J. Fixed Point Theory Appl. (2025) 27:62
https://doi.org/10.1007/s11784-025-01215-1

Journal of Fixed Point Theory and Applications

Concentration of ground states for quasilinear Kirchhoff type equations at critical growth

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Abstract. We are concerned with the existence and concentrating behavior of positive ground state solutions for a quasilinear Kirchhoff equation involving critical Sobolev exponent with competing potentials

$$\left(\epsilon^2 a + \epsilon b \int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 \mathrm{d}x\right)\left[-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2\right] + V(x)u$$
$$= Q(x)h(u) + K(x)|G(u)|^4 G(u)g(u),\ x \in \mathbb{R}^3,$$

where $a, b > 0$ are constants, $\epsilon > 0$ is a small parameter, and g is an even differential function related to the quasilinear term, such that $G(t) = \int_0^t g(s)\mathrm{d}s$. Under some suitable assumptions on V, Q, K and h, we conclude that this equation admits a positive ground state solution for all sufficiently small $\epsilon > 0$ using variational methods, where the decay rate of the obtained solution as $|x| \to +\infty$ and its concentration on the set of minimal points of V and the sets of maximal points of Q and K as $\epsilon \to 0^+$ are also considered. In particular, we also investigate the nonexistence of ground state solutions.

Mathematics Subject Classification. 35J20, 35Q55.

Keywords. Ground state solution, generalized quasilinear Kirchhoff equation, critical Sobolev exponent, concentration, exponential decay, variational method.

Contents

Birkhäuser

Published online: 03 July 2025

1. Introduction and main results

The aim of this article is to consider the existence and concentrating behavior of positive ground state solutions to a quasilinear Kirchhoff equation with critical growth

$$\begin{aligned}&\left(\epsilon^2 a+\epsilon b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2\mathrm{d}x\right)\left[-\mathrm{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2\right]+V(x)u\\&\quad =Q(x)h(u)+K(x)|G(u)|^4G(u)g(u),\ x\in\mathbb{R}^3,\end{aligned}\tag{1.1}$$

where $a, b > 0$ are constants and $\epsilon > 0$ is a small parameter. The function g is supposed to satisfy

- (g) $g \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ is an even positive function with $g(0) = 1$ and $g'(t) \geq 0$ for all $t \geq 0$.

For the potentials V, Q, K and the nonlinearity h, we make the following assumptions:

- (V) $V \in \mathcal{C}^0(\mathbb{R}^3,\mathbb{R})$ and $0 < V_0 \triangleq \inf_{x\in\mathbb{R}^3} V(x) < \liminf_{|x|\to\infty} V(x) \triangleq V_\infty < +\infty$.
- (Q) $Q \in \mathcal{C}^0(\mathbb{R}^3,\mathbb{R})$, $\lim_{|x|\to\infty} Q(x) = Q_\infty \in (0,\infty)$ and $Q(x) \geq Q_\infty$ for all $x \in \mathbb{R}^3$.
- (K) $K \in \mathcal{C}^0(\mathbb{R}^3,\mathbb{R})$, $\lim_{|x|\to\infty} K(x) = K_\infty \in (0,\infty)$ and $K(x) \geq K_\infty$ for all $x \in \mathbb{R}^3$.
- (K_1) There exist some constants $\mathcal{C}_0 > 0$, $\delta_0 > 0$ and $\beta \in [1, 3)$, such that
$$|K(x)-K(x_0)|\leq \mathcal{C}_0|x-x_0|^\beta \text{ whenever } |x-x_0|<\delta_0,$$
where $x_0 \in \mathbb{R}^3$ satisfies $K(x_0) = \max_{x\in\mathbb{R}^3} K(x) \triangleq K_0 < +\infty$;
- (K_2) $\Theta \cap \Theta_1 \cap \Theta_2 \neq \emptyset$, where
$$\Theta \triangleq \{x\in\mathbb{R}^3 : V(x)=V_0\},$$
$$\Theta_1 \triangleq \{x\in\mathbb{R}^3 : K(x)=K_0 \triangleq \max_{x\in\mathbb{R}^3} K(x)\},$$
$$\Theta_2 \triangleq \{x\in\mathbb{R}^3 : Q(x)=Q_0 \triangleq \max_{x\in\mathbb{R}^3} Q(x)\}.$$

Since we are interested in positive solutions, without loss of generality, we assume that $h \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$ vanishes in $(-\infty, 0)$ and satisfies the following conditions:

- (H_1) $h(t) \in \mathcal{C}^0(\mathbb{R},\mathbb{R}^+)$ and $h(t) = o(t)$ as $t \to 0^+$;
- (H_2) $h(t) = o(t)$ as $t \to \infty$;
- (H_3) The map $t \to \dfrac{h(t)}{g(t)G^3(t)}$ is strictly increasing on $(0, +\infty)$;
- (H_4) There exist two constants $\mathcal{C}_1 > 0$ and $p \in (3, 5)$, such that $h(t) \geq \mathcal{C}_1 G^p(t)g(t)$.

When $a = 1$ and $b = 0$ in Eq. (1.1), solutions of this type are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\epsilon\partial_t z=-\epsilon^2\Delta z+W(x)z-k(x,z)-\omega\epsilon^2\Delta l(|z|^2)l'(|z|^2)z,\ x\in\mathbb{R}^N,\tag{1.2}$$

where ω is a real constant, $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is an external potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. If we set $z(t,x) = \exp(-iEt)u(x)$ with $E \in \mathbb{R}$ and $\epsilon = 1$ in (1.2), one obtains a corresponding quasilinear Schrödinger equation

$$-\Delta u + a(x)u - \Delta l(u^2)l'(u^2)u = k(x,u),\ x \in \mathbb{R}^N. \tag{1.3}$$

Recently, authors in [11,35] studied the following generalized quasilinear Schrödinger equation:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + a(x)u = k(x,u),\ x \in \mathbb{R}^N. \tag{1.4}$$

The reason we call Eq. (1.4) a generalized one is that if one takes $g^2(u) = 1 + \frac{[l'(u^2)]^2}{2}$, then it reduces to Eq. (1.3). There are many interesting and meaningful results on Eqs. (1.2), (1.3) and (1.4), we refer the reader to [1,6,7,10,11,20,24,26,30,32] and their references therein.

If $\omega = 0$ in (1.2), it is closely related to the singularly perturbed semilinear problem

$$-\epsilon^2\Delta u + V(x)u = k(x,u),\ x \in \mathbb{R}^N. \tag{1.5}$$

Since solutions of Eq. (1.5) are known as semiclassical states which can be used to describe the transition from quantum to classical mechanics for every $\epsilon > 0$ small, a lot of mathematicians pay considerable attentions to this problem. The authors in [28,29] depended on the Lyapunov–Schmidt reduction to get single and multiple spike solutions. Whereas, the essential feature of the Lyapunov–Schmidt reduction method is the uniqueness or non-degeneracy of ground state solutions of the corresponding limiting equation. To deal with it, Rabinowitz [31] first proposed the assumption

$(\bar{V})$ $\liminf_{|x|\to+\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) > 0$

to obtain the existence of solutions of (1.5) for $\epsilon > 0$ small. In [36], Wang studied the concentrating phenomenon of solutions. In [8], del Pino–Felmer showed a localized version of the results in [31,36] by introducing the penalization approach. Regarding the other interesting results on the singularly perturbed problems, see [2,3,5,9,15,18,33,34] for example.

Therefore, Eq. (1.1) can be regarded as a Kirchhoff type of the generalized quasilinear Schrödinger equation (1.4) because of the appearance of the nonlocal term $\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2dx$. Indeed, if we choose $\epsilon = 1$ and $g(t) = 1$ for all $t \in \mathbb{R}$ and $Q(x)h(u) + K(x)|G(u)|^4G(u)g(u) = k(x,u)$, then Eq. (1.1) transforms to the following classical Kirchhoff type equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = k(x,u),\ x \in \mathbb{R}^3, \tag{1.6}$$

which is called degenerate if $a = 0$ and non-degenerate otherwise. Equation (1.6) arises in an interesting physical context. Let $V \equiv 0$ and replace $\mathbb{R}^3$ with a bounded domain $\Omega \subset \mathbb{R}^3$ in (1.3), then we get the following Dirichlet problem of Kirchhoff type:

$$-\left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = k(x,u)$$

which is related to the stationary analog of the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}-\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=0 \tag{1.7}$$

proposed in [19] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, where ρ, P_0, h, E_0 and L are constants. The Kirchhoff model takes the changes in length of the string produced by transverse vibrations into account. Due to a pioneer work [21] developing the abstract functional analysis framework to (1.7), this problem has been widely contemplated in extensive literatures; see, e.g., [14,16,17,25,27,38,39]. In view of this, it is also reasonable to consider the generalized quasilinear Schrödinger equation of Kirchhoff type. In particular, it seems that the existence and concentrating behavior of ground state solutions for Eq. (1.1) have not been obtained yet. In this article, we shall introduce sone interesting analyses to fill this blank.

It follows from the standard procedure of studying the existence and concentrating behavior of solutions for Eq. (1.1), namely by performing $\epsilon z = x$, one has that:

$$\begin{aligned}\left(a+b\int_{\mathbb{R}^3}g^2(u)|\nabla u|^2\mathrm{d}x\right)&\left[-\mathrm{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2\right]+V(\epsilon x)u\\&=Q(\epsilon x)h(u)+K(\epsilon x)|G(u)|^4G(u)g(u),\ x\in\mathbb{R}^3.\end{aligned} \tag{1.8}$$

Denoting $H(u)=\int_0^u h(t)dt$, it is clear to observe that the corresponding variational functional

$$\begin{aligned}I_\epsilon(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left[ag^2(u)|\nabla u|^2+V(\epsilon x)u^2\right]\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}g^2(u)|\nabla u|^2\mathrm{d}x\right)^2\\-\int_{\mathbb{R}^3}Q(\epsilon x)H(u)\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}K(\epsilon x)|G(u)|^6\mathrm{d}x\end{aligned}$$

associated with Eq. (1.8) might be not well defined in $H^1(\mathbb{R}^3)$. To overcome this difficulty, motivated by [35], we are able to make a change of variable

$$v=G(u)=\int_0^u g(\tau)\mathrm{d}\tau.$$

As a consequence, it allows us to rewrite $J_\epsilon(v)\triangleq I_\epsilon(G^{-1}(v))$ as follows:

$$\begin{aligned}J_\epsilon(v)=\frac{1}{2}\int_{\mathbb{R}^3}\left[a|\nabla v|^2+V(\epsilon x)|G^{-1}(v)|^2\right]\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2\\-\int_{\mathbb{R}^3}Q(\epsilon x)H(G^{-1}(v))\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}K(\epsilon x)|v|^6\mathrm{d}x.\end{aligned} \tag{1.9}$$

Since g is a nondecreasing positive function, we obtain $|G^{-1}(v)|\le|v|/g(0)$. Moreover, one further shows that J_ϵ is well defined in $H^1(\mathbb{R}^3)$ and $J_\epsilon\in\mathcal{C}^1$ if Q and h satisfy the assumptions (Q) and (H_2), respectively.

If u is a nontrivial solution of Eq. (1.8), then it satisfies

$$\begin{aligned}&\left(a+b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}\left[g^2(u)\nabla u\nabla\varphi+g(u)g'(u)|\nabla u|^2\varphi\right]\mathrm{d}x\\&\quad+\int_{\mathbb{R}^3}\left[V(\epsilon x)u-Q(\epsilon x)h(u)-K(\epsilon x)|G(u)|^4G(u)g(u)\right]\varphi\mathrm{d}x=0\end{aligned}\tag{1.10}$$

for any $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^3)$. Let $\varphi=\psi/g(u)$, one finds that (1.10) is equivalent to

$$\begin{aligned}J_\epsilon'(v)(\psi)&=\left(a+b\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}\nabla v\nabla\psi\mathrm{d}x\\&\quad+\int_{\mathbb{R}^3}\left[V(\epsilon x)\frac{G^{-1}(v)}{g(G^{-1}(v))}-Q(\epsilon x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))}-K(\epsilon x)|v|^4v\right]\psi\mathrm{d}x\\&=0,\ \forall\psi\in\mathcal{C}_0^\infty(\mathbb{R}^3).\end{aligned}\tag{1.11}$$

Therefore, to find the nontrivial solutions of (1.1) which is equivalent to

$$-\left(\epsilon^2a+\epsilon b\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)\Delta v+V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}=Q(x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))}+K(x)|v|^4v,\tag{1.12}$$

then we consider (1.8) and it suffices to study the existence of the nontrivial solutions of the following equation:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)\Delta v+V(\epsilon x)\frac{G^{-1}(v)}{g(G^{-1}(v))}=Q(\epsilon x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))}+K(\epsilon x)|v|^4v.\tag{1.13}$$

The reader is invited to observe that the nontrivial critical points of J_ϵ are the nontrivial solutions of Eq. (1.13). Motivated by all the works described above, particularly, by the results in [16,17,25,39], we intend to obtain the existence and concentrating behavior of positive ground state solutions of problem (1.13).

The main result in the present paper can be stated as follows:

Theorem 1.1. *Suppose* (g), (V), (Q), (K), $(K_1)-(K_2)$, *and* $(H_1)-(H_4)$, *then there is a constant* $\epsilon^*>0$, *such that Eq.* (1.1) *possesses a positive ground state solution* $u_\epsilon\in H^1(\mathbb{R}^3)$ *for all* $\epsilon\in(0,\epsilon^*)$. *Furthermore, we obtain the following conclusions:*

(1) If $\mathcal{L}_\epsilon$ *denotes the set of ground state solutions to Eq.* (1.1), *then* $\mathcal{L}_\epsilon$ *is compact in* $H^1(\mathbb{R}^3)$;

(2) $G(u_\epsilon)$ *possesses a maximum point* $\gamma_\epsilon\in\mathbb{R}^3$, *such that*

$$\lim_{\epsilon\to0^+}V(\gamma_\epsilon)=V_0,\ \lim_{\epsilon\to0^+}Q(\gamma_\epsilon)=Q_0,\ \lim_{\epsilon\to0^+}K(\gamma_\epsilon)=K_0,$$

and $\gamma_\epsilon\to x^*\in\Theta\cap\Theta_1\cap\Theta_2$ *as* $\epsilon\to0^+$;

(3) If we set $\tilde{u}_\epsilon(x)=u_\epsilon(\epsilon x+\gamma_\epsilon)$, *going to a subsequence if necessary, we have* $\tilde{u}_\epsilon\to\tilde{u}$ *in* $H^1(\mathbb{R}^3)$ *as* $\epsilon\to0^+$, *and* $\tilde{u}$ *is a ground state solution of*

the equation below

$$\left(a+b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2\mathrm{d}x\right)\left[-\mathrm{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2\right]+V(x^*)u$$
$$=Q(x^*)h(u)+K(x^*)|G(u)|^4G(u)g(u),\ x\in\mathbb{R}^3. \tag{1.14}$$

(4) *There are two positive constants* $\bar{c}$ *and* $\hat{c}$ *independent of* $\epsilon\in(0,\epsilon^*)$, *such that*

$$u_\epsilon(x)\le \bar{c}\exp\left(-\hat{c}\frac{|x-\gamma_\epsilon|}{\epsilon}\right),\ \forall x\in\mathbb{R}^3.$$

Remark 1.2. Since $\Theta\cap\Theta_1\cap\Theta_2\neq\emptyset$, without loss of generality, we are supposing that $0\in\Theta\cap\Theta_1\cap\Theta_2$, i.e., $V(0)=V_0$, $K(0)=K_0$ and $Q(0)=Q_0$, where $V(0)=V_0$ will be used in Lemma 3.2. Obviously, V, Q, and K are bounded below and above by some positive constants, we shall adopt these properties directly without mentioning them anymore.

Remark 1.3. In contrast to [17,25,39], the novelties of Theorem 1.1 are threefold: the restriction on $h\in\mathcal{C}^1$ can be relaxed to $h\in\mathcal{C}^0$; a unified approach to contemplate the singularly perturbed Kirchhoff equation with critical Sobolev exponent is presented and the extension from classic Kirchhoff equation to generalized quasilinear Kirchhoff equation is obtained.

In view of Theorem 1.1, it seems natural to wonder whether Eq. (1.1) always has a ground state solution or not. To this end, we investigate the following nonexistence result.

Theorem 1.4. *Suppose* (g) *and* $(H_1)-(H_4)$. *If in addition*

($\bar{H}$) *the continuous functions* V, K, *and* Q *satisfy* $V(x)\ge V_\infty=V_0$, $K(x)\le K_\infty$ *and* $Q(x)\le Q_\infty$, *respectively, where one of the strictly inequalities holds on a positive measure subset.*

Then, for any $\epsilon>0$, *Eq.* (1.1) *has no ground state solution.*

Remark 1.5. As far as we are concerned, it seems that those results in Theorems 1.1 and 1.4 are the first attempts regarding the existence and concentrating behavior of positive ground state solutions and nonexistence results of ground state solutions for Eq. (1.1) when g is no longer a constant.

The proofs of Theorems 1.1 and 1.4 are relied on variational methods. Compared with the previous works, there are two main difficulties in concluding the proof of Theorem 1.1:

(1) Because of the appearance of the quasilinear terms in the nonlocal term of Kirchhoff type
$$-\left(\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2\mathrm{d}x\right)\mathrm{div}(g^2(u)\nabla u),$$
there exist some interesting calculations introduced in this article to address the issues, see Lemma 4.2 below for example.

(2) Due to the unboundedness of the domain $\mathbb{R}^3$ and the nonlinearity involving the critical growth as well as the the nonlocal term, we are confronted with the lack of compactness of J_ϵ.

To conclude this section, we sketch the main ideas to prove Theorems 1.1 and 1.4.

In the proof of Theorem 1.1, in the spirit of [35], the quasilinear problem (1.1) can be transformed into a semilinear problem (1.12). It is worth pointing out here that the nonlinearity in our problem is not only more general, but also seems more complicated (see Lemma 2.2). Then, a standard method to the semilinear problems is adopted to study the properties of the corresponding variational functional. At last, we make full use of the celebrated concentration–compactness principle developed by Lions [22,23] to show that the (PS) condition holds at the mountain-pass energy level after pulling the energy level down below some critical level by the well-known Brézis–Nirenberg argument [4]. Therefore, with aid of some delicate calculations, we are able to reach the proof of Theorem 1.1. In the proof of Theorem 1.4, we mainly obtain Lemma 5.1 to finish the proof by a contradiction argument.

The outline of this paper is as follows. At first, some preliminary results are presented for Theorem 1.1 in Sect. 2 and the existence of positive ground state solution is obtained in Sect. 3. Then, we deduce the concentration of the ground state solutions in Sect. 4. Finally, Sect. 5 is devoted to the nonexistence of ground state solution.

Notations. From now on in this paper, otherwise mentioned, we utilize the following notations:

- $C, C_1, C_2, \ldots$ denote any positive constant, whose value is not relevant and $\mathbb{R}^+ \triangleq (0, +\infty)$.
- Let $(Z, \|\cdot\|_Z)$ be a Banach space with dual space $(Z^{-1}, \|\cdot\|_{Z^{-1}})$, and Ψ be functional on Z.
- The (PS) sequence at a level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ sequence in short) corresponding to Φ means that $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ in Z^{-1} as $n \to \infty$, where $(x_n) \subset Z$.
- If for any $(\mathrm{PS})_c$ sequence (x_n) in Z, there exists a subsequence (x_{n_k}), such that $x_{n_k} \to x_0$ in Z for some $x_0 \in Z$, then we say that the functional Ψ satisfies the so-called $(\mathrm{PS})_c$ condition.
- $L^p(\mathbb{R}^3)$ $(1 \le p \le +\infty)$ is the usual Lebesgue space with the standard norm $|u|_p$ and $|u|_{L^p(\Omega)}$ means $|u|_p$ restricts to a subset Ω in $\mathbb{R}^3$;
- $D^{1,2}(\mathbb{R}^3) \triangleq \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$ and the best Sobolev constant
$$S \triangleq \{|\nabla u|_2^2 : u \in D^{1,2}(\mathbb{R}^3) \text{ and } |u|_6 = 1\} \tag{1.15}$$
- "$\to$" and "$\rightharpoonup$" denote the strong and weak convergence in the related function space, respectively;
- For any $\rho > 0$ and any $x \in \mathbb{R}^3$, $B_\rho(x) \triangleq \{y \in \mathbb{R}^3 : |y - x| < \rho\}$.

2. Variational setting and preliminaries

In this section, we introduce some preliminary results. Throughout the whole paper, for all fixed $a > 0$, we consider the Hilbert space $H^1(\mathbb{R}^3)$ endowed with the inner product and the norm

$$(u,v)=\int_{\mathbb{R}^3}[a\nabla u\nabla v+V(x)uv]\,\mathrm{d}x \text{ and } \|u\|=\sqrt{(u,u)},\ \forall u,v\in H^1(\mathbb{R}^3).$$

For any $\epsilon>0$, let

$$E_\epsilon\triangleq\left\{u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}V(\epsilon x)u^2\mathrm{d}x<+\infty\right\}$$

be the Sobolev space equipped with the inner product and norm

$$(u,v)_\epsilon=\int_{\mathbb{R}^3}[a\nabla u\nabla v+V(\epsilon x)uv]\,\mathrm{d}x \text{ and } \|u\|_\epsilon=\sqrt{(u,u)_\epsilon}.$$

Clearly, by (V), one knows that $\|\cdot\|$ and $\|\cdot\|_\epsilon$ are uniformly equivalent norms on E_ϵ for each $\epsilon>0$. Moreover, E_ϵ is continuously imbedded into $L^m(\mathbb{R}^3)$ and compactly imbedded into $L^m_{\mathrm{loc}}(\mathbb{R}^3)$ for all $2\le m<6$. Therefore, there exist constants $d_m>0$ independent of $\epsilon>0$, such that

$$|u|_m\le d_m\|u\|_\epsilon \text{ for any } 2\le m\le 6 \text{ and } u\in E_\epsilon. \tag{2.1}$$

Lemma 2.1. *Suppose (g), then the following conclusions hold true:*

(1) Both G and G^{-1} are odd, and for all $s\ge0$, $t\ge0$, there hold
$$G(t)\le g(t)t,\ \ s/g(G^{-1}(s))\le G^{-1}(s)\le s.$$
(2) For all $s\ge0$, $G^{-1}(s)/s$ is non-increasing and
$$\lim_{s\to0}\frac{G^{-1}(s)}{s}=\frac{1}{g(0)}=1 \text{ and } \lim_{s\to\infty}\frac{G^{-1}(s)}{s}=\begin{cases}\frac{1}{g(\infty)}, & \text{if } g \text{ is bounded},\\ 0, & \text{if } g \text{ is unbounded}.\end{cases}$$
(3) For all $s\ge0$, $G^{-1}(s)/\big(s^3g(G^{-1}(s))\big)$ is non-increasing.

(4) For all $s\ge0$, $\big(G^{-1}(s)\big)^2-G^{-1}(s)s/g(G^{-1}(s))$ is increasing.

Proof. Points (1)–(3) are trivial, see [11,35] for example. By Point-(1), for all $s>0$, we have

$$\begin{aligned}&\Big[\big(G^{-1}(s)\big)^2-G^{-1}(s)s/g(G^{-1}(s))\Big]'_s\\&\quad=\frac{G^{-1}(s)\big[g(G^{-1}(s))\big]^2-sg(G^{-1}(s))+sG^{-1}(s)g'(G^{-1}(s))}{\big(g(G^{-1}(s))\big)^3}\ge0.\end{aligned}$$

Hence, Point-(4) concludes. The proof is completed. □

Let us denote

$$f(x,s)\triangleq Q(x)\frac{h(G^{-1}(s))}{g(G^{-1}(s))}+V(x)s-V(x)\frac{G^{-1}(s)}{g(G^{-1}(s))} \tag{2.2}$$

and

$$F(x,s)\triangleq Q(x)H(G^{-1}(s))+\frac{1}{2}V(x)s^2-\frac{1}{2}V(x)|G^{-1}(s)|^2. \tag{2.3}$$

Lemma 2.2. *Suppose (g), (V), (Q), (K), and $(H_1)-(H_4)$, then*

(1) $f(x,s)\ge0$ for all $x\in\mathbb{R}^3$ and $s\ge0$;
(2) $f(x,s)=o(s)$ and $F(x,s)=o(s^2)$ as $s\to0^+$ uniformly in $x\in\mathbb{R}^3$;

(3) $f(x,s)=o(s^5)$ and $F(x,s)=o(s^6)$ as $s\to+\infty$ uniformly in $x\in\mathbb{R}^3$;
(4) $\frac{1}{4}f(x,s)s-F(x,s)+\frac{1}{4}V(x)s^2$ is increasing on $s\geq 0$ for all $x\in\mathbb{R}^3$;
(5) $\left[f(x,s)-V(x)s\right]/s^3$ is increasing on $s\geq 0$ for all $x\in\mathbb{R}^3$.

Proof. Points (1)–(3) are obvious; see [11,35] for example.

Since $h(t)/(g(t)G^3(t))$ is increasing on $(0,\infty)$, for all $0<t_1<t_2<+\infty$, we have

$$H(t_2)-H(t_1)=\int_{t_1}^{t_2}\frac{h(\sigma)}{g(\sigma)G^3(\sigma)}g(\sigma)G^3(\sigma)d\sigma\leq\frac{h(t_2)G^4(t_2)}{4g(t_2)}-\frac{h(t_1)G^4(t_1)}{4g(t_1)},$$

which implies that $h(t)G(t)/(4g(t))-H(t)$ is increasing on $(0,\infty)$. Therefore, by Lemma 2.1-(4), we can conclude that

$$\begin{aligned}&\frac{1}{4}f(x,s)s-F(x,s)+\frac{1}{4}V(x)s^2\\&\quad=Q(x)\left[\frac{h(G^{-1}(s))s}{4g(G^{-1}(s))}-H(G^{-1}(s))\right]+\frac{1}{4}V(x)|G^{-1}(s)|^2\\&\qquad+\frac{1}{4}V(x)\left[|G^{-1}(s)|^2-\frac{G^{-1}(s)s}{g(G^{-1}(s))}\right]\end{aligned}\tag{2.4}$$

is increasing on $s\geq 0$ for all $x\in\mathbb{R}^3$, and then, Point-(4) is obtained. Point-(5) is a direct consequence of Lemma 2.1-(3). The proof is completed. □

By (2.2) and (2.3), we rewrite Eq. (1.13) and its corresponding variational functional (1.9) as

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)\Delta v+V(\epsilon x)v=f(\epsilon x,v)+K(\epsilon x)|v|^4v\tag{2.5}$$

and

$$J_\epsilon(v)=\frac{1}{2}\|v\|_\epsilon^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2-\int_{\mathbb{R}^3}F(\epsilon x,v)\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}K(\epsilon x)|v|^6\mathrm{d}x,\tag{2.6}$$

respectively. In the sequel, one calls v a ground state solution of (1.13) if it is a critical point of J_ϵ and verifies

$$J_\epsilon(v)=\inf_{w\in\mathcal{N}_\epsilon}J_\epsilon(w),$$

where $\mathcal{N}_\epsilon\triangleq\left\{w\in E_\epsilon\backslash\{0\}:J_\epsilon'(w)(w)=0\right\}$.

Lemma 2.3. *Suppose* (g), (V), (Q), (K), *and* $(H_1)-(H_4)$, *then*

(i) *Given a* $v\in E_\epsilon\backslash\{0\}$, *there is a unique* $t_v>0$, *such that* $t_vv\in\mathcal{N}_\epsilon$ *and* $J_\epsilon(t_vv)=\max\limits_{t\geq0}J_\epsilon(tv)$.
(ii) *For any* $v\in\mathcal{N}_\epsilon$, *there exists a* $C>0$ *and* ϵ *independent of* v, *such that* $\|v\|_\epsilon\geq C>0$.
(iii) *Let* (v_n) *satisfy* $J_\epsilon'(v_n)(v_n)\to0$ *and* $\|v_n\|_\epsilon\to a_0>0$, *then, going to a subsequence if necessary, there exists a constant* $t_n>0$, *such that*

$$J_\epsilon'(t_nv_n)(t_nv_n)=0\text{ and }t_n\to1\text{ as }n\to\infty.$$

(iv) *Let* (v_n) *be a* $(\mathrm{PS})_c$ *sequence of* J_ϵ, *then, going to a subsequence if necessary, there exists a* $v\in E_\epsilon$, *such that* $v_n\rightharpoonup v$ *and* $J_\epsilon'(v)=0$.

Proof. (i) Taking Lemma 2.2-(5) into account, then the proof is standard and we omit it here.

(ii) With aid of Lemma 2.2-(2) and (3), for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$, such that

$$|f(x,t)| \le \varepsilon|t| + C_\varepsilon|t|^{p-1}, \ \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{2.7}$$

for some $2 \le p \le 6$. Analogously, one has that

$$|F(x,t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^p, \ \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.8}$$

Due to the definition of $\mathcal{N}_\epsilon$, we choose $\varepsilon = \frac{1}{2d_2^2} > 0$ to find that

$$\|v\|_\epsilon^2 \le \frac{1}{2}\|v\|_\epsilon^2 + C_1\|v\|_\epsilon^p + C_2\|v\|_\epsilon^6.$$

Since $p > 2$, we can obtain the desired result immediately.

(iii) First of all, we claim that there exists a constant $\tilde{a}_0 \in (0, a_0)$, such that $|\nabla v_n|_2^2 \to \tilde{a}_0 > 0$. In fact, suppose it by contradiction that $|\nabla v_n|_2^2 \to 0$, then we are derived from (1.15) that

$$\int_{\mathbb{R}^3} f(\epsilon x, v_n)v_n \mathrm{d}x \to 0 \text{ and } \int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x \to 0$$

which together with $J'_\epsilon(v_n)(v_n) \to 0$ gives that $\|v_n\|_\epsilon \to 0$, a contradiction. Therefore, the claim is true. By Point-(i) above, there exists a constant $t_n > 0$, such that $t_n v_n \in \mathcal{N}_\epsilon$, that is,

$$\begin{aligned} t_n^2\|v_n\|_\epsilon^2 &+ bt_n^4\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x\right)^2 \\ &- \int_{\mathbb{R}^3} f(\epsilon x, t_n v_n)t_n v_n \mathrm{d}x - t_n^6\int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x = 0. \end{aligned}$$

We claim that (t_n) is uniformly bounded. Otherwise, we could assume $t_n \to +\infty$. It follows from $J'_\epsilon(v_n)(v_n) \to 0$ and $J'_\epsilon(t_n v_n)(t_n v_n) = 0$ as well as Lemma 2.2-(5) that:

$$\begin{aligned} o_n(1) &= \left(1 - \frac{1}{t_n^2}\right) a\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x + (t_n^2 - 1)\int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} \left[\frac{f(\epsilon x, t_n v_n) - V(\epsilon x)t_n v_n}{(t_n v_n)^3} - \frac{f(\epsilon x, v_n) - V(\epsilon x)v_n}{v_n^3}\right] v_n^4 \mathrm{d}x \\ &\ge \left(1 - \frac{1}{t_n^2}\right) a\int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x = a\tilde{a}_0 + o_n(1), \end{aligned}$$

a contradiction. Next, we verify that (t_n) is bounded below by some positive constant. Actually, we adopt (2.7) to see that

$$t_n^2\|v_n\|_\epsilon^2 \le \frac{1}{2}t_n^2\|v_n\|_\epsilon^2 + C_1 t_n^p\|v_n\|_\epsilon^p + C_2 t_n^6\|v_n\|_\epsilon^6.$$

Recalling point-(ii) above, there is a $T_0 > 0$ is independent on ϵ and n such that $t_n \ge T_0$. In summary, there is a $t_0 > 0$, such that $t_n \to t_0$

along a subsequence. Finally, we shall conclude that $t_0 \equiv 1$. On the one hand, if $0 < t_0 < 1$, using Lemma 2.2-(5) again

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^3} \left[\frac{f(\epsilon x, t_0 v_n) - V(\epsilon x) t_0 v_n}{(t_0 v_n)^3} - \frac{f(\epsilon x, v_n) - V(\epsilon x) v_n}{v_n^3} \right] v_n^4 \mathrm{d}x \\ &= \left(\frac{1}{t_0^2} - 1 \right) a \int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x + (1 - t_0^2) \int_{\mathbb{R}^3} K(\epsilon x) |v_n|^6 \mathrm{d}x + o_n(1) \\ &\geq \left(\frac{1}{t_0^2} - 1 \right) a \int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x + o_n(1) = \left(\frac{1}{t_0^2} - 1 \right) a \tilde{a}_0 + o_n(1), \end{aligned}$$

a contradiction. Similarly, we can derive a contradiction when $t_0 > 1$. Therefore, we obtain that $t_0 \equiv 1$.

(iv). In view of (2.4), since (v_n) is a $(\mathrm{PS})_c$ sequence of J_ϵ, it holds that

$$c + \|v_n\|_\epsilon + o_n(1) \geq \frac{a}{4} \int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} V(\epsilon x) |G^{-1}(v_n)|^2 \mathrm{d}x. \tag{2.9}$$

In light of the function g is increasing, then we depend on Lemma 2.1-(1) to derive

$$\begin{aligned} \int_{\mathbb{R}^3} V(\epsilon x) v_n^2 \mathrm{d}x &= \int_{|G^{-1}(v_n)|>1} V(\epsilon x) v_n^2 \mathrm{d}x + \int_{|G^{-1}(v_n)|\leq 1} V(\epsilon x) v_n^2 \mathrm{d}x \\ &\leq C \int_{\mathbb{R}^3} v_n^6 \mathrm{d}x + g^2(1) \int_{|G^{-1}(v_n)|\leq 1} V(\epsilon x) \frac{v_n^2}{g^2(G^{-1}(v_n))} \mathrm{d}x \\ &\leq C \int_{\mathbb{R}^3} v_n^6 \mathrm{d}x + g^2(1) \int_{\mathbb{R}^3} V(\epsilon x) |G^{-1}(v_n)|^2 \mathrm{d}x. \end{aligned} \tag{2.10}$$

Combining (1.15) and (2.9)–(2.10), one sees that (v_n) is uniformly bounded in E_ϵ. Up to a subsequence if necessary, there is a $v \in E_\epsilon$, such that $v_n \rightharpoonup v$ in E_ϵ. Moreover, the Fatou's lemma gives us that

$$|\nabla v|_2^2 \leq \liminf_{n\to\infty} |\nabla v_n|_2^2 \triangleq A^2.$$

We claim that $|\nabla v|_2^2 = A^2$. If not, we would suppose that $|\nabla v|_2^2 < A^2$. Then, for all $\psi \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned} 0 &= \liminf_{n\to\infty} J_\epsilon'(v_n)(\psi) \\ &= \int_{\mathbb{R}^3} \left[a \nabla v \nabla \psi + V(\epsilon x) v \psi \right] \mathrm{d}x + b A^2 \int_{\mathbb{R}^3} \nabla v \nabla \psi \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \left[f(\epsilon x, v) + K(\epsilon x) |v|^4 v \right] \psi \mathrm{d}x. \end{aligned} \tag{2.11}$$

As a consequence, we conclude that $J_\epsilon'(v)(v) < 0$ by taking $\psi = v$ in the above formula. According to (2.7), we will easily show that $J_\epsilon'(tv)(tv) > 0$ for some sufficiently small $t > 0$. Therefore, there is a $\hat{t} \in (0, 1)$, such that $J_\epsilon'(\hat{t}v)(\hat{t}v) = 0$ which is $\hat{t}v \in \mathcal{N}_\epsilon$. Therefore, Lemma 2.2-(4) indicates that

$$\begin{aligned} c &\leq J_\epsilon(\hat{t}v) = J_\epsilon(\hat{t}v) - \frac{1}{4} J_\epsilon'(\hat{t}v)(\hat{t}v) \\ &= \frac{a}{4} \hat{t}^2 \int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(\epsilon x, \hat{t}v) \hat{t}v - F(\epsilon x, \hat{t}v) + \frac{1}{4} V(\epsilon x) (\hat{t}v)^2 \right] \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\frac{\hat{t}^6}{12}\int_{\mathbb{R}^3}K(\epsilon x)|v|^6\mathrm{d}x\\
&<\frac{a}{4}\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left[\frac{1}{4}f(\epsilon x,v)v-F(\epsilon x,v)+\frac{1}{4}V(\epsilon x)v^2\right]\mathrm{d}x\\
&+\frac{1}{12}\int_{\mathbb{R}^3}K(\epsilon x)|v|^6\mathrm{d}x\\
&\le\liminf_{n\to\infty}\left[J_\epsilon(v_n)-\frac{1}{4}J'_\epsilon(v_n)(v_n)\right]=\lim_{n\to\infty}J_\epsilon(v_n)=c,
\end{aligned}
$$

which is an absurd. Hence, $|\nabla v|_2^2=A^2$ holds true. Inserting to into (2.11), we have that $J'_\epsilon(v)(\psi)=0$ which yields that $J'_\epsilon(v)=0$, since ψ is arbitrary. The proof is completed.

□

Lemma 2.4. *Suppose* (g), (V), (Q), (K), *and* $(H_1)-(H_4)$, *then the variational functional* J_ϵ *satisfies the mountain-pass geometry around* $0\in E_\epsilon$,

(i) there are constants $\rho,\varrho>0$ *independent of* ϵ, *such that* $J_\epsilon(v)\ge\rho>0$ *when* $\|v\|_\epsilon=\varrho>0$;

(ii) there exists $e\in E_\epsilon$ *independent of* ϵ *with* $\|e\|_\epsilon>\rho$, *such that* $J_\epsilon(e)<0$.

Proof. (i). It is essentially similar to the proof of Lemma 2.3-(ii), we omit it here.

(ii). In view of (H_4), for any $v\in E_\epsilon$, one easily finds that

$$
\begin{aligned}
\frac{J_\epsilon(tv)}{t^4}&=\frac{1}{2t^2}\|v\|_\epsilon^2+\frac{b}{4}\left(\int_{\mathbb{R}^2}|\nabla v|^2\mathrm{d}x\right)^2-\int_{\mathbb{R}^2}\frac{F(\epsilon x,tv)}{(tv)^4}v^4\mathrm{d}x\\
&\quad-\frac{t^2}{6}\int_{\mathbb{R}^3}K(\epsilon x)|v|^6\mathrm{d}x\to-\infty
\end{aligned}
$$

as $t\to+\infty$ uniformly in $\epsilon>0$. Therefore, we choose $e=t_0v$ with a sufficiently large $t_0>0$. The proof is completed. □

With Lemma 2.4 in hands, we are able to apply [37, Theorem 2.10] to look for a (PS) sequence of J_ϵ at the level

$$
c_\epsilon\triangleq\inf_{\eta\in\Gamma}\max_{t\in[0,1]}J_\epsilon(\eta(t))>0,\tag{2.12}
$$

where the set of paths is defined as

$$
\Gamma\triangleq\big\{\eta\in C([0,1],E_\epsilon):\eta(0)=0,J_\epsilon(\eta(1))<0\big\}.
$$

It is similar to the idea used in [37, Lemma 4.1] that

$$
c_\epsilon=\inf_{v\in\mathcal{N}_\epsilon}J_\epsilon(v)=\inf_{v\in E_\epsilon\setminus\{0\}}\max_{t\ge0}J_\epsilon(tv).\tag{2.13}
$$

The following concentration–compactness principle due to Lions [22,23] shall play a crucial role in verifying the (PS) condition associated with J_ϵ in this paper.

Lemma 2.5. (see [22,23]) *Let* (ρ_n) *be a sequence of nonnegative functions satisfying* $|\rho_n|_1=\lambda$ *and* $\lambda>0$ *is fixed, then there exists a subsequence, still denoted by* (ρ_n), *satisfying one of the following three possibilities:*

(i) (Vanishing) *for any fixed* $R > 0$*, there holds*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n(x)\mathrm{d}x = 0;$$

(ii) (Compactness) *there exists* $(y_n) \subset \mathbb{R}^N$*, such that for any* $\varepsilon > 0$*, there exists* $R > 0$ *satisfying*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} \rho_n(x)\mathrm{d}x \geq \lambda - \varepsilon;$$

(iii) (Dichotomy) *there exist* $\alpha \in (0, \lambda)$ *and* $(y_n) \subset \mathbb{R}^N$*, such that for any* $\varepsilon > 0$*, there exists* $R > 0$*, for all* $r_1 \geq R$ *and* $r_2 \geq R$*, it holds*

$$\limsup_{n\to\infty} \left(\left|\alpha - \int_{B_r(y_n)} \rho_n(x)\mathrm{d}x\right| + \left|(\lambda - \alpha) - \int_{B^c_{r_2}(y_n)} \rho_n(x)\mathrm{d}x\right| \right) < \varepsilon.$$

To study Eq. (1.13) well, we need to introduce the following "limit problem":

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)\Delta v + V_\infty v = f(\infty, v) + K_\infty |v|^4 v, \ x \in \mathbb{R}^3, \quad (2.14)$$

where

$$f(\infty, s) \triangleq Q_\infty \frac{h(G^{-1}(s))}{g(G^{-1}(s))} + V_\infty s - V_\infty \frac{G^{-1}(s)}{g(G^{-1}(s))}.$$

As described above, to find a weak solution to Eq. (2.14), it suffices to seek for a critical point of the variational functional $J_\infty : E_\epsilon \to \mathbb{R}$ given by

$$J_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla v|^2 + V_\infty v^2\right)\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} F(\infty, v)\mathrm{d}x - \frac{K_\infty}{6}\int_{\mathbb{R}^3} |v|^6 \mathrm{d}x,$$

where and in the sequel $F(\infty, t) = \int_0^t f(\infty, s)ds$. Let us define

$$m_\infty \triangleq \inf_{v\in\mathcal{N}_\infty} J_\infty(v), \quad (2.15)$$

where

$$\mathcal{N}_\infty \triangleq \left\{v \in H^1(\mathbb{R}^3)\backslash\{0\} : J'_\infty(v)(v) = 0\right\}.$$

In what follows, we are going to show that the variational functional J_ϵ satisfies the (PS) condition at some particular level.

Lemma 2.6. *Suppose* (g)*,* (V)*,* (Q)*,* (K) *and* $(H_1) - (H_4)$*, then* J_ϵ *satisfies the* $(PS)_{c_\epsilon}$ *condition with* $c_\epsilon < \min\{m_\infty, c^*\}$*, where*

$$c^* \triangleq \frac{abS^3}{4K_0} + \frac{\left(b^2 S^4 + 4K_0 aS\right)^{3/2}}{24K_0^2} + \frac{b^3 S^6}{24K_0^2},$$

$K_0 > 0$ *and* $S > 0$ *are given by* (K_2) *and* (1.15)*, respectively.*

Proof. Let (v_n) be a (PS) sequence of J_ϵ at the level c_ϵ, that is

$$J_\epsilon(v_n) \to c_\epsilon \text{ and } J'_\epsilon(v_n) \to 0. \tag{2.16}$$

In view of the proof of Lemma 2.3-(iv), we know that $\{v_n\}$ is bounded in E_ϵ. Setting

$$\rho_n(x) \triangleq \frac{a}{4}|\nabla v_n|^2 + \left[\frac{1}{4}f(\epsilon x, v_n)v_n - F(\epsilon x, v_n) + \frac{1}{4}V(\epsilon x)v_n^2\right] + \frac{1}{12}K(\epsilon x)|v_n|^6.$$

Clearly, (ρ_n) is bounded in $L^1(\mathbb{R}^3)$ and we can assume, choosing a subsequence if necessary,

$$\Phi(v_n) = \int_{\mathbb{R}^3} \rho_n(x)dx \to l \text{ as } n \to \infty.$$

Obviously, we have $l = c_\epsilon > 0$ and (ρ_n) satisfies the assumptions in Lemma 2.5. Next, if Vanishing or Dichotomy does not occur, we can get the compactness of (ρ_n).

Vanishing does not occur.

Indeed, if (ρ_n) vanishes, then there exists $R > 0$, such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} v_n^2 \mathrm{d}x = 0.$$

By means of [37, Lemma 1.21], one concludes $v_n \to 0$ in $L^m(\mathbb{R}^3)$ for $2 < m < 6$. It follows from (2.7) and (2.8) that:

$$\int_{\mathbb{R}^3} F(\epsilon x, v_n)\mathrm{d}x \to 0 \text{ and } \int_{\mathbb{R}^3} f(\epsilon x, v_n)v_n \mathrm{d}x \to 0,$$

which together with (2.16) indicate that

$$J_\epsilon(v_n) = \frac{1}{2}\|v_n\|_\epsilon^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x\right)^2 - \frac{1}{6}\int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x = c_\epsilon + o_n(1)$$

and

$$J'_\epsilon(v_n)(v_n) = \|v_n\|_\epsilon^2 + b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x = o_n(1).$$

Without loss of generality, we could assume that

$$\lim_{n\to\infty} \|v_n\|_\epsilon^2 = l_\epsilon^1 \text{ and } \lim_{n\to\infty} b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x\right)^2 = l_\epsilon^2$$

form where one has that

$$c_\epsilon = \frac{1}{3}l_\epsilon^1 + \frac{1}{12}l_\epsilon^2 \text{ and } \lim_{n\to\infty} \int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x = l_\epsilon^1 + l_\epsilon^2.$$

It follows from (1.15) that:

$$\begin{aligned}\int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x &\le K_0 \int_{\mathbb{R}^3} |v_n|^6 \mathrm{d}x \le K_0 S^{-3}\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x\right)^3 \\ &\le K_0 a^{-3} S^{-3} \|v_n\|_\epsilon^6\end{aligned}$$

and

$$\int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x \le K_0 b^{-\frac{3}{2}} S^{-3} \left[b \left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x \right)^2 \right]^{\frac{3}{2}}$$

which imply that

$$l_\epsilon^1 \ge K_0^{-\frac{1}{3}} aS \left(l_\epsilon^1 + l_\epsilon^2 \right)^{\frac{1}{3}} \text{ and } l_\epsilon^2 \ge K_0^{-\frac{1}{3}} bS^2 \left(l_\epsilon^1 + l_\epsilon^2 \right)^{\frac{2}{3}}.$$

Consequently, we have that that

$$(l_\epsilon^1 + l_\epsilon^2)^{\frac{1}{3}} \ge \frac{bS^2 + \sqrt{b^2 S^4 + 4K_0 aS}}{2K_0^{\frac{2}{3}}}.$$

Therefore we derive

$$c_\epsilon = \frac{1}{3} l_\epsilon^1 + \frac{1}{12} l_\epsilon^2 \ge c^*,$$

a contradiction. Therefore, Vanishing does not occur.

Dichotomy does not occur.

Arguing it indirectly, we could suppose that there exist $\alpha \in (0, l)$ and $(y_n) \subset \mathbb{R}^3$, such that for every $\varepsilon_n \to 0^+$, one can choose $(R_n) \subset \mathbb{R}^+$ with $R_n \to +\infty$ to satisfy

$$\limsup_{n\to\infty} \left(\left| \alpha - \int_{B_{R_n}(y_n)} \rho_n(x) \mathrm{d}x \right| + \left| (l-\alpha) - \int_{B^c_{2R_n}(y_n)} \rho_n(x) \mathrm{d}x \right| \right) < \varepsilon_n. \tag{2.17}$$

Let $\xi(s) : \mathbb{R}^+ \to [0,1]$ be a cut-off function satisfying $\xi(s) \equiv 1$ for $s \le 1$, $\xi(s) \equiv 0$ for $s \ge 2$ and $|\xi'(s)| \le 2$ for any $s \in \mathbb{R}^+$. Setting

$$v_n^1(x) \triangleq \xi \left(\frac{|x-y_n|}{R_n} \right) v_n(x) \text{ and } v_n^2(x) \triangleq \left[1 - \xi \left(\frac{|x-y_n|}{R_n} \right) \right] v_n(x),$$

then by (2.17) and the definitions of v_n^1 and v_n^2, we can see that

$$\liminf_{n\to\infty} \Phi(v_n^1) \ge \alpha \text{ and } \liminf_{n\to\infty} \Phi(v_n^2) \ge l - \alpha. \tag{2.18}$$

Denoting $\Omega_n = B_{2R_n}(y_n) \backslash B_{R_n}(y_n)$, then by (2.17), one has

$$\int_{\Omega_n} \rho_n \mathrm{d}x = \int_{\mathbb{R}^3} \rho_n \mathrm{d}x - \int_{B_{R_n}(y_n)} \rho_n \mathrm{d}x - \int_{B^c_{2R_n}(y_n)} \rho_n \mathrm{d}x \to 0,$$

which implies that

$$\int_{\Omega_n} |\nabla v_n|^2 \mathrm{d}x = o_n(1) \text{ and } \int_{\Omega_n} |v_n|^6 \mathrm{d}x = o_n(1). \tag{2.19}$$

By the definition of ξ, it is simple to verify that

$$\begin{cases} \int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} |\nabla v_n^1|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla v_n^2|^2 \mathrm{d}x + o_n(1), \\ \int_{\mathbb{R}^3} V(\epsilon x)|v_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} V(\epsilon x)\left|v_n^1\right|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V(\epsilon x)\left|v_n^2\right|^2 \mathrm{d}x + o_n(1), \\ \int_{\mathbb{R}^3} F(\epsilon x, v_n)\mathrm{d}x = \int_{\mathbb{R}^3} F(\epsilon x, v_n^1)\mathrm{d}x + \int_{\mathbb{R}^3} F(\epsilon x, v_n^2)\mathrm{d}x + o_n(1), \\ \int_{\mathbb{R}^3} f(\epsilon x, v_n)v_n \mathrm{d}x = \int_{\mathbb{R}^3} f(\epsilon x, v_n^1)v_n^1 \mathrm{d}x + \int_{\mathbb{R}^3} f(\epsilon x, v_n^2)v_n^2 \mathrm{d}x + o_n(1), \\ \int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6 \mathrm{d}x = \int_{\mathbb{R}^3} K(\epsilon x)\left|v_n^1\right|^6 \mathrm{d}x + \int_{\mathbb{R}^3} K(\epsilon x)\left|v_n^2\right|^6 \mathrm{d}x + o_n(1), \end{cases} \tag{2.20}$$

and

$$\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x\right)^2 \geq \left(\int_{\mathbb{R}^3} |\nabla v_n^1|^2 \mathrm{d}x\right)^2 + \left(\int_{\mathbb{R}^3} |\nabla v_n^2|^2 \mathrm{d}x\right)^2 + o_n(1). \tag{2.21}$$

Hence, we are derived from (2.20) and (2.21) that $\Phi(v_n) \geq \Phi(v_n^1) + \Phi(v_n^2) + o_n(1)$ which together with (2.18) implies that

$$l = \lim_{n\to\infty} \Phi(v_n) \geq \liminf_{n\to\infty} \Phi(v_n^1) + \liminf_{n\to\infty} \Phi(v_n^2) \geq \alpha + l - \alpha = l.$$

Furthermore, we have that

$$\lim_{n\to\infty} \Phi(v_n^1) = \alpha \text{ and } \lim_{n\to\infty} \Phi(v_n^2) = l - \alpha. \tag{2.22}$$

It follows from (2.16) and (2.20)–(2.21) that:

$$0 = J'_\epsilon(v_n)(v_n) + o_n(1) \geq J'_\epsilon(v_n^1)(v_n^1) + J'_\epsilon(v_n^2)(v_n^2) + o_n(1). \tag{2.23}$$

Now, we distinguish the following two cases.

Case 1. Up to a subsequence if necessary, we assume that either $J'_\epsilon(v_n^1)(v_n^1) \leq 0$ or $J'_\epsilon(v_n^2)(v_n^2) \leq 0$. Without loss of generality, we suppose that $J'_\epsilon(v_n^1)(v_n^1) \leq 0$, that is,

$$\|v_n^1\|_\epsilon^2 + b\left(\int_{\mathbb{R}^3} |\nabla v_n^1|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} f(\epsilon x, v_n^1)v_n^1 \mathrm{d}x - \int_{\mathbb{R}^3} K(\epsilon x)|v_n^1|^6 \mathrm{d}x \leq 0.$$

By Lemma 2.3-(i), there exists $t_n \in (0, 1]$, such that $t_n v_n^1 \in \mathcal{N}_\epsilon$, and so, Lemma 2.2-(6) reveals that

$$\begin{aligned} c_\epsilon \leq J_\epsilon(t_n v_n^1) &= J_\epsilon(t_n v_n^1) - \frac{1}{4} J'_\epsilon(t_n v_n^1)(t_n v_n^1) \\ &\leq \frac{a}{4}\int_{\mathbb{R}^3} |\nabla v_n^1|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(\epsilon x, v_n^1)v_n^1 - F(\epsilon x, v_n^1) + \frac{1}{4} V(\epsilon x)|v_n^1|^2\right] \mathrm{d}x \\ &\quad + \frac{1}{12}\int_{\mathbb{R}^3} K(\epsilon x)|v_n^1|^6 \mathrm{d}x \\ &= \Phi(v_n^1) \to \alpha < c_\epsilon, \end{aligned}$$

a contradiction, where we have used Lemma 2.2-(4).

Case 2. Going to a subsequence if necessary, we assume that both $J'_\epsilon(v_n^1)(v_n^1) > 0$ and $J'_\epsilon(v_n^2)(v_n^2) > 0$. Thereby, taking (2.23) into account, we derive

$J'_\epsilon(v_n^1)(v_n^1) \to 0$ and $J'_\epsilon(v_n^2)(v_n^2) \to 0$ which together with (2.20)-(2.21) give us that

$$J_\epsilon(v_n) \geq J_\epsilon(v_n^1) + J_\epsilon(v_n^2) + o_n(1). \tag{2.24}$$

If (y_n) is bounded, there exists $R' > 0$, such that $|y_n| < R'$. By the assumptions (V), (Q) and (K), for any $\sigma > 0$, there exists $R_0 > 0$, such that

$$V(\epsilon x) - V_\infty > -\sigma,\ |K(\epsilon x) - K_\infty| < \sigma,\ |Q(\epsilon x) - Q_\infty| < \sigma \text{ whenever } |x| \geq \frac{R_0}{\epsilon}.$$

Since $R_n \to +\infty$, we can find that $B_{R_0/\epsilon}(0) \subset B_{R_n - R'}(0) \subset B_{R_n}(y_n)$ for sufficiently large n, that is, $B^c_{R_n}(y_n) \subset B^c_{R_n - R'}(0) \subset B^c_{R_0/\epsilon}(0)$ for sufficiently large n, and so, the definition of v_n^2 indicates that

$$\begin{aligned} \int_{\mathbb{R}^3} [V(\epsilon x) - V_\infty] \left|G^{-1}(v_n^2)\right|^2 \mathrm{d}x &= \int_{B^c_{R_n}(y_n)} [V(\epsilon x) - V_\infty] \left|G^{-1}(v_n^2)\right|^2 \mathrm{d}x \\ &\geq -\sigma \int_{\mathbb{R}^3} \left|G^{-1}(v_n^2)\right|^2 \mathrm{d}x. \end{aligned}$$

Since $\sigma > 0$ is arbitrary and (v_n^2) is bounded in $L^2(\mathbb{R}^3)$, then one has

$$\int_{\mathbb{R}^3} [V(\epsilon x) - V_\infty] \left|G^{-1}(v_n^2)\right|^2 \mathrm{d}x \geq o_n(1).$$

Similarly, we have

$$\int_{\mathbb{R}^3} [K(\epsilon x) - K_\infty] |v_n^2|^6 \mathrm{d}x = o_n(1)$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3} [Q(\epsilon x) - Q_\infty] H(G^{-1}(v_n^2)) \mathrm{d}x = o_n(1) \text{ and} \\ &\int_{\mathbb{R}^3} [Q(\epsilon x) - Q_\infty] \frac{h(G^{-1}(v_n^2)) v_n^2}{g(G^{-1}(v_n^2))} \mathrm{d}x = o_n(1). \end{aligned}$$

Thus, we are able to deduce that

$$J_\epsilon(v_n^2) \geq J_\infty(v_n^2) + o_n(1) \text{ and } o_n(1) = J'_\epsilon(v_n^2)(v_n^2) \geq J'_\infty(v_n^2)(v_n^2) + o_n(1). \tag{2.25}$$

On the one hand, if $J'_\infty(v_n^2)(v_n^2) \leq 0$, then Lemma 2.3-(1) permits us to look for a $t_n^\infty \in (0, 1]$, such that $t_n^\infty v_n^2 \in \mathcal{N}_\infty$. In this scenario, taking $J'_\epsilon(v_n^1)(v_n^1) \to 0$ and $J'_\epsilon(v_n^2)(v_n^2) \to 0$ into account, we adopt (2.24) and (2.25) to reach

$$\begin{aligned} J_\epsilon(v_n) &\geq J_\epsilon(v_n^1) + J_\epsilon(v_n^2) + o_n(1) \\ &\geq J_\epsilon(v_n^2) + o_n(1) \\ &= J_\epsilon(v_n^2) - \frac{1}{4} J'_\epsilon(v_n^2)(v_n^2) + o_n(1) \\ &\geq J_\infty(v_n^2) - \frac{1}{4} J'_\infty(v_n^2)(v_n^2) + o_n(1) \\ &\geq J_\infty(t_n^\infty v_n^2) - \frac{1}{4} J'_\infty(t_n^\infty v_n^2)(t_n^\infty v_n^2) + o_n(1) \\ &= J_\infty(t_n^\infty v_n^2) + o_n(1) \geq m_\infty + o_n(1) \end{aligned}$$

which contradicts with $c_\epsilon < m_\infty$. On the one hand, if $J'_\infty(v_n^2)(v_n^2) > 0$, then we have $J'_\infty(v_n^2)(v_n^2) = o_n(1)$ by (2.25). Owing to (2.22), we can suppose that $\|v_n^2\| \to a_1 > 0$ and $\|v_n^1\|_\epsilon \to a_2 > 0$. With aid of $J'_\epsilon(v_n^1)(v_n^1) = o_n(1)$, due to Lemma 2.3-(iii), there exist two sequences $(t_n) \subset \mathbb{R}^+$ and $(s_n) \subset \mathbb{R}^+$ satisfying $t_n \to 1$ and $s_n \to 1$ as $n \to \infty$, respectively, such that $t_n v_n^2 \in \mathcal{N}_\infty$ and $s_n v_n^1 \in \mathcal{N}_\epsilon$. Hence, by using (2.25) again,

$$J_\epsilon(v_n^2) \geq J_\infty(v_n^2) + o_n(1) = J_\infty(t_n v_n^2) + o_n(1) \geq m_\infty + o_n(1)$$

and

$$J_\epsilon(v_n^1) = J_\epsilon(s_n v_n^1) + o_n(1) \geq c_\epsilon + o_n(1).$$

By (2.24) and the above two formulas, we have that $c_\epsilon \geq m_\infty + c_\epsilon > m_\infty$, a contradiction.

If $(y_n) \subset \mathbb{R}^3$ is unbounded, without loss of generality, we can choose R_n to satisfy $|y_n| \geq 3R_n$ for sufficiently large n and so $B_{2R_n}(y_n) \subset B^c_{R_n}(0) \subset B^c_{R_0/\epsilon}(0)$. Some similar calculations above provide us that

$$\int_{\mathbb{R}^3} [V(\epsilon x) - V_\infty] \left|G^{-1}(v_n^1)\right|^2 \mathrm{d}x \geq o_n(1) \text{ and}$$
$$\int_{\mathbb{R}^3} [K(\epsilon x) - K_\infty] |v_n^1|^6 \mathrm{d}x = o_n(1)$$

and

$$\int_{\mathbb{R}^3} [Q(\epsilon x) - Q_\infty] H(G^{-1}(v_n^1)) \mathrm{d}x = o_n(1) \text{ and}$$
$$\int_{\mathbb{R}^3} [Q(\epsilon x) - Q_\infty] \frac{h(G^{-1}(v_n^1)) v_n^1}{g(G^{-1}(v_n^1)} \mathrm{d}x = o_n(1).$$

Repeating some very similar calculations in (2.25) for (v_n^1), there would be also a contradiction. In a word, Dichotomy can never occur.

Hence, the sequence (ρ_n) is compact, that is, there exists $(y_n) \subset \mathbb{R}^3$, such that for any $\varepsilon > 0$, there is $R > 0$ satisfying

$$\int_{B_R^c(y_n)} \rho_n(x) \mathrm{d}x < \varepsilon. \tag{2.26}$$

We claim that (y_n) is bounded. Otherwise, we could follow the idea of showing (2.25) to get:

$$J_\epsilon(v_n) \geq J_\infty(v_n) + o(1) \text{ and } o_n(1) = J'_\epsilon(v_n)(v_n) \geq J'_\infty(v_n)(v_n) + o_n(1).$$

Either $J'_\infty(v_n)(v_n) \leq 0$ or $J'_\infty(v_n)(v_n) > 0$, we would arrive at a contradiction. Thus, (y_n) is bounded. Since (v_n) is bounded in E_ϵ, passing to a subsequence if necessary, there exists $v \in E_\epsilon$, such that $v_n \rightharpoonup v$ in E_ϵ, $v_n \to v$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ for $1 \leq r < 6$ and $v_n \to v$ a.e. in $\mathbb{R}^3$. Recalling (2.26) and (y_n) is bounded, we have that $v_n \to v$ in $L^r(\mathbb{R}^3)$ for $2 < r < 6$.

With $v_n \to v$ in $L^r(\mathbb{R}^3)$ for $2 < r < 6$ in hands, we define $\widetilde{v}_n = v_n - v$, then

$$\begin{cases} \|v_n\|_\epsilon^2 = \|\widetilde{v}_n\|_\epsilon^2 + \|v\|_\epsilon^2 + o_n(1) \\ \displaystyle\int_{\mathbb{R}^3} F(\epsilon x, v_n)\mathrm{d}x = \int_{\mathbb{R}^3} F(\epsilon x, v)\mathrm{d}x + o_n(1), \\ \displaystyle\int_{\mathbb{R}^3} f(\epsilon x, v_n)v_n\mathrm{d}x = \int_{\mathbb{R}^3} f(\epsilon x, v)v\mathrm{d}x + o_n(1), \\ \displaystyle\int_{\mathbb{R}^3} K(\epsilon x)|v_n|^6\mathrm{d}x = \int_{\mathbb{R}^3} K(\epsilon x)|\widetilde{v}_n|^6 dx + \int_{\mathbb{R}^3} K(\epsilon x)|v|^6\mathrm{d}x + o_n(1), \end{cases} \tag{2.27}$$

In view of Lemma 2.3-(iv), we have

$$J_\epsilon'(v) = 0 \text{ and } J_\epsilon(v) \geq 0. \tag{2.28}$$

By means of (2.27), we can deduce that

$$\begin{aligned} c_\epsilon - J_\epsilon(v) = \frac{1}{2}\|\widetilde{v}_n\|_\epsilon^2 + \frac{b}{4}\left[\left(\int_{\mathbb{R}^3} |\nabla\widetilde{v}_n|^2\mathrm{d}x\right)^2 + 2\int_{\mathbb{R}^3} |\nabla\widetilde{v}_n|^2\mathrm{d}x \int_{\mathbb{R}^3} |\nabla v|^2\mathrm{d}x\right] \\ - \frac{1}{6}\int_{\mathbb{R}^3} K(\epsilon x)|\widetilde{v}_n|^6\mathrm{d}x, \end{aligned}$$

and by $o_n(1) = J_\epsilon'(v_n)(v_n) - \langle J_\epsilon'(v)(v)$ together with (2.28),

$$\begin{aligned} \|\widetilde{v}_n\|_\epsilon^2 + b\left[\left(\int_{\mathbb{R}^3} |\nabla\widetilde{v}_n|^2\mathrm{d}x\right)^2 + 2\int_{\mathbb{R}^3} |\nabla\widetilde{v}_n|^2\mathrm{d}x \int_{\mathbb{R}^3} |\nabla v|^2\mathrm{d}x\right] \\ - \frac{1}{6}\int_{\mathbb{R}^3} K(\epsilon x)|\widetilde{v}_n|^6\mathrm{d}x = o_n(1). \end{aligned}$$

Up to a subsequence if necessary, we may assume that

$$\begin{aligned} &\lim_{n\to\infty} \|\widetilde{v}_n\|_\epsilon^2 = \widetilde{l_\epsilon^1} \text{ and} \\ &\lim_{n\to\infty} b\left[\left(\int_{\mathbb{R}^3} |\nabla\widetilde{v}_n|^2\mathrm{d}x\right)^2 + 2\int_{\mathbb{R}^3} |\nabla\widetilde{v}_n|^2\mathrm{d}x \int_{\mathbb{R}^3} |\nabla v|^2\mathrm{d}x\right] = \widetilde{l_\epsilon^2}. \end{aligned}$$

If $\widetilde{l_\epsilon^1} > 0$, we can get a contradiction $c_\epsilon \geq c_\epsilon - J_\epsilon(v) \geq c^*$ as the proof of "Vanishing does not occur". Hence, $\widetilde{l_\epsilon^1} = 0$. The proof is completed. $\square$

To apply Lemma 2.6, according to the Brézis–Nirenberg argument in [4], we shall pull the mountain-pass energy c_ϵ down the particular level. For this purpose, we introduce a well-known fact that the minimization problem (1.15) has a solution given by

$$w_\delta(x) = \frac{(3\delta)^{\frac{1}{4}}}{(\delta + |x - x_0/\delta|)^{\frac{1}{2}}},$$

and

$$|\nabla w_\delta|_2^2 = |w_\delta|_6^6 = S^{\frac{3}{2}},$$

where $\delta > 0$ can be arbitrarily chosen and $x_0 \in \mathbb{R}^3$ is given by (H_1).

Let $\varphi \in C_0^\infty(\mathbb{R}^3, [0,1])$ be a radial cut-off function, such that $\varphi(x) = 1$ for $|x| \le \rho_\delta$ and $\varphi(x) = 0$ for $|x| \ge 2\rho_\delta$, where $\rho_\delta = \delta^\tau$ with $\tau \in (1/4, 1/2)$. Set

$$\psi_\delta(x) = \varphi(x) w_\delta(x), \tag{2.29}$$

then we get the following estimations:

Lemma 2.7. (See e.g. [4]) *As $\delta \to 0$, $\psi_\delta(x)$ verifies the following estimations:*

$$\int_{\mathbb{R}^3} |\nabla\psi_\delta|^2 \mathrm{d}x = S^{\frac{3}{2}} + O(\delta^{\frac{1}{2}}), \ \int_{\mathbb{R}^3} |\psi_\delta|^6 \mathrm{d}x = S^{\frac{3}{2}} + O(\delta^{\frac{3}{2}}),$$

and

$$\int_{\mathbb{R}^3} |\psi_\delta|^2 \mathrm{d}x = O(\delta^{\frac{1}{2}}), \ \int_{\mathbb{R}^3} |\psi_\delta|^q \mathrm{d}x = O(\delta^{\frac{6-q}{4}}) \text{ with } q \in (3,6).$$

Lemma 2.8. *Suppose (g), (V), (Q), (K), and $(H_1)-(H_4)$, then the mountain-pass value $c_\epsilon < c^*$ for all $\epsilon > 0$, where c^* is given by Lemma 2.6.*

Proof. We claim first that for $\delta > 0$ small enough, there exists a constant $t_\delta > 0$, such that

$$J_\epsilon(t_\delta\psi_\delta) = \max_{t \ge 0} J_\epsilon(t\psi_\delta)$$

and

$$0 < t_0 < t_\delta < t_1 < +\infty \text{ for all } \delta > 0 \text{ sufficiently small}, \tag{2.30}$$

where t_0 and t_1 are constants independent of δ and ϵ. In fact, since $J_\epsilon(0) = 0$ and $\lim_{t\to+\infty} J_\epsilon(t\psi_\delta) = -\infty$, there exists $t_\delta > 0$, such that

$$J_\epsilon(t_\delta\psi_\delta) = \max_{t\ge 0} J_\epsilon(t\psi_\delta) \text{ and } \left.\frac{dJ_\epsilon(t\psi_\delta)}{dt}\right|_{t=t_\delta} = 0.$$

Thus, we have

$$t_\delta^2\|\psi_\delta\|_\epsilon^2 + bt_\delta^4\left(\int_{\mathbb{R}^3} |\nabla\psi_\delta|^2 \mathrm{d}x\right)^2 = \int_{\mathbb{R}^3} f(\epsilon x, t_\delta\psi_\delta) t_\delta\psi_\delta \mathrm{d}x + t_\delta^6 \int_{\mathbb{R}^3} K(\epsilon x)|\psi_\delta|^6 \mathrm{d}x \tag{2.31}$$

which together with Lemma 2.2-(1) and Lemma 2.7 as well as (K) gives that

$$a\int_{\mathbb{R}^3} |\nabla\psi_\delta|^2 \mathrm{d}x + O(\delta^{\frac{1}{2}}) + bt_\delta^2\left(\int_{\mathbb{R}^3} |\nabla\psi_\delta|^2 \mathrm{d}x\right)^2 \ge K_\infty t_\delta^4 \int_{\mathbb{R}^3} |\psi_\delta|^6 \mathrm{d}x.$$

Therefore, we can conclude that

$$\begin{aligned} t_\delta &\le \sqrt{\frac{b|\nabla\psi_\delta|_2^4 + \left(b^2|\nabla\psi_\delta|_2^8 + 4aK_\infty|\nabla\psi_\delta|_2^2|\psi_\delta|_6^6\right)^{1/2}}{2}} \\ &< t_1 < +\infty \text{ if } \delta \text{ is sufficiently small.} \end{aligned}$$

Taking Lemma 2.2-(2) and (3) into account, for any $\sigma > 0$, there exists $C > 0$, such that

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{f(\epsilon x, t_\delta\psi_\delta)t_\delta\psi_\delta}{t_\delta^2|\psi_\delta|_6^6} \mathrm{d}x &\le \int_{\mathbb{R}^3} \frac{\sigma t_\delta^6\psi_\delta^6 + Ct_\delta^2\psi_\delta^2}{t_\delta^2|\psi_\delta|_6^6} \mathrm{d}x = \sigma t_\delta^4 + C\frac{|\psi_\delta|_2^2}{|\psi_\delta|_6^6} \\ &= \sigma t_\delta^4 + C\big(S^{\frac{3}{2}} + O(\delta^{\frac{3}{2}})\big)^{-1}|\psi_\delta|_2^2 \end{aligned}$$

$$\leq \sigma t_\delta^4 + CS^{-\frac{3}{2}} O(\delta^{\frac{1}{2}}) = \sigma t_\delta^4 + o_\delta(1) \text{ as } \delta \to 0,$$

which together with (2.31) implies that

$$1 \leq \sigma t_\delta^4 + o_\delta(1) + K_0 t_\delta^4 \text{ as } \delta \to 0.$$

Thus

$$t_\delta \geq (2(\sigma + K_0))^{-\frac{1}{4}} \triangleq t_0 > 0 \text{ if } \delta \text{ is sufficiently small.}$$

Let us define

$$\xi(t) \triangleq \frac{a}{2} t^2 \int_{\mathbb{R}^3} |\nabla \psi_\delta|^2 \mathrm{d}x + \frac{b}{4} t^4 \left(\int_{\mathbb{R}^3} |\nabla \psi_\delta|^2 \mathrm{d}x \right)^2 - \frac{K_0}{6} t^6 \int_{\mathbb{R}^3} |\psi_\delta|^6 \mathrm{d}x, \ \forall t > 0,$$

then by Lemma 2.7 and some elementary computations, we derive

$$\max_{t \geq 0} \xi(t) = \frac{abS^3}{4K_0} + \frac{b^3 S^6}{24K_0^2} + \frac{(b^2 S^4 + 4aK_0 S)^{\frac{3}{2}}}{24K_0^2} + O(\delta^{\frac{1}{2}}).$$

On the other hand, using (H_1), we have that

$$\int_{\mathbb{R}^3} [K(x_0) - K(\epsilon x)] \, |\psi_\delta|^6 \mathrm{d}x \leq C\delta^{\frac{1}{2}}, \text{ if } \delta \text{ is sufficiently small.}$$

Finally, we can take an estimate for $J_\epsilon(t_\delta \psi_\delta)$ below

$$\begin{aligned}
J_\epsilon(t_\delta \psi_\delta) &= \frac{a}{2} t_\delta^2 \int_{\mathbb{R}^3} |\nabla \psi_\delta|^2 \mathrm{d}x + \frac{b}{4} t_\delta^4 \left(\int_{\mathbb{R}^3} |\nabla \psi_\delta|^2 \mathrm{d}x \right)^2 - \frac{K_0}{6} \int_{\mathbb{R}^3} |\psi_\delta|^6 \mathrm{d}x \\
&\quad + \frac{t_\delta^2}{2} \int_{\mathbb{R}^3} V(\epsilon x) |\psi_\delta|^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(\epsilon x, t_\delta \psi_\delta) \mathrm{d}x \\
&\quad + \frac{1}{6} \int_{\mathbb{R}^3} [K(x_0) - K(\epsilon x)] \, |\psi_\delta|^6 \mathrm{d}x \\
&\leq c^* + O(\delta^{\frac{1}{2}}) + C \int_{\mathbb{R}^3} |\psi_\delta|^2 \mathrm{d}x - \int_{\mathbb{R}^3} H\left(G^{-1}(t_\delta \psi_\delta)\right) \mathrm{d}x \\
&\leq c^* + CO(\delta^{\frac{1}{2}}) - C \int_{\mathbb{R}^3} |\psi_\delta|^{p+1} \mathrm{d}x \leq c^* + CO(\delta^{\frac{1}{2}}) - CO(\delta^{\frac{5-p}{4}}) < c^*
\end{aligned}$$

for sufficiently small $\delta > 0$ since $p > 3$. In view of (2.13), we have $c_\epsilon < c^*$. The proof is completed. □

3. Existence of positive ground state

Although we have showed that $c_\epsilon < c^*$ in Lemma 2.8, it seems unavailable to exploit Lemma 2.6 to find a ground state solution to Eq. (1.1). Simply speaking, we have to investigate the relation $c_\epsilon < m_\infty$.

First of all, we have the following lemma whose proof can be found below.

Lemma 3.1. *Suppose* (g) *and* $(H_1) - (H_4)$*, then the following equation*

$$\begin{aligned}
&\left(a + b \int_{\mathbb{R}^3} g^2(u) |\nabla u|^2 \mathrm{d}x \right) \left[-\operatorname{div}(g^2(u) \nabla u) + g(u) g'(u) |\nabla u|^2 \right] + Au \\
&\quad = Bh(u) + D|G(u)|^4 G(u) g(u), \ x \in \mathbb{R}^3
\end{aligned}$$

which is equivalent to

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)\Delta v+A\frac{G^{-1}(v)}{g(G^{-1}(v))}=B\frac{h(G^{-1}(v))}{g(G^{-1}(v))}+D|v|^4v,\ x\in\mathbb{R}^3$$

admits at least a positive ground state solution, where $A,B,D>0$ are constants.

Proof. Let us define the variational functional $\mathcal{J}_{A,B,D}:H^1(\mathbb{R}^3)\to\mathbb{R}$ by

$$\mathcal{J}_{A,B,D}(v)=\frac{1}{2}\int_{\mathbb{R}^3}\left[a|\nabla v|^2+A\left|G^{-1}(v)\right|^2\right]\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2$$
$$-B\int_{\mathbb{R}^3}H(G^{-1}(v))\mathrm{d}x-\frac{D}{6}\int_{\mathbb{R}^3}|v|^6\mathrm{d}x.$$

Clearly, it is of class $\mathcal{C}^1$. First of all, by Lemma 2.4, there exists a (PS) sequence (w_n) of $\mathcal{J}_{A,B,D}$ at the level

$$c_{A,B,D}\triangleq\inf_{\eta\in\Gamma_{A,B,D}}\max_{t\in[0,1]}\mathcal{J}_{A,B,D}(\eta(t))>0,$$

with $\Gamma_{A,B,D}\triangleq\left\{\eta\in C([0,1],H^1(\mathbb{R}^3)):\eta(0)=0,\mathcal{J}_{A,B,D}(\eta(1))<0\right\}$. Moreover, it simply has that

$$c_{A,B,D}=\inf_{v\in\mathcal{N}_{A,B,D}}J_{A,B,D}(v)=\inf_{v\in H^1(\mathbb{R}^2)\backslash\{0\}}\max_{t\geq 0}\mathcal{J}_{A,B,D}(tv),$$

where $\mathcal{N}_{A,B,D}=\{v\in H^1(\mathbb{R}^3)\backslash\{0\}:J'_{A,B,D}(v)(v)=0\}$. The similar calculations in Lemma 2.8 reveal

$$c_{A,B,D}<\frac{abS^3}{4}+\frac{\left(b^2S^4+4aS\right)^{3/2}}{24}+\frac{b^3S^6}{24}.$$

Thus, Lemma 2.6 shows that there exists a w, such that $w_n\to w$ in $H^1(\mathbb{R}^3)$ along a subsequence, and so, w is a nontrivial solution. As to its positivity, we postpone it later. The proof is completed. □

Lemma 3.2. *Suppose (g), (V), (Q), (K), and $(H_1)-(H_4)$, then there exists a constant $\epsilon^*>0$, such that $c_\epsilon<m_\infty$ for any $\epsilon\in(0,\epsilon^*)$.*

Proof. Let $\omega\in\mathbb{R}$ be a fixed constant satisfying $V_0<\omega<V_\infty$ according to (V) and set the functional $J_\omega:H^1(\mathbb{R}^3)\to\mathbb{R}$ as follows:

$$J_\omega(v)=\frac{1}{2}\int_{\mathbb{R}^3}\left[a|\nabla v|^2+\omega\left|G^{-1}(v)\right|^2\right]\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2$$
$$-Q_\infty\int_{\mathbb{R}^3}H(G^{-1}(v))\mathrm{d}x-\frac{K_\infty}{6}\int_{\mathbb{R}^3}|v|^6\mathrm{d}x.$$

Moreover, we define $m_\omega=\inf_{v\in\mathcal{N}_\omega}J_\omega(v)$ with $\mathcal{N}_\omega=\left\{v\in H^1(\mathbb{R}^3)\backslash\{0\}:J'_\omega(v)(v)=0\right\}$. We claim that $m_\omega<m_\infty$. Actually, by means of Lemma 3.1, there exists $v\in H^1(\mathbb{R}^3)\backslash\{0\}$, such that $v\in\mathcal{N}_\infty$ and $J_\infty(v)=m_\infty$. In view of Lemma 2.3-(i), we shall conclude that $m_\infty=J_\infty(v)=\max_{t\geq 0}J_\infty(tv)$ and there exists a $t_\omega>0$, such that $t_\omega v\in N_\omega$ and $J_\omega(t_\omega v)=\max_{t\geq 0}J_\omega(tt_\omega v)=\max_{t\geq 0}J_\omega(tv)$. Thus, it holds that

$$m_\infty\geq J_\infty(t_\omega v)=J_\omega(t_\omega v)+\frac{(V_\infty-\omega)}{2}\int_{\mathbb{R}^3}\left|G^{-1}(t_\omega v)\right|^2\mathrm{d}x$$

$$> J_\omega(t_\omega v) \geq m_\omega,$$

so the claim concludes.

Due to Lemma 3.1, there exists a $v_\omega \in H^1(\mathbb{R}^3)\backslash\{0\}$, such that $v_\omega \in \mathcal{N}_\omega$ and $J_\omega(v_\omega) = m_\omega$. Let $\varphi(x) : \mathbb{R}^3 \to [0,1]$ be a cut-off function to satisfy $\varphi(x) \equiv 1$ when $|x| \leq 1$, $\varphi(x) \equiv 0$ when $|x| \geq 2$ and $|\varphi'(x)| \leq 2$ on $\mathbb{R}^3$. For every $R > 0$, we define $v_{\omega,R}(x) \triangleq \varphi\left(\frac{x}{R}\right) v_\omega(x)$. Then, the definition of $\varphi_{\omega,R}$ and the Lebesgue theorem show that

$$v_{\omega,R} \to v_\omega \text{ in } H^1(\mathbb{R}^3) \text{ as } R \to \infty.$$

Since $v_{\omega,R} \in H^1(\mathbb{R}^3)\backslash\{0\}$, then there exists an $t_{\omega,R} > 0$, such that $t_{\omega,R}v_{\omega,R} \in \mathcal{N}_\omega$ by Lemma 2.3-(i). We claim that there exists an $R_0 > 0$, such that $J_\omega(t_{\omega,R_0}v_{\omega,R_0}) < m_\infty$. Otherwise, we could suppose that $J_\omega(t_{\omega,R}v_{\omega,R}) \geq m_\infty$ for all $R > 0$. By gathering these facts that $t_{\omega,R}v_{\omega,R} \in \mathcal{N}_\omega$, $v_\omega \in \mathcal{N}_\omega$ and $v_{\omega,R} \to v_\omega$ in $H^1(\mathbb{R}^3)$ as $R \to +\infty$, we can proceed as Lemma 2.3-(iii) to deduce that $t_{\omega,R} \to 1$ as $R \to +\infty$. and so

$$m_\infty \leq \liminf_{R\to+\infty} J_\omega(t_{\omega,R}v_{\omega,R}) = J_\omega(v_w) = m_\omega < m_\infty,$$

a contradiction. Thus, the claim is true. Due to the definition of v_{ω,R_0}, one has $\operatorname{supp} v_{\omega,R_0} \subset B_{2R_0}(0)$, where $\operatorname{supp} v_{\omega,R_0}$ denotes the support of v_{ω,R_0}. Thereby, for all $x \in B_{2R_0}(0)$, there exists a constant $\epsilon^* > 0$, such that $w \geq V(\epsilon x)$ for all $\epsilon \in (0,\epsilon^*)$ using the fact that $\omega > V_0$. As a consequence, we apply $t_{R_0}v_{R_0} \in \mathcal{N}_w$ to reach

$$m_\infty > J_\omega(t_{R_0}v_{R_0}) = \max_{t\geq 0} J_\omega(tv_{R_0}) \geq \max_{t\geq 0} J_\epsilon(tv_{R_0}) \geq \inf_{v\in E_\epsilon} \max_{t\geq 0} J_\epsilon(tv) = c_\epsilon$$

for any $\epsilon \in (0,\epsilon^*)$. The proof is completed. ☐

Proposition 3.3. *Suppose* (g), (V), (Q), (K), *and* $(H_1)-(H_4)$, *then Eq.* (1.13) *admits a positive ground state solution for any* $\epsilon \in (0,\epsilon^*)$, *where* $\epsilon^* > 0$ *is determined by Lemma 3.2.*

Proof. According to Lemma 2.4, there is a (PS) sequence (v_n) for the functional J_ϵ at the level c_ϵ. In view of Lemmas 2.6, 2.8, and 3.2, one sees that J_ϵ admits a strongly convergent subsequence for all $\epsilon \in (0,\epsilon^*)$. Going to a subsequence if necessary, there is a $v \in E_\epsilon$, such that $v_n \to v \in E_\epsilon$, and so, $J'_\epsilon(v) = 0$ and $J_\epsilon(v) = c_\epsilon$. Combining (2.12) and (2.13), we know that v is a nontrivial ground state solution to Eq. (1.13). We postpone the positivity of v below. The proof is completed. ☐

Due to Proposition 3.3, for any $\epsilon \in (0,\epsilon^*)$, there is a positive solution to Eq. (1.13) whose energy is c_ϵ. Now, if we regard the energy c_ϵ as a sequence, what happens about $\lim\limits_{\epsilon\to 0^+} c_\epsilon$? To deal with it, we introduce the following problem:

$$\begin{aligned}&\left(a + b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 \mathrm{d}x\right)\left[-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2\right] + V_0 u\\ &\quad = Q_0 h(u) + K_0|G(u)|^4 G(u)g(u),\ x \in \mathbb{R}^3,\end{aligned}$$

which is equivalent to

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)\Delta v+V_0\frac{G^{-1}(v)}{g(G^{-1}(v))}=Q_0\frac{h(G^{-1}(v))}{g(G^{-1}(v))}+K_0|v|^4v,\ x\in\mathbb{R}^3.$$

The corresponding variational functional $J_0 : H^1(\mathbb{R}^2) \to \mathbb{R}$ is given by

$$\begin{aligned}J_0(v)=&\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla v|^2+V_0v^2\right)\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2\\&-\int_{\mathbb{R}^3}F(0,v)\mathrm{d}x-\frac{K_0}{6}\int_{\mathbb{R}^3}|v|^6\mathrm{d}x,\end{aligned}$$

where

$$F(0,s)\triangleq Q_0H(G^{-1}(s))+\frac{1}{2}V_0s^2-\frac{1}{2}V_0|G^{-1}(s)|^2.$$

We also set

$$\mathcal{N}_0=\left\{v\in H^1(\mathbb{R}^3)\backslash\{0\}:J_0'(v)(v)=0\right\}\text{ and }m_0=\inf_{v\in\mathcal{N}_0}J_0(v).$$

Lemma 3.4. *Suppose (g), (V), (Q), (K) and $(H_1)-(H_4)$, then* $\lim_{\epsilon\to 0^+}c_\epsilon = m_0$ *along a subsequence.*

Proof. To prove $\liminf_{\epsilon\to 0^+}c_\epsilon \geq m_0$, it suffices to show that

$$c_\epsilon\geq m_0,\ \forall\epsilon\in(0,\epsilon^*).\tag{3.1}$$

Otherwise, we could suppose that there exists some $\epsilon_0 \in (0, \epsilon^*)$, such that $c_{\epsilon_0} < m_0$. It follows from Proposition 3.3 that there exists a positive ground state v_{ϵ_0}, such that:

$$\max_{t\geq 0}J_{\epsilon_0}(tv_{\epsilon_0})=J_{\epsilon_0}(v_{\epsilon_0})=c_{\epsilon_0}<m_0,$$

where the variational functional $J_{\epsilon_0} : H^1(\mathbb{R}^2) \to \mathbb{R}$ is given by

$$\begin{aligned}J_{\epsilon_0}(v)=&\frac{1}{2}\int_{\mathbb{R}^3}\left[a|\nabla v|^2+V(\epsilon_0x)v^2\right]\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2\\&-\int_{\mathbb{R}^3}F(\epsilon_0x,v)\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}K(\epsilon_0x)|v|^6\mathrm{d}x.\end{aligned}$$

Arguing as Lemma 2.3-(i), there exists $t_{\epsilon_0} > 0$, such that $t_{\epsilon_0}v_{\epsilon_0} \in \mathcal{N}_0$ and $J_0(t_{\epsilon_0}v_{\epsilon_0}) = \max_{t\geq 0} J_0(tv_{\epsilon_0})$ which together with the definition of m_0 indicates that $m_0 \leq \max_{t\geq 0} J_0(tv_{\epsilon_0})$. Since $V(\epsilon_0x) \geq V_0$, $K(\epsilon_0x) \leq K_0$ and $Q(\epsilon_0x) \leq Q_0$, then it holds that

$$m_0\leq\max_{t\geq 0}J_0(tv_{\epsilon_0})\leq\max_{t\geq 0}J_{\epsilon_0}(tv_{\epsilon_0})=c_{\epsilon_0}<m_0,$$

a contradiction. Hence, we see that (3.1) holds true.

In view of Lemma 3.1, there exists $v_0 \in \mathcal{N}_0$, such that $J_0(v_0) = m_0$. Let $\varphi(x) : \mathbb{R}^3 \to [0, 1]$ be a cut-off function satisfying $\varphi(x) \equiv 1$ when $|x| \leq 1$, $\varphi(x) \equiv 0$ when $|x| \geq 2$ and $|\varphi'(x)| \leq 2$ on $\mathbb{R}^3$. For any $R > 0$, we set $v_R(x) \triangleq \varphi(x/R)v_0(x)$. By the definition of $\varphi(x)$ and the Lebesgue theorem, one has $v_R \to v_0$ in $H^1(\mathbb{R}^3)$ as $R \to \infty$. For all $\epsilon, R > 0$, arguing as Lemma 2.3-(i), there exists $t_{\epsilon,R} > 0$, such that $t_{\epsilon,R} v_R \in \mathcal{N}_\epsilon$ and

$$J_\epsilon(t_{\epsilon,R} v_R) = \max_{t \geq 0} J_\epsilon(t t_{\epsilon,R} v_R) = \max_{t \geq 0} J_\epsilon(t v_R). \tag{3.2}$$

Thus, by $\operatorname{supp} v_R \subset B_{2R}(0)$, it follows from $t_{\epsilon,R} v_R \in \mathcal{N}_\epsilon$ and Lemma (2.2)-(1) that:

$$\begin{aligned}
&\frac{1}{t_{\epsilon,R}^4} \int_{B_{2R}(0)} \left[|\nabla v_R|^2 + \left(\max_{x \in B_{2R}(0)} V(x) \right) v_R^2 \right] \mathrm{d}x \\
&\quad \geq \frac{1}{t_{\epsilon,R}^4} \int_{\mathbb{R}^3} |\nabla v_R|^2 + V(\epsilon x) v_R^2 \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} K(\epsilon x) |v_R|^6 \mathrm{d}x + \frac{1}{t_{\epsilon,R}^6} \int_{\mathbb{R}^3} f(\epsilon x, t_{\epsilon,R} v_R) t_{\epsilon,R} v_R \mathrm{d}x \\
&\quad \geq \int_{\mathbb{R}^3} K_\infty |v_R|^6 \mathrm{d}x,
\end{aligned}$$

which indicates that there exists a $\bar{T}_R < +\infty$ independent of ϵ, such that $t_{\epsilon,R} \leq \bar{T}_R < +\infty$. Proceeding as Lemma 2.3-(ii), one shall search for $\bar{T}'_R > 0$ independent of ϵ, such that $t_{\epsilon,R} \geq \bar{T}'_R > 0$. Consequently, passing to a subsequence if necessary, we have $\lim_{\epsilon \to 0^+} t_{\epsilon,R} = t_R \in (0, +\infty)$ which implies that

$$\begin{aligned}
J_\epsilon(t_{\epsilon,R} v_R) &= \frac{t_{\epsilon,R}^2}{2} \int_{B_{2R}(0)} \left[a|\nabla v_R|^2 + V(\epsilon x) v_R^2 \right] dx + \frac{b}{4} t_{\epsilon,R}^4 \left(\int_{B_{2R}(0)} |\nabla v_R|^2 \mathrm{d}x \right)^2 \\
&\quad - \int_{B_{2R}(0)} F(\epsilon x, t_{\epsilon,R} v_R) \mathrm{d}x - \frac{t_{\epsilon,R}^6}{6} \int_{B_{2R}(0)} K(\epsilon x) |v_R|^6 \mathrm{d}x \\
&\to \frac{t_R^2}{2} \int_{B_{2R}(0)} \left[a|\nabla v_R|^2 dx + V_0 u_R^2 \right] \mathrm{d}x + \frac{b}{4} t_R^4 \left(\int_{B_{2R}(0)} |\nabla v_R|^2 \mathrm{d}x \right)^2 \\
&\quad - \int_{B_{2R}(0)} F(0, t_R v_R) \mathrm{d}x - \frac{t_R^6}{6} \int_{B_{2R}(0)} K_0 |v_R|^5 \mathrm{d}x \\
&= J_0(t_R v_R) \text{ as } \epsilon \to 0^+.
\end{aligned} \tag{3.3}$$

Adopting $t_{\epsilon,R} v_R \in \mathcal{N}_\epsilon$, it is similar to (3.3) that

$$\begin{aligned}
&t_R^2 \int_{\mathbb{R}^3} a|\nabla v_R|^2 + V_0 v_R^2 dx + b t_R^4 \left(\int_{\mathbb{R}^3} |\nabla v_R|^2 dx \right)^2 \\
&\quad = \int_{\mathbb{R}^3} f(0, t_R v_R) t_R v_R dx + t_R^6 \int_{\mathbb{R}^3} K_0 |v_R|^5 dx,
\end{aligned}$$

which implies that $t_R v_R \in \mathcal{N}_0$, and hence, $J_0(t_R v_R) = \max_{t \geq 0} J_0(t t_R v_R) = \max_{t \geq 0} J_0(t v_R)$. We gather $v_0 \in \mathcal{N}_0$, $t_R v_R \in \mathcal{N}_0$ and $v_R \to v_0$ in $H^1(\mathbb{R}^3)$ as $R \to \infty$ to conclude that $t_R \to 1$ as $R \to \infty$ along a subsequence, where the arguments in Lemma 2.3-(iii) are used. Then, it holds that

$$\|t_R v_R - v_0\| \leq |t_R - 1| \cdot \|v_R\| + \|v_R - v_0\| \to 0 \text{ as } R \to \infty. \tag{3.4}$$

It follows from the definition of c_ϵ and (3.2) that:

$$c_\epsilon = \inf_{v\in E_\epsilon\setminus\{0\}} \max_{t\geq 0} J_\epsilon(tv) \leq \max_{t\geq 0} J_\epsilon(tv_R) = J_\epsilon(t_{\epsilon,R}v_R).$$

which together with (3.3) indicates that

$$\limsup_{\epsilon\to 0^+} c_\epsilon \leq J_0(t_R v_R). \tag{3.5}$$

By (3.4) and (3.5), letting $R\to\infty$, then $\limsup_{\epsilon\to 0^+} c_\epsilon \leq J_0(v_0) = m_0$. The proof is completed. □

4. Concentration of ground states

In this section, we dispose of the concentrating behavior of ground state solutions to Eq. (1.1). As a consequence of Proposition 3.3, there exists a $\epsilon^* > 0$, such that for each $\epsilon \in (0,\epsilon^*)$, Eq. (1.13) possesses a positive ground state solution $\bar{v}_\epsilon(x) = v_\epsilon(\epsilon x) \in H^1(\mathbb{R}^3)$ satisfying $J_\epsilon(\bar{v}_\epsilon) = c_\epsilon > 0$ and $J'_\epsilon(\bar{v}_\epsilon) = 0$, where v_ϵ is a ground state solution to Eq. (1.12). Before we study the concentrating behavior of v_ϵ, it is simple to find that any minimizing sequence of $m_0 = \inf_{v\in\mathcal{N}_0} J_0(v)$ is bounded in $H^1(\mathbb{R}^3)$ and we have the following key lemma.

Lemma 4.1. *Suppose (g) and (H_1)–(H_4). If $(v_n)\subset\mathcal{N}_0$ satisfies $J_0(v_n)\to m_0$ and $v_n \rightharpoonup v_0 \neq 0$ in $H^1(\mathbb{R}^3)$ as $n\to\infty$, then $v_n\to v_0$ in $H^1(\mathbb{R}^3)$. In particular, $J'_0(v_0)=0$ in $H^1(\mathbb{R}^3)$ and $J_0(v_0)=m_0$.*

Proof. Owing to the Ekeland's variational principle [13], there is a sequence $(w_n)\subset\mathcal{N}_0$, such that

$$J_0(w_n)\to c_0,\ \big((J_0)|_{\mathcal{N}_0}\big)'(w_n)\to 0,\ \|w_n - v_n\|\to 0 \text{ as } n\to\infty.$$

We claim that $J'_0(w_n)\to 0$ in $(H^1(\mathbb{R}^3))^{-1}$ as $n\to\infty$. Otherwise, there exist a constant $\sigma>0$ and a subsequence still denoted by itself, such that

$$\|J'_0(w_n)\| > \sigma,\ \forall n\in\mathbb{N}. \tag{4.1}$$

Given a $\varphi\in H^1(\mathbb{R}^3)$, one sees $|[J'_0(w_n)-J'_0(w)](\varphi)| \leq C_0\|w_n-w\|\|\varphi\|$. Taking the supremum over $\|\varphi\|\leq 1$, then it yields that $\|J'_0(w_n)-J'_0(w)\|\leq C_0\|w_n-w\|$ for any $w\in H^1(\mathbb{R}^3)$. Therefore, for any $\delta_1>0$, we have $\|J'_0(w_n)-J'_0(w)\|<\delta_1$ if $\|w_n-w\|\leq\min\{1,\delta_1/C_0\}\triangleq 3\delta$. Therefore, by (4.1),

$$\|J'_0(w)\| > \|J'_0(w_n)\| - \delta_1 > \sigma - \delta_1 \text{ if } \|w_n-w\|\leq 3\delta.$$

Choosing $\delta_1=\sigma/2$, we derive $\|J'_0(w)\|>\sigma/2$ for each $w\in B_{3\delta}(w_n)$. Let $\varepsilon = \min\{\frac{m_0}{2},\frac{\sigma\delta}{16}\}>0$ and $S=B_\delta(w_n)$, then [37, Lemma 2.3] enjoys a deformation $\eta\in C\big([0,1]\times H^1(\mathbb{R}^3),H^1(\mathbb{R}^3)\big)$, such that

(i) $\eta(t,u)=u$ if $t=0$, or $u\notin J_0^{-1}([m_0-2\varepsilon,m_0+2\varepsilon])\cap S_{2\delta}$;
(ii) $\eta\big(1,J_0^{m_0+\epsilon}\cap B_\delta(w_n)\big)\subset J_0^{m_0-\varepsilon}$;
(iii) $J_0(\eta(1,u))\leq J_0(u)$, $\forall u\in H^1(\mathbb{R}^3)$;
(iv) $\eta(1,u)$ is a homeomorphism of $H^1(\mathbb{R}^3)$.

For a sufficiently large $L > 0$, we set $\gamma(t) = \eta(1, tLw_n\big)$ and then by (iii),

$$J_0(\gamma(1)) = J_0(\eta(1, Lw_n) \le J_0(Lw_n) \to -\infty \text{ as } L \to +\infty,$$

which indicates that $\gamma(t) \in \Gamma_0 \triangleq \big\{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : J_0(0) = 0, J_0(\gamma(1)) < 0\big\}$. Thereby, we take advantage of (ii) to deduce that

$$c_0 \triangleq \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} J_0(\gamma(t)) \le \max_{t\in[0,1]} J_0(\eta(1, Ltw_n) = \max_{t>0} J_0(\eta(1, tw_n)) < m_0 - \varepsilon. \tag{4.2}$$

However, arguing as (2.13), we have $c_0 = m_0$, it contradicts with (4.2). Therefore, we have showed that $J_0'(w_n) \to 0$ in $(H^1(\mathbb{R}^2))^{-1}$ and then $J_0'(n_n) \to 0$ in $(H^1(\mathbb{R}^2))^{-1}$. Now, we are capable of using Lemma 2.3-(iv) to have that $J_0'(v_0) = 0$ in $H^1(\mathbb{R}^3)$. Since $v_0 \ne 0$, one sees

$$\begin{aligned} m_0 &\le J_0(v_0) - \frac{1}{4}J_0'(v_0)(v_0) \le \liminf_{n\to\infty} \left[J_0(v_n) - \frac{1}{4}J_0'(v_n)(v_n)\rangle\right] \\ &= \lim_{n\to\infty} J_0(v_n) = m_0, \end{aligned}$$

which yields that $v_n \to v_0$ in $H^1(\mathbb{R}^3)$ and then $J_0(v_0) = m_0$. The proof is completed. □

Recalling the definition of $\bar{v}_\epsilon$, that is, $J_\epsilon(\bar{v}_\epsilon) = c_\epsilon$ and $J_\epsilon'(\bar{v}_\epsilon) = 0$, one deduces that $(\bar{v}_\epsilon)$ is a special $(\mathrm{PS})_{c_\epsilon}$ sequence of the variational functional J_ϵ. With aid of Lemma 2.8, we proceed as the same ideas of "Vanishing does not occur" in Lemma 2.6 to conclude that for all $\epsilon \in (0, \epsilon^*)$, there exist a family $(y_\epsilon) \subset \mathbb{R}^3$ and $r, \varrho > 0$, such that

$$\int_{B_r(y_\epsilon)} |\bar{v}_\epsilon|^2 \mathrm{d}x \ge \varrho > 0. \tag{4.3}$$

Lemma 4.2. *Suppose (g), (V), (Q), (K), and $(H_1)-(H_4)$, then (ϵy_ϵ) in (4.3) is uniformly bounded in $\mathbb{R}^3$. Furthermore if we take x^* as the limit of the sequence of $(\epsilon_n y_{\epsilon_n})$, then one has $x^* \in \Theta \cap \Theta_1 \cap \Theta_2$, where $(\epsilon_n y_{\epsilon_n})$ is a subsequence of (ϵy_ϵ).*

Proof. Arguing it by contradiction, we suppose that $\epsilon_n \to 0$ and $|\epsilon_n y_{\epsilon_n}| \to +\infty$ as $n \to \infty$. We take $y_n \triangleq y_{\epsilon_n}$ and $\bar{v}_n \triangleq \bar{v}_{\epsilon_n}$ for simplicity and set $w_n(\cdot) \triangleq \bar{v}_n(\cdot + y_n) \ge 0$, then

$$\begin{aligned} &-\left(a + b\int_{\mathbb{R}^3} |\nabla w_n|^2 \mathrm{d}x\right)\Delta w_n + V(\epsilon_n x + \epsilon_n y_n)w_n \\ &\quad = f(\epsilon_n x + \epsilon_n y_n, w_n) + K(\epsilon_n x + \epsilon_n y_n)|w_n|^4 w_n \text{ in } \mathbb{R}^3, \end{aligned} \tag{4.4}$$

and we are derived (4.3) that

$$\int_{B_r(0)} w_n^2 \mathrm{d}x \ge \varrho > 0. \tag{4.5}$$

Obviously, $\|w_n\| = \|v_n\|$, then (w_n) is bounded in $H^1(\mathbb{R}^3)$ and $w_n \rightharpoonup w_0$ in $H^1(\mathbb{R}^3)$ in the sense of a subsequence. Moreover, $w_0 \ge 0$ and we see $w_0 \ne 0$ from (4.5).

For every $n \in \mathbb{N}$, there exists $t_n > 0$, such that $t_n w_n \in \mathcal{N}_0$ which together with $w_n \neq 0$ implies that $J_0(t_n w_n) \geq m_0$, and then, $\liminf_{n\to\infty} J_0(t_n w_n) \geq m_0$. On the other hand, using Lemma 3.4

$$\begin{aligned} J_0(t_n w_n) &\leq \frac{t_n^2}{2}\int_{\mathbb{R}^3}\left[a|w_n|^2 + V(\epsilon_n x + \epsilon_n y_n)w_n^2\right]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla w_n|^2\mathrm{d}x\right)^2 \\ &\quad - \int_{\mathbb{R}^3} F(\epsilon_n x + \epsilon_n y_n, w_n)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} K(\epsilon_n x + \epsilon_n y_n)|w_n|^6\mathrm{d}x \\ &= J_{\epsilon_n}(t_n\bar{v}_n) \leq \max_{t\geq 0} J_{\epsilon_n}(t\bar{v}_n) = J_{\epsilon_n}(\bar{v}_n) = c_{\epsilon_n} = m_0 + o_n(1), \end{aligned}$$

which implies that $\limsup_{n\to\infty} J_0(t_n w_n) \leq m_0$ and then $\lim_{n\to\infty} J_0(t_n w_n) = m_0$. Simply, one can conclude that (t_n) is bounded, passing to a subsequence if necessary, we are able to assume that $\lim_{n\to\infty} t_n = t_0 \geq 0$. If $t_0 = 0$, since $\{w_n\}$ is bounded, then $|t_n w_n\| = t_n\|w_n\| \to 0$ which is $t_n w_n \to 0$ in $H^1(\mathbb{R}^2)$ and so $0 = \lim_{n\to\infty} J_0(t_n w_n) = m_0 > 0$, a contradiction. Therefore, $\lim_{n\to\infty} t_n = t_0 > 0$. By the uniqueness of the weak limit, we derive $t_n w_n \rightharpoonup t_0 w_0 \neq 0$ in $H^1(\mathbb{R}^3)$. In summary, we have concluded that $t_n w_n \in \mathcal{N}_0$, $\lim_{n\to\infty} J_0(t_n w_n) = m_0$ and $t_n w_n \rightharpoonup t_0 w_0 \neq 0$, and then, by Lemma 4.1, one sees $t_n w_n \to t_0 w_0 \neq 0$ in $H^1(\mathbb{R}^3)$, which implies that $t_0 w_0 \in \mathcal{N}_0$. Using Fatou's lemma and Lemma 3.4, it holds that

$$\begin{aligned} m_0 &\leq J_0(t_0 w_0) < J_\infty(t_0 w_0) = J_\infty(t_0 w_0) - \frac{1}{4}J_0'(t_0 w_0)(t_0 w_0) \\ &= \frac{a}{4}\int_{\mathbb{R}^3}|\nabla(t_0 w_0)|^2\mathrm{d}x + \int_{\mathbb{R}^3}\left(\frac{V_\infty}{2} - \frac{V_0}{4}\right)\left|G^{-1}(t_0 w_0)\right|^2\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\left[\frac{Q_0}{4}\frac{h(G^{-1}(t_0 w_0))t_0 w_0}{g(G^{-1}(t_0 w_0))} - Q_\infty H(G^{-1}(t_0 w_0))\right]\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\left(\frac{K_0}{4} - \frac{K_\infty}{6}\right)|t_0 w_0|^6\mathrm{d}x \\ &= \liminf_{n\to\infty}\left\{\frac{a}{4}\int_{\mathbb{R}^3}|\nabla(t_n w_n)|^2\mathrm{d}x + \int_{\mathbb{R}^3}\left(\frac{V(\epsilon_n x + \epsilon_n y_n)}{2} - \frac{V_0}{4}\right)\left|G^{-1}(t_n w_n)\right|^2\mathrm{d}x\right. \\ &\quad + \int_{\mathbb{R}^3}\left[\frac{Q_0}{4}\frac{h(G^{-1}(t_n w_n))t_n w_n}{g(G^{-1}(t_n w_n))} - Q(\epsilon_n x + \epsilon_n y_n)H(G^{-1}(t_n w_n n))\right]\mathrm{d}x \\ &\quad \left. + \int_{\mathbb{R}^3}\left(\frac{K_0}{4} - \frac{K(\epsilon_n x + \epsilon_n y_n)}{6}\right)|t_n w_n|^6\mathrm{d}x\right\} \\ &= \liminf_{n\to\infty}\left[J_{\epsilon_n}(t_n\bar{v}_n) - \frac{1}{4}J_0'(t_n w_n)(t_n w_n)\right] = \liminf_{n\to\infty} J_{\epsilon_n}(t_n\bar{v}_n) \\ &\leq \liminf_{n\to\infty}\max_{t\geq 0} J_{\epsilon_n}(t\bar{v}_n) = \liminf_{n\to\infty} J_{\epsilon_n}(\bar{v}_n) = \liminf_{n\to\infty} c_{\epsilon_n} = m_0, \end{aligned} \tag{4.6}$$

a contradiction. Therefore, (ϵy_ϵ) is bounded in $\mathbb{R}^3$. Passing to a subsequence if necessary, we shall suppose that $\epsilon_n y_{\epsilon_n} \to x^*$ in $\mathbb{R}^3$ as $n \to \infty$. We define the variational functional $J_{x^*} : H^1(\mathbb{R}^2) \to \mathbb{R}$ as follows:

$$\begin{aligned} J_{x^*}(v) &= \frac{1}{2}\int_{\mathbb{R}^3}\left[a|\nabla v|^2 + V(x^*)v^2\right]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2 \\ &\quad - \int_{\mathbb{R}^3} F(x^*, v)\mathrm{d}x - \frac{K(x^*)}{6}\int_{\mathbb{R}^3}|v|^6\mathrm{d}x. \end{aligned}$$

If $x^* \notin \Theta \cap \Theta_1 \cap \Theta_2$, without loss of generality, we can assume $x^* \notin \Theta$. By the definitions of Θ and V_0, we conclude $V(x^*) > V_0$. Replacing J_∞ with J_{x^*} in (4.6) and proceeding the similar arguments as above, one has a contradiction. Therefore, $x^* \in \Theta$. Similarly, we can obtain $x^* \in \Theta_1$ and $x^* \in \Theta_2$. Hence, $x^* \in \Theta \cap \Theta_1 \cap \Theta_2$. The proof is completed. $\square$

Lemma 4.3. *Suppose* (g), (V), (Q), (K), *and* $(H_1) - (H_4)$. *Let* $\epsilon \in (0, \epsilon^*)$ *be fixed, decreasing* ϵ^* *if necessary, then* $\bar{v}_\epsilon$ *possesses a maximum* z_ϵ *satisfying*

$$\lim_{\epsilon \to 0^+} V(\epsilon z_\epsilon) = V(x^*), \quad \lim_{\epsilon \to 0^+} K(\epsilon z_\epsilon) = K(x^*), \quad \lim_{\epsilon \to 0^+} Q(\epsilon z_\epsilon) = Q(x^*). \tag{4.7}$$

Moreover, there exist positive constants $\bar{c}$ *and* $\hat{c}$ *independent of* ϵ, *such that*

$$\bar{v}_\epsilon(x) \le \bar{c} \exp\left(-\hat{c}|x - z_\epsilon|\right), \tag{4.8}$$

for all $\epsilon \in (0, \epsilon^*)$ *and* $x \in \mathbb{R}^3$.

Proof. First, we analyze some properties of w_ϵ. Since $w_\epsilon(\cdot) = \bar{v}_\epsilon(\cdot + y_\epsilon)$, according to the proof of Lemma 4.2, we have showed that $t_\epsilon w_\epsilon \to t_0 w_0 \neq 0$ in $H^1(\mathbb{R}^3)$ and $t_\epsilon \to t_0$ with $t_0 > 0$. Thus, it has that

$$t_0\|w_\epsilon - w_0\| = \|t_0 w_\epsilon - t_\epsilon w_\epsilon + t_\epsilon w_\epsilon - t_0 w_0\| \le |t_\epsilon - t_0| \cdot \|w_\epsilon\| + \|t_\epsilon w_\epsilon - t_0 w_0\| \to 0,$$

which indicates that $w_\epsilon \to w_0$ in $H^1(\mathbb{R}^3)$ as $\epsilon \to 0^+$. Combining (4.4) and $\epsilon y_\epsilon \to x^*$, we shall observe that w_0 is a ground state solution of the equation below

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x\right) \Delta v + V(x^*)v = f(x^*, v) + K(x^*)|v|^4 v \text{ in } \mathbb{R}^3. \tag{4.9}$$

We postpone the detailed proofs in Lemma A.2 in the Appendix to give that $|w_0|_\infty, |w_\epsilon|_\infty \le C$ for some $C > 0$ independent of $\epsilon \in (0, \epsilon^*)$, $w_\epsilon \in \mathcal{C}^{1,\chi}_{\mathrm{loc}}(\mathbb{R}^3)$ for some $\chi \in (0, 1)$ as well as as well as

$$|w_\epsilon|_\infty \ge \tau \text{ and } \lim_{|x| \to +\infty} w_\epsilon(x) = 0 \text{ uniformly in } \epsilon \in (0, \epsilon^*).$$

where $\tau > 0$ is independent of $\epsilon \in (0, \epsilon^*)$.

Second, we verify that there exist $\bar{c}', \hat{c}' > 0$ independent of ϵ, such that $w_\epsilon(x) \le \bar{c}' \exp(-\hat{c}'|x|)$ for all $\epsilon \in (0, \epsilon^*)$ and $x \in \mathbb{R}^3$, see Lemma A.3 in the Appendix in detail.

Finally, let k_ϵ be a maximum of w_ϵ, we have that $|w_\epsilon(k_\epsilon)|_\infty \ge \tau$. Since $\lim_{|x| \to \infty} w_\epsilon(x) = 0$ uniformly in ϵ, there exists an $R > 0$ independent of ϵ, such that $|k_\epsilon| \le R$. Recalling $w_\epsilon(\cdot) = \bar{v}_\epsilon(\cdot + y_\epsilon)$, then $y_\epsilon + k_\epsilon$ acts as a maximum of of $\bar{v}_\epsilon$. Define $z_\epsilon = y_\epsilon + k_\epsilon$, according to Lemma 4.2 and $|k_\epsilon| \le R$, we are derived that $\epsilon z_\epsilon \to x^*$ as $\epsilon \to 0^+$ and hence (4.7) holds true by the continuities of V, Q and K. Moreover, since $w_\varepsilon(x) \le \bar{c} \exp(-\hat{c}|x|)$ for all $x \in \mathbb{R}^3$ and $|k_\epsilon| \le R$, there holds

$$\begin{aligned}\bar{v}_\epsilon(x) = w_\epsilon(x - y_\epsilon) &\le \bar{c}' \exp(-\hat{c}'|x - y_\epsilon|) \\ &= \bar{c}' \exp(-\hat{c}'|x - z_\epsilon + k_\epsilon|) \le \bar{c} \exp(-\hat{c}|x - z_\epsilon|)\end{aligned}$$

for all $\epsilon \in (0, \epsilon^*)$ and $x \in \mathbb{R}^3$. The proof is completed. $\square$

At this stage, we are in a position to exhibit the proof of Theorem 1.1 in detail.

Proof of Theorem 1.1. It follows from Proposition 3.3 that there exists some positive constant ϵ^*, such that Eq. (1.13) admits at least a positive ground state solution $\bar{v}_\epsilon(x) = v_\epsilon(\epsilon x)$ for every $\epsilon \in (0, \epsilon^*)$, and hence, v_ϵ is positive ground state solution to Eq. (1.12) for any $\epsilon \in (0, \epsilon^*)$, where the positivity can be found in Lemma A.1 in the Appendix. Next, we shall show the proof one by one:

(1) For any bounded sequence $(u_n) \subset \mathcal{L}_\epsilon$, denoting $v_n = G(u_n)$, then $J_\epsilon(v_n) = c_\epsilon$ and $J'_\epsilon(v_n)(v_n) = 0$. Going to a subsequence if necessary, there exists $v \in E_\epsilon$, such that $v_n \rightharpoonup v$ in E_ϵ and $J'_\epsilon(v) = 0$ by Lemma 2.3-(iv). Similar to (4.3), it holds that

$$\int_{B_r(y_n)} v_n^2 \mathrm{d}x \geq \varrho > 0.$$

Arguing as the same arguments in Lemma 2.6, one can verify that (y_n) is bounded in $\mathbb{Z}^N$. Hence, we have $v \neq 0$ and so $v \in \mathcal{N}_\epsilon$ which together with the Fatou's lemma implies that

$$\begin{aligned} c_\epsilon \leq J_\epsilon(v) &= J_\epsilon(v) - \frac{1}{4} J'_\epsilon(v)(v) \leq \liminf_{n\to\infty} \left[J_\epsilon(v_n) - \frac{1}{4} J'_\epsilon(v_n)(v_n) \right] \\ &= \lim_{n\to\infty} J_\epsilon(v_n) = c_\epsilon. \end{aligned}$$

Thus, we deduce that $v_n \to v$ in E_ϵ, and so, $v_n \to v$ in $H^1(\mathbb{R}^3)$. In view of Lemma 2.1-(1), one sees that $u_n \to u$ in $H^1(\mathbb{R}^3)$, so $\mathcal{L}_\epsilon$ is compact.

(2) In view of Lemma 4.3 and $w_\epsilon(x) = \bar{v}_\epsilon(x + y_\epsilon)$, we have deduced that $\bar{v}_\epsilon$ possesses a maximum point $z_\epsilon = y_\epsilon + k_\epsilon$. The reader is invited to recall that $v_\epsilon(\cdot) = \bar{v}_\epsilon\left(\frac{\cdot}{\epsilon}\right)$, then $v_\epsilon(\cdot) = G(u_\epsilon(\cdot))$ admits a global maximum point $\gamma_\epsilon = \epsilon z_\epsilon$. Due to (4.7), the proof of this case is done.

(3) By the above facts, we know that

$$\tilde{v}_\epsilon(x) = v_\epsilon(\epsilon x + \gamma_\epsilon) = v_\epsilon(\epsilon x + \epsilon z_\epsilon) = \bar{v}_\epsilon(x + z_\epsilon) = \bar{v}_\epsilon(x + y_\epsilon + k_\epsilon) = w_\epsilon(x + k_\epsilon).$$

In view of the proof of Lemma 4.3, we have that $w_\epsilon \to w_0$ in $H^1(\mathbb{R}^3)$ as $\epsilon \to 0^+$ and w_0 is a positive ground state solution of Eq. (4.9). Since $|k_\epsilon| \leq R$, then $\tilde{v}_\epsilon \to \tilde{v}$ in $H^1(\mathbb{R}^3)$ as $\epsilon \to 0^+$ with $\tilde{v} = w_0$. By Lemma 2.1-(1), $G^{-1}(\tilde{v}_\epsilon) \to G^{-1}(\tilde{v})$ in $H^1(\mathbb{R}^3)$ as $\epsilon \to 0^+$ which is $\tilde{u}_\epsilon \to \tilde{u}$ in $H^1(\mathbb{R}^3)$ as $\epsilon \to 0^+$. Clearly, $\tilde{u}$ is a positive ground state solution of Eq. (1.14).

(4) Using (4.8) with $\gamma_\epsilon = \epsilon z_\epsilon$, we have

$$v_\epsilon(x) = \bar{v}_\epsilon\left(\frac{x}{\epsilon}\right) \leq \bar{c} \exp\left(-\hat{c}\left|\frac{x}{\epsilon} - z_\epsilon\right|\right) = \bar{c} \exp\left(-\hat{c}\frac{|x - \gamma_\epsilon|}{\epsilon}\right)$$

for all $x \in \mathbb{R}^3$ and $\epsilon \in (0, \epsilon^*)$.

Recalling $u_\epsilon = G^{-1}(v_\epsilon) \leq v_\epsilon$ by Lemma 2.1-(1), the proof is completed. □

5. Nonexistence of ground states

In this section, the nonexistence of ground state solutions to Eq. (1.1) will be investigated. As what we have done before, to this aim, we shall dispose of the nonexistence of ground state solutions to Eq. (1.12). Explaining it more clearly, we are going to demonstrate that the ground state energy c_ϵ cannot be attained for all $\epsilon > 0$. For simplicity, the notations of this section shall remain unchanged from those in the previous sections.

Let us start with the following lemma.

Lemma 5.1. *Suppose* (g), $(H_1)-(H_4)$ *and* $(\bar{H})$, *then* $c_\epsilon = m_\infty$ *for all* $\epsilon > 0$.

Proof. According to the definition of c_ϵ, for all $\sigma > 0$, there exists a $v_\sigma \in \mathcal{N}_\epsilon$, such that $c_\epsilon \le J_\epsilon(v_\sigma) < c_\epsilon + \sigma$. In view of Lemma 2.3-(ii), one shall conclude that $J_\epsilon(v_\sigma) = \max_{t\ge 0} J_\epsilon(tv_\sigma)$. Moreover, a similar argument in Lemma 2.3-(ii) guarantees a $t_\sigma > 0$, such that $t_\sigma v_\sigma \in \mathcal{N}_\infty$ and $J_\infty(t_\sigma v_\sigma) = \max_{t\ge 0} J_\infty(tv_\sigma)$. With these discussions, it follows from $(\bar{H})$ that:

$$m_\infty \le J_\infty(t_\sigma v_\sigma) = \max_{t\ge 0} J_\infty(tv_\sigma) \le \max_{t\ge 0} J_\epsilon(tv_\sigma) = J_\epsilon(v_\sigma) < c_\epsilon + \sigma$$

yielding that $m_\infty \le c_\epsilon$ by tending $\sigma \to 0^+$. In the following, we are going to verify that $c_\epsilon \le m_\infty$.

By Lemma 3.1, Eq. (2.14) possesses a ground state solution $v_\infty \in \mathcal{N}_\infty$, such that $J_\infty(v_\infty) = m_\infty$. Let $(x_n) \subset \mathbb{R}^3$ satisfy $|x_n| \to \infty$ as $n \to \infty$ and set $v_n(x) \triangleq v_\infty(x - x_n)$. Owing to the assumptions that $\liminf_{|x|\to\infty} V(x) = V_\infty$, $\lim_{|x|\to\infty} Q(x) = Q_\infty$ and $\lim_{|x|\to\infty} K(x) = K_\infty$, we are derived from the Lebesgue theorem that

$$\begin{cases} \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^3} [V(\epsilon x + \epsilon x_n) - V_\infty]|G^{-1}(v_\infty)|^2 \mathrm{d}x = 0, \\ \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^3} [Q(\epsilon x + \epsilon x_n) - Q_\infty]\frac{h(G^{-1}(v_\infty))}{g(G^{-1}(v_\infty))} v_\infty \mathrm{d}x = 0, \\ \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^3} [K(\epsilon x + \epsilon x_n) - K_\infty]|v_\infty|^6 \mathrm{d}x = 0. \end{cases} \tag{5.1}$$

Therefore, it holds that $J'_\epsilon(v_n)(v_n) = o_n(1)$. By exploiting Lemma 2.3-(iii), there exists a $t_n > 0$, such that $t_n v_n \in \mathcal{N}_\epsilon$ and $\lim_{n\to\infty} t_n = 1$. Proceeding as (5.1), one has that

$$\begin{cases} \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^3} [V(\epsilon x + \epsilon x_n) - V_\infty]|G^{-1}(t_n v_\infty)|^2 \mathrm{d}x = 0, \\ \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^3} [Q(\epsilon x + \epsilon x_n) - Q_\infty] H(G^{-1}(t_n v_\infty)) \mathrm{d}x = 0, \end{cases}$$

from where it follows that:

$$\begin{aligned} c_\epsilon &\le J_\epsilon(t_n v_n) \\ &= J_\infty(t_n v_\infty) + \frac{1}{2}\int_{\mathbb{R}^3} \big[V(\epsilon x + \epsilon x_n) - V_\infty\big]|G^{-1}(t_n v_\infty)|^2 \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \big[Q(\epsilon x + \epsilon x_n) - Q_\infty\big] H(G^{-1}(t_n v_\infty)) \mathrm{d}x \\ &\quad - \frac{t_n^6}{6}\int_{\mathbb{R}^3} \big[K(\epsilon x + \epsilon x_n) - K_\infty\big]|v_\infty|^6 \mathrm{d}x \to J_\infty(v_\infty) = m_\infty, \end{aligned}$$

showing that $c_\epsilon \le m_\infty$. The proof is completed. □

With the help of Lemma 5.1, we are ready to present the proof of Theorem 1.4.

Proof of Theorem 1.4. Suppose it by a contradiction, we would assume that there exist $\epsilon_0 > 0$ and $v_0 \in H^1(\mathbb{R}^3)$, such that $J_{\epsilon_0}(v_0) = c_{\epsilon_0}$ and $J'_{\epsilon_0}(v_0) = 0$ as well as $J_{\epsilon_0}(v_0) = \max_{t\geq 0} J_{\epsilon_0}(tv_0)$. Proceeding as Lemma 2.3-(ii), there exists a constant $t_0 > 0$, such that $t_0 v_0 \in \mathcal{N}_\infty$. In view of Lemma 5.1, that is, $m_\infty = c_{\epsilon_0}$, one has

$$\begin{aligned} m_\infty \leq J_\infty(t_0 v_0) \leq J_{\epsilon_0}(t_0 v_0) &\leq \max_{t\geq 0} J_{\epsilon_0}(tv_0) \\ &= J_{\epsilon_0}(v_0) = c_{\epsilon_0} = m_\infty, \end{aligned}$$

which indicates that $J_\infty(t_0 v_0) = J_{\epsilon_0}(t_0 v_0)$. Alternatively, we can apply $(\bar{H})$ to get

$$\begin{aligned} J_\infty(t_0 v_0) = J_{\epsilon_0}(t_0 v_0) &+ \frac{t_0^2}{2}\int_{\mathbb{R}^3} [V_\infty - V(\epsilon_0 x)] \left|G^{-1}(v_0)\right|^2 \mathrm{d}x \\ &+ \int_{\mathbb{R}^3} [Q(\epsilon_0 x) - Q_\infty] H(G^{-1}(t_0 v_0))\mathrm{d}x \\ &+ \frac{t_0^6}{6}\int_{\mathbb{R}^3} [K(\epsilon_0 x) - K_\infty] |v_0|^6 \mathrm{d}x \\ &< J_{\epsilon_0}(t_0 v_0) = J_\infty(t_0 v_0), \end{aligned}$$

a contradiction. The proof is completed. □

Funding Liejun Shen was partially supported by NSFC (Grant No. 12201565) and ZJNSF (Grant No. LMS25A010006). Marco Squassina is member of Gruppo Nazionale per l'Analisi Matematica, la Probabilita e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Appendix A. Some technical lemmas

In this section, we mainly show some results whose detailed proofs have been left above.

Lemma A.1. *Suppose* (g), (V), (Q), (K), *and* $(H_1)-(H_4)$. *If* $v_0 \in H^1(\mathbb{R}^2)$ *is a nontrivial solution of Eq.* (1.13) *for all* $\epsilon > 0$, *the* $v_0(x) > 0$ *for all* $x \in \mathbb{R}^3$.

Proof. Proceeding as the very similar arguments used in Lemma 3.1, we are able to deduce that v_0 can be obtained by looking for critical point of the variational functional $\mathcal{J}_\epsilon : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\mathcal{J}_\epsilon(v) = \frac{1}{2}\int_{\mathbb{R}^3}\left[a|\nabla v|^2 + V(\epsilon x)\left|G^{-1}(v)\right|^2\right]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2$$
$$-\int_{\mathbb{R}^3} Q(\epsilon x)H(G^{-1}(v))\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} K(\epsilon x)|v^+|^6\mathrm{d}x,$$

where $v^+ = \max\{v, 0\}$. Let us define $v_0^- = \min\{v_0, 0\}$, then $\mathcal{J}_\epsilon'(v)(v_0^-) = 0$ implies that

$$\int_{\mathbb{R}^2}\left[a|\nabla v_0^-|^2 + V(\epsilon x)\frac{G^{-1}(v_0^-)}{g(G^{-1}(v_0^-))}v_0^-\right]\mathrm{d}x + b\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)\left(\int_{\mathbb{R}^3}|\nabla v_0^-|^2\mathrm{d}x\right)$$
$$= 0,$$

where we have used the fact that h vanishes in $(-\infty, 0]$. Recalling (V) and Lemma 2.1-(1), it holds that

$$\int_{\mathbb{R}^2}\left(a|\nabla v_0^-|^2 + V_0|v_0^-|^2\right)\mathrm{d}x = 0$$

yielding that $v_0^- \equiv 0$, and hence, $v_0 = v_0^+$ is a nonnegative solution of Eq. (1.13). Consequently, the strong maximum principle and the fact $v_0 \neq 0$ imply that $v_0 > 0$ in $\mathbb{R}^3$. The proof is completed. □

Lemma A.2. *Suppose (g), (V), (Q), (K), and $(H_1) - (H_4)$. Let w_ϵ be defined as Lemma 4.2 for all $\epsilon \in (0, \epsilon^*)$, then there is a constant $C > 0$ independent of $\epsilon \in (0, \epsilon^*)$ such that $|w_\epsilon|_\infty \leq C$. Moreover, we have that $w_\epsilon \in \mathcal{C}^{1,\chi}_{loc}(\mathbb{R}^3)$ for some $\chi \in (0, 1)$ as well as*

$$|w_\epsilon|_\infty \geq \tau \text{ and } \lim_{|x|\to+\infty} w_\epsilon(x) = 0 \text{ uniformly in } \epsilon \in (0, \epsilon^*),$$

where $\tau > 0$ is independent of $\epsilon \in (0, \epsilon^*)$.

Proof. For every $R > 0$ and $0 < r \leqslant \frac{R}{2}$, we choose a cut-off function $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^3, [0, 1])$, such that $\eta(x) = 1$ if $|x| \geqslant R$, and $\eta(x) = 0$ if $|x| \leqslant R - r$ as well as $|\nabla\eta| \leqslant \frac{2}{r}$. Given $\epsilon \in (0, \epsilon^*)$ and $L > 1$, define

$$w_{\epsilon,L}(x) = \begin{cases} w_\epsilon(x), & w_\epsilon(x) \leqslant L, \\ L, & w_\epsilon(x) \geqslant L, \end{cases}$$

and

$$\varphi_{\epsilon,L} = \eta^2(w_{\epsilon,L})^{2(\vartheta-1)}w_\epsilon \text{ and } \psi_{\epsilon,L} = \eta w_\epsilon(w_{\epsilon,L})^{\vartheta-1}$$

with $\vartheta \geq 1$ to be determined later. Some simple calculations show that

$$|\nabla\psi_{\epsilon,L}|^2 \leq C_1\vartheta^2\left(\eta^2(w_{\epsilon,L})^{2(\vartheta-1)}|\nabla w_\epsilon|^2 + |\nabla\eta|^2(w_{\epsilon,L})^{2(\vartheta-1)}|w_\epsilon|^2\right). \tag{A.1}$$

and

$$\int_{\mathbb{R}^3}\nabla w_\epsilon\nabla\varphi_{\varepsilon,L}\mathrm{d}x = \int_{\mathbb{R}^3}\eta^2(w_{\epsilon,L})^{2(\vartheta-1)}|\nabla w_\epsilon|^2\mathrm{d}x$$
$$+ 2\int_{\mathbb{R}^N}\eta(w_{\epsilon,L})^{2(\vartheta-1)}w_\varepsilon\nabla w_\epsilon\nabla\eta\mathrm{d}x.$$
$$+ 2(\vartheta-1)\int_{\mathbb{R}^3}(w_{\epsilon,L})^{2\vartheta-3}\eta^2 w_\epsilon\nabla w_\epsilon\nabla w_{\varepsilon,L}\mathrm{d}x. \tag{A.2}$$

In view of (2.7) with $\varepsilon = V_0 > 0$ and $p = \tilde{q} \in (2,6)$ which is determined later, there holds

$$f(\epsilon x + \epsilon y_\epsilon, w_\epsilon) w_\epsilon \le V_0 |w_\varepsilon|^2 + C_{\tilde{q}} |w_\varepsilon|^{\tilde{q}}. \tag{A.3}$$

Denoting $\mathbb{A}_\epsilon \triangleq a + b \int_{\mathbb{R}^3} |\nabla w_\epsilon|^2 \mathrm{d}x \ge a$, then we are derived from (4.4) that

$$-\Delta w_\epsilon = \frac{f(\epsilon x + \epsilon y_\epsilon, w_\epsilon) - V(\epsilon x + \epsilon y_\epsilon) w_\epsilon}{\mathbb{A}_\epsilon} + \frac{K(\epsilon x + \epsilon y_\epsilon)|w_\epsilon|^4 w_\epsilon}{\mathbb{A}_\epsilon} \text{ in } \mathbb{R}^3. \tag{A.4}$$

Taking $\varphi_{\varepsilon,L}$ as a test function in (A.4), we apply (A.2) and (A.3) to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |\nabla w_\epsilon|^2 \mathrm{d}x &\le 2 \int_{\mathbb{R}^3} \eta (w_{\epsilon,L})^{2(\vartheta-1)} w_\epsilon |\nabla w_\epsilon| |\nabla \eta| \mathrm{d}x \\ &+ \frac{C_{\tilde{q}}}{a} \int_{\mathbb{R}^2} \eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |w_\epsilon|^{\tilde{q}} \mathrm{d}x + \frac{K_0}{a} \int_{\mathbb{R}^3} \eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |w_\epsilon|^6 \mathrm{d}x,\end{aligned}$$

from where it follows the Young's inequality that:

$$\begin{aligned}\int_{\mathbb{R}^3} \eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |\nabla w_\epsilon|^2 \mathrm{d}x &\le 4 \int_{\mathbb{R}^3} (w_{\epsilon,L})^{2(\vartheta-1)} |w_\varepsilon|^2 |\nabla \eta|^2 \mathrm{d}x \\ &+ \frac{2C_{\tilde{q}}}{a} \int_{\mathbb{R}^3} \eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |w_\epsilon|^{\tilde{q}} \mathrm{d}x + \frac{2K_0}{a} \int_{\mathbb{R}^3} \eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |w_\epsilon|^6 \mathrm{d}x.\end{aligned} \tag{A.5}$$

We gather (1.15), (A.1), and (A.5) to conclude that

$$\begin{aligned}\left(\int_{\mathbb{R}^3} |\psi_{\epsilon,L}|^6 \mathrm{d}x\right)^{\frac{1}{3}} &\le C_1 S^{-1} \vartheta^2 \int_{\mathbb{R}^3} \left(\eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |\nabla w_\epsilon|^2 + |\nabla \eta|^2 (w_{\epsilon,L})^{2(\vartheta-1)} |w_\epsilon|^2\right) \mathrm{d}x \\ &\le C_2 \vartheta^2 \Bigg(\int_{\mathbb{R}^3} (w_{\epsilon,L})^{2(\vartheta-1)} |w_\varepsilon|^2 |\nabla \eta|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |w_\epsilon|^{\tilde{q}} \mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^3} \eta^2 (w_{\epsilon,L})^{2(\vartheta-1)} |w_\epsilon|^6 \mathrm{d}x \Bigg) \\ &\le C_2 \vartheta^2 \Bigg(\int_{R-r \le |x| \le R} |w_\epsilon|^{2\vartheta} \mathrm{d}x + \int_{|x| \ge R-r} (w_\epsilon)^{2(\vartheta-1)} |w_\epsilon|^{\tilde{q}} \mathrm{d}x \\ &\qquad + L^{2(\vartheta-1)} \int_{|x| \ge R-r} |w_\epsilon|^6 \mathrm{d}x \Bigg).\end{aligned}$$

In what follows, we shall fix $t = \sqrt[3]{r} > \frac{3}{2}$, $\vartheta = \frac{3(t-1)}{t} > 1$ and $\tilde{q} = \frac{2(1+t)}{t}$. As a consequence

$$\begin{aligned}\left(\int_{\mathbb{R}^3} |\psi_{\epsilon,L}|^6 \mathrm{d}x\right)^{\frac{1}{3}} &\le C_2 \vartheta^2 \Bigg\{ \left(\int_{R-r \le |x| \le R} |w_\epsilon|^6 \mathrm{d}x\right)^{\frac{t-1}{t}} \left(\int_{R-r \le |x| \le R} \mathrm{d}x\right)^{\frac{1}{t}} \\ &\qquad + \left(\int_{|x| \ge R-r} |w_\epsilon|^6 \mathrm{d}x\right)^{\frac{t-1}{t}} \left(\int_{|x| \ge R-r} |w_\epsilon|^2 \mathrm{d}x\right)^{\frac{1}{t}} \\ &\qquad + L^{2(\vartheta-1)} \int_{|x| \ge R-r} |w_\epsilon|^6 \mathrm{d}x \Bigg\} \\ &\le C_{2,r} \vartheta^2 \left[\left(\int_{|x| \ge R-r} |w_\epsilon|^6 \mathrm{d}x\right)^{\frac{t-1}{t}} + L^{2(\vartheta-1)} \int_{|x| \ge R-r} |w_\epsilon|^6 \mathrm{d}x \right],\end{aligned} \tag{A.6}$$

where we have used the fact that $|w_\epsilon|_2^2$ is uniformly bounded associated with $\epsilon \in (0, \epsilon^*)$.

Since $\psi_{\epsilon,L} = \eta w_\epsilon (w_{\epsilon,L})^{\vartheta-1}$, we are derived from (A.6) that

$$\begin{aligned}\left(\int_{|x|\geq R} |w_{\epsilon,L}|^{6\vartheta}\mathrm{d}x\right)^{\frac{1}{3}} &\leq \left(\int_{|x|\geq R} \eta^6 |w_\epsilon|^6 |w_{\epsilon,L}|^{6(\vartheta-1)}\mathrm{d}x\right)^{\frac{1}{3}} \\ &\leq \left(\int_{\mathbb{R}^3} |\psi_{\epsilon,L}|^6 \mathrm{d}x\right)^{\frac{1}{3}} \\ \leq C_{2,r}\vartheta^2 &\left[\left(\int_{|x|\geq R-r} |w_\epsilon|^6 \mathrm{d}x\right)^{\frac{t-1}{t}} + L^{2(\vartheta-1)} \int_{|x|\geq R-r} |w_\epsilon|^6 \mathrm{d}x\right]. \quad \text{(A.7)}\end{aligned}$$

Recalling the proof of Lemma 4.3, we have showed that $w_\epsilon \to w_0$ in $H^1(\mathbb{R}^3)$. Therefore, for the given $L > 1$, we can increase R sufficiently large to satisfy

$$\int_{|x|\geq R-r} |w_\epsilon|^6 \mathrm{d}x \leq \frac{1}{L^{2\vartheta}}. \quad \text{(A.8)}$$

Combining (A.7) with $\vartheta = \frac{3(t-1)}{t} > 1$ and (A.8), by tending $L \to +\infty$, we find that

$$\left(\int_{|x|\geq R} |w_\epsilon|^{\vartheta^2 s}\mathrm{d}x\right)^{\frac{1}{\vartheta^2 s}} \leq C_{2,r}^{\frac{1}{2\vartheta}} \vartheta^{\frac{1}{\vartheta}} \left(\int_{|x|\geq R-r} |w_\epsilon|^{\vartheta s}\mathrm{d}x\right)^{\frac{1}{\vartheta s}},$$

where $s = \frac{2t}{t-1}$. Therefore, proceeding this iteration procedure m times and multiplying these $m+1$ formulas,

$$|w_\epsilon|_{L^{\vartheta^{m+1}s}(|x|\geqslant R)} \leqslant C_{2,r}^{\sum_{i=1}^m \vartheta^{-i}} \vartheta^{\sum_{i=1}^m i\vartheta^{-i}} |w_\epsilon|_{L^6(|x|\geqslant R-r)},$$

and so

$$|w_\epsilon|_{L^\infty(|x|\geqslant R)} \leqslant C_{2,r}^{\sum_{i=1}^m \vartheta^{-i}} \vartheta^{\sum_{i=1}^m i\vartheta^{-i}} |w_\epsilon|_{L^6(|x|\geqslant R-r)}. \quad \text{(A.9)}$$

Adopting again $w_\epsilon \to w_0$ in $H^1(\mathbb{R}^3)$, one observes that $w_\epsilon(x) \to 0$ as $|x| \to \infty$ uniformly in $\epsilon \to 0^+$ if we let $R \to \infty$ in (A.9). Analogously, there exists a $\tau > 0$, such that $|w_\epsilon|_\infty \geq \tau$. Otherwise, we could suppose that $|w_\epsilon|_\infty \to 0$ as $\varepsilon \to 0^+$ in some sense of a subsequence, and so, $w_0 = 0$, which contradicts with $w_0 \neq 0$ concluded in the proof of Lemma 4.2. Finally, we take some very similar calculations exhibited above to prove that $|w_\epsilon|_\infty \leq C$ for some $C > 0$ independent of $\epsilon \in (0, \epsilon^*)$. In addition, we are able to follow [12] to conclude that $w_\epsilon \in \mathcal{C}^{1,\chi}_{\mathrm{loc}}(\mathbb{R}^3)$ for some $\chi \in (0, 1)$. The proof is completed. □

Lemma A.3. *Suppose* (g), (V), (Q), (K), *and* $(H_1)-(H_4)$. *Let* w_ϵ *be defined as Lemma 4.2 for all* $\epsilon \in (0, \epsilon^*)$, *then there exist* $\bar{c}', \hat{c}' > 0$ *independent of* ϵ, *such that*

$$w_\epsilon(x) \leq \bar{c}' \exp(-\hat{c}'|x|)$$

for all $\epsilon \in (0, \epsilon^*)$ *and* $x \in \mathbb{R}^3$.

Proof. Without loss of generality, we could assume that there is a constant $\mathcal{C} > 0$ independent of $\epsilon \in (0, \epsilon^*)$, such that $\|w_\epsilon\|^2 \leq \mathcal{C}$. According to (2.7) and (K), we apply $\lim_{|x|\to+\infty} w_\epsilon(x) = 0$ uniformly in $\epsilon \in (0, \epsilon^*)$ to reach

$$\lim_{|x|\to\infty} \left| \frac{f(\epsilon x + \epsilon y_\epsilon, w_\epsilon) + K(\epsilon x + \epsilon y_\epsilon)|w_\epsilon|^4 w_\epsilon}{w_\epsilon} \right| = 0 \text{ uniformly in } \epsilon \in (0, \epsilon^*).$$

Therefore, there is an $R > 0$ which is independent of $\epsilon \in (0, \epsilon^*)$, such that

$$f(\epsilon x+\epsilon y_\epsilon, w_\epsilon)+K(\epsilon x+\epsilon y_\epsilon)|w_\epsilon|^4 w_\epsilon \leq \frac{a^2 V_0}{2(a^2 + b\mathcal{C})} w_\epsilon, \ \forall \epsilon \in (0, \epsilon^*) \text{ and } |x| \geq R. \tag{A.10}$$

In view of (A.4), since $a \leq \mathbb{A}_\epsilon \leq \dfrac{a^2 + b\mathcal{C}}{a}$ for all $\epsilon \in (0, \epsilon^*)$, it follows from (A.10) and (V_0) that:

$$-\Delta w_\epsilon + \frac{aV_0}{2(a^2 + b\mathcal{C})} w_\epsilon \leq 0, \ \forall \epsilon \in (0, \epsilon^*) \text{ and } |x| \geq R.$$

Let $\psi(x) = \bar{c}' \exp(-\hat{c}'|x|)$ with $\bar{c}', \hat{c}' > 0$, such that $(\hat{c}')^2 < \dfrac{aV_0}{2(a^2 + b\mathcal{C})}$ and $w_\epsilon(x) \leq \bar{c}' \exp(-\hat{c}' R)$ for all $|x| = R$. Some simple calculations provide us that

$$-\Delta\psi + \frac{aV_0}{2(a^2 + b\mathcal{C})}\psi = \psi \left[\frac{aV_0}{2(a^2 + b\mathcal{C})} - (\hat{c}')^2 + \frac{2\hat{c}'}{|x|} \right] > 0, \text{ for all } |x| \geq R.$$

We define $\Sigma = \{|x| \geq R\} \cap \{w_\epsilon > \psi\}$ and choose $\phi = \max\{w_\epsilon - \psi, 0\} \in H_0^1(\mathbb{R}^3 \backslash B_R(0))$ as a test function in

$$-\Delta(w_\epsilon - \psi) + \frac{aV_0}{2(a^2 + b\mathcal{C})} (w_\epsilon - \psi) \leq 0, \text{ for all } |x| \geq R$$

to conclude that

$$0 \geq \int_\Sigma \left(|\nabla w_\epsilon - \nabla\psi|^2 + \frac{aV_0}{2(a^2 + b\mathcal{C})}|w_\epsilon - \psi|^2 \right) \mathrm{d}x \geq 0.$$

Therefore, the set $\Sigma \equiv \emptyset$. From which, we know that $w_\epsilon \leq \psi(x)$ for all $|x| \geq R$ and

$$w_\epsilon \leq \psi(x) = \bar{c}' \exp(-\hat{c}'|x|) \text{ for all } |x| \geq R.$$

Exploiting Lemma A.2 again, $|w_\epsilon|_\infty \leq C$, and so, the above inequality holds true for the whole space $\mathbb{R}^3$ by increasing $\bar{c}'$ to be large if necessary. The proof is completed. □

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Accepted: June 16, 2025.