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Symmetry of n-mode positive solutions for two-dimensional Hénon type systems

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ABSTRACT

We provide a symmetry result for n-mode positive solutions of a general class of semilinear elliptic systems under cooperative conditions on the nonlinearities. Moreover, we apply the result to a class of Hénon systems and provide the existence of multiple n-mode positive solutions.

1. Introduction and results

Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ and, for $m \in \mathbb{N}$ with $m \geq 2$, consider the system

$$\begin{cases} \Delta u^i + f^i(|x|, u^i) = 0 & \text{in } D, \\ u^i = 0 & \text{on } \partial D, \end{cases} \quad \text{for } i = 1, \dots, m, \tag{1.1}$$

where f^i are smooth functions over $(0, 1) \times (0, \infty)^m$. Semi-linear elliptic systems as (1.1) arise naturally in many physical and biological contests; see e.g. [8,15,16,18,20] and the references therein. As far as the symmetry of positive solutions is concerned and the functions f^i are decreasing in the radial variable, the celebrated moving plane method [9] can be applied when the system is cooperative namely $\partial f^i/\partial u^j \geq 0$ for every $i \neq j$ [6,13,21]. The aim of this note is to establish a general symmetry result (Theorem 1.1) for n-mode ($2\pi/n$-rotation invariant) solutions, namely solutions $(u^1, \dots, u^m)$ such that each component $u^i : \overline{D} \to \mathbb{R}$, in polar coordinates, satisfies

$$u^i(r, \theta) = u^i(r, \theta + 2\pi/n), \quad \text{for all } (r, \theta) \in [0, 1] \times \mathbb{R},$$

as well as provide a meaningful application of it (Theorem 1.2) to the system of Hénon type

$$\begin{cases} \Delta u + \dfrac{2p}{p+q}|x|^\alpha u^{p-1} v^q = 0 & \text{in } D, \\ \Delta v + \dfrac{2q}{p+q}|x|^\alpha u^p v^{q-1} = 0 & \text{in } D, \\ u > 0, \quad v > 0 & \text{in } D, \\ u = v = 0 & \text{on } \partial D. \end{cases} \tag{1.2}$$

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Quite recently, these systems were carefully investigated in [22,23] (see also [1,10,11] and references therein) and they can be considered as a vectorial counterpart of the celebrated equation

$$\begin{cases} \Delta u + |x|^{\alpha} u^{p-1} = 0 & \text{in } D, \\ u > 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

first studied in [17] after being introduced by Hénon in [12] in connection with the research of rotating stellar structures. We shall say that u is of class C^n at the origin if u is of class C^{n-1} in a neighborhood of the origin and each $(n-1)$-th partial derivative is totally differentiable at the origin. Then we prove the following

Theorem 1.1. *Let $m, n \in \mathbb{N}$ with $m, n \geq 2$ and $f^1, \dots, f^m \in C((0, 1) \times (0, \infty)^m, \mathbb{R})$ such that*

(i) *for each $i \in \{1, \dots, m\}$ and $(u^1, \dots, u^m) \in (0, \infty)^m$, the map*

$$r \mapsto r^{2-2n} f^i(r, u^1, \dots, u^m) : (0, 1) \to \mathbb{R}$$

is nonincreasing;

(ii) *for each $i \in \{1, \dots, m\}$ and $r \in (0, 1)$, $f^i(r, \cdot, \dots, \cdot) \in C^1((0, \infty)^m, \mathbb{R})$;*

(iii) *for each $i, j \in \{1, \dots, m\}$ with $i \neq j$ and $(r, u^1, \dots, u^m) \in (0, 1) \times (0, \infty)^m$,*

$$\frac{\partial f^i}{\partial u^j}(r, u^1, \dots, u^m) \geq 0;$$

(iv) *for each $i, j \in \{1, \dots, m\}$, $r_0 \in (0, 1)$ and $M \in (0, \infty)$,*

$$\sup\left\{ \left| \frac{\partial f^i}{\partial u^j}(r, u^1, \dots, u^m) \right| : (r, u^1, \dots, u^m) \in (r_0, 1) \times (0, M]^m \right\} < \infty.$$

Let $(u^1, \dots, u^m) \in C^2(D \setminus \{0\}) \cap C(\overline{D})$ be a solution of (1.1) such that each u^i is n-mode, positive and of class C^n at the origin. Then, each u^i is radially symmetric and $\frac{\partial u^i}{\partial r}(|x|) < 0$ for $r = |x|$.

For scalar equations, this result was obtained in [19]. Due to the recent interest of the community for the symmetry issues for elliptic systems, we believe that the statement above is of interest. Also, it admits some interesting consequences; see for instance Theorem 1.2 below. Of course, system (1.1) includes both variational and nonvariational problems or systems of Hamiltonian type; see e.g. [7] for a wide overview. We point out, in particular, that the weakly coupled semi-linear Schrödinger systems, see [14] and the references therein, which come from physically relevant situations and have recently received much attention, satisfy conditions (ii)–(iv).

For the sake of completeness, we refer the reader to [3–5] for recent partial (foliated Schwarz symmetry) symmetry results for the smooth solutions to (1.1) in rotationally invariant domains and for possibly sign-changing solutions and where the maps $r \mapsto f^i(r, s_1, \dots, s^m)$ are possibly nondecreasing and some convexity assumptions are assumed on the s_i variables.

For every $\alpha \geq 0$ and $p, q > 1$, let us set

$$R_{\alpha,p,q}(u, v) := \frac{\int_D (|\nabla u|^2 + |\nabla v|^2)}{\left(\int_D |x|^{\alpha} |u|^p |v|^q \right)^{\frac{2}{p+q}}}, \quad \text{for any } u, v \in H_0^1(D).$$

Moreover, for each $\gamma > 0$, we shall denote by $\lceil \gamma \rceil$ the smallest integer greater than or equal to γ. Then, we have the following

Theorem 1.2. *The following facts hold.*

(I) *If $\alpha \in (0, \infty)$, $p, q \in (1, \infty)$ and (u, v) is an n-mode solution of (1.2) with $n \geq 1 + \lceil \alpha/2 \rceil$, then u, v are radially symmetric.*

(II) *For each $\alpha \in (2, \infty)$ and $p, q \in (1, \infty)$, if $n_\alpha \geq 1$ then (1.2) has a nonradial n-mode solution (u_n, v_n) for $n = 1, \dots, n_\alpha$ such that*

$$R_{\alpha,p,q}(u_1, v_1) < \cdots < R_{\alpha,p,q}(u_{n_\alpha}, v_{n_\alpha}), \tag{1.3}$$

where n_α is the greatest integer less than

$$\left(\frac{\alpha + 2}{2\alpha} \right)^{\frac{4}{p+q-2}} \left(\frac{\alpha - 2}{\alpha} \right)^{\frac{2\alpha}{p+q-2}} \left(1 + \frac{\alpha}{2} \right). \tag{1.4}$$

In particular, the following facts hold.

(i) *For each $\alpha \in (2, \infty)$, if at least one of $p, q \in (1, \infty)$ is large enough, then $n_\alpha = \lceil \alpha/2 \rceil$, that is (1.2) has a nonradial n-mode solution for $n = 1, \dots, \lceil \alpha/2 \rceil$ satisfying (1.3).*

(ii) *For each $p, q \in (1, \infty)$ and $\ell \in \mathbb{N}$, if $\alpha \in (2, \infty)$ is large enough (1.2) has a nonradial n-mode solution (u_n, v_n) for $n = 1, \dots, \ell$ such that*
$$R_{\alpha,p,q}(u_1, v_1) < \cdots < R_{\alpha,p,q}(u_\ell, v_\ell).$$
In particular, for each $p, q \in (1, \infty)$, the number of nonradial solutions of (1.2) tends to infinity as $\alpha \to \infty$.

Hence, as far as α gets large, the symmetry breaking phenomenon occurs and we can find as many n-mode positive solutions as we want. In Section 2 we shall prove Theorem 1.1, while in Section 3 we shall provide the proof of Theorem 1.2.

2. Proof of Theorem 1.1

By using the planar polar coordinates, for each $i = 1, \dots, m$, we define the function $\widetilde{u}^i : \overline{D} \to \mathbb{R}$ by setting $\widetilde{u}^i(r, \theta) := u^i(r^{1/n}, \theta/n)$, for every $(r, \theta) \in \overline{D}$. Since $(u^1, \dots, u^m)$ satisfies system (1.1) and each u^i is n-mode, we can see that $\widetilde{u}^i \in C^2(D \setminus \{0\}) \cap C(\overline{D})$ and $(\widetilde{u}^1, \dots, \widetilde{u}^m)$ satisfies the system

$$\begin{cases} \Delta \widetilde{u}^i + \widetilde{f}^i(|x|, \widetilde{u}^1, \dots, \widetilde{u}^m) = 0 & \text{in } D \setminus \{0\}, \\ \widetilde{u}^i = 0 & \text{on } \partial D, \end{cases} \quad \text{for } i = 1, \dots, m. \tag{2.1}$$

Here, $\widetilde{f}^i \in C((0, 1) \times (0, \infty)^m, \mathbb{R})$ is the function defined by

$$\widetilde{f}^i(r, t^1, \dots, t^m) := n^{-2} r^{(2-2n)/n} f^i(r^{1/n}, t^1, \dots, t^m),$$

for every $(r, t^1, \dots, t^m) \in (0, 1) \times (0, \infty)^m$ and $\widetilde{f}^i$ satisfies

$$\text{for each } (t^1, \dots, t^m) \in (0, \infty)^m, r \mapsto \widetilde{f}^i(r, t^1, \dots, t^m) \text{ is nonincreasing.} \tag{2.2}$$

Indeed, from

$$\begin{aligned} \Delta \widetilde{u}^i(r, \theta) + \widetilde{f}^i(r, \widetilde{u}^1(r, \theta), \dots, \widetilde{u}^m(r, \theta)) &= \frac{1}{n^2} r^{\frac{2-2n}{n}} \left(u^i_{rr}\left(r^{\frac{1}{n}}, \frac{\theta}{n} \right) + \frac{1}{r^{\frac{1}{n}}} u^i_r\left(r^{\frac{1}{n}}, \frac{\theta}{n} \right) + \frac{1}{r^{\frac{2}{n}}} u^i_{\theta\theta}\left(r^{\frac{1}{n}}, \frac{\theta}{n} \right) \right. \\ &\quad \left. + f^i\left(r^{\frac{1}{n}}, u^1\left(r^{\frac{1}{n}}, \frac{\theta}{n} \right), \dots, u^m\left(r^{\frac{1}{n}}, \frac{\theta}{n} \right) \right) \right) = 0, \end{aligned}$$

we deduce (2.1), and we can easily see that (2.2) holds as well, in light of assumption (i). For each $\lambda \in (0, 1)$, we set $\Sigma_\lambda = \{x \in D : x_1 > \lambda\}$ and we define the map $h_\lambda : \overline{\Sigma_\lambda} \to \overline{D}$ by $h_\lambda(x) = (2\lambda - x_1, x_2)$ for $x = (x_1, x_2) \in \overline{\Sigma_\lambda}$. We note that h_λ satisfies

$$|h_\lambda(x)| < |x| \quad \text{for each } \lambda \in (0, 1) \text{ and } x \in \Sigma_\lambda \cup \mathrm{Int}_{\partial D}(\overline{\Sigma_\lambda} \cap \partial D). \tag{2.3}$$

Here, for a subset E of ∂D, we denote by $\mathrm{Int}_{\partial D} E$, the interior set of E with respect to the relative topology of ∂D. We set $x_\lambda = (2\lambda, 0)$ for $\lambda \in (0, 1)$. We can see

$$x_\lambda \in \begin{cases} \Sigma_\lambda & \text{for each } \lambda \in \left(0, \frac{1}{2}\right), \\ \partial \Sigma_\lambda & \text{for } \lambda = \frac{1}{2}, \\ \mathbb{R}^2 \setminus \overline{\Sigma_\lambda} & \text{for each } \lambda \in \left(\frac{1}{2}, 1\right) \end{cases} \tag{2.4}$$

and

$$h_\lambda(x_\lambda) = 0 \quad \text{for each } \lambda \in (0, 1/2]. \tag{2.5}$$

For the sake of completeness, we note that $\Sigma_\lambda \setminus \{x_\lambda\} = \Sigma_\lambda$ for each $\lambda \in [\frac{1}{2}, 1)$ and $\overline{\Sigma_\lambda} \setminus \{x_\lambda\} = \overline{\Sigma_\lambda}$ for each $\lambda \in (\frac{1}{2}, 1)$. For each $i, j = 1, \dots, m$, we define $v^i_\lambda \in C^2(\Sigma_\lambda \setminus \{x_\lambda\}) \cap C(\overline{\Sigma_\lambda})$ and $c^{ij}_\lambda \in L^\infty(\Sigma_\lambda)$ by setting

$$v^i_\lambda(x) := \widetilde{u}^i(x) - \widetilde{u}^i(h_\lambda(x)), \quad \text{for } x \in \overline{\Sigma_\lambda}, \tag{2.6}$$

and

$$c^{ij}_\lambda(x) := -\int_0^1 \frac{\partial \widetilde{f}^i}{\partial u^j} \big(|x|, s\widetilde{u}^1(x) + (1-s)\widetilde{u}^1(h_\lambda(x)), \dots, s\widetilde{u}^m(x) + (1-s)\widetilde{u}^m(h_\lambda(x)) \big)\, ds. \tag{2.7}$$

By the assumptions of Theorem 1.1, we can see that $c_\lambda^{ij} \le 0$ if $i \ne j$ for $x \in \Sigma_\lambda$ and

$$\sup_{r<\lambda<1} \sup_{x\in\Sigma_\lambda} |c_\lambda^{ij}(x)| < \infty, \quad \text{for each } r \in (0,1) \text{ and } i,j = 1,\dots,m. \tag{2.8}$$

Therefore, it holds

$$-\Delta v_\lambda^i(x) + \sum_{j=1}^m c_\lambda^{ij}(x) v_\lambda^j(x) \le 0 \quad \text{for } \lambda \in (0,1), x \in \Sigma_\lambda \setminus \{x_\lambda\} \text{ and } i = 1,\dots,m. \tag{2.9}$$

Indeed, (2.9) can be obtained as follows:

$$\begin{aligned} 0 &= \Delta\widetilde{u}^i(h_\lambda(x)) + \widetilde{f}^i(|h_\lambda(x)|, \widetilde{u}^1(h_\lambda(x)),\dots,\widetilde{u}^m(h_\lambda(x))) - \Delta\widetilde{u}^i(x) - \widetilde{f}^i(|x|,\widetilde{u}^1(x),\dots,\widetilde{u}^m(x)) \\ &\ge -\Delta v_\lambda^i(x) + \widetilde{f}^i(|x|, \widetilde{u}^1(h_\lambda(x)),\dots,\widetilde{u}^m(h_\lambda(x))) - \widetilde{f}^i(|x|,\widetilde{u}^1(x),\dots,\widetilde{u}^m(x)) \\ &= -\Delta v_\lambda^i(x) + \sum_{j=1}^m c_\lambda^{ij}(x) v_\lambda^j(x). \end{aligned}$$

We set

$$\begin{aligned} &A_1 = \{\lambda \in [1/2, 1) : v_\lambda^i(x) < 0 \text{ for each } x \in \Sigma_\lambda \text{ and } i \in \{1,\dots,m\}\}, \\ &\mu_1 = \inf_{\lambda\in A_1} \lambda. \end{aligned} \tag{2.10}$$

We now claim that $A_1 \ne \emptyset$. Let $i \in \{1,\dots,m\}$ and $\lambda \in [1/2,1)$ such that λ is sufficiently close to 1. Then we can easily see $v_\lambda^i(x) \le 0$ for $x \in \partial\Sigma_\lambda$ and $v_\lambda^i(x) < 0$ for $x \in \mathrm{Int}_{\partial D}(\partial D \cap \partial\Sigma_\lambda)$ from (2.3). Since $|\Sigma_\lambda| \ll 1$ and (2.9) holds, by [2, Corollary 14.1], we have $v_\lambda^i \le 0$ on $\overline{\Sigma_\lambda}$. By

$$-\Delta v_\lambda^i(x) + c_\lambda^{ii}(x) v_\lambda^i(x) \le -\sum_{j\ne i}^m c_\lambda^{ij}(x) v_\lambda^j(x) \le 0 \quad \text{in } \Sigma_\lambda \setminus \{x_\lambda\} \tag{2.11}$$

and the strong maximum principle, we have $v_\lambda^i < 0$ in Σ_λ. Since i is any element of $\{1,\dots,m\}$, we have shown $\lambda \in A_1$, which proves the claim.

We now claim that $\mu_1 = 1/2 \in A_1$. Let $i \in \{1,\dots,m\}$. We have $v_{\mu_1}^i(x) \le 0$ for $x \in \Sigma_{\mu_1}$. Since (2.11) holds with $\lambda = \mu_1$ and $v_{\mu_1}^i(x) < 0$ for $x \in \mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_1} \cap \partial D)$ from (2.3), by the strong maximum principle, we have $v_{\mu_1}^i(x) < 0$ for $x \in \Sigma_{\mu_1}$. Since i is an arbitrary element of $\{1,\dots,m\}$, we have $\mu_1 \in A_1$. We will show $\mu_1 = 1/2$. Suppose not, namely $\mu_1 > 1/2$. Again let $i \in \{1,\dots,m\}$. Let G be an open set such that $\overline{G} \subset \Sigma_{\mu_1}$ and $|\Sigma_{\mu_1} \setminus \overline{G}| \ll 1$. We have $\max_{x\in\overline{G}} v_{\mu_1}^i(x) < 0$. Let $0 < \varepsilon \ll 1$. Then we have $\max_{x\in\overline{G}} v_{\mu_1-\varepsilon}^i(x) < 0$ and $|\Sigma_{\mu_1-\varepsilon} \setminus \overline{G}| \ll 1$. Since (2.9) holds with $\lambda = \mu_1 - \varepsilon$ and $v_{\mu_1-\varepsilon}^i(x) \le 0$ for $x \in \partial(\Sigma_{\mu_1-\varepsilon} \setminus \overline{G})$, we have $v_{\mu_1-\varepsilon}^i(x) \le 0$ for $x \in \Sigma_{\mu_1-\varepsilon}$ by [2, Corollary 14.1]. From $v_{\mu_1-\varepsilon}^i(x) < 0$ for $x \in (\mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_1-\varepsilon} \cap \partial D)) \cup \partial G$ and the strong maximum principle, we have $v_{\mu_1-\varepsilon}^i(x) < 0$ for $x \in \Sigma_{\mu_1-\varepsilon}$. Since i is an arbitrary element of $\{1,\dots,m\}$, we have $\mu_1 - \varepsilon \in A_1$. This is a contradiction. Hence $\mu_1 = 1/2 \in A_1$.

We now set

$$\begin{aligned} &A_2 := \{\lambda \in (0, 1/2) : \ v_\lambda^i(x) < 0 \text{ for each } x \in \Sigma_\lambda \text{ and } i \in \{1,\dots,m\}\}, \\ &\mu_2 := \inf_{\lambda\in A_2} \lambda. \end{aligned} \tag{2.12}$$

We now claim that $A_2 \ne \emptyset$. Let $i \in \{1,\dots,m\}$. We note that $x_{1/2} = (1,0)$. Let G be an open set such that $\overline{G} \subset \Sigma_{1/2}$ and $|\Sigma_{1/2} \setminus G| \ll 1$. From $1/2 \in A_1$ and $\overline{G} \subset \Sigma_{1/2}$, we have $\max_{x\in\overline{G}} v_{1/2}^i(x) < 0$. Let $\lambda \in (0,1/2)$ such that λ is sufficiently close to $1/2$. We note $|\Sigma_\lambda \setminus \overline{G}| \ll 1$ and x_λ is close to $(1,0)$. We choose a sufficiently small open neighborhood U of x_λ with $\overline{U} \subset \Sigma_\lambda$, and we set $H = G \cup U$. Then we have $v_\lambda^i(x) < 0$ for $x \in \overline{H}$, $v_\lambda^i(x) \le 0$ for $x \in \partial\Sigma_\lambda \cup \partial H$ and $|\Sigma_\lambda \setminus \overline{H}| \ll 1$. Since (2.9) holds on $\Sigma_\lambda \setminus \overline{H}$, by [2, Corollary 14.1], we have $v_\lambda^i \le 0$ on Σ_λ. From (2.11) and the strong maximum principle, we have $v_\lambda^i < 0$ on Σ_λ. Since i is an arbitrary element of $\{1,\dots,m\}$, we have shown $\lambda \in A_2$.

Recalling that u is of class C^n at the origin, arguing exactly as in [19, Lemma 4] we get

$$\frac{\partial(\widetilde{u}^i \circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = 0 \quad \text{for each } i = 1,\dots,m. \tag{2.13}$$

We now claim that $\mu_2 = 0$. Suppose not. Let $i \in \{1,\dots,m\}$. Then we have $\mu_2 \in (0,1/2)$ by the previous claim and we can see $v_{\mu_2}^i \le 0$ on $\overline{\Sigma_{\mu_2}}$. We will show $v_{\mu_2}^i < 0$ on $\Sigma_{\mu_2} \setminus \{x_{\mu_2}\}$. We have $v_{\mu_2}^i(x) < 0$ for $x \in \mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_2} \cap \partial D)$ from (2.3).

By (2.11) with $\lambda = \mu_2$ and the strong maximum principle, we have $v^i_{\mu_2} < 0$ on $\Sigma_{\mu_2} \setminus \{x_{\mu_2}\}$. Next, we will show $v^i_{\mu_2}(x_{\mu_2}) < 0$. Suppose $v^i_{\mu_2}(x_{\mu_2}) < 0$ does not hold, i.e., $v^i_{\mu_2}(x_{\mu_2}) = 0$. Let $\nu_1 = (-1, 0)$ and $\nu_2 = (1, 0)$. From (2.13), we have

$$\frac{\partial v^i_{\mu_2}}{\partial \nu_1}(x_{\mu_2}) = -\frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}) + \frac{\partial(\widetilde{u}^i \circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = -\frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}),$$
$$\frac{\partial v^i_{\mu_2}}{\partial \nu_2}(x_{\mu_2}) = \frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}) - \frac{\partial(\widetilde{u}^i \circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = \frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}).$$

By Hopf's lemma, we obtain

$$-\frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}) < 0 \quad \text{and} \quad \frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}) < 0,$$

which is a contradiction. So we have shown $v^i_{\mu_2}(x_{\mu_2}) < 0$. Thus we have $v^i_{\mu_2} < 0$ on Σ_{μ_2}. Since i is an arbitrary element of $\{1, \dots, m\}$, we have $\mu_2 \in A_2$. Again let $i \in \{1, \dots, m\}$. We choose an open set G such that $\overline{G} \subset \Sigma_{\mu_2}$ and $|\Sigma_{\mu_2} \setminus \overline{G}| \ll 1$. We have $\max_{\overline{G}} v^i_{\mu_2} < 0$. Let $0 < \varepsilon \ll 1$. Then we have $|\Sigma_{\mu_2-\varepsilon} \setminus \overline{G}| \ll 1$ and $\max_{\overline{G}} v^i_{\mu_2-\varepsilon} < 0$. Since (2.9) holds with $\lambda = \mu_2 - \varepsilon$, by [2, Corollary 14.1], we have $v^i_{\mu_2-\varepsilon}(x) \le 0$ for $x \in \Sigma_{\mu_2-\varepsilon} \setminus \overline{G}$. By (2.11) with $\lambda = \mu_2 - \varepsilon$ and the strong maximum principle, we have $v^i_{\mu_2-\varepsilon}(x) < 0$ for $x \in \Sigma_{\mu_2-\varepsilon} \setminus \overline{G}$. Hence we have shown $v^i_{\mu_2-\varepsilon}(x) < 0$ for $x \in \Sigma_{\mu_2-\varepsilon}$. Since i is an arbitrary element of $\{1, \dots, m\}$, we have $\mu_2 - \varepsilon \in A_2$, which is a contradiction. Therefore we obtain $\mu_2 = 0$.

We can finally conclude the proof of Theorem 1.1. Let $i \in \{1, \dots, m\}$. By the conclusions above, we can infer that $\widetilde{u}^i$ is radially symmetric and $\frac{\partial \widetilde{u}^i}{\partial r}(|x|) < 0$ for $r = |x| \in (0, 1)$. From the definition of $\widetilde{u}^i$, we can find u^i is also radially symmetric and $\frac{\partial u^i}{\partial r}(|x|) < 0$.

3. Proof of Theorem 1.2

Let us first prove assertion ($\mathbb{I}$) of Theorem 1.2. Assume that (u, v) is an n-mode solution to system (1.2) such that $n \ge 1 + \lceil \alpha/2 \rceil$. Then, we may choose $\hat{n}, m \in \mathbb{N}$ such that $m/\hat{n} \in \mathbb{N}$, $(\alpha + 2)\hat{n} \le 2n < 2(\alpha + 2)\hat{n}$ and

$$m > \max\left\{\frac{n\hat{n}}{(\alpha+2)\hat{n} - n}, \frac{2n}{\alpha+2}\right\}.$$

Setting $\hat{u}(r, \theta) := u(r^{m/n}, m\theta/n)$ and $\hat{v}(r, \theta) := v(r^{m/n}, m\theta/n)$, it is readily seen that $\hat{u}$ and $\hat{v}$ are both $\hat{m} = m/\hat{n}$-mode and solve, in $D \setminus \{0\}$, the system

$$\begin{cases} \Delta \hat{u} + \dfrac{2pm^2}{(p+q)n^2}|x|^{\frac{m(\alpha+2)-2n}{n}} \hat{u}^{p-1}\hat{v}^q = 0 & \text{in } D \setminus \{0\}, \\ \Delta \hat{v} + \dfrac{2pm^2}{(p+q)n^2}|x|^{\frac{m(\alpha+2)-2n}{n}} \hat{u}^{p}\hat{v}^{q-1} = 0 & \text{in } D \setminus \{0\}, \\ \hat{u} > 0, \qquad \hat{v} > 0 & \text{in } D \setminus \{0\}, \\ \hat{u} = \hat{v} = 0 & \text{on } \partial D. \end{cases}$$

We need to show that $(\hat{u}, \hat{v})$ is a solution of the corresponding system on D, namely

$$\begin{cases} \Delta \hat{u} + \dfrac{2pm^2}{(p+q)n^2}|x|^{\frac{m(\alpha+2)-2n}{n}} \hat{u}^{p-1}\hat{v}^q = 0 & \text{in } D, \\ \Delta \hat{v} + \dfrac{2pm^2}{(p+q)n^2}|x|^{\frac{m(\alpha+2)-2n}{n}} \hat{u}^{p}\hat{v}^{q-1} = 0 & \text{in } D. \end{cases} \tag{3.1}$$

To this aim, let $\varphi \in C^\infty_c(D)$ a function and let $\varepsilon \in (0, 1)$. Then, if D_ε denotes the ball centered at zero with radius ε, we get

$$\begin{aligned} 0 &= \int_{D\setminus D_\varepsilon} \Delta \hat{u}\varphi + \frac{2pm^2}{(p+q)n^2}\int_{D\setminus D_\varepsilon} |x|^{\frac{m(\alpha+2)-2n}{n}} \hat{u}^{p-1}\hat{v}^q\varphi \\ &= -\int_{\partial D_\varepsilon} \frac{\partial \hat{u}}{\partial r}\varphi dS - \int_{D\setminus D_\varepsilon} \nabla \hat{u}\nabla\varphi + \frac{2pm^2}{(p+q)n^2}\int_{D\setminus D_\varepsilon} |x|^{\frac{m(\alpha+2)-2n}{n}} \hat{u}^{p-1}\hat{v}^q\varphi, \\ 0 &= \int_{D\setminus D_\varepsilon} \Delta \hat{v}\varphi + \frac{2pm^2}{(p+q)n^2}\int_{D\setminus D_\varepsilon} |x|^{\frac{m(\alpha+2)-2n}{n}} \hat{u}^{p}\hat{v}^{q-1}\varphi \\ &= -\int_{\partial D_\varepsilon} \frac{\partial \hat{v}}{\partial r}\varphi dS - \int_{D\setminus D_\varepsilon} \nabla \hat{v}\nabla\varphi + \frac{2pm^2}{(p+q)n^2}\int_{D\setminus D_\varepsilon} |x|^{\frac{m(\alpha+2)-2n}{n}} \hat{u}^{p}\hat{v}^{q-1}\varphi. \end{aligned}$$

Since also $u \in C^1(\overline{D})$, the functions $\frac{\partial u}{\partial r}(r,\theta),\ \frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta),\ \frac{\partial \varphi}{\partial r}(r,\theta),\ \frac{1}{r}\frac{\partial \varphi}{\partial \theta}(r,\theta)$ are bounded on D. From $\bar{u}(r,\theta) = u(r^{\frac{m}{n}}, \frac{m}{n}\theta)$, there exists some positive constant C such that

$$\left|\frac{\partial \bar{u}}{\partial r}(r,\theta)\right| \le Cr^{\frac{m}{n}-1}, \qquad \left|\frac{\partial \bar{u}}{\partial \theta}(r,\theta)\right| \le Cr,$$

for each $(r,\theta) \in D$. Hence, we have

$$\left|\int_{|x|=\varepsilon} \frac{\partial \bar{u}}{\partial r}\varphi\, dS\right| \le C\varepsilon^{\frac{m}{n}},$$
$$\left|\int_{\{\varepsilon<|x|<1\}} \nabla\bar{u}\,\nabla\varphi - \int_D \nabla\bar{u}\,\nabla\varphi\right| \le C\varepsilon^{\frac{m}{n}+1} + C\varepsilon^2.$$

Hence, letting $\varepsilon \to +0$, we conclude that $(\hat{u},\hat{v})$ is a weak (and hence a strong) solution to (3.1). Since $(m(\alpha+2)-2n)/n \ge \hat{m}-1$, it follows that $\hat{u}, \hat{v} \in C^{\hat{m}}(\overline{D})$. Furthermore, since $(\alpha+2)\hat{n} \le 2n$ the map $r \mapsto r^{2-2\hat{m}+(m(\alpha+2)-2n)/n}$ is nonincreasing. In turn, by applying Theorem 1.1, it follows that $\hat{u}$ and $\hat{v}$ are radially symmetric and hence u and v are radially symmetric, concluding the first part of the proof.

We now come to the proof of assertion (III). We set $H_n = \{u \in H_0^1(D) : u \text{ is } n\text{-mode}\}$ for all $n \in \mathbb{N}$ and $H_\infty = \{u \in H_0^1(D) : u \text{ is radially symmetric}\}$. For any $p, q > 1$, $\alpha \ge 0$ and $n \in \mathbb{N} \cup \{\infty\}$, set

$$S_{\alpha,p,q,n} = \inf\{R_{\alpha,p,q}(u,v) : u, v \in H_n \setminus \{0\}\}.$$

From the proof of [23, Proposition 2.5], we can find

$$S_{\alpha,p,q,\infty} \ge S_{0,p,q,1}\left(\frac{\alpha+2}{2}\right)^{1+\frac{2}{p+q}}. \tag{3.2}$$

Next, let φ be any element of $C_0^\infty(D)$. Since we can consider $\varphi \in C_0^\infty(\mathbb{R}^2)$ by the trivial extension, we can define $\varphi_\alpha \in C_0^\infty(D)$ by $\varphi_\alpha((x_1,x_2)) = \varphi(\alpha(x_1-(1-1/\alpha)), \alpha x_2)$ for $(x_1,x_2) \in D$. We set $D_1 = D$ and

$$D_n = \{(r,\theta) : 0 < r < 1, -\pi/n < \theta < \pi/n\} \quad \text{for } n \in \mathbb{N} \setminus \{1\}.$$

For each $n \in \mathbb{N}$ and $\alpha > 0$ with $\operatorname{supp}\varphi_\alpha \subset D_n$, we will show

$$S_{\alpha,p,q,n} \le S_{0,p,q,1}\, n^{1-\frac{2}{p+q}}\alpha^{\frac{4}{p+q}}\left(\frac{\alpha}{\alpha-2}\right)^{\frac{2\alpha}{p+q}}. \tag{3.3}$$

We define $P_n : D \to D$ by $P_n(r,\theta) = (r, \theta + 2\pi/n)$ for $(r,\theta) \in [0,1) \times \mathbb{R}$. We set $\widetilde{\varphi}(x) = \varphi_\alpha(x) + \varphi_\alpha(P_n(x)) + \cdots + \varphi_\alpha(P_n^{n-1}(x))$ for $x \in D$. Since we have

$$\int_D |\nabla\varphi_\alpha|^2 = \int_D |\nabla\varphi|^2$$

and

$$\int_D |x|^\alpha |\varphi_\alpha|^p |\varphi_\alpha|^q \ge \alpha^{-2}\left(1-\frac{2}{\alpha}\right)^\alpha \int_D |\varphi|^p|\varphi|^q,$$

we obtain

$$S_{\alpha,p,q,n} \le R_{\alpha,p,q}(\widetilde{\varphi},\widetilde{\varphi}) \le \frac{n\int_D(|\nabla\varphi|^2 + |\nabla\varphi|^2)}{\left(n\alpha^{-2}\left(1-\frac{2}{\alpha}\right)^\alpha \int_D |\varphi|^p|\varphi|^q\right)^{\frac{2}{p+q}}}.$$

Since $\varphi \in C_0^\infty(D)$ is arbitrary, we have shown (3.3). From (3.2) and (3.3), we can see that $n \le n_\alpha$ is a sufficient condition for $S_{\alpha,p,q,n} < S_{\alpha,p,q,\infty}$. We will show that if $n > 1$ and $S_{\alpha,p,q,n} < S_{\alpha,p,q,\infty}$ then $S_{\alpha,p,q,1} < \cdots < S_{\alpha,p,q,n}$. Let $n > 1$ and $S_{\alpha,p,q,n} < S_{\alpha,p,q,\infty}$. We can choose $u, v \in H_n \setminus \{0\}$ such that $R_{\alpha,p,q}(u,v) = S_{\alpha,p,q,n}$ and $u, v \ge 0$. We note that $u, v \notin H_\infty$ and (u,v) is a positive solution of (1.2). Let $m \in \{1,\ldots,n-1\}$. We define $\bar{u}, \bar{v} \in H_m$ by $\bar{u}(r,\theta) = u(r, m\theta/n)$ and $\bar{v}(r,\theta) = v(r, m\theta/n)$ for $(r,\theta) \in [0,1) \times \mathbb{R}$. Since we can see

$$\int_D |x|^\alpha |\bar{u}|^p|\bar{v}|^q = \int_D |x|^\alpha |u|^p|v|^q,$$
$$\int_D |\nabla\bar{u}|^2 = \int_0^{2\pi}\int_0^1 \left(\left|\frac{\partial u}{\partial r}\right|^2 + \frac{m^2}{n^2r^2}\left|\frac{\partial u}{\partial \theta}\right|^2\right) r\,drd\theta < \int_D |\nabla u|^2$$

and $\int_D |\nabla \bar{v}|^2 < \int_D |\nabla v|^2$, we have

$$S_{\alpha,p,q,m} \leq R_{\alpha,p,q}(\bar{u}, \bar{v}) < R_{\alpha,p,q}(u, v) = S_{\alpha,p,q,n}.$$

By a similar argument, we conclude that $S_{\alpha,p,q,1} < \cdots < S_{\alpha,p,q,n}$. Hence we infer that if $S_{\alpha,p,q,n} < S_{\alpha,p,q,\infty}$, then for each $\ell = 1, \ldots, n$, there exists a nonradial positive solution $(u_\ell, v_\ell) \in H_\ell \times H_\ell$ of (1.2) satisfying $R_{\alpha,p,q}(u_\ell, v_\ell) = S_{\alpha,p,q,\ell}$. We set the number in (1.4) as $\eta(\alpha, p, q)$. For a fixed $\alpha \in (2, \infty)$, we have $\eta(\alpha, p, q) \to 1 + \alpha/2$ as $p + q \to \infty$, which yields (i). For a fixed $p, q \in (1, \infty)$, we have $\eta(\alpha, p, q) \to \infty$ as $\alpha \to \infty$, yielding (ii). Hence, we finish the proof of Theorem 1.2.

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