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ON THE STABILITY OF STANDING WAVES OF KLEIN-GORDON EQUATIONS IN A SEMICLASSICAL REGIME

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Abstract. We investigate the orbital stability and instability of standing waves for two classes of Klein-Gordon equations in the semi-classical regime.

1. Introduction and results.

The nonlinear Klein-Gordon equation

$$\varepsilon^2 u_{tt} - \varepsilon^2 \Delta u + mu - |u|^{p-1}u = 0 \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

where $\varepsilon, m > 0$, $p > 1$ for $N = 1, 2$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$, arises in many physical contexts, e.g. in particle physics. It is also a model case for the mathematical study of nonlinear partial differential equations. We are interested in the study of the nonlinear Klein Gordon equation in presence of a potential depending on the space variable. Two different choices are viable. We can simply add a potential term $W(x)u$ to equation (1). This case has been studied, for the linear wave equation, for example, by Beals and Strauss in [5]. This approach leads us to consider the equation

$$\varepsilon^2 u_{tt} - \varepsilon^2 \Delta u + mu - Wu - |u|^{p-1}u = 0, \qquad \text{in } \mathbb{R}^N. \tag{2}$$

Otherwise, typically when dealing with quantum electrodynamics, the interaction between u and an external electromagnetic field is described substituting in (1) the usual time and space derivatives with the so called Weyl derivative, that is $D_t = \partial_t + iV(x)$, $D_{x_j} = \partial_{x_j} - iA_j(x)$. Here V and $(A_j)_j$ are the potentials of the electric and the magnetic external fields. This approach is classical in the linear

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theory of electromagnetic waves, and can be extended to the nonlinear setting, as in [14, section 7.5.1]. The nonlinear problem is called in literature Klein-Gordon-Maxwell nonlinear problem and in the last ten years it gained the attention of the mathematical community. See for example [6], [13] and the references therein and [4,12,25,36,39]. We will consider the case of zero magnetic potential, that leads us to consider the equation

$$\varepsilon^2 u_{tt} + 2i\varepsilon V u_t - \varepsilon^2 \Delta u + mu - V^2 u - |u|^{p-1}u = 0, \qquad \text{in } \mathbb{R}^N. \tag{3}$$

In this paper, we shall state all the results simultaneously for equations (2) and (3) by studying the problem

$$\begin{cases} \varepsilon^2 u_{tt} + 2i\varepsilon V u_t - \varepsilon^2 \Delta u + mu - Wu - |u|^{p-1}u = 0, & \text{in } \mathbb{R}^N, \\ u(0,x) = u_0 \in H^1(\mathbb{R}^N), \\ u_t(0,x) = u_1 \in L^2(\mathbb{R}^N), \end{cases} \tag{4}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, and V, W are real valued potential functions. Equation (4) yields to (3) for the choice $W = V^2$ as well as to (2) when $V = 0$.

We are interested in standing wave solutions of problem (4). Standing waves are solutions of the form $u(t,x) = e^{i\omega t/\varepsilon}\varphi_\omega(x/\varepsilon)$, which solve (4) with initial data $u_0(x) = \varphi_\omega(x/\varepsilon)$, $u_1(x) = i\omega/\varepsilon\varphi_\omega(x/\varepsilon)$ where $\omega \in \mathbb{R}$ and φ_ω satisfies

$$-\Delta\varphi_\omega + \big(m - \omega^2 - 2\omega V(\varepsilon y) - W(\varepsilon y)\big)\varphi_\omega - |\varphi_\omega|^{p-1}\varphi_\omega = 0, \qquad \text{in } \mathbb{R}^N. \tag{5}$$

We shall study the stability of standing waves of this equation in the semiclassical regime $\varepsilon \to 0$. To ensure existence of solutions to (5) for ε close to 0, we assume the following. The potentials V and W satisfy

$$V, W \in \mathcal{C}^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N). \tag{6}$$

For the function

$$Z(y) := m - \omega^2 - 2\omega V(y) - W(y), \quad y \in \mathbb{R}^N$$

there exists $x_0 \in \mathbb{R}^N$ such that

$$\nabla Z(x_0) = 0, \qquad \nabla^2 Z(x_0) \text{ is non-degenerate.} \tag{7}$$

Furthermore, we assume that

$$\inf_{x\in\mathbb{R}^N} Z(x) > 0. \tag{8}$$

Under these hypotheses, it is well-known (see e.g. [2] or [3, Section 8.2]) that when ε is close to 0 the equation (5) admits a family of positive, exponentially decaying, solutions $\varphi_\omega \subset H^1(\mathbb{R}^N)$ (hiding the dependence upon ε). More precisely, there exist $\xi_\varepsilon \in \mathbb{R}^N$ and $\psi_\omega \in H^1(\mathbb{R}^N)$ such that $\varphi_\omega(\cdot) = \psi_\omega(\cdot - \xi_\varepsilon) + \mathcal{O}(\varepsilon^2)$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0$, where $\xi_\varepsilon = x_0 + o(\varepsilon)$ and ψ_ω is the unique positive and radial solution of

$$-\Delta\psi_\omega + Z(x_0)\psi_\omega = |\psi_\omega|^{p-1}\psi_\omega, \qquad \text{in } \mathbb{R}^N. \tag{9}$$

The rate of exponential decay is uniform in ε for sufficiently small ε. Indeed, let $\lambda_0 := \inf_{x\in\mathbb{R}^N} Z(x)$. By assumption (8), we have $\lambda_0 > 0$. Then there exists $C_0 > 0$ depending only on λ_0 such that $|\varphi_\omega(x)| \le C_0 e^{-\sqrt{\lambda_0}|x|/2}$ (see e.g. [8, Chapter 3]).

In what follows, we will need the following assumption on the dependence in ω of the family (φ_ω).

$$\omega \mapsto \varphi_\omega \in H^1(\mathbb{R}^N) \text{ is } \mathcal{C}^1 \text{ uniformly in } \epsilon. \tag{10}$$

Actually, since the family φ_ω is build upon (ψ_ω), which is $\mathcal{C}^1$ in ω, the statement (10) could probably be obtained by rewriting the proofs of [2,3] by using parameter depending versions of the various results used. Since it is not our main concern in this paper we leave this issue aside and simply assume (10).

A standing wave of (4) is said to be *(orbitally) stable* if any solution of (4) starting close to the standing wave remains close for all time, up to the invariances of the equation. More precisely, for fixed ε, we say that $e^{\frac{i\omega t}{\varepsilon}}\varphi_\omega\left(\frac{x}{\varepsilon}\right)$ is stable if for all $\eta>0$ there exists $\delta>0$ such that for all $(u_0,u_1)\in H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$ verifying

$$\left\|u_0-\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{H^1}+\left\|u_1-i\frac{\omega}{\varepsilon}\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2}<\delta$$

the solution $u(t,x)$ of (4) with initial data (u_0,u_1) satisfies

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R}}\left(\left\|u-e^{i\theta}\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{H^1}+\left\|u_t-ie^{i\theta}\frac{\omega}{\varepsilon}\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2}\right)<\eta. \tag{11}$$

Since the pioneering works [7, 10, 15, 16, 37, 38], the study of orbital stability for standing waves of dispersive PDE has attracted a lot of attention. Among many others, one can refer to [18,19,22]; see also the books and surveys [9,21,33,35] and the references therein. Relatively few works [17, 23, 26] are concerned with stability at the semi-classical limit for Schrödinger type equations. For Klein-Gordon equations, after the ground works [30, 31] revisited some years ago in [34], there has been a recent interest for instability by blow-up [24, 27–29].

We study stability within the framework of Grillakis-Shatah-Strauss Theory [15, 16]. We first rewrite (4) in Hamiltonian form

$$\varepsilon\frac{\partial U}{\partial t}=JE'(U), \tag{12}$$

where $U=\begin{pmatrix}u\\v\end{pmatrix}$, $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, and the energy E is defined for $U\in H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$ by

$$E(U)=\frac12\|v-iVu\|_{L^2}^2+\frac{\varepsilon^2}{2}\|\nabla u\|_{L^2}^2+\frac{m}{2}\|u\|_{L^2}^2-\frac12\int_{\mathbb{R}^N}W|u|^2dx-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

It is easy to see that if u solves (4) and v is defined by $v:=\varepsilon u_t+iVu$, then $U=\begin{pmatrix}u\\v\end{pmatrix}$ solves (12). Due to the Hamiltonian form, the energy E is (at least formally) a conserved quantity for the flow of (12). The invariance with respect to phase shift (i.e. if U solves (12), then for any fixed $\theta\in\mathbb{R}$ the function $e^{i\theta}U$ also solves (12)) generates another conserved quantity, the charge Q, which is defined by

$$Q(U)=\Im\int_{\mathbb{R}^N}\bar{u}v\,dx.$$

In this Hamiltonian formulation, a standing wave $u=e^{i\omega t/\varepsilon}\varphi_\omega(x/\varepsilon)$ becomes $U=e^{i\omega t/\varepsilon}\Phi_\omega(x/\varepsilon)$ for $\Phi_\omega=\begin{pmatrix}\varphi_\omega\\i(\omega+V)\varphi_\omega\end{pmatrix}$. Note that $\Phi_\omega(\cdot/\varepsilon)$ is a critical point of the functional $E-\omega Q$. The energy and the charge for a standing wave are given by

$$\begin{aligned}E(\varphi_\omega):=E(\Phi_\omega(\cdot/\varepsilon))=\varepsilon^N\Big(&\frac12\|\nabla\varphi_\omega\|_{L^2}^2-\frac12\int_{\mathbb{R}^N}W(\varepsilon y)|\varphi_\omega|^2dy\\&+\frac{m+\omega^2}{2}\|\varphi_\omega\|_{L^2}^2-\frac{1}{p+1}\|\varphi_\omega\|_{L^{p+1}}^{p+1}\Big),\end{aligned} \tag{13}$$

$$Q(\varphi_\omega):=Q(\Phi_\omega(\cdot/\varepsilon))=\varepsilon^N\left(\omega\|\varphi_\omega\|_{L^2}^2+\int_{\mathbb{R}^N}V(\varepsilon y)|\varphi_\omega|^2\right). \tag{14}$$

To work in the context of the theory developed by Grillakis, Shatah and Strauss in [15,16], three assumptions have to be satisfied. First, the Cauchy Problem has to be locally well-posed in $H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$. This follows from standard results when $V\equiv 0$ and we shall assume it otherwise. Indeed, when W verifies (6), the local well-posedness of the Cauchy Problem for (2) follows from a simple adaptation of classical methods (see e.g. [32]). No well-posedness result is available for the specific case of (3), see nevertheless [1] for results on a Klein-Gordon equation with a damping term and [11,20] for the Maxwell-Klein-Gordon equation. Second, the map $\omega\to\varphi_\omega$ has to be $\mathcal{C}^1$, which is granted by assumption (10). Third, the spectrum of the linearized operator

$$H_\varepsilon := E''(\Phi_\omega(\cdot/\varepsilon)) - \omega Q''(\Phi_\omega(\cdot/\varepsilon))$$

must decompose into a finite number of negative eigenvalues, a nondegenerate kernel (i.e. containing only the eigenvectors due to the invariances of the equation), and positive spectrum away from 0. This will be proved in Proposition 2. Under these three assumptions the stability of the standing waves depends on two informations. The first one is a slope information given by the sign of the quantity $\frac{\partial^2}{\partial\omega^2}d(\omega)$, where $d(\omega)=E(\varphi_\omega)-\omega Q(\varphi_\omega)$. Note that it is easy to verify that $d'(\omega)=-Q(\varphi_\omega)$ (see e.g. [15, Eq. (2.20)]). The second information is related to the number of negative eigenvalues of the linearized operator H_ε.

According to the theory developed in [15, 16], a standing wave $e^{\frac{i\omega t}{\varepsilon}}\varphi_\omega\left(\frac{x}{\varepsilon}\right)$ is stable if two conditions are satisfied.

(i) The *Slope Condition*: $\dfrac{\partial}{\partial\omega}Q(\varphi_\omega)<0$.

(ii) The *Spectral Condition*: H_ε has exactly one negative eigenvalue.

On the other hand, denote by $n(H_\varepsilon)$ the number of negative eigenvalues of H_ε and set $p(\omega)=0$ if $\dfrac{\partial}{\partial\omega}Q(\varphi_\omega)>0$, $p(\omega)=1$ if $\dfrac{\partial}{\partial\omega}Q(\varphi_\omega)<0$. Then the standing wave is unstable if

(iii) *Instability Condition*: $n(H_\varepsilon)-p(\omega)$ is odd.

In [16], it was proved that when (iii) is satisfied, then the instability of the standing waves follows from a linear mechanism, in the sense that the 0 solution of the linearized equation around the standing wave is unstable. Note that when $n(H_\varepsilon)-p(\omega)$ is even, the question of stability or instability is still open.

Our main result is the following.

Theorem 1.1. *Assume that conditions (6)-(8), (10) hold. Then, there exists $\varepsilon_0>0$ such that for any $0<\varepsilon<\varepsilon_0$ we have the following facts.*

1. *If $p<1+4/N$, then the Slope Condition $\frac{\partial}{\partial\omega}Q(\varphi_\omega)<0$ is fulfilled if*

$$Z(x_0)<(\omega+V(x_0))^2\Big(\frac{4}{p-1}-N\Big) \qquad \textit{(non-critical case)}$$

or if

$$\begin{cases} Z(x_0)=(\omega+V(x_0))^2\Big(\frac{4}{p-1}-N\Big),\\ \Big(\Delta Z(x_0)-\Delta V(x_0)\Big(1+\frac{2(\omega+V(x_0))}{Z(x_0)}\Big)\Big)<0, \end{cases} \qquad \textit{(critical case).}$$

2. *If $p\geq 1+4/N$, then we always have $\frac{\partial}{\partial\omega}Q(\varphi_\omega)>0$.*
3. *We have the equality $n(H_\varepsilon)=n(\nabla^2 Z(x_0))+1$, where $n(\nabla^2Z(x_0))$ is the number of negative eigenvalues of $\nabla^2Z(x_0)$.*

From the theory of [15,16] we infer the following corollary on the stability of the standing wave in the particular case where x_0 is a minimum of Z.

Corollary 1. *Assume that* (4) *is locally well-posed in* $H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$*, conditions* (6)-(8)*,* (10) *hold and that* x_0 *is non-degenerate local minimum of* Z*. Then there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *the standing waves* $e^{i\omega t}\varphi_\omega$ *are stable if* $p < 1 + 4/N$ *and*

$$Z(x_0) < (\omega + V(x_0))^2\Big(\frac{4}{p-1} - N\Big)$$

and unstable if

$$Z(x_0) > (\omega + V(x_0))^2\Big(\frac{4}{p-1} - N\Big)$$

or if $p \geq 1 + 4/N$*.*

Note that, conversely to what was happening in the case of Schrödinger equations studied in [23], the values of the potentials V and W at x_0 come into play for the Slope Condition even in the noncritical case. Note also that only the local behavior of Z around x_0 influences the stability or instability.

Notations. Most of the objects we consider will depend both on ε and ω. We will emphasize the most important parameter by indicating it as a subscript, the dependence in the other parameter being understood.

2. **Proofs.**

In this section, we prove Theorem 1.1 and Corollary 1. We start by focusing on the Slope Condition and then we study the Spectral Condition. We finish by the proof of Corollary 1. For the sake of simplicity in notations and without loss of generality, in the rest of this section we assume that $x_0 = 0$.

2.1. **The Slope Condition.**

We start with the noncritical case.

2.1.1. *Noncritical case.*

We assume that

$$Z(0) \neq (\omega + V(0))^2\Big(\frac{4}{p-1} - N\Big).$$

We first rewrite $Q(\varphi_\omega)$ by expanding $V(\varepsilon y)$ and using the exponential decay of φ_ω:

$$Q(\varphi_\omega) = \varepsilon^N\,(\omega + V(0))\,\|\varphi_\omega\|_{L^2}^2 + \mathcal{O}(\varepsilon^{N+1}).$$

Therefore, since

$$\frac{\partial}{\partial\omega}Q(\varphi_\omega) = \varepsilon^N\|\varphi_\omega\|_{L^2}^2 + \varepsilon^N(\omega + V(0))\frac{\partial}{\partial\omega}\|\varphi_\omega\|_{L^2}^2 + \mathcal{O}(\varepsilon^{N+1}), \tag{15}$$

to evaluate the sign of the map $\omega \mapsto \frac{\partial}{\partial\omega}Q(\varphi_\omega)$ one should compute the quantity

$$\frac{\partial}{\partial\omega}\|\varphi_\omega\|_{L^2}^2 = 2\int_{\mathbb{R}^N} R_\omega\varphi_\omega, \tag{16}$$

where $R_\omega(x) := \frac{\partial\varphi_\omega}{\partial\omega}(x)$. We remark that differentiation of (5) with respect to ω easily yields

$$L_\varepsilon R_\omega = 2(\omega + V(\varepsilon y))\varphi_\omega, \tag{17}$$

where the linearized operator L_ε is defined by

$$L_\varepsilon := -\Delta + Z(\varepsilon y) - p|\varphi_\omega|^{p-1}.$$

If we now introduce the rescaling $\varphi_\omega(x) = \lambda^{\frac{1}{p-1}}\varphi_\lambda(\sqrt{\lambda}x)$, it follows that φ_λ satisfies

$$-\Delta\varphi_\lambda + \lambda^{-1} Z\left(\frac{\varepsilon y}{\sqrt{\lambda}}\right)\varphi_\lambda - |\varphi_\lambda|^{p-1}\varphi_\lambda = 0, \qquad \text{in } \mathbb{R}^N. \tag{18}$$

Now, differentiating equation (18) with respect to λ and denoting $T_\lambda = \frac{\partial \varphi_\lambda}{\partial \lambda}_{|\lambda=1}$ yields

$$L_\varepsilon T_\lambda - Z(\varepsilon y)\varphi_\omega - \frac{1}{2}\varepsilon y \cdot \nabla Z(\varepsilon y)\varphi_\omega = 0. \tag{19}$$

Since 0 is a critical point of Z, a Taylor expansion gives

$$Z(\varepsilon y) = Z(0) + \mathcal{O}(\varepsilon^2|y|^2), \tag{20}$$

$$\frac{1}{2}\varepsilon y \cdot \nabla Z(\varepsilon y) = \mathcal{O}(\varepsilon^2|y|^2). \tag{21}$$

Then, from (19), as $\varepsilon \to 0$ we have

$$L_\varepsilon T_\lambda = Z(0)\varphi_\omega + \mathcal{O}(\varepsilon^2|y|^2\varphi_\omega), \quad \text{in } \mathbb{R}^N. \tag{22}$$

Then, in turn, taking into account identity (17) we get

$$\begin{aligned}
Z(0)\int_{\mathbb{R}^N} R_\omega\varphi_\omega &= \int_{\mathbb{R}^N} R_\omega L_\varepsilon T_\lambda + \mathcal{O}(\varepsilon^2) \\
&= \int_{\mathbb{R}^N} L_\varepsilon R_\omega T_\lambda + \mathcal{O}(\varepsilon^2) \\
&= \int_{\mathbb{R}^N} 2(\omega + V(\varepsilon y))\varphi_\omega T_\lambda + \mathcal{O}(\varepsilon^2) \\
&= 2(\omega + V(0))\int_{\mathbb{R}^N} \varphi_\omega T_\lambda + \mathcal{O}(\varepsilon) \\
&= (\omega + V(0))\frac{\partial}{\partial\lambda}\|\varphi_\lambda\|^2_{L^2|\lambda=1} + \mathcal{O}(\varepsilon) \\
&= (\omega + V(0))\Big(\frac{N}{2} - \frac{2}{p-1}\Big)\|\varphi_\omega\|^2_{L^2} + \mathcal{O}(\varepsilon).
\end{aligned} \tag{23}$$

In conclusion, by combining (15), (16) and (22), we have

$$\frac{\partial}{\partial\omega}Q(\varphi_\omega) = \varepsilon^N\left(1 + \frac{(\omega+V(0))^2}{Z(0)}\Big(N - \frac{4}{p-1}\Big)\right)\|\varphi_\omega\|^2_{L^2} + \mathcal{O}(\varepsilon^{N+1}).$$

Then, taking into account the fact that $Z(0) > 0$ and that φ_ω converges to ψ_ω in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0$, the sign of $\frac{\partial}{\partial\omega}Q(\varphi_\omega)$ is the sign of

$$Z(0) - (\omega + V(0))^2\Big(\frac{4}{p-1} - N\Big).$$

2.1.2. *Critical case.* We assume now that

$$Z(0) = (\omega + V(0))^2\Big(\frac{4}{p-1} - N\Big). \tag{24}$$

In the critical case, the term of order ε^N in the expansion of $\frac{\partial}{\partial\omega}Q(\varphi_\omega)$ vanishes and we need to calculate the expansion at a higher order. We first refine (20)-(21).

$$Z(\varepsilon y) = Z(0) + \frac{\varepsilon^2}{2}\nabla^2 Z(0)(y,y) + \mathcal{O}(\varepsilon^3|y|^3)$$

$$\frac{1}{2}\varepsilon y \cdot \nabla Z(\varepsilon y) = \frac{\varepsilon^2}{2}\nabla^2 Z(0)(y,y) + \mathcal{O}(\varepsilon^3|y|^3).$$

Then (19) gives

$$L_\varepsilon T_\lambda = Z(0)\varphi_\omega + \varepsilon^2 \nabla^2 Z(0)(y,y)\varphi_\omega + \mathcal{O}(\varepsilon^3|y^3|)\varphi_\omega.$$

Now, we have

$$Z(0)\int_{\mathbb{R}^N} R_\omega \varphi_\omega = \int_{\mathbb{R}^N} R_\omega L_\varepsilon T_\lambda - \varepsilon^2 \int_{\mathbb{R}^N} \nabla^2 Z(0)(y,y) R_\omega \varphi_\omega + \mathcal{O}(\varepsilon^3). \tag{25}$$

From (17), we obtain

$$\int_{\mathbb{R}^N} R_\omega L_\varepsilon T_\lambda = \int_{\mathbb{R}^N} L_\varepsilon R_\omega T_\lambda = \int_{\mathbb{R}^N} 2(\omega + V(\varepsilon y))\varphi_\omega T_\lambda. \tag{26}$$

Expanding the potential V we get

$$\begin{aligned}\int_{\mathbb{R}^N} 2V(\varepsilon y)\varphi_\omega T_\lambda = \int_{\mathbb{R}^N} 2V(0)\varphi_\omega T_\lambda + 2\varepsilon \int_{\mathbb{R}^N} y\cdot \nabla V(0)\varphi_\omega T_\lambda \\ + \varepsilon^2 \int_{\mathbb{R}^N} \nabla^2 V(0)(y,y)\varphi_\omega T_\lambda + \mathcal{O}(\varepsilon^3).\end{aligned} \tag{27}$$

Note that since $\varphi_\omega = \psi_\omega(\cdot - \xi_\varepsilon) + \mathcal{O}(\varepsilon^2)$ and $\xi_\varepsilon = o(\varepsilon)$, we have

$$2\varepsilon \int_{\mathbb{R}^N} y\cdot\nabla V(0)\varphi_\omega T_\lambda = 2\varepsilon \int_{\mathbb{R}^N} y\cdot \nabla V(0)\psi_\omega T_\lambda + o(\varepsilon^2) = o(\varepsilon^2) \tag{28}$$

where the last cancellation comes from the fact that ψ_ω is radial. Coming back to (26) and as in (23), we have

$$\begin{aligned}\int_{\mathbb{R}^N} R_\omega L_\varepsilon T_\lambda = (\omega + V(0))\Big(\frac{N}{2} - \frac{2}{p-1}\Big)\|\varphi_\omega\|_{L^2}^2 \\ + \varepsilon^2 \int_{\mathbb{R}^N} \nabla^2 V(0)(y,y)\varphi_\omega T_\lambda + o(\varepsilon^2).\end{aligned} \tag{29}$$

It remains to compute the integrals involving the Hessians in (25) and (29). Since our problem is invariant by an orthonormal transformation, we can assume without loss of generality that $\nabla^2 V(0) = \mathrm{diag}(b_1,\dots,b_N)$. Hence $\nabla^2 V(0)(y,y) = \sum_{j=1}^N b_j y_j^2$. Recall also that T_λ can be computed explicitly to have

$$T_\lambda = -\frac{1}{p-1}\varphi_\omega - \frac{1}{2} y\cdot\nabla\varphi_\omega.$$

Therefore,

$$\int_{\mathbb{R}^N} b_j y_j^2 \varphi_\omega T_\lambda = -\frac{b_j}{p-1}\int_{\mathbb{R}^N} y_j^2\varphi_\omega^2 - \frac{b_j}{2}\sum_{k=1}^N \int_{\mathbb{R}^N} y_j^2 y_k \varphi_\omega \frac{\partial \varphi_\omega}{\partial y_k}.$$

We have after integration by parts

$$2\sum_{k=1}^N \int_{\mathbb{R}^N} y_j^2 y_k\varphi_\omega \frac{\partial\varphi_\omega}{\partial y_k} = -\sum_{k=1}^N \int_{\mathbb{R}^N} (y_j^2 + 2\delta_{jk} y_j^2)\varphi_\omega^2 = -(N+2)\int_{\mathbb{R}^N} y_j^2\varphi_\omega^2.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla^2 V(0)(y,y)\varphi_\omega T_\lambda &= \sum_{j=1}^N \int_{\mathbb{R}^N} b_j y_j^2 \varphi_\omega T_\lambda \\ &= -\Big(\frac{1}{p-1} - \frac{N+2}{4}\Big)\sum_{j=1}^N b_j \int_{\mathbb{R}^N} y_j^2\varphi_\omega^2.\end{aligned}$$

Recall the following expansion in ε for R_ω and φ_ω.

$$\varphi_\omega = \psi_\omega + o(\varepsilon), \qquad R_\omega = \frac{\partial \psi_\omega}{\partial \omega} + o(\varepsilon).$$

Therefore, since ψ_ω is radial,

$$\int_{\mathbb{R}^N} y_j^2 \varphi_\omega^2 = \int_{\mathbb{R}^N} y_j^2 \psi_\omega^2 + o(\varepsilon) = \frac{1}{N} \||y|\psi_\omega\|_{L^2}^2 + o(\varepsilon),$$

and so

$$\int_{\mathbb{R}^N} \nabla^2 V(0)(y,y)\varphi_\omega T_\lambda = -\left(\frac{1}{p-1} - \frac{N+2}{4}\right) \frac{1}{N} \||y|\psi_\omega\|_{L^2}^2 \Delta V(0) + o(\varepsilon). \tag{30}$$

Similarly, we have

$$\int_{\mathbb{R}^N} \nabla^2 Z(0)(y,y) R_\omega \varphi_\omega = -\left(\frac{1}{p-1} - \frac{N+2}{4}\right) \frac{1}{N} \||y|\psi_\omega\|_{L^2}^2 \Delta Z(0) + o(\varepsilon). \tag{31}$$

Summarizing, using successively (25), (29), (30), (31) and (24) we have obtained

$$\begin{aligned} \int_{\mathbb{R}^N} R_\omega \varphi_\omega = &-\frac{1}{2(\omega + V(0))} \|\varphi_\omega\|_{L^2}^2 \\ &+ \varepsilon^2 \frac{(\Delta Z(0) - \Delta V(0))}{N Z(0)} \left(\frac{1}{p-1} - \frac{N+2}{4}\right) \||y|\psi_\omega\|_{L^2}^2 + o(\varepsilon^2). \end{aligned} \tag{32}$$

Now, we compute $\frac{\partial Q(\varphi_\omega)}{\partial \omega}$. First, recall that, coming back to the definition (14) of Q, we have

$$\varepsilon^{-N} \frac{\partial Q(\varphi_\omega)}{\partial \omega} = \|\varphi_\omega\|_{L^2}^2 + 2\omega \int_{\mathbb{R}^N} R_\omega \varphi_\omega + 2 \int_{\mathbb{R}^N} V(\varepsilon y) R_\omega \varphi_\omega.$$

As in (27), (28), and (30) we can expand in ε and get

$$\begin{aligned} 2 \int_{\mathbb{R}^N} V(\varepsilon y) R_\omega \varphi_\omega = 2V(0) &\int_{\mathbb{R}^N} R_\omega \varphi_\omega \\ &- \varepsilon^2 \left(\frac{1}{p-1} - \frac{N+2}{4}\right) \frac{1}{N} \||y|\psi_\omega\|_{L^2}^2 \Delta V(0) + o(\varepsilon^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \varepsilon^{-N} \frac{\partial Q(\varphi_\omega)}{\partial \omega} = \|\varphi_\omega\|_{L^2}^2 + 2(\omega + V(0)) &\int_{\mathbb{R}^N} R_\omega \varphi_\omega \\ &- \varepsilon^2 \left(\frac{1}{p-1} - \frac{N+2}{4}\right) \frac{1}{N} \||y|\psi_\omega\|_{L^2}^2 \Delta V(0) + o(\varepsilon^2). \end{aligned}$$

Using (32), we finally get

$$\begin{aligned} \varepsilon^{-N} \frac{\partial Q(\varphi_\omega)}{\partial \omega} = \varepsilon^2 &\left(\frac{1}{p-1} - \frac{N+2}{4}\right) \frac{1}{N} \||y|\psi_\omega\|_{L^2}^2 \times \\ &\times \left(\Delta Z(0) - \Delta V(0) \left(1 + \frac{2(\omega + V(0))}{Z(0)}\right)\right) + o(\varepsilon^2). \end{aligned}$$

2.2. **The spectral condition.** We first give some preliminary considerations on the scalar operator

$$L_\varepsilon := -\Delta + Z(\varepsilon y) - p|\varphi_\omega|^{p-1}.$$

The analysis of the spectrum of the operator L_ε was essential in the case of Schrödinger equations [17,23] to determine the spectral stability condition. It turns out that it will play also an important role in the analysis for the Klein-Gordon equation.

We define the operator $L_0 := -\Delta + Z(0) - p\psi_\omega^{p-1}$ (recall that ψ_ω solves (9)). It is well known (see e.g. [3]) that the spectrum of L_0 consists of one negative eigenvalue, a N-dimensional kernel (generated by $\frac{\partial \psi_\omega}{\partial y_j}$ for $j = 1, \dots, N$) and the rest of the spectrum is bounded away from 0. When ε is close to 0, the spectrum of L_ε will be close to the spectrum of L_0. In particular, the 0 eigenvalue, of multiplicity N, will transform into N possibly different eigenvalues close to 0 but shifted either to the positive or to the negative side of the real axis, depending on the sign of the eigenvalues of the Hessian of Z at 0. More precisely, the following proposition was proved in [23] (see [17] for a detailed justification).

Proposition 1. *The spectrum of L_ε consists of positive spectrum away from 0 and a set of $N+1$ simple eigenvalues $\{\lambda_0, \lambda_1, \dots, \lambda_N\}$ such that*

$$\lambda_0 < \lambda_1 \le \cdots \le \lambda_N.$$

As $\varepsilon \to 0$, we have $\lambda_0 < 0$ and the following asymptotic expansion holds for the other eigenvalues:

$$\lambda_j = c_j \varepsilon^2 + o(\varepsilon^2), \qquad j = 1, ..., N,$$

where $c_j = \frac{1}{2} \frac{\|\psi_\omega\|_{L^2}^2}{\|\frac{\partial \psi_\omega}{\partial x_j}\|_{L^2}^2} a_j$ and $\{a_1, \dots, a_N\}$ are the eigenvalues of the Hessian matrix $\nabla^2 Z(0)$.

In the following proposition, we establish the spectral decomposition for H_ε and we relate the number of negative eigenvalues of H_ε with the number of negative eigenvalues of L_ε.

Proposition 2. *The spectrum of H_ε consists into positive essential spectrum away from 0, a finite number of eigenvalues and a nondegenerate kernel, i.e.*

$$\ker(H_\varepsilon) = \operatorname{span}\left\{ \begin{pmatrix} i\varphi_\omega(\cdot/\varepsilon) \\ -(\omega + V)\varphi_\omega(\cdot/\varepsilon) \end{pmatrix} \right\}.$$

In addition, the number $n(H_\varepsilon)$ of negative eigenvalues of H_ε is identical to the number of negative eigenvalues $n(L_\varepsilon)$ of the operator L_ε.

Therefore, (3) in Theorem 1.1 is a direct consequence of Propositions 1 and 2. In particular, the spectral condition for stability will be satisfied if and only if 0 is a non-degenerate local minimum of Z.

Proof of Proposition 2. Explicitly, $H_\varepsilon = E''(\varphi_\omega) - \omega Q''(\varphi_\omega)$ is given by

$$\begin{pmatrix} \begin{array}{r} -\varepsilon^2\Delta + m - W(x) + V(x)^2 - p|\varphi_\omega(x/\varepsilon)|^{p-1}\Re(\cdot) \\ -i|\varphi_\omega(x/\varepsilon)|^{p-1}\Im(\cdot) \end{array} & i(\omega + V(x)) \\ -i(\omega + V(x)) & 1 \end{pmatrix}.$$

For notational convenience we change variables and replace x/ε by y. We denote $V(\varepsilon y)$ and $W(\varepsilon y)$ by V_ε and W_ε. Then H_ε becomes $H_\varepsilon = \varepsilon^N \tilde{H}_\varepsilon$, where $\tilde{H}_\varepsilon$ is the operator

$$\begin{pmatrix} -\Delta + m - W_\varepsilon + V_\varepsilon^2 - p|\varphi_\omega|^{p-1}\Re(\cdot) - i|\varphi_\omega|^{p-1}\Im(\cdot) & i(\omega + V_\varepsilon) \\ -i(\omega + V_\varepsilon) & 1 \end{pmatrix}. \tag{33}$$

Therefore, to find the spectrum of H_ε it is enough to find the spectrum of $\tilde{H}_\varepsilon$. Due to exponential localization of φ_ω, this operator is a compact perturbation of

$$L := \begin{pmatrix} -\Delta + m - W_\varepsilon + V_\varepsilon^2 & i(\omega + V_\varepsilon) \\ -i(\omega + V_\varepsilon) & 1 \end{pmatrix},$$

and therefore by Weyl's Theorem they share the same essential spectrum. For $U = \begin{pmatrix} u \\ v \end{pmatrix}$ we have

$$\langle LU, U\rangle = \|\nabla u\|_{L^2}^2 + m\|u\|_{L^2}^2 - \int_{\mathbb{R}^N} (W_\varepsilon - V_\varepsilon^2)|u|^2 dx + 2\Re \int_{\mathbb{R}^N} i(\omega + V_\varepsilon) v\bar{u} dx + \|v\|_{L^2}^2,$$

which can easily be factorized into

$$\begin{aligned} \langle LU, U\rangle &= \|\nabla u\|_{L^2}^2 + (m - \omega^2)\|u\|_{L^2}^2 - \int_{\mathbb{R}^N} (W_\varepsilon + 2\omega V_\varepsilon)|u|^2 dx \\ &\qquad + \|v - i(\omega + V_\varepsilon)u\|_{L^2}^2, \\ &= \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^N} Z(\varepsilon y)|u|^2 dx + \|v - i(\omega + V_\varepsilon)u\|_{L^2}^2, \\ &\geq \|\nabla u\|_{L^2}^2 + \lambda_0\|u\|_{L^2}^2 + \|v - i(\omega + V_\varepsilon)u\|_{L^2}^2, \end{aligned} \tag{34}$$

where the last inequality follows from the assumption $\lambda_0 = \inf_{x\in\mathbb{R}^N} Z(x) > 0$. We claim that there exists $\delta > 0$ such that

$$\langle LU, U\rangle \geq \delta\|U\|_{H^1\times L^2}^2. \tag{35}$$

Indeed, assume by contradiction that there exists $(U_n) = \begin{pmatrix} u_n \\ v_n \end{pmatrix} \subset H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ such that $\|U_n\|_{H^1\times L^2}^2 = 1$ and $\langle LU_n, U_n\rangle \to 0$ as $n \to +\infty$. From (34) this implies that $u_n \to 0$ in $H^1(\mathbb{R}^N)$ and $\|v_n - i(\omega + V)u_n\|_{L^2} \to 0$. Therefore $\|U_n\|_{H^1\times L^2}^2 \to 0$, which is a contradiction and proves (35). From (35), we infer that the spectrum of L is contained into $[\delta, +\infty)$. In particular, this implies that the essential spectrum of $\tilde{H}_\varepsilon$ is contained into $[\delta, +\infty)$. Let us now treat the eigenvalues of $\tilde{H}_\varepsilon$. Recall the definition of L_ε and also define another operator R_ε by

$$\begin{aligned} L_\varepsilon &:= -\Delta + Z(\varepsilon y) - p|\varphi_\omega|^{p-1}, \\ R_\varepsilon &:= -\Delta + Z(\varepsilon y) - |\varphi_\omega|^{p-1}. \end{aligned}$$

Then a similar factorization as in (34) gives for $U = \begin{pmatrix} u \\ v \end{pmatrix} \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$

$$\left\langle \tilde{H}_\varepsilon U, U \right\rangle = \langle L_\varepsilon \Re(u), \Re(u)\rangle + \langle R_\varepsilon \Im(u), \Im(u)\rangle + \|v - i(\omega + V)u\|_{L^2}^2 \tag{36}$$

First we remark that due to the boundedness of φ_ω, there exists $C > 0$ such that for any $U \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ we have

$$\left\langle \tilde{H}_\varepsilon U, U \right\rangle \geq -C\|U\|_{H^1\times L^2}^2$$

Therefore, in $(-\infty, \delta/2)$, the spectrum of $\tilde{H}_\varepsilon$ consists of a finite number of eigenvalues. We claim that the eigenvalues of $\tilde{H}_\varepsilon$ will be distributed on one side or the other of the real axis in the same fashion as the eigenvalues of $(L_\varepsilon, R_\varepsilon)$. Indeed, the number of negative eigenvalues of $\tilde{H}_\varepsilon$ is given by

$$\max\Big\{d\in\mathbb{N};\ \Big\langle \tilde{H}_\varepsilon U, U\Big\rangle < 0 \qquad \text{for all } U\in\mathcal{M}\subset H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N), U\neq 0,\ \dim(\mathcal{M})=d\Big\},$$

which, in view of (36), is exactly the same number as

$$\max\Big\{d\in\mathbb{N};\ \langle L_\varepsilon\Re(u),\Re(u)\rangle + \langle R_\varepsilon\Im(u),\Im(u)\rangle < 0 \qquad \text{for all } u\in\mathcal{M}\subset H^1(\mathbb{R}^N), u\neq 0,\ \dim(\mathcal{M})=d\Big\},$$

Since $R_\varepsilon\varphi_\omega = 0$ and $\varphi_\omega > 0$, R_ε has a first simple eigenvalue at 0 with eigenvector φ_ω, and the rest of its spectrum is positive. Therefore R_ε has no negative eigenvalue, and we can conclude that

$$n(\tilde{H}_\varepsilon) = n(L_\varepsilon).$$

Consider now the kernel of $\tilde{H}_\varepsilon$. We readily see from (33) that $U = \begin{pmatrix} u \\ v\end{pmatrix}$ belongs to the kernel if and only if $v = i(\omega+V)u$ and u belongs to the kernel of

$$-\Delta + m - \omega^2 - W - 2\omega V - p|\varphi_\omega|^{p-1}\Re(\cdot) - i|\varphi_\omega|^{p-1}\Im(\cdot),$$

in other words $(\Re(u),\Im(u))$ belongs to the kernel of $(L_\varepsilon, R_\varepsilon)$.We already know that $\ker(R_\varepsilon) = \operatorname{span}\{\varphi_\omega\}$. From Proposition 1 and the nondegeneracy (7) of $\nabla^2 Z(0)$ we infer that L_ε has no kernel for ε small. Therefore, the kernel of $\tilde{H}_\varepsilon$ is given by

$$\ker(\tilde{H}_\varepsilon) = \operatorname{span}\left\{\begin{pmatrix} i\varphi_\omega \\ -(\omega+V_\varepsilon)\varphi_\omega\end{pmatrix}\right\}.$$

Coming back to the original variable $x=\varepsilon y$ implies the desired result on the kernel of H_ε. □

Proof of Corollary 1. The definition of stability given by the theory of [15,16] is the following. The standing wave $U = e^{i\omega t/\varepsilon}\Phi_\omega(x/\varepsilon)$ is stable if for any $\eta>0$ there exists $\delta>0$ such that for all $U_0 = \begin{pmatrix} u_0 \\ v_0\end{pmatrix}$ verifying

$$\left\| U_0 - \Phi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{H^1\times L^2} < \delta$$

the solution U of (12) with initial data U_0 satisfies

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R}}\left\| U(t) - e^{i\theta}\Phi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{H^1\times L^2} < \eta.$$

More explicitly, we have

$$\left\| U - e^{i\theta}\Phi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|^2_{H^1\times L^2} = \left\| u - e^{i\theta}\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|^2_{H^1} + \left\| \varepsilon u_t + iVu - ie^{i\theta}(\omega+V)\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|^2_{L^2}.$$

This definition is not exactly the same as the one we use (stated in (11)), but our stability is implied by this definition. Indeed, we have from the triangle inequality

$$\begin{aligned}\Big\|u_t - ie^{i\theta}\frac{\omega}{\varepsilon}\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\Big\|_{L^2} &\leq \varepsilon^{-1}\Big\|\varepsilon u_t - ie^{i\theta}\omega\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\Big\|_{L^2} \leq \\ \varepsilon^{-1}\Big\|\varepsilon u_t + iVu - ie^{i\theta}(\omega+V)\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\Big\|_{L^2} &+ \varepsilon^{-1}\Big\|V\left(u - e^{i\theta}\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\right)\Big\|_{L^2} \leq \\ C\varepsilon^{-1}\left(\Big\|u_t - ie^{i\theta}\frac{\omega}{\varepsilon}\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\Big\|_{L^2}\right. &\left. + \Big\|u - e^{i\theta}\varphi_\omega\left(\frac{\cdot}{\varepsilon}\right)\Big\|_{L^2}\right),\end{aligned}$$

where the last inequality follows from the boundedness in $L^\infty(\mathbb{R}^N)$ of V. With similar arguments, it is rather easy to check that instability in the sense of [15, 16] also implies instability in our sense (11). Hence, the corollary simply follows from Theorem 1.1 and a direct application of the theory developed by Grillakis, Shatah, and Strauss in [15, 16]. □

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