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Asymptotic behavior of the eigenvalues of the $p(x)$-Laplacian

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Abstract. We obtain asymptotic estimates for the eigenvalues of the $p(x)$-Laplacian defined consistently with a homogeneous notion of first eigenvalue recently introduced in the literature.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$ and let $p \in C(\overline{\Omega}, (1, \infty))$. The purpose of this paper is to study the asymptotic behavior of the eigenvalues of the problem

$$- \operatorname{div}\left(\left|\frac{\nabla u}{K(u)}\right|^{p(x)-2} \frac{\nabla u}{K(u)}\right) = \lambda\, S(u) \left|\frac{u}{k(u)}\right|^{p(x)-2} \frac{u}{k(u)}, \quad u \in W_0^{1,p(x)}(\Omega), \tag{1.1}$$

where

$$K(u) = \|\nabla u\|_{p(x)}, \qquad k(u) = \|u\|_{p(x)}, \qquad S(u) = \frac{\displaystyle\int_\Omega \left|\frac{\nabla u(x)}{K(u)}\right|^{p(x)} dx}{\displaystyle\int_\Omega \left|\frac{u(x)}{k(u)}\right|^{p(x)} dx}.$$

The equation in (1.1) was derived by Franzina and Lindqvist in [5] as the Euler–Lagrange equation arising from minimizing the Rayleigh quotient $K(u)/k(u)$ over $W_0^{1,p(x)}(\Omega)\setminus\{0\}$. It was shown there that the first eigenvalue $\lambda_1 > 0$ and has an associated eigenfunction $\varphi_1 > 0$.

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We recall that the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ consists of all measurable functions u on Ω with the Luxemburg norm

$$\|u\|_{p(x)} := \inf\left\{\nu > 0 : \int_\Omega \left|\frac{u(x)}{\nu}\right|^{p(x)} \frac{dx}{p(x)} \le 1\right\} < \infty.$$

The Sobolev space $W^{1,p(x)}(\Omega)$ consists of functions $u \in L^{p(x)}(\Omega)$ with a distributional gradient $\nabla u \in L^{p(x)}(\Omega)$, and the norm in this space is $\|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. The space $W_0^{1,p(x)}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm above, and has the equivalent norm $\|\nabla u\|_{p(x)}$. We refer the reader to Diening et al. [2] for details on these spaces.

It was shown in [5] that

$$\big(K'(u), v\big) = \frac{\displaystyle\int_\Omega \left|\frac{\nabla u(x)}{K(u)}\right|^{p(x)-2} \frac{\nabla u(x)}{K(u)} \cdot \nabla v(x)\, dx}{\displaystyle\int_\Omega \left|\frac{\nabla u(x)}{K(u)}\right|^{p(x)} dx}, \quad u, v \in W_0^{1,p(x)}(\Omega)$$

and

$$\big(k'(u), v\big) = \frac{\displaystyle\int_\Omega \left|\frac{u(x)}{k(u)}\right|^{p(x)-2} \frac{u(x)}{k(u)}\, v(x)\, dx}{\displaystyle\int_\Omega \left|\frac{u(x)}{k(u)}\right|^{p(x)} dx}, \quad u, v \in W_0^{1,p(x)}(\Omega),$$

so the eigenvalues and eigenfunctions of (1.1) on the manifold

$$\mathcal{M} = \big\{u \in W_0^{1,p(x)}(\Omega) : k(u) = 1\big\}$$

coincide with the critical values and critical points of $\widetilde{K} := K|_{\mathcal{M}}$. In the next section we will show that $\widetilde{K}$ satisfies the (PS) condition, so we can define an increasing and unbounded sequence of eigenvalues of (1.1) by a minimax scheme. Although the standard scheme uses Krasnoselskii's genus, we prefer to use a cohomological index as shown in [12] by the first author since this gives additional Morse theoretic information that is often useful in applications.

Let us recall the definition of the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [3]. Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$. For $M \in \mathcal{F}$, let $\overline{M} = M/\mathbb{Z}_2$ be the quotient space of M with each u and $-u$ identified, let $f : \overline{M} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{M}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{M})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. Then the cohomological index of M is defined by

$$i(M) = \begin{cases} \sup\big\{m \ge 1 : f^*(\omega^{m-1}) \ne 0\big\}, & M \ne \emptyset \\ 0, & M = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \ge 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \le m-1$, so $i(S^{m-1}) = m$.

Set

$$\lambda_j := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq j}} \sup_{u \in M} \widetilde{K}(u), \quad j \geq 1. \tag{1.2}$$

Then $\left(\lambda_j\right)$ is a sequence of eigenvalues of (1.1) and $\lambda_j \nearrow \infty$. Moreover,

$$\lambda_j < \lambda \leq \lambda_{j+1} \implies i(\widetilde{K}^\lambda) = j,$$

where $\widetilde{K}^\lambda = \left\{u \in \mathcal{M} : \widetilde{K}(u) < \lambda\right\}$, so

$$i(\widetilde{K}^\lambda) = \#\left\{j : \lambda_j < \lambda\right\} \qquad \forall \lambda \in \mathbb{R} \tag{1.3}$$

(see Propositions 3.52 and 3.53 of Perera et al. [13]). Our main result is the following.

Theorem 1.1 *If* $1 < p^- \leq p(x) \leq p^+ < \infty$ *for all* $x \in \Omega$ *and*

$$\sigma := n\left(\frac{1}{p^-} - \frac{1}{p^+}\right) < 1, \qquad \tau := \left(\frac{1}{p^-} - \frac{1}{p^+}\right)|\Omega| < 1,$$

then there are constants $C_1, C_2 > 0$ *depending only on* n *and* $p^\pm$ *such that*

$$C_1\,|\Omega|\,(\lambda/\kappa)^{n/(1+\sigma)} \leq \#\left\{j : \lambda_j < \lambda\right\} \leq C_2\,|\Omega|\,(\kappa\,\lambda)^{n/(1-\sigma)} \qquad \text{for } \lambda > 0 \text{ large},$$

where $|\Omega|$ *is the Lebesgue measure of* Ω *and* $\kappa = (1+\tau)^{1/p^-}/(1-\tau)^{1/p^+}$.

This result is a contribution towards understanding the spectrum of the $p(x)$-Laplacian, which many researchers have recently found to be somewhat puzzling. For example, it is currently unknown if the first eigenvalue is simple, or if a given positive eigenfunction is automatically a first eigenfunction. Affirmative answers were given to both of these questions for the usual eigenvalue problem for the p-Laplacian,

$$-\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = \lambda\,|u|^{p-2}\,u, \quad u \in W_0^{1,p}(\Omega), \tag{1.4}$$

where $p > 1$ is a constant, in Lindqvist [8,9] (see also [10]). It should be noted that, in the case when p is constant, (1.1) reduces, not to the problem (1.4), which is homogeneous of degree $p-1$, but rather to the nonlocal problem

$$-\operatorname{div}\left(\frac{|\nabla u|^{p-2}\,\nabla u}{\|\nabla u\|_p^{p-1}}\right) = \lambda\,\frac{|u|^{p-2}\,u}{\|u\|_p^{p-1}}, \quad u \in W_0^{1,p}(\Omega)$$

that has been normalized to be homogeneous of degree 0. The estimate

$$C_1\,|\Omega|\,\lambda^n \leq \#\left\{j : \lambda_j < \lambda\right\} \leq C_2\,|\Omega|\,\lambda^n \qquad \text{for } \lambda > 0 \text{ large}$$

that Theorem 1.1 gives for the eigenvalues of this problem should be compared with the estimate

$$C_1\,|\Omega|\,\lambda^{n/p} \leq \#\left\{j : \lambda_j < \lambda\right\} \leq C_2\,|\Omega|\,\lambda^{n/p} \qquad \text{for } \lambda > 0 \text{ large}$$

obtained by Friedlander in [6] for (1.4) (see also García Azorero and Peral Alonso [7]). Caliari and Squassina [1] have recently developed a numerical method to compute the first eigenpair of the problem (1.1) and investigate the symmetry breaking phenomena with respect to the constant case.

In the course of proving Theorem 1.1, we will also establish the same asymptotic estimates for the eigenvalues of the problem

$$-\operatorname{div}\left(\left|\frac{\nabla u}{L(u)}\right|^{p(x)-2}\frac{\nabla u}{L(u)}\right)=\mu\, T(u)\left|\frac{u}{l(u)}\right|^{p(x)-2}\frac{u}{l(u)},\quad u\in W^{1,p(x)}(\Omega),\tag{1.5}$$

where

$$L(u)=\|\nabla u\|_{p(x)},\qquad l(u)=\|u\|_{p(x)},\qquad T(u)=\frac{\displaystyle\int_\Omega\left|\frac{\nabla u(x)}{L(u)}\right|^{p(x)}dx}{\displaystyle\int_\Omega\left|\frac{u(x)}{l(u)}\right|^{p(x)}dx}$$

(which coincide with K, k, and S, respectively, on $W_0^{1,p(x)}(\Omega)$). The eigenvalues and eigenfunctions of this problem on

$$\mathcal{N}=\left\{u\in W^{1,p(x)}(\Omega):l(u)=1\right\}$$

coincide with the critical values and critical points of $\widetilde{L}:=L|_{\mathcal{N}}$. Let $\mathcal{G}$ denote the class of symmetric subsets of $\mathcal{N}$ and set

$$\mu_j:=\inf_{\substack{N\in\mathcal{G}\\ i(N)\geq j}}\ \sup_{u\in N}\ \widetilde{L}(u),\quad j\geq 1.$$

Then (μ_j) is a sequence of eigenvalues of (1.5), $\mu_j\nearrow\infty$, and

$$i(\widetilde{L}^{\mu})=\#\left\{j:\mu_j<\mu\right\}\qquad\forall\mu\in\mathbb{R},$$

where $\widetilde{L}^{\mu}=\left\{u\in\mathcal{N}:\widetilde{L}(u)<\mu\right\}$. Since $W^{1,p(x)}(\Omega)\supset W_0^{1,p(x)}(\Omega)$ and $l|_{W_0^{1,p(x)}(\Omega)}=k$, we have $\mathcal{N}\supset\mathcal{M}$, and $\widetilde{L}|_{\mathcal{M}}=\widetilde{K}$, so $\mu_j\leq\lambda_j$ for all j. We will see that, under the hypotheses of Theorem 1.1,

$$C_1\,|\Omega|\,(\mu/\kappa)^{n/(1+\sigma)}\leq\#\left\{j:\mu_j<\mu\right\}\leq C_2\,|\Omega|\,(\kappa\,\mu)^{n/(1-\sigma)}\qquad\text{for }\mu>0\text{ large.}$$

Finally, for the sake of completeness, let us also mention that a different notion of first eigenvalue for the $p(x)$-Laplacian, that does not make use of the Luxemburg norm, has been considered in the past literature, namely,

$$\lambda_1^*=\inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega|\nabla u|^{p(x)}\,dx}{\displaystyle\int_\Omega|u|^{p(x)}\,dx}.$$

In this framework, $\lambda \in \mathbb{R}$ and $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ are an eigenvalue and an eigenfunction of the $p(x)$-Laplacian, respectively, if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega |u|^{p(x)-2} uv \, dx \qquad \forall v \in W_0^{1,p(x)}(\Omega)$$

(this should be compared with (1.1)). Let Λ denote the set of all eigenvalues of this problem. If the function $p(x)$ is a constant $p > 1$, then it is well-known that this problem admits an increasing sequence of eigenvalues, $\sup \Lambda = +\infty$, and $\inf \Lambda = \lambda_{1,p} > 0$, the first eigenvalue of the p-Laplacian (see Lindqvist [8–10]). For general $p(x)$, Λ is a nonempty infinite set, $\sup \Lambda = +\infty$, and $\inf \Lambda = \lambda_1^*$ (see Fan et al. [4]). In contrast to the situation when minimizing the Rayleigh quotient with respect to the Luxemburg norm, one often has $\lambda_1^* = 0$, and $\lambda_1^* > 0$ only under special conditions. In [4], the authors provide sufficient conditions for λ_1^* to be zero or positive. In particular, if $p(x)$ has a strict local minimum (or maximum) in Ω, then $\lambda_1^* = 0$. If $n > 1$ and there is a vector $\ell \neq 0$ in $\mathbb{R}^n$ such that for every $x \in \Omega$, the map $t \mapsto p(x + t\ell)$ is monotone on $\{t : x + t\ell \in \Omega\}$, then $\lambda_1^* > 0$. Finally, if $n = 1$, then $\lambda_1^* > 0$ if and only if the function $p(x)$ is monotone.

2. Compactness

In this section we will show that $\widetilde{K}$ satisfies the (PS) condition. Here and in the next section we will make use of the well-known Young's inequality

$$ab \leq \left(1 - \frac{1}{p}\right) a^{p/(p-1)} + \frac{1}{p} b^p \qquad \forall a, b \geq 0, \; p > 1. \tag{2.1}$$

Lemma 2.1 *For $u \neq 0$ in $L^{p(x)}(\Omega)$ and all $v \in L^{p(x)}(\Omega)$,*

$$\left|\left(k'(u), v\right)\right| \leq \|v\|_{p(x)}. \tag{2.2}$$

Proof. Equality holds in (2.2) if $v = 0$, so suppose $v \neq 0$. We have

$$\left|\left(k'(u), v\right)\right| \leq \frac{\displaystyle\int_\Omega \left|\frac{u(x)}{k(u)}\right|^{p(x)-1} |v(x)| \, dx}{\displaystyle\int_\Omega \left|\frac{u(x)}{k(u)}\right|^{p(x)} dx}. \tag{2.3}$$

Taking $a = |u(x)/k(u)|^{p(x)-1}$, $b = |v(x)/k(v)|$, $p = p(x)$ in (2.1) and integrating over Ω gives

$$\int_\Omega \left|\frac{u(x)}{k(u)}\right|^{p(x)-1} \left|\frac{v(x)}{k(v)}\right| dx \leq \int_\Omega \left|\frac{u(x)}{k(u)}\right|^{p(x)} dx - \int_\Omega \left|\frac{u(x)}{k(u)}\right|^{p(x)} \frac{dx}{p(x)}$$
$$+ \int_\Omega \left|\frac{v(x)}{k(v)}\right|^{p(x)} \frac{dx}{p(x)}.$$

The last two integrals are both equal to 1, so this shows that the right-hand side of (2.3) is less than or equal to $k(v) = \|v\|_{p(x)}$. $\square$

Lemma 2.2 *K' is a mapping of type (S_+), i.e., if $u_j \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and*

$$\varlimsup_{j\to\infty} \big(K'(u_j), u_j - u\big) \le 0,$$

then $u_j \to u$ in $W_0^{1,p(x)}(\Omega)$.

Proof. Since

$$\big(K'(u_j), u_j\big) = K(u_j) = \left\|\nabla u_j\right\|_{p(x)}$$

and

$$\big(K'(u_j), u\big) = \big(k'(\nabla u_j), \nabla u\big) \le \|\nabla u\|_{p(x)}$$

by Lemma 2.1,

$$\begin{aligned}\varlimsup_{j\to\infty} \left\|\nabla u_j\right\|_{p(x)} &\le \varlimsup_{j\to\infty} \big(K'(u_j), u_j - u\big) + \|\nabla u\|_{p(x)} \\ &\le \|\nabla u\|_{p(x)} \le \varliminf_{j\to\infty} \left\|\nabla u_j\right\|_{p(x)},\end{aligned}$$

so that $\left\|\nabla u_j\right\|_{p(x)} \to \|\nabla u\|_{p(x)}$. The conclusion follows since $W_0^{1,p(x)}(\Omega)$ is uniformly convex. □

Lemma 2.3 *For all $c \in \mathbb{R}$, $\widetilde{K}$ satisfies the $(\mathrm{PS})_c$ condition, i.e., every sequence $\big(u_j\big) \subset \mathcal{M}$ such that $\widetilde{K}(u_j) \to c$ and $\widetilde{K}'(u_j) \to 0$ has a convergent subsequence.*

Proof. We have

$$K(u_j) \to c, \qquad K'(u_j) - c_j\, k'(u_j) \to 0 \tag{2.4}$$

for some sequence $\big(c_j\big) \subset \mathbb{R}$. Since $\big(K'(u_j), u_j\big) = K(u_j)$ and $\big(k'(u_j), u_j\big) = k(u_j) = 1$, $c_j \to c$. Since $\big(u_j\big)$ is bounded in $W_0^{1,p(x)}(\Omega)$, for a renamed subsequence and some $u \in W_0^{1,p(x)}(\Omega)$, $u_j \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $u_j \to u$ in $L^{p(x)}(\Omega)$. By Lemma 2.1,

$$\left|\big(k'(u_j), u_j - u\big)\right| \le \left\|u_j - u\right\|_{p(x)} \to 0,$$

so the second limit in (2.4) now gives $\big(K'(u_j), u_j - u\big) \to 0$ as $j \to \infty$. Then we conclude that $u_j \to u$ strongly in $W_0^{1,p(x)}(\Omega)$, in light of Lemma 2.2. □

3. Proof of Theorem 1.1

Let σ, τ, and κ be as in Theorem 1.1.

Lemma 3.1 *We have*

$$\frac{\|u\|_{p^-}}{(1+\tau)^{1/p^-}} \le \|u\|_{p(x)} \le \frac{\|u\|_{p^+}}{(1-\tau)^{1/p^+}} \qquad \forall u \in L^{p^+}(\Omega), \tag{3.1}$$

and hence

$$\frac{1}{\kappa}\,\frac{\|\nabla u\|_{p^-}}{\|u\|_{p^+}} \le \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \le \kappa\,\frac{\|\nabla u\|_{p^+}}{\|u\|_{p^-}} \qquad \forall u \in W^{1,p^+}(\Omega)\setminus\{0\}.$$

Proof. Equality holds throughout (3.1) if $u = 0$, so suppose $u \ne 0$. Taking $a = 1$, $b = |u(x)/\|u\|_{p(x)}|^{p^-}$, $p = p(x)/p^-$ in (2.1), dividing by p^-, and integrating over Ω gives

$$\frac{1}{\|u\|_{p(x)}^{p^-}} \int_\Omega |u(x)|^{p^-}\,\frac{dx}{p^-} \le \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) dx + \int_\Omega \left|\frac{u(x)}{\|u\|_{p(x)}}\right|^{p(x)} \frac{dx}{p(x)}.$$

The first integral is equal to $\|u\|_{p^-}^{p^-}$ and the last integral is equal to 1, so this gives the first inequality in (3.1). Now taking $a = 1$, $b = |u(x)/\|u\|_{p(x)}|^{p(x)}$, $p = p^+/p(x)$ in (2.1), dividing by $p(x)$, and integrating over Ω gives

$$\int_\Omega \left|\frac{u(x)}{\|u\|_{p(x)}}\right|^{p(x)} \frac{dx}{p(x)} \le \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right) dx + \frac{1}{\|u\|_{p(x)}^{p^+}} \int_\Omega |u(x)|^{p^+}\,\frac{dx}{p^+}.$$

The first integral is equal to 1 and the last integral is equal to $\|u\|_{p^+}^{p^+}$, so this gives the second inequality in (3.1). □

Recall that the genus and the cogenus of $M \in \mathcal{F}$ are defined by

$$\gamma(M) = \inf\left\{m \ge 1 : \exists \text{ an odd continuous map } g : M \to S^{m-1}\right\}$$

and

$$\widetilde{\gamma}(M) = \sup\left\{\widetilde{m} \ge 1 : \exists \text{ an odd continuous map } \widetilde{g} : S^{\widetilde{m}-1} \to M\right\},$$

respectively. If there are odd continuous maps $S^{\widetilde{m}-1} \to M \to S^{m-1}$, then $\widetilde{m} \le i(M) \le m$ by the monotonicity of the index, so $\widetilde{\gamma}(M) \le i(M) \le \gamma(M)$. Since $\widetilde{K}^\lambda \subset \widetilde{L}^\lambda$, this gives

$$\widetilde{\gamma}(\widetilde{K}^\lambda) \le i(\widetilde{K}^\lambda) \le i(\widetilde{L}^\lambda) \le \gamma(\widetilde{L}^\lambda) \qquad \forall \lambda \in \mathbb{R}. \tag{3.2}$$

Set

$$\widehat{K}(u) := \|\nabla u\|_{p^+}, \quad u \in \widehat{\mathcal{M}} := \left\{u \in W_0^{1,p^+}(\Omega) : \|u\|_{p^-} = 1\right\}$$

and

$$\widehat{L}(u) := \|\nabla u\|_{p^-}, \quad u \in \widehat{\mathcal{N}} := \left\{u \in W^{1,p^+}(\Omega) : \|u\|_{p^+} = 1\right\},$$

and let $\widehat{K}^\lambda = \left\{u \in \widehat{\mathcal{M}} : \widehat{K}(u) < \lambda\right\}$ and $\widehat{L}^\mu = \left\{u \in \widehat{\mathcal{N}} : \widehat{L}(u) < \mu\right\}$.

Lemma 3.2 *We have*

$$\widetilde{\gamma}(\widehat{K}^{\lambda/\kappa}) \le \widetilde{\gamma}(\widetilde{K}^{\lambda}), \qquad \gamma(\widetilde{L}^{\lambda}) \le \gamma(\widehat{L}^{\kappa\lambda}) \qquad \forall \lambda \in \mathbb{R}.$$

Proof. Lemma 3.1 gives the odd continuous maps

$$\widehat{K}^{\lambda/\kappa} \to \widetilde{K}^{\lambda}, \quad u \mapsto \frac{u}{\|u\|_{p(x)}}, \qquad \widetilde{L}^{\lambda} \cap W^{1,p^+}(\Omega) \to \widehat{L}^{\kappa\lambda}, \quad u \mapsto \frac{u}{\|u\|_{p^+}},$$

and the inclusion $\widetilde{L}^{\lambda} \cap W^{1,p^+}(\Omega) \subset \widetilde{L}^{\lambda}$ is a homotopy equivalence by Palais [11, Theorem 17] since $W^{1,p^+}(\Omega)$ is a dense linear subspace of $W^{1,p(x)}(\Omega)$, so the conclusion follows. □

Lemma 3.3 *Let* $0 < \delta < 1$*, consider the homothety* $\Omega \to \delta\,\Omega$, $x \mapsto \delta x =: y$*, and write* $u(x) = v(y)$*. Then*

$$\frac{\|\nabla v\|_{p^+}}{\|v\|_{p^-}} = \delta^{-\sigma-1}\,\frac{\|\nabla u\|_{p^+}}{\|u\|_{p^-}}, \qquad \frac{\|\nabla v\|_{p^-}}{\|v\|_{p^+}} = \delta^{\sigma-1}\,\frac{\|\nabla u\|_{p^-}}{\|u\|_{p^+}} \quad \forall u \in W^{1,p^+}(\Omega)\setminus\{0\}\,.$$

Proof. Straightforward. □

Lemma 3.4 *If* Ω_1 *and* Ω_2 *are disjoint subdomains of* Ω *such that* $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$*, then*

$$\widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega_1}) + \widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega_2}) \le \widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega}), \qquad \gamma(\widehat{L}^{\lambda}_{\Omega}) \le \gamma(\widehat{L}^{\lambda'}_{\Omega_1}) + \gamma(\widehat{L}^{\lambda'}_{\Omega_2}) \qquad \forall \lambda < \lambda',$$

where the subscripts indicate the corresponding domains.

Proof. Since $\widehat{K}^{\lambda}_{\Omega}$ contains $\widehat{K}^{\lambda}_{\Omega_1}$ and $\widehat{K}^{\lambda}_{\Omega_2}$, if $\widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega_1})$ or $\widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega_2})$ is infinite, then so is $\widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega})$ and hence the first inequality holds. So let $\widetilde{m}_i := \widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega_i}) < \infty$ and let $\widetilde{g}_i : S^{\widetilde{m}_i - 1} \to \widehat{K}^{\lambda}_{\Omega_i}$ be an odd continuous map for $i = 1, 2$. Write $y \in S^{\widetilde{m}_1 + \widetilde{m}_2 - 1}$ as $y = (y_1, y_2) \in \mathbb{R}^{\widetilde{m}_1} \oplus \mathbb{R}^{\widetilde{m}_2}$, set $|y_2| = t$, and let

$$\widetilde{g}(y) = \begin{cases} \widetilde{g}_1(y_1), & t = 0 \\ \dfrac{(1-t)\,\widetilde{g}_1(y_1/\sqrt{1-t^2}) + t\,\widetilde{g}_2(y_2/t)}{\left\|(1-t)\,\widetilde{g}_1(y_1/\sqrt{1-t^2}) + t\,\widetilde{g}_2(y_2/t)\right\|_{p^-}}, & 0 < t < 1 \\ \widetilde{g}_2(y_2), & t = 1. \end{cases}$$

Clearly, $\widetilde{g}(y) \in \widehat{K}^{\lambda}_{\Omega}$ for $t = 0, 1$. For $0 < t < 1$,

$$\widehat{K}_{\Omega}(\widetilde{g}(y)) < \lambda\, \frac{\left[(1-t)^{p^+} + t^{p^+}\right]^{1/p^+}}{\left[(1-t)^{p^-} + t^{p^-}\right]^{1/p^-}} \le \lambda$$

since $p \mapsto [(1-t)^p + t^p]^{1/p}$ on $(1, \infty)$ is nonincreasing. So $\widetilde{g} : S^{\widetilde{m}_1 + \widetilde{m}_2 - 1} \to \widehat{K}^{\lambda}_{\Omega}$ is an odd continuous map and hence $\widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega}) \ge \widetilde{m}_1 + \widetilde{m}_2$.

Since the second inequality holds if $\gamma(\widehat{L}^{\lambda'}_{\Omega_1})$ or $\gamma(\widehat{L}^{\lambda'}_{\Omega_2})$ is infinite, let $m_i := \gamma(\widehat{L}^{\lambda'}_{\Omega_i}) < \infty$ and let $g_i : \widehat{L}^{\lambda'}_{\Omega_i} \to S^{m_i - 1}$ be an odd continuous map for $i = 1, 2$. For $u \in \widehat{L}^{\lambda}_{\Omega}$, let $u_i = u|_{\Omega_i}$, $\rho_i = \|u_i\|_{p^+}$, and $\widetilde{u}_i = u_i/\rho_i$ if $\rho_i \ne 0$. Fix $\lambda'' \in (\lambda, \lambda')$

such that $(\lambda/\lambda'')^{p^+} \geq 1/2$, take smooth cutoff functions $\eta,\ \zeta : [0,\infty) \to [0,1]$ such that $\eta = 0$ near zero, $\eta = 1$ on $[[1-(\lambda/\lambda'')^{p^+}]^{1/p^+}, \infty)$, $\zeta = 1$ on $[0, \lambda'']$ and $\zeta = 0$ on $[\lambda', \infty)$, and let

$$g(u) = \frac{\big(\eta(\rho_1)\, \zeta(\widehat{L}_{\Omega_1}(\widetilde{u}_1))\, g_1(\widetilde{u}_1),\ \eta(\rho_2)\, \zeta(\widehat{L}_{\Omega_2}(\widetilde{u}_2))\, g_2(\widetilde{u}_2)\big)}{\sqrt{\eta(\rho_1)^2\, \zeta(\widehat{L}_{\Omega_1}(\widetilde{u}_1))^2 + \eta(\rho_2)^2\, \zeta(\widehat{L}_{\Omega_2}(\widetilde{u}_2))^2}}, \tag{3.3}$$

with the understanding that $\eta(\rho_i)\, \zeta(\widehat{L}_{\Omega_i}(\widetilde{u}_i))\, g_i(\widetilde{u}_i) = 0$ if $\rho_i = 0$. We claim that the denominator is greater than or equal to 1. The claim is clearly true if $u_1 = 0$ or $u_2 = 0$, so suppose $u_1 \neq 0$ and $u_2 \neq 0$. Since $\rho_1^{p^+} + \rho_2^{p^+} = \|u\|_{p^+}^{p^+} = 1$, either $\rho_1 \geq 1/2^{1/p^+}$ or $\rho_2 \geq 1/2^{1/p^+}$, and since $1/2^{1/p^+} \geq [1-(\lambda/\lambda'')^{p^+}]^{1/p^+}$, then either $\eta(\rho_1) = 1$ or $\eta(\rho_2) = 1$. Moreover, if $\widehat{L}_{\Omega_i}(\widetilde{u}_i) \geq \lambda$ for $i = 1, 2$, then

$$1 = \rho_1^{p^+} + \rho_2^{p^+} \leq \rho_1^{p^-} + \rho_2^{p^-} \leq \frac{\|\nabla u_1\|_{p^-}^{p^-} + \|\nabla u_2\|_{p^-}^{p^-}}{\lambda^{p^-}} = \frac{\|\nabla u\|_{p^-}^{p^-}}{\lambda^{p^-}} < 1,$$

a contradiction, so either $\zeta(\widehat{L}_{\Omega_1}(\widetilde{u}_1)) = 1$ or $\zeta(\widehat{L}_{\Omega_2}(\widetilde{u}_2)) = 1$. Consequently, we are done if $\eta(\rho_1) = 1$ and $\eta(\rho_2) = 1$, so assume that one of them, say $\eta(\rho_2)$, is less than 1. Then $\eta(\rho_1) = 1$. Moreover, $\rho_2 < [1-(\lambda/\lambda'')^{p^+}]^{1/p^+}$ and hence

$$\widehat{L}_{\Omega_1}(\widetilde{u}_1) = \frac{\|\nabla u_1\|_{p^-}}{\rho_1} \leq \frac{\|\nabla u\|_{p^-}}{(1-\rho_2^{p^+})^{1/p^+}} < \lambda'',$$

so $\zeta(\widehat{L}_{\Omega_1}(\widetilde{u}_1)) = 1$. Thus, the denominator in (3.3) is greater than or equal to $\eta(\rho_1)\, \zeta(\widehat{L}_{\Omega_1}(\widetilde{u}_1)) = 1$. So $g : \widehat{L}_\Omega^\lambda \to S^{m_1+m_2-1}$ is an odd continuous map and hence $\gamma(\widehat{L}_\Omega^\lambda) \leq m_1 + m_2$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Continuously extend p to the whole space, with the same bounds p^- and p^+, using the Tietze extension theorem. Let Q be the unit cube in $\mathbb{R}^n$, fix $\lambda_0 > \max\left\{\inf \widehat{K}_Q, \inf \widehat{L}_Q\right\}$, and set

$$r = \widetilde{\gamma}(\widehat{K}_Q^{\lambda_0}), \qquad s = \gamma(\widehat{L}_Q^{\lambda_0}).$$

Then for $\lambda' > \lambda > \lambda_0$ and any two cubes Q_{a_λ} and $Q_{b_{\lambda'}}$ of sides $a_\lambda = (\lambda_0/\lambda)^{1/(1+\sigma)}$ and $b_{\lambda'} = (\lambda_0/\lambda')^{1/(1-\sigma)}$, respectively, Lemma 3.3 gives the odd homeomorphisms

$$\widehat{K}_Q^{\lambda_0} \to \widehat{K}_{Q_{a_\lambda}}^{\lambda}, \quad u \mapsto \frac{v}{\|v\|_{p^-}}, \qquad \widehat{L}_Q^{\lambda_0} \to \widehat{L}_{Q_{b_{\lambda'}}}^{\lambda'}, \quad u \mapsto \frac{v}{\|v\|_{p^+}},$$

so

$$\widetilde{\gamma}(\widehat{K}_{Q_{a_\lambda}}^{\lambda}) = r, \qquad \gamma(\widehat{L}_{Q_{b_{\lambda'}}}^{\lambda'}) = s.$$

Now it follows from Lemma 3.4 that if Q_a is a cube of side $a > 0$, then

$$r\left[\frac{a}{a_\lambda}\right]^n \leq \widetilde{\gamma}(\widehat{K}_{Q_a}^{\lambda}), \qquad \gamma(\widehat{L}_{Q_a}^{\lambda}) \leq s\left(\left[\frac{a}{b_{\lambda'}}\right] + 1\right)^n,$$

where $[\cdot]$ denotes the integer part. Thus, there are constants $C_1, C_2 > 0$, independent of a, λ, and λ', such that

$$C_1\, a^n \lambda^{n/(1+\sigma)} \le \widetilde{\gamma}(\widehat{K}^{\lambda}_{Q_a}), \quad \gamma(\widehat{L}^{\lambda}_{Q_a}) \le C_2\, a^n (\lambda')^{n/(1-\sigma)}, \quad \lambda < \lambda' \text{ large.} \tag{3.4}$$

Let $\varepsilon > 0$ and let Ω_ε, Ω^ε be unions of cubes with pairwise disjoint interiors such that $\Omega_\varepsilon \subset \Omega \subset \Omega^\varepsilon$ and $|\Omega^\varepsilon \backslash \Omega_\varepsilon| < \varepsilon$. Then

$$C_1\, |\Omega_\varepsilon|\, \lambda^{n/(1+\sigma)} \le \widetilde{\gamma}(\widehat{K}^{\lambda}_{\Omega_\varepsilon}) \le \widetilde{\gamma}(\widehat{K}^{\lambda}), \qquad \gamma(\widehat{L}^{\lambda}) \le \gamma(\widehat{L}^{\lambda}_{\Omega^\varepsilon}) \le C_2\, |\Omega^\varepsilon|\, (\lambda')^{n/(1-\sigma)}$$

by (3.4) and Lemma 3.4. Letting $\varepsilon \searrow 0$, $\lambda' \searrow \lambda$, and combining with (1.3), (3.2), and Lemma 3.2 yields the conclusion. □

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