CORON PROBLEM FOR FRACTIONAL EQUATIONS

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Abstract. We prove that the critical problem for the fractional Laplacian in an annular type domain admits a positive solution provided that the inner hole is sufficiently small.

1. Introduction

Let $N \geq 3$ and Ω be a smooth bounded domain of $\mathbb{R}^N$. The classical formulation of Coron problem goes back to 1984 and says that if there is a point $x_0 \in \mathbb{R}^N$ and radii $R_2 > R_1 > 0$ such that

$$\{R_1 \leq |x - x_0| \leq R_2\} \subset \Omega, \qquad \{|x - x_0| \leq R_1\} \not\subset \overline{\Omega}, \tag{1.1}$$

then the critical elliptic problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

admits a solution provided that R_2/R_1 is sufficiently large [6]. A few years later Bahri and Coron [1], in a seminal paper, considerably improved this existence result by showing, via sofisticated topological arguments based upon homology theory, that (1.2) admits a solution provided that $H_m(\Omega, \mathbb{Z}_2) \neq \{0\}$ for some $m > 0$. Furthermore, in [8, 11, 14] the authors show that existence of a solution is possible also in some contractible domains. Let $N \geq 2$ and $s \in (0, 1)$ with $N > 2s$, and consider the nonlocal fractional problem

$$\begin{cases} (-\Delta)^s u = u^{\frac{N+2s}{N-2s}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

involving the fractional Laplacian $(-\Delta)^s$. Here, for smooth functions φ, $(-\Delta)^s\varphi$ is defined by

$$(-\Delta)^s \varphi(x) = C(N, s) \lim_{\varepsilon \to 0} \int_{\complement B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}}\, dy, \qquad x \in \mathbb{R}^N,$$

where

$$C(N, s) = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \zeta_1}{|\zeta|^{N+2s}}\, d\zeta \right)^{-1}; \tag{1.4}$$

see [10]. Fractional Sobolev spaces are introduced in the middle part of the last century, especially in the framework of harmonic analysis. More recently, after the paper of Caffarelli and Silvestre [2], a large amount of papers were written on problems which involve the fractional diffusion $(-\Delta)^s$, $0 < s < 1$. Due to its nonlocal character, working on bounded domains imposes that an appropriate variational formulation of the problem is to consider functions on $\mathbb{R}^N$ with the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ replacing the boundary condition $u = 0$ on $\partial\Omega$. We set $X_0 = \{u \in \dot{H}^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$ and we consider the formulation

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi\, dx = \int_\Omega u^{\frac{N+2s}{N-2s}} \varphi\, dx \qquad \text{for all } \varphi \in X_0.$$

It has been proved recently [15, Corollary 1.3] that if Ω is a star-shaped domain, then problem (1.3) does not admit solutions and that for exponents larger that $(N + 2s)/(N - 2s)$ the problem does not admit any nontrivial solution thus dropping the positivity requirement. It is then natural to think that, as in

2010 *Mathematics Subject Classification.* 34K37, 58K05.

Key words and phrases. Fractional equation, critical embedding, Coron problem.

S. Secchi is supported by 2012 FIRB *Dispersive equations and Fourier analysis.* N. Shioji is partially supported by the Grant-in-Aid for Scientific Research (C) (No. 21540214 and No. 26400160) from Japan Society for the Promotion of Science. M. Squassina is supported by 2009 PRIN *Variational and Topological Methods in the Study of Nonlinear Phenomena.*

the local case $s = 1$, by assuming suitable geometrical or topological conditions on Ω one can get the existence of nontrivial solutions. We note that Capella [3] studies the problem for the particular case $s = 1/2$ by using the Caffarelli reduction to transform the problem in a local form and that Servadei and Valdinoci [16] studies the Brezis-Nirenberg problem with the fractional Laplacian.

The main result of the paper is the following Coron type result in the fractional setting.

Theorem 1.1. *If (1.1) holds, then (1.3) admits a weak solution in X_0 for R_2/R_1 sufficiently large.*

We roughly recall Coron's argument [6] for the case $s = 1$. Although the corresponding Rayleigh quotient does not attain the infimum value, say $\mathbb{S}$, the global compactness theorem due to Struwe [17] implies that it satisfies the Palais-Smale condition at each level in $(\mathbb{S}, 2^{2/N}\mathbb{S})$. He introduced a test function defined on a small ball which contains the small hole of Ω, and he showed that under assumption (1.1), the maximum value of the test function is less than $2^{2/N}\mathbb{S}$. If there is no critical point of the Rayleigh quotient in $(\mathbb{S}, 2^{2/N}\mathbb{S})$, he showed that the small ball can be retracted into its boundary, which is a contradiction. For the case $s \in (0, 1)$, one of the main difficulties that one has to face is to get a uniform estimate for the energy of truncations of the family of functions

$$U_{\varepsilon,z}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2}\Big)^{\frac{N-2s}{2}}, \qquad z \in \mathbb{R}^N,\ \varepsilon > 0, \tag{1.5}$$

which are precisely obtained in Propositions 2.1-2.2. We note that $U_{\varepsilon,z}$ satisfies $(-\Delta)^s u = u^{(N+2s)/(N-2s)}$ in $\mathbb{R}^N$ up to a constant, and $U_{\varepsilon,z}$ with the constant factor is called Talenti function for the fractional Laplacian. The other difficultly for the case $s \in (0, 1)$ is global compactness. We give a compactness result which is sufficient for our arguments.

2. Preliminary results

We define

$$\dot H^s(\mathbb{R}^N) = \{u \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N) : \|u\| < \infty\},$$

where

$$\|u\| = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy\Big)^{\frac12}.$$

We also define

$$\langle u, v\rangle = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy \quad \text{for each } u, v \in \dot H^s(\mathbb{R}^N).$$

Then we know that $\dot H^s(\mathbb{R}^N)$ is a Hilbert space with the inner product above, it is continuously embedded into $L^{2N/(N-2s)}(\mathbb{R}^N)$ and it holds

$$\langle u, v\rangle = \frac{2}{C(N,s)} \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v\, dx \quad \text{for each } u, v \in \dot H^s(\mathbb{R}^N), \tag{2.1}$$

where $C(N, s)$ is the constant given in (1.4); see [10]. We set

$$X_0 = \{u \in \dot H^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Since it is a closed subspace of $\dot H^s(\mathbb{R}^N)$, X_0 itself is also a Hilbert space, and we use the same symbols $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for its inner product and norm. We note

$$\|u\| = \Big(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\Big)^{1/2} \quad \text{for each } u \in X_0,$$

where $Q = \mathbb{R}^{2N} \setminus (\complement\Omega \times \complement\Omega)$.

Since we have (2.1), for the sake of simplicity, we will find a positive weak solution of

$$\begin{cases} (-\Delta)^s u = \frac{C(N,s)}{2}|u|^{\frac{4s}{N-2s}}u & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.2}$$

which is equivalent to find a weak solution of (1.3). Here, we say $u \in X_0$ is a weak solution to (2.2) if it satisfies

$$\iint_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dxdy = \int_\Omega |u|^{\frac{4s}{N-2s}} u\varphi\, dx \qquad \text{for each } \varphi \in X_0.$$

Without loss of generality, we may assume (1.1) with $x_0 = 0 \notin \overline{\Omega}$, R_2 is fixed with $R_2 > 10$ and $R_1 = \delta \in (0, 1/20]$ which will be fixed later. We set $B_r = \{x \in \mathbb{R}^N : |x| \le r\}$ for $r > 0$. Without loss of generality, we may also assume $\Omega \cap B_\delta = \emptyset$ and $B_5 \setminus B_{3\delta/2} \subset \Omega$. Let $\varphi_\delta : \mathbb{R}^N \to [0,1]$ be a smooth radially symmetric function such that

$$\varphi_\delta(x) = \begin{cases} 0 & \text{if } 0 \le |x| \le 2\delta \text{ and } |x| \ge 4, \\ 1 & \text{if } 4\delta \le |x| \le 3, \end{cases}$$

$$|\nabla \varphi_\delta(x)| \le \delta^{-1}, \quad \text{for } x \in B_{4\delta}, \qquad |\nabla \varphi_\delta(x)| \le 2, \quad \text{for } x \in \complement B_3.$$

For δ, $\varepsilon \in (0, 1/20]$ and $z \in B_1$, we set

$$u_{\delta,\varepsilon,z}(x) = \varphi_\delta(x) U_{\varepsilon,z}(x),$$

where the $U_{\varepsilon,z}$ were defined in (1.5). The next estimates will be crucial for the proof of Theorem 1.1.

Proposition 2.1. *There exists $C_1 > 0$ such that*

$$\|u_{\delta,\varepsilon,z}\|^2 \le \|U_{\varepsilon,z}\|^2 + C_1 \left(\left(\frac{\delta}{\varepsilon}\right)^{N-2s} + \left(\frac{\delta}{\varepsilon}\right)^{N+2-2s} + \varepsilon^{N-2s} \right) \tag{2.3}$$

for each δ, $\varepsilon \in (0, 1/20]$ and $z \in B_1$, and

$$\|u_{\delta,\varepsilon,z}\|^2 \le \|U_{\varepsilon,z}\|^2 + C_1 \varepsilon^{N-2s}(1 + \delta^{-2s}) \tag{2.4}$$

for each δ, $\varepsilon \in (0, 1/20]$ and $z \in B_1 \setminus B_{1/2}$.

Proof. Let δ, $\varepsilon \in (0, 1/20]$ and $z \in B_1$. We define

$$D = \{(x,y) \in (B_4 \times \complement B_3) \cup (\complement B_3 \times B_4) : |x-y| > 1\},$$
$$E = \{(x,y) \in (B_4 \times \complement B_3) \cup (\complement B_3 \times B_4) : |x-y| \le 1\},$$
$$\widetilde{D} = \{(x,y) \in (B_{4\delta} \times (B_4 \setminus B_{4\delta})) \cup ((B_4 \setminus B_{4\delta}) \times B_{4\delta}) : |x-y| > \delta\}$$

and

$$\widetilde{E} = (B_{4\delta} \times B_{4\delta}) \cup \{(x,y) \in (B_{4\delta} \times (B_4 \setminus B_{4\delta})) \cup ((B_4 \setminus B_{4\delta}) \times B_{4\delta}) : |x-y| \le \delta\}.$$

Then we have

$$\mathbb{R}^{2N} = \widetilde{E} \cup \widetilde{D} \cup E \cup D \cup ((B_3 \setminus B_{4\delta}) \times (B_3 \setminus B_{4\delta})) \cup (\complement B_4 \times \complement B_4).$$

We remark that this is *not* a disjoint union. We can easily see that

$$\int_{(B_3 \setminus B_{4\delta}) \times (B_3 \setminus B_{4\delta})} \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy = 0$$

and

$$\int_{\complement B_4 \times \complement B_4} \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \le 0.$$

We shall denote by C generic positive constants, possibly varying from line to line, and which do not depend on $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1$. For each $(x,y) \in \mathbb{R}^{2N}$, we have

$$\begin{aligned} &\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \\ &= \frac{(u_{\delta,\varepsilon,z}(x) + U_{\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y) - U_{\varepsilon,z}(y))(u_{\delta,\varepsilon,z}(x) - U_{\varepsilon,z}(x) + U_{\varepsilon,z}(y) - u_{\delta,\varepsilon,z}(y))}{|x-y|^{N+2s}} \\ &\le \frac{2U_{\varepsilon,z}(x)U_{\varepsilon,z}(y)}{|x-y|^{N+2s}}. \end{aligned} \tag{2.5}$$

From $z \in B_1$, we have

$$\begin{aligned} &\int_{B_4 \times \complement B_3,\, |x-y|>1} \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \\ &\le \int_{B_4 \times \complement B_3,\, |x-y|>1} \frac{2U_{\varepsilon,z}(x)U_{\varepsilon,z}(y)}{|x-y|^{N+2s}} dxdy \le C\varepsilon^{\frac{N-2s}{2}} \int_{B_4 \times \complement B_3,\, |x-y|>1} \frac{\left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2}\right)^{\frac{N-2s}{2}}}{|x-y|^{N+2s}} dxdy \\ &\le C\varepsilon^{N-2s} \int_{|\xi| \le 5} \frac{d\xi}{(\varepsilon^2 + |\xi|^2)^{\frac{N-2s}{2}}} \int_{|\eta|>1} \frac{d\eta}{|\eta|^{N+2s}} = C\varepsilon^{N-2s}\varepsilon^{2s} \int_{|\xi| \le 5/\varepsilon} \frac{d\xi}{(1 + |\xi|^2)^{\frac{N-2s}{2}}} \le C\varepsilon^{N-2s}. \end{aligned}$$

So we can infer

$$\int_D \left(\frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x)-U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \le C\varepsilon^{N-2s}.$$

We note

$$\nabla U_{\varepsilon,z}(x) = -(N-2s)\left(\frac{\varepsilon}{\varepsilon^2+|x-z|^2}\right)^{\frac{N-2s}{2}} \frac{x-z}{\varepsilon^2+|x-z|^2},$$

and

$$\frac{|x-z|}{\varepsilon^2+|x-z|^2} \le \frac{1}{2\varepsilon}.$$

Since $|\nabla\varphi_\delta(x)| \le 2$ for $|x| \ge 4\delta$, $z \in B_1$ and $|tx+(1-t)y| \ge 2$ for each $(x,y) \in E$ and $t \in [0,1]$,

$$\begin{aligned}
&\int_E \left(\frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x)-U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \\
&\quad \le \int_E \frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}}\, dxdy = \int_E \frac{|\int_0^1 (\nabla u_{\delta,\varepsilon,z})(tx+(1-t)y)\cdot(x-y)\, dt|^2}{|x-y|^{N+2s}}\, dxdy \\
&\quad \le \int_E \frac{\int_0^1 (8|U_{\varepsilon,z}(tx+(1-t)y)|^2 + 2|(\nabla U_{\varepsilon,z})(tx+(1-t)y)|^2)\, dt}{|x-y|^{N+2s-2}}\, dxdy \\
&\quad \le C\varepsilon^{N-2s} \int_E \frac{dxdy}{|x-y|^{N+2s-2}} \\
&\quad \le C\varepsilon^{N-2s} \int_{|\xi|\le 4} d\xi \int_{|\eta|\le 1} \frac{d\eta}{|\eta|^{N+2s-2}} = C\varepsilon^{N-2s}.
\end{aligned}$$

From (2.5), we also have

$$\begin{aligned}
&\int_{\widetilde{D}} \left(\frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x)-U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \\
&\quad \le 2\int_{\widetilde{D}} \frac{U_{\varepsilon,z}(x)U_{\varepsilon,z}(y)}{|x-y|^{N+2s}}\, dxdy \le \frac{2}{\varepsilon^{N-2s}} \int_{\widetilde{D}} \frac{dxdy}{|x-y|^{N+2s}} \\
&\quad \le \frac{C}{\varepsilon^{N-2s}} \int_{|\xi|\le 4\delta} d\xi \int_{|\eta|>\delta} \frac{d\eta}{|\eta|^{N+2s}} \le C\left(\frac{\delta}{\varepsilon}\right)^{N-2s}.
\end{aligned}$$

Since $|\nabla\varphi_\delta(x)| \le 1/\delta$ for $x \in B_{4\delta}$, we have

$$\begin{aligned}
&\int_{\widetilde{E}} \left(\frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x)-U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \\
&\quad \le \int_{\widetilde{E}} \frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}}\, dxdy \\
&\quad = \int_{\widetilde{E}} \frac{|\int_0^1 (\nabla u_{\delta,\varepsilon,z})(tx+(1-t)y)\cdot(x-y)\, dt|^2}{|x-y|^{N+2s}}\, dxdy \\
&\quad \le \int_{\widetilde{E}} \frac{\int_0^1 ((1/\delta)^2|U_{\varepsilon,z}(tx+(1-t)y)|^2 + |(\nabla U_{\varepsilon,z})(tx+(1-t)y)|^2)\, dt}{|x-y|^{N+2s-2}}\, dxdy \\
&\quad \le C\left(\frac{1}{\delta^2\varepsilon^{N-2s}} + \frac{1}{\varepsilon^{N-2s}}\cdot\frac{1}{\varepsilon^2}\right) \int_{\widetilde{E}} \frac{dxdy}{|x-y|^{N+2s-2}} \\
&\quad \le C\left(\frac{1}{\delta^2\varepsilon^{N-2s}} + \frac{1}{\varepsilon^{N+2-2s}}\right) \int_{|\xi|\le 4\delta} d\xi \int_{|\eta|\le\delta} \frac{d\eta}{|\eta|^{N+2s-2}} \\
&\quad = C\left(\left(\frac{\delta}{\varepsilon}\right)^{N-2s} + \left(\frac{\delta}{\varepsilon}\right)^{N+2-2s}\right).
\end{aligned}$$

By the inequalities above, we obtain (2.3). Let $z \in B_1 \setminus B_{1/2}$. In order to obtain (2.4) we need to consider the integrals on $\widetilde{D}$ and $\widetilde{E}$. We have

$$\begin{aligned}
&\int_{(B_4\setminus B_{4\delta})\times B_{4\delta},\, |x-y|>\delta} \left(\frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x)-U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \\
&\le \int_{(B_4\setminus B_{4\delta})\times B_{4\delta},\, |x-y|>\delta} \frac{2U_{\varepsilon,z}(x)U_{\varepsilon,z}(y)}{|x-y|^{N+2s}}\, dxdy \le C\varepsilon^{\frac{N-2s}{2}} \int_{(B_4\setminus B_{4\delta})\times B_{4\delta},\, |x-y|>\delta} \frac{\left(\frac{\varepsilon}{\varepsilon^2+|x-z|^2}\right)^{\frac{N-2s}{2}}}{|x-y|^{N+2s}}\, dxdy \\
&\le C\varepsilon^{N-2s} \int_{|\xi|\le 5} \frac{d\xi}{(\varepsilon^2+|\xi|^2)^{\frac{N-2s}{2}}} \int_{|\eta|>\delta} \frac{d\eta}{|\eta|^{N+2s}} \\
&= C\varepsilon^{N-2s}\delta^{-2s}\cdot\varepsilon^{2s} \int_{|\xi|\le 5/\varepsilon} \frac{d\xi}{(1+|\xi|^2)^{\frac{N-2s}{2}}} = C\varepsilon^{N-2s}\delta^{-2s}.
\end{aligned}$$

Hence

$$\int_{\widetilde{D}} \left(\frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x)-U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \le C\varepsilon^{N-2s}\delta^{-2s}.$$

Since $|\nabla\varphi_\delta(x)| \le 1/\delta$ for $x \in \mathbb{R}^N$, $z \in B_1 \setminus B_{1/2}$ and $|tx+(1-t)y| \le 5\delta \le 1/4$ for each $(x,y) \in \widetilde{E}$ and $t \in [0,1]$, we have

$$\begin{aligned}
&\int_{\widetilde{E}} \left(\frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x)-U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dxdy \\
&\le \int_{\widetilde{E}} \frac{|u_{\delta,\varepsilon,z}(x)-u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}}\, dxdy \\
&= \int_{\widetilde{E}} \frac{|\int_0^1 (\nabla u_{\delta,\varepsilon,z})(tx+(1-t)y)\cdot(x-y)\, dt|^2}{|x-y|^{N+2s}}\, dxdy \\
&\le 2\int_{\widetilde{E}} \frac{\int_0^1 ((1/\delta)^2 |U_{\varepsilon,z}(tx+(1-t)y)|^2 + |(\nabla U_{\varepsilon,z})(tx+(1-t)y)|^2)\, dt}{|x-y|^{N+2s-2}}\, dxdy \\
&\le C\varepsilon^{N-2s}\delta^{-2} \int_{\widetilde{E}} \frac{dxdy}{|x-y|^{N+2s-2}} \\
&\le C\varepsilon^{N-2s}\delta^{-2} \int_{|\xi|\le 4\delta} d\xi \int_{|\eta|\le\delta} \frac{d\eta}{|\eta|^{N+2s-2}} = C(\varepsilon\delta)^{N-2s} \le C\varepsilon^{N-2s}.
\end{aligned}$$

Thus, we obtain the second desired inequality. □

Proposition 2.2. *There exists $C_2 > 0$ such that*

$$\int_{\mathbb{R}^N} |u_{\delta,\varepsilon,z}|^{\frac{2N}{N-2s}}\, dx \ge \int_{\mathbb{R}^N} |U_{\varepsilon,z}|^{\frac{2N}{N-2s}}\, dx - C_2\left(\left(\frac{\delta}{\varepsilon}\right)^N + \varepsilon^N\right) \tag{2.6}$$

for each δ, $\varepsilon \in (0,1/20]$ and $z \in B_1$, and

$$\int_{\mathbb{R}^N} |u_{\delta,\varepsilon,z}|^{\frac{2N}{N-2s}}\, dx \ge \int_{\mathbb{R}^N} |U_{\varepsilon,z}|^{\frac{2N}{N-2s}}\, dx - C_2\varepsilon^N \tag{2.7}$$

for each δ, $\varepsilon \in (0,1/20]$ and $z \in B_1 \setminus B_{1/2}$.

Proof. Let δ, $\varepsilon \in (0,1/20]$ and $z \in B_1$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |U_{\varepsilon,z}|^{\frac{2N}{N-2s}}\, dx - \int_{\mathbb{R}^N} |u_{\delta,\varepsilon,z}|^{\frac{2N}{N-2s}}\, dx \\
&\le \int_{|x|\le 4\delta} \left(\frac{\varepsilon}{\varepsilon^2+|x-z|^2}\right)^N dx + \int_{|x|\ge 3} \left(\frac{\varepsilon}{\varepsilon^2+|x-z|^2}\right)^N dx \le C\left(\frac{\delta}{\varepsilon}\right)^N + C\varepsilon^N,
\end{aligned}$$

which yields (2.6). By a similar calculation, we can obtain (2.7) as well. □

Let $I : \dot{H}^s(\mathbb{R}^N) \to \mathbb{R}$ be given by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{N-2s}{2N}\int_{\mathbb{R}^N} |u|^{2N/(N-2s)}\, dx \quad \text{for } u \in \dot{H}^s(\mathbb{R}^N),$$

and let $I_0 : X_0 \to \mathbb{R}$ be its restriction to X_0, i.e.,

$$I_0(u) = I(u) \quad \text{for } u \in X_0.$$

Next, let us define $\mathscr{R}\colon \dot H^s(\mathbb{R}^N)\setminus\{0\} \to \mathbb{R}$ by

$$\mathscr{R}(u) = \frac{\|u\|^2}{\mathscr{N}(u)},$$

where

$$\mathscr{N}(u) = \Big(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}}\,dx\Big)^{\frac{N-2s}{N}}. \tag{2.8}$$

We also define $\mathscr{N}_0$ and $\mathscr{R}_0$ by the restrictions of $\mathscr{N}$ and $\mathscr{R}$ to $X_0\setminus\{0\}$, respectively. That is,

$$\mathscr{N}_0(u) = \mathscr{N}(u) \quad \text{and} \quad \mathscr{R}_0(u) = \mathscr{R}(u) \quad \text{for } u \in X_0\setminus\{0\}.$$

Lemma 2.3. *$\mathscr{R}_0 \in C^1(X_0\setminus\{0\})$, and if $\mathscr{R}_0'(v) = 0$ with $v \in X_0$, then $I_0'(\lambda v) = 0$ with some $\lambda > 0$.*

Proof. We can easily see $\mathscr{R}_0 \in C^1(X_0\setminus\{0\})$. Let $v \in X_0$. Since we have

$$\mathscr{R}_0'(v)(\varphi) = \frac{2\mathscr{N}_0(v)\iint_Q \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy - 2\|v\|^2\mathscr{N}_0(v)^{-\frac{2s}{N-2s}}\int_\Omega |v|^{\frac{4s}{N-2s}} v\varphi\,dx}{\mathscr{N}_0(v)^2}$$

for every $\varphi \in X_0$, we have $\mathscr{R}_0'(v) = 0$ if and only if, for every $\varphi \in X_0$,

$$\iint_Q \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy = \frac{\|v\|^2}{\int_\Omega |v|^{\frac{2N}{N-2s}}\,dx}\int_\Omega |v|^{\frac{4s}{N-2s}} v\varphi\,dx.$$

Setting λ by

$$\lambda^{\frac{4s}{N-2s}} = \frac{\|v\|^2}{\int_\Omega |v|^{\frac{2N}{N-2s}}\,dx}, \tag{2.9}$$

we have $I_0'(\lambda v) = 0$. This concludes the proof. □

We define a manifold of codimension one by setting

$$\mathscr{M} = \Big\{u \in X_0 : \int_\Omega |u|^{\frac{2N}{N-2s}}\,dx = 1\Big\}. \tag{2.10}$$

Lemma 2.4. *Let $\{v_n\}_n \subset \mathscr{M}$ be a Palais-Smale sequence for $\mathscr{R}_0$ at level c. Then*

$$u_n = \lambda_n v_n, \qquad \lambda_n = \mathscr{R}_0(v_n)^{(N-2s)/(4s)}$$

is a Palais-Smale sequence for I_0 at level $(s/N)c^{N/(2s)}$.

Proof. By following the computations of Lemma 2.3, if λ_n is defined as in (2.9), we have

$$\frac{1}{2}\mathscr{R}_0'(v_n)(\varphi) = \iint_Q \frac{(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dxdy - \lambda_n^{\frac{4s}{N-2s}}\int_\Omega |v_n|^{\frac{4s}{N-2s}} v_n\varphi\,dx$$

for every $\varphi \in X_0$. Hence, in turn, by multiplying this identity by λ_n, we conclude that

$$I_0'(u_n)(\varphi) = \iint_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dxdy - \int_\Omega |u_n|^{\frac{4s}{N-2s}} u_n\varphi\,dx$$

for every $\varphi \in X_0$. Recalling (2.8) and (2.9), we have

$$\lambda_n = \|v_n\|^{\frac{N-2s}{2s}} = \mathscr{R}_0(v_n)^{\frac{N-2s}{4s}}.$$

From $\mathscr{R}_0(v_n) = c + o(1)$ and $\{v_n\}_n \subset \mathscr{M}$, $\{v_n\}_n$ is bounded in X_0 and so is $\{\lambda_n\}_n$. In particular, it follows that $I_0'(u_n) \to 0$ in X_0' as $n \to \infty$. Moreover, $\{u_n\}_n$ is bounded in X_0 as well, yielding

$$o(1) = I_0'(u_n)(u_n) = \|u_n\|^2 - \int_\Omega |u_n|^{\frac{2N}{N-2s}}\,dx.$$

These facts imply that

$$\lim_{n\to\infty} I_0(u_n) = \frac{s}{N}\lim_{n\to\infty}\int_\Omega |u_n|^{\frac{2N}{N-2s}}\,dx = \frac{s}{N}\lim_{n\to\infty}\lambda_n^{\frac{2N}{N-2s}} = \frac{s}{N}\Big(\lim_{n\to\infty}\mathscr{R}_0(v_n)^{\frac{N-2s}{4s}}\Big)^{\frac{2N}{N-2s}} = \frac{s}{N}c^{N/(2s)},$$

concluding the proof. □

Let us set

$$\mathbb{S} = \inf\{\mathscr{R}(u) : u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}\}.$$

By [7], we know that

$$\mathscr{R}(U_{\varepsilon,z}) = \mathbb{S} \quad \text{for each } \varepsilon > 0 \text{ and } z \in \mathbb{R}^N,$$

only these functions with any nonzero constant factor attain the infimum,

$$\mathbb{S} = \inf\{\mathscr{R}_0(u) : u \in X_0 \setminus \{0\}\},$$

and the infimum is never attained in the latter case. We also have the following result for sign-changing weak solutions.

Lemma 2.5. *Let $u \in X_0$ be a sign-changing weak solution to (2.2), then $\|u\|^2 \geq 2\mathbb{S}^{N/(2s)}$. Moreover, the same conclusion holds for sign-changing critical points of I.*

Proof. We have $u^\pm \in X_0 \setminus \{0\}$ and

$$|u(x) - u(y)|^2 = |u^+(x) - u^+(y)|^2 + |u^-(x) - u^-(y)|^2 + 2u^+(y)u^-(x) + 2u^+(x)u^-(y)$$

for every $x, y \in \mathbb{R}^N$, where $u^-(x) = -\min\{u(x), 0\}$. This, in turn, implies

$$\|u\|^2 = \|u^+\|^2 + \|u^-\|^2 + 4\iint_Q \frac{u^+(y)u^-(x)}{|x-y|^{N+2s}}\,dxdy.$$

By multiplying equation (2.2) by $u^\pm$ easily yields

$$\|u^+\|^2 + 2\iint_Q \frac{u^+(y)u^-(x)}{|x-y|^{N+2s}}\,dxdy = \int_\Omega |u^+|^{\frac{2N}{N-2s}}\,dx,$$

$$\|u^-\|^2 + 2\iint_Q \frac{u^+(y)u^-(x)}{|x-y|^{N+2s}}\,dxdy = \int_\Omega |u^-|^{\frac{2N}{N-2s}}\,dx.$$

Combining these equalities with $\mathbb{S}\|u^\pm\|^2_{L^{2N/(N-2s)}} \leq \|u^\pm\|^2$, yields $\int_\Omega |u^\pm|^{2N/(N-2s)} \geq \mathbb{S}^{N/(2s)}$, concluding the proof. □

Now, we show the following compactness result. In order to show it, we follow the arguments in [18, Section 8.3], which treat the case $s = 1$.

Proposition 2.6. *Let $\{u_n\}_n \subset X_0$ be a Palais-Smale sequence for I_0 at level c with*

$$\frac{s}{N}\mathbb{S}^{N/(2s)} \leq c < \frac{2s}{N}\mathbb{S}^{N/(2s)}.$$

If $(s/N)\mathbb{S}^{N/(2s)} < c < (2s/N)\mathbb{S}^{N/(2s)}$, then $\{u_n\}_n$ converges strongly to a nontrivial constant-sign weak solution to problem (2.2) up to a subsequence, and if $c = (s/N)\mathbb{S}^{N/(2s)}$, then there exist a nontrivial constant-sign weak solution $v \in \dot{H}^s(\mathbb{R}^N)$ to problem

$$(-\Delta)^s v = \frac{C(N,s)}{2}|v|^{\frac{4s}{N-2s}}v \quad \text{in } \mathbb{R}^N, \tag{2.11}$$

$\{x_n\}_n \subset \Omega$ and $\{r_n\}_n \subset (0,\infty)$ with $r_n \to 0$ such that $\{u_n - r_n^{(2s-N)/2}v((\cdot - x_n)/r_n)\}_n$ converges strongly to 0 in $\dot{H}^s(\mathbb{R}^N)$ up to a subsequence.

Proof. First, we note that $\{u_n\}_n$ is bounded and $I_0(u_n) = (s/N)\|u_n\|^2 + o(1)$. We may assume that $\{u_n\}_n$ converges weakly to u in X_0. Then u is a possibly trivial solution to (2.2) and

$$\|u\|^2 \leq \varliminf_{n\to\infty} \|u_n\|^2 = \frac{N}{s}\lim_{n\to\infty} I_0(u_n) < 2\mathbb{S}^{N/(2s)}.$$

From Lemma 2.5, u is not sign-changing. By a similar argument as in [18, Lemma 8.10], we have

$$I'(u_n - u) \to 0, \quad I(u_n - u) \to c - I_0(u) \quad \text{and} \quad \|u_n - u\|^2 \to \frac{Nc}{s} - \|u\|^2.$$

If $\|u_n - u\|_{L^{2N/(N-2s)}} \to 0$, we can infer that $\|u_n - u\| \to 0$, $(s/N)\mathbb{S}^{N/(2s)} < c < (2s/N)\mathbb{S}^{N/(2s)}$ and u is a nontrivial constant-sign solution to (2.2). From here, we consider the case $\|u_n - u\|_{L^{2N/(N-2s)}} \not\to 0$. Taking small $\delta > 0$, we may assume that $\int_{\mathbb{R}^N} |u_n - u|^{2N/(N-2s)}dx \geq \delta$, for each $n \in \mathbb{N}$. As in the proof of [18, 2) and 3) of Theorem 8.13], we can choose appropriate sequences $\{x_n\}_n \subset \Omega$ and $\{r_n\}_n \subset (0,\infty)$ such that the sequence $\{v_n\}_n \subset \dot{H}^s(\mathbb{R}^N)$ defined by

$$v_n(x) = r_n^{(N-2s)/2}(u_n - u)(r_n x + x_n)$$

converges weakly to $v \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$. We have

$$\|v\|^2 \leq \varliminf_{n\to\infty} \|v_n\|^2 = \lim_{n\to\infty} \|u_n - u\|^2 = \frac{Nc}{s} - \|u\|^2 < 2\mathbb{S}^{N/(2s)} - \|u\|^2. \tag{2.12}$$

By the boundedness of Ω and $v \neq 0$, we may assume $r_n \to 0$ and $x_n \to x_0 \in \overline{\Omega}$. We may also assume that $\{\mathrm{dist}(x_n, \partial\Omega)/r_n\}_n$ has a limit value in $[0, \infty]$. Assume that this limit value is finite. Then v is a solution to the problem

$$(-\Delta)^s v = \frac{C(N,s)}{2} |v|^{\frac{4s}{N-2s}} v$$

in a half-space. From [9, Theorem 1.1 and Remark 4.2], v is locally bounded (although the boundedness of a domain is assumed in [9], the proof works for our case). Then, in light of [4, Corollary 3] (see also [12, Corollary 1.6]), we know that the above problem in any half-space does not admit a nontrivial constant-sign solution. So v must be sign-changing, but then by a similar proof of Lemma 2.5, we have $\|v\|^2 \geq 2\mathbb{S}^{N/(2s)}$, which contradicts (2.12). So we find that $\mathrm{dist}(x_n, \partial\Omega)/r_n \to \infty$. Then we can see that v is a nontrivial solution of (2.11). Using (2.12) again, we find that v is constant-sign and u is trivial. Setting

$$w_n(x) = u_n(x) - r_n^{(2s-N)/2} v((x - x_n)/r_n),$$

we have

$$I'(w_n) \to 0, \quad I(w_n) \to c - I(v) \quad \text{and} \quad \|w_n\|^2 \to \frac{Nc}{s} - \mathbb{S}^{N/(2s)} < \mathbb{S}^{N/(2s)}.$$

If $\|w_n\|_{L^{2N/(N-2s)}} \not\to 0$, repeating the argument above, we can obtain a contradiction. Hence, we have $\|w_n\| \to 0$ and $c = (s/N)\mathbb{S}^{N/(2s)}$. Therefore, we have shown our assertion. ☐

We define $\mathscr{Y}_0, \mathscr{Z}_0 : X_0 \setminus \{0\} \to X_0$ by

$$\mathscr{Y}_0(u) = \frac{\nabla \mathscr{N}_0(u)}{\|\nabla \mathscr{N}_0(u)\|}, \qquad \mathscr{Z}_0(u) = \nabla \mathscr{R}_0(u) - \langle \nabla \mathscr{R}_0(u), \mathscr{Y}_0(u)\rangle \, \mathscr{Y}_0(u) \quad \text{for each } u \in X_0 \setminus \{0\}.$$

Here, $\nabla \mathscr{N}_0(u)$ and $\nabla \mathscr{R}_0(u)$ are the elements of X_0 respectively obtained from $\mathscr{N}_0'(u)$ and $\mathscr{R}_0'(u)$ by the Riesz representation theorem. We note that

$$\langle \mathscr{Z}_0(u), \nabla \mathscr{N}_0(u)\rangle = 0 \quad \text{and} \quad \langle \mathscr{Z}_0(u), \nabla \mathscr{R}_0(u)\rangle = \|\mathscr{Z}_0(u)\|^2 \quad \text{for each } u \in \mathscr{M}. \tag{2.13}$$

The next proposition essentially says that $\mathscr{R}_0|_{\mathscr{M}}$ satisfies the Palais-Smale condition at any level in $(\mathbb{S}, 2^{2s/N}\mathbb{S})$. In the last section, we give a negative gradient flow of $\mathscr{R}_0|_{\mathscr{M}}$; see (3.2).

Proposition 2.7. *Let $\{v_n\}_n \subset \mathscr{M}$ which satisfies $\mathscr{Z}_0(v_n) \to 0$ in X_0 and $\mathscr{R}_0(v_n) \to c \in (\mathbb{S}, 2^{2s/N}\mathbb{S})$. Then $\{v_n\}_n$ has a convergent subsequence.*

Proof. For each $u \in \mathscr{M}$, we have

$$\begin{aligned} \|\mathscr{Z}_0(u)\|^2 &= \|\nabla \mathscr{R}_0(u) - \langle \nabla \mathscr{R}_0(u), \mathscr{Y}_0(u)\rangle \mathscr{Y}_0(u)\|^2 = \|\nabla \mathscr{R}_0(u)\|^2 - \langle \nabla \mathscr{R}_0(u), \mathscr{Y}_0(u)\rangle^2 \\ &\geq \|\nabla \mathscr{R}_0(u)\|^2 \frac{\langle \mathscr{Y}_0(u), u\rangle^2}{\|u\|^2} = \frac{2\|\nabla \mathscr{R}_0(u)\|^2}{\|u\|^2 \|\nabla \mathscr{N}_0(u)\|^2}. \end{aligned} \tag{2.14}$$

From our assumptions, we can infer that $\nabla \mathscr{R}_0(v_n) \to 0$. By virtue of Lemma 2.4, the sequence $u_n = \lambda_n v_n$, where λ_n is defined as in (2.9), is a Palais-Smale sequence for I_0 at level $(s/N)c^{N/(2s)}$. According to Proposition 2.6, there exists a subsequence of $\{u_n\}_n$ which converges strongly in X_0. Since we have $\lambda_n \to c^{(N-2s)/(4s)}$ from Lemma 2.4, we can see that our assertion holds. ☐

For the reader's convenience, we give the following lemma.

Lemma 2.8. *Let $\eta > 0$ and $u \in \mathscr{M}$ with $\mathscr{R}_0(u) \leq \mathbb{S} + \eta$. Then there exists $v \in \mathscr{M}$ such that $\|u - v\| \leq \sqrt{\eta}$, $\mathscr{R}_0(v) \leq \mathscr{R}_0(u)$ and $\|\mathscr{R}_0'(v)\| \leq \sqrt{\eta}(1 + 1/\sqrt{\mathbb{S}})$.*

Proof. By Ekeland's variational principle, we can find $v \in \mathscr{M}$ such that $\|u - v\| \leq \sqrt{\eta}$, $\mathscr{R}_0(v) \leq \mathscr{R}_0(u)$ and $\mathscr{R}_0(w) \geq \mathscr{R}_0(v) - \sqrt{\eta}\|w - v\|$ for each $w \in \mathscr{M}$. Fix $z \in X_0$ with $\|z\| = 1$. For each $s \in \mathbb{R}$ with $v + sz \neq 0$, there exists unique $t(s) > 0$ satisfying $t(s)(v + sz) \in \mathscr{M}$. Then we can easily see

$$t'(0) = -\int_\Omega |v|^{\frac{4s}{N-2s}} vz \, dx.$$

From

$$\mathscr{R}_0(v + sz) - \mathscr{R}_0(v) = \mathscr{R}_0(t(s)(v + sz)) - \mathscr{R}_0(v) \geq -\sqrt{\eta}\|t(s)(v + sz) - t(s)v + t(s)v - v\|,$$

we obtain

$$|\mathscr{R}_0'(v)(z)| \le \sqrt{\eta}\|z + t'(0)v\| \le \sqrt{\eta}(1 + 1/\sqrt{\mathbb{S}}),$$

which yields $\|\mathscr{R}_0'(v)\| \le \sqrt{\eta}(1 + 1/\sqrt{\mathbb{S}})$. □

3. Proof of Theorem 1.1 concluded

In the following proof, we will repeatedly use the fact that $\mathscr{R}(\sigma u) = \mathscr{R}(u)$ for every $\sigma > 0$ and every $u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$. We write, for $u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$,

$$\Pi(u) = \frac{u}{\|u\|_{L^{2N/(N-2s)}}}.$$

From Propositions 2.1 and 2.2, we can find $C_3 > 0$ with

$$\mathscr{R}(\Pi(u_{\delta,\varepsilon,z})) \le \frac{\|U_{1,0}\|^2 + C_1\varepsilon^{N-2s}}{\left(\int_{\mathbb{R}^N} |U_{1,0}|^{\frac{2N}{N-2s}}\,dx - C_2\varepsilon^N\right)^{\frac{N-2s}{N}}} \le \mathscr{R}(U_{1,0}) + C_3\varepsilon^{N-2s}$$

for each $\varepsilon \in (0, 1/20]$, $\delta \in (0, \varepsilon^2]$ and $z \in B_1$. Hence, we can find $\bar{\varepsilon} \in (0, 1/20]$ such that

$$\mathscr{R}(\Pi(u_{\bar{\varepsilon}^2,\bar{\varepsilon},z})) \le \varpi 2^{2s/N}\mathbb{S} \quad \text{for each } z \in B_1,$$

where $2^{-\frac{2s}{N}} < \varpi < 1$. Now, we fix $\delta = \bar{\varepsilon}^2$ and we define a kind of barycenter mapping

$$\beta(u) = \int_{\mathbb{R}^N} 1_{B_K}(x)\, x|u(x)|^{\frac{2N}{N-2s}}\,dx \quad \text{for each } u \in \dot{H}^s(\mathbb{R}^N) \text{ with } \|u\|_{L^{2N/(N-2s)}} = 1,$$

where $K = \sup\{|x| : x \in \Omega\} + 1$ and 1_{B_K} is the characteristic function for B_K. We also define

$$\bar{c} = \inf\{\mathscr{R}_0(u) : u \in \mathscr{M}, \beta(u) = 0\}.$$

Then, $\bar{c} > \mathbb{S}$. If not, there is a sequence $\{v_n\}_n \subset \mathscr{M}$ such that $\beta(v_n) = 0$ and $\mathscr{R}_0(v_n) \to \mathbb{S}$. From Lemma 2.8, we have $\mathscr{R}_0'(v_n) \to 0$. Then by Proposition 2.6, taking a subsequence if necessary, there exist $\{\lambda_n\}_n \subset (0, 1)$ and $\{z_n\}_n \subset \Omega$ such that $\lambda_n \to 0$, $z_n \to z \in \overline{\Omega}$ and

$$\text{either} \quad \|v_n - \Pi(U_{\lambda_n,z_n})\| = o(1) \quad \text{or} \quad \|v_n + \Pi(U_{\lambda_n,z_n})\| = o(1) \quad \text{as } n \to \infty.$$

From $\beta(v_n) = 0$ and $\beta(v_n) \to z$, we obtain $0 \in \overline{\Omega}$, which is a contradiction. Now, from Propositions 2.1 and 2.2, we can find a map $f\colon B_1 \to \mathscr{M}$ which satisfies

$$\mathscr{R}_0(f(z)) \le \varpi 2^{2s/N}\mathbb{S} \quad \text{for each } z \in B_1,$$

$$\mathscr{R}_0(f(z)) \le \frac{\mathbb{S} + \bar{c}}{2} < \bar{c} \quad \text{for each } z \in \partial B_1$$

and

$$|\beta(f(z)) - z| \le \frac{1}{2} \quad \text{for each } z \in \partial B_1. \tag{3.1}$$

Such f can be obtained by setting $f(z) = u_{\bar{\varepsilon}^2, h_\varepsilon(|z|), z}$ with sufficiently small $\varepsilon > 0$, where

$$h_\varepsilon(t) = \begin{cases} \bar{\varepsilon} & \text{for } 0 \le t \le 1/2, \\ 2(1-t)\bar{\varepsilon} + (2t-1)\varepsilon & \text{for } 1/2 \le t \le 1, \end{cases}$$

and we can show (3.1) by a similar argument above which shows $\bar{c} > \mathbb{S}$. Then, for each $t \in [0, 1]$ and $z \in \partial B_1$, we have $|(1-t)z + t\beta(f(z))| \ge |z| - t|\beta(f(z)) - z| \ge 1/2$. So by using Brouwer's degree theory, we have $\deg(\beta \circ f, \mathrm{Int}(B_1), 0) = 1$. Defining

$$c = \inf_{g \in G} \max_{x \in B_1} \mathscr{R}_0(g(x)), \qquad G = \{g \in C(B_1, \mathscr{M}) : g = f \text{ on } \partial B_1 \text{ and } \deg(\beta \circ g, \mathrm{Int}(B_1), 0) = 1\},$$

we have

$$\mathbb{S} < \bar{c} \le c \le \varpi 2^{2s/N}\mathbb{S}.$$

Now, we will show there is $u \in \mathscr{M}$ such that $\nabla\mathscr{R}_0(u) = 0$ and $\mathscr{R}_0(u) = c$. Assume not. By Proposition 2.7, we can choose a positive constant $\eta > 0$ such that $(\mathbb{S}+c)/2 < c - 2\eta$, $c + 2\eta < \varpi 2^{2s/N}\mathbb{S}$ and $\mathscr{Z}_0(u) \ne 0$ for each $u \in \mathscr{M}$ with $|\mathscr{R}_0(u) - c| \le 3\eta$. We also choose a locally Lipschitz function $\alpha : \mathscr{M} \to [0, 1]$ such that

$$\alpha(u) = \begin{cases} 1 & \text{for each } u \in \mathscr{M} \text{ with } |\mathscr{R}_0(u) - c| \le \eta, \\ 0 & \text{for each } u \in \mathscr{M} \text{ with } |\mathscr{R}_0(u) - c| \ge 2\eta. \end{cases}$$

Then we can define $\gamma : [0,1] \times \mathscr{M} \to \mathscr{M}$ by

$$\gamma(0,u) = u \quad \text{and} \quad \frac{d}{dt}\gamma(t,u) = -\frac{2\eta\alpha(\gamma(t,u))}{\|\mathscr{Z}_0(\gamma(t,u))\|^2}\mathscr{Z}_0(\gamma(t,u)); \tag{3.2}$$

see (2.13) and (2.14). Let $g \in G$ such that $\max_{z\in B_1} \mathscr{R}_0(g(z)) < c + \eta$. Then we can easily see $\gamma(t, g(z)) = g(z)$ for each $(t, z) \in [0,1] \times \partial B_1$, which yields $\deg(\beta(\gamma(1, g(\cdot))), \mathrm{Int}(B_1), 0) = 1$. Moreover, we can find $\mathscr{R}_0(\gamma(1, g(z))) \leq c - \eta$ for each $z \in B_1$, which contradicts the definition of c. From Proposition 2.6, we can find that this contradiction proves the existence of a nonnegative weak solution to (2.2). By [13, Theorem 2.5], the obtained solution is positive in Ω. □

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