x}{m_i}\frac{|\phi_i^\varepsilon|}{\varepsilon^{N/2}}\right|^2 \mathrm{d}x + m_i\left|\xi_i - \frac{\int p_i^\varepsilon\,\mathrm{d}x}{m_i}\right|^2 \\ &\quad = \varepsilon^N \int \frac{|p_i^\varepsilon|^2}{|\phi_i^\varepsilon|^2}\,\mathrm{d}x - \frac{2}{m_i}\left[\int p_i^\varepsilon\,\mathrm{d}x\right]^2 + \frac{[\int p_i^\varepsilon]^2 \int |\phi_i^\varepsilon|^2}{\varepsilon^N m_i^2} + m_i\xi_i^2 + \frac{[\int p_i^\varepsilon]^2}{m_i} - 2\xi_i \int p_i^\varepsilon\,\mathrm{d}x \end{aligned}$$

which, by (4.5) is equal to (4.10). In turn, (4.9) implies that

$$\int \left|\varepsilon^{N/2}\frac{p_i^\varepsilon(x,t)}{|\phi_i^\varepsilon|} - \frac{\int p_i^\varepsilon\,\mathrm{d}x}{m_i}\frac{|\phi_i^\varepsilon|}{\varepsilon^{N/2}}\right|^2 + m_i\left|\xi_i - \frac{\int p_i^\varepsilon\,\mathrm{d}x}{m_i}\right|^2 \leqslant \alpha^\varepsilon(t) + \eta^\varepsilon(t) + \mathcal{O}(\varepsilon^2). \tag{4.11}$$

In order to prove the assertion, we need to estimate $\rho_i^\varepsilon(t)$ for $i = 1, 2$, where

$$\rho_i^\varepsilon(t) = \left|\frac{1}{\varepsilon^N}\int \psi(x)|\phi_i^\varepsilon|^2\,\mathrm{d}x - m_i\psi(y_1^\varepsilon)\right| + \left|\int P^\varepsilon(x,t)\psi(x)\,\mathrm{d}x - M(t)\psi(y_1^\varepsilon)\right| \tag{4.12}$$

for every function ψ in C^2 such that $\|\psi\|_{C^2} \leqslant 1$. From the definition of (3.5) it holds

$$\begin{aligned}
&\left|\int P^\varepsilon(x,t)\psi(x)\,\mathrm{d}x - M(t)\psi(y_1^\varepsilon)\right| \\
&\quad\leqslant \left|\int P^\varepsilon(x,t)\left[\psi(x) - \psi(y_1^\varepsilon)\right]\mathrm{d}x\right| + \left|\psi(y_1^\varepsilon)\left(\int P^\varepsilon(x,t)\,\mathrm{d}x - M(t)\right)\right| \\
&\quad\leqslant \left|\int P^\varepsilon(x,t)\left[\psi(x) - \psi(y_1^\varepsilon)\right]\mathrm{d}x\right| + \left|\alpha_1^\varepsilon(t) + \alpha_2^\varepsilon(t)\right| \\
&\quad\leqslant \sum_{i=1}^2 \frac{1}{m_i}\left|\int p_i^\varepsilon(x,t)\,\mathrm{d}x\right|\left|\int \frac{\psi(x)|\phi_i^\varepsilon(x,t)|^2}{\varepsilon^N}\,\mathrm{d}x - m_i\psi(y_1^\varepsilon)\right| \\
&\qquad + \sum_{i=1}^2\left|\int \psi(x)\left[p_i^\varepsilon(x,t) - \frac{1}{m_i}\left(\int p_i^\varepsilon(x,t)\,\mathrm{d}x\right)\frac{|\phi_i^\varepsilon(x,t)|^2}{\varepsilon^N}\right]\mathrm{d}x\right| \\
&\qquad + \left|\alpha_1^\varepsilon(t) + \alpha_2^\varepsilon(t)\right| + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and for every $t \in [0, T_*^\varepsilon]$. Taking into account that $\int p_i^\varepsilon\,\mathrm{d}x$ is uniformly bounded and that, of course,

$$\int \left[p_i^\varepsilon(x,t) - \frac{1}{m_i}\left(\int p_i^\varepsilon(x,t)\,\mathrm{d}x\right)\frac{|\phi_i^\varepsilon(x,t)|^2}{\varepsilon^N}\right]\mathrm{d}x = 0,$$

there exists a positive constant C_0 such that, if we set $\tilde{\psi}(x) = \psi(x) - \psi(y_1^\varepsilon)$, it holds

$$\begin{aligned}
\rho_i^\varepsilon(t) &\leqslant \frac{1}{\varepsilon^N}\int |\tilde{\psi}(x)|\left|\phi_i^\varepsilon(x,t)\right|^2\mathrm{d}x + \sum_{i=1}^2 \frac{C_0}{\varepsilon^N}\int |\tilde{\psi}(x)|\left|\phi_i^\varepsilon(x,t)\right|^2\mathrm{d}x \\
&\quad + \sum_{i=1}^2 \int |\tilde{\psi}(x)|\left|p_i^\varepsilon(x,t) - \frac{1}{m_i}\left(\int p_i^\varepsilon(x,t)\,\mathrm{d}x\right)\frac{|\phi_i^\varepsilon(x,t)|^2}{\varepsilon^N}\right|\mathrm{d}x \\
&\quad + \left|\alpha_1^\varepsilon(t) + \alpha_2^\varepsilon(t)\right| + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

From Young inequality and (4.11) it follows (from now on C_0 will denote a constant that can vary from line to line)

$$\begin{aligned}
\rho_i^\varepsilon(t) &\leqslant \frac{1}{\varepsilon^N}\sum_{i=1}^2 \int \Big[C_0|\tilde\psi(x)| + \frac{1}{2}|\tilde\psi(x)|^2\Big]|\phi_i^\varepsilon(x,t)|^2\,\mathrm{d}x \\
&\quad + \frac{1}{2}\sum_{i=1}^2 \int \bigg|\frac{p_i^\varepsilon(x,t)\varepsilon^{N/2}}{|\phi_i^\varepsilon(x,t)|} - \frac{1}{m_i}\Big(\int p_i^\varepsilon(x,t)\,\mathrm{d}x\Big)\frac{|\phi_i^\varepsilon(x,t)|}{\varepsilon^{N/2}}\bigg|^2 \\
&\quad + \big|\alpha_1^\varepsilon(t)+\alpha_2^\varepsilon(t)\big| + \mathcal{O}(\varepsilon^2) \\
&\leqslant \frac{1}{\varepsilon^N}\sum_{i=1}^2 \int \Big[C_0|\tilde\psi(x)| + \frac{1}{2}|\tilde\psi(x)|^2\Big]|\phi_i^\varepsilon(x,t)|^2 \\
&\quad + \big|\alpha_1^\varepsilon(t)+\alpha_2^\varepsilon(t)\big| + \frac{1}{2}\big[\alpha^\varepsilon(t)+\eta^\varepsilon(t)+\mathcal{O}(\varepsilon^2)\big].
\end{aligned}$$

Using the elementary inequality $a^2 \leqslant 2b^2 + 2(a-b)^2$ with

$$a = \frac{\phi_i^\varepsilon(x,t)}{\varepsilon^{N/2}}, \qquad b = \frac{1}{\varepsilon^{N/2}} r_i\Big(\frac{x-y_1^\varepsilon}{\varepsilon}\Big),$$

and recalling that $\tilde\psi$ is a uniformly bounded function we derive

$$\begin{aligned}
\rho_i^\varepsilon(t) &\leqslant \frac{1}{\varepsilon^N}\sum_{i=1}^2 \int \big[C_0|\tilde\psi(x)| + |\tilde\psi(x)|^2\big] r_i^2\Big(\frac{x-y_1^\varepsilon}{\varepsilon}\Big)\,\mathrm{d}x \\
&\quad + \frac{C_0}{\varepsilon^N}\sum_{i=1}^2 \int \bigg|\phi_i^\varepsilon(x,t) - r_i\Big(\frac{x-y_1^\varepsilon}{\varepsilon}\Big)\bigg|^2\,\mathrm{d}x \\
&\quad + \big|\alpha_1^\varepsilon(t)+\alpha_2^\varepsilon(t)\big| + \frac{1}{2}\big[\alpha^\varepsilon(t)+\eta^\varepsilon(t)+\mathcal{O}(\varepsilon^2)\big]
\end{aligned}$$

for every $\varepsilon \in (0,\varepsilon_0)$ and for every $t \in [0,T_*^\varepsilon]$. Notice that $\tilde\psi$ satisfies the hypothesis of Lemma 3.3 and $\tilde\psi(y_1^\varepsilon) = 0$, then by virtue of inequality (4.4) we obtain the conclusion. $\square$

4.3. Location estimates for y_1^ε

In the next results we start the study of the asymptotic behavior of y_1^ε.

Lemma 4.5. *Let us define the function*

$$\gamma^\varepsilon(t) = |\gamma_1^\varepsilon(t)| + |\gamma_2^\varepsilon(t)|, \quad \textit{with } \gamma_i^\varepsilon(t) = m_i x_i(t) - \frac{1}{\varepsilon^N}\int x\chi(x)|\phi_i^\varepsilon(x,t)|^2\,\mathrm{d}x, \tag{4.13}$$

where $\chi(x)$ is the characteristic function defined in (3.7). *Then $\gamma_i^\varepsilon(t)$ is a continuous function with respect to t and $|\gamma_i^\varepsilon(0)| = \mathcal{O}(\varepsilon^2)$ for $i = 1, 2$.*

Proof. The continuity of γ^ε immediately follows from the properties of the functions χ and ϕ_i^ε. In order to complete the proof, note that Lemma 3.3 implies

$$\left|\gamma_i^\varepsilon(0)\right| = \left|m_i\tilde{x} - \int(\tilde{x}+\varepsilon y)\chi(\tilde{x}+\varepsilon y)\left|r_i(y)\right|^2 \mathrm{d}y\right| \leqslant C_0\varepsilon^2 + \left|m_i\tilde{x} - \tilde{x}\chi(\tilde{x})\|r_i\|_{L^2}^2\right|$$

and as $\chi(\tilde{x}) = 1$ we reach the conclusion. □

Lemma 4.6. *Let T_*^ε be the time introduced in* (4.6). *There exist positive constants h_0 and ε_0 such that, if $|\eta_i^\varepsilon| \leqslant h_0$ and $\varepsilon \in (0, \varepsilon_0)$ there is a positive constant L_2 such that*

$$\left|x_1(t) - y_1^\varepsilon(t)\right| \leqslant L_2\left[\hat{\alpha}^\varepsilon(t) + \eta^\varepsilon(t) + \gamma^\varepsilon(t) + \mathcal{O}\left(\varepsilon^2\right)\right]$$

for every $t \in [0, T_*^\varepsilon]$.

Proof. First we show that there exist $T_0 > 0$ and $B > 0$ such that

$$\left|y_1^\varepsilon(t)\right| \leqslant B \tag{4.14}$$

for every $t \in [0, T_*^\varepsilon]$ with $T_*^\varepsilon \leqslant T_0$. Let us first prove that

$$\|\delta_{y_1^\varepsilon(t_2)} - \delta_{y_1^\varepsilon(t_1)}\|_{C^{2*}} < B \quad \text{for all } t_1, t_2 \in \left[0, T_*^\varepsilon\right].$$

Let $\psi \in C^2$ with $\|\psi\|_{C^2} \leqslant 1$ and pick $t_1, t_2 \in [0, T_*^\varepsilon]$ with $t_2 > t_1$. From identity (3.9) and integrating by parts, we obtain

$$\begin{aligned}
\frac{1}{\varepsilon^N}\int \psi(x)\left(\left|\phi_i^\varepsilon(x,t_2)\right|^2 - \left|\phi_i^\varepsilon(x,t_1)\right|^2\right)\mathrm{d}x &= \frac{1}{\varepsilon^N}\int_{t_1}^{t_2}\mathrm{d}t\int\psi(x)\frac{\partial|\phi_i^\varepsilon(x,t)|^2}{\partial t}\,\mathrm{d}x \\
&= -\int_{t_1}^{t_2}\mathrm{d}t\int\psi(x)\operatorname{div} p_i^\varepsilon(x,t)\,\mathrm{d}x \\
&\leqslant \|\nabla\psi\|_\infty\int_{t_1}^{t_2}\mathrm{d}t\int\left|p_i^\varepsilon(x,t)\right|\mathrm{d}x.
\end{aligned}$$

It is readily seen from the L^2 estimate of $\nabla\phi_i^\varepsilon$ that the last integral on the right-hand side is uniformly bounded, so that there exists a positive constant C_0 such that

$$\left\|\left|\phi_i^\varepsilon(x,t_2)\right|^2/\varepsilon^N\,\mathrm{d}x - \left|\phi_i^\varepsilon(x,t_1)\right|^2/\varepsilon^N\,\mathrm{d}x\right\|_{C^{2*}} \leqslant C_0|t_2 - t_1| \leqslant C_1T_0,$$

with $C_1 = 2C_0$. Then Lemmas 4.3, 3.5, 3.7 and 4.5 imply that the following inequality holds for sufficiently small ε and h_0 (the quantity α^ε should be replaced by $\hat{\alpha}^\varepsilon$ in the definition of T_*^ε)

$$\begin{aligned}
m_1\|\delta_{y_1^\varepsilon(t_2)} - \delta_{y_1^\varepsilon(t_1)}\|_{C^{2*}} &\leqslant C_1T_0 + L\left[\hat{\alpha}^\varepsilon(t_2) + \eta^\varepsilon(t_2) + \hat{\alpha}^\varepsilon(t_1) + \eta^\varepsilon(t_1) + \mathcal{O}\left(\varepsilon^2\right)\right] \\
&\leqslant C_1\left[T_0 + \mathcal{O}\left(\varepsilon^2\right) + h_0\right].
\end{aligned}$$

Here we fix T_0 and then ε_0, h_0 so small that $C_1[T_0+\mathcal{O}(\varepsilon^2)+h_0] < MK_0$ where K_0 is the constant fixed in Lemma 3.4 and from this lemma it follows

$$\left|y_1^\varepsilon(t_2) - y_1^\varepsilon(t_1)\right| \leqslant C_2K_0$$

for every $t_1, t_2 \leqslant T_0$, and since $y_1^\varepsilon(0) = \tilde{x}$ we obtain (4.14) for $B = C_2K_0 + |\tilde{x}|$. In view of property (4.14) we can now prove the assertion. Let us first observe that the properties of the function χ imply

$$\begin{aligned}\left|x_1(t) - y_1^\varepsilon(t)\right| &= \frac{1}{m_1}\left|m_1x_1(t) - m_1y_1^\varepsilon(t)\right| \\ &\leqslant \frac{1}{m_1}\left|\gamma_1^\varepsilon(t)\right| + \frac{1}{m_1}\left|\int \frac{1}{\varepsilon^N} x\chi(x)\left|\phi_1^\varepsilon(x,t)\right|^2 - m_1y_1^\varepsilon(t)\right|.\end{aligned}$$

Using (4.14) and (3.7) we obtain that $\chi(y_1^\varepsilon) = 1$, so that there exists a positive constant C^0 such that

$$\left|x_1(t) - y_1^\varepsilon(t)\right| \leqslant C_0\|x\chi\|_{C^2}\left\|\left|\phi_1^\varepsilon\right|^2/\varepsilon^N\,\mathrm{d}x - m_i\delta_{y_1^\varepsilon}\right\|_{C^{2*}} + C_0\gamma^\varepsilon(t).$$

This and Lemma 4.3 give the conclusion. □

In the previous lemma we have fixed T_0 such that also Lemma 4.3 and Theorem 4.1 hold and now we are able to prove Theorem 2.2.

Proof of Theorem 2.2. We start the proof from the second conclusion of Theorem 4.1. By Theorem 4.1, the family of continuous functions $\varrho^\varepsilon : \mathbb{R}^+ \to \mathbb{R}$,

$$\varrho^\varepsilon(t) = \hat{L}\left[\hat{\alpha}^\varepsilon(t) + \eta^\varepsilon(t) + \gamma^\varepsilon(t)\right], \quad \hat{L} = \max\{L, L_1, L_2\}, \tag{4.15}$$

is such that $\varrho^\varepsilon(0) = \mathcal{O}(\varepsilon^2)$ and it satisfies

$$\left\|\Phi^\varepsilon - \left(\mathrm{e}^{(\mathrm{i}/\varepsilon)(x\xi_1+\theta_1^\varepsilon)} r_1\left(\frac{x-y_1^\varepsilon}{\varepsilon}\right), \mathrm{e}^{(\mathrm{i}/\varepsilon)(x\xi_2+\theta_2^\varepsilon)} r_2\left(\frac{x-y_1^\varepsilon}{\varepsilon}\right)\right)\right\|_{\mathbb{H}_\varepsilon}^2 \leqslant \varrho^\varepsilon(t). \tag{4.16}$$

Moreover, Lemma 4.6 implies $|\varepsilon\tilde{y}^\varepsilon| = |x_1 - y_1^\varepsilon| \leqslant \varrho^\varepsilon(t)$, so that $|\tilde{y}^\varepsilon| = \frac{\varrho^\varepsilon(t)}{\varepsilon}$. Also,

$$\left\|r_i(\cdot) - r_i\left(\cdot - \tilde{y}^\varepsilon\right)\right\|_{H^1}^2 \leqslant \left|\tilde{y}^\varepsilon\right|^2\|\nabla r_i\|_{H^1}^2 \leqslant C\left(\frac{\varrho^\varepsilon(t)}{\varepsilon}\right)^2 \quad \text{for all } i = 1, 2.$$

Then

$$\left\|\Phi^\varepsilon - \left(\mathrm{e}^{(\mathrm{i}/\varepsilon)(x\cdot\xi_1+\theta_1^\varepsilon)} r_1\left(\frac{x-x_1(t)}{\varepsilon}\right), \mathrm{e}^{(\mathrm{i}/\varepsilon)(x\cdot\xi_2+\theta_2^\varepsilon)} r_2\left(\frac{x-x_1(t)}{\varepsilon}\right)\right)\right\|_{\mathbb{H}_\varepsilon}^2 \leqslant \tilde{\varrho}^\varepsilon(t),$$

where we have set

$$\tilde{\varrho}^\varepsilon(t) = \varrho^\varepsilon(t) + C\left(\varrho^\varepsilon(t)/\varepsilon\right)^2.$$

Since $\tilde{\varrho}^\varepsilon(0) = \mathcal{O}(\varepsilon^2)$, the assertion follows. □

Proof of Theorem 2.4. In view of definition (4.15), the assertion immediately follows by combining Lemmas 4.3, 4.6 and 3.4. □

4.4. Smallness estimates for $\hat{\alpha}^\varepsilon, \eta^\varepsilon, \gamma^\varepsilon$

In the next lemma, under the assumptions of Theorem 1.5, we complete the study of the asymptotic behaviour of system (S_ε) by obtaining the vanishing rate of the functions $\hat{\alpha}^\varepsilon$, η^ε and γ^ε as ε vanishes. The time T_0 is the one chosen in the proof of Lemma 4.6.

Lemma 4.7. *Consider the framework of Theorem* 1.5, *that is $V = W$ and $\tilde{\xi}_1 = \tilde{\xi}_2 = \tilde{\xi}$. Then there exists a positive constant $\bar{L}$ such that*

$$\hat{\alpha}^\varepsilon(t) + \eta^\varepsilon(t) + \gamma^\varepsilon(t) \leqslant \bar{L}(T^0)\varepsilon^2 \quad \textit{for every } t \in [0, T_0].$$

Proof. By the definition of $\alpha^\varepsilon(t)$ (see formula (4.3)) and taking into account that under the assumptions of Theorem 1.5 it holds $\xi_1 = \xi_2 = \xi$ (with respect to the notations of Theorem 4.1), there exists a positive constant C such that, for $t > 0$,

$$\hat{\alpha}^\varepsilon(t) = \alpha^\varepsilon(t) + \left|\alpha_1^\varepsilon(t) + \alpha_2^\varepsilon(t)\right| \leqslant \left(1 + \left|\xi(t)\right|\right)\left|\alpha_1^\varepsilon(t) + \alpha_2^\varepsilon(t)\right| \leqslant C\left|\alpha_1^\varepsilon(t) + \alpha_2^\varepsilon(t)\right|.$$

Hence, without loss of generality, we can replace in the previous theorems (in particular Theorem 1.5) the quantities $\alpha^\varepsilon(t)$ and $\hat{\alpha}^\varepsilon(t)$ with the absolute value $|\alpha_1^\varepsilon(t) + \alpha_2^\varepsilon(t)|$. In a similar fashion, it is possible to replace the quantity $\gamma^\varepsilon(t)$ defined in formula (4.13) with the value $|\gamma_1^\varepsilon(t) + \gamma_2^\varepsilon(t)|$. We will prove the desired assertion via Gronwall lemma, so that we will first show that there exists a positive constant $\bar{L}$ such that, for all $t \in [0, T_*^\varepsilon]$,

$$\hat{\alpha}^\varepsilon(t) \leqslant \mathcal{O}(\varepsilon^2) + \bar{L}\int_0^t \left[\hat{\alpha}^\varepsilon(t) + \eta^\varepsilon(t) + \gamma^\varepsilon(t)\right] \mathrm{d}t, \tag{4.17}$$

$$\eta^\varepsilon(t) \leqslant \mathcal{O}(\varepsilon^2) + \bar{L}\int_0^t \left[\hat{\alpha}^\varepsilon(t) + \eta^\varepsilon(t) + \gamma^\varepsilon(t)\right] \mathrm{d}t, \tag{4.18}$$

$$\gamma^\varepsilon(t) \leqslant \mathcal{O}(\varepsilon^2) + \bar{L}\int_0^t \left[\hat{\alpha}^\varepsilon(t) + \eta^\varepsilon(t) + \gamma^\varepsilon(t)\right] \mathrm{d}t. \tag{4.19}$$

Now, identity (3.10) of Lemma 3.8 yield

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_1^\varepsilon + \alpha_2^\varepsilon\right)(t)\right| \leqslant \|\nabla V\|_{C^2}\left\||\phi_1^\varepsilon|^2/\varepsilon^N - m_1\delta_{x(t)}\right\|_{C^{2*}} + \|\nabla V\|_{C^2}\left\||\phi_2^\varepsilon|^2/\varepsilon^N - m_2\delta_{x(t)}\right\|_{C^{2*}}.$$

Hence, using Lemmas 4.3 and 4.6 one obtains, for all $t \in [0, T_*^\varepsilon]$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_1^\varepsilon + \alpha_2^\varepsilon\right)(t)\right| \leqslant L_1\left[\hat{\alpha}^\varepsilon + \eta^\varepsilon + \gamma^\varepsilon + \mathcal{O}(\varepsilon^2)\right]$$

for some positive constant A_1, yielding inequality (4.17). As far as concern η^ε, using (3.9) and Lemmas 4.3 and 4.6 one has for $t \in [0, T_*^\varepsilon]$ that there exists a positive constant A_2 such that, for all

$t \in [0, T_*^\varepsilon]$,

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}(\eta_1^\varepsilon + \eta_2^\varepsilon)(t)\right| \leqslant \bigg| & m_1 \nabla V\big(x(t)\big) \cdot \xi(t) + m_2 \nabla V\big(x(t)\big) \cdot \xi(t) \\ & + \int \chi(x) V(x) \operatorname{div}_x p_1^\varepsilon(x,t) + \int \chi(x) V(x) \operatorname{div}_x p_2^\varepsilon(x,t)\bigg| \\ = & \left| \int \big[\nabla(\chi V)(p_1^\varepsilon + p_2^\varepsilon)(x,t) - \nabla V\big(x(t)\big) \cdot \big(m_1 \xi(t) + m_2 \xi(t)\big)\big]\, \mathrm{d}x \right| \\ \leqslant & \big\|\nabla(\chi V)\big\|_{C^2} \big\|P^\varepsilon(x,t)\,\mathrm{d}x - M(t)\delta_{x(t)}\big\|_{C^{2*}} \\ \leqslant & A_2\big[\hat{\alpha}^\varepsilon + \eta^\varepsilon + \gamma^\varepsilon + \mathcal{O}\big(\varepsilon^2\big)\big].\end{aligned}$$

Let us now come to γ^ε. By the properties of the function χ, identity (3.9), Lemmas 4.3 and 4.6 it follows that there exists a positive constant A_3 such that, for all $t \in [0, T_*^\varepsilon]$,

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1^\varepsilon + \gamma_2^\varepsilon)(t)\right| & = \left| \int \big[\nabla(x\chi) \cdot p_1^\varepsilon(x,t) + \nabla(x\chi) \cdot p_2^\varepsilon(x,t)\big]\, \mathrm{d}x - m_1 \xi(t) - m_2 \xi(t) \right| \\ & = \left| \int \big[\nabla(x\chi) \cdot P^\varepsilon(x,t) - \nabla(x\chi) \cdot M(t) \delta_{x(t)}\big]\, \mathrm{d}x \right| \\ & \leqslant \big\|\nabla(x\chi)\big\|_{C^2} \big\|P^\varepsilon(x,t)\,\mathrm{d}x - M(t)\delta_{x(t)}\big\|_{C^{2*}} \\ & \leqslant A_3\big[\hat{\alpha}^\varepsilon + \eta^\varepsilon + \gamma^\varepsilon + \mathcal{O}\big(\varepsilon^2\big)\big].\end{aligned}$$

Then inequalities (4.17)–(4.19) immediately follow from Lemmas 3.5, 3.7 and 4.5. The conclusion on $[0, T_*^\varepsilon]$ is now a simple consequence of the Gronwall lemma over $[0, T_*^\varepsilon]$. By the definition of T_*^ε and the continuity of α^ε, $\hat{\alpha}^\varepsilon$ and η^ε we have that $T_*^\varepsilon = T_0$ provided ε is chosen sufficiently small. To have this, one also has to take into account that, by construction (cf. formula (4.7)) and by the uniform smallness inequalities that we have just obtained over $[0, T_*^\varepsilon]$, we reach

$$\varGamma_{\Upsilon^\varepsilon(t)} \leqslant L\big[\alpha^\varepsilon(t) + \eta^\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big)\big] \leqslant \mathcal{O}\big(\varepsilon^2\big) \quad \text{for all } t \in \big[0, T_*^\varepsilon\big]. \qquad \square$$

$\square$

5. Proofs of the main results

Proof of Theorem 1.5. In light of Lemma 4.7 we have $\varrho^\varepsilon(t)$, $\tilde{\varrho}^\varepsilon(t) \leqslant \bar{L}\varepsilon^2$ for any $t \in [0, T_0]$. Hence, the conclusions hold in $[0, T_0]$ as a direct consequence of Theorem 2.2. Finally, taking as new initial data

$$\phi_i^0(x) := r_i\left(\frac{x - x(T_0)}{\varepsilon}\right) \mathrm{e}^{(\mathrm{i}/\varepsilon) x \cdot \xi(T_0)},$$

and taking as a new a guiding Hamiltonian system

$$\begin{cases} \dot{\bar{x}}(t) = \bar{\xi}(t), \\ \dot{\bar{\xi}}(t) = -\nabla V\big(\bar{x}(t)\big), \\ \bar{x}(0) = x(T_0), \qquad \bar{\xi}(0) = \xi(T_0), \end{cases}$$

the assertion is valid over $[T_0, 2T_0]$. Reiterating (T_0 only depends on the problem) the argument yields the assertion locally uniformly in time. □

Proof of Theorem 1.9. Combining definition (4.15) with the assertions of Lemmas 4.3 and 4.7, we obtain the property over the interval $[0, T_0]$. Then we can argue as in the proof of Theorem 1.5 to achieve the conclusion locally uniformly in time. □

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