GENERALIZED POLYA-SZEGÖ INEQUALITY AND APPLICATIONS TO SOME QUASI-LINEAR ELLIPTIC PROBLEMS

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Abstract. We generalize Polya-Szegö inequality to integrands depending on u and its gradient. Under additional assumptions, we establish equality cases in this generalized inequality. We give relevant applications of our study to a class of quasi-linear elliptic equations and systems.

1. Introduction

The Polya-Szegö inequality asserts that the L^2 norm of the gradient of a positive function u in $W^{1,p}(\mathbb{R}^N)$ cannot increase under Schwarz symmetrization,

$$\int_{\mathbb{R}^N} |\nabla u^*|^2 dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx. \tag{1.1}$$

The Schwarz rearrangement of u is denoted here by u^*. Inequality (1.1) has numerous applications in physics. It was first used in 1945 by G. Polya and G. Szegö to prove that the capacity of a condenser diminishes or remains unchanged by applying the process of Schwarz symmetrization (see [31]). Inequality (1.1) was also the key ingredients to show that, among all bounded bodies with fixed measure, balls have the minimal capacity (see [27, Theorem 11.17]). Finally (1.1) has also played a crucial role in the solution of the famous Choquard's conjecture (see [26]). It is heavily connected to the isoperimetric inequality and to Riesz-type rearrangement inequalities. Moreover, it turned out that (1.1) is extremely helpful in establishing the existence of ground states solutions of the nonlinear Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t \Phi + \Delta\Phi + f(|x|, \Phi) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ \Phi(x, 0) = \Phi_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

A ground state solution of equation (1.2) is a positive solution to the following associated variational problem

$$\inf \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(|x|, u) dx : \, u \in H^1(\mathbb{R}^N), \ \|u\|_{L^2} = 1 \right\}, \tag{1.3}$$

where $F(|x|, s)$ is the primitive of $f(|x|, \cdot)$ with $F(|x|, 0) = 0$. Inequality (1.1) together with the generalized Hardy-Littlewood inequality were crucial to prove that (1.3) admits a radial and radially decreasing solution. Furthermore, under appropriate regularity

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assumptions on the nonlinearity F, there exists a Lagrange multiplier λ such that any minimizer of (1.3) is a solution of the following semi-linear elliptic PDE

$$-\Delta u + f(|x|, u) + \lambda u = 0, \quad \text{in } \mathbb{R}^N.$$

We refer the reader to [21] for a detailed analysis. The same approach applies to the more general quasi-linear PDE

$$-\Delta_p u + f(|x|, u) + \lambda u = 0, \quad \text{in } \mathbb{R}^N.$$

where $\Delta_p u$ means $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, and we can derive similar properties of ground state solutions since (1.1) extends to gradients that are in $L^p(\mathbb{R}^N)$ in place of $L^2(\mathbb{R}^N)$, namely

$$\int_{\mathbb{R}^N} |\nabla u^*|^p dx \le \int_{\mathbb{R}^N} |\nabla u|^p dx. \tag{1.4}$$

Due to the multitude of applications in physics, rearrangement inequalities like (1.1) and (1.4) have attracted a huge number of mathematicians from the middle of the last century. Different approaches were built up to establish these inequalities such as heat-kernel methods, slicing and cut-off techniques and two-point rearrangement.

A generalization of inequality (1.4) to suitable convex integrands $A : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\int_{\mathbb{R}^N} A(|\nabla u^*|) dx \le \int_{\mathbb{R}^N} A(|\nabla u|) dx, \tag{1.5}$$

was first established by Almgren and Lieb (see [1]). Inequality (1.5) is important in studying the continuity and discontinuity of Schwarz symmetrization in Sobolev spaces (see e.g. [1, 11]). It also permits us to study symmetry properties of variational problems involving integrals of type $\int_{\mathbb{R}^N} A(|\nabla u|)dx$. Extensions of Polya-Szegö inequality to more general operators of the form

$$j(s, \xi) = b(s)A(|\xi|), \quad s \in \mathbb{R},\ \xi \in \mathbb{R}^N,$$

on bounded domains have been investigated by Kawohl, Mossino and Bandle. More precisely, they proved that

$$\int_{\Omega^*} b(u^*)A(|\nabla u^*|) dx \le \int_{\Omega} b(u)A(|\nabla u|) dx, \tag{1.6}$$

where Ω^* denotes the ball in $\mathbb{R}^N$ centered at the origin having the Lebesgue measure of Ω, under suitably convexity, monotonicity and growth assumptions (see e.g. [3, 25, 30]). Numerous applications of (1.6) have been discussed in the above references. In [37], Tahraoui claimed that a general integrand $j(s, \xi)$ with appropriate properties can be written in the form

$$\sum_{i=1}^{\infty} b_i(s)A_i(|\xi|) + R_1(s) + R_2(\xi), \quad s \in \mathbb{R},\ \xi \in \mathbb{R}^N,$$

where b_i and A_i are such that inequality (1.6) holds. However, there are some mistakes in [37] and we do not believe that this density type result holds true. Until quite recently there were no results dealing with the generalized Polya-Szegö inequality, namely

$$\int_{\Omega^*} j(u^*, |\nabla u^*|) dx \le \int_{\Omega} j(u, |\nabla u|) dx. \tag{1.7}$$

While writing down this paper we have learned about a very recent survey by F. Brock [6] who was able to prove (1.7) under continuity, monotonicity, convexity and growth conditions.

Following a different approach, we prove (1.7) without requiring any growth conditions on j. As it can be easily seen it is important to drop these conditions to the able to cover some relevant applications. Our approach is based upon a suitable approximation of the Schwarz symmetrized u^* of a function u. More precisely, if $(H_n)_{n\geq 1}$ is a dense sequence in the set of closed half spaces H containing 0 and $u \in L^p_+(\mathbb{R}^N)$, there exists a sequence (u_n) consisting of iterated polarizations of the H_ns which converges to u^* in $L^p(\mathbb{R}^N)$ (see [18, 39]). On the other hand, a straightforward computation shows that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla u_0\|_{L^p(\mathbb{R}^N)} = \cdots = \|\nabla u_n\|_{L^p(\mathbb{R}^N)}, \quad \text{for all } n \in \mathbb{N}.$$

Combining these properties with the weak lower semicontinuity of the functional $J(u) = \int j(u, |\nabla u|)dx$ enable us to conclude (see Theorem 3.1). Note that (1.5) was proved using coarea formula; however this approach does not apply to integrands depending both on u and its gradient since one has to apply simultaneously the coarea formula to $|\nabla u|$ and to decompose u with the Layer-Cake principle.

Detailed applications of our results concerning (1.7) are given in Section 4, where we determine a suitable class of assumptions that allow us to solve the (vector) problem of minimizing the functional $J : W^{1,p}(\mathbb{R}^N, \mathbb{R}^m) \to \mathbb{R}$, $m \geq 1$,

$$J(u) = \sum_{k=1}^{m} \int_{\mathbb{R}^N} j_k(u_k, |\nabla u_k|)dx - \int_{\mathbb{R}^N} F(|x|, u_1, \ldots, u_m)dx.$$

on the constraint of functions $u = (u_1, \ldots, u_m) \in W^{1,p}(\mathbb{R}^N, \mathbb{R}^m)$ such that

$$G_k(u_k),\ j_k(u_k, |\nabla u_k|) \in L^1(\mathbb{R}^N) \quad \text{and} \quad \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k)dx = 1.$$

Notice that Brock's method is based on an intermediate maximization problem and cannot yield to the establishment of equality cases. Our approximation approach was also fruitful in determining the relationship between u and u^* such that

$$\int_{\mathbb{R}^N} j(u^*, |\nabla u^*|)dx = \int_{\mathbb{R}^N} j(u, |\nabla u|)dx. \tag{1.8}$$

Indeed, under suitable assumptions, we prove that (1.8) yields

$$\int_{\mathbb{R}^N} |\nabla u^*|^p dx = \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

This is very useful, as for $j(\xi) = |\xi|^p$, identity cases were completely studied in the breakthrough paper of Brothers and Ziemer [10].

The paper is organized as follows.

Section 2 is dedicated to some preliminary stuff, especially the ones concerning the invariance of a class of functionals under polarization. These observations are crucial, in Section 3, to establish in a simple way the generalized Polya-Szegö inequality. With the help of this, we then study in Section 4 a class of variational problems involving quasi-linear operators. We first prove that our variational problem (4.1) always admits a Schwarz symmetric minimizer. Then, using the result we have established in Corollary 3.8, under suitable assumptions we show that all minimizers u of problem (4.1) are radially symmetric and radially decreasing, up to a translation in $\mathbb{R}^N$, provided the set of critical points of u^* has zero measure. Another meaningful variant of the main application, related to a recent paper of the second author, is also stated in Theorem 4.7.

Notations.

(1) For $N \in \mathbb{N}$, $N \geq 1$, we denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^N$.
(2) $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) is the set of positive (resp. negative) real values.
(3) μ denotes the Lebesgue measure in $\mathbb{R}^N$.
(4) $M(\mathbb{R}^N)$ is the set of measurable functions in $\mathbb{R}^N$.
(5) For $p > 1$ we denote by $L^p(\mathbb{R}^N)$ the space of f in $M(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |f|^p dx < \infty$.
(6) The norm $(\int_{\mathbb{R}^N} |f|^p dx)^{1/p}$ in $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$.
(7) For $p > 1$ we denote by $W^{1,p}(\mathbb{R}^N)$ the Sobolev space of functions f in $L^p(\mathbb{R}^N)$ having generalized partial derivatives $D_i f$ in $L^p(\mathbb{R}^N)$, for $i = 1, \dots, N$.
(8) $D^{1,p}(\mathbb{R}^N)$ is the space of measurable functions whose gradient is in $L^p(\mathbb{R}^N)$.
(9) $L^p_+(\mathbb{R}^N)$ is the cone of positive functions of $L^p(\mathbb{R}^N)$.
(10) $W^{1,p}_+(\mathbb{R}^N)$ is the cone of positive functions of $W^{1,p}(\mathbb{R}^N)$.
(11) For $R > 0$, $B(0, R)$ is the ball in $\mathbb{R}^N$ centered at zero with radius R.

2. Preliminary stuff

In the following H will design a closed half-space of $\mathbb{R}^N$ containing the origin, $0_{\mathbb{R}^N} \in H$. We denote by $\mathcal{H}$ the set of closed half-spaces of $\mathbb{R}^N$ containing the origin. We shall equip $\mathcal{H}$ with a topology ensuring that $H_n \to H$ as $n \to \infty$ if there is a sequence of isometries $i_n : \mathbb{R}^N \to \mathbb{R}^N$ such that $H_n = i_n(H)$ and i_n converges to the identity as $n \to \infty$.

We first recall some basic notions. For more details, we refer the reader to [12].

Definition 2.1. *A reflection $\sigma : \mathbb{R}^N \to \mathbb{R}^N$ with respect to H is an isometry such that the following properties hold*

(1) $\sigma \circ \sigma(x) = x$, *for all* $x \in \mathbb{R}^N$;
(2) *the fixed point set of σ separates $\mathbb{R}^N$ in H and $\mathbb{R}^N \setminus H$ (interchanged by σ);*
(3) $|x - y| < |x - \sigma(y)|$, *for all* $x, y \in H$.

Given $x \in \mathbb{R}^N$, the reflected point $\sigma_H(x)$ will also be denoted by x^H.

Definition 2.2. *Let H be a given half-space in $\mathbb{R}^N$. The two-point rearrangement (or polarization) of a nonnegative real valued function $u : \mathbb{R}^N \to \mathbb{R}_+$ with respect to a given reflection σ_H (with respect to H) is defined as*

$$u^H(x) := \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & \text{for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, & \text{for } x \in \mathbb{R}^N \setminus H. \end{cases}$$

Definition 2.3. *We say that a nonnegative measurable function u is symmetrizable if $\mu(\{x \in \mathbb{R}^N : u(x) > t\}) < \infty$ for all $t > 0$. The space of symmetrizable functions is denoted by F_N and, of course, $L^p_+(\mathbb{R}^N) \subset F_N$. Also, two functions u, v are said to be equimeasurable (and we shall write $u \sim v$) when*

$$\mu(\{x \in \mathbb{R}^N : u(x) > t\}) = \mu(\{x \in \mathbb{R}^N : v(x) > t\}),$$

for all $t > 0$.

Definition 2.4. *For a given u in F_N, the Schwarz symmetrization u^* of u is the unique function with the following properties (see e.g. [20])*

1. *u and u^* are equimeasurable;*
2. *$u^*(x) = h(|x|)$, where $h : (0, \infty) \to \mathbb{R}_+$ is a continuous and decreasing function.*

In particular, u, u^H and u^* are all equimeasurable functions (see e.g. [2]).

Lemma 2.5. *Let $u \in W^{1,p}_+(\mathbb{R}^N)$ and let H be a given half-space with $0 \in H$. Then $u^H \in W^{1,p}_+(\mathbb{R}^N)$ and, setting*

$$v(x) := u(x^H), \quad w(x) := u^H(x^H), \qquad x \in \mathbb{R}^N,$$

the following facts hold:

1. *We have*

$$\nabla u^H(x) = \begin{cases} \nabla u(x) & \text{for } x \in \{u > v\} \cap H, \\ \nabla v(x) & \text{for } x \in \{u \le v\} \cap H, \end{cases}$$

$$\nabla w(x) = \begin{cases} \nabla v(x) & \text{for } x \in \{u > v\} \cap H, \\ \nabla u(x) & \text{for } x \in \{u \le v\} \cap H. \end{cases}$$

2. *Let $j : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Then, if $j(u, |\nabla u|) \in L^1(\mathbb{R}^N)$, it follows that $j(u^H, |\nabla u^H|) \in L^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} j(u, |\nabla u|)dx = \int_{\mathbb{R}^N} j(u^H, |\nabla u^H|)dx. \tag{2.1}$$

Proof. Observe that, for all $x \in H$, we have

$$u^H(x) = v(x) + (u(x) - v(x))^+, \qquad w(x) = u(x) - (u(x) - v(x))^+.$$

Moreover, it follows that the functions u^H, v, w belong to $W^{1,p}_+(\mathbb{R}^N)$ (see [34, Proposition 2.3]). Assertion (1) follows by a simple direct computation. Concerning (2), assume that $j(u, |\nabla u|) \in L^1(\mathbb{R}^N)$. Writing σ_H as $\sigma_H(x) = x_0 + Rx$, where R is an orthogonal linear transformation (symmetric, as reflection), taking into account that $|\det R| = 1$ and

$$|\nabla v(x)| = |\nabla(u(\sigma_H(x)))| = |R(\nabla u(\sigma_H(x)))| = |(\nabla u)(\sigma_H(x))|,$$

we have, by a change of variable,

$$\begin{aligned}\int_{\mathbb{R}^N} j(u,|\nabla u|)dx &= \int_H j(u,|\nabla u|)dx + \int_{\mathbb{R}^N\setminus H} j(u,|\nabla u|)dx \\ &= \int_H j(u,|\nabla u|)dx + \int_H j(u(\sigma_H(x)),|(\nabla u)(\sigma_H(x))|)dx \\ &= \int_H j(u,|\nabla u|)dx + \int_H j(v,|\nabla v|)dx.\end{aligned}$$

In particular, $j(v,|\nabla v|)\in L^1(H)$. In a similar fashion, we have

$$\begin{aligned}\int_{\mathbb{R}^N} j(u^H,|\nabla u^H|)dx &= \int_H j(u^H,|\nabla u^H|)dx + \int_H j(u^H(\sigma_H(x)),|(\nabla u^H)(\sigma_H(x))|)dx \\ &= \int_H j(u^H,|\nabla u^H|)dx + \int_H j(w,|\nabla w|)dx \\ &= \int_{\{u>v\}\cap H} j(u,|\nabla u|)dx + \int_{\{u>v\}\cap H} j(v,|\nabla v|)dx \\ &+ \int_{\{u\le v\}\cap H} j(v,|\nabla v|)dx + \int_{\{u\le v\}\cap H} j(u,|\nabla u|)dx \\ &= \int_H j(u,|\nabla u|)dx + \int_H j(v,|\nabla v|)dx.\end{aligned}$$

Hence $j(u^H,|\nabla u^H|)\in L^1(\mathbb{R}^N)$, and we have the desired identity, concluding the proof. □

3. Generalized Polya-Szegö inequality

The first main result of the paper is the following

Theorem 3.1. *Let $\varrho:\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}_+$ be a continuous function. For any function $u\in W^{1,p}_+(\mathbb{R}^N)$, let us set*

$$J(u)=\int_{\mathbb{R}^N}\varrho(u,\nabla u)dx.$$

Moreover, let $(H_n)_{n\ge 1}$ be a dense sequence in the set of closed half spaces containing $0_{\mathbb{R}^N}$. For $u\in W^{1,p}_+(\mathbb{R}^N)$, define a sequence (u_n) by setting

$$\begin{cases} u_0=u \\ u_{n+1}=u_n^{H_1\dots H_{n+1}}.\end{cases}$$

Assume that the following conditions hold:

(1)
$$-\infty<J(u)<+\infty;$$

(2)
$$\liminf_n J(u_n)\le J(u); \tag{3.1}$$

(3) *if (u_n) converges weakly to some v in $W^{1,p}_+(\mathbb{R}^N)$, then*

$$J(v)\le\liminf_n J(u_n).$$

Then

$$J(u^*) \leq J(u).$$

Proof. By the (explicit) approximation results contained in [18, 39], we know that $u_n \to u^*$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$. Moreover, by Lemma 2.5 applied with $j(s, |\xi|) = |\xi|^p$, we have

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla u_0\|_{L^p(\mathbb{R}^N)} = \cdots = \|\nabla u_n\|_{L^p(\mathbb{R}^N)}, \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

In particular, up to a subsequence, (u_n) is weakly convergent to some function v in $W^{1,p}(\mathbb{R}^N)$. By uniqueness of the weak limit in $L^p(\mathbb{R}^N)$ one can easily check that $v = u^*$, namely $u_n \rightharpoonup u^*$ in $W^{1,p}(\mathbb{R}^N)$. Hence, using assumption (3) and (3.1), we have

$$J(u^*) \leq \liminf_n J(u_n) \leq J(u), \tag{3.3}$$

concluding the proof. □

Remark 3.2. A quite large class of functionals J which satisfy assumption (3.1) of the previous Theorem is provided by Lemma 2.5.

Corollary 3.3. *Let $j : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying the assumptions:*

(1) *$j(s,t)$ is continuous;*
(2) *$j(s,\cdot)$ is convex for all $s \in \mathbb{R}_+$;*
(3) *$j(s,\cdot)$ is nondecreasing for all $s \in \mathbb{R}_+$.*

Then, for all function $u \in W^{1,p}_+(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} j(u, |\nabla u|)dx < \infty,$$

we have

$$\int_{\mathbb{R}^N} j(u^*, |\nabla u^*|)dx \leq \int_{\mathbb{R}^N} j(u, |\nabla u|)dx.$$

Proof. The assumptions on j imply that $\{\xi \mapsto j(s, |\xi|)\}$ is convex so that the weak lower semicontinuity assumption of Theorem 3.1 holds. We refer the reader e.g. to the papers [22, 23] by A. Ioffe for even more general assumptions. Also, assumption (3.1) of Theorem 3.1 is provided by means of Lemma 2.5. □

Remark 3.4. In [6, Theorem 4.3], F. Brock proved Corollary 3.3 for Lipschitz functions having compact support. In order to prove the most interesting cases in the applications, the inequality has to hold for functions u in $W^{1,p}_+(\mathbb{R}^N)$. This forces him to assume some growth conditions of the Lagrangian j, for instance to assume that there exists a positive constant K and $q \in [p, p^*]$ such that

$$|j(s, |\xi|)| \leq K(s^q + |\xi|^p), \quad \text{for all } s \in \mathbb{R}_+ \text{ and } \xi \in \mathbb{R}^N.$$

In our approach, instead, we can include integrands such as

$$j(s, |\xi|) = \frac{1}{2}(1 + s^{2\alpha})|\xi|^p, \quad \text{for all } s \in \mathbb{R}_+ \text{ and } \xi \in \mathbb{R}^N,$$

for some $\alpha > 0$, which have meaningful physical applications (for instance quasi-linear Schrödinger equations, see [28] and references therein). We also stress that the approach of [6] cannot yield the establishment of equality cases (see Theorem 3.5).

Theorem 3.5. *In addition to the assumptions of Theorem 3.1, assume that*

$$J(u_n) \to J(u^*) \text{ as } n \to \infty \text{ implies that } u_n \to u^* \text{ in } D^{1,p}(\mathbb{R}^N) \text{ as } n \to \infty, \tag{3.4}$$

where we recall that $u_n \rightharpoonup u^*$ in $D^{1,p}(\mathbb{R}^N)$. Then

$$J(u) = J(u^*) \implies \|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla u^*\|_{L^p(\mathbb{R}^N)}.$$

Proof. Assume that we have $J(u) = J(u^*)$. Therefore, by assumption (3.1), along a subsequence, we obtain

$$J(u^*) = \lim_n J(u_n) = J(u).$$

In turn, by assumption, $u_n \to u^*$ in $D^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. Then, taking the limit inside equalities (3.2), we conclude the assertion. □

Remark 3.6. Assume that $\{\xi \mapsto j(s, |\xi|)\}$ is strictly convex for any $s \in \mathbb{R}_+$ and there exists $\nu' > 0$ such that $j(s, |\xi|) \geq \nu'|\xi|^p$ for all $s \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^N$. Then, in many cases, assumption (3.4) is fulfilled for $J(u) = \int_{\mathbb{R}^N} j(u, |\nabla u|)dx$. We refer to [40, Section 2].

Remark 3.7. Equality cases of the type $\|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla u^*\|_{L^p(\mathbb{R}^N)}$ have been completely characterized in the breakthrough paper by Brothers and Ziemer [10].

Let us now set

$$M = \text{esssup}_{\mathbb{R}^N} u = \text{esssup}_{\mathbb{R}^N} u^*, \qquad C^* = \{x \in \mathbb{R}^N : \nabla u^*(x) = 0\}.$$

Corollary 3.8. *Assume that $\{\xi \mapsto j(s, |\xi|)\}$ is strictly convex and there exists a positive constant ν' such that*

$$j(s, |\xi|) \geq \nu'|\xi|^p, \quad \textit{for all } s \in \mathbb{R} \textit{ and } \xi \in \mathbb{R}^N.$$

Moreover, assume that (3.4) holds and

$$\int_{\mathbb{R}^N} j(u, |\nabla u|)dx = \int_{\mathbb{R}^N} j(u^*, |\nabla u^*|)dx, \quad \mu(C^* \cap (u^*)^{-1}(0, M)) = 0.$$

Then there exists $x_0 \in \mathbb{R}^N$ such that

$$u(x) = u^*(x - x_0), \quad \textit{for all } x \in \mathbb{R}^N,$$

namely u is radially symmetric after a translation in $\mathbb{R}^N$.

Proof. It is sufficient to combine Theorem 3.5 with [10, Theorem 1.1]. □

4. Applications to minimization problems

In this section we shall study a minimization problem of the following form

$$T = \inf \big\{ J(u) : \ u \in \mathcal{C} \big\}, \tag{4.1}$$

where

$$\mathcal{C} = \Big\{ u \in W^{1,p}(\mathbb{R}^N, \mathbb{R}^m) : G_k(u_k),\ j_k(u_k, |\nabla u_k|) \in L^1 \text{ for any } k,\ \sum_{k=1}^m \int_{\mathbb{R}^N} G_k(u_k)dx = 1 \Big\},$$

where J is the functional defined, for $u = (u_1, \dots, u_m)$, by

$$J(u) = \sum_{k=1}^{m} \int_{\mathbb{R}^N} j_k(u_k, |\nabla u_k|)dx - \int_{\mathbb{R}^N} F(|x|, u_1, \dots, u_m)dx.$$

Under suitable additional regularity assumptions on j_k, F and G_k the solutions to (4.1) yields a nontrivial solution to the system on $\mathbb{R}^N$

$$\begin{cases} -\mathrm{div}(D_\xi j_k(u_k, |\nabla u_k|)) + D_s j_k(u_k, |\nabla u_k|) + \gamma D_s G_k(u_k) = D_{s_k} F(|x|, u_1, \dots, u_m), \\ k = 1, \dots, m, \end{cases}$$

for some Lagrange multiplier $\gamma \in \mathbb{R}$.

4.1. **Assumptions on j_k, F, G_k.** Before stating the main results of the section, we collect here the assumptions we take.

4.2. **Assumptions on j_k.** Let $m \geq 1$, $1 < p < N$ and let

$$j_k : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+, \quad \text{for } k = 1, \dots, m$$

be continuous functions, convex and increasing with respect to the second argument and such that there exist $\nu > 0$ and a continuous and increasing function $\beta_k : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\nu|\xi|^p \leq j_k(s, |\xi|)| \leq \beta_k(|s|)|\xi|^p, \quad \text{for } k = 1, \dots, m, \tag{4.2}$$

for all $s \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^N$. We also consider the following assumptions:

$$j_k(-s, |\xi|) \leq j_k(s, |\xi|), \quad \text{for all } s \in \mathbb{R}_- \text{ and all } \xi \in \mathbb{R}^N. \tag{4.3}$$

Moreover, there exists $\alpha \geq p$ such that

$$j_k(ts, t|\xi|) \leq t^\alpha j_k(s, |\xi|) \quad \text{for all } t \geq 1,\ s \in \mathbb{R}_+ \text{ and } \xi \in \mathbb{R}^N. \tag{4.4}$$

4.3. **Assumptions on F.** Let us consider a function

$$F : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R},$$

of variables $(r, s_1, \dots, s_m)$, measurable with respect r and continuous with respect to $(s_1, \dots, s_m) \in \mathbb{R}^N$ with $F(r, 0, \dots, 0) = 0$ for any r. We assume that

$$F(r, s + he_i + ke_j) + F(r, s) \geq F(r, s + he_i) + F(r, s + ke_j), \tag{4.5}$$

$$F(r_1, s + he_i) + F(r_0, s) \leq F(r_1, s) + F(r_0, s + he_i), \tag{4.6}$$

for every $i \neq j$, $i, j = 1, \dots, m$ where e_i denotes the i-th standard basis vector in $\mathbb{R}^m$, $r > 0$, for all $h, k > 0$, $s = (s_1, \dots, s_m) \in \mathbb{R}^m_+$ and r_0, r_1 such that $0 < r_0 < r_1$.

The regularity assumptions on F could be further relaxed via the notion of Borel measurability (see [12]). Conditions (4.5)-(4.6) are also known as cooperativity conditions and, in general, they are necessary conditions for rearrangement inequalities to hold (see [9]). Moreover, we assume that

$$\limsup_{(s_1,\dots,s_m)\to(0,\dots,0)^+} \frac{F(r,s_1,\dots,s_m)}{\sum_{k=1}^m s_k^p} < \infty, \tag{4.7}$$

$$\lim_{|(s_1,\dots,s_m)|\to+\infty} \frac{F(r,s_1,\dots,s_m)}{\sum_{k=1}^m s_k^{p+\frac{p^2}{N}}} = 0, \tag{4.8}$$

uniformly with respect to r.

There exist $r_0 > 0$, $\delta > 0$, $\mu_k > 0$, $\tau_k \in [0,p)$ and $\sigma_k \in [0, \frac{p(p-\tau_k)}{N})$ such that $F(r,s_1,\dots,s_m) \geq 0$ for $|r| \leq r_0$ and

$$F(r,s_1,\dots,s_m) \geq \sum_{k=1}^m \mu_k r^{-\tau_k} s_k^{\sigma_k+p}, \qquad \text{for } r > r_0 \text{ and } s \in \mathbb{R}^m_+ \text{ with } |s| \leq \delta. \tag{4.9}$$

Also,

$$\lim_{\substack{r\to+\infty \\ (s_1,\dots,s_m)\to(0,\dots,0)^+}} \frac{F(r,s_1,\dots,s_m)}{\sum_{k=1}^m s_k^p} = 0. \tag{4.10}$$

Finally, we consider the following assumptions:

$$F(r,s_1,\dots,s_m) \leq F(r,|s_1|,\dots,|s_m|), \tag{4.11}$$

for all $r > 0$ and $(s_1,\dots,s_m) \in \mathbb{R}^m$ and

$$F(r,ts_1,\dots,ts_m) \geq t^\alpha F(r,s_1,\dots,s_m), \tag{4.12}$$

for all $r > 0$, $t \geq 1$ and $(s_1,\dots,s_m) \in \mathbb{R}^m_+$, where $\alpha \geq p$ is the values which appears in condition (4.4).

Remark 4.1. We stress that a condition from below on F like (4.9) was firstly considered by C.A. Stuart in [36].

Remark 4.2. As a variant, in place of the growth assumptions (4.7)-(4.8), one could directly assume that there exist $m \geq 1$ constants

$$0 < \sigma_k < \frac{p^2}{N}, \quad k = 1,\dots,m,$$

and a positive constant C such that

$$0 \leq F(r,s_1,\dots,s_m) \leq C\sum_{k=1}^m s_k^p + C\sum_{k=1}^m s_k^{p+\sigma_k},$$

for all $r > 0$ and $(s_1,\dots,s_m) \in \mathbb{R}^m_+$. Indeed, conclusion (4.21) can be reached again by slightly modifying the Gagliardo-Nirenberg inequality (4.20).

Remark 4.3. For instance, take

$$\beta \geq 0, \quad \tau \in [0,p), \quad \sigma \in [0, \tfrac{p(p-\tau)}{N}),$$

and a continuous and decreasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$a(|x|) = \mathcal{O}\left(|x|^{-\tau}\right) \quad \text{as } |x| \to \infty.$$

Consider the function

$$F(|x|, s_1, \dots, s_m) = \frac{a(|x|)}{p+\sigma}\sum_{k=1}^m |s_k|^{p+\sigma} + \frac{2\beta a(|x|)}{p+\sigma}\sum_{\substack{i,j=1\\ i<j}}^m |s_i|^{\frac{p+\sigma}{2}}|s_j|^{\frac{p+\sigma}{2}}.$$

Hence (4.5)-(4.12) are fulfilled. This allows to treat elliptic systems of the type

$$\begin{cases} -\Delta_p u_k + \gamma u_k^{p-1} = a(|x|)|u_k|^{p+\sigma-2}u_k + \beta a(|x|)\displaystyle\sum_{i\neq k}|u_i|^{\frac{p+\sigma}{2}}|u_k|^{\frac{p+\sigma-4}{2}}u_k, \quad \text{in } \mathbb{R}^N, \\ \quad k = 1, \dots, m. \end{cases}$$

In the particular case $m = 2$, $p = 2$ and $a(s) = 1$ (thus $\tau = 0$), the above system reduces to the important class of physical systems, systems of weakly coupled Schrödinger equations

$$\begin{cases} -\Delta u + \gamma u = |u|^\sigma u + \beta|u|^{\frac{\sigma-2}{2}}|v|^{\frac{\sigma+2}{2}}u, & \text{in } \mathbb{R}^N \\ -\Delta v + \gamma v = |v|^\sigma u + \beta|v|^{\frac{\sigma-2}{2}}|u|^{\frac{\sigma+2}{2}}v, & \text{in } \mathbb{R}^N, \end{cases} \qquad 0 < \sigma < 4/N.$$

These problems, particularly in the case where $\sigma = 2$ (thus in the range $\sigma < 4/N$ only for $N = 1$) have been deeply investigated in the last few years, mainly with respect to the problem of existence of bound and ground state depending on the values of β (see e.g. [33] and references therein).

Remark 4.4. In general, the upper bound $\sigma \leq p^2/N$ in the growth conditions on F is a necessary condition for the minimization problem (4.1) to be well posed, otherwise $T = -\infty$. In fact, assume that $(w_1, \dots, w_m)$ is an element of $\mathcal{C}$. Then we have that $(w_1^\delta, \dots, w_m^\delta) \in \mathcal{C}$ for all $\delta \in (0,1]$, where $w_j^\delta(x) = \delta^{-N/p}w_j(x/\delta)$. Hence, taking for instance j_k such that there exists a positive constant C with $j_k(s, |\xi|) \leq C|\xi|^p$ and

$$F(s_1, \dots, s_m) = \frac{1}{p+\sigma}\sum_{k=1}^m |s_k|^{p+\sigma} + \frac{2}{p+\sigma}\sum_{\substack{k,h=1\\ h<k}}^m |s_h|^{\frac{p+\sigma}{2}}|s_k|^{\frac{p+\sigma}{2}},$$

by a simple change of scale we find

$$T \leq J(w_1^\delta, \dots, w_m^\delta) \leq \frac{C}{\delta^p}\sum_{k=1}^m \int_{\mathbb{R}^N} |\nabla w_k|^p dx - \frac{1}{\delta^{\frac{N\sigma}{p}}}\int_{\mathbb{R}^N} F(w_1, \dots, w_m)dx,$$

which, letting $\delta \to 0^+$, yields $T = -\infty$, provided that $\frac{N\sigma}{p} > p$, hence $\sigma > \frac{p^2}{N}$. In some cases, instead, T is $-\infty$ for larger values of σ. Consider, for instance, the case

$$j_k(s, |\xi|) = (1 + s^{2\alpha_k})|\xi|^p, \quad s \in \mathbb{R},\ \xi \in \mathbb{R}^N.$$

for $\alpha_k > 0$, $k = 1, \dots, m$. Therefore, after scaling, it follows that

$$\sum_{k=1}^{m} \int_{\mathbb{R}^N} j_k(w_k^\delta, |\nabla w_k^\delta|)dx \le \frac{C}{\delta^p} + \sum_{k=1}^{m} \frac{C'}{\delta^{\frac{2\alpha_k N + p^2}{p}}} - \frac{C''}{\delta^{\frac{N\sigma}{p}}},$$

where C, C', C'' are positive contants. But then $T = -\infty$ if

$$\frac{N\sigma}{p} > \max_{k=1,\dots,m} \{p, \frac{2\alpha_k N + p^2}{p}\} = \max_{k=1,\dots,m} \frac{2\alpha_k N + p^2}{p},$$

namely $\sigma > 2\alpha_{\max} + p^2/N$, where $\alpha_{\max} = \max\{\alpha_k : k = 1, \dots, m\}$. In fact, the presence of powers of u in front of the gradient term $|\nabla u|^p$ allows to recover some regularity on u as soon as the functional is finite (see e.g. [28]) and improve the growth conditions we assumed for the nonlinearity F at infinity.

4.4. **Assumptions on G_k.** Consider $m \ge 1$ continuous and p-homogeneous functions

$$G_k : \mathbb{R} \to \mathbb{R}_+, \quad G_k(0) = 0, \quad \text{for } k = 1, \dots, m$$

such that there exists $\gamma > 0$ such that

$$G_k(s) \ge \gamma |s|^p, \quad \text{for all } s \in \mathbb{R}. \tag{4.13}$$

4.5. **Statement of the results.** In the above framework, the main results of the section are the following

Theorem 4.5. *Assume that conditions (4.2)-(4.8) and (4.13) hold. Then the minimum problem (4.1) admits a radially symmetric and radially decreasing nonnegative solution.*

For a vector function $(u_1, \dots, u_m)$, let us set

$$M_i = \mathrm{esssup}_{\mathbb{R}^N} u_i = \mathrm{esssup}_{\mathbb{R}^N} u_i^*, \qquad C_i^* = \{x \in \mathbb{R}^N : \nabla u_i^*(x) = 0\}.$$

Theorem 4.6. *Assume that conditions (4.2)-(4.8) and (4.13) hold and that the function $\{\xi \mapsto j(s, |\xi|)\}$ is strictly convex and (3.4) holds. Then, for any nonnegative solution u to problem (4.1) such that*

$$\mu(C^* \cap (u_i^*)^{-1}(0, M_i)) = 0, \quad \textit{for some } i \in \{1, \dots, m\}. \tag{4.14}$$

the component u_i is radially symmetric and radially decreasing, after a suitable translation in $\mathbb{R}^N$. In particular, if (4.14) holds for any $i = 1, \dots, m$, the solution u is radially symmetric and radially decreasing, after suitable translations in $\mathbb{R}^N$.

We point out that there are situations where condition (4.3)-(4.4) can be replaced by a monotonicity conditions on j with respect to s. A well established sign condition for these type of operators, which is often involved in both *existence* and *regularity* questions (see e.g. [14, 32, 35, 38]) is the following: there exists $R \ge 0$ such that

$$s \frac{\partial j}{\partial s}(s, t) \ge 0, \quad \text{for all } t \in \mathbb{R}_+ \text{ and } s \in \mathbb{R} \text{ with } |s| \ge R. \tag{4.15}$$

There are also counterexamples in the literature showing that, for $j = j(x, u, \nabla u)$, if condition (4.15) is *not* fulfilled (for instance if (4.4) is satisfied), the solutions of the Euler-Lagrange equation might be unbounded (see [17]).

In the last two results of this section we provide the existence and symmetry properties of least energy solutions for a class of quasi-linear elliptic problems by assuming, among other things, condition (4.15). This problem has recently been investigated in [24] by the second author jointly with L. Jeanjean via a combination of tools from non-smooth analysis and recent results on the symmetry properties for homogeneous constrained minimization problems (see [13, 29]). Here we obtain the result as an application of Corollary 3.8. The prize one has to pay is that an additional information on the measure of the critical set for the Schwarz rearrangement of a solution is needed. However this condition on u^* is quite natural and, as it is shown in [10], without this additional assumption there are counterexample to equality cases.

Theorem 4.7. *Assume that $m = 1$, $1 < p < N$, $F = 0$ and that $G_1 = G : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 with $G' = g$ and*

$$\limsup_{s\to 0} \frac{G(s)}{|s|^{p^*}} \leq 0,$$

$$\lim_{s\to\infty} \frac{g(s)}{|s|^{p^*-1}} = 0.$$

Moreover, assume that $j(s, |\xi|) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is a function of class C^1 in s and ξ and denote by j_s and j_t the derivatives of j with respect of s and $t = |\xi|$ respectively. We assume that, for any $s \in \mathbb{R}$,

the map $\{\xi \mapsto j(s, |\xi|\}$ is strictly convex, increasing and p-homogeneous;

Moreover, there exist positive constants c_1, c_2, c_3, c_4 and R such that

$$c_1|\xi|^p \leq j(s, |\xi|) \leq c_2|\xi|^p, \qquad \text{for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N;$$

$$|j_s(s, |\xi|)| \leq c_3|\xi|^p, \quad |j_t(s, |\xi|)| \leq c_4|\xi|^{p-1}, \qquad \text{for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N;$$

$$j_s(s, |\xi|)s \geq 0, \qquad \text{for all } s \in \mathbb{R} \text{ with } |s| \geq R \text{ and } \xi \in \mathbb{R}^N.$$

Then equation

$$-\operatorname{div}(D_\xi j(u, |\nabla u|)) + j_s(u, |\nabla u|) = g(u), \quad \text{in } \mathbb{R}^N \tag{4.16}$$

admits positive, radially symmetric and radially decreasing least energy solutions. Furthermore, if (3.4) holds, any least energy solution u of (4.16) such that

$$\mu(C^* \cap (u^*)^{-1}(0, M)) = 0, \tag{4.17}$$

is positive, radially symmetric and radially decreasing, up to a translation in $\mathbb{R}^N$.

4.6. **Proof of Theorem 4.5.**

Proof. Let $u^h = (u_1^h, \dots, u_m^h) \subset \mathcal{C}$ be a minimizing sequence for $J|_{\mathcal{C}}$. Then

$$\lim_h \left(\sum_{k=1}^m \int_{\mathbb{R}^N} j_k(u_k^h, |\nabla u_k^h|)dx - \int_{\mathbb{R}^N} F(|x|, u_1^h, \dots, u_m^h)dx \right) = T, \tag{4.18}$$

$$G_k(u_k^h),\ j_k(u_k^h, |\nabla u_k^h|) \in L^1(\mathbb{R}^N), \quad \sum_{k=1}^m \int_{\mathbb{R}^N} G_k(u_k^h)dx = 1, \quad \text{for all } h \in \mathbb{N}.$$

Taking into account assumption (4.11), we have

$$F(|x|, u_1^h, \dots, u_m^h) \leq F(|x|, |u_1^h|, \dots, |u_m^h|), \quad \text{for all } h \in \mathbb{N}.$$

Moreover, by the fact that $|\nabla u_k^h(x)| = |\nabla |u_k^h(x)||$ for a.e. $x \in \mathbb{R}^N$, for all $k = 1, \dots, m$ and $h \in \mathbb{N}$, in light of assumption (4.3), it holds

$$j_k(|u_k^h|, |\nabla |u_k^h||) \leq j_k(u_k^h, |\nabla u_k^h|), \quad \text{for all } k = 1, \dots, m \text{ and } h \in \mathbb{N}.$$

In conclusion, we have

$$J(|u_1^h|, \dots, |u_m^h|) \leq J(u_1^h, \dots, u_m^h), \quad \text{for all } h \in \mathbb{N},$$

so that we may assume, without loss of generality, that $u_k^h \geq 0$ a.e., for all $k = 1, \dots, m$ and $h \in \mathbb{N}$. Let us now prove that (u^h) is bounded in $W^{1,p}(\mathbb{R}^N, \mathbb{R}^m)$. Indeed, as $(u^h) \subset \mathcal{C}$, by assumption (4.13) on G_k, it follows that the sequence (u^h) is uniformly bounded in $L^p(\mathbb{R}^N)$. By combining the growth assumptions (4.7)-(4.8), for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(r, s_1, \dots, s_m) \leq C_\varepsilon \sum_{k=1}^m s_k^p + \varepsilon \sum_{k=1}^m s_k^{p+\frac{p^2}{N}}, \quad \text{for all } r, s_1, \dots, s_m \in (0, \infty). \tag{4.19}$$

Therefore, in view of the Gagliardo-Nirenberg inequality

$$\|u_k^h\|_{L^{p+\frac{p^2}{N}}(\mathbb{R}^N)}^{p+\frac{p^2}{N}} \leq C \|u_k^h\|_{L^p(\mathbb{R}^N)}^{\frac{p^2}{N}} \|\nabla u_k^h\|_{L^p(\mathbb{R}^N)}^p, \quad \text{for all } h \in \mathbb{N}, \tag{4.20}$$

for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, for all $h \in \mathbb{N}$,

$$\begin{aligned}
\int_{\mathbb{R}^N} F(|x|, u_1^h, \dots, u_m^h) dx &\leq C_\varepsilon \sum_{k=1}^m \|u_k^h\|_{L^p(\mathbb{R}^N)}^p + \varepsilon \sum_{k=1}^m \|u_k^h\|_{L^{p+\frac{p^2}{N}}(\mathbb{R}^N)}^{p+\frac{p^2}{N}} \\
&\leq CC_\varepsilon + C\varepsilon \sum_{k=1}^m \|u_k^h\|_{L^p(\mathbb{R}^N)}^{\frac{p^2}{N}} \|\nabla u_k^h\|_{L^p(\mathbb{R}^N)}^p \\
&\leq CC_\varepsilon + C\varepsilon \sum_{k=1}^m \|\nabla u_k^h\|_{L^p(\mathbb{R}^N)}^p.
\end{aligned}$$

In turn, by combining assumption (4.2) with (4.18), fixed $\varepsilon_0 \in (0, \frac{\nu}{C})$, it follows

$$(\nu - C\varepsilon_0) \sum_{k=1}^m \|\nabla u_k^h\|_{L^p(\mathbb{R}^N)}^p \leq CC_{\varepsilon_0} + T, \qquad \text{for all } h \in \mathbb{N}. \tag{4.21}$$

yielding the desired boundedness of (u^h) in $W^{1,p}(\mathbb{R}^N, \mathbb{R}^m)$. Hence, after extracting a subsequence, which we still denote by (u^h), we get for any $k = 1, \dots, m$

$$u_k^h \rightharpoonup u_k \text{ in } L^{p^*}(\mathbb{R}^N), \; Du_k^h \rightharpoonup Du_k \text{ in } L^p(\mathbb{R}^N), \; u_k^h(x) \to u_k(x) \quad \text{a.e. } x \in \mathbb{R}^N. \tag{4.22}$$

For any $k = 1, \dots, m$ and $h \in \mathbb{N}$, let us denote by u_k^{*h} the Schwarz symmetric rearrangement of u_k^h. By means of [12, Theorem 1], we have

$$\int_{\mathbb{R}^N} F(|x|, u_1^h, \dots, u_m^h) dx \leq \int_{\mathbb{R}^N} F(|x|, u_1^{*h}, \dots, u_m^{*h}) dx. \tag{4.23}$$

Moreover, by Corollary 3.3, we have

$$\int_{\mathbb{R}^N} j_k(u_k^{*h}, |\nabla u_k^{*h}|)dx \le \int_{\mathbb{R}^N} j_k(u_k^h, |\nabla u_k^h|)dx,$$

Finally, as it is well-known, we have

$$G_k(u_k^{*h}),\ j_k(u_k^{*h}, |\nabla u_k^{*h}|) \in L^1(\mathbb{R}^N), \quad \sum_{k=1}^m \int_{\mathbb{R}^N} G_k(u_k^{*h})dx = 1, \quad \text{for all } h \in \mathbb{N}.$$

Hence, since

$$J(u^{*h}) \le J(u^h), \quad u^{*h} \in \mathcal{C}, \quad \text{for all } h \in \mathbb{N},$$

it follows that $u^{*h} = (u_1^{*h}, \dots, u_m^{*h})$ is also a positive minimizing sequence for $J|_{\mathcal{C}}$, which is now radially symmetric and radially decreasing. In what follows, we shall denote it back to $u^h = (u_1^h, \dots, u_m^h)$. Taking into account that u_k^h is bounded in $L^p(\mathbb{R}^N)$, it follows that (see [4, Lemma A.IV])

$$u_k^h(x) \le c_k |x|^{-\frac{N}{p}}, \qquad \text{for all } x \in \mathbb{R}^N \setminus \{0\} \text{ and } h \in \mathbb{N}, \tag{4.24}$$

for a positive constant c_k, independent of h. In turn, by virtue of condition (4.10), for all $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that

$$|F(|x|, u_1^h(|x|), \dots, u_m^h(|x|))| \le \varepsilon \sum_{k=1}^m |u_k^h(|x|)|^p, \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \ge \rho_\varepsilon,$$

yielding, via the boundedness of (u_k^h) in $L^p(\mathbb{R}^N)$,

$$\Big| \int_{\mathbb{R}^N \setminus B(0,\rho_\varepsilon)} F(|x|, u_1^h, \dots, u_m^h)dx \Big| \le \varepsilon \sum_{k=1}^m \|u_k^h\|_p^p \le \varepsilon C. \tag{4.25}$$

Since (4.24) holds also for the pointwise limit $(u_1, \dots, u_m)$, analogously it follows

$$\Big| \int_{\mathbb{R}^N \setminus B(0,\rho_\varepsilon)} F(|x|, u_1, \dots, u_m)dx \Big| \le \varepsilon C.$$

On the other hand, by the growth assumption (4.8) and the local strong convergence of (u^h) to u in L^m with $m < p^*$, for this ρ_ε we obtain

$$\lim_h \int_{B(0,\rho_\varepsilon)} F(|x|, u_1^h, \dots, u_m^h)dx = \int_{B(0,\rho_\varepsilon)} F(|x|, u_1, \dots, u_m)dx.$$

Then, by (4.25), we have

$$\lim_h \int_{\mathbb{R}^N} F(|x|, u_1^h, \dots, u_m^h)dx = \int_{\mathbb{R}^N} F(|x|, u_1, \dots, u_m)dx. \tag{4.26}$$

Also as $j(s,t)$ is positive, convex and increasing in the t-argument (and thus $\xi \mapsto j(s, |\xi|)$ is convex), by well known lower semicontinuity results (cf. [22, 23], see e.g. [15, Theorem 3.23]), for any $k = 1, \dots, m$ it follows

$$\int_{\mathbb{R}^N} j_k(u_k, |Du_k|)dx \le \liminf_h \int_{\mathbb{R}^N} j_k(u_k^h, |Du_k^h|)dx, \tag{4.27}$$

where the right hand side is uniformly bounded, in view of (4.18) and (4.26). Hence, in conclusion we have $j_k(u_k, |Du_k|) \in L^1(\mathbb{R}^N)$ for any $k = 1, \dots, m$ and

$$J(u) \leq \liminf_h J(u^h) = \lim_h J(u^h) = T. \tag{4.28}$$

Then, to conclude the proof, it is sufficient to show that the limit u satisfies the constraint. Let us first prove that $T < 0$. For any $\theta \in (0, 1]$, let us consider the function

$$\Upsilon_k^\theta(x) = \frac{\theta^{N/p^2}}{d_k^{1/p}} e^{-\theta|x|^p}, \quad d_k = m \int_{\mathbb{R}^N} G_k(e^{-|x|^p})dx, \quad k = 1, \dots, m.$$

Therefore $(\Upsilon_1^\theta, \dots, \Upsilon_m^\theta)$ belongs to $\mathcal{C}$ with $\Upsilon_k^\theta \in L^\infty(\mathbb{R}^N, \mathbb{R}_+)$ for all $k = 1, \dots, m$ since by the p-homogeneity of any G_k and a simple change of scale we get

$$\sum_{k=1}^m \int_{\mathbb{R}^N} G_k(\Upsilon_k^\theta(x))dx = \sum_{k=1}^m \frac{\theta^{N/p}}{d_k} \int_{\mathbb{R}^N} G_k(e^{-\theta|x|^p})dx = \sum_{k=1}^m \frac{1}{d_k} \int_{\mathbb{R}^N} G_k(e^{-|x|^p})dx = 1.$$

Notice that

$$|\nabla \Upsilon_k^\theta(x)|^p = p^p \frac{\theta^{N/p+p}}{d_k} e^{-p\theta|x|^p} |x|^{p(p-1)}, \quad x \in \mathbb{R}^N, \quad k = 1, \dots, m.$$

Recalling that the function β_k is continuous, we have

$$\Lambda_k = \sup_{x \in \mathbb{R}^N} \sup_{\theta \in [0,1]} \beta_k(\Upsilon_k^\theta(x)) < \infty.$$

By virtue of the growth condition (4.2) and a simple change of variable, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} j_k(\Upsilon_k^\theta(x), |\nabla \Upsilon_k^\theta(x)|)dx &\leq \int_{\mathbb{R}^N} \beta_k(\Upsilon_k^\theta(x))|\nabla \Upsilon_k^\theta(x)|^p dx \leq \Lambda_k \int_{\mathbb{R}^N} |\nabla \Upsilon_k^\theta(x)|^p dx \\ &\leq \frac{\Lambda_k p^p \theta^{N/p+p}}{d_k} \int_{\mathbb{R}^N} e^{-p\theta|x|^p} |x|^{p(p-1)} dx = \theta C_k,\end{aligned}$$

where we have set

$$C_k = \frac{\Lambda_k p^p}{d_k} \int_{\mathbb{R}^N} e^{-p|x|^p} |x|^{p(p-1)} dx, \quad k = 1, \dots, m.$$

In light of assumption (4.9), since of course $0 \leq \Upsilon_k^\theta(x) \leq \theta^{N/p^2}/d_k^{1/p} \leq \delta$ for θ sufficiently small and all $k = 1, \dots, k$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} F(|x|, \Upsilon_1^\theta(x), \dots, \Upsilon_m^\theta(x))dx &\geq \sum_{k=1}^m \frac{\mu_k}{d_k^{\frac{\sigma_k+p}{p}}} \theta^{\frac{N(\sigma_k+p)}{p^2}} \int_{\{|x| \geq r_0\}} |x|^{-\tau_k} e^{-\theta(\sigma_k+p)|x|^p} dx \\ &\geq \sum_{k=1}^m \theta^{\frac{N\sigma_k+p\tau_k}{p^2}} C_k',\end{aligned}$$

where we have set

$$C_k' = \frac{\mu_k}{d_k^{\frac{\sigma_k+p}{p}}} \int_{\{|x| \geq r_0\}} |x|^{-\tau_k} e^{-(\sigma_k+p)|x|^p} dx, \quad k = 1, \dots, m.$$

In conclusion, collecting the previous inequalities, for $\theta > 0$ sufficiently small,

$$\begin{aligned} T &\leq \sum_{k=1}^{m} \int_{\mathbb{R}^N} j_k(\Upsilon_k^\theta(x), |\nabla \Upsilon_k^\theta(x)|)dx - \int_{\mathbb{R}^N} F(|x|, \Upsilon_1^\theta(x), \dots, \Upsilon_m^\theta(x))dx \\ &\leq \theta \sum_{k=1}^{m} \Big(C_k - \theta^{\frac{N\sigma_k + p\tau_k - p^2}{p^2}} C_k' \Big) < 0, \end{aligned}$$

as $N\sigma_k + p\tau_k - p^2 < 0$, yielding the desired assertion. Now, of course, we have

$$\sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k)dx \leq \liminf_{h\to\infty} \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k^h)dx = 1.$$

In particular it holds $G_k(u_k) \in L^1(\mathbb{R}^N)$, for every $k = 1, \dots, m$. Notice also that we have $(u_1, \dots, u_m) \neq (0, \dots, 0)$, otherwise we would get a contradiction by combining inequality (4.28) with $T < 0$. Choosing the positive number

$$\tau := \Big(\sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k)dx \Big)^{-1/p} \geq 1,$$

via the p-homogeneity of G_k it follows that $(\tau u_1, \dots, \tau u_m)$ belongs $\mathcal{C}$ as

$$\sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(\tau u_k)dx = \tau^p \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k)dx = 1.$$

Therefore, by taking into account conditions (4.4) and (4.12), it follows from (4.28) that

$$\begin{aligned} T &\leq \sum_{k=1}^{m} \int_{\mathbb{R}^N} j_k(\tau u_k, \tau|\nabla u_k|)dx - \int_{\mathbb{R}^N} F(|x|, \tau u_1, \dots, \tau u_m)dx \\ &\leq \tau^\alpha \Big(\sum_{k=1}^{m} \int_{\mathbb{R}^N} j_k(u_k, |\nabla u_k|)dx - \int_{\mathbb{R}^N} F(|x|, u_1, \dots, u_m)dx \Big) \\ &= \tau^\alpha J(u) \leq \tau^\alpha T. \end{aligned}$$

This, being $T < 0$, yields $\tau = 1$ so that that $(u_1, , \dots, u_m) \in \mathcal{C}$, concluding the proof. □

Remark 4.8. Assume that the map

$$\Big\{ \xi \mapsto \sum_{k=1}^{m} j_k(s_k, |\xi_k|) \Big\} \tag{4.29}$$

is strictly convex and there exists $\nu > 0$ such that

$$\sum_{k=1}^{m} j_k(s_k, |\xi_k|) \geq \nu \sum_{k=1}^{m} |\xi_k|^p, \quad \text{for all } s \in \mathbb{R}^m \text{ and } \xi \in \mathbb{R}^{mN}.$$

From the proof of Theorem 4.5 we know that the weak limit $(u_1,\dots,u_m)$ of the minimizing sequence satisfies the constraint. Then, recalling (4.26) we have

$$\begin{aligned}
T &= \sum_{k=1}^m \int_{\mathbb{R}^N} j_k(u_k^h,|\nabla u_k^h|)dx - \int_{\mathbb{R}^N} F(|x|,u_1^h,\dots,u_m^h)dx + o(1) \\
&= \int_{\mathbb{R}^N} \sum_{k=1}^m (j_k(u_k^h,|\nabla u_k^h|) - j_k(u_k,|\nabla u_k|))dx \\
&+ \sum_{k=1}^m \int_{\mathbb{R}^N} j_k(u_k,|\nabla u_k|)dx - \int_{\mathbb{R}^N} F(|x|,u_1,\dots,u_m)dx + o(1) \\
&\geq T + \int_{\mathbb{R}^N} \sum_{k=1}^m (j_k(u_k^h,|\nabla u_k^h|) - j_k(u_k,|\nabla u_k|))dx + o(1),
\end{aligned}$$

as $h \to \infty$. Taking into account the weak lower semicontinuity, along a subsequence,

$$\int_{\mathbb{R}^N} \sum_{k=1}^m j_k(u_k^h,|\nabla u_k^h|)dx = \int_{\mathbb{R}^N} \sum_{k=1}^m j_k(u_k,|\nabla u_k|)dx + o(1), \quad \text{as } h \to \infty.$$

In turn, by the strict convexity of (4.29), whenever a condition as (3.4) holds, we have

$$\lim_h \sum_{k=1}^m \|\nabla u_k^h\|_p^p = \sum_{k=1}^m \|\nabla u_k\|_p^p.$$

Recalling (4.13), since we have $G_k(u_k^h) \to G_k(u_k)$ a.e. as $h \to \infty$,

$$\sum_{k=1}^m \int_{\mathbb{R}^N} G_k(u_k^h)dx = \sum_{k=1}^m \int_{\mathbb{R}^N} G_k(u_k)dx \quad \text{and} \quad \gamma \sum_{k=1}^m |u_k^h|^p \leq \sum_{k=1}^m G_k(u_k^h),$$

we conclude that $(u_1^h,\dots,u_m^h)$ converges strongly to $(u_1,\dots,u_m)$ in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^m)$.

4.7. **Proof of Theorem 4.6.**

Proof. We know from Theorem 4.5 that problem (4.1) admits at least a radially symmetry and radially decreasing positive solution u. Assume now that $v=(v_1,\dots,v_m)$ is another positive solution to problem (4.1). Hence, if v_k^* denotes the Schwarz symmetrization of v_k, as $\int_{\mathbb{R}^N} G_k(v_k^*)dx = \int_{\mathbb{R}^N} G_k(v_k)dx$ for all $k=1,\dots,m$, we have

$$\begin{aligned}
T &\leq \sum_{k=1}^m \int_{\mathbb{R}^N} j_k(v_k^*,|\nabla v_k^*|)dx - \int_{\mathbb{R}^N} F(|x|,v_1^*,\dots,v_m^*)dx \\
&\leq \sum_{k=1}^m \int_{\mathbb{R}^N} j_k(v_k,|\nabla v_k|)dx - \int_{\mathbb{R}^N} F(|x|,v_1,\dots,v_m)dx = T.
\end{aligned}$$

In turn, equality must holds, and since

$$\int_{\mathbb{R}^N} j_k(v_k^*,|\nabla v_k^*|)dx \leq \int_{\mathbb{R}^N} j_k(v_k,|\nabla v_k|)dx, \quad \text{for all } k=1,\dots,m,$$

$$\int_{\mathbb{R}^N} F(|x|,v_1,\dots,v_m)dx \leq \int_{\mathbb{R}^N} F(|x|,v_1^*,\dots,v_m^*)dx,$$

in particular one has

$$\int_{\mathbb{R}^N} j_k(v_k^*, |\nabla v_k^*|)dx = \int_{\mathbb{R}^N} j_k(v_k, |\nabla v_k|)dx, \quad \text{for all } k = 1, \dots, m.$$

Therefore, in light of Corollary 3.8, any component u_i of the solution which satisfies $\mu(C^* \cap (u_i^*)^{-1}(0, M_i)) = 0$ is automatically radially symmetric and radially decreasing, after suitable translations in $\mathbb{R}^N$. □

4.8. **Proof of Theorem 4.7.**

Proof. It is sufficient to follow the proof of [24, Lemma 2.11] to show that there exists a positive solution u of the minimization problem

$$\min \Big\{ \int_{\mathbb{R}^N} j(u, |\nabla u|)dx : u \in D^{1,p}(\mathbb{R}^N),\, G(u) \in L^1(\mathbb{R}^N),\, \int_{\mathbb{R}^N} G(u)dx = 1 \Big\}. \tag{4.30}$$

Since $\int_{\mathbb{R}^N} G(u^*)dx = 1$ and $\int_{\mathbb{R}^N} j(u^*, |\nabla u^*|)dx \le \int_{\mathbb{R}^N} j(u, |\nabla u|)dx$ by Corollary 3.3, one can assume that u is radially symmetric and radially decreasing. By Corollary 3.8 it follows that every solution w of the problem (which is positive by [13, Proposition 5]) which satisfies $\mu(C^* \cap (w^*)^{-1}(0, M)) = 0$ is radially symmetric and radially decreasing after a suitable translation in $\mathbb{R}^N$. By means of [13, Lemma 1] (more precisely a simple generalization of the lemma to cover general operators $j(u, |\nabla u|)$ which are p-homogeneous in the gradient), minimizers of (4.30) and least energy solutions to (4.16) correspond via scaling. In order to apply [13, Lemma 1], one also needs that the properties ($\mathbf{C_2}$) and ($\mathbf{C_3}$) indicated therein are satisfied. This properties have been proved in [24, Lemma 2.13 and Lemma 2.16]. This concludes the proof of Theorem 4.7. □

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