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Nonlinear Differential Equations and Applications NoDEA

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Nonautonomous fractional problems with exponential growth

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Abstract. We study a class of nonlinear nonautonomous nonlocal equations with subcritical and critical exponential nonlinearity. The involved potential can vanish at infinity.

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1. Introduction and main results

We consider existence of positive solutions for the following class of equations

$$(-\Delta)^{1/2}u + u = K(x)g(u) \quad \text{ in } \mathbb{R}. \tag{1.1}$$

Here $(-\Delta)^{1/2}$ stands for the 1/2-Laplacian, $K : \mathbb{R} \to \mathbb{R}$ is a positive function and g is a continuous function with exponential subcritical or critical growth in the sense of the Trudinger-Moser embedding due to Ozawa [17]. Recently, a great attention has been focused on the study of nonlocal operators. These arise in thin obstacle problems, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, water waves, etc. [16]. The fractional laplacian $(-\Delta)^s$ for $s \in (0,1)$ of a function $u : \mathbb{R} \to \mathbb{R}$ is defined by $\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi)$, where $\mathcal{F}$ is the Fourier transform. Since the problem is set on the whole space one has to

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tackle compactness issues, which can be overcome by considering suitable assumptions of the vanishing behaviour of K at infinity. Recently the problem in $\mathbb{R}^N$ with $N > 2s, s \in (0,1), 2_s^* = 2N/(N-2s)$,

$$(-\Delta)^s u + V(x)u = K(x)g(u) + \lambda |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^N.$$

where g has subcritical growth has been investigated in [9] inspired by some arguments of [4]. The aim of this paper is to extend the achievements of [9] to cover the case where the nonlinearity is allowed to grow at an exponential rate. As is pointed out in [14], nonlocal problems with linear fractional diffusion involving exponential growth should be set in $\mathbb{R}$. In that manuscript the authors prove existence results for problems involving critical and subcritical exponential growth nonlinearities and 1/2-Laplacian in a bounded domain. The main ingredient is the Trudinger-Moser type inequality [17] (see Proposition 2.3). For related problems involving Moser-Trudinger embeddings we would like to mention the celebrated works [15,18] as well as [1–3,8,10–12] and the references therein. As known, Caffarelli and Silvestre [7] developed a local interpretation of the fractional Laplacian by considering a Neumann type operator in $\mathbb{R}_+^{N+1} = \{(x,t) \in \mathbb{R}^{N+1} : t > 0\}$ (see also [6] for bounded domains of $\mathbb{R}^N$, with $N > 1$). For the case $N = 1$ the main reference is the work by Frank and Lenzmann [13]. The space $\dot{H}^{1/2}(\mathbb{R})$ is the completion of $C_0^\infty(\mathbb{R})$ under

$$[u]_{1/2} := \left(\int_{\mathbb{R}} |\xi||\mathcal{F}u|^2 \,\mathrm{d}\xi\right)^{1/2} = \left(\int_{\mathbb{R}} |(-\Delta)^{1/4}u|^2 \,\mathrm{d}x\right)^{1/2},$$

while $H^{1/2}(\mathbb{R})$ is the Hilbert space of $u \in L^2(\mathbb{R})$ such that $[u]_{H^{1/2}} < \infty$, endowed with the norm

$$\|u\|_{1/2} = \left(\|u\|_{L^2}^2 + [u]_{1/2}^2\right)^{1/2}.$$

The space $X^1(\mathbb{R}_+^2)$ is defined as the completion of $C_0^\infty(\overline{\mathbb{R}_+^2})$ under the seminorm

$$\|w\|_{X^1} := \left(\int_{\mathbb{R}_+^2} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

For a function $u \in \dot{H}^{1/2}(\mathbb{R})$, the solution $w \in X^1(\mathbb{R}_+^2)$ to

$$\begin{cases} -\mathrm{div}(\nabla w) = 0 & \text{in } \mathbb{R}_+^2 \\ w = u & \text{on } \mathbb{R} \times \{0\} \end{cases} \tag{1.2}$$

is called harmonic extension $w = E_{1/2}(u)$ of u and it is proved in [5,7] for $N > 1$ and in [13] for $N = 1$, that, up to some constant,

$$\lim_{y \to 0^+} \frac{\partial w}{\partial y}(x,y) = -(-\Delta)^{1/2}u(x).$$

Also, up to a constant, $[u]_{1/2} = \|w\|_{X^1}$, see [5]. Our problem (1.1) will be studied in the half-space,

$$\begin{cases} -\mathrm{div}(\nabla w) = 0 & \text{in } \mathbb{R}_+^2 \\ -\frac{\partial w}{\partial \nu} = -u + K(x)g(u) & \text{on } \mathbb{R} \times \{0\}, \end{cases} \tag{1.3}$$

where $\frac{\partial w}{\partial \nu} = \lim_{y\to 0^+} \frac{\partial w}{\partial y}(x,y)$. We look for positive solutions in the Hilbert space E defined by

$$E := \left\{ w \in X^1(\mathbb{R}^2_+) : \int_{\mathbb{R}} w(x,0)^2 \, \mathrm{d}x < \infty \right\},$$

endowed with the norm

$$\|w\| := \left(\int_{\mathbb{R}^2_+} |\nabla w|^2 \, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} w(x,0)^2 \, \mathrm{d}x \right)^{1/2}.$$

We mention that E is precisely the space introduced in [13, formula (3.13)]. Consider now the energy functional $J : E \to \mathbb{R}$ associated to (1.3) given by

$$J(w) := \frac{1}{2}\|w\|^2 - \int_{\mathbb{R}} K(x)G(w(x,0)\, \mathrm{d}x, \qquad G(s) := \int_0^s g(t)\, \mathrm{d}t, \tag{1.4}$$

which, under suitable assumptions, is C^1 (see Proposition 2.9) and, for all $w, v \in E$,

$$\begin{aligned} J'(w)(v) = \int_{\mathbb{R}^2_+} \nabla w \cdot \nabla v \, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} w(x,0)v(x,0)\, \mathrm{d}x \\ - \int_{\mathbb{R}} K(x)g(w(x,0))v(x,0)\, \mathrm{d}x. \end{aligned} \tag{1.5}$$

We now formulate assumptions for K and g in order to be able to solve (1.1).
• *Assumption on K.* We assume $K \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ with $K > 0$. Furthermore, if $\{A_n\}$ is a sequence of measurable sets of $\mathbb{R}$ with $|A_n| \le R$ for some $R > 0$,

$$\lim_{r\to\infty} \int_{A_n \cap B_r^c(0)} K(x)\, \mathrm{d}x = 0, \quad \text{uniformly with respect to } n \in \mathbb{N}. \tag{1.6}$$

• *Assumptions on g-subcritical case*

(g1) (behaviour at zero). $g : \mathbb{R} \to \mathbb{R}^+$ is continuous with $g = 0$ on $\mathbb{R}^-$ and

$$\limsup_{s\to 0^+} \frac{g(s)}{s} = 0.$$

(g2) (subcritical growth). it holds

$$\limsup_{s\to+\infty} \frac{g(s)}{e^{\alpha s^2} - 1} = 0, \quad \text{for all } \alpha > 0.$$

(g3) (super-quadraticity). $\frac{g(s)}{s}$ is non-decreasing in $\mathbb{R}^+$ and

$$\limsup_{s\to+\infty} \frac{G(s)}{s^2} = +\infty.$$

Under assumption (1.6) on K, we have the following

Theorem 1.1. *Assume* (g1)–(g3). *Then* (1.1) *admits a positive solution* $u \in H^{1/2}(\mathbb{R})$.

• *Assumptions on g-critical case*

(g2)′ (critical growth). there exists $\omega \in (0,\pi]$ and $\alpha_0 \in (0,\omega]$ (cf. Proposition 2.3)

$$\limsup_{s\to+\infty} \frac{g(s)}{e^{\alpha s^2}-1} = 0, \quad \text{for all } \alpha > \alpha_0,$$
$$\limsup_{s\to+\infty} \frac{g(s)}{e^{\alpha s^2}-1} = +\infty, \quad \text{for all } \alpha < \alpha_0.$$

(g3)′ (super-quadraticity). $\frac{g(s)}{s}$ is non-decreasing in $\mathbb{R}^+$ and there exists $q > 2$ such that

$$G(s) \geq C_q s^q, \quad \text{for all } s \in \mathbb{R}^+,$$

where $C_q > 0$ is sufficiently large.

(AR) (Ambrosetti-Rabinowitz). there exists $\vartheta > 2$ such that

$$\vartheta G(s) \leq sg(s), \qquad \text{for all } s \in \mathbb{R}^+. \tag{AR}$$

Under assumption (1.6) on K, we also have the following

Theorem 1.2. *Assume* (g1)–(g2)′–(g3)′ *and (AR). Then* (1.1) *has a positive solution* $u \in H^{1/2}(\mathbb{R})$ *provided that the constant* C_q *in condition* (g3)′ *satisfies*

$$C_q > \Big(\frac{2\vartheta}{\vartheta-2}\frac{q-2}{2q}\Big)^{\frac{q-2}{2}} \frac{\mathbb{S}_q^q}{qL},$$

with

$$\mathbb{S}_q = \frac{\|\psi\|}{\|\psi(\cdot,0)\|_{L^q}} > 0, \qquad L = \inf_{\mathrm{supt}(\psi)} K > 0,$$

where ψ *is a fixed smooth and compactly supported function on* $\mathbb{R}$.

The above results extend the existence results obtained in [14] in the case where the problem is set on the whole $\mathbb{R}$ and compactness issues have to be tackled. Also, they constitute an extension to the results of [9] to the case where the nonlinearity is allowed for an exponential growth, critical or subcritical with respect to the Trudinger-Moser inequality (2.8). As potentials K satisfying (1.6), one can consider Ks with $K(x) \to 0$ as $|x| \to \infty$. As examples of nonlinearities satisfying the above assumptions, define $g : \mathbb{R} \to \mathbb{R}^+$ by setting $g(t) = 0$ for all $t \leq 0$ and

$$g(t) = \begin{cases} t^q & \text{if } 0 \leq t \leq 1, \\ t^q e^{t^r-1} & \text{if } t \geq 1, \end{cases} \qquad 1 < r < 2, \quad q > 1,$$

This function satisfies (g1)–(g3). Define $g : \mathbb{R} \to \mathbb{R}^+$ by setting $g(t) = 0$ for all $t \leq 0$ and

$$g(t) = C_q \begin{cases} t^q & \text{if } 0 \leq t \leq 1, \\ t^q e^{\alpha_0(t^2-1)} & \text{if } t > 1, \end{cases} \qquad q > 2,$$

where $\alpha_0 \in (0,\omega]$ and $C_q > 0$ is sufficiently large. This map satisfies (g1)–(g2)′–(g3)′ and (AR).

2. Preliminary results

In this section we provide some preliminary stuff. Consider the weighted Banach space

$$L^p_K(\mathbb{R}) = \left\{u : \mathbb{R} \to \mathbb{R} \text{ measurable: } \int_{\mathbb{R}} K(x)|u|^p \,\mathrm{d}x < \infty\right\}, \quad p \in (1, \infty),$$

endowed with the norm

$$\|u\|_{L^p_K} = \left(\int_{\mathbb{R}} K(x)|u|^p \,\mathrm{d}x\right)^{1/p}.$$

Remark 2.1. Let $\Omega \subset \mathbb{R}$ be a bounded domain. Then the space $H^{1/2}(\mathbb{R})$ is compactly embedded into $L^q(\Omega)$ for any $q \geq 2$. Firstly, [16, Theorem 7.1] provides the compactness in $L^2(\Omega)$. Then the assertion follows by interpolation, since $H^{1/2}(\mathbb{R})$ is continuously embedded into $L^m(\mathbb{R})$ for any $m \geq 2$, by [16, Theorem 6.9]. Finally, in view of [13, Proposition 3.6], also E is compactly embedded into $L^q(\Omega)$ for any $q \geq 2$.

The first result, is a compact injection for the space E.

Proposition 2.2. *E is compactly embedded into $L^q_K(\mathbb{R})$ for all $q \in (2, \infty)$.*

Proof. Let $q > 2, r > q$ and $\varepsilon > 0$. Then, there exist $0 < s_0(\varepsilon) < s_1(\varepsilon)$, a positive constant $C(\varepsilon)$ and C_0 depending only on K, such that

$$K(x)|s|^q \leq \varepsilon C_0(|s|^2 + |s|^r) + C(\varepsilon)K(x)\chi_{[s_0(\varepsilon),s_1(\varepsilon)]}(|s|)|s|^q, \quad x, s \in \mathbb{R}. \tag{2.1}$$

Therefore we obtain, for every $w \in E$ and $r > 0$,

$$\int_{B^c_r(0)} K(x)|w(x,0)|^q \,\mathrm{d}x \leq \varepsilon Q(w) + C(\varepsilon)s_1(\varepsilon)^q \int_{A_\varepsilon \cap B^c_r(0)} K(x) \,\mathrm{d}x, \tag{2.2}$$

where we have set

$$Q(w) := C_0\|w(\cdot,0)\|^2_{L^2} + C_0\|w(\cdot,0)\|^r_{L^r}, \quad A_\varepsilon := \{x \in \mathbb{R} : s_0(\varepsilon) \leq |w(x,0)| \leq s_1(\varepsilon)\}. \tag{2.3}$$

If $(w_n) \subset E$ is such that $w_n \rightharpoonup w$ weakly in E for some $w \in E$, there exists $M > 0$ such that

$$\begin{gathered}\int_{\mathbb{R}^2_+} |\nabla w_n|^2 \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} |w_n(x,0)|^2 \,\mathrm{d}x \leq M, \\ \int_{\mathbb{R}} |w_n(x,0)|^r \,\mathrm{d}x \leq M, \quad \text{for all } r \geq 2.\end{gathered} \tag{2.4}$$

The second inequality is due to the continuous injection of $H^{1/2}(\mathbb{R})$ in an arbitrary $L^r(\mathbb{R})$ space with $r \geq 2$, see [16, Theorem 6.9]. Hence $Q(w_n)$ is bounded. On the other hand, if

$$A^n_\varepsilon := \big\{x \in \mathbb{R} : s_0(\varepsilon) \leq |w_n(x,0)| \leq s_1(\varepsilon)\big\},$$

we get

$$s_0(\varepsilon)^q|A^n_\varepsilon| \leq \int_{A^n_\varepsilon} |w_n(x,0)|^q \,\mathrm{d}x \leq \int_{\mathbb{R}^N} |w_n(x,0)|^q \,\mathrm{d}x \leq M, \quad \text{for all } n \in \mathbb{N}.$$

which implies that $\sup_{n\in\mathbb{N}} |A_\varepsilon^n| < +\infty$.

Then, in light of (1.6), there exists $r(\varepsilon) > 0$ such that

$$\int_{A_\varepsilon^n \cap B^c_{r(\varepsilon)}(0)} K(x)\, \mathrm{d}x < \frac{\varepsilon}{C(\varepsilon)s_1(\varepsilon)^q}, \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

Whence, in light of (2.2), we conclude

$$\int_{B^c_{r(\varepsilon)}(0)} K(x)|w_n(x,0)|^q\, \mathrm{d}x \leq (2C_0M+1)\varepsilon. \tag{2.6}$$

On account of Remark 2.1, we have

$$\lim_n \int_{B_{r(\varepsilon)}(0)} K(x)|w_n(x,0)|^q\, \mathrm{d}x = \int_{B_{r(\varepsilon)}(0)} K(x)|w(x,0)|^q\, \mathrm{d}x. \tag{2.7}$$

Combining (2.6)–(2.7), yields

$$\lim_n \int_{\mathbb{R}} K(x)|w_n(x,0)|^q\, \mathrm{d}x = \int_{\mathbb{R}} K(x)|w(x,0)|^q\, \mathrm{d}x.$$

This concludes the proof. □

Let us now recall the Trudinger Moser type inequality of [17].

Proposition 2.3. *There exists $0 < \omega \leq \pi$ such that, for all $\alpha \in (0,\omega)$, there exists $H_\alpha > 0$ with*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, \mathrm{d}x \leq H_\alpha \|u\|^2_{L^2}, \tag{2.8}$$

for all $u \in H^{1/2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|^2_{L^2} \leq 1$.

Next, we state a useful Trudinger-Moser type bound for bounded sequences of E.

Lemma 2.4. *Let $(w_n) \subset E$ be a bounded sequence and set $\sup_{n\in\mathbb{N}} \|w_n\| = M$. Then*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} (e^{\alpha w_n(x,0)^2} - 1)\, \mathrm{d}x < \infty, \quad \textit{for every } 0 < \alpha < \frac{\omega}{M^2};$$

In particular, if $M \in (0,1)$, there exists $\alpha_M > \omega$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} (e^{\alpha_M w_n(x,0)^2} - 1)\, \mathrm{d}x < \infty.$$

Proof. Let $0 < \alpha M^2 < \omega$. Then, setting $u_n(x) = w_n(x,0)$, by virtue of Proposition 2.3, we have

$$\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1\right) \mathrm{d}x \leq \int_{\mathbb{R}} \left(e^{\alpha M^2 \left(\frac{u_n}{\|w_n\|}\right)^2} - 1\right) \mathrm{d}x \leq H_{\alpha M^2} \frac{\|u_n\|^2_{L^2}}{\|w_n\|^2} \leq H_{\alpha M^2}, \tag{2.9}$$

since $\|(-\Delta)^{1/4}u_n\|w_n\|^{-1}\|^2_{L^2} = \|(-\Delta)^{1/4}u_n\|^2_{L^2}/\|w_n\|^2 = \|w_n\|^2_{X^1}/\|w_n\|^2 \leq 1$. Concerning the last assertion, there exists $\alpha_M > \omega$ with $\alpha_M M^2 < \omega$ and the conclusion follows. □

Lemma 2.5. *Let $\alpha > 0$ and let $(w_n) \subset E$ be such that $w_n \to w$ strongly in E. Then*

$$\lim_n \int_{\mathbb{R}} \left(e^{\alpha w_n(x,0)^2} - 1\right) \mathrm{d}x = \int_{\mathbb{R}} \left(e^{\alpha w(x,0)^2} - 1\right) \mathrm{d}x.$$

Proof. By applying Lagrange's theorem to the function $s \mapsto e^{\alpha s^2}$, we get

$$\begin{aligned} \left|\left(e^{\alpha w_n(x,0)^2} - 1\right) - \left(e^{\alpha w(x,0)^2} - 1\right)\right| &\le 2\alpha(|w_n(x,0) - w(x,0)| + |w(x,0)|) \\ &\quad \times e^{2\alpha|w_n(x,0)-w(x,0)|^2} e^{2\alpha|w(x,0)|^2} |w_n(x,0) - w(x,0)|. \end{aligned}$$

The right-hand side splits into several terms. We shall handle one of them, namely

$$\begin{aligned} &(|w_n(x,0) - w(x,0)| + |w(x,0)|)(e^{2\alpha|w_n(x,0)-w(x,0)|^2} - 1) \\ &\quad\times(e^{2\alpha|w(x,0)|^2} - 1)|w_n(x,0) - w(x,0)|, \end{aligned}$$

since the other terms can be handled in a similar fashion. Then one applies Hölder inequality with four terms with exponents $r_1, r_4 \ge 2$ and $r_2, r_3 > 1$ such that $1/r_1 + 1/r_2 + 1/r_3 + 1/r_4 = 1$. Recall that $(e^x - 1)^r \le (e^{rx} - 1)$ holds for $r > 1$ and $x \ge 0$. For the first term, $\|w_n - w\|_{L^{r_1}} + \|w\|_{L^{r_1}} \le C$ by the continuous Sobolev embedding in any L^r space with $r \ge 2$. For the second term, since $\|w_n - w\| \to 0$, one can apply Lemma 2.4 (this is the key point of the proof) and deduce

$$\int_{\mathbb{R}} \left(e^{2r_2\alpha|w_n(x,0)-w(x,0)|^2} - 1\right) \mathrm{d}x \le C.$$

For the third term we have

$$\int_{\mathbb{R}} \left(e^{2r_3\alpha|w(x,0)|^2} - 1\right) \mathrm{d}x < \infty.$$

Here we used that $e^{u^2} - 1 \in L^1(\mathbb{R})$ for $u \in H^{1/2}(\mathbb{R})$, see the argument in [14, Proposition 2.5]. Finally the last term is estimated with $\|w_n - w\|_{L^{r_4}}$, which goes to zero and conclude the proof. $\square$

The next is a straightforward application of the generalized Dominated Convergence Theorem.

Lemma 2.6. *Let $f_n, g_n, h_n : \mathbb{R} \to \mathbb{R}^+$ sequences of nonnegative measurable functions. Assume that f_n converges pointwisely to 0 and that g_n, h_n converge pointwisely to $g, h : \mathbb{R} \to \mathbb{R}^+$. Assume also that, for every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that*

$$f_n \le \varepsilon g_n + C(\varepsilon)h_n, \quad n \in \mathbb{N}, \qquad \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} g_n \,\mathrm{d}x < \infty, \quad \lim_n \int_{\mathbb{R}} h_n \,\mathrm{d}x = \int_{\mathbb{R}} h \,\mathrm{d}x.$$

Then $f_n \to 0$ in $L^1(\mathbb{R})$.

We can now state the following compactness result for the subcritical growth case.

Proposition 2.7 (Compactness I-subcritical case). *Assume* (g1)–(g3). *Let* $(w_n) \subset E$ *be a bounded sequence and* $w_n \rightharpoonup w$ *in* E. *Then, up to a subsequence, the following facts hold:*

$$\lim_n \int_{\mathbb{R}} K(x)G(w_n(x,0))\,\mathrm{d}x = \int_{\mathbb{R}} K(x)G(w(x,0))\,\mathrm{d}x; \tag{2.10}$$

$$\lim_n \int_{\mathbb{R}} K(x)w_n(x,0)g(w_n(x,0))\,\mathrm{d}x = \int_{\mathbb{R}} K(x)w(x,0)g(w(x,0))\,\mathrm{d}x; \tag{2.11}$$

$$\lim_n \int_{\mathbb{R}} K(x)g(w_n(x,0))v(x,0)\,\mathrm{d}x = \int_{\mathbb{R}} K(x)g(w(x,0)v(x,0))\,\mathrm{d}x, \text{ for all } v \in E. \tag{2.12}$$

Proof. Let us prove (2.10) and (2.11). Let $\sup_{n\in\mathbb{N}} \|w_n\| =: M$. Let us also fix $\varepsilon > 0, q > 2$ and $0 < \alpha < \omega/M^2$, according to Lemma 2.4. In light of (g2), we learn that

$$\limsup_{s\to+\infty} \frac{g(s)s}{e^{\alpha s^2}-1} = \limsup_{s\to+\infty} \frac{G(s)}{e^{\alpha s^2}-1} = 0, \quad \limsup_{s\to 0^+} \frac{g(s)s}{s^2} = \limsup_{s\to 0^+} \frac{G(s)}{s^2} = 0.$$

Then there exist $0 < s_0(\varepsilon) < s_1(\varepsilon), C(\varepsilon) > 0$ and C_0 depending only upon K, such that

$$|K(x)G(s)| \le \varepsilon C_0(s^2 + e^{\alpha s^2} - 1) + C(\varepsilon)K(x)\chi_{[s_0(\varepsilon),s_1(\varepsilon)]}(|s|)|s|^q, \quad \text{for all } s \in \mathbb{R}, \tag{2.13}$$

$$|K(x)g(s)s| \le \varepsilon C_0(s^2 + e^{\alpha s^2} - 1) + C(\varepsilon)K(x)\chi_{[s_0(\varepsilon),s_1(\varepsilon)]}(|s|)|s|^q, \quad \text{for all } s \in \mathbb{R}. \tag{2.14}$$

By virtue of Lemma 2.4 we find $E > 0$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} (e^{\alpha w_n(x,0)^2} - 1)\,\mathrm{d}x \le E, \qquad \int_{\mathbb{R}} |w_n(x,0)|^2\,\mathrm{d}x \le E. \tag{2.15}$$

Notice again that, by means of (1.6), there exists $r(\varepsilon) > 0$ such that

$$\int_{A_\varepsilon^n \cap B^c_{r(\varepsilon)}(0)} K(x)\,\mathrm{d}x \le \frac{\varepsilon}{C(\varepsilon)s_1(\varepsilon)^q}, \quad \text{for all } n \in \mathbb{N}. \tag{2.16}$$

Now, combining the above inequality with (2.13)–(2.14), we have

$$\int_{B^c_{r(\varepsilon)}(0)} K(x)G(w_n(x,0))\,\mathrm{d}x \le (2C_0E + 1)\varepsilon, \quad \text{for all } n \in \mathbb{N}, \tag{2.17}$$

$$\int_{B^c_{r(\varepsilon)}(0)} K(x)g(w_n(x,0))w_n(x,0)\,\mathrm{d}x \le (2C_0E + 1)\varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{2.18}$$

Notice that we have

$$\begin{aligned} |K(x)(G(w_n(x,0)) - G(w(x,0))| \le{} & \varepsilon(w_n(x,0)^2 + e^{\alpha w_n(x,0)^2} - 1 + w(x,0)^2 \\ & + e^{\alpha w(x,0)^2} - 1) \\ & + C(\varepsilon)(|w_n(x,0)|^q + |w(x,0)|^q). \end{aligned}$$

A similar estimation holds for $K(x)g(s)s$. Hence, by (2.15) and since $w_n(x,0) \to w(x,0)$ in $L^q(B_{r(\varepsilon)(0)})$ on account of Remark 2.1, Lemma 2.6, allows to conclude that

$$\lim_n \int_{B_{r(\varepsilon)}(0)} K(x)G(w_n(x,0))\,\mathrm{d}x = \int_{B_{r(\varepsilon)}(0)} K(x)G(w(x,0))\,\mathrm{d}x,$$

$$\lim_n \int_{B_{r(\varepsilon)}(0)} K(x)g(w_n(x,0))w_n(x,0)\,\mathrm{d}x = \int_{B_{r(\varepsilon)}(0)} K(x)g(w(x,0)w(x,0)\,\mathrm{d}x.$$

Combining these with (2.17)–(2.18) we conclude the proof.

Let us now prove (2.12). The sequence $(\sqrt{K(x)}g(w_n(x,0))\chi_{\{|w_n(x,0)|\le 1\}})$ is bounded in $L^2(\mathbb{R})$ as by (g1)

$$|\sqrt{K(x)}g(w_n(x,0))\chi_{\{|w_n(x,0)|\le 1\}}|^2 \le C|w_n(x,0)|^2.$$

This, by pointwise convergence, yields for every $\varphi \in L^2(\mathbb{R})$

$$\lim_n \int_{\mathbb{R}} \sqrt{K(x)}g(w_n(x,0))\chi_{\{|w_n(x,0)|\le 1\}}\varphi(x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} \sqrt{K(x)}g(w(x,0))\chi_{\{|w(x,0)|\le 1\}}\varphi(x)\,\mathrm{d}x.$$

Given $v \in E$, it follows $\sqrt{K(x)}v(x,0) \in L^2(\mathbb{R})$, yielding

$$\lim_n \int_{\mathbb{R}} K(x)g(w_n(x,0))\chi_{\{|w_n(x,0)|\le 1\}}v(x,0)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} K(x)g(w(x,0))\chi_{\{|w(x,0)|\le 1\}}v(x,0)\,\mathrm{d}x.$$

Moreover, by $(g2), (K(x)g(w_n(x,0))\chi_{\{|w_n(x,0)|\ge 1\}})$ is bounded in $L^m(\mathbb{R})$ by Lemma 2.4 as

$$|K(x)g(w_n(x,0))\chi_{\{|w_n(x,0)|\ge 1\}}|^m \le C(e^{m\alpha w_n(x,0)^2} - 1), \quad \text{for } m\alpha < \omega/M^2.$$

Here $m > 1$ is taken close to 1. Then, for all $v \in E \subset L^{m'}(\mathbb{R})$ (notice that $m' > 2$), we get

$$\lim_n \int_{\mathbb{R}} K(x)g(w_n(x,0))\chi_{\{|w_n(x,0)|\ge 1\}}v(x,0)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} K(x)g(w(x,0))\chi_{\{|w(x,0)|\ge 1\}}v(x,0)\,\mathrm{d}x.$$

This concludes the proof of (2.12). □

From now on, in assumption (g2)′, we can assume that $\alpha_0 = \omega$, without loss of generality. We can state the following for the critical growth case.

Proposition 2.8 (Compactness II-critical case)**.** *Assume* (g1)–(g2)′–(g3)′. *Let* $(w_n) \subset E$ *a bounded sequence and* $w_n \rightharpoonup w$ *in* E *such that*

$$\sup_{n\in\mathbb{N}} \|w_n\| \in (0,1).$$

Then, up to a subsequence, the following facts hold:

$$\lim_n \int_{\mathbb{R}} K(x)G(w_n(x,0))\,\mathrm{d}x = \int_{\mathbb{R}} K(x)G(w(x,0))\,\mathrm{d}x; \tag{2.19}$$

$$\lim_n \int_{\mathbb{R}} K(x)w_n(x,0)g(w_n(x,0))\,\mathrm{d}x = \int_{\mathbb{R}} K(x)w(x,0)g(w(x,0))\,\mathrm{d}x; \tag{2.20}$$

$$\lim_n \int_{\mathbb{R}} K(x)g(w_n(x,0))v(x,0)\,\mathrm{d}x = \int_{\mathbb{R}} K(x)g(w(x,0)v(x,0))\,\mathrm{d}x, \quad \textit{for all } v\in E. \tag{2.21}$$

Proof. Let us prove (2.19) and (2.20). Let $\sup_{n\in\mathbb{N}}\|w_n\| =: M \in (0,1)$. By virtue of Lemma 2.4 there are $\alpha_M > \omega$ and $E > 0$ with

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} (e^{\alpha_M w_n(x,0)^2} - 1)\,\mathrm{d}x \le E, \qquad \int_{\mathbb{R}} |w_n(x,0)|^2\,\mathrm{d}x \le E \tag{2.22}$$

Let us fix $\varepsilon > 0$ and $q > 2$. By virtue of (g1) and (g2)$'$ we know that

$$\limsup_{s\to+\infty} \frac{g(s)s}{e^{\alpha_M s^2}-1} = \limsup_{s\to+\infty} \frac{G(s)}{e^{\alpha_M s^2}-1} = 0,$$
$$\limsup_{s\to 0^+} \frac{g(s)s}{s^2} = \limsup_{s\to 0^+} \frac{G(s)}{s^2} = 0.$$

Then there exist $0 < s_0(\varepsilon) < s_1(\varepsilon), C(\varepsilon) > 0$ and C_0 depending only upon K, with

$$|K(x)G(s)| \le \varepsilon C_0(|s|^2 + e^{\alpha_M s^2} - 1) + C(\varepsilon)K(x)\chi_{[s_0(\varepsilon),s_1(\varepsilon)]}(|s|)|s|^q, \quad \text{for all } s \in \mathbb{R}, \tag{2.23}$$

$$|K(x)g(s)s| \le \varepsilon C_0(|s|^2 + e^{\alpha_M s^2} - 1) + C(\varepsilon)K(x)\chi_{[s_0(\varepsilon),s_1(\varepsilon)]}(|s|)|s|^q, \quad \text{for all } s \in \mathbb{R}. \tag{2.24}$$

Notice again that, by means of (1.6), there exists $r(\varepsilon) > 0$ such that

$$\int_{A_\varepsilon^n \cap B^c_{r(\varepsilon)}(0)} K(x)\,\mathrm{d}x \le \frac{\varepsilon}{C(\varepsilon)s_1(\varepsilon)^q}, \quad \text{for all } n \in \mathbb{N}.$$

Now, combining the above inequality with (2.22) and (2.23)–(2.24), we have

$$\int_{B^c_{r(\varepsilon)}(0)} K(x)G(w_n(x,0))\,\mathrm{d}x \le (2C_0E+1)\varepsilon, \quad \text{for all } n \in \mathbb{N},$$
$$\int_{B^c_{r(\varepsilon)}(0)} K(x)g(w_n(x,0))w_n(x,0)\,\mathrm{d}x \le (2C_0E+1)\varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

The rest of the proof for (2.19) and (2.20) follows an in Proposition 2.7, with α replaced by α_M. Concerning (2.21), since $\alpha_M M^2 < \omega$ there exists $m > 1$ very close to 1 such that $m\alpha_M M^2 < \omega$. Then $(K(x)g(w_n(x,0))\chi_{\{|w_n(x,0)|\ge 1\}})$ is bounded in $L^m(\mathbb{R})$ by Lemma 2.4 since

$$|K(x)g(w_n(x,0))\chi_{\{|w_n(x,0)|\ge 1\}}|^m \le C(e^{m\alpha_M w_n(x,0)^2} - 1).$$

Then, for all $v \in E \subset L^{m'}(\mathbb{R})$ (as $m' > 2$), we get

$$\lim_n \int_{\mathbb{R}} K(x)g(w_n(x,0))\chi_{\{|w_n(x,0)|\geq 1\}} v(x,0)\,\mathrm{d}x = \int_{\mathbb{R}} K(x)g(w(x,0))\chi_{\{|w(x,0)|\geq 1\}} v(x,0)\,\mathrm{d}x.$$

This concludes the proof of (2.21). □

Proposition 2.9. $J \in C^1(E, \mathbb{R})$.

Proof. Let $(w_n) \subset E$ with $w_n \to w$ strongly in E. There exist $C > 0$ and $\alpha > 0$ such that

$$|g(s)|^2 \leq C(|s|^2 + e^{\alpha s^2} - 1), \qquad \text{for all } s \in \mathbb{R}.$$

This choice fits both the subcritical and critical case. Hence

$$|g(w_n(x,0)) - g(w(x,0))|^2 \leq C(|w_n(x,0)|^2 + e^{\alpha w_n(x,0)^2} - 1) + C(|w(x,0)|^2 + e^{\alpha w(x,0)^2} - 1).$$

Taking into account Lemma 2.5, we have

$$\lim_n \int_{\mathbb{R}} (|w_n(x,0)|^2 + e^{\alpha w_n(x,0)^2} - 1)\,\mathrm{d}x = \int_{\mathbb{R}} (|w(x,0)|^2 + e^{\alpha w(x,0)^2} - 1)\,\mathrm{d}x.$$

Then, by the Generalized Dominated Convergence Theorem, $\|g(w_n) - g(w)\|_{L^2} \to 0$. In turn,

$$\sup_{\|v\|\leq 1} \left| \int_{\mathbb{R}} K(x)(g(w_n(x,0)) - g(w(x,0)))v\,\mathrm{d}x \right| \leq C\|g(w_n) - g(w)\|_{L^2} \sup_{\|v\|\leq 1} \|v\|_{L^2(\mathbb{R})} \leq C o_n(1),$$

which concludes the proof. □

Next, we show that J satisfies the Mountain Pass geometry.

Lemma 2.10. *The functional J satisfies*

(1) *There exists $\beta, \rho > 0$ such that $J(w) \geq \beta$ if $w \in E$ and $\|w\| = \rho$;*
(2) *There exists $e \in E \setminus \{0\}$ with $\|e\| > \rho$ such that $J(e) \leq 0$;*

Proof. Assertion (2) is straightforward due to the superquadraticity assumptions. For (1), let us consider $w \in E$ with $\|w\| = \rho < 1$ and $\omega < \alpha < \omega/\rho^2$. By the growth conditions on g (both critical and subcritical), there exist $r > 1$ so close to 1 that $r\alpha < \omega/\rho^2, q > 2$ and $C > 0$ with

$$G(s) \leq \frac{1}{4}s^2 + C(e^{r\alpha s^2} - 1)^{1/r} s^q, \quad \text{for all } s \in \mathbb{R}^+.$$

Then, taking into account inequality (2.9), we have

$$
\begin{aligned}
J(w) &\geq \frac{1}{2}\|w\|^2 - \frac{1}{4}\|w(\cdot,0)\|_{L^2}^2 - C\int_{\mathbb{R}} \left(e^{r\alpha|w(x,0)|^2}-1\right)^{1/r}|w(x,0)|^q\,\mathrm{d}x \\
&\geq \frac{1}{4}\|w\|^2 - C\left(\int_{\mathbb{R}}\left(e^{r\alpha|w(x,0)|^2}-1\right)\mathrm{d}x\right)^{1/r}\left(\int_{\mathbb{R}}|w(x,0)|^{r'q}\,\mathrm{d}x\right)^{1/r'} \\
&\geq \frac{1}{4}\|w\|^2 - C\|w\|^q = \frac{1}{4}\rho^2 - C\rho^q = \beta > 0,
\end{aligned}
$$

for every $\rho > 0$ sufficiently small. □

Therefore, there exists a sequence $(w_n) \subset E$, so called *Cerami sequence* such that

$$
J(w_n) \to c, \quad (1 + \|w_n\|)\|J'(w_n)\| \to 0, \tag{2.25}
$$

where c is given by

$$
c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),
$$

with

$$
\Gamma = \big\{\gamma \in C([0,1],E) : \gamma(0) = 0 \quad \text{and} \quad J(\gamma(1)) \leq 0\big\}.
$$

We have the following result

Lemma 2.11. *The Cerami sequence $(w_n) \subset E$ is bounded and $\|w_n^-\| \to 0$ as $n \to \infty$.*

Proof. If g has critical growth, the assertion is obvious since the Ambrosetti-Rabinowitz condition (AR) is assumed. On the contrary, in the subcritical case, the proof follows by mimicking the argument in the first part of the proof of [9, Lemma 2.3], which is based upon monotonicity of $\mathcal{H}(s) = sg(s) - 2G(s)$, holding since $\frac{g(s)}{s}$ is non-decreasing in $\mathbb{R}^+$, and the application of (2.10). □

To handle the case where g is at critical growth, we shall need the following result.

Lemma 2.12. *Let $(w_n) \subset E$ be a bounded Palais-Smale sequence for the functional J at the Mountain Pass energy level c. Then*

$$
\sup_{n\in\mathbb{N}} \|w_n\| \in (0,1),
$$

provided that the constant $C_q > 0$ which appears in (g3)$'$ *is sufficiently large.*

Proof. Let $q > 2$ and $C_q > 0$ as in assumption (g3)$'$, to be chosen later sufficiently large. Let us fix a cut-off function $\psi \in C_c^\infty(\mathbb{R},\mathbb{R}^+)\setminus\{0\}$ and let us denote

$$
L := \inf_{\mathrm{supt}(\psi)} K > 0, \qquad \mathbb{S}_q := \frac{\|\psi\|}{\|\psi(\cdot,0)\|_{L^q}}.
$$

Also, let $\omega_q > 0$ be such that $J(\omega_q\psi) < 0$. This is possible, since

$$J(\omega\psi) = \frac{\omega^2}{2}\|\psi\|^2 - \int_{\mathbb{R}} K(x)G(\omega\psi(x,0))\,\mathrm{d}x \\ \leq \frac{\omega^2}{2}\|\psi\|^2 - \omega^q C_q L\|\psi(x,0)\|_{L^q}^q\,\mathrm{d}x < 0,$$

for every $\omega > 0$ sufficiently large, say $\omega = \omega_q$. Then, $\gamma \in C([0,1], E)$ defined by $\gamma(t) := t\omega_q\psi \in \Gamma$ for $t \in [0,1]$ belongs to the class of continuous paths Γ. Hence, we get

$$\begin{aligned} c &= \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \leq \max_{t\in[0,1]} J(t\omega_q\psi) \leq \max_{t\in\mathbb{R}^+} J(t\psi) \\ &= \max_{t\geq 0}\left(\frac{t^2}{2}\|\psi\|^2 - C_qLt^q\|\psi(\cdot,0)\|_{L^q}^q\right) \\ &= \max_{t\geq 0}\left(\frac{\mathbb{S}_q^2}{2}t^2\|\psi(\cdot,0)\|_{L^q}^2 - C_qLt^q\|\psi(\cdot,0)\|_{L^q}^q\right) \\ &= \max_{t\geq 0}\left(\frac{\mathbb{S}_q^2}{2}t^2 - C_qLt^q\right) = \frac{q-2}{2q}\frac{\mathbb{S}_q^{\frac{2q}{q-2}}}{(qC_qL)^{\frac{2}{q-2}}}. \end{aligned}$$

On the other hand, since (w_n) is a Palais-Smale sequence, we get

$$c = \limsup_n \left(J(w_n) - \frac{1}{\vartheta}J'(w_n)(w_n)\right) \geq \frac{\vartheta-2}{2\vartheta}\limsup_n\|w_n\|^2.$$

In turn, by combining the above inequalities, we get

$$\limsup_n\|w_n\|^2 \leq \frac{2\vartheta}{\vartheta-2}\frac{q-2}{2q}\frac{\mathbb{S}_q^{\frac{2q}{q-2}}}{(qC_qL)^{\frac{2}{q-2}}} < 1,$$

provided that C_q satisfies the condition in the statement of Theorem 1.2. □

3. Proof of the main results

3.1. Proof of Theorem 1.1 completed

In light of Lemma 2.10, there exists a *Cerami* sequence $\{w_n\} \subset E$ for J at the Mountain Pass level $c > 0$. From Lemma 2.11 it follows that $\{w_n\}$ is bounded, $w_n^- \to 0$ in E, and thus it admits a nonnegative weak limit $w \in E$. By (2.12) of Proposition 2.7, it follows that

$$\int_{\mathbb{R}^2_+}\nabla w\cdot\nabla v\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} w(x,0)v(x,0)\,\mathrm{d}x = \int_{\mathbb{R}} K(x)g(w(x,0))v(x,0)\,\mathrm{d}x, \quad \forall v \in E. \tag{3.1}$$

Then, we have a weak solution $u \in H^{1/2}(\mathbb{R})$ to (1.1). We have $u > 0$ if $u \neq 0$, arguing as in [9]. We prove that $w = E_{1/2}(u) \not\equiv 0$. In fact, (w_n) converges to w strongly in E, as $n \to \infty$. Indeed, since $J'(w_n)(w_n) = o_n(1)$, we have, by

(2.11) of Proposition 2.7,

$$\lim_n \|w_n\|^2 = \lim_n \int_{\mathbb{R}} K(x)g(w_n(x,0))w_n(x,0)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} K(x)g(w(x,0))w(x,0)\,\mathrm{d}x = \|w\|^2,$$

that is, $w_n \to w$ in E. Hence $J(w) = c > 0$ by continuity, yielding $w \not\equiv 0$. □

3.2. Proof of Theorem 1.2 completed

In light of Lemma 2.10, there exists a *Cerami* sequence $\{w_n\} \subset E$ for J at the Mountain Pass level $c > 0$. From Lemma 2.11 it follows that $\{w_n\}$ is bounded, $w_n^- \to 0$ in E, and thus it admits a nonnegative weak limit $w \in E$. By taking C_q sufficiently large in assumption $(\mathrm{g}3)'$, in light of Lemma 2.12, it follows that $\sup_{n\in\mathbb{N}} \|w_n\| \in (0,1)$. Then, we are allowed to apply the assertions of Proposition 2.8. By (2.21) of Proposition 2.8, it follows that (3.1) is satisfied. Then, we have a weak solution $u \in H^{1/2}(\mathbb{R})$ to (1.1). We have $u > 0$ if $u \neq 0$, arguing as in [9]. Indeed $u \neq 0$. In fact $w = E_{1/2}(u) \not\equiv 0$. Suppose by contradiction that $w = 0$. Then, since $w_n \rightharpoonup 0$, we have by (2.19) and (2.20) of Proposition 2.8

$$\lim_n \int_{\mathbb{R}} K(x)G(w_n(x,0)\,\mathrm{d}x = \lim_n \int_{\mathbb{R}} K(x)g(w_n(x,0)w_n(x,0)\,\mathrm{d}x = 0.$$

Then, from

$$\frac{1}{2}\|w_n\|^2 - \int_{\mathbb{R}} K(x)G(w_n(x,0))\,\mathrm{d}x = c + o_n(1),$$
$$\|w_n\|^2 - \int_{\mathbb{R}} K(x)g(w_n(x,0))w_n(x,0)\,\mathrm{d}x = o_n(1)$$

we get a contradiction, since $c > 0$. The proof is complete. □

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