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ON THE LOGARITHMIC SCHRÖDINGER EQUATION

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In the framework of the nonsmooth critical point theory for lower semi-continuous functionals, we propose a direct variational approach to investigate the existence of infinitely many weak solutions for a class of semi-linear elliptic equations with logarithmic nonlinearity arising in physically relevant situations. Furthermore, we prove that there exists a unique positive solution which is radially symmetric and nondegenerate.

Keywords: Logarithmic Schrödinger equation; existence; multiplicity; qualitative properties.

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1. Introduction

The logarithmic Schrödinger equation

$$i\partial_t \phi + \Delta\phi + \phi \log|\phi|^2 = 0, \quad \phi : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}, \quad n \geq 3, \tag{1.1}$$

admits applications to quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose–Einstein condensation (see [30] and the references therein). We refer to [11–13] for a study of existence and uniqueness of the solutions of the associated Cauchy problem in a suitable functional framework as well as to a study of the asymptotic behavior of its solutions and their orbital stability, in the spirit of [14], with respect to radial perturbations, of the so-called Gausson solution (see [5]). In this paper we are interested in the existence, multiplicity and qualitative properties of the standing waves solution of (1.1), i.e. solution in the form ($\phi = e^{i\omega t}u(x)$), where ($\omega \in \mathbb{R}$) and (u) is a real-valued function which has to solve the following semi-linear elliptic problem

$$-\Delta u + \omega u = u \log u^2, \quad u \in H^1(\mathbb{R}^n). \tag{1.2}$$

It is well known (see [5, 6]) that the Gausson

$$\mathfrak{g}(x) = e^{-|x|^2/2}$$

solves (1.2) for $\omega = -n$. We emphasize that if u is a solution of (1.2), then λu, $\lambda \neq 0$, is a solution of $-\Delta v + \omega' v = v \log v^2$ with $\omega' = \omega + \log \lambda^2$. This fact allows us to name the solution $\exp\{(\omega+n-|x|^2)/2\}$, *Gausson* for (1.2). Moreover, without loss of generality, we can restrict to the case $\omega > 0$, even if our results hold for every $\omega \in \mathbb{R}$. From a variational point of view, the search of nontrivial solutions to (1.2) can be formally associated with the study of critical points of the functional on $H^1(\mathbb{R}^n)$ defined by

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{\omega+1}{2}\int u^2 - \frac{1}{2}\int u^2 \log u^2. \tag{1.3}$$

Due to the logarithmic Sobolev inequality

$$\int u^2 \log u^2 \le \frac{a^2}{\pi}\|\nabla u\|_2^2 + (\log \|u\|_2^2 - n(1+\log a))\|u\|_2^2, \qquad \text{for } u \in H^1(\mathbb{R}^n) \text{ and } a > 0, \tag{1.4}$$

(see, e.g., [25]), we have $J(u) > -\infty$ for all $u \in H^1(\mathbb{R}^n)$, but there are elements $u \in H^1(\mathbb{R}^n)$ such that $\int u^2 \log u^2 = -\infty$. Thus, in general, J fails to be finite and C^1 on $H^1(\mathbb{R}^n)$. Due to this loss of smoothness, in order to study existence of solutions to (1.2), to the best of our knowledge, two indirect approaches were followed so far in the literature. On the one hand, in [11], the idea is to work on the Banach space

$$\begin{aligned} W &= \left\{ u \in H^1(\mathbb{R}^n) \,\middle|\, \int u^2|\log u^2| < \infty \right\}, \\ \|u\|_W &= \|u\|_{H^1} + \inf\left\{ \gamma > 0 \,\middle|\, \int A(\gamma^{-1}|u|) \le 1 \right\}, \end{aligned} \tag{1.5}$$

where $A(s) = -s^2 \log s^2$ on $[0, e^{-3}]$ and $A(s) = 3s^2 + 4e^{-3}s - e^{-6}$ on $[e^{-3}, \infty)$. In fact, it turns out that, in this framework $J : W \to \mathbb{R}$ is well defined and C^1 smooth (see [11, Proposition 2.7]). On the other hand, in [23], the authors penalize the nonlinearity around the origin and try to obtain *a priori* estimates to get a nontrivial solution at the limit. However, the drawback of these indirect approaches is that the Palais–Smale condition cannot be obtained, due to a loss of coercivity of the functional J, and, in general, no multiplicity result can be obtained by the Lusternik–Schnirelmann category theory. In this paper we introduce a direct approach to study the existence of infinitely many weak solutions to (1.2), in the framework of the nonsmooth critical point theory developed by Degiovanni–Zani in [19, 20] (see also [10]) for suitable classes of lower semi-continuous functionals, and based on the notion of weak slope (see [18, 17]). In fact, it is easy to see that the functional $J : H^1_{\rm rad}(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous (see Proposition 2.2) and that it satisfies the Palais–Smale condition in the sense of weak slope (see Proposition 2.3). More precisely, we shall prove the following theorem.

Theorem 1.1. *Problem* (1.2) *has a sequence of solutions* $u_k \in H^1_{\rm rad}(\mathbb{R}^n)$ *with* $J(u_k) \to +\infty$ *as* $k \to +\infty$.

To the best of our knowledge this is the first multiplicity result for (1.2) on the space $H^1(\mathbb{R}^n)$ and it is obtained directly *without penalizing* the functional and *without changing the topology* of the space. It should also be noticed that, due to the behavior around zero, our logarithmic nonlinearity does not fit into the framework of the classical papers by Berestycki and Lions [2, 3]. We also point out that, even without working in the restricted space of radial functions $H^1_{\rm rad}(\mathbb{R}^n)$, since J decreases under polarization of nonnegative functions of $H^1(\mathbb{R}^n)$, we can obtain the existence of a Palais–Smale sequence $\{u_k\} \subset H^1(\mathbb{R}^n)$ with the additional information that $\|u_k - |u_k|^*\|_{L^{2^*}(\mathbb{R}^n)} \to 0$ as $h \to \infty$, namely $\{u_k\}$ is *almost* radially symmetric and decreasing (see [28, Theorem 3.10]). An analogous multiplicity result is expected to hold both for the problem

$$-\Delta_p u + \omega|u|^{p-2}u = |u|^{p-2}u \log u^2, \quad u \in W^{1,p}(\mathbb{R}^n),$$

by applying the L^p-logarithmic Sobolev inequality (see [21]), and for the *fractional logarithmic Schrödinger equation.*

In the last section we study some qualitative properties of the solutions of (1.2). We are able to prove that the nonnegative solutions are strictly positive and that they are smooth. By exploiting the moving plane method (we outline that our nonlinearity is not C^1 in $[0, \infty)$), we show the following theorem.

Theorem 1.2. *Up to translations, the Gausson for* (1.2) *is the unique strictly positive C^2-solution such that* $u(x) \to 0$ *as* $|x| \to \infty$.

Then we get that the first solution u_1 in Theorem 1.1 is the Gausson for (1.2). Moreover we prove the following theorem.

Theorem 1.3. *The Gausson* $\mathfrak{g}$ *is nondegenerate, that is* $\mathrm{Ker}(L) = \mathrm{span}\{\partial_{x_h}\mathfrak{g}\}$, *where* $Lu = -\Delta u + (|x|^2 - n - 2)u$ *is the linearized operator for* $-\Delta u - nu = u \log u^2$ *at* $\mathfrak{g}$.

Finally, in Theorem 3.1, we also obtain a variational characterization of ground state solutions (namely minima of J on the Nehari manifold) of the problem as minima on the L^2-sphere. We believe that the nondegeneracy of $\mathfrak{g}$ and the connection between the minimization on the Nehari manifold and on the L^2-sphere can be useful in the study of the stability properties of the logarithmic Schrödinger equation (1.1), possibly in presence of an external driving potential (see, e.g., [7]).

Notations.

(1) $L^\infty_c(\mathbb{R}^n)$ denotes the space of functions in $L^\infty(\mathbb{R}^n)$ with compact support;
(2) $H^1_{\mathrm{rad}}(\mathbb{R}^n)$ denotes the space of $H^1(\mathbb{R}^n)$ functions that are radially symmetric;
(3) C denotes a generic positive constant which can change from line to line.

2. The Multiplicity Result

The aim of this section is to prove Theorem 1.1.

2.1. *Recalls of nonsmooth critical point theory*

Let us recall some notions useful in the following. For a more complete treatment of these arguments we refer the reader to [10, 18, 20]. Let $(X, \|\cdot\|_X)$ be a Banach space and $f : X \to \bar{\mathbb{R}}$ be a function. We consider

$$\mathrm{epi}(f) = \{(x, \lambda) \in X \times \mathbb{R} \,|\, f(x) \leq \lambda\}$$

endowed with the metric induced by the norm $\|\cdot\|_{X\times\mathbb{R}} = (\|\cdot\|_X^2 + |\cdot|^2)^{1/2}$ of $X \times \mathbb{R}$ and we denote with $B_\delta(x, \lambda)$ the open ball of center (x, λ) and radius $\delta > 0$. Moreover we give the following definitions. First we give the notion of weak slope for continuous functions.

Definition 2.1. Let $f : X \to \mathbb{R}$ be continuous. For every $x \in X$, we denote by $|df|(x)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(x) \times [0, \delta] \to X$, satisfying

$$\|\mathcal{H}(w, t) - w\|_X \leq t, \quad f(\mathcal{H}(w, t)) \leq f(w) - \sigma t,$$

whenever $w \in B_\delta(x)$ and $t \in [0, \delta]$. The extended real number $|df|(x)$ is called the *weak slope* of f at x.

Now, let us consider the function $\mathcal{G}_f := (x, \lambda) \in \mathrm{epi}(f) \mapsto \lambda \in \mathbb{R}$. The function $\mathcal{G}_f$ is continuous and Lipschitzian of constant 1 and it allows to generalize the notion of weak slope for noncontinuous functions f as follows.

Definition 2.2. For all $x \in X$ with $f(x) \in \mathbb{R}$

$$|df|(x) := \begin{cases} \dfrac{|d\mathcal{G}_f|(x, f(x))}{\sqrt{1 - |d\mathcal{G}_f|(x, f(x))^2}} & \text{if } |d\mathcal{G}_f|(x, f(x)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_f|(x, f(x)) = 1. \end{cases}$$

We also need the following definition.

Definition 2.3. Let $c \in \mathbb{R}$. The function f satisfies $(\mathrm{epi})_c$ condition if there exists $\varepsilon > 0$ such that

$$\inf\{|d\mathcal{G}_f|(x,\lambda) \mid f(x) < \lambda, |\lambda - c| < \varepsilon\} > 0.$$

Definition 2.4. $x \in X$ is a (lower) critical point of f if $f(x) \in \mathbb{R}$ and $|df|(x) = 0$.

Definition 2.5. Let $c \in \mathbb{R}$. A sequence $\{x_k\} \subset X$ is a Palais–Smale sequence for f at level c if $f(x_k) \to c$ and $|df|(x_k) \to 0$. Moreover f satisfies the Palais–Smale condition at level c if every Palais–Smale sequence for f at level c admits a convergent subsequence in X.

Definition 2.6. Let f be even with $f(0) \in \mathbb{R}$. For every $\lambda \geq f(0)$, we denote by $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\lambda)$ the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : (B_\delta(0,\lambda) \cap \mathrm{epi}(f)) \times [0,\delta] \to \mathrm{epi}(f)$, satisfying

$$\|\mathcal{H}((w,\mu),t) - (w,\mu)\|_{X\times\mathbb{R}} \leq t, \quad \mathcal{H}_2((w,\mu),t) \leq \mu - \sigma t,$$

$$\mathcal{H}_1((-w,\mu),t) = -\mathcal{H}_1((w,\mu),t),$$

whenever $(w,\mu) \in B_\delta(0,\lambda) \cap \mathrm{epi}(f)$ and $t \in [0,\delta]$.

We will apply the following abstract result (see [20]).

Theorem 2.1. *Let X be a Banach space and $f : X \to \bar{\mathbb{R}}$ be a lower semicontinuous even functional. Assume that $f(0) = 0$ and there exists a strictly increasing sequence $\{V_k\}$ of finite-dimensional subspaces of X with the following properties*:

(GH1) *there exist a closed subspace Z of X, $\rho > 0$ and $\alpha > 0$ such that $X = V_0 \oplus Z$ and for every $x \in Z$ with $\|x\|_X = \rho$, $f(x) \geq \alpha$;*
(GH2) *there exists a sequence $\{R_k\} \subset]\rho,+\infty[$ such that for any $x \in V_k$ with $\|x\|_X \geq R_k$, $f(x) \leq 0$.*

Moreover, *assume that*

(PSH) *for every $c \geq \alpha$, the function f satisfies the Palais–Smale condition at level c and $(epi)_c$ condition*;
(WSH) $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\lambda) \neq 0$, *whenever $\lambda \geq \alpha$.*

Then there exists a sequence $\{x_k\}$ of critical points of f such that $f(x_k) \to +\infty$.

2.2. *The Palais–Smale condition*

In this subsection we prove the properties of the functional J that will be useful in the last part of section. First we establish the relation between the weak slope of the functional J and its directional derivatives (along *admissible* directions). In the following, we shall denote by g and G the extensions by continuity of the functions $s \log s^2$ and $s^2 \log s^2$ respectively and G_1 and G_2 the continuous functions

$$G_1(s) := (s^2 \log s^2)^- \quad \text{and} \quad G_2(s) := (s^2 \log s^2)^+.$$

Observe that, if $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$, then for every $v \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, $g(u)v \in L^1(\mathbb{R}^n)$, since

$$\int |g(u)v| \leq C \left(1 + \int_{\operatorname{supp} v \cap \{|u|>1\}} |u|^{1+\delta}\right) < +\infty, \quad \text{for some } \delta \in (0, 2^* - 1]$$

and so, in particular, $g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $u \in H^1(\mathbb{R}^n)$ and $v \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, we can consider

$$\langle J'(u), v\rangle := \int \nabla u \cdot \nabla v + \omega \int uv - \int uv \log u^2. \tag{2.1}$$

We have the following proposition.

Proposition 2.1. *Let $u \in H^1(\mathbb{R}^n)$ with $J(u) \in \mathbb{R}$ and $|dJ|(u) < +\infty$. Then the following facts hold*:

(1) *$g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap H^{-1}(\mathbb{R}^n)$ and for any $v \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, we have*

$$|\langle J'(u), v\rangle| \leq |dJ|(u)\|v\|; \tag{2.2}$$

(2) *if $v \in H^1(\mathbb{R}^n)$ is such that $(g(u)v)^+ \in L^1(\mathbb{R}^n)$ or $(g(u)v)^- \in L^1(\mathbb{R}^n)$, then $g(u)v \in L^1(\mathbb{R}^n)$ and identity* (2.1) *holds, identifying $J'(u)$ as an element in $H^{-1}(\mathbb{R}^n)$.*

Proof. Recalling the notion of subdifferential in [10] and, by [10, Theorem 4.13], we have $\partial J(u) \neq \emptyset$ and $|dJ|(u) \geq \min\{\|\alpha\|_* \,|\, \alpha \in \partial J(u)\}$ where $\|\cdot\|_*$ is the norm in $H^{-1}(\mathbb{R}^n)$. Now let

$$T(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{\omega+1}{2}\|u\|_2^2 \quad \text{and} \quad Q(u) = -\frac{1}{2}\int u^2 \log u^2.$$

By [10, Corollary 5.3] we have $\partial J(u) \subset \partial T(u) + \partial Q(u)$ and, since $\partial J(u) \neq \emptyset$, then $\partial Q(u)$ is nonempty too. Hence, in light of [20, (b) of Theorem 3.1], we get that $-u - g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap H^{-1}(\mathbb{R}^n)$, and then $g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap H^{-1}(\mathbb{R}^n)$, and $\partial Q(u) = \{-u \log u^2 - u\}$. Thus, taking into account that $\partial J(u) = \{J'(u)\}$, with $J'(u)$ as in (2.1), we get (2.2). Assertion (2) follows by the result of [8]. $\square$

Proposition 2.2. *The functional J is lower semicontinuous.*

Proof. Assume that $\{u_k\} \subset H^1(\mathbb{R}^n)$ is a sequence converging to some u. Up to a subsequence, $G_1(u_k)$ converges pointwise to $G_1(u)$. Hence, by virtue of Fatou's

Lemma, we get

$$\int G_1(u) \le \liminf_k \int G_1(u_k).$$

On the other hand, taking into account that, for any $\delta \in (0, 2^+ - 2]$, there exists $C_\delta > 0$ such that $G_2(s) \le C_\delta |s|^{2+\delta}$ for all $s \in \mathbb{R}$ and that $u_k \to u$ strongly in $L^{2+\delta}(\mathbb{R}^N)$, we conclude that

$$\int G_2(u) = \lim_k \int G_2(u_k).$$

Hence, as $G(s) = G_2(s) - G_1(s)$, the desired conclusion follows. ◻

Proposition 2.3. $J|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)}$ *satisfies the Palais–Smale condition at level c for every $c \in \mathbb{R}$.*

Proof. Let us first prove that the Palais–Smale sequences of J are bounded in $H^1(\mathbb{R}^n)$. Let $\{u_k\} \subset H^1(\mathbb{R}^n)$ be a Palais–Smale sequence of J, namely $J(u_k) \to c$ and $|dJ|(u_k) \to 0$. By Proposition 2.1, we have that $\langle J'(u_k), v\rangle = o(1)\|v\|$ for any $v \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, namely $J'(u_k) \to 0$ in $H^{-1}(\mathbb{R}^n)$ as $k \to \infty$. Now, notice that, if $u \in H^1(\mathbb{R}^n)$, then $(u^2 \log u^2)^+ \in L^1(\mathbb{R}^n)$. Thus, by virtue of (2) of Proposition 2.1, we are allowed to choose u_k as admissible test functions in Eq. (2.1) and

$$\|u_k\|_2^2 = 2J(u_k) - \langle J'(u_k), u_k\rangle \le 2c + o(1)\|u_k\|. \tag{2.3}$$

By (1.4) for $a > 0$ small, (2.3) and the boundedness of $\{J(u_k)\}$, for $\delta > 0$ small, we have that

$$\|u_k\|^2 \le C + C(1 + o(1)\|u_k\|)^{1+\delta} + o(1)\|u_k\|$$

and so $\{u_k\}$ is bounded in $H^1(\mathbb{R}^n)$. Let $\{u_k\}$ now be a Palais–Smale sequence for J in $H^1_{\mathrm{rad}}(\mathbb{R}^n)$. The above argument shows that $\{u_k\}$ is bounded in $H^1_{\mathrm{rad}}(\mathbb{R}^n)$. Then, up to a subsequence, there is $u \in H^1_{\mathrm{rad}}(\mathbb{R}^n)$ with

$$u_k \rightharpoonup u \text{ in } H^1(\mathbb{R}^n), \quad u_k \to u \text{ in } L^p(\mathbb{R}^n),\ 2 < p < 2^*, \quad u_k \to u \text{ a.e. in } \mathbb{R}^n.$$

We want to prove that

$$\int \nabla u \cdot \nabla v + \omega \int uv = \int uv \log u^2 \quad \text{for all } v \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n). \tag{2.4}$$

So, fixed $v \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, let us consider $\vartheta_R(u_k)v$, where, given $R > 0$, $\vartheta_R : \mathbb{R} \to [0,1]$ is smooth, $\vartheta_R(s) = 1$ for $|s| \le R$, $\vartheta_R(s) = 0$ for $|s| \ge 2R$ and $|\vartheta_R'(s)| \le C/R$ in $\mathbb{R}$. Obviously we have that $\vartheta_R(u_k)v \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$. Thus, by (2.1) and taking into account the boundedness of $\{u_k\}$, we have

$$\left|\int \vartheta_R(u_k)\nabla u_k \nabla v + \omega \int \vartheta_R(u_k)u_k v - \int \vartheta_R(u_k)u_k v \log u_k^2 - \langle J'(u_k), \vartheta_R(u_k)v\rangle\right|$$
$$\leq \frac{C}{R}.$$

Passing to the limit as $k \to +\infty$, since $\vartheta_R(u_k)\nabla v \to \vartheta_R(u)\nabla v$ in $L^2(\mathbb{R}^n, \mathbb{R}^n)$, $\vartheta_R(u_k)u_k \log u_k^2 \to \vartheta_R(u)u \log u^2$ a.e. in $\mathbb{R}^n$ and taking into account that $\{\vartheta_R(u_k)u_k \log u_k^2\}$ is bounded in $L^2_{\text{loc}}(\mathbb{R}^n)$, we have

$$\left|\int \vartheta_R(u)\nabla u \nabla v + \omega \int \vartheta_R(u)uv - \int \vartheta_R(u)uv \log u^2\right| \leq \frac{C}{R}.$$

Thus we pass to the limit as $R \to +\infty$ and we get (2.4). Moreover, as in the proof of Proposition 2.2, we have that

$$\limsup_k \int u_k^2 \log u_k^2 \leq \int u^2 \log u^2.$$

Hence, since $\langle J'(u_k), u_k\rangle \to 0$ and choosing, by (2) of Proposition 2.1, $v = u$ in (2.4), we get

$$\limsup_k (\|\nabla u_k\|_2^2 + \omega\|u_k\|_2^2) = \limsup_k \int u_k^2 \log u_k^2 \leq \int u^2 \log u^2 = \|\nabla u\|_2^2 + \omega\|u\|_2^2,$$

which implies the convergence of $u_k \to u$ in $H^1_{\text{rad}}(\mathbb{R}^n)$. □

2.3. *Proof of Theorem* 1.1

To prove the existence of sequence $\{u_k\} \subset H^1_{\text{rad}}(\mathbb{R}^n)$ of (weak) solutions to (1.2) with $J(u_k) \to +\infty$, we will apply Theorem 2.1 with $X = H^1_{\text{rad}}(\mathbb{R}^n)$. In light of Proposition 2.3, J satisfies the Palais–Smale condition. Moreover J satisfies $(\text{epi})_c$ and (WSH) conditions (see [20, Theorem 3.4]). Hence, it remains to check that J satisfies also the geometrical assumptions. Obviously, $J(0) = 0$. Moreover by the logarithmic Sobolev inequality (1.4), we have that

$$J(u) \geq \frac{1}{2}\left(1 - \frac{a^2}{\pi}\right)\|\nabla u\|_2^2 + \frac{1}{2}(\omega + 1 + n(1 + \log a) - \log \|u\|_2^2)\|u\|_2^2$$
$$\geq c\|u\|^2,$$

for a suitable a and if $\|u\|$ are sufficiently small. Then, if we take $Z = X = H^1_{\text{rad}}(\mathbb{R}^n)$ and $V_0 = \{0\}$ we have (GH1). Finally, let us consider a strictly increasing sequence $\{V_k\}$ of finite-dimensional subspaces of $H^1_{\text{rad}}(\mathbb{R}^n)$ constituted by bounded functions (for instance, one can consider the eigenvectors of $-\Delta + |x|^2$, see [4, Chap. 3]). Since any norm is equivalent on any V_k, if $\{u_m\} \subset V_k$ is such that $\|u_m\| \to +\infty$, then also $\mu_m = \|u_m\|_2 \to +\infty$. Write now $u_m = \mu_m w_m$, where $w_m = \|u_m\|_2^{-1} u_m$. Thus $\|w_m\|_2 = 1$, $\|\nabla w_m\|_2 \le C$ and $\|w_m\|_\infty \le C$, yielding in turn

$$J(u_m) = \frac{\mu_m^2}{2}\left(\|\nabla w_m\|_2^2 + \omega + 1 - \log \mu_m^2 - \int w_m^2 \log w_m^2\right)$$

$$\le \frac{\mu_m^2}{2}(C - \log \mu_m^2) \to -\infty.$$

Thus, there exist $\{R_k\} \subset]\rho, +\infty[$ such that for $u \in V_k$ with $\|u\| \ge R_k$, $J(u) \le 0$. Hence, also (GH2) is satisfied and the assertion follows as, by Proposition 2.1, the critical points of J in the sense of weak slope correspond to solutions to (1.2).

3. Qualitative Properties of the Nonnegative Solutions

3.1. *Positivity and regularity of solutions*

If we take $\beta(s) = \omega s - s \log s^2$, since β is continuous, nondecreasing for s small, $\beta(0) = 0$ and $\beta(\sqrt{e^\omega}) = 0$, by [29, Theorem 1] we have that each solution $u \ge 0$ of (1.2) such that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Delta u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ in the sense of distribution is either trivial or strictly positive. Moreover, observe that any given nonnegative solution to Eq. (1.2) satisfies the inequality

$$-\Delta u + \omega u \le (u \log u^2)^+.$$

In particular, for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$-\Delta u \le \ell(u), \quad \ell(s) = -\omega s + C_\delta s^{1+\delta}.$$

Since we have $|\ell(s)| \le C(1 + |s|^{1+\delta})$ for all $s \in \mathbb{R}$ and some $C > 0$, by repeating the argument of the proof of [27, Lemma B.3], it is possible to prove that $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for every $q < \infty$. Then, by standard regularity arguments, the C^2 smoothness of u readily follows.

3.2. *Uniqueness of positive solutions*

In this subsection we prove Theorem 1.2.

Proof of Theorem 1.2. First of all, by means of the moving plane method [22], we prove that each positive and vanishing classical solution to (1.2) has to be radially symmetric about some point. Let $u \in C^2(\mathbb{R}^n)$ be a solution to Eq. (1.2) with $u > 0$ and $u(x) \to 0$ as $|x| \to \infty$, $\lambda \in \mathbb{R}$, $\Sigma_\lambda := \{x \in \mathbb{R}^n \,|\, x_1 < \lambda\}$, $x_\lambda := (2\lambda - x_1, x_2, \ldots, x_n)$, $u_\lambda(x) := u(x_\lambda)$ and $w_\lambda := u_\lambda - u$. Then, it is easy to verify that

$$-\Delta w_\lambda + c_\lambda(x) w_\lambda = 0,$$

where we have set

$$c_\lambda(x) := -\int_0^1 (2 - \omega + \log(\sigma u_\lambda(x) + (1-\sigma)u(x))^2) d\sigma.$$

Notice that $x \mapsto c_\lambda(x)$ is possibly unbounded from above, but it is bounded from below. Since u goes to zero at infinity we notice that there exists $R > 0$ such that $u(x) < \sqrt{e^{\omega-2}}$ in $B_R^c(0)$. We claim that for every $\lambda \in \mathbb{R}$ we have that $w_\lambda \geq 0$ in $B_R^c(0)$. Indeed, assume by contradiction that there exist $\lambda \in \mathbb{R}$ and points in $B_R^c(0)$ at which $w_\lambda < 0$. Let $\bar{x} \in B_R^c(0)$ a negative minimum point of w_λ. Then, we have

$$0 < -(2 - \omega + \log u^2(\bar{x})) \leq c_\lambda(\bar{x}) \leq -(2 - \omega + \log u_\lambda^2(\bar{x})),$$

and so $-\Delta w_\lambda(\bar{x}) \geq 0$ that is a contradiction. Thus, if $\lambda < -R$, we have that $\Sigma_\lambda \subset B_R^c(0)$ and then $w_\lambda(x) \geq 0$ for every $x \in \Sigma_\lambda$. Now we want to move the hyperplane $\partial\Sigma_\lambda$ to the right (i.e. increasing the value of λ) preserving the inequality $w_\lambda \geq 0$ to the limit position. Let $\lambda_0 := \sup\{\lambda < 0 \,|\, w_\lambda \geq 0 \text{ in } \Sigma_\lambda\}$. First of all we observe that, by continuity, $w_{\lambda_0} \geq 0$ in Σ_{λ_0}. Then by the maximum principle (see [16, Theorem 7.3.3], where one can assume that the function $c(x)$ is merely bounded from below), we have that either $w_{\lambda_0} \equiv 0$ in Σ_{λ_0} or $w_{\lambda_0} > 0$ in the interior of Σ_{λ_0}. We claim that if $\lambda_0 < 0$ then $w_{\lambda_0} \equiv 0$. We show that $w_{\lambda_0} > 0$ in the interior of Σ_{λ_0} implies that

$$\exists\, \delta_0 > 0 \text{ such that } \forall\, \delta \in (0, \delta_0) : w_{\lambda_0+\delta} \geq 0 \quad \text{in } \Sigma_{\lambda_0+\delta}, \tag{3.1}$$

violating the definition of λ_0. Indeed, assume by contradiction that (3.1) is not true. Then we can consider a sequence $\delta_k \to 0$ such that for every k there exists a negative minimum point $\bar{x}_k$ of $w_{\lambda_0+\delta_k}$ in $\Sigma_{\lambda_0+\delta_k}$. Then $\bar{x}_k \in \bar{B}_R(0) \cap \Sigma_{\lambda_0+\delta_k}$ and $\nabla w_{\lambda_0+\delta_k}(\bar{x}_k) = 0$. The boundedness of the sequence $\{\bar{x}_k\}$ implies that, up to a subsequence, $\bar{x}_k \to \bar{x}$ and

$$w_{\lambda_0}(\bar{x}) = \lim_k w_{\lambda_0+\delta_k}(\bar{x}_k) \leq 0, \quad \bar{x} \in \overline{\Sigma}_{\lambda_0} \tag{3.2}$$

and

$$\nabla w_{\lambda_0}(\bar{x}) = \lim_k \nabla w_{\lambda_0+\delta_k}(\bar{x}_k) = 0. \tag{3.3}$$

Then, by (3.2) we have that $\bar{x} \in \partial\Sigma_{\lambda_0}$ and $w_{\lambda_0}(\bar{x}) = 0$. Therefore, by the Hopf lemma (see again [16, Theorem 7.3.3]) we have that

$$\frac{\partial w_{\lambda_0}}{\partial \mathbf{n}}(\bar{x}) < 0,$$

which contradicts (3.3). If $\lambda_0 = 0$, then we can carry out the above procedure in the opposite direction, namely, we move the hyperplane $x_1 = \lambda$ with $\lambda > 0$ in the negative direction. If the infimum of values of such λ's is strictly positive, we get the symmetry and monotonicity as in the case $\lambda_0 < 0$. If such infimum is 0 we get obviously the symmetry and monotonicity with respect to the hyperplane $x_1 = 0$. By the arbitrariness of the choice of the x_1 direction, we can conclude that the solution u must be radially symmetric about some point. Finally, Serrin and Tang [26] prove that there exists at most one nonnegative nontrivial C^1 distribution solution of (1.2) in the class of radial functions which tends to zero at infinity. Then, up to translations, such a solution is $\mathfrak{g}$. $\square$

3.3. *Gausson's nondegeneracy*

We have shown that $\mathfrak{g}$ is the unique radial positive solution of the equation

$$-\Delta u - nu = u \log u^2. \tag{3.4}$$

In this subsection we prove Theorem 1.3. The linearized operator L for (3.4) at $\mathfrak{g}$ is found to be

$$Lu = -\Delta u + (|x|^2 - n - 2)u,$$

acting on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$. To prove that $\mathrm{Ker}(L) = \mathrm{span}\{\partial_{x_i}\mathfrak{g}\}$, we introduce the following notations. We set

$$r = |x|, \quad \vartheta = \frac{x}{|x|} \in \mathbb{S}^{n-1},$$

and we denote by Δ_r the Laplace operator in radial coordinates and with $\Delta_{\mathbb{S}^{n-1}}$ the Laplace–Beltrami operator. Let us consider the spherical harmonics $Y_{k,h}(\vartheta)$, satisfying

$$-\Delta_{\mathbb{S}^{n-1}} Y_{k,h} = \lambda_k Y_{k,h}. \tag{3.5}$$

We recall that (3.5) admits a sequence of eigenvalues $\lambda_k = k(k+n-2)$, $k \in \mathbb{N}$ whose multiplicity is given by $\mu_k - \mu_{k-2}$ where

$$\mu_k := \begin{cases} \dfrac{(n+k-1)!}{(n-1)!k!} & \text{for } k \geq 0, \\ 0 & \text{for } k < 0 \end{cases}$$

(see, e.g., [4]). In particular $\lambda_0 = 0$ and $\lambda_1 = n-1$ have, respectively, multiplicity 1 and n. For every $u \in H^1(\mathbb{R}^n)$, we have that

$$\begin{aligned} u(x) &= \sum_{k\in\mathbb{N}} \sum_{h=1}^{\mu_k - \mu_{k-2}} \psi_{k,h}(r) Y_{k,h}(\vartheta), \quad \text{where } \psi_{k,h}(r) \\ &:= \int_{\mathbb{S}^{n-1}} u(r\vartheta) Y_{k,h}(\vartheta) d\vartheta \end{aligned} \tag{3.6}$$

and, for every $k \in \mathbb{N}$ and $h \in \{1, \ldots, \mu_k - \mu_{k-2}\}$,

$$\Delta(\psi_{k,h} Y_{k,h}) = Y_{k,h}(\vartheta)\Delta_r \psi_{k,h}(r) + \frac{1}{r^2}\psi_{k,h}(r)\Delta_{\mathbb{S}^{n-1}} Y_{k,h}(\vartheta). \tag{3.7}$$

Therefore, by combining formulas (3.5), (3.6) and (3.7), we have that $u \in \mathrm{Ker}(L)$ if and only if, for every $k \in \mathbb{N}$ and all $h \in \{1, \ldots, \mu_k - \mu_{k-2}\}$,

$$A_k(\psi_{k,h}) = 0, \tag{3.8}$$

where

$$A_k(\psi) = -\psi'' - \frac{n-1}{r}\psi' + \left(r^2 + \frac{\lambda_k}{r^2} - n - 2\right)\psi.$$

For the spectral properties of this kind of operators we refer the reader to [4]. Now, as usual in this kind of proofs (see, e.g., [1, 15]), we proceed by showing the following three claims.

Claim 1. *For* $k = 0$, *Eq.* (3.8) *has only the trivial solution in* $H^1(\mathbb{R}_+)$.

Claim 2. *For* $k = 1$, *the solutions of* (3.8) *in* $H^1(\mathbb{R}_+)$ *are of the form* $c\mathfrak{g}'$, *for* $c \in \mathbb{R}$.

Claim 3. *For* $k \geq 2$, *Eq.* (3.8) *has only the trivial solution in* $H^1(\mathbb{R}_+)$.

Proof of Claim 1. Let $k = 0$ and $\psi_0 \in H^1(\mathbb{R}_+)$ be a nonzero solution of (3.8). The relation $A_0(\mathfrak{g}) = -2\mathfrak{g}$ and the positivity of $\mathfrak{g}$ imply that the Gausson is the first eigenfunction and then ψ_0 has to change sign. Thus, by Sturm–Liouville theory ψ_0 is unbounded and we get the contradiction.

Proof of Claim 2. First notice that an easy calculation shows that $A_1(\mathfrak{g}') = 0$ and $\mathfrak{g}' \in H^1(\mathbb{R}_+)$. If we look for a second solution of the equation $A_1(\psi) = 0$ in the form $\psi(r) = c(r)\mathfrak{g}'(r)$ we have that the function c has to satisfy

$$rc'' + (n + 1 - 2r^2)c' = 0$$

and then, in turn,

$$c(r) = c_1\Phi(r) + c_2, \quad \text{where } \Phi \text{ is primitive of } r \mapsto r^{-n-1}e^{r^2}.$$

Then $c(r)\mathfrak{g}'(r) \to +\infty$ as $r \to +\infty$ if $c_1 \neq 0$ and this implies that the unique possible choice to get solutions of the form $c(r)\mathfrak{g}'(r)$ is to take $c(r)$ constant, proving the claim.

Proof of Claim 3. Since $\lambda_k = \lambda_1 + \delta_k$ with $\delta_k > 0$ and, by Claim 2, the operator A_1 is a nonnegative operator, then, if $k \geq 2$, $A_k = A_1 + \frac{\delta_k}{r^2}$ is a positive operator and so $A_k(\psi) = 0$ implies that $\psi = 0$. Thus for every $k \geq 2$ and $h \in \{1, \ldots, \mu_k - \mu_{k-2}\}$, we have that $\psi_{k,h} = 0$.

Finally we can conclude observing that, summarizing the previous results, we have that

$$\operatorname{Ker}(L) = \operatorname{span}\{\mathfrak{g}'Y_{1,h}\} = \operatorname{span}\{\partial_{x_h}\mathfrak{g}\}.$$

3.4. *Minimization on L^2-spheres*

Let J be as in (1.3), W as in (1.5) and set

$$\mathcal{M}_\nu := \{u \in W \mid \|u\|_2^2 = \nu\},$$
$$\mathcal{N}_\omega := \left\{u \in W\backslash\{0\} \mid \|\nabla u\|_2^2 + \omega\|u\|_2^2 = \int u^2 \log u^2\right\},$$
$$E(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int u^2 \log u^2.$$

We recall that the functionals J and E are of class C^1 in W (see [11]). We say that a *ground state solution* u of (1.2) is a solution of the following minimization problem:

$$J(u) = m_\omega = \inf_{\mathcal{N}_\omega} J. \tag{3.9}$$

We also set

$$c_\nu = \inf_{\mathcal{M}_\nu} E. \tag{3.10}$$

Consider now the sequence $\{u_k\}$ of solutions of (1.2) found in Theorem 1.1. Proceeding as in [24] we have that the first element u_1 of such sequence solves problem (3.9). Furthermore, by [9], it follows that u_1 has a fixed sign. Then u_1 is the Gausson for (1.2) and it belongs to W. We have the following property that can be useful in the study of the orbital stability of ground states for Eq. (1.1).

Theorem 3.1. *For every $\omega \in \mathbb{R}$, we have $\inf_{\mathcal{N}_\omega} J = \inf_{\mathcal{M}_{2m_\omega}} J$.*

This result follows a particular case of the following lemma.

Lemma 3.1. *The critical levels of J on $\mathcal{N}_\omega$ are one-to-one with the critical levels of E on $\mathcal{M}_\nu$.*

Proof. Let m be a critical level of J on $\mathcal{N}_\omega$. We prove that m uniquely detects a critical level c of E on $\mathcal{M}_\nu$ and we have

$$c = \frac{\nu}{2}\left(\log \frac{2m}{\nu} - \omega\right). \tag{3.11}$$

Let u be a constrained critical point of J on the manifold $\mathcal{N}_\omega$ such that $J(u) = m$; then

$$\|\nabla u\|_2^2 + (\omega + 1)\|u\|_2^2 = \int u^2 \log u^2 + 2m.$$

Moreover

$$\|\nabla u\|_2^2 + \omega\|u\|_2^2 = \int u^2 \log u^2$$

and, by the Pohozaev identity we have

$$\frac{n-2}{n}\|\nabla u\|_2^2 + (\omega + 1)\|u\|_2^2 = \int u^2 \log u^2.$$

The three identities above give the following action ripartition

$$\|u\|_2^2 = 2m, \quad \|\nabla u\|_2^2 = nm, \quad \int u^2 \log u^2 = (2\omega + n)m.$$

Let us consider $u_\mu(x) = \mu u(x)$ with $\mu \in \mathbb{R}^*$. We notice that u_μ solves

$$-\Delta u + (\omega + \log \mu^2)u = u \log u^2.$$

Moreover $u_\mu \in \mathcal{M}_\nu$ if $2\mu^2 m = \nu$ and then we obtain (3.11) concluding the proof. $\square$

Proof of Theorem 3.1. For the minimization problems (3.9) and (3.10), choosing $\nu = 2m_\omega$, it follows that $c_\nu = -\omega\nu/2$, which yields immediately the last assertion. $\square$

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