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Locating the Peaks of Semilinear Elliptic Systems

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Abstract

We consider a system of weakly coupled singularly perturbed semilinear elliptic equations. First, we obtain a Lipschitz regularity result for the associated ground energy function Σ as well as representation formulas for the left and the right derivatives. Then, we show that the concentration points of the solutions locate close to the critical points of Σ in the sense of subdifferential calculus.

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1 Introduction and main results

In the asymptotic analysis of the singularly perturbed elliptic equation

$$-\varepsilon^2\Delta u + u = f(x,u) \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n, \tag{P_ε}$$

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there are well known situations where the associated ground energy function Σ (cf. [30]) is C^1-smooth and around its nondegenerate critical points the solutions u_ε of P_ε exhibit a spike-like profile as ε goes to zero. This is the case, for instance, for the power nonlinearity

$$f(x,u) = K(x)u^q, \qquad 1 < q < \tfrac{n+2}{n-2}, \quad n \geq 3,$$

where $K(x)$ is a suitable C^1 function (see e.g. [3, 4] and references therein). It turns out that the C^1 (and higher) smoothness of Σ is related to the crucial fact that, for every fixed $z \in \mathbb{R}^n$, the limiting autonomous equation

$$-\Delta u + u = f(z,u) \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n, \tag{P_0}$$

admits a unique solution, up to translations [30]. However, unfortunately, the uniqueness feature for P_0 is a delicate matter and it is currently available only under rather restrictive assumptions on f (cf. e.g. [16]). What is known, in general, is only that Σ is a locally Lipschitz continuous function which admits representation formulas for the left and right derivatives (cf. [30, Lemma 2.3]). Motivated by these facts, recently, some conditions for locating the concentration points for P_ε in presence of a more general nonlinearity f, not necessarily of power type, have been investigated (see [23] and also [24]). The underlying philosophy is that when the limit problem P_0 lacks of uniqueness up to translations, then the ground energy function Σ could lose its additional regularity properties.

Nevertheless, in this (possibly nonsmooth) framework, it turns out that a necessary condition for the solutions u_ε to concentrate (in a suitable sense) around a given point z is that it is critical for Σ in the sense of the Clarke subdifferential ∂_C, that is $0 \in \partial_C\Sigma(z)$, or in a even weaker sense. The main theme of this note is the search of suitable conditions for locating the spikes, as $\varepsilon \to 0$, of the solutions to the semilinear model system

$$\begin{cases} -\varepsilon^2\Delta u + u = K(x)v^q, & \text{in } \mathbb{R}^n, \\ -\varepsilon^2\Delta v + v = Q(x)u^p, & \text{in } \mathbb{R}^n, \\ u,\, v > 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{S_ε}$$

where $p > 1$ and $q > 1$ are lying below the so called "critical hyperbola"

$$\mathscr{C}_n = \big\{(p,q) \in (1,\infty) \times (1,\infty) : \tfrac{1}{p+1} + \tfrac{1}{q+1} = 1 - \tfrac{2}{n}\big\}, \qquad n \geq 3,$$

which naturally arises in the study of this problem and constitutes the borderline between existence and nonexistence results (cf. e.g. [10, 15]).

Now, according to the above discussion, the interest in looking for conditions for the spike location of the solutions to (S_ε) is mainly motivated by the following simple observation: contrary to the scalar case, there is no uniqueness result available in the literature for the (radial) solutions to the (limiting) system associated with (S_ε)

$$\begin{cases} -\Delta u + u = K(z)v^q, & \text{in } \mathbb{R}^n, \\ -\Delta v + v = Q(z)u^p, & \text{in } \mathbb{R}^n, \\ u,\, v > 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{S_z}$$

where $z \in \mathbb{R}^n$ is frozen and acts as a parameter. As a consequence, in the vectorial case, we do not know whether the (suitably defined) ground energy map Σ associated with (S_ε) (cf. Definition 1.2) is C^1-smooth and admits an explicit representation formula. Hence, the necessary conditions in terms of Clarke subdifferential (or weaker) appear here even more natural than in the case of a single equation. See Section 1.2 for the statements of the main results, Theorems 1.1 and 1.2. As far as we are aware, other criteria for the concentration have been established so far, but all of them consider the scalar case. We refer the reader e.g. to [3,20,29,30] for the case of power-like nonlinearities and to [23,24] for more general classes of nonlinearities.

Semilinear systems like (S_ε) naturally arise in the study of various kinds of nonlinear phenomena such as population evolution, pattern formation, chemical reaction, etc., being u and v the concentrations of different species in the process (see also [32] and references therein). Visibly, the interest in the study of the various qualitative properties of (S_ε) has steadily increased in recent times. In a smooth bounded domain Ω, (S_1) was extensively studied by Clement, Costa, De Figueiredo, Felmer, Hulshof, Magalhães, van der Vorst in [10–12, 14, 15]. The asymptotic analysis with respect to ε has been very recently performed both with Dirichlet and Neumann boundary conditions by Pistoia-Ramos [18, 19] and Ramos-Yang [22]. In the whole space $\mathbb{R}^n$, the existence of least energy solutions to (S_ε) has been investigated by Alves-Carrião-Miyagaki, De Figueiredo, Yang and Sirakov in [1, 2, 13, 26, 27, 31], whereas the asymptotic behavior with respect to ε has been pursued by Alves-Soares-Yang in [2]. Finally, for the exponential decay, the radial symmetry and the regularity properties of the solutions to (S_z), we refer the reader to the quite recent achievements of Busca-Sirakov and Sirakov [6, 27].

The outline of the paper is as follows: in Sections 1.1-1.2 we provide preliminary stuff such as the (dual) variational framework and the (dual) ground energy function Σ and we state the main results of the paper. Throughout Section 2 we deal with the $\mathrm{Lip}_{\mathrm{loc}}$ regularity and the representation formulas of the directional derivatives for Σ. Finally, in Section 3 we complete the proofs of the main results.

1.1 The dual variational functional

It is known, if e.g. p and q are both less than $\frac{n+2}{n-2}$, then system (S_ε) admits a natural variational structure (of Hamiltonian type) which is based on the strongly indefinite functional $f_\varepsilon : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R}$,

$$f_\varepsilon(u, v) = \int_{\mathbb{R}^n} \varepsilon^2 \nabla u \cdot \nabla v + uv - \tfrac{1}{q+1} \int_{\mathbb{R}^n} K(x)|v|^{q+1} - \tfrac{1}{p+1} \int_{\mathbb{R}^n} Q(x)|u|^{p+1}.$$

However, as already done in [1, 2], for our purposes, as well as for dealing with possibly supercritical values of p or q, we consider a corresponding dual variational structure, mainly relying on the Legendre-Fenchel transformation (see e.g. [8, 9, 17] and references therein). In the following, we just briefly recall some of the core ingredients, referring to [1, Section 2] for expanded details on this framework. For $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}$, consider the linear operators

$$T_1 : L^{\frac{q+1}{q}}(\mathbb{R}^n) \to W^{2,\frac{q+1}{q}}(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n),$$
$$T_2 : L^{\frac{p+1}{p}}(\mathbb{R}^n) \to W^{2,\frac{p+1}{p}}(\mathbb{R}^n) \hookrightarrow L^{q+1}(\mathbb{R}^n),$$

defined as

$$T_1 = T_2 = (-\Delta + \mathrm{Id})^{-1}.$$

Notice that T_1 and T_2 are continuous. Then, we consider the linear operator (take into account the proper Sobolev embeddings)

$$T : L^{\frac{p+1}{p}}(\mathbb{R}^n) \times L^{\frac{q+1}{q}}(\mathbb{R}^n) \to L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n), \qquad T = \begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix},$$

explicitly defined by

$$\langle T\eta, \xi\rangle = \xi_1 T_1 \eta_2 + \xi_2 T_2 \eta_1, \qquad \forall \eta = (\eta_1, \eta_2),\ \forall \xi = (\xi_1, \xi_2).$$

Finally we introduce the Banach space $(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$,

$$\mathscr{H} = L^{\frac{p+1}{p}}(\mathbb{R}^n) \times L^{\frac{q+1}{q}}(\mathbb{R}^n), \qquad \|\eta\|^2_{\mathscr{H}} = \|\eta_1\|^2_{L^{\frac{p+1}{p}}(\mathbb{R}^n)} + \|\eta_2\|^2_{L^{\frac{q+1}{q}}(\mathbb{R}^n)}$$

and the (dual) C^1 functional $J_\varepsilon : \mathscr{H} \to \mathbb{R}$ defined as

$$J_\varepsilon(\eta) = \tfrac{p}{p+1} \int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\varepsilon x)} + \tfrac{q}{q+1} \int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\varepsilon x)} - \tfrac{1}{2} \int_{\mathbb{R}^n} \langle T\eta, \eta\rangle.$$

If $\eta^\varepsilon = (\eta_1^\varepsilon, \eta_2^\varepsilon)$ is a critical point of J_ε, then $(u_\varepsilon(x), v_\varepsilon(x)) = (\bar{u}_\varepsilon(\frac{x}{\varepsilon}), \bar{v}_\varepsilon(\frac{x}{\varepsilon}))$, with

$$(\bar{u}_\varepsilon, \bar{v}_\varepsilon) = (T_1 \eta_2^\varepsilon, T_2 \eta_1^\varepsilon) \in W^{2,\frac{q+1}{q}} \times W^{2,\frac{p+1}{p}}, \tag{1.1}$$

corresponds to a solution to (S_ε) with $u_\varepsilon(x), v_\varepsilon(x) \to 0$ for $|x| \to \infty$ (see [1, p.677]). In light of the above summability, we have $f_\varepsilon(u_\varepsilon, v_\varepsilon) \in \mathbb{R}$ for all $\varepsilon > 0$. Analogously, associated with (S_z), we introduce the limiting functional $I_z : \mathscr{H} \to \mathbb{R}$

$$I_z(\eta) = \tfrac{p}{p+1} \int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(z)} + \tfrac{q}{q+1} \int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(z)} - \tfrac{1}{2} \int_{\mathbb{R}^n} \langle T\eta, \eta\rangle.$$

From the viewpoint of our investigation, the main advantage of exploiting the dual variational functional I_z is that it admits a mountain-pass geometry and the mountain-pass value corresponds to the least possible energy of system (S_z). As we shall see in the next section, this allows to provide in the vectorial framework a suitable definition of ground energy function with nice features, similar to those available in the scalar case.

1.2 Preliminaries and the main statements

In order to state the main achievements of the paper, we need some preparatory material. For the sake of self-containedness we shall also recall a few pretty well known notions from nonsmooth calculus (see e.g. [7]).

Definition 1.1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function near a point $z \in \mathbb{R}^n$. The *Clarke subdifferential* of f at z is defined by

$$\partial_C f(z) := \big\{\eta \in \mathbb{R}^n : \ f^0(z, w) \geq \eta \cdot w, \ \text{ for every } w \in \mathbb{R}^n\big\},$$

where $f^0(z, w)$ is the generalized derivative of f at z along $w \in \mathbb{R}^n$, defined by

$$f^0(z; w) := \limsup_{\substack{\xi \to z \\ \lambda \to 0+}} \frac{f(\xi + \lambda w) - f(\xi)}{\lambda}.$$

Definition 1.2 The (dual) *ground energy function* $\Sigma : \mathbb{R}^n \to \mathbb{R}$ of (S_z) is given by

$$\Sigma(z) := \inf_{\eta \in \mathcal{N}_z} I_z(\eta),$$

where $\mathcal{N}_z$ is the *Nehari manifold* of I_z, that is

$$\mathcal{N}_z = \big\{\eta \in \mathscr{H} : \ \eta \neq (0, 0) \text{ and } I_z'(\eta)[\eta] = 0\big\}.$$

We shall denote by $\mathcal{K} \subset \mathbb{R}^n$ the set of *Clarke critical points* of Σ, namely

$$\mathcal{K} := \big\{z \in \mathbb{R}^n : 0 \in \partial_C \Sigma(z)\big\}.$$

Definition 1.3 We say that the pair $(u_\varepsilon, v_\varepsilon)$ is a *strong solution* to system (S_ε) if it is a distributional solution and $(u_\varepsilon, v_\varepsilon) \in W^{2,(q+1)/q}(\mathbb{R}^n) \times W^{2,(p+1)/p}(\mathbb{R}^n)$. We say that the pair $\eta^\varepsilon = (\eta_1^\varepsilon, \eta_2^\varepsilon)$ corresponding to $(u_\varepsilon, v_\varepsilon)$ through (1.1) is the related *dual solution.*

Definition 1.4 We set

$$\begin{aligned}\mathcal{E} := \big\{z \in \mathbb{R}^n :\ & \text{there exists a sequence of strong solutions } (u_{\varepsilon_h}, v_{\varepsilon_h}) \text{ of } (S_\varepsilon) \text{ with} \\ & |u_{\varepsilon_h}(z)|, |v_{\varepsilon_h}(z)| \geq \delta \text{ for some } \delta > 0,\ |u_{\varepsilon_h}(z + \varepsilon_h x)|,\ |v_{\varepsilon_h}(z + \varepsilon_h x)| \to 0 \\ & \text{as } |x| \to \infty \text{ uniformly w.r.t. } h, \text{ and } {\varepsilon_h}^{-n} f_{\varepsilon_h}(u_{\varepsilon_h}, v_{\varepsilon_h}) \to \Sigma(z) \text{ as } h \to \infty\big\}.\end{aligned}$$

We say that $\mathcal{E}$ is the *energy concentration set* for (S_ε).

Assume that $K, Q \in C^1(\mathbb{R}^n)$ and

$$\alpha \leq K(x) \leq \beta, \quad \alpha \leq Q(x) \leq \beta, \qquad \text{for all } x \in \mathbb{R}^n, \tag{1.2}$$

$$|\nabla K(x)|, |\nabla Q(x)| \leq C e^{M|x|}, \qquad \text{for all } x \in \mathbb{R}^n \text{ with } |x| \text{ large} \tag{1.3}$$

for some positive constants α, β, C and M.

The main result of the paper, linking the energy concentration set $\mathcal{E}$ with the set $\mathcal{K}$ of Clarke critical set of Σ, is provided by the following

Theorem 1.1 *Assume that $K, Q \in C^1(\mathbb{R}^n)$ and that (1.2)-(1.3) hold. Then $\mathcal{E} \subset \mathcal{K}$.*

Remark 1.1 By [2, Theorem 1], under suitable assumptions, if there exists an absolute minimum (or maximum) point z_* for Σ, then $z_* \in \mathcal{E} \neq \emptyset$.

Remark 1.2 As a straightforward combination of Theorem 1.1 with the well known convex hull characterization of $\partial_C \Sigma(z)$, if z is a concentration point for (S_ε), then

$$0 \in \mathrm{Co}\big\{ \lim_j \nabla\Sigma(\xi_j) : \xi_j \notin \Omega \text{ and } \xi_j \to z \big\},$$

where $\mathrm{Co}\{X\}$ denotes the convex hull of X and Ω is any null set containing the set of points at which Σ fails to be differentiable.

Corollary 1.1 *Under the (unproved) assumption that, for all $z \in \mathbb{R}^n$, system (S_z) admits a unique positive solution (up to translations), Σ is C^1-smooth and*

$$\mathcal{E} \subset \mathrm{Crit}\big(Q^{\frac{q+1}{pq-1}}\, K^{\frac{p+1}{pq-1}}\big),$$

where Crit*(f) denotes the set of (classical) critical points of f.*

In the following definition we consider solutions which concentrate close to a point z, with bounded energy but not necessary stabilizing towards $\Sigma(z)$.

Definition 1.5 Let $m \geq 1$. We set

$$\begin{aligned}\mathcal{E}_m := \big\{ & z \in \mathbb{R}^n : \text{there exists a sequence of strong solutions } (u_{\varepsilon_h}, v_{\varepsilon_h}) \text{ of } (S_\varepsilon) \text{ with} \\ & |u_{\varepsilon_h}(z)|, |v_{\varepsilon_h}(z)| \geq \delta \text{ for some } \delta > 0,\ |u_{\varepsilon_h}(z+\varepsilon_h x)|,\ |v_{\varepsilon_h}(z+\varepsilon_h x)| \to 0 \\ & \text{as } |x| \to \infty \text{ uniformly w.r.t. } h, \text{ and } {\varepsilon_h}^{-n} f_{\varepsilon_h}(u_{\varepsilon_h}, v_{\varepsilon_h}) \to m \text{ as } h \to \infty \big\}.\end{aligned}$$

We say that $\mathcal{E}_m$ is the *concentration set* for (S_ε) at the energy level m.

Definition 1.6 Let $m \geq 1$ and $z \in \mathbb{R}^n$. For every $w \in \mathbb{R}^n$ we define $\Gamma^{\mp}_{z,m}(w)$ by

$$\Gamma^{-}_{z,m}(w) := \sup_{\eta \in \mathbb{G}_m(z)} \Big[-\frac{1}{p+1}\frac{\partial Q}{\partial w}(z)\int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(z)} - \frac{1}{q+1}\frac{\partial K}{\partial w}(z)\int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(z)} \Big],$$

$$\Gamma^{+}_{z,m}(w) := -\inf_{\eta \in \mathbb{G}_m(z)} \Big[-\frac{1}{p+1}\frac{\partial Q}{\partial w}(z)\int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(z)} - \frac{1}{q+1}\frac{\partial K}{\partial w}(z)\int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(z)} \Big],$$

where $\mathbb{G}_m(z)$ denotes the set of all the nontrivial, radial, exponentially decaying solutions of (S_z) having energy equal to m.

It is readily seen that $\Gamma^{\mp}_{z,m}(w) \in \mathbb{R}$ for all z, w in $\mathbb{R}^n$ (see the proof of (2.11)). It is also straightforward to check that, for any $z \in \mathbb{R}^n$, the functions $\{w \mapsto \Gamma^{\mp}_{z,m}(w)\}$ are convex.

Definition 1.7 Let $m \geq 1$. We set

$$\mathcal{K}_m := \Big\{ z \in \mathbb{R}^n : \ 0 \in \partial \Gamma^-_{z,m}(0) \cap \partial \Gamma^+_{z,m}(0) \Big\},$$

where ∂ stands for the subdifferential of convex functions,

$$\partial \Gamma^{\mp}_{z,m}(0) = \Big\{ \xi \in \mathbb{R}^n : \ \Gamma^{\mp}_{z,m}(w) \geq \xi \cdot w, \ \text{ for every } w \in \mathbb{R}^n \Big\}.$$

It is known by standard convex analysis that $\partial \Gamma^{\mp}_{z,m}(0) \neq \emptyset$, for every $z \in \mathbb{R}^n$. Observe that $z \in \mathcal{K}_m$ if and only if 0 is a critical point for both $\Gamma^-_{z,m}$ and $\Gamma^+_{z,m}$. Of course, if $\mathbb{G}_m(z) = \{\eta_0\}$ was a singleton, then $z \in \mathcal{K}_m$ if and only if

$$\Gamma^-_{z,m}(w) = \Gamma^+_{z,m}(w) = \frac{\partial I_z}{\partial w}(\eta_0) = 0, \qquad \forall w \in \mathbb{R}^n.$$

Without forcing the energy levels of the solutions to approach the least energy of the limit system, we get the following correlation between the sets $\mathcal{E}_m$ and $\mathcal{K}_m$.

Theorem 1.2 *Assume that $K, Q \in C^1(\mathbb{R}^n)$ and (1.2)-(1.3) hold. Then $\mathcal{E}_m \subset \mathcal{K}_m$.*

2 Properties of the ground energy function

Before coming to the proof of the results, we need some preliminary stuff.

2.1 Some preparatory lemmas

The next proposition is well known (see e.g. [31]); on the other hand, for the sake of completeness and self-containedness, we give a brief proof.

Proposition 2.1 *Let $z \in \mathbb{R}^n$. Then $(u,v) \in W^{2,\frac{q+1}{q}}(\mathbb{R}^n) \times W^{2,\frac{p+1}{p}}(\mathbb{R}^n)$ is a solution to (S_z) if and only if $\eta = (\eta_1, \eta_2) = (T_2^{-1}v, T_1^{-1}u) \in \mathscr{H}$ is a critical point of I_z. Moreover, there holds $f_z(u,v) = I_z(\eta_1,\eta_2)$, where f_z is the functional defined as*

$$f_z(u,v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v + uv - \tfrac{1}{q+1} \int_{\mathbb{R}^n} K(z)|v|^{q+1} - \tfrac{1}{p+1} \int_{\mathbb{R}^n} Q(z)|u|^{p+1}.$$

Proof. Observe first that, if $(u,v) \in W^{2,\frac{q+1}{q}}(\mathbb{R}^n) \times W^{2,\frac{p+1}{p}}(\mathbb{R}^n)$ solves (S_z), taking into account the Sobolev embedding, the value $f_z(u,v)$ is indeed finite (cf. (1.1)). Let (u,v) be a solution to (S_z). Then, since

$$\eta_1 = T_2^{-1}v, \qquad \eta_2 = T_1^{-1}u,$$

we have

$$\begin{cases} \eta_2 = T_1^{-1}u = -\Delta u + u = K(z)v^q, \\ \eta_1 = T_2^{-1}v = -\Delta v + v = Q(z)u^p. \end{cases}$$

Therefore, we get

$$T_2\eta_1 = v = \frac{\eta_2^{\frac{1}{q}}}{K^{\frac{1}{q}}(z)} \qquad \text{and} \qquad T_1\eta_2 = u = \frac{\eta_1^{\frac{1}{p}}}{Q^{\frac{1}{p}}(z)}, \tag{2.1}$$

and so (η_1, η_2) is a critical point of I_z. Vice versa, if (η_1, η_2) is a critical point of I_z, it is readily seen that (2.1) hold, so that $(T_1\eta_2, T_2\eta_1) = (u, v)$ is a solution to (S_z) (cf. [1, p.677]). Furthermore, on the solutions to (S_z), we have

$$f_z(u,v) = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right)\int_{\mathbb{R}^n} Q(z)u^{p+1} + \left(\tfrac{1}{2} - \tfrac{1}{q+1}\right)\int_{\mathbb{R}^n} K(z)v^{q+1}.$$

Then, in light of (2.1), we have

$$\begin{aligned}
I_z(\eta) &= \tfrac{p}{p+1}\int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(z)} + \tfrac{q}{q+1}\int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(z)} - \tfrac{1}{2}\int_{\mathbb{R}^n} \langle T\eta, \eta\rangle \\
&= \left(\tfrac{p}{p+1} - \tfrac{1}{2}\right)\int_{\mathbb{R}^n} \eta_1\, T_1\eta_2 + \left(\tfrac{q}{q+1} - \tfrac{1}{2}\right)\int_{\mathbb{R}^n} \eta_2\, T_2\eta_1 \\
&= \left(\tfrac{p}{p+1} - \tfrac{1}{2}\right)\int_{\mathbb{R}^n} (-\Delta v + v)u + \left(\tfrac{q}{q+1} - \tfrac{1}{2}\right)\int_{\mathbb{R}^n} (-\Delta u + u)v \\
&= \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right)\int_{\mathbb{R}^n} Q(z)u^{p+1} + \left(\tfrac{1}{2} - \tfrac{1}{q+1}\right)\int_{\mathbb{R}^n} K(z)v^{q+1} = f_z(u,v),
\end{aligned}$$

which concludes the proof.

Definition 2.1 We say that $\eta \in \mathscr{H}$ is a *dual solution* to (S_z) if it is a critical point of I_z. We say that η is a *dual least energy solution* to (S_z) if it is a dual solution and, in addition, $I_z(\eta) = \Sigma(z)$.

The next property, classical in the scalar case, will be pretty useful for our purposes.

Lemma 2.1 *For every $z \in \mathbb{R}^n$, let us set*

$$b_1(z) := \inf_{\eta\in\mathscr{H}\setminus\{0\}} \sup_{t\geq 0} I_z(t\eta),$$

$$b_2(z) := \inf_{\eta\in\mathcal{N}_z} I_z(\eta) = \Sigma(z),$$

$$b_3(z) := \inf\left\{I_z(\eta) : \eta \in \mathscr{H}\setminus\{0\} \textit{ is a dual solution to } (S_z)\right\}.$$

Then $b_1(z) = b_2(z) = b_3(z)$. Moreover $\{z \mapsto \Sigma(z)\}$ is continuous.

Proof. The first equality follows from [1, Lemma 2]. Moreover in [1] it is proved that $b_1(z) = b_2(z)$ is a critical value so that also $b_2(z) = b_3(z)$ follows. Finally, by virtue of [2, Lemma 1], we know that Σ is continuous.

Lemma 2.2 *Let $z \in \mathbb{R}^n$ and define the (nonempty) set*

$$\mathscr{H}_+ := \Big\{ \eta \in \mathscr{H} : \int_{\mathbb{R}^n} \langle T\eta, \eta \rangle > 0 \Big\}.$$

Then, for every $\eta \in \mathscr{H}_+$, there exists a unique maximum point $t_\eta > 0$ of the map $\phi \colon t \in (0, \infty) \mapsto I_z(t\eta)$. In particular, $t_\eta \eta \in \mathcal{N}_z$.

Proof. Let us observe that if $\phi'(t) = 0$, then

$$\int_{\mathbb{R}^n} \langle T\eta, \eta \rangle = t^{\frac{1-p}{p}} \int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(z)} + t^{\frac{1-q}{q}} \int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(z)}.$$

Since the function $g(t) = At^{\frac{1-q}{q}} + Bt^{\frac{1-p}{p}}$ with $A, B > 0$ is strictly decreasing for $t > 0$, then ϕ has at most one critical value. It is easy to see that for all $\eta \in \mathscr{H}$, $\phi(t) > 0$ for t small, while if $\eta \in \mathscr{H}_+$, it is readily seen that $\phi(t) < 0$ for t large.

2.2 Conjecturing the representation of Σ

Consider for a moment the equation

$$-\varepsilon^2 \Delta u + V(x)u = K(x)u^p, \qquad \text{in } \mathbb{R}^n, \tag{2.2}$$

with p subcritical and V and K potentials functions bounded away from zero. By the results of [16], we know that there is uniqueness (up to translation) of positive solutions for

$$-\Delta u + u = u^p, \qquad \text{in } \mathbb{R}^n,$$

and, by a suitable change of variable, also for the "limit" problem at $x = z$ of (2.2)

$$-\Delta u + V(z)u = K(z)u^p, \qquad \text{in } \mathbb{R}^n.$$

This allows to give an explicit representation for the ground state function associated with (2.2), merely depending on the potentials V and K (see for example [4, 30]):

$$\Sigma(z) = \Gamma \frac{V^{\frac{p+1}{p-1} - \frac{n}{2}}(z)}{K^{\frac{2}{p-1}}(z)}, \tag{2.3}$$

for a suitable positive constant Γ. On the contrary, as already observed, to our knowledge there is no (known) uniqueness result for the elliptic system

$$-\Delta \xi + \xi = \zeta^q, \qquad -\Delta \zeta + \zeta = \xi^p, \qquad \text{in } \mathbb{R}^n, \tag{2.4}$$

and so, in general, we cannot provide an explicit expression for Σ for (S_z). Slightly more generally, if V is smooth and $\alpha \le V(x) \le \beta$, consider the system

$$\begin{cases} -\Delta u + V(z)u = K(z)v^q, & \text{in } \mathbb{R}^n, \\ -\Delta v + V(z)v = Q(z)u^p, & \text{in } \mathbb{R}^n, \\ u, \, v > 0, & \text{in } \mathbb{R}^n. \end{cases} \tag{2.5}$$

Assuming that (2.4) has a unique solution (ξ, ζ), then we claim that

$$\Sigma(z) = \Gamma \frac{V^{\frac{(p+1)(q+1)}{pq-1} - \frac{n}{2}}(z)}{Q^{\frac{q+1}{pq-1}}(z)\, K^{\frac{p+1}{pq-1}}(z)} \tag{2.6}$$

for a suitable positive constant Γ. Indeed, by rescaling

$$u(x) = \varpi_1 \xi(\mu x) \quad \text{and} \quad v(x) = \varpi_2 \zeta(\mu x),$$

where we have set

$$\begin{aligned} \mu = \mu(z) &:= V^{\frac{1}{2}}(z), \\ \varpi_1 = \varpi_1(z) &:= \frac{V^{\frac{q+1}{pq-1}}(z)}{Q^{\frac{q}{pq-1}}(z)\, K^{\frac{1}{pq-1}}(z)}, \\ \varpi_2 = \varpi_2(z) &:= \frac{V^{\frac{p+1}{pq-1}}(z)}{Q^{\frac{1}{pq-1}}(z)\, K^{\frac{p}{pq-1}}(z)}, \end{aligned}$$

it is easy to see that (u, v) is the unique solution of the system (2.5). Hence, by a straightforward calculation, we reach (2.6). Let us observe that the exponent of $V(z)$ in (2.6) is equal to zero if, and only if, the pair (p, q) belongs to $\mathscr{C}_n$. Then, for problems with powers p, q close to the set $\mathscr{C}_n$, the potential V is expected to have a weak influence in the location of concentration points. Notice that the same phenomenon appears in the scalar case (cf. formula (2.3)), since $\frac{p+1}{p-1} - \frac{n}{2} \sim 0$ if and only if $p \sim \frac{n+2}{n-2} = 2^* - 1$, where 2^* is the critical Sobolev exponent for H^1. Finally, we just wish to mention that, incidentally, the exponents

$$\theta_1 = \frac{p+1}{pq-1}, \qquad \theta_2 = \frac{q+1}{pq-1}$$

in formula (2.6) also arise in the study of the blow-up rates for the parabolic system

$$u_t = \Delta u + v^q, \qquad v_t = \Delta v + u^p, \qquad x \in \Omega,\ t > 0,$$

with initial data $u(x, 0) = u_0(x) \geq 0$, $v(x, 0) = v_0(x) \geq 0$ and Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$. Here Ω is a ball in $\mathbb{R}^n$ and u_0 and v_0 are continuous which vanish on the boundary. If u_0, v_0 are nontrivial C^1 functions, the solution (u, v) blows up at a finite time $T < \infty$, and $u_t \geq 0$, $v_t \geq 0$ on $\Omega \times (0, T)$, then there exist two constants $C > c > 0$ with

$$\frac{c}{(T-t)^{\theta_1}} \leq \max_{\overline{\Omega}} u(x, t) \leq \frac{C}{(T-t)^{\theta_1}}, \qquad \frac{c}{(T-t)^{\theta_2}} \leq \max_{\overline{\Omega}} v(x, t) \leq \frac{C}{(T-t)^{\theta_2}},$$

for all $t \in (0, T)$. We refer the interested reader, e.g., to [28].

2.3 Local lipschitzian property of Σ

In the case of a single semilinear elliptic equation, it is known [30] that the ground energy map enjoys a basic regularity property, in addition to the continuity, namely it is locally Lipschitz continuous (hence differentiable a.e. by virtue of Rademacher's theorem). Analogously, for system (S_ε), we obtain the following:

Theorem 2.1 $\Sigma \in Lip_{loc}(\mathbb{R}^n)$.

Proof. Let $\rho_0 > 0$ and $\mu \in \mathbb{R}^n$ with $|\mu| \le \rho_0$ and let η^μ be a (dual) solution to (S_μ) such that $I_\mu(\eta^\mu) = \Sigma(\mu)$ (we already know that such a solution does exist, see [1]). Then, the corresponding (direct) solution (u_μ, v_μ) satisfies

$$-\Delta u + u = K(\mu)v^q, \qquad -\Delta v + v = Q(\mu)u^p, \qquad \text{in } \mathbb{R}^n. \tag{2.7}$$

We also know that u_μ and v_μ are radially symmetric, radially decreasing with respect to, say, the origin, and exponentially decaying (see [6, 13, 27], in particular [6, Theorem 2] and [27, Theorem 1(a)]). We claim that there exist $\varpi_1 > 0$ and $\varpi_2 = \varpi_2(\rho_0) > 0$ independent of μ such that

$$\varpi_1 \le \|u_\mu\|_{L^{p+1}} \le \varpi_2 \qquad \text{and} \qquad \varpi_1 \le \|v_\mu\|_{L^{q+1}} \le \varpi_2. \tag{2.8}$$

Let us prove first the estimates from below. By multiplying the first equation of (2.7) by u_μ and taking into account (1.2), we get

$$\|u_\mu\|_{H^1}^2 = \int_{\mathbb{R}^n} K(\mu)v_\mu^q u_\mu \le \beta\|v_\mu\|_{L^{q+1}}^q \|u_\mu\|_{L^{q+1}} \le \beta S\|v_\mu\|_{L^{q+1}}^q \|u_\mu\|_{H^1}, \tag{2.9}$$

where S is the Sobolev constant. Now, by multiplying the first equation of system (2.7) by v_μ and the second equation by u_μ, and comparing the resulting equations, we have

$$\|v_\mu\|_{L^{q+1}}^q \le \left(\tfrac{\beta}{\alpha}\right)^{q/(q+1)} \|u_\mu\|_{L^{p+1}}^{q(p+1)/(q+1)}. \tag{2.10}$$

By combining inequalities (2.9) and (2.10), and using again the Sobolev inequality, the assertion follows. The proof of the estimate from below for $\|v_\mu\|_{L^{q+1}}$ is similar. To prove the inequalities from above, we simply observe that Σ is continuous and

$$\begin{aligned}\max_{|\mu|\le\rho_0} \Sigma(\mu) &= \max_{|\mu|\le\rho_0} I_\mu(\eta^\mu) = \max_{|\mu|\le\rho_0} f_\mu(u_\mu, v_\mu)\\ &\ge \left(\tfrac{\alpha}{2} - \tfrac{\alpha}{q+1}\right)\|v_\mu\|_{L^{q+1}}^{q+1} + \left(\tfrac{\alpha}{2} - \tfrac{\alpha}{p+1}\right)\|u_\mu\|_{L^{p+1}}^{p+1}.\end{aligned}$$

Thus (2.8) follows. As a consequence, according to the definition of the dual norm $\|\cdot\|_{\mathscr{H}}$, we immediately obtain

$$\alpha\sqrt{\varpi_1^{2p} + \varpi_1^{2q}} \le \max_{|\mu|\le\rho_0} \|\eta^\mu\|_{\mathscr{H}} \le \beta\sqrt{\varpi_2^{2p}(\rho_0) + \varpi_2^{2q}(\rho_0)}. \tag{2.11}$$

Now, since $\eta^\mu \in \mathcal{N}_\mu$, we get

$$\int_{\mathbb{R}^n} \langle T\eta^\mu, \eta^\mu\rangle = \int_{\mathbb{R}^n} \frac{|\eta_1^\mu|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\mu)} + \int_{\mathbb{R}^n} \frac{|\eta_2^\mu|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\mu)} > 0. \tag{2.12}$$

Hence $\eta^\mu \in \mathscr{H}_+$ and, by means of Lemma 2.2, there exists precisely one positive number $\theta(\mu,\xi)$ such that $\theta(\mu,\xi)\eta^\mu \in \mathcal{N}_\xi$. By definition, this means that

$$\int_{\mathbb{R}^n} \langle T\eta^\mu, \eta^\mu\rangle = \theta(\mu,\xi)^{\frac{1-p}{p}} \int_{\mathbb{R}^n} \frac{|\eta_1^\mu|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\xi)} + \theta(\mu,\xi)^{\frac{1-q}{q}} \int_{\mathbb{R}^n} \frac{|\eta_2^\mu|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\xi)}. \tag{2.13}$$

Moreover, we have $\theta(\mu,\mu)=1$. Collecting these facts, we see that, by the implicit function theorem, θ is differentiable with respect to the variable ξ. Moreover, in light of (2.11), it follows that $\theta(\mu,\xi)$ remains bounded for μ and ξ varying in a bounded set. Indeed, by combining (2.12) and (2.13), supposing for example that $p\leq q$, we have

$$\theta(\mu,\xi)^{\frac{p-1}{p}}\left[\int_{\mathbb{R}^n}\frac{|\eta_1^\mu|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\mu)}+\int_{\mathbb{R}^n}\frac{|\eta_2^\mu|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\mu)}\right]=\int_{\mathbb{R}^n}\frac{|\eta_1^\mu|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\xi)}+\theta(\mu,\xi)^{\frac{1}{q}-\frac{1}{p}}\int_{\mathbb{R}^n}\frac{|\eta_2^\mu|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\xi)}.$$

Then the (local) boundedness of $\theta(\mu,\xi)$ follows immediately by (2.11) and by the fact that $\frac{1}{q}-\frac{1}{p}\leq 0$. Let us now observe that

$$\begin{aligned}I_\xi(\theta(\mu,\xi)\eta^\mu)=\theta(\mu,\xi)^{\frac{p+1}{p}}\frac{p}{p+1}\int_{\mathbb{R}^n}\frac{|\eta_1^\mu|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\xi)}\\ +\theta(\mu,\xi)^{\frac{q+1}{q}}\frac{q}{q+1}\int_{\mathbb{R}^n}\frac{|\eta_2^\mu|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\xi)}\\ -\frac{\theta(\mu,\xi)^2}{2}\int_{\mathbb{R}^n}\langle T\eta^\mu,\eta^\mu\rangle.\end{aligned}$$

The gradient of the function $\left\{\xi\mapsto I_\xi(\theta(\mu,\xi)\eta^\mu)\right\}$ is thus given by

$$\begin{aligned}\nabla_\xi I_\xi(\theta(\mu,\xi)\eta^\mu)=&-\frac{\theta(\mu,\xi)^{\frac{p+1}{p}}}{p+1}\nabla_\xi Q(\xi)\int_{\mathbb{R}^n}\frac{|\eta_1^\mu|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(\xi)}\\ &-\frac{\theta(\mu,\xi)^{\frac{q+1}{q}}}{q+1}\nabla_\xi K(\xi)\int_{\mathbb{R}^n}\frac{|\eta_2^\mu|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(\xi)}\\ &+\nabla_\xi\theta(\mu,\xi)\Bigg[\theta(\mu,\xi)^{\frac{1}{p}}\int_{\mathbb{R}^n}\frac{|\eta_1^\mu|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\xi)}\\ &+\theta(\mu,\xi)^{\frac{1}{q}}\int_{\mathbb{R}^n}\frac{|\eta_2^\mu|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\xi)}\\ &-\theta(\mu,\xi)\int_{\mathbb{R}^n}\langle T\eta^\mu,\eta^\mu\rangle\Bigg],\end{aligned}$$

and so, since $\theta(\mu,\xi)\eta^\mu\in\mathcal{N}_\xi$, in turn we get

$$\begin{aligned}\nabla_\xi I_\xi(\theta(\mu,\xi)\eta^\mu)=&-\frac{\theta(\mu,\xi)^{\frac{p+1}{p}}}{p+1}\nabla_\xi Q(\xi)\int_{\mathbb{R}^n}\frac{|\eta_1^\mu|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(\xi)}\\ &-\frac{\theta(\mu,\xi)^{\frac{q+1}{q}}}{q+1}\nabla_\xi K(\xi)\int_{\mathbb{R}^n}\frac{|\eta_2^\mu|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(\xi)}.\end{aligned}\tag{2.14}$$

From this representation formula, the Mean-Value Theorem and the local boundedness of θ, the assertion readily follows.

2.4 Left and right derivatives of Σ

Let us define $\mathbb{S}(z)$ as the set of all the positive (dual) solutions (η_1, η_2) of (S_z) at the energy level $\Sigma(z)$. The representation formulas for the (left and right) directional derivatives of Σ are provided in the following

Theorem 2.2 *The directional derivatives from the left and the right of Σ at every point $z \in \mathbb{R}^n$ along any $w \in \mathbb{R}^n$ exist and*

$$\left(\frac{\partial \Sigma}{\partial w}\right)^{-}(z) = \sup_{\eta \in \mathbb{S}(z)} \nabla_z I_z(\eta) \cdot w,$$

$$\left(\frac{\partial \Sigma}{\partial w}\right)^{+}(z) = \inf_{\eta \in \mathbb{S}(z)} \nabla_z I_z(\eta) \cdot w.$$

Explicitly, we have

$$\left(\frac{\partial \Sigma}{\partial w}\right)^{-}(z) = \sup_{\eta \in \mathbb{S}(z)} \left[-\frac{1}{p+1}\frac{\partial Q}{\partial w}(z) \int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(z)} - \frac{1}{q+1}\frac{\partial K}{\partial w}(z) \int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(z)} \right]$$

$$\left(\frac{\partial \Sigma}{\partial w}\right)^{+}(z) = \inf_{\eta \in \mathbb{S}(z)} \left[-\frac{1}{p+1}\frac{\partial Q}{\partial w}(z) \int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(z)} - \frac{1}{q+1}\frac{\partial K}{\partial w}(z) \int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(z)} \right]$$

for every $z, w \in \mathbb{R}^n$.

Proof. Let $\{\mu_j\} \subset \mathbb{R}^n$ be a sequence converging to μ_0 and let $\eta^j = \eta^{\mu_j}$ be a sequence of (dual) solutions of least energy $\Sigma(\mu_j)$. We want to prove that, up to a subsequence,

$$\eta^{\mu_j} \to \eta^0, \qquad \text{strongly in } \mathscr{H}, \qquad \eta^0 \in \mathbb{S}(\mu_0). \tag{2.15}$$

Consider the corresponding (direct) solutions (u_{μ_j}, v_{μ_j}) (resp. (u_0, v_0)) of (2.7) with $\mu = \mu_j$ (resp. $\mu = \mu_0$). Since (u_{μ_j}, v_{μ_j}) is bounded in $W^{2,(q+1)/q}(\mathbb{R}^n) \times W^{2,(p+1)/p}(\mathbb{R}^n)$ (cf. [2]), up to a subsequence, it converges weakly to a pair (u_0, v_0). In addition, since K and Q are uniformly bounded, by virtue of the Schauder local regularity estimates (cf. [25]), (u_{μ_j}, v_{μ_j}) is bounded in $C^{2,\beta}_{\text{loc}}(\mathbb{R}^n)$ for some $\beta > 0$ and

$$u_{\mu_j} \to u_0 \quad \text{and} \quad v_{\mu_j} \to v_0, \qquad \text{locally in } C^2\text{-sense}, \tag{2.16}$$

so that (u_0, v_0) solves (2.7) with $\mu = \mu_0$. We claim that $u_0 > 0$ and $v_0 > 0$. By [6, Theorem 2], for every $j \geq 1$, u_{μ_j} and v_{μ_j} are radially symmetric and radially decreasing with respect to some point, say the origin, that is

$$u_{\mu_j}(x) = u_j(r), \quad v_{\mu_j}(x) = v_j(r), \quad \frac{d}{dr}u_j(r) < 0, \quad \frac{d}{dr}v_j(r) < 0, \tag{2.17}$$

for every $r > 0$. Hence, for every $j \geq 1$, we have

$$u_{\mu_j}(0) \leq -\Delta u_{\mu_j}(0) + u_{\mu_j}(0) = K(\mu_j) v^q_{\mu_j}(0) \leq \beta v^q_{\mu_j}(0),$$

$$v_{\mu_j}(0) \leq -\Delta v_{\mu_j}(0) + v_{\mu_j}(0) = Q(\mu_j) u^p_{\mu_j}(0) \leq \beta u^p_{\mu_j}(0).$$

It follows that, for every $j \geq 1$,

$$u_{\mu_j}(0) \leq \beta^{q+1} u_{\mu_j}^{pq}(0).$$

Then there exists $\hat{\delta} > 0$ such that $u_{\mu_j}(0) \geq \hat{\delta}$ for every $j \geq 1$. Similarly, $v_{\mu_j}(0) \geq \hat{\delta}$ for every $j \geq 1$. Hence, letting $j \to \infty$, by (2.16), we conclude that $u_0(0) \geq \hat{\delta}$ and $v_0(0) \geq \hat{\delta}$, which entails $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$. Since we have $u_0 \geq 0$, $v_0 \geq 0$, $K(\mu_0), Q(\mu_0) > 0$ and

$$-\Delta u_0 + u_0 \geq 0 \quad \text{ and } \quad -\Delta v_0 + v_0 \geq 0,$$

the claim just follows by a straightforward application of the maximum principle.

Observe that, by the continuity of Σ and by Fatou's Lemma, we get

$$\Sigma(\mu_0) = \lim_{j\to\infty} \Sigma(\mu_j) = \lim_{j\to\infty} I_{\mu_j}(\eta^j) \geq I_{\mu_0}(\eta^0) \geq \Sigma(\mu_0).$$

Hence

$$\lim_{j\to\infty} I_{\mu_j}(\eta^j) = I_{\mu_0}(\eta^0) = \Sigma(\mu_0),$$

which reads as

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \frac{|\eta_1^j|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\mu_j)} = \int_{\mathbb{R}^n} \frac{|\eta_1^0|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\mu_0)}, \qquad \lim_{j\to\infty} \int_{\mathbb{R}^n} \frac{|\eta_2^j|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\mu_j)} = \int_{\mathbb{R}^n} \frac{|\eta_2^0|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\mu_0)}.$$

In particular, taking into account (1.2), for any $\delta > 0$, there exists $\rho > 0$ such that

$$\int_{\{|x|\geq\rho\}} |\eta_1^j|^{\frac{p+1}{p}} < \delta, \qquad \int_{\{|x|\geq\rho\}} |\eta_2^j|^{\frac{q+1}{q}} < \delta,$$

for every $j \geq 1$ sufficiently large. Moreover, of course

$$\lim_{j\to\infty} \int_{\{|x|\leq\rho\}} |\eta_1^j|^{\frac{p+1}{p}} = \int_{\{|x|\leq\rho\}} |\eta_1^0|^{\frac{p+1}{p}}, \qquad \lim_{j\to\infty} \int_{\{|x|\leq\rho\}} |\eta_2^j|^{\frac{q+1}{q}} = \int_{\{|x|\leq\rho\}} |\eta_2^0|^{\frac{q+1}{q}}.$$

Then we have $\eta^{\mu_j} \to \eta^0$ strongly in $\mathscr{H}$, namely (2.15) holds true.

Without loss of generality, we can prove the formula of the right derivative of Σ in the case $n = 1$, $z = 0$ and $w = 1$. For any $\eta^0 \in \mathbb{S}(0)$, we get

$$\begin{aligned} \Sigma(\rho) - \Sigma(0) &\leq I_\rho(\vartheta(\rho, 0)\eta^0) - I_0(\eta^0) \\ &= \rho \nabla_\xi I_\xi(\vartheta(\xi, 0)\eta^0)|_{\xi=\mu\in[0,\rho]}. \end{aligned}$$

Whence, by virtue of (2.14) and the arbitrariness of $\eta^0 \in \mathbb{S}(0)$,

$$\limsup_{\rho\to 0^+} \frac{\Sigma(\rho) - \Sigma(0)}{\rho} \leq \inf_{\eta^0\in\mathbb{S}(0)} \Big[-\frac{Q'(0)}{p+1} \int_{\mathbb{R}^n} \frac{|\eta_1^0|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(0)} - \frac{K'(0)}{q+1} \int_{\mathbb{R}^n} \frac{|\eta_2^0|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(0)} \Big].$$

Moreover, similarly, we get

$$\begin{aligned} \Sigma(\rho) - \Sigma(0) &\geq I_\rho(\vartheta(\rho, \rho)\eta^\rho) - I_0(\vartheta(0, \rho)\eta^\rho) \\ &= \rho \nabla_\xi I_\xi(\vartheta(\xi, \rho)\eta^\rho)|_{\xi=\mu\in[0,\rho]}, \end{aligned}$$

so that, by exploiting (2.14) and (2.15), we conclude

$$\liminf_{\rho\to 0^+}\frac{\Sigma(\rho)-\Sigma(0)}{\rho}\geq \inf_{\eta^0\in\mathbb{S}(0)}\Big[-\frac{Q'(0)}{p+1}\int_{\mathbb{R}^n}\frac{|\eta_1^0|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(0)}-\frac{K'(0)}{q+1}\int_{\mathbb{R}^n}\frac{|\eta_2^0|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(0)}\Big].$$

Then the desired formula for the right derivative of Σ follows. A very similar argument provides the corresponding formula for the left derivative.

Remark 2.1 Nowadays, further regularity of Σ is, to our knowledge, an open problem. Actually, not even in the case of a single equation is the situation very well understood. For instance, on one hand, if we consider the problem

$$-\varepsilon^2\Delta u+V(x)u=K(x)u^p \quad \text{in } \mathbb{R}^n, \qquad u>0 \quad \text{in } \mathbb{R}^n,$$

then $\Sigma\in C^m(\mathbb{R}^n)$ provided that both the potentials V and K belong to $C^m(\mathbb{R}^n)$, with $m\geq 1$. On the other hand, if f is not a power (and does not satisfy conditions ensuring uniqueness up to translations), for the equation

$$-\varepsilon^2\Delta u+V(x)u=K(x)f(u) \quad \text{in } \mathbb{R}^n, \qquad u>0 \quad \text{in } \mathbb{R}^n,$$

we do not know which regularity beyond $\mathrm{Lip}_{\mathrm{loc}}$ can be achieved by Σ. Even though we do not have any specific counterexample, our feeling is that there exist functions f for which the associated Σ fails to be C^1 smooth. It is evident by the (left and right) derivative formulas of Σ that its further regularity is related to the uniqueness of positive radial solutions to $-\Delta u+u=f(u)$ in $\mathbb{R}^n$, $u>0$ in $\mathbb{R}^n$, which occurs just for very particular nonlinearities f. Based upon these considerations, for semilinear systems, the further regularity of Σ seems an *ever harder* matter, since as already stressed nothing is known, so far, about the uniqueness of solutions to the system

$$-\Delta u+u=f(v), \qquad -\Delta v+v=g(u), \quad \text{in } \mathbb{R}^n, \qquad u,v>0 \quad \text{in } \mathbb{R}^n,$$

not even with the particular choices $f(v)=v^q$ and $g(u)=u^p$.

3 Proofs of results

3.1 Proof of Theorem 1.1

Let $z\in\mathcal{E}$ and let $(u_{\varepsilon_h},v_{\varepsilon_h})\in W^{2,\frac{q+1}{q}}(\mathbb{R}^n)\times W^{2,\frac{p+1}{p}}(\mathbb{R}^n)$ be a corresponding a sequence of strong solutions to (S_ε) with $|u_{\varepsilon_h}(z)|,|v_{\varepsilon_h}(z)|\geq\delta$ for some $\delta>0$, $|u_{\varepsilon_h}(z+\varepsilon_h x)|\to 0$, $|v_{\varepsilon_h}(z+\varepsilon_h x)|\to 0$ as $|x|\to\infty$ uniformly w.r.t. h, and ${\varepsilon_h}^{-n}f_{\varepsilon_h}(u_{\varepsilon_h},v_{\varepsilon_h})\to\Sigma(z)$ as $h\to\infty$. Let us set:

$$\varphi_h(x)=u_{\varepsilon_h}(z+\varepsilon_h x) \qquad \text{and} \qquad \psi_h(x)=v_{\varepsilon_h}(z+\varepsilon_h x),$$

for all $h\geq 1$. Then, since $(u_{\varepsilon_h},v_{\varepsilon_h})$ is a solution (S_ε), (φ_h,ψ_h) is solution of

$$-\Delta\varphi_h+\varphi_h=K(z+\varepsilon_h x)\psi_h^q, \qquad -\Delta\psi_h+\psi_h=Q(z+\varepsilon_h x)\varphi_h^p. \tag{3.1}$$

By arguing as in the proof of Theorem 2.2, it is readily proved that, up to a subsequence, (φ_h) and (ψ_h) converge weakly in $W^{2,(q+1)/q}(\mathbb{R}^n)\times W^{2,(p+1)/p}(\mathbb{R}^n)$ to some φ_0 and ψ_0 respectively. Let us now prove that there exist $\Theta>0$, $\rho>0$ and $h_0\geq 1$ such that

$$\varphi_h(x)\leq ce^{-\Theta|x|} \quad \text{and} \quad \psi_h(x)\leq ce^{-\Theta|x|}, \qquad \text{for all } |x|\geq\rho \text{ and } h\geq h_0. \tag{3.2}$$

We follow the line of [13]. Since $z\in\mathcal{E}$, then the functions φ_h and ψ_h decay to zero at infinity, uniformly with respect to h. Hence, since $p,q>1$, we can find $\rho>0$, $\Theta>0$ and $h_0\geq 1$ such that

$$K(z+\varepsilon_h x)\psi_h^q\leq(1-\Theta^2)\psi_h,$$
$$Q(z+\varepsilon_h x)\varphi_h^p\leq(1-\Theta^2)\varphi_h,$$

for all $|x|>\rho$ and $h\geq h_0$. Let us set

$$\xi(x)=\mu e^{-\Theta(|x|-\rho)}, \qquad \mu=\max_{|x|=\rho}\max_{h\geq h_0}(\psi_h+\varphi_h),$$

and introduce the set

$$A=\bigcup_{R>\rho}D_R,$$

where, for any $R>\rho$, we put

$$D_R=\big\{\rho<|x|<R:\ \psi_h(x)+\varphi_h(x)>\xi(x)\ \text{ for some } h\geq h_0\big\}.$$

If $A=\emptyset$, we are done. Instead, if A is nonempty, there exists $R_*>\rho$ such that

$$\begin{aligned}\Delta(\xi-\psi_h-\varphi_h)&\leq\Big[\Theta^2-\frac{\Theta(n-1)}{|x|}\Big]\xi(x)-\Theta^2\psi_h-\Theta^2\varphi_h\\&\leq\Theta^2(\xi-\psi_h-\varphi_h)<0, \qquad \text{on } D_R \text{ for all } R\geq R_*.\end{aligned}$$

Hence, by the maximum principle, since $(\xi-\psi_h-\varphi_h)|_{\{|x|=\rho\}}\geq 0$, we get

$$\xi-\psi_h-\varphi_h\geq\min\big\{0,\min_{|x|=R}(\xi-\psi_h-\varphi_h)\big\}, \qquad \text{for all } R\geq R_*$$

so that, letting $R\to\infty$, yields, for any $\rho>0$, $\psi_h(x)+\varphi_h(x)\leq\xi(x)$ for $|x|>\rho$, which contradicts the definition of $D_{R_*}\neq\emptyset$.

By virtue of the Schauder interior estimates (see e.g. [25]), $\varphi_h\to\varphi_0$ and $\psi_h\to\psi_0$ locally in C^2 sense, so that (φ_0,ψ_0) is a (nontrivial, radial, decaying) solution to (S_z). Moreover, in light of the exponential barriers provided by (3.2), since $z\in\mathcal{E}$, it is not difficult to see that $(\varphi_0,\psi_0)\in\mathbb{S}(z)$, for we have

$$\begin{aligned}\Sigma(z)&=\big(\tfrac{1}{2}-\tfrac{1}{q+1}\big)\int_{\mathbb{R}^n}K(z)|\psi_0|^{q+1}+\big(\tfrac{1}{2}-\tfrac{1}{p+1}\big)\int_{\mathbb{R}^n}Q(z)|\varphi_0|^{p+1}\\&=f_z(\varphi_0,\psi_0)=I_z(\eta^0),\end{aligned}$$

where η^0 is the dual solution corresponding to (φ_0,ψ_0).

Let us now consider the Lagrangian $\mathscr{L} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$\mathscr{L}(x, s_1, s_2, \xi_1, \xi_2) = \xi_1 \cdot \xi_2 + s_1 s_2 - \tfrac{1}{q+1} K(z + \varepsilon_h x) s_2^{q+1} - \tfrac{1}{p+1} Q(z + \varepsilon_h x) s_1^{p+1}.$$

Then system (3.1) rewrites as

$$\begin{cases} -\mathrm{div}\,(\partial_{\xi_2} \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h)) + \partial_{s_2} \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h) = 0, & \text{in } \mathbb{R}^n, \\ -\mathrm{div}\,(\partial_{\xi_1} \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h)) + \partial_{s_1} \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h) = 0, & \text{in } \mathbb{R}^n, \\ \varphi_h,\, \psi_h > 0, & \text{in } \mathbb{R}^n. \end{cases}$$

By the Pucci-Serrin identity for systems [21, see §5], we have

$$\begin{aligned} &\sum_{i,\,l=1}^{n} \int_{\mathbb{R}^n} \partial_i \boldsymbol{q}^l \partial_{(\xi_2)_i} \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h) \partial_l \psi_h \\ &\quad + \sum_{i,\,l=1}^{n} \int_{\mathbb{R}^n} \partial_i \boldsymbol{q}^l \partial_{(\xi_1)_i} \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h) \partial_l \varphi_h \\ &= \int_{\mathbb{R}^n} \big[(\mathrm{div}\,\boldsymbol{q}) \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h) + \boldsymbol{q} \cdot \partial_x \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h)\big], \end{aligned}$$

for all $\boldsymbol{q} \in C^1_{\mathrm{c}}(\mathbb{R}^n, \mathbb{R}^n)$. Let us take, for $\lambda > 0$,

$$\boldsymbol{q}(x) = (\Upsilon(\lambda x),\, 0, \ldots, 0),$$

and $\Upsilon \in C^1_{\mathrm{c}}(\mathbb{R}^n)$ such that $\Upsilon(x) = 1$ if $|x| \le 1$ and $\Upsilon(x) = 0$ if $|x| \ge 2$. Then,

$$\begin{aligned} &\sum_{i=1}^{n} \int_{\mathbb{R}^n} \lambda \partial_i \Upsilon(\lambda x) \partial_i \varphi_h \, \partial_1 \psi_h + \sum_{i=1}^{n} \int_{\mathbb{R}^n} \lambda \partial_i \Upsilon(\lambda x) \partial_i \psi_h \, \partial_1 \varphi_h \\ &= \int_{\mathbb{R}^n} \lambda \partial_1 \Upsilon(\lambda x) \mathscr{L}(x, \varphi_h, \psi_h, \nabla\varphi_h, \nabla\psi_h) \\ &+ \int_{\mathbb{R}^n} \varepsilon_h \Upsilon(\lambda x) \left[-\tfrac{1}{q+1} \partial_1 K(z + \varepsilon_h x) \psi_h^{q+1} - \tfrac{1}{p+1} \partial_1 Q(z + \varepsilon_h x) \varphi_h^{p+1} \right]. \end{aligned}$$

By the arbitrariness of $\lambda > 0$, letting $\lambda \to 0$ and keeping h fixed, we obtain

$$\int_{\mathbb{R}^n} \left[-\tfrac{1}{q+1} \partial_1 K(z + \varepsilon_h x) \psi_h^{q+1} - \tfrac{1}{p+1} \partial_1 Q(z + \varepsilon_h x) \varphi_h^{p+1} \right] = 0.$$

Therefore, letting now $h \to \infty$, since in light of (1.3) we get

$$|\nabla K(z + \varepsilon_h x)|, |\nabla Q(z + \varepsilon_h x)| \le c e^{M \varepsilon_h |x|}, \qquad \text{for } |x| \text{ large},$$

by virtue of (3.2), we have

$$\int_{\mathbb{R}^n} \left[-\tfrac{1}{q+1} \partial_1 K(z) \psi_0^{q+1} - \tfrac{1}{p+1} \partial_1 Q(z) \varphi_0^{p+1} \right] = 0.$$

Analogously, we can show that, for all $w \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \left[-\tfrac{1}{q+1} \nabla K(z) \psi_0^{q+1} - \tfrac{1}{p+1} \nabla Q(z) \varphi_0^{p+1} \right] \cdot w = 0.$$

Hence

$$-\tfrac{1}{p+1} \frac{\partial Q}{\partial w}(z) \int_{\mathbb{R}^n} \frac{|\eta_1^0|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(z)} - \tfrac{1}{q+1} \frac{\partial K}{\partial w}(z) \int_{\mathbb{R}^n} \frac{|\eta_2^0|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(z)} = 0. \tag{3.3}$$

Since $\eta^0 \in \mathbb{S}(z)$, by Theorem 2.2 we have

$$\begin{aligned}\left(\frac{\partial \Sigma}{\partial w}\right)^+(z) &= \inf_{\eta \in \mathbb{S}(z)} \nabla_z I_z(\eta) \cdot w \\ &\leq -\tfrac{1}{p+1} \frac{\partial Q}{\partial w}(z) \int_{\mathbb{R}^n} \frac{|\eta_1^0|^{\frac{p+1}{p}}}{Q^{\frac{p+1}{p}}(z)} - \tfrac{1}{q+1} \frac{\partial K}{\partial w}(z) \int_{\mathbb{R}^n} \frac{|\eta_2^0|^{\frac{q+1}{q}}}{K^{\frac{q+1}{q}}(z)} = 0.\end{aligned}$$

Then, by the very definition of $(-\Sigma)^0(z; w)$ (see Definition 1.1), we get

$$(-\Sigma)^0(z; w) \geq \left(\frac{\partial(-\Sigma)}{\partial w}\right)^+(z) \geq 0, \quad \text{for every } w \in \mathbb{R}^n.$$

Then $0 \in \partial_C(-\Sigma)(z)$ and, since $\partial_C(-\Sigma)(z) = -\partial_C \Sigma(z)$ (cf. [7]), we obtain $z \in \mathcal{K}$.

3.2 Proof of Corollary 1.1

It suffices to combine Theorems 1.1 and 2.2, taking into account what discussed in Section 2.2 about the conjectured explicit representation formula for Σ.

3.3 Proof of Theorem 1.2

Let $m \geq 1$ and $z \in \mathcal{E}_m$. The assertion follows by mimicking the various steps in the proof of Theorem 1.1 with $\mathcal{E}_m$ in place of $\mathcal{E}$, and combining formula (3.3) with the definitions of $\Gamma^{\mp}_{z,m}$ and $\mathcal{K}_m$, taking into account that $\eta^0 \in \mathbb{G}_m(z)$, as it holds $I_z(\eta^0) = m$, being η^0 the strong limit of η^{ε_j}. Indeed, by (3.3), we have

$$\Gamma^+_{z,m}(w) \geq 0, \quad \forall w \in \mathbb{R}^n, \qquad \Gamma^-_{z,m}(w) \geq 0, \quad \forall w \in \mathbb{R}^n,$$

so that $0 \in \partial\Gamma^+_{z,m}(0) \cap \partial\Gamma^-_{z,m}(0)$, yielding $z \in \mathcal{K}_m$.

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