Calculus of Variations

CrossMark

The Brezis–Nirenberg problem for the fractional *p*-Laplacian

Sunra Mosconi[1] · **Kanishka Perera**[2] ·
Marco Squassina[1] · **Yang Yang**[3]

Abstract We obtain nontrivial solutions to the Brezis–Nirenberg problem for the fractional *p*-Laplacian operator, extending some results in the literature for the fractional Laplacian. The quasilinear case presents two serious new difficulties. First an explicit formula for a minimizer in the fractional Sobolev inequality is not available when $p \neq 2$. We get around this difficulty by working with certain asymptotic estimates for minimizers recently obtained in (Brasco et al., Cal. Var. Partial Differ Equations 55:23, 2016). The second difficulty is the lack of a direct sum decomposition suitable for applying the classical linking theorem. We use an abstract linking theorem based on the cohomological index proved in (Yang and Perera, Ann. Sci. Norm. Super. Pisa Cl. Sci. doi:10.2422/2036-2145.201406_004, 2016) to overcome this difficulty.

Mathematics Subject Classification Primary 35R11 · 35J92 · 35B33; Secondary 35A15

Communicated by P. Rabinowitz.

✉ Marco Squassina
  marco.squassina@univr.it

  Sunra Mosconi
  sunrajohannes.mosconi@univr.it

  Kanishka Perera
  kperera@fit.edu

  Yang Yang
  yynjnu@126.com

1 Dipartimento di Informatica, Università degli Studi di Verona, Verona, Italy

2 Department of Mathematical Sciences, Florida Institute of Technology,
   150 W University Blvd, Melbourne, FL 32901, USA

3 School of Science, Jiangnan University, Wuxi 214122, Jiangsu, China

🖄 Springer

1 Introduction and main result

For $1 < p < \infty$, $s \in (0, 1)$, and $N > sp$, the fractional p-Laplacian $(-\Delta)_p^s$ is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s \, u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon(x)^c} \frac{|u(x) - u(y)|^{p-2} \, (u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N.$$

This definition is consistent, up to a normalization constant depending on N and s, with the usual definition of the linear fractional Laplacian operator $(-\Delta)^s$ when $p = 2$. There is, currently, a rapidly growing literature on problems involving these nonlocal operators. In particular, fractional p-eigenvalue problems have been studied in Brasco et al. [7], Brasco and Parini [6], Franzina and Palatucci [21], Iannizzotto and Squassina [30], and Lindgren and Lindqvist [35]. Regularity of solutions was obtained in Brasco and Lindgren [5], Di Castro et al. [16,17], Iannizzotto et al. [29], Kuusi et al. [32], and Lindgren [34]. Existence via Morse theory was investigated in Iannizzotto et al. [28]. This operator appears in some recent works, see [2,31] as well as [9] for the motivations, that led to its introduction.

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. We consider the problem

$$\begin{cases} (-\Delta)_p^s \, u = \lambda \, |u|^{p-2} \, u + |u|^{p_s^*-2} \, u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $p_s^* = Np/(N - sp)$ is the fractional critical Sobolev exponent. Let us recall the weak formulation of problem (1.1). Let

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx dy \right)^{1/p}$$

be the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = \left(|u|_p^p + [u]_{s,p}^p \right)^{1/p},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$. We work in the closed linear subspace

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\},$$

equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$, which is a uniformly convex Banach space. The imbedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, p_s^*]$ and compact for $r \in [1, p_s^*)$. A function $u \in W_0^{s,p}(\Omega)$ is a weak solution of problem (1.1) if

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} \, (u(x) - u(y)) \, (v(x) - v(y))}{|x - y|^{N+sp}} \, dx dy = \lambda \int_\Omega |u|^{p-2} \, uv \, dx$$

$$+ \int_\Omega |u|^{p_s^*-2} \, uv \, dx, \quad \forall v \in W_0^{s,p}(\Omega).$$

See [28] and the references therein for further details for this framework. In the semilinear case $p = 2$ problem (1.1) reduces to the critical fractional Laplacian problem

$$\begin{cases} (-\Delta)^s \, u = \lambda u + |u|^{2_s^*-2} \, u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.2}$$

where $\lambda > 0$ and $2_s^* = 2N/(N-2s)$. This nonlocal problem generalizes the well-known Brezis–Nirenberg problem, which has been extensively studied beginning with the seminal paper [8] (see, e.g., [1,10–13,18,22,24–27,45–47,49] and references therein). Consequently, many results known in the local case $s = 1$ have been extended to problem (1.2). In particular, Servadei [41,42] and Servadei and Valdinoci [43,44] have shown that problem (1.2) has a nontrivial weak solution in the following cases:

 (i) $2s < N < 4s$ and λ is sufficiently large;
 (ii) $N = 4s$ and λ is not an eigenvalue of $(-\Delta)^s$ in Ω;
(iii) $N > 4s$.

This extends to the fractional setting some well-known results of Brezis and Nirenberg [8], Capozzi et al. [10], Zhang [49], and Gazzola and Ruf [24] for critical Laplacian problems. In the present paper we consider the quasilinear case $p \neq 2$ of problem (1.1). This presents us with two serious new difficulties. Let

$$\dot{W}^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}$$

endowed with the norm $\| \cdot \| = [\cdot]$, and let

$$S = \inf_{u \in \dot{W}^{s,p}(\mathbb{R}^N) \backslash \{0\}} \frac{\|u\|^p}{|u|_{p_s^*}^p}, \tag{1.3}$$

which is positive by the fractional Sobolev inequality. Our first major difficulty is the lack of an explicit formula for a minimizer for S. It has been conjectured that all minimizers are of the form $c \, U(|x - x_0|/\varepsilon)$, where

$$U(x) = \frac{1}{\left(1 + |x|^{p'}\right)^{(N-sp)/p}}, \quad x \in \mathbb{R}^N,$$

$p' = p/(p-1)$ is the Hölder conjugate of p, $c \neq 0$, $x_0 \in \mathbb{R}^N$, and $\varepsilon > 0$. This has been proved in Lieb [33] for $p = 2$, but for $p \neq 2$ it is not even known if these functions are minimizers. We will get around this difficulty by working with certain asymptotic estimates for minimizers recently obtained in Brasco et al. [4].

Our second main difficulty, which is common to nonlinear eigenvalue problems, is that the linking arguments based on eigenspaces of $(-\Delta)^s$ used in the case $p = 2$ do *not* work when $p \neq 2$ since the nonlinear operator $(-\Delta)_p^s$ does not have linear eigenspaces. We will use a more general construction based on sublevel sets as in Perera and Szulkin [39] (see also Perera et al. [37, Proposition 3.23]). A similar construction based on the notion of cohomological linking was used in Degiovanni and Lancelotti [15]. Moreover, the standard sequence of variational eigenvalues of $(-\Delta)_p^s$ based on the genus does not give enough information about the structure of the sublevel sets to carry out this linking construction. Therefore we will use a different sequence of eigenvalues introduced in Iannizzotto et al. [28] that is based on the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [20].

Let us recall the definition of the cohomological index. Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \backslash \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{A}$,

and let $f^* : H^*(\mathbb{R}P^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} \sup\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\}, & \text{if } A \neq \emptyset \\ 0, & \text{if } A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}P^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega]$.

Example 1.1 The classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}P^{m-1} \subset \mathbb{R}P^\infty$, which induces isomorphisms on the cohomology groups H^q for $q \leq m-1$, so $i(S^{m-1}) = m$.

The following proposition summarizes the basic properties of this index.

Proposition 1.2 (Fadell-Rabinowitz [20]) *The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:*

(i_1) *Definiteness: $i(A) = 0$ if and only if $A = \emptyset$;*
(i_2) *Monotonicity: If there is an odd continuous map from A to B (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism;*
(i_3) *Dimension: $i(A) \leq \dim W$;*
(i_4) *Continuity: If A is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of A such that $i(N) = i(A)$. When A is compact, N may be chosen to be a δ-neighborhood $N_\delta(A) = \{u \in W : dist(u, A) \leq \delta\}$;*
(i_5) *Subadditivity: If A and B are closed, then $i(A \cup B) \leq i(A) + i(B)$;*
(i_6) *Stability: If SA is the suspension of $A \neq \emptyset$, obtained as the quotient space of $A \times [-1, 1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, then $i(SA) = i(A) + 1$;*
(i_7) *Piercing property: If A, A_0 and A_1 are closed, and $\varphi : A \times [0, 1] \to A_0 \cup A_1$ is a continuous map such that $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in A \times [0, 1]$, $\varphi(A \times [0, 1])$ is closed, $\varphi(A \times \{0\}) \subset A_0$ and $\varphi(A \times \{1\}) \subset A_1$, then $i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A)$;*
(i_8) *Neighborhood of zero: If U is a bounded closed symmetric neighborhood of 0, then $i(\partial U) = \dim W$.*

The Dirichlet spectrum of $(-\Delta)^s_p$ in Ω consists of those $\lambda \in \mathbb{R}$ for which the problem

$$\begin{cases} (-\Delta)^s_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.4}$$

has a nontrivial weak solution. Although a complete description of the spectrum is not known when $p \neq 2$, we can define an increasing and unbounded sequence of variational eigenvalues via a suitable minimax scheme. The standard scheme based on the genus does not give the index information necessary for our purposes here, so we will use the following scheme based on the cohomological index as in Iannizzotto et al. [28] (see also Perera [36]). Let

$$\Psi(u) = \frac{1}{|u|^p_p}, \quad u \in \mathcal{M} = \{u \in W^{s,p}_0(\Omega) : \|u\| = 1\}.$$

Then eigenvalues of problem (1.4) coincide with critical values of Ψ. We use the standard notation

$$\Psi^a = \{u \in \mathcal{M} : \Psi(u) \leq a\}, \quad \Psi_a = \{u \in \mathcal{M} : \Psi(u) \geq a\}, \quad a \in \mathbb{R}$$

for the sublevel sets and superlevel sets, respectively. Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, and set

$$\lambda_k := \inf_{M \in \mathcal{F},\, i(M) \geq k} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$

Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$ is a sequence of eigenvalues of problem (1.4), and

$$\lambda_k < \lambda_{k+1} \implies i(\Psi^{\lambda_k}) = i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}}) = k \tag{1.5}$$

(see Iannizzotto et al. [28, Proposition 2.4]). The asymptotic behavior of these eigenvalues was recently studied in Iannizzotto and Squassina [30]. Making essential use of the index information in (1.5), we will prove the following theorem.

Theorem 1.3 (Nonlocal Brezis–Nirenberg problem) *Let $1 < p < \infty$, $s \in (0, 1)$, $N > sp$, and $\lambda > 0$. Then problem (1.1) has a nontrivial weak solution in the following cases:*

  (i) $N = sp^2$ *and* $\lambda < \lambda_1$;
 (ii) $N > sp^2$ *and* λ *is not one of the eigenvalues* λ_k;
(iii) $N^2/(N + s) > sp^2$;
 (iv) $(N^3 + s^3 p^3)/N (N + s) > sp^2$ *and* $\partial\Omega \in C^{1,1}$.

This theorem extends to the fractional setting some well-known results of García Azorero and Peral Alonso [23], Egnell [19], Guedda and Véron [27], Arioli and Gazzola [3], and Degiovanni and Lancelotti [15] for critical p-Laplacian problems.

Weak solutions of problem (1.1) coincide with critical points of the C^1-functional

$$I_\lambda(u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{p} |u|_p^p - \frac{1}{p_s^*} |u|_{p_s^*}^{p_s^*}, \quad u \in W_0^{s,p}(\Omega). \tag{1.6}$$

Proof of Theorem 1.3 will be based on the following abstract critical point theorem proved in Yang and Perera (cf. [48, Theorem 2.2]).

Theorem 1.4 *Let W be a Banach space, let $S = \{u \in W : \|u\| = 1\}$ be the unit sphere in W, and let $\pi : W \setminus \{0\} \to S$, $u \mapsto u/\|u\|$ be the radial projection onto S. Let I be a C^1-functional on W and let A_0 and B_0 be disjoint nonempty closed symmetric subsets of S such that*

$$i(A_0) = i(S \setminus B_0) < \infty.$$

Assume that there exist $R > r > 0$ and $v \in S \setminus A_0$ such that

$$\sup I(A) \leq \inf I(B), \quad \sup I(X) < \infty,$$

where

$$A = \{tu : u \in A_0,\ 0 \leq t \leq R\} \cup \{R\pi((1 - t) u + tv) : u \in A_0,\ 0 \leq t \leq 1\},$$
$$B = \{ru : u \in B_0\},$$
$$X = \{tu : u \in A,\ \|u\| = R,\ 0 \leq t \leq 1\}.$$

Let $\Gamma = \{\gamma \in C(X, W) : \gamma(X) \text{ is closed and } \gamma|_A = id_A\}$, and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} I(u).$$

Then

$$\inf I(B) \leq c \leq \sup I(X), \tag{1.7}$$

in particular, c is finite. If, in addition, I satisfies the $(PS)_c$ *condition, then c is a critical value of I.*

Theorem 1.4 generalizes the linking theorem of Rabinowitz [40]. The linking construction in its proof was also used in Perera and Szulkin [39] to obtain nontrivial solutions of p-Laplacian problems with nonlinearities that interact with the spectrum. A similar construction based on the notion of cohomological linking was given in Degiovanni and Lancelotti [14]. See also Perera et al. [37, Proposition 3.23].

The following compactness result, proved in Perera et al. [38, Proposition 3.1], will be crucial for applying Theorem 1.4 to our functional I_λ.

Proposition 1.5 *Let* $1 < p < \infty$, $s \in (0, 1)$, $N > sp$, *and let S be as in* (1.3). *Then for any* $\lambda \in \mathbb{R}$, I_λ *satisfies the* $(PS)_c$ *condition for all* $c < \dfrac{s}{N} S^{N/sp}$.

Notations We use the following notations throughout the paper. For $a \in \mathbb{R}$ and $q > 0$, we write $a^q = |a|^{q-1} a$. For $1 \le q \le \infty$, $|\cdot|_q$ denotes the norm in $L^q(\Omega)$ and

$$q' = \begin{cases} \infty, & \text{if } q = 1 \\ q/(q-1), & \text{if } 1 < q < \infty \\ 1, & \text{if } q = \infty \end{cases}$$

is the Hölder conjugate of q.

2 Preliminaries

2.1 Minimizers for the Sobolev inequality

We have the following proposition from Brasco et al. [4] regarding the minimization problem (1.3).

Proposition 2.1 *Let* $1 < p < \infty$, $s \in (0, 1)$, $N > sp$, *and let S be as in* (1.3). *Then*

(i) *There exists a minimizer for S;*
(ii) *For every minimizer U, there exist* $x_0 \in \mathbb{R}^N$ *and a constant sign monotone function* $u : \mathbb{R} \to \mathbb{R}$ *such that* $U(x) = u(|x - x_0|)$;
(iii) *For every minimizer U, there exists* $\lambda_U > 0$ *such that*

$$\int_{\mathbb{R}^{2N}} \frac{(U(x) - U(y))^{p-1} (v(x) - v(y))}{|x - y|^{N+sp}} \, dx dy = \lambda_U \int_{\mathbb{R}^N} U^{p_s^* - 1} v \, dx \quad \forall v \in \dot{W}^{s,p}(\mathbb{R}^N).$$

In the following, we shall fix a radially symmetric nonnegative decreasing minimizer $U = U(r)$ for S. Multiplying U by a positive constant if necessary, we may assume that

$$(-\Delta)_p^s U = U^{p_s^* - 1}. \tag{2.1}$$

Testing this equation with U and using (1.3) shows that

$$\|U\|^p = |U|_{p_s^*}^{p_s^*} = S^{N/sp}. \tag{2.2}$$

For any $\varepsilon > 0$, the function

$$U_\varepsilon(x) = \frac{1}{\varepsilon^{(N-sp)/p}} U\left(\frac{|x|}{\varepsilon}\right) \tag{2.3}$$

is also a minimizer for S satisfying (2.1) and (2.2), so after a rescaling we may assume that $U(0) = 1$. Henceforth, U will denote such a normalized (with respect to constant multiples and rescaling) minimizer and U_ε will denote the associated family of minimizers given by (2.3). In the absence of an explicit formula for U, we will use the following asymptotic estimates.

Lemma 2.2 *There exist constants $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r \geq 1$,*

$$\frac{c_1}{r^{(N-sp)/(p-1)}} \leq U(r) \leq \frac{c_2}{r^{(N-sp)/(p-1)}} \tag{2.4}$$

and

$$\frac{U(\theta r)}{U(r)} \leq \frac{1}{2}. \tag{2.5}$$

Proof The inequalities in (2.4) were proved in Brasco et al. [4]. They imply

$$\frac{U(\theta r)}{U(r)} \leq \frac{c_2}{c_1} \frac{1}{\theta^{(N-sp)/(p-1)}},$$

and (2.5) follows for sufficiently large θ.   □

2.2 Regularity estimates

Weak solutions of the equation $(-\Delta)_p^s u = f(x)$ enjoy the natural L^q-estimates given in the following lemma.

Lemma 2.3 *Let $f \in L^q(\Omega)$, $1 < q \leq \infty$ and let $u \in W_0^{s,p}(\Omega)$ be a weak solution of $(-\Delta)_p^s u = f$ in Ω. Then*

$$|u|_r \leq C |f|_q^{1/(p-1)}, \tag{2.6}$$

where

$$r = \begin{cases} \dfrac{N(p-1)q}{N-spq}, & \text{if } 1 < q < \dfrac{N}{sp} \\[2mm] \infty, & \text{if } \dfrac{N}{sp} < q \leq \infty \end{cases}$$

and $C = C(N, \Omega, p, s, q) > 0$. In particular, if $f \in L^\infty(\Omega)$, then

$$|u|_\infty \leq C |f|_\infty^{1/(p-1)}.$$

Proof For $k > 0$, $t \in \mathbb{R}$, and $\alpha > 0$, set $t_k = \max\{-k, \min\{t, k\}\}$ and consider the nondecreasing function $g(t) = t_k^\alpha$. Using Brasco and Parini [6, Lemma A.2] and testing the equation $(-\Delta)_p^s u = f$ with $g(u) \in W_0^{s,p}(\Omega)$ gives

$$\|G(u)\|^p \leq \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1} (g(u(x)) - g(u(y)))}{|x-y|^{N+sp}} \, dx dy = \int_\Omega f(x) \, g(u(x)) \, dx,$$

where

$$G(t) = \int_0^t g'(\tau)^{1/p} \, d\tau = \frac{\alpha^{1/p} \, p}{\alpha + p - 1} t_k^{(\alpha+p-1)/p}.$$

Using the Sobolev inequality on the left and the Hölder inequality on the right we get

$$\left| u_k^{(\alpha+p-1)/p} \right|_{p_s^*}^p \leq C |f|_q \left| u_k^\alpha \right|_{q'}. \tag{2.7}$$

If $1 < q < N/sp$, take

$$\alpha = \frac{(p-1)\, p_s^*}{pq' - p_s^*} = \frac{N\,(p-1)\,(q-1)}{N - spq} > 0,$$

so that

$$\frac{\alpha + p - 1}{p}\, p_s^* = \alpha\, q' =: r.$$

Then $r = N\,(p-1)\,q/(N - spq)$ and (2.7) gives

$$|u_k|_r^{pr/p_s^*} \le C\,|f|_q\,|u_k|_r^{r/q'},$$

so

$$|u_k|_r \le C\,|f|_q^{1/(p-1)}.$$

Letting $k \to +\infty$ gives (2.6) for this case. If $N/sp < q \le \infty$, then (2.6) follows from Brasco and Parini [6, Theorem 3.1].   □

We also have the following Caccioppoli-type inequality.

Lemma 2.4 *Let* $f \in L^q(\Omega)$, $1 < q \le \infty$ *and let* $u \in W_0^{s,p}(\Omega)$ *be a weak solution of* $(-\Delta)_p^s\, u = f$ *in* Ω. *If* $u\,|\varphi|^p \in W_0^{s,p}(\Omega)$, *then*

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p\,|\varphi(x)|^p}{|x-y|^{N+sp}}\, dx dy \le 2 \int_\Omega f(x)\,u(x)\,|\varphi(x)|^p\, dx$$
$$+ C \int_{\mathbb{R}^{2N}} \frac{|u(y)|^p\,|\varphi(x) - \varphi(y)|^p}{|x-y|^{N+sp}}\, dx dy, \qquad (2.8)$$

where $C = C(p) > 0$.

Proof Testing the equation $(-\Delta)_p^s\, u = f$ with $u\,|\varphi|^p$ gives

$$\int_\Omega f(x)\,u(x)\,|\varphi(x)|^p\, dx$$
$$= \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1}\,(u(x)\,|\varphi(x)|^p - u(y)\,|\varphi(y)|^p)}{|x-y|^{N+sp}}\, dx dy$$
$$= \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p\,|\varphi(x)|^p}{|x-y|^{N+sp}}\, dx dy$$
$$+ \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1}\,u(y)\,(|\varphi(x)|^p - |\varphi(y)|^p)}{|x-y|^{N+sp}}\, dx dy. \qquad (2.9)$$

By the elementary inequality $||a|^p - |b|^p| \leq p\,|a-b|\,(|a|^{p-1}+|b|^{p-1})$ valid for all $a, b \in \mathbb{R}$ and the Young's inequality,

$$
-\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}\,u(y)\,(|\varphi(x)|^p - |\varphi(y)|^p)}{|x-y|^{N+sp}}\,dxdy
$$

$$
\leq p\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-1}\,|u(y)|\,|\varphi(x)-\varphi(y)|\,(|\varphi(x)|^{p-1}+|\varphi(y)|^{p-1})}{|x-y|^{N+sp}}\,dxdy
$$

$$
\leq \frac{1}{4}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p\,(|\varphi(x)|^p + |\varphi(y)|^p)}{|x-y|^{N+sp}}\,dxdy
$$

$$
+ C\int_{\mathbb{R}^{2N}} \frac{|u(y)|^p\,|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}}\,dxdy
$$

$$
= \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p\,|\varphi(x)|^p}{|x-y|^{N+sp}}\,dxdy + C\int_{\mathbb{R}^{2N}} \frac{|u(y)|^p\,|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}}\,dxdy.
$$

Combining this with (2.9) gives (2.8). $\square$

As a consequence of Lemmas 2.3 and 2.4, we have the following lemma.

Lemma 2.5 *Let* $f \in L^q(\Omega)$, $N/sp < q \leq \infty$ *and let* $u \in W_0^{s,p}(\Omega)$ *be a weak solution of* $(-\Delta)_p^s\,u = f(x)$ *in* Ω. *Then*

$$
\|u\varphi\|^p \leq C\,|f|_q^{p/(p-1)}\left(|\varphi|_{pq'}^p + \|\varphi\|^p\right) \quad \forall \varphi \in L^{pq'}(\Omega) \cap W_0^{s,p}(\Omega), \tag{2.10}
$$

where $C = C(N, \Omega, p, s, q) > 0$.

Proof Setting $t_k = \max\{-k, \min\{t, k\}\}$ for $k > 0$ and $t \in \mathbb{R}$, noting that $u\,|\varphi_k|^p \in W_0^{s,p}(\Omega)$, and applying Lemma 2.4 gives

$$
\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p\,|\varphi_k(x)|^p}{|x-y|^{N+sp}}\,dxdy \leq 2\int_{\Omega} f(x)\,u(x)\,|\varphi_k(x)|^p\,dx
$$

$$
+ C\int_{\mathbb{R}^{2N}} \frac{|u(y)|^p\,|\varphi_k(x)-\varphi_k(y)|^p}{|x-y|^{N+sp}}\,dxdy. \tag{2.11}
$$

Since $N/sp < q \leq \infty$,

$$
|u|_\infty \leq C\,|f|_q^{1/(p-1)} \tag{2.12}
$$

by Lemma 2.3. By (2.11), (2.12), and the Hölder inequality,

$$
\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p\,|\varphi_k(x)|^p}{|x-y|^{N+sp}}\,dxdy \leq C\,|f|_q^{p/(p-1)}\left(|\varphi_k|_{pq'}^p + \|\varphi_k\|^p\right)
$$

$$
\leq C\,|f|_q^{p/(p-1)}\left(|\varphi|_{pq'}^p + \|\varphi\|^p\right),
$$

and letting $k \to +\infty$ gives

$$
\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p\,|\varphi(x)|^p}{|x-y|^{N+sp}}\,dxdy \leq C\,|f|_q^{p/(p-1)}\left(|\varphi|_{pq'}^p + \|\varphi\|^p\right). \tag{2.13}
$$

Since

$$
\int_{\mathbb{R}^{2N}} \frac{|u(x)\,\varphi(x) - u(y)\,\varphi(y)|^p}{|x-y|^{N+sp}}\,dxdy \leq C\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p\,|\varphi(x)|^p}{|x-y|^{N+sp}}\,dxdy \right.
$$

$$
\left. + \int_{\mathbb{R}^{2N}} \frac{|u(y)|^p\,|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}}\,dxdy\right),
$$

(2.10) readily follows from (2.13) and (2.12).   □

Now let θ be as in Lemma 2.2, let $\eta \in C^\infty(\mathbb{R}^N, [0, 1])$ be such that

$$\eta(x) = \begin{cases} 0, & \text{if } |x| \le 2\theta \\ 1, & \text{if } |x| \ge 3\theta, \end{cases}$$

and let $\eta_\delta(x) = \eta\left(\dfrac{x}{\delta}\right)$ for $\delta > 0$.

Lemma 2.6 *Assume that $0 \in \Omega$. Then there exists a constant $C = C(N, \Omega, p, s) > 0$ such that for any $v \in W_0^{s,p}(\Omega)$ such that $(-\Delta)_p^s v \in L^\infty(\Omega)$ and $\delta > 0$ such that $B_{5\theta\delta}(0) \subset \Omega$,*

$$\|v\eta_\delta\|^p \le \|v\|^p + C \left|(-\Delta)_p^s v\right|_\infty^{p/(p-1)} \delta^{N-sp}.$$

Proof We have

$$\|v\eta_\delta\|^p \le \int_{A_1} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}}\, dxdy + \int_{A_2} \frac{|v(x)\,\eta_\delta(x) - v(y)\,\eta_\delta(y)|^p}{|x - y|^{N+sp}}\, dxdy$$
$$+ 2 \int_{A_3} \frac{|v(x)\,\eta_\delta(x) - v(y)|^p}{|x - y|^{N+sp}}\, dxdy =: I_1 + I_2 + 2I_3, \qquad (2.14)$$

where

$$A_1 = B_{3\theta\delta}(0)^c \times B_{3\theta\delta}(0)^c, \quad A_2 = B_{4\theta\delta}(0) \times B_{4\theta\delta}(0), \quad A_3 = B_{3\theta\delta}(0) \times B_{4\theta\delta}(0)^c.$$

Clearly, $I_1 \le \|v\|^p$. To estimate I_2, let $\varphi \in C_0^\infty(B_{5\theta}(0), [0, 1])$ with $\varphi = \eta$ in $B_{4\theta}(0)$ and let $\varphi_\delta(x) = \varphi(x/\delta)$. Then

$$I_2 = \int_{A_2} \frac{|v(x)\,\varphi_\delta(x) - v(y)\,\varphi_\delta(y)|^p}{|x - y|^{N+sp}}\, dxdy \le \|v\varphi_\delta\|^p \le C \left|(-\Delta)_p^s v\right|_\infty^{p/(p-1)} \|\varphi_\delta\|^p$$

by Lemma 2.5 applied to φ_δ with $q = \infty$, and $\|\varphi_\delta\|^p = \delta^{N-sp}\|\varphi\|^p$. Since $|x - y| \ge |y| - 3\theta\delta \ge |y|/4$ on A_3,

$$I_3 \le C\,|v|_\infty^p \int_{A_3} \frac{dxdy}{|y|^{N+sp}} \le C \left|(-\Delta)_p^s v\right|_\infty^{p/(p-1)} \delta^{N-sp}$$

by Lemma 2.3.   □

2.3 Auxiliary estimates

We now construct some auxiliary functions and estimate their norms. In what follows θ is the universal constant in Lemma 2.2 that depends only on N, p, and s. We may assume without loss of generality that $0 \in \Omega$. For $\varepsilon, \delta > 0$, let

$$m_{\varepsilon,\delta} = \frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\delta)},$$

let

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0, & \text{if } 0 \le t \le U_\varepsilon(\theta\delta) \\ m_{\varepsilon,\delta}^p\,(t - U_\varepsilon(\theta\delta)), & \text{if } U_\varepsilon(\theta\delta) \le t \le U_\varepsilon(\delta) \\ t + U_\varepsilon(\delta)\,(m_{\varepsilon,\delta}^{p-1} - 1), & \text{if } t \ge U_\varepsilon(\delta), \end{cases}$$

and let

$$
G_{\varepsilon,\delta}(t) = \int_0^t g'_{\varepsilon,\delta}(\tau)^{1/p} \, d\tau =
\begin{cases}
0, & \text{if } 0 \leq t \leq U_\varepsilon(\theta\delta) \\
m_{\varepsilon,\delta}\,(t - U_\varepsilon(\theta\delta)), & \text{if } U_\varepsilon(\theta\delta) \leq t \leq U_\varepsilon(\delta) \\
t, & \text{if } t \geq U_\varepsilon(\delta).
\end{cases} \tag{2.15}
$$

The functions $g_{\varepsilon,\delta}$ and $G_{\varepsilon,\delta}$ are nondecreasing and absolutely continuous. Consider the radially symmetric nonincreasing function

$$
u_{\varepsilon,\delta}(r) = G_{\varepsilon,\delta}(U_\varepsilon(r)),
$$

which satisfies

$$
u_{\varepsilon,\delta}(r) =
\begin{cases}
U_\varepsilon(r), & \text{if } r \leq \delta \\
0, & \text{if } r \geq \theta\delta.
\end{cases} \tag{2.16}
$$

We have the following estimates for $u_{\varepsilon,\delta}$.

Lemma 2.7 *There exists a constant $C = C(N, p, s) > 0$ such that for any $\varepsilon \leq \delta/2$,*

$$
\|u_{\varepsilon,\delta}\|^p \leq S^{N/sp} + C \left(\frac{\varepsilon}{\delta}\right)^{(N-sp)/(p-1)}, \tag{2.17}
$$

$$
|u_{\varepsilon,\delta}|_p^p \geq
\begin{cases}
\dfrac{1}{C}\, \varepsilon^{sp} \log\left(\dfrac{\delta}{\varepsilon}\right), & \text{if } N = sp^2 \\[2ex]
\dfrac{1}{C}\, \varepsilon^{sp}, & \text{if } N > sp^2,
\end{cases} \tag{2.18}
$$

$$
|u_{\varepsilon,\delta}|_{p_s^*}^{p_s^*} \geq S^{N/sp} - C \left(\frac{\varepsilon}{\delta}\right)^{N/(p-1)}. \tag{2.19}
$$

Proof Using Brasco and Parini [6, Lemma A.2] and testing the equation $(-\Delta)_p^s \, U_\varepsilon = U_\varepsilon^{p_s^*-1}$ with $g_{\varepsilon,\delta}(U_\varepsilon) \in W_0^{s,p}(\Omega)$ gives

$$
\begin{aligned}
\|G_{\varepsilon,\delta}(U_\varepsilon)\|^p &\leq \int_{\mathbb{R}^{2N}} \frac{(U_\varepsilon(x) - U_\varepsilon(y))^{p-1} \, (g_{\varepsilon,\delta}(U_\varepsilon(x)) - g_{\varepsilon,\delta}(U_\varepsilon(y)))}{|x - y|^{N+sp}} \, dx\,dy \\
&= \int_{\mathbb{R}^N} U_\varepsilon(x)^{p_s^*-1} \, g_{\varepsilon,\delta}(U_\varepsilon(x)) \, dx \\
&= |U_\varepsilon|_{p_s^*}^{p_s^*} + \int_{\mathbb{R}^N} (g_{\varepsilon,\delta}(U_\varepsilon(x)) - U_\varepsilon(x)) \, U_\varepsilon(x)^{p_s^*-1} \, dx.
\end{aligned}
$$

We have $|U_\varepsilon|_{p_s^*}^{p_s^*} = S^{N/sp}$ by (2.2),

$$
\begin{aligned}
g_{\varepsilon,\delta}(t) - t &\leq U_\varepsilon(\delta) \, m_{\varepsilon,\delta}^{p-1} = \frac{1}{\varepsilon^{(N-sp)/p}} \, U\left(\frac{\delta}{\varepsilon}\right) \left[1 - U\left(\frac{\theta\delta}{\varepsilon}\right) \middle/ U\left(\frac{\delta}{\varepsilon}\right)\right]^{-(p-1)} \\
&\leq 2^{p-1} \, c_2 \, \frac{\varepsilon^{(N-sp)/p(p-1)}}{\delta^{(N-sp)/(p-1)}}, \quad \forall t \geq 0
\end{aligned}
$$

by (2.4) and (2.5),

$$
\int_{\mathbb{R}^N} U_\varepsilon(x)^{p_s^*-1} \, dx = \varepsilon^{(N-sp)/p} \int_{\mathbb{R}^N} U(x)^{p_s^*-1} \, dx,
$$

and the last integral is finite by (2.4) again, so (2.17) follows. Using (2.16),

$$
\int_{\mathbb{R}^N} u_{\varepsilon,\delta}(x)^p \, dx \geq \int_{B_\delta(0)} u_{\varepsilon,\delta}(x)^p \, dx = \int_{B_\delta(0)} U_\varepsilon(x)^p \, dx = \varepsilon^{sp} \int_{B_{\delta/\varepsilon}(0)} U(x)^p \, dx,
$$

and the last integral is greater than or equal to

$$\int_1^{\delta/\varepsilon} U(r)^p \, r^{N-1} \, dr \geq c_1^p \int_1^{\delta/\varepsilon} r^{-(N-sp^2)/(p-1)-1} \, dr$$

by (2.4). A direct evaluation of the integral on the right gives (2.18) since $\delta/\varepsilon \geq 2$. Using (2.16) again,

$$\int_{\mathbb{R}^N} u_{\varepsilon,\delta}(x)^{p_s^*} \, dx \geq \int_{B_\delta(0)} u_{\varepsilon,\delta}(x)^{p_s^*} \, dx = \int_{B_\delta(0)} U_\varepsilon(x)^{p_s^*} \, dx$$

$$= S^{N/sp} - \int_{B_{\delta/\varepsilon}(0)^c} U(x)^{p_s^*} \, dx$$

by (2.2). By (2.4), the last integral is less than or equal to

$$c_2^{p_s^*} \int_{\delta/\varepsilon}^\infty r^{-N/(p-1)-1} \, dr = \frac{(p-1) \, c_2^{p_s^*}}{N} \left(\frac{\varepsilon}{\delta}\right)^{N/(p-1)},$$

so (2.18) follows.   □

We note that Lemma 2.7 gives the following estimate for

$$S_{\varepsilon,\delta}(\lambda) := \frac{\|u_{\varepsilon,\delta}\|^p - \lambda \, |u_{\varepsilon,\delta}|_p^p}{|u_{\varepsilon,\delta}|_{p_s^*}^p} :$$

there exists a constant $C = C(N, p, s) > 0$ such that for any $\varepsilon \leq \delta/2$,

$$S_{\varepsilon,\delta}(\lambda) \leq \begin{cases} S - \dfrac{\lambda}{C} \, \varepsilon^{sp} \, \log\left(\dfrac{\delta}{\varepsilon}\right) + C \left(\dfrac{\varepsilon}{\delta}\right)^{sp}, & \text{if } N = sp^2 \\[2ex] S - \dfrac{\lambda}{C} \, \varepsilon^{sp} + C \left(\dfrac{\varepsilon}{\delta}\right)^{(N-sp)/(p-1)}, & \text{if } N > sp^2. \end{cases} \tag{2.20}$$

3 Proof of the main result

In this section we prove Theorem 1.3. For $0 < \lambda < \lambda_1$, mountain pass theorem and (2.20) will give us a positive critical level of I_λ below the threshold level for compactness given in Proposition 1.5. For $\lambda \geq \lambda_1$, we will use the abstract linking theorem, Theorem 1.4.

3.1 Case 1: $N \geq sp^2$ and $0 < \lambda < \lambda_1$

We have

$$I_\lambda(u) \geq \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^p - \frac{1}{p_s^* \, S^{p_s^*/p}} \|u\|^{p_s^*},$$

so the origin is a strict local minimizer of I_λ. Fix $\delta > 0$ so small that $B_{\theta\delta}(0) \subset\subset \Omega$, so that $\operatorname{supp} u_{\varepsilon,\delta} \subset \Omega$ by (2.16). Noting that

$$I_\lambda(R u_{\varepsilon,\delta}) = \frac{R^p}{p} \left(\|u_{\varepsilon,\delta}\|^p - \lambda \, |u_{\varepsilon,\delta}|_p^p\right) - \frac{R^{p_s^*}}{p_s^*} |u_{\varepsilon,\delta}|_{p_s^*}^{p_s^*} \to -\infty \quad \text{as } R \to +\infty,$$

fix $R_0 > 0$ so large that $I_\lambda(R_0 u_{\varepsilon,\delta}) < 0$. Then let

$$\Gamma = \left\{\gamma \in C([0, 1], W_0^{s,p}(\Omega)) : \gamma(0) = 0, \; \gamma(1) = R_0 u_{\varepsilon,\delta}\right\}$$

and set

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0.$$

Since $t \mapsto t R_0 u_{\varepsilon,\delta}$ is a path in Γ,

$$c \leq \max_{t \in [0,1]} I_\lambda(t R_0 u_{\varepsilon,\delta}) = \frac{s}{N} \left(\frac{\|u_{\varepsilon,\delta}\|^p - \lambda |u_{\varepsilon,\delta}|_p^p}{|u_{\varepsilon,\delta}|_{p_s^*}^p} \right)^{N/sp} = \frac{s}{N} S_{\varepsilon,\delta}(\lambda)^{N/sp}. \qquad (3.1)$$

By (2.20),

$$S_{\varepsilon,\delta}(\lambda) \leq \begin{cases} S + \left(C - \dfrac{\lambda}{C} |\log \varepsilon| \right) \varepsilon^{sp}, & \text{if } N = sp^2 \\[2mm] S - \left(\dfrac{\lambda}{C} - C \, \varepsilon^{(N-sp^2)/(p-1)} \right) \varepsilon^{sp}, & \text{if } N > sp^2, \end{cases}$$

so $S_{\varepsilon,\delta}(\lambda) < S$ if $\varepsilon > 0$ is sufficiently small. So

$$c < \frac{s}{N} S^{N/sp}$$

by (3.1), and hence I_λ satisfies the (PS)$_c$ condition by Proposition 1.5. Then c is a critical level of I_λ by the mountain pass theorem.

3.2 Case 2: $N > sp^2$ and $\lambda > \lambda_1$ is not one of the eigenvalues λ_k

We have $\lambda_k < \lambda < \lambda_{k+1}$ for some $k \in \mathbb{N}$, and then $i(\Psi^{\lambda_k}) = i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}}) = k$ by (1.5). In what follows

$$\pi(u) = \frac{u}{\|u\|}, \quad \pi_p(u) = \frac{u}{|u|_p}, \quad u \in W_0^{s,p}(\Omega) \setminus \{0\}$$

are the radial projections onto

$$\mathcal{M} = \left\{ u \in W_0^{s,p}(\Omega) : \|u\| = 1 \right\}, \quad \mathcal{M}_p = \left\{ u \in W_0^{s,p}(\Omega) : |u|_p = 1 \right\},$$

respectively.

Proposition 3.1 *If $\lambda_k < \lambda_{k+1}$, then Ψ^{λ_k} has a compact symmetric subset E with $i(E) = k$ such that*

$$\left| (-\Delta)_p^s v \right|_\infty \leq C \quad \forall v \in E,$$

where $C = C(N, \Omega, p, s, k) > 0$. In particular,

$$|v|_\infty \leq C \quad \forall v \in E.$$

Proof For $w \in L^q(\Omega)$ with $q \geq \max \left\{ 1, (p-1)(p_s^*)' \right\}$, the equation $(-\Delta)_p^s u = |w|^{p-2} w$ has a unique weak solution $u = \mathcal{B}(w) \in W_0^{s,p}(\Omega)$. By Lemma 2.3,

$$|\mathcal{B}(w)|_{\gamma(q)} \leq C(q) |w|_q, \qquad (3.2)$$

where

$$\gamma(q) = \begin{cases} \dfrac{N(p-1)q}{N(p-1) - spq}, & \text{if } \dfrac{q}{p-1} < \dfrac{N}{sp} \\[3mm] \infty, & \text{if } \dfrac{N}{sp} < \dfrac{q}{p-1} \leq \infty. \end{cases}$$

For $w \in \mathcal{M}_p$, let $J(w) = \pi_p(u) \in \mathcal{M}_p$, where $u = \mathcal{B}(w)$. Testing $(-\Delta)_p^s u = |w|^{p-2} w$ with u, w and using the Hölder inequality gives

$$\|u\|^p = \int_\Omega |w|^{p-2} wu \, dx \leq |w|_p^{p-1} |u|_p = |u|_p \,,$$

$$1 = |w|_p^p = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1} (w(x) - w(y))}{|x - y|^{N+sp}} \, dxdy \leq \|u\|^{p-1} \|w\| \,,$$

respectively, so

$$\|J(w)\| = \frac{\|u\|}{|u|_p} \leq \frac{1}{\|u\|^{p-1}} \leq \|w\| \,, \quad |\mathcal{B}(w)|_p = |u|_p \geq \|u\|^p \geq \frac{1}{\|w\|^{p/(p-1)}}. \quad (3.3)$$

Let

$$A = \pi_p(\Psi^{\lambda_k}) = \{w \in \mathcal{M}_p : \|w\|^p \leq \lambda_k\}.$$

Then $i(A) = i(\Psi^{\lambda_k}) = k$ by the monotonicity of the index and (1.5), and A is strongly compact in $L^p(\Omega)$. By (3.3), $J(A) \subset A$ and

$$|\mathcal{B}(w)|_p \geq \frac{1}{\lambda_k^{1/(p-1)}} \quad \forall w \in A. \quad (3.4)$$

For $w \in A$, if $p/(p-1) > N/sp$, then $\gamma(p) = \infty$ and hence

$$|J(w)|_\infty = \frac{|\mathcal{B}(w)|_\infty}{|\mathcal{B}(w)|_p} \leq C(p) \lambda_k^{1/(p-1)} |w|_p = C(p) \lambda_k^{1/(p-1)}$$

by (3.2) and (3.4). Otherwise, take $\max\{1, (p-1)(p_s^*)'\} \leq q_0 < p$ and define the sequence (q_i) recursively by setting $q_i = \gamma(q_{i-1})$ if $q_{i-1}/(p-1) < N/sp$, in which case

$$q_i - q_{i-1} = \frac{sp \, q_{i-1}^2}{N(p-1) - sp \, q_{i-1}} \geq \frac{sp}{N(p-1) - sp} > 0.$$

Hence q_0 may be chosen so that $q_{n-1}/(p-1) < N/sp < q_n/(p-1)$ for some $n \geq 1$. Iterating (3.2) and (3.4), and using the Hölder inequality at the last step then gives

$$\left|J^n(w)\right|_\infty = \frac{\left|\mathcal{B}(J^{n-1}(w))\right|_\infty}{\left|\mathcal{B}(J^{n-1}(w))\right|_p} \leq C(q_n) \lambda_k^{1/(p-1)} \left|J^{n-1}(w)\right|_{q_n} \quad (3.5)$$

$$\leq \cdots \leq C(q_n) \cdots C(q_0) \lambda_k^{(n+1)/(p-1)} |w|_{q_0} \leq C \lambda_k^{(n+1)/(p-1)}.$$

Let $\widetilde{A} = J^{n+1}(A) \subset A$. For each $\widetilde{v} \in \widetilde{A}$, there exists $\widetilde{w} \in J^n(A) \subset A$ such that $\widetilde{v} = J(\widetilde{w}) = u/|u|_p$, where $u = \mathcal{B}(\widetilde{w})$. Then

$$(-\Delta)_p^s \, \widetilde{v} = \frac{(-\Delta)_p^s u}{|u|_p^{p-1}} = \frac{|\widetilde{w}|^{p-2} \widetilde{w}}{|\mathcal{B}(\widetilde{w})|_p^{p-1}},$$

so

$$\left|(-\Delta)_p^s \, \widetilde{v}\right|_\infty = \frac{|\widetilde{w}|_\infty^{p-1}}{|\mathcal{B}(\widetilde{w})|_p^{p-1}} \leq C \lambda_k^{n+2} \quad (3.6)$$

by (3.4) and (3.5). Since the imbedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact and J is an odd continuous map from $L^p(\Omega)$ to $W_0^{s,p}(\Omega)$, $\widetilde{A}$ is a compact set and $i(\widetilde{A}) = i(A) = k$.

Let $E = \pi(\widetilde{A})$ and note that E is compact with $i(E) = i(\widetilde{A}) = k$. For each $v \in E$, there exists $\widetilde{v} \in \widetilde{A} \subset A$ such that $v = \widetilde{v}/\|\widetilde{v}\|$. Then

$$\Psi(v) = \frac{\|\widetilde{v}\|^p}{|\widetilde{v}|_p^p} \le \lambda_k,$$

so $E \subset \Psi^{\lambda_k}$. Since $1 = |\widetilde{v}|_p \le C\|\widetilde{v}\|$,

$$\left|(-\Delta)_p^s\, v\right|_\infty = \frac{\left|(-\Delta)_p^s\, \widetilde{v}\right|_\infty}{\|\widetilde{v}\|^{p-1}} \le C\,\lambda_k^{n+2}$$

by (3.6). $\qquad\qquad\square$

For $v \in E$, let $v_\delta = v\eta_\delta$, where η_δ is the cut-off function in Lemma 2.6, and let

$$E_\delta = \{\pi(v_\delta) : v \in E\}.$$

Proposition 3.2 *There exists a constant $C = C(N, \Omega, p, s, k) > 0$ such that for all sufficiently small $\delta > 0$,*

$$\frac{1}{C} \le |w|_q \le C \quad \forall w \in E_\delta, \quad 1 \le q \le \infty, \tag{3.7}$$

$$\sup_{w \in E_\delta} \Psi(w) \le \lambda_k + C\delta^{N-sp}, \tag{3.8}$$

$E_\delta \cap \Psi_{\lambda_{k+1}} = \emptyset$, $i(E_\delta) = k$, *and* $\operatorname{supp} w \subset B_{2\theta\delta}(0)^c$ *for all* $w \in E_\delta$. *In particular, the supports of w and $\pi(u_{\varepsilon,\delta})$ are disjoint and hence $\pi(u_{\varepsilon,\delta}) \notin E_\delta$.*

Proof Let $v \in E$ and let $w = \pi(v_\delta)$. We have

$$\int_\Omega |v|^q\,dx = \int_{\Omega\setminus B_{3\theta\delta}(0)} |v_\delta|^q\,dx + \int_{B_{3\theta\delta}(0)} |v|^q\,dx \le \int_\Omega |v_\delta|^q\,dx + C\,|v|_\infty^q\,\delta^N,$$

so

$$\int_\Omega |v_\delta|^q\,dx \ge \int_\Omega |v|^q\,dx - C\delta^N \tag{3.9}$$

by Proposition 3.1. In particular, $|v_\delta|_1 \ge |v|_1 - C\delta^N$. On the other hand,

$$1 = \|v\|^p = \int_\Omega v\,(-\Delta)_p^s\, v\,dx \le \left|(-\Delta)_p^s\, v\right|_\infty \int_\Omega |v|\,dx \le C\,|v|_1$$

by Proposition 3.1 again, so $|v_\delta|_1 \ge 1/C - C\delta^N$. Since

$$\|v_\delta\|^p \le 1 + C\delta^{N-sp} \tag{3.10}$$

by Lemma 2.6 and Proposition 3.1, then

$$|w|_1 = \frac{|v_\delta|_1}{\|v_\delta\|} \ge \frac{\dfrac{1}{C} - C\delta^N}{1 + C\delta^{(N-sp)/p}} = \frac{1}{C} + \mathrm{O}(\delta^{(N-sp)/p}),$$

which together with the Hölder inequality gives the first half of (3.7). By (3.9) with $q = p$,

$$|v_\delta|_p^p \ge |v|_p^p - C\delta^N \ge \frac{1}{\lambda_k} - C\delta^N \tag{3.11}$$

since $E \subset \Psi^{\lambda_k}$. So $|v_\delta|_p$, and hence also $\|v_\delta\|$, is bounded away from zero. Since $|v|_\infty$ is bounded by Proposition 3.1 and $0 \leq \eta_\delta \leq 1$, $|v_\delta|_\infty$ is bounded, so this shows that $|w|_\infty = |v_\delta|_\infty / \|v_\delta\|$ is bounded, which gives the second half of (3.7).

Combining (3.10) and (3.11) gives

$$\Psi(w) = \frac{\|v_\delta\|^p}{|v_\delta|_p^p} \leq \frac{1 + C\delta^{N-sp}}{\frac{1}{\lambda_k} - C\delta^N} = \lambda_k + O(\delta^{N-sp}).$$

Fix $\delta > 0$ so small that $\lambda_k + C\delta^{N-sp} < \lambda_{k+1}$. Then $E_\delta \subset \mathcal{M} \backslash \Psi_{\lambda_{k+1}}$ by (3.8), and hence $i(E_\delta) \leq i(\mathcal{M} \backslash \Psi_{\lambda_{k+1}}) = k$ by the monotonicity of the index and (1.5). On the other hand, $E \to E_\delta$, $v \mapsto \pi(v_\delta)$ is an odd continuous map and hence $i(E_\delta) \geq i(E) = k$. So $i(E_\delta) = k$.

Finally, $\operatorname{supp} \pi(v_\delta) = \operatorname{supp} v_\delta \subset \operatorname{supp} \eta_\delta \subset B_{2\theta\delta}(0)^c$ for all $v \in E$, and

$$\operatorname{supp} \pi(u_{\varepsilon,\delta}) = \operatorname{supp} u_{\varepsilon,\delta} \subset \overline{B_{\theta\delta}(0)},$$

by virtue of (2.16). □

We are now ready to apply Theorem 1.4 to obtain a nontrivial critical point of I_λ in the case where $\lambda > \lambda_1$ is not one of the eigenvalues λ_k. Fix λ' such that $\lambda_k < \lambda' < \lambda < \lambda_{k+1}$, and let $\delta > 0$ be so small that the conclusions of Proposition 3.2 hold with $\lambda_k + C\delta^{N-sp} < \lambda'$, in particular,

$$\Psi(w) < \lambda' \quad \forall w \in E_\delta. \tag{3.12}$$

Then take $A_0 = E_\delta$ and $B_0 = \Psi_{\lambda_{k+1}}$, and note that A_0 and B_0 are disjoint nonempty closed symmetric subsets of $\mathcal{M}$ such that

$$i(A_0) = i(\mathcal{M} \backslash B_0) = k$$

by Proposition 3.2 and (1.5). Now let $0 < \varepsilon \leq \delta/2$, let $R > r > 0$, let $v_0 = \pi(u_{\varepsilon,\delta}) \in \mathcal{M} \backslash E_\delta$, and let A, B and X be as in Theorem 1.4.

For $u \in \Psi_{\lambda_{k+1}}$,

$$I_\lambda(ru) \geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) r^p - \frac{1}{p_s^* S^{p_s^*/p}} r^{p_s^*}.$$

Since $\lambda < \lambda_{k+1}$, it follows that $\inf I_\lambda(B) > 0$ if r is sufficiently small.

Next we show that $I_\lambda \leq 0$ on A if R is sufficiently large. For $w \in E_\delta$ and $t \geq 0$,

$$I_\lambda(tw) \leq \frac{t^p}{p}\left(1 - \frac{\lambda}{\Psi(w)}\right) \leq 0$$

by (3.12). Now let $w \in E_\delta$ and $0 \leq t \leq 1$, and set $u = \pi((1-t)w + tv_0)$. Clearly, $\|(1-t)w + tv_0\| \leq 1$, and since the supports of w and v_0 are disjoint by Proposition 3.2,

$$|(1-t)w + tv_0|_{p_s^*}^{p_s^*} = (1-t)^{p_s^*}|w|_{p_s^*}^{p_s^*} + t^{p_s^*}|v_0|_{p_s^*}^{p_s^*}.$$

In view of (3.7) and since

$$|v_0|_{p_s^*}^{p_s^*} = \frac{|u_{\varepsilon,\delta}|_{p_s^*}^{p_s^*}}{\|u_{\varepsilon,\delta}\|^{p_s^*}} \geq \frac{1}{S^{N/(N-sp)}} + O(\varepsilon^{(N-sp)/(p-1)}) \tag{3.13}$$

by Lemma 2.7, it follows that

$$|u|_{p_s^*}^{p_s^*} = \frac{|(1-t)w + tv_0|_{p_s^*}^{p_s^*}}{\|(1-t)w + tv_0\|^{p_s^*}} \geq \frac{1}{C}$$

if ε is sufficiently small, where $C = C(N, \Omega, p, s, k) > 0$. Then

$$I_\lambda(Ru) \leq \frac{R^p}{p} - \frac{R^{p_s^*}}{p_s^*} |u|_{p_s^*}^{p_s^*} \leq \frac{R^p}{p} - \frac{R^{p_s^*}}{p_s^* C} \leq 0$$

if R is sufficiently large. In view of (1.7) and Proposition 1.5, it only remains to show that

$$\sup I_\lambda(X) < \frac{s}{N} S^{N/sp},$$

if ε is sufficiently small. Noting that

$$X = \{\rho \, \pi((1 - t) \, w + t v_0) : w \in E_\delta, \, 0 \leq t \leq 1, \, 0 \leq \rho \leq R\},$$

let $w \in E_\delta$ and $0 \leq t \leq 1$, and set $u = \pi((1 - t) \, w + t v_0)$. Then

$$\sup_{0 \leq \rho \leq R} I_\lambda(\rho u) \leq \sup_{\rho \geq 0} \left[\frac{\rho^p}{p} \left(1 - \lambda \, |u|_p^p\right) - \frac{\rho^{p_s^*}}{p_s^*} |u|_{p_s^*}^{p_s^*} \right] = \frac{s}{N} \left[\frac{\left(1 - \lambda \, |u|_p^p\right)^+}{|u|_{p_s^*}^p} \right]^{N/sp}$$

$$= \frac{s}{N} \left[\frac{\left(\|(1 - t) \, w + t v_0\|^p - \lambda \, |(1 - t) \, w + t v_0|_p^p\right)^+}{|(1 - t) \, w + t v_0|_{p_s^*}^p} \right]^{N/sp}. \quad (3.14)$$

Since $w = 0$ in $B_{2\theta\delta}(0)$ by Proposition 3.2 and $v_0 = 0$ in $B_{\theta\delta}(0)^c$ by (2.16),

$$\|(1 - t) \, w + t v_0\|^p$$
$$\leq (1 - t)^p \int_{A_1} \frac{|w(x) - w(y)|^p}{|x - y|^{N+sp}} \, dx dy + t^p \int_{A_2} \frac{|v_0(x) - v_0(y)|^p}{|x - y|^{N+sp}} \, dx dy$$
$$+ 2 \int_{A_3} \frac{|(1 - t) \, w(x) - t v_0(y)|^p}{|x - y|^{N+sp}} \, dx dy =: (1 - t)^p \, I_1 + t^p \, I_2 + 2 I_3, \quad (3.15)$$

where

$$A_1 = B_{\theta\delta}(0)^c \times B_{\theta\delta}(0)^c, \quad A_2 = B_{2\theta\delta}(0) \times B_{2\theta\delta}(0), \quad A_3 = B_{2\theta\delta}(0)^c \times B_{\theta\delta}(0).$$

We estimate I_3 using the following elementary inequality: given $\kappa > 1$ and $p - 1 < q < p$, there exists a constant $C = C(\kappa, q) > 0$ such that

$$|a + b|^p \leq \kappa \, |a|^p + |b|^p + C \, |a|^{p-q} \, |b|^q \quad \forall a, b \in \mathbb{R}.$$

Taking $\kappa = \lambda/\lambda'$ and, thanks to $N > sp^2$, choosing $q \in \,]N(p - 1)/(N - sp), p[$, we get

$$I_3 \leq \frac{\lambda}{\lambda'} (1 - t)^p \int_{A_3} \frac{|w(x) - w(y)|^p}{|x - y|^{N+sp}} \, dx dy + t^p \int_{A_3} \frac{|v_0(x) - v_0(y)|^p}{|x - y|^{N+sp}} \, dx dy$$
$$+ C \int_{A_3} \frac{|w(x)|^{p-q} \, v_0(y)^q}{|x - y|^{N+sp}} \, dx dy =: \frac{\lambda}{\lambda'} (1 - t)^p \, I_4 + t^p \, I_5 + C J_q. \quad (3.16)$$

Clearly, $I_1 + 2 I_4 \leq \|w\|^p = 1$ and $I_2 + 2 I_5 \leq \|v_0\|^p = 1$. By (3.7) and since

$$|x - y| \geq |x| - \theta\delta \geq |x|/2, \quad \text{on } A_3,$$

we have

$$J_q \leq \frac{C}{\|u_{\varepsilon,\delta}\|^q} \int_{A_3} \frac{u_{\varepsilon,\delta}(y)^q}{|x|^{N+sp}} \, dx dy \leq \frac{C}{\delta^{sp}} \int_{\mathbb{R}^N} u_{\varepsilon,\delta}(y)^q \, dy$$

since (2.18) implies that $\left|u_{\varepsilon,\delta}\right|_{p_s^*}$, and hence also $\left\|u_{\varepsilon,\delta}\right\|$, is bounded away from zero if ε is sufficiently small. Recalling (2.15), it holds $G_{\varepsilon,\delta}(t) \leq t$ for all $t \geq 0$, and thus

$$\int_{\mathbb{R}^N} u_{\varepsilon,\delta}(y)^q \, dy \leq \int_{\mathbb{R}^N} U_\varepsilon(y)^q \, dy = \varepsilon^{N-(N-sp)\,q/p} \int_{\mathbb{R}^N} U(y)^q \, dy,$$

and the last integral is finite by (2.4) since $q > N(p-1)/(N-sp)$. So combining (3.15) and (3.16) gives

$$\|(1-t)\,w + tv_0\|^p \leq \frac{\lambda}{\lambda'}\,(1-t)^p + t^p + C\,\varepsilon^{N-(N-sp)\,q/p}. \tag{3.17}$$

On the other hand, since the supports of w and v_0 are disjoint,

$$|(1-t)\,w + tv_0|_p^p = (1-t)^p\,|w|_p^p + t^p\,|v_0|_p^p,$$
$$|(1-t)\,w + tv_0|_{p_s^*}^{p_s^*} = (1-t)^{p_s^*}\,|w|_{p_s^*}^{p_s^*} + t^{p_s^*}\,|v_0|_{p_s^*}^{p_s^*}. \tag{3.18}$$

By (3.12), $|w|_p^p = 1/\Psi(w) > 1/\lambda'$. By (3.7), $|w|_{p_s^*}$ is bounded away from zero, and (3.13) implies that so is $|v_0|_{p_s^*}$ if ε is sufficiently small, so the last expression in (3.18) is bounded away from zero. It follows from (3.17) and (3.18) that

$$\frac{\|(1-t)\,w + tv_0\|^p - \lambda\,|(1-t)\,w + tv_0|_p^p}{|(1-t)\,w + tv_0|_{p_s^*}^p} \leq \frac{1-\lambda\,|v_0|_p^p}{|v_0|_{p_s^*}^p} + C\,\varepsilon^{N-(N-sp)\,q/p}.$$

Since $v_0 = u_{\varepsilon,\delta}/\left\|u_{\varepsilon,\delta}\right\|$, the right-hand side is less than or equal to

$$S_{\varepsilon,\delta}(\lambda) + C\,\varepsilon^{N-(N-sp)\,q/p} \leq S - \left(\frac{\lambda}{C} - C\,\varepsilon^{(N-sp^2)/(p-1)} - C\,\varepsilon^{(N-sp)(1-q/p)}\right)\varepsilon^{sp}$$

by (2.20). Since $N > sp^2$ and $q < p$, it follows from this that the last expression in (3.14) is strictly less than $\frac{s}{N}\,S^{N/sp}$ if ε is sufficiently small.

3.3 Case 3: $N^2/(N+s) > sp^2$ and $\lambda = \lambda_k$

Let $\lambda = \lambda_k < \lambda_{k+1}$, let $\delta > 0$ be so small that the conclusions of Proposition 3.2 hold with $\lambda_k + C\delta^{N-sp} < \lambda_{k+1}$, in particular, $\Psi(w) < \lambda_{k+1}$ for all $w \in E_\delta$, and take $A_0 = E_\delta$ and $B_0 = \Psi_{\lambda_{k+1}}$ as in the last subsection. Then let $0 < \varepsilon \leq \delta/2$, let $R > r > 0$, let $v_0 = \pi(u_{\varepsilon,\delta}) \in \mathcal{M}\backslash E_\delta$, and let A, B and X be as in Theorem 1.4. As before, $\inf I_\lambda(B) > 0$ if r is sufficiently small and

$$I_\lambda(R\,\pi((1-t)\,w + tv_0)) \leq 0 \quad \forall w \in E_\delta, \quad 0 \leq t \leq 1$$

if R is sufficiently large. On the other hand,

$$I_\lambda(tw) \leq \frac{t^p}{p}\left(1 - \frac{\lambda_k}{\Psi(w)}\right) \leq CR^p\delta^{N-sp} \quad \forall w \in E_\delta, \quad 0 \leq t \leq R$$

by (3.8), where C denotes a generic positive constant independent of ε and δ. It follows that

$$\sup I_\lambda(A) \leq CR^p\delta^{N-sp} < \inf I_\lambda(B)$$

if δ is sufficiently small. As in the last proof, it only remains to show that (see (3.14))

$$\sup_{(w,t)\in E_\delta\times[0,1]} \frac{\|(1-t)\,w + tv_0\|^p - \lambda_k\,|(1-t)\,w + tv_0|_p^p}{|(1-t)\,w + tv_0|_{p_s^*}^p} < S \tag{3.19}$$

if ε and δ are suitably small. We estimate the integral I_3 in (3.15) using the elementary inequality

$$|a + b|^p \leq |a|^p + |b|^p + C \left(|a|^{p-1} |b| + |a| \, |b|^{p-1} \right) \quad \forall a, b \in \mathbb{R} \tag{3.20}$$

to get

$$I_3 \leq (1 - t)^p \int_{A_3} \frac{|w(x) - w(y)|^p}{|x - y|^{N+sp}} \, dxdy + t^p \int_{A_3} \frac{|v_0(x) - v_0(y)|^p}{|x - y|^{N+sp}} \, dxdy$$

$$+ C \, (1-t)^{p-1} \int_{A_3} \frac{|w(x)|^{p-1} \, v_0(y)}{|x-y|^{N+sp}} \, dxdy + C \, (1-t) \int_{A_3} \frac{|w(x)| \, v_0(y)^{p-1}}{|x-y|^{N+sp}} \, dxdy$$

$$=: (1 - t)^p \, I_4 + t^p \, I_5 + C \, (1 - t)^{p-1} J_1 + C \, (1 - t) J_{p-1}. \tag{3.21}$$

As before, $I_1 + 2I_4$, $I_2 + 2I_5 \leq 1$ and for $q = 1, p - 1$,

$$J_q := \int_{A_3} \frac{|w(x)|^{p-q} \, v_0(y)^q}{|x - y|^{N+sp}} \, dxdy \leq C \int_{A_3} \frac{u_{\varepsilon,\delta}(y)^q}{|x|^{N+sp}} \, dxdy \leq \frac{C}{\delta^{sp}} \int_{B_{\theta\delta}(0)} U_\varepsilon(y)^q \, dy$$

$$\leq \frac{C \, \varepsilon^{N-(N-sp) \, q/p}}{\delta^{sp}} \int_{B_{\theta\delta/\varepsilon}(0)} U(y)^q \, dy.$$

We take $\delta = \varepsilon^\alpha$ with $\alpha \in (0, 1)$ and use (2.4) to estimate the last integral to get

$$J_q \leq C \, \varepsilon^{(N-sp)[p \, (p-q-1) \, \alpha + q]/p \, (p-1)}.$$

So combining (3.15) and (3.21) gives

$$\|(1 - t) \, w + t v_0\|^p \leq (1 - t)^p + t^p + \widetilde{J}_1 + \widetilde{J}_{p-1}, \tag{3.22}$$

where

$$\widetilde{J}_q := C \, (1 - t)^{p-q} \, J_q \leq C \, (1 - t)^{p-q} \, \varepsilon^{(N-sp)[p \, (p-q-1) \, \alpha + q]/p \, (p-1)}.$$

Young's inequality then gives

$$\widetilde{J}_q \leq \frac{\kappa}{3} \, (1 - t)^{p_s^*} + C \, \varepsilon^{sp+\beta_q(\alpha)} \kappa^{-\gamma_q} \tag{3.23}$$

for any $\kappa > 0$, where

$$\beta_q(\alpha) = \frac{[N^2 - sp^2 \, (N + s)](p - 1)(p - q) - Np \, (N - sp)(p - q - 1)(\alpha_0 - \alpha)}{[(N - sp) \, q + sp^2](p - 1)},$$

and

$$\alpha_0 = \frac{N - sp^2}{N - sp}, \quad \gamma_q = \frac{(N - sp)(p - q)}{Np - (N - sp)(p - q)}.$$

Then

$$\|(1-t) \, w + t v_0\|^p \leq (1-t)^p + t^p + \frac{2\kappa}{3} \, (1 - t)^{p_s^*} + C \, \varepsilon^{sp} \left(\frac{\varepsilon^{\beta_1(\alpha)}}{\kappa^{-\gamma_1}} + \frac{\varepsilon^{\beta_{p-1}(\alpha)}}{\kappa^{-\gamma_{p-1}}} \right) \tag{3.24}$$

by (3.22) and (3.23). Using $N^2/(N+s) > sp^2$, we fix $\alpha < \alpha_0$ so close to α_0 that $\beta_q(\alpha) > 0$ for $q = 0, 1, p - 1, p$. By (3.8) and Young's inequality,

$$\lambda_k \, (1-t)^p \, |w|_p^p \geq (1-t)^p \left(1 - C \, \varepsilon^{(N-sp) \, \alpha} \right)$$

$$\geq (1-t)^p - \frac{\kappa}{3} \, (1 - t)^{p_s^*} - C \, \varepsilon^{sp+\beta_0(\alpha)} \kappa^{-\gamma_0}. \tag{3.25}$$

By (3.24), (3.18), and (3.25), the quotient $Q(w, t)$ in (3.19) satisfies

$$Q(w, t) \leq \frac{\left(1 - \lambda_k |v_0|_p^p\right) t^p + \kappa (1 - t)^{p_s^*} + C \varepsilon^{sp + \beta(\alpha)} \kappa^{-\gamma}}{\left[(1 - t)^{p_s^*} |w|_{p_s^*}^{p_s^*} + t^{p_s^*} |v_0|_{p_s^*}^{p_s^*}\right]^{p/p_s^*}}, \tag{3.26}$$

where

$$\beta(\alpha) = \min \left\{\beta_0(\alpha), \beta_1(\alpha), \beta_{p-1}(\alpha)\right\} > 0, \quad \gamma = \max \left\{\gamma_0, \gamma_1, \gamma_{p-1}\right\} = \frac{N}{sp} - 1.$$

As before, the denominator is bounded away from zero if ε is sufficiently small, so it follows that

$$\sup_{(w,t) \in E_{\varepsilon^\alpha} \times [0, t_0)} Q(w, t) \leq C(t_0^p + \kappa + \varepsilon^{sp + \beta(\alpha)} \kappa^{-\gamma}) < S$$

for some $t_0 > 0$ if κ and ε are sufficiently small. For $t \geq t_0$, rewriting the right-hand side of (3.26) as

$$\frac{\dfrac{1 - \lambda_k |v_0|_p^p}{|v_0|_{p_s^*}^p} + \dfrac{\kappa (1 - t)^{p_s^*} + C \varepsilon^{sp + \beta(\alpha)} \kappa^{-\gamma}}{t^p |v_0|_{p_s^*}^p}}{\left[\dfrac{|w|_{p_s^*}^{p_s^*}}{t^{p_s^*} |v_0|_{p_s^*}^{p_s^*}} (1 - t)^{p_s^*} + 1\right]^{p/p_s^*}}$$

gives $Q(w, t) \leq g((1 - t)^{p_s^*})$, where

$$g(\tau) = \frac{S_{\varepsilon, \varepsilon^\alpha}(\lambda_k) + C \left(\kappa \tau + \varepsilon^{sp + \beta(\alpha)} \kappa^{-\gamma}\right)}{(1 + C^{-1} \tau)^{p/p_s^*}}, \quad C = C(N, p, s, t_0).$$

Since $0 \leq (1 - t)^{p_s^*} < 1$, then

$$Q(w, t) \leq S_{\varepsilon, \varepsilon^\alpha}(\lambda_k) + C\left(\kappa + \varepsilon^{sp + \beta(\alpha)} \kappa^{-\gamma}\right).$$

If $S_{\varepsilon_j, \varepsilon_j^\alpha}(\lambda_k) < S/2$ for some sequence $\varepsilon_j \to 0$, then the right-hand side is less than S for sufficiently small κ and $\varepsilon = \varepsilon_j$ with sufficiently large j, so we may assume that $S_{\varepsilon, \varepsilon^\alpha}(\lambda_k) \geq S/2$ for all sufficiently small ε. Then it is easily seen that if $\kappa \leq (p/p_s^*) S/2 (C + 1)$, then $g'(\tau) \leq 0$ for all $\tau \in [0, 1]$ and hence the maximum of $g((1 - t)^{p_s^*})$ on $[t_0, 1]$ occurs at $t = 1$. So, we reach

$$Q(w, t) \leq S_{\varepsilon, \varepsilon^\alpha}(\lambda_k) + C \varepsilon^{sp + \beta(\alpha)} \kappa^{-\gamma} \leq S - \left(\frac{\lambda_k}{C} - C \varepsilon^{\beta_p(\alpha)} - C \varepsilon^{\beta(\alpha)} \kappa^{-\gamma}\right) \varepsilon^{sp}$$

by (2.20), and the desired conclusion follows for sufficiently small κ and ε.

3.4 Case 4: $(N^3 + s^3 p^3)/N (N + s) > sp^2$, $\partial\Omega \in C^{1,1}$, and $\lambda = \lambda_k$

By Iannizzotto et al. [29, Theorem 4.4], there exists a constant $C = C(N, \Omega, p, s) > 0$ such that for any $v \in W_0^{s,p}(\Omega)$ with $(-\Delta)_p^s v \in L^\infty(\Omega)$,

$$|v(x)| \leq C \left|(-\Delta)_p^s v\right|_\infty^{1/(p-1)} d^s(x) \quad \forall x \in \mathbb{R}^N, \tag{3.27}$$

where $d(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega)$.

Lemma 3.3 *Assume that* $\partial\Omega \in C^{1,1}$. *Given* $\alpha, \beta > 1$, *there exists a constant* $C = C(N, \Omega, p, s, \alpha, \beta) > 0$ *such that if* $B_{\beta r}(0) \subset \{x \in \Omega : d(x) < \alpha r\}$, *then for any* $v \in W_0^{s,p}(\Omega)$ *with* $(-\Delta)_p^s v \in L^\infty(\Omega)$,

$$|v(x) - v(y)| \leq C \left|(-\Delta)_p^s v\right|_\infty^{1/(p-1)} |x - y|^s \quad \forall x \in B_r(0), \quad y \in \Omega \backslash B_{\beta r}(0).$$

Proof By (3.27),

$$|v(x) - v(y)| \leq C \left|(-\Delta)_p^s v\right|_\infty^{1/(p-1)} (d^s(x) + d^s(y)).$$

Since $d(x) \leq \alpha r$ and $|x - y| \geq (\beta - 1) r$,

$$d(x) \leq \frac{\alpha}{\beta - 1} |x - y|,$$

and since $d(y) \leq d(x) + |x - y|$ by the triangle inequality and $s < 1$,

$$d^s(y) \leq d^s(x) + |x - y|^s.$$

So the desired inequality holds with the constant $C \left(2\alpha^s/(\beta - 1)^s + 1\right)$. $\qquad\square$

Let η_δ be the cut-off function in Lemma 2.6.

Lemma 3.4 *Assume that* $\partial\Omega \in C^{1,1}$. *Then there exists a constant* $C = C(N, \Omega, p, s) > 0$ *such that for any* $v \in W_0^{s,p}(\Omega)$ *such that* $(-\Delta)_p^s v \in L^\infty(\Omega)$ *and* $\delta > 0$ *such that* $B_{6\theta\delta}(0) \subset \{x \in \Omega : d(x) < 12\theta\delta\}$,

$$\|v\eta_\delta\|^p \leq \|v\|^p + C \left|(-\Delta)_p^s v\right|_\infty^{p/(p-1)} \delta^N. \tag{3.28}$$

Proof Set $f = (-\Delta)_p^s v$ and $K = |f|_\infty < \infty$. Then

$$|v(x)| \leq CK^{1/(p-1)} \delta^s \quad \forall x \in B_{6\theta\delta}(0) \tag{3.29}$$

by (3.27), and for $k = 3, 5$,

$$|v(x) - v(y)| \leq CK^{1/(p-1)} |x - y|^s \quad \forall x \in B_{k\theta\delta}(0), \quad y \in \Omega \backslash B_{(k+1)\theta\delta}(0) \tag{3.30}$$

by Lemma 3.3. We proceed splitting $\|v\eta_\delta\|^p$ as in the proof of Lemma 2.6, and estimate the integral

$$I_3 = \int_{A_3} \frac{|v(x) - v(y) + v(x)(\eta_\delta(x) - 1)|^p}{|x - y|^{N+sp}} \, dx dy$$

in (2.14) using the elementary inequality

$$|a + b|^p \leq |a|^p + C \left(|a|^{p-1}|b| + |b|^p\right) \quad \forall a, b \in \mathbb{R}$$

to get

$$I_3 \leq \int_{A_3} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} \, dx dy + C \int_{A_3} \frac{|v(x) - v(y)|^{p-1} |v(x)|}{|x - y|^{N+sp}} \, dx dy$$
$$+ C \int_{A_3} \frac{|v(x)|^p}{|x - y|^{N+sp}} \, dx dy =: I_4 + C I_5 + C I_6.$$

We have $I_1 + 2I_4 \leq \|v\|^p$. By (3.29) and (3.30), and since $|x - y| \geq |y|/4$ on A_3,

$$I_5 \leq C K^{p/(p-1)} \delta^s \int_{A_3} \frac{dxdy}{|y|^{N+s}} = C K^{p/(p-1)} \delta^N,$$

$$I_6 \leq C K^{p/(p-1)} \delta^{sp} \int_{A_3} \frac{dxdy}{|y|^{N+sp}} = C K^{p/(p-1)} \delta^N.$$

To estimate I_2, let φ_δ be as in the proof of Lemma 2.6. Since $\varphi_\delta = \eta_\delta$ in A_2,

$$I_2 \leq C \left(\int_{A_2} \frac{|v(x)|^p \, |\varphi_\delta(x) - \varphi_\delta(y)|^p}{|x - y|^{N+sp}} \, dxdy + \int_{A_2} \frac{|v(x) - v(y)|^p \, \varphi_\delta(y)^p}{|x - y|^{N+sp}} \, dxdy \right)$$
$$=: C \, (I_7 + I_8).$$

By (3.29) and $\|\varphi_\delta\|^p = \delta^{N-ps} \, \|\varphi\|^p$ by scaling, we get

$$I_7 \leq C K^{p/(p-1)} \delta^{sp} \, \|\varphi_\delta\|^p = C K^{p/(p-1)} \delta^N. \tag{3.31}$$

By Lemma 2.4,

$$I_8 \leq 2 \int_\Omega f(x) \, v(x) \, \varphi_\delta(x)^p \, dx + C \int_{\mathbb{R}^{2N}} \frac{|v(x)|^p \, |\varphi_\delta(x) - \varphi_\delta(y)|^p}{|x - y|^{N+sp}} \, dxdy =: 2I_9 + C I_{10}.$$

Since $\varphi = 0$ outside $B_{5\theta\delta}(0)$,

$$I_9 \leq \int_{B_{5\theta\delta}(0)} |f(x)| \, |v(x)| \, dx \leq C K^{p/(p-1)} \delta^{N+s}$$

by (3.29) again. Changing variables gives

$$I_{10} = \delta^{N-sp} \int_{\mathbb{R}^{2N}} \frac{|v(\delta x)|^p \, |\varphi(x) - \varphi(y)|^p}{|x - y|^{N+sp}} \, dxdy.$$

We have $|v(\delta x)| \leq C K^{1/(p-1)} d^s(\delta x)$ by (3.27), and $d(\delta x) \leq d(0) + \delta \, |x| \leq C\delta$ since $d(0) \leq 6\theta\delta$ and Ω is bounded, so the last integral is less than or equal to $C K^{p/(p-1)} \delta^{sp} \, \|\varphi\|^p$. Hence $I_{10} \leq C K^{p/(p-1)} \delta^N$. □

Since $\partial\Omega \in C^{1,1}$, for all sufficiently small $\delta > 0$, the ball $B_{6\theta\delta}(0)$ is contained in $\{x \in \Omega : d(x) < 12\theta\delta\}$ after a translation. Then by Lemma 3.4 and Proposition 3.1,

$$\|v_\delta\|^p \leq 1 + C\delta^N \quad \forall v \in E,$$

and using this inequality in place of (3.10) in the proof of Proposition 3.2 shows that (3.8) can now be strengthened to

$$\sup_{w \in E_\delta} \Psi(w) \leq \lambda_k + C\delta^N. \tag{3.32}$$

Proceeding as in the last subsection, we have to verify (3.19) for suitably small ε and δ. Since the argument is similar, we only point out where it differs. Let $v \in E$ and let

$$w = \pi(v_\delta) = v_\delta / \|v_\delta\|.$$

As noted in the proof of Proposition 3.2, $\|v_\delta\|$ is bounded away from zero, so

$$J_q \leq C \int_{A_3} \frac{|v_\delta(x)|^{p-q} \, u_{\varepsilon,\delta}(y)^q}{|x - y|^{N+sp}} \, dxdy,$$

where $A_3 = B_{2\theta\delta}(0)^c \times B_{\theta\delta}(0)$. By Lemma 3.3, (3.27), and Proposition 3.1, and since

$$|x - y| \geq |x|/2 \geq \theta\delta, \quad \text{on } A_3,$$

we get

$$|v_\delta(x)|^{p-q} \leq |v(x)|^{p-q} \leq C \left(|v(x) - v(y)|^{p-q} + |v(y)|^{p-q} \right)$$
$$\leq C \left(|x - y|^{s(p-q)} + \delta^{s(p-q)} \right) \leq C \, |x - y|^{s(p-q)},$$

so

$$J_q \leq C \int_{A_3} \frac{u_{\varepsilon,\delta}(y)^q}{|x|^{N+sq}} \, dx dy \leq \frac{C}{\delta^{sq}} \int_{B_{\theta\delta}(0)} U_\varepsilon(y)^q \, dy$$
$$\leq C \, \varepsilon^{\{p\,[(p-q-1)\,N + sq]\,\alpha + (N-sp)\,q\}/p\,(p-1)}.$$

Then (3.23) holds with

$$\beta_q(\alpha) = \frac{[N^3 + s^3 p^3 - sp^2 N\,(N+s)]\,(p-1)\,(p-q) - Np\,(N-sp)\,[N\,(p-q-1) + sq]\,(\alpha_0 - \alpha)}{(N-sp)[(N-sp)\,q + sp^2]\,(p-1)},$$

and so does (3.25) by (3.32). Using

$$(N^3 + s^3 p^3)/N\,(N+s) > sp^2,$$

we fix $\alpha < \alpha_0$ so close to α_0 that $\beta_q(\alpha) > 0$ for $q = 0, 1, p - 1, p$ and proceed as before.

References

1. Ambrosetti, A., Struwe, M.: A note on the problem $-\Delta u = \lambda u + u|u|^{2^*-2}$. Manuscr. Math. **54**, 373–379 (1986)
2. Andreu, F., Mazón, J.M., Rossi, J.D., Toledo, J.: A nonlocal $p-$Laplacian evolution equation with non-homogeneous Dirichlet boundary conditions. SIAM J. Math. Anal. **40**, 1815–185 (2009)
3. Arioli, G., Gazzola, F.: Some results on p-Laplace equations with a critical growth term. Differ. Integr. Equations **11**, 311–326 (1998)
4. Brasco, L., Mosconi, S., Squassina, M.: Optimal decay of extremal functions for the fractional Sobolev inequality, Cal. Var. Partial Differ. Equations **55**, 23 (2016)
5. Brasco, L., Lindgren, E.: Higher Sobolev regularity for the fractional p-Laplace equation in the superquadratic case. Adv. Math. (2016). doi:10.1016/j.aim.2016.03.039 (to appear)
6. Brasco, L., Parini, E.: The second eigenvalue of the fractional p-Laplacian. Adv. Calc. Var. (2016). doi:10.1515/acv-2015-0007 (to appear)
7. Brasco, L., Parini, E., Squassina, M.: Stability of variational eigenvalues for the fractional p-Laplacian. Discrete Contin. Dyn. Syst. Ser. A **36**, 1813–1845 (2016)
8. Brézis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. **36**, 437–477 (1983)
9. Caffarelli, L.: Non-local diffusions, drifts and games, In: Nonlinear partial differential equations, vol. 7 of Abel Symposia, pp. 37–52 (2012)
10. Capozzi, A., Fortunato, D., Palmieri, G.: An existence result for nonlinear elliptic problems involving critical Sobolev exponent. Ann. Inst. H. Poincaré Anal. Non Linéaire **2**, 463–470 (1985)
11. Cerami, G., Fortunato, D., Struwe, M.: Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**, 341–350 (1984)
12. Comte, M.: Solutions of elliptic equations with critical Sobolev exponent in dimension three. Nonlinear Anal. **17**, 445–455 (1991)
13. Costa, D.G., Silva, E.A.: A note on problems involving critical Sobolev exponents. Differ. Integr. Equations **8**, 673–679 (1995)

14. Degiovanni, M., Lancelotti, S.: Linking over cones and nontrivial solutions for p-Laplace equations with p-superlinear nonlinearity. Ann. Inst. H. Poincaré Anal. Non Linéaire **24**, 907–919 (2007)
15. Degiovanni, M., Lancelotti, S.: Linking solutions for p-Laplace equations with nonlinearity at critical growth. J. Funct. Anal. **256**, 3643–3659 (2009)
16. Di Castro, A., Kuusi, T., Palatucci, G.: Local behavior of fractional p-minimizers. Ann. Inst. H. Poincaré Anal. Non Linéaire (2016). doi:10.1016/j.anihpc.2015.04.003 (to appear)
17. Di Castro, A., Kuusi, T., Palatucci, G.: Nonlocal Harnack inequalities. J. Funct. Anal. **267**, 1807–1836 (2014)
18. Drábek, P., Huang, Y.X.: Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbb{R}^N$ with critical Sobolev exponent. J. Differ. Equations **140**, 106–132 (1997)
19. Egnell, H.: Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents. Arch. Ration. Mech. Anal. **104**, 57–77 (1998)
20. Fadell, E.R., Rabinowitz, P.H.: Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. Invent. Math. **45**, 139–174 (1978)
21. Franzina, G., Palatucci, G.: Fractional p-eigenvalues. Riv. Mat. Univ. Parma **5**, 373–386 (2014)
22. García Azorero, J.P., Peral Alonso, I.: Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. Trans. Am. Math. Soc. **323**, 877–895 (1991)
23. García Azorero, J.P., Peral Alonso, I.: Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues. Comm. Partial Differ. Equations **12**, 1389–1430 (1987)
24. Gazzola, F., Ruf, B.: Lower-order perturbations of critical growth nonlinearities in semilinear elliptic equations. Adv. Differ. Equations **2**, 555–572 (1997)
25. Ghoussoub, N., Yuan, C.: Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. Trans. Am. Math. Soc. **352**, 5703–5743 (2000)
26. Gonçalves, J.V., Alves, C.O.: Existence of positive solutions for m-Laplacian equations in $\mathbb{R}^N$ involving critical Sobolev exponents. Nonlinear Anal. **32**, 53–70 (1998)
27. Guedda, M., Véron, L.: Quasilinear elliptic equations involving critical Sobolev exponents. Nonlinear Anal. **13**, 879–902 (1989)
28. Iannizzotto, A., Liu, S., Perera, K., Squassina, M.: Existence results for fractional p-Laplacian problems via Morse theory. Adv. Calc. Var. **9**, 101–125 (2016)
29. Iannizzotto, A., Mosconi, S., Squassina, M.: Global Hölder regularity for the fractional p-Laplacian. Rev. Mat. Iberoam. (2016). arXiv:1411.2956 (to appear)
30. Iannizzotto, A., Squassina, M.: Weyl-type laws for fractional p-eigenvalue problems. Asymptot. Anal. **88**, 233–245 (2014)
31. Ishii, H., Nakamura, G.: A class of integral equations and approximation of $p-$Laplace equations. Calc. Var. Partial Differ. Equations **37**, 485–522 (2010)
32. Kuusi, T., Mingione, G., Sire, Y.: Nonlocal equations with measure data. Comm. Math. Phys. **337**(3), 1317–1368 (2015)
33. Lieb, E.H.: Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. Math. **118**(2), 349–374 (1983)
34. Lindgren, E.: Hölder estimates for viscosity solutions of equations of fractional p-Laplace type (2016). arXiv:1405.6612 (preprint)
35. Lindgren, E., Lindqvist, P.: Fractional eigenvalues. Calc. Var. Partial Differ. Equations **49**, 795–826 (2014)
36. Perera, K.: Nontrivial critical groups in p-Laplacian problems via the Yang index. Topol. Methods Nonlinear Anal. **21**, 301–309 (2003)
37. Perera, K., Agarwal, R.P., O'Regan, D.: Morse theoretic aspects of p-Laplacian type operators. Mathematical Surveys and Monographs, vol. 161. American Mathematical Society, Providence (2010)
38. Perera, K., Squassina, M., Yang, Y.: Bifurcation and multiplicity results for critical fractional p-Laplacian problems. Math. Nachr. **289**, 332–342 (2015)
39. Perera, K., Szulkin, A.: p-Laplacian problems where the nonlinearity crosses an eigenvalue. Discrete Contin. Dyn. Syst. **13**, 743–753 (2005)
40. Rabinowitz, P.H.: Some critical point theorems and applications to semilinear elliptic partial differential equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **5**(4), 215–223 (1978)
41. Servadei, R.: The Yamabe equation in a non-local setting. Adv. Nonlinear Anal. **2**, 235–270 (2013)
42. Servadei, R.: A critical fractional Laplace equation in the resonant case. Topol. Methods Nonlinear Anal. **43**, 251–267 (2014)
43. Servadei, R., Valdinoci, E.: A Brezis-Nirenberg result for non-local critical equations in low dimension. Commun. Pure Appl. Anal. **12**, 2445–2464 (2013)
44. Servadei, R., Valdinoci, E.: The Brezis-Nirenberg result for the fractional Laplacian. Trans. Am. Math. Soc. **367**, 67–102 (2005)

45. Silva, E.A.B., Soares, S.H.M.: Quasilinear Dirichlet problems in $\mathbb{R}^n$ with critical growth. Nonlinear Anal. **43**, 1–20 (2001)
46. Silva, E.A.B., Xavier, M.S.: Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents. Ann. Inst. H. Poincaré Anal. Non Linéaire **20**, 341–358 (2003)
47. Wei, Z.H., Wu, X.M.: A multiplicity result for quasilinear elliptic equations involving critical Sobolev exponents. Nonlinear Anal. **18**, 559–567 (1992)
48. Yang, Y., Perera, K.: N-Laplacian problems with critical Trudinger-Moser nonlinearities. Ann. Sci. Norm. Super. Pisa Cl. Sci. (5) (2016). doi:10.2422/2036-2145.201406_004 (to appear)
49. Zhang, D.: On multiple solutions of $\Delta u + \lambda u + |u|^{4/(n-2)} u = 0$. Nonlinear Anal. **13**, 353–372 (1989)