SOME CHARACTERIZATIONS OF MAGNETIC SOBOLEV SPACES

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ABSTRACT. The aim of this note is to survey recent results contained in [30–33, 39], where the authors extended to the magnetic setting several characterizations of Sobolev and BV functions.

1. INTRODUCTION

Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of radial mollifiers, i.e. $\rho_n(x) = \rho_n(|x|)$, such that

$$\rho_n \geq 0, \quad \int_0^\infty \rho_n(r)r^{N-1}dx = 1, \quad \text{and} \quad \lim_{n\to\infty}\int_\delta^\infty \rho_n(r)r^{N-1}dr = 0 \quad \text{for every } \delta > 0.$$

Let Ω be a smooth bounded open subset of $\mathbb{R}^N$ and let $p \geq 1$. In [5], Bourgain, Brézis, and Mironescu proved that, if $u \in L^p(\Omega)$ and

$$\sup_{n\in\mathbb{N}} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p}\, \rho_n(|x-y|)\, dx\, dy \leq C,$$

for some positive constant C, then $u \in W^{1,p}(\Omega)$ if $p > 1$ and $u \in BV(\Omega)$ if $p = 1$. Moreover, one has

$$(1.1) \qquad \lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p}\, \rho_n(|x-y|)\, dx\, dy = pQ_{p,N} \int_\Omega |\nabla u|^p\, dx.$$

Here

$$(1.2) \qquad Q_{p,N} := \frac{1}{p}\int_{\mathbb{S}^{N-1}} |\boldsymbol{\omega}\cdot\sigma|^p d\sigma,$$

where $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ denotes the unit sphere and $\boldsymbol{\omega}$ stands for an arbitrary unit vector of $\mathbb{R}^N$. Assertion (1.1) is established by Bourgain, Brézis, and Mironescu in [5] for $u \in W^{1,p}(\Omega)$ with $p \geq 1$. Assertion (1.1) with $p = 1$ and $u \in BV(\Omega)$ is obtained by Davila [13]. In particular, we have the following celebrated Bourgain-Brézis-Mironescu (BBM) formula, for every $u \in W^{1,p}(\Omega)$,

$$(1.3) \qquad \lim_{s\to 1^-} (1-s)\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy = Q_{p,N}\int_\Omega |\nabla u|^p\, dx.$$

Other properties related to the BBM formula can be found in [9, 11, 12]. In the spirit of (1.3), Maz'ya and Shaposhnikova proved in [22] that for any $p \in [1,\infty)$,

$$\lim_{s\searrow 0} s \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy = \frac{4\pi^{N/2}}{p\Gamma(N/2)}\|u\|^p_{L^p(\mathbb{R}^N)},$$

whenever $u \in D_0^{s,p}(\mathbb{R}^N)$ for some $s \in (0,1)$. Here Γ denotes the Gamma function and the space $D_0^{s,p}(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the Gagliardo semi-norm.

Other characterizations of Sobolev spaces and BV functions which are somewhat related to the one of Bourgain, Brézis, and Mironescu are established in [7, 24]. For example, in the case $p = 2$, the following characterization of $H^1(\Omega)$ is given in [7, 24]. Set

$$I_\delta(u) := \iint_{\substack{\Omega \times \Omega \\ \{|u(y)-u(x)|>\delta\}}} \frac{\delta^2}{|x-y|^{N+2}} dx\, dy, \quad \text{for } u \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \delta > 0.$$

Then for any $u \in L^2(\Omega)$, $u \in H^1(\Omega)$ if and only if $\sup_{0<\delta<1} I_\delta(u) < \infty$. Moreover, for every $u \in H^1(\Omega)$

$$\lim_{\delta \searrow 0} I_\delta(u) = Q_{2,N} \int_\Omega |\nabla u|^2 dx,$$

where $Q_{2,N}$ is the same positive constant appearing in (1.3) for $p = 2$. Other results related to the nonlocal operator I_δ can be found in [10, 26–29]. The aim of this note is to survey recent results contained in [30–33, 39], where the authors have extended the aforementioned results to the magnetic setting. We refer the interested reader to these papers for the proofs and further details.

2. Magnetic Sobolev and BV spaces

An important role in the study of particles which interact with a magnetic field $B = \nabla \times A$, $A : \mathbb{R}^3 \to \mathbb{R}^3$, is played by an *extension* of the Laplacian, known as *magnetic Laplacian* $(\nabla - iA)^2$ (see [3, 37]). Nonlinear magnetic Schrödinger equations like

$$-(\nabla - iA)^2 u + u = f(u)$$

have been extensively studied (see e.g. [2, 3, 16, 38] and the references therein). The functional framework to work with these equations is the magnetic Sobolev spaces which will be now recalled, see [21] for a concise introduction to the topic. For $p \geq 1$, let us endow the vector space $\mathbb{C}^N$ with the norm

$$|z|_p := (|(\Re z_1, \ldots, \Re z_N)|^p + |(\Im z_1, \ldots, \Im z_N)|^p)^{1/p},$$

where $\Re a$, $\Im a$ denote the real and imaginary parts of $a \in \mathbb{C}$ respectively, and $|\cdot|$ the Euclidean norm of $\mathbb{R}^N$. We notice that $|z|_p = |z|$ whenever $z \in \mathbb{R}^N$. We warn the reader that in the non Hilbert case $p \neq 2$, this choice for a norm on $\mathbb{C}^N$ is different from the standard one. Continuing with the notation, we will denote the imaginary unit by i, and we denote by $L^p(\Omega, \mathbb{C})$ the Lebesgue space naturally associated to $|\cdot|_p$.

We are ready to introduce

Definition 2.1. *Let $p \geq 1$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ be a measurable function. The magnetic Sobolev space $W_A^{1,p}(\Omega)$ is given by*

$$W_A^{1,p}(\Omega) := \left\{ u \in L^p(\Omega, \mathbb{C}) : [u]_{W_A^{1,p}(\Omega)} < \infty \right\},$$

where

$$[u]_{W_A^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u - iA(x)u|_p^p dx \right)^{1/p}.$$

The space $W_A^{1,p}(\Omega)$ is equipped with the following norm

$$\|u\|_{W_A^{1,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + [u]_{W_A^{1,p}(\Omega)}^p \right)^{1/p}.$$

We can also define the space $W_{0,A}^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W_A^{1,p}(\Omega)$. As a notational remark, as it is customary, when $p = 2$ we will denote the magnetic Sobolev space $W_A^{1,2}(\Omega)$ by $H_A^1(\Omega)$.

A possibility to define a suitable notion of *fractional magnetic Sobolev space* is to use the energy space of a non-local operator on $\mathbb{R}^N$, see [15, 18]. There are at least three possible notions of *magnetic fractional Laplacian* which are in general not equivalent, see the survey of Ichinose in [18]. The most frequently used operator is $(-\Delta)_A^s$, which is defined as the gradient of the non-local energy functional

$$u \mapsto \iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{N+2s}} dx\, dy,$$

namely

$$(-\Delta)_A^s u(x) = c(N,s) \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon^c(x)} \frac{u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{N+2s}} dy,$$

where

$$\lim_{s \nearrow 1} \frac{c(N,s)}{1-s} = \frac{4N\Gamma(N/2)}{2\pi^{N/2}}.$$

Recently, the operator $(-\Delta)_A^s$ has been investigated in several directions. Here is a brief (and far from being complete) list of references: [4, 14, 17, 23, 41]

We are ready to introduce the non-local counterpart of the magnetic Sobolev spaces:

Definition 2.2. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a locally bounded measurable function and let $\Omega \subset \mathbb{R}^N$ be an open set. For any $s \in (0,1)$ and $p \geq 1$, the magnetic Gagliardo semi-norm is defined as*

$$[u]_{W_A^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|_p^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}.$$

The fractional magnetic Sobolev space $W_A^{s,p}(\Omega)$ is given by

$$W_A^{s,p}(\Omega) := \left\{ u \in L^p(\Omega, \mathbb{C}) : [u]_{W_A^{s,p}(\Omega)} < \infty \right\},$$

and it is equipped with the norm

$$\|u\|_{W_A^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + [u]_{W_A^{s,p}(\Omega)}^p \right)^{1/p}.$$

We stress that for $A \equiv 0$ and u real-valued, the above definition is consistent with the usual fractional Sobolev space $W^{s,p}(\Omega)$ endowed with the classical norm

$$\| \cdot \|_{W^{s,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|_p^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}.$$

Remark 2.1. As it is pointed out in [18], in place of the magnetic norm defined via the simple midpoint prescription $(x,y) \mapsto A((x+y)/2)$, other prescriptions are viable in applications such as the averaged one

$$(x,y) \mapsto \int_0^1 A\left((1-\vartheta)x + \vartheta y\right) d\vartheta =: A_\sharp(x,y).$$

If $(-\Delta)_A^s$ and $(-\Delta)_{A_\sharp}^s$ are the fractional operators associated with $A((x+y)/2)$ and $A_\sharp(x,y)$ respectively, it follows that $(-\Delta)_{A_\sharp}^s$ is Gauge covariant, which is relevant for Schrödinger operators, i.e. for all $\phi \in \mathscr{S}(\mathbb{R}^n)$

$$(-\Delta)_{(A+\nabla\phi)_\sharp}^s = e^{i\phi}(-\Delta)_{A_\sharp}^s e^{-i\phi},$$

see e.g. [18, Proposition 2.8].

We present now the notion of magnetic bounded variation functions introduced in [32].

Definition 2.3 ($A-$bounded variation functions). *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a locally bounded function. A function $u \in L^1(\Omega, \mathbb{C})$ is said to be of A-bounded variation and we write $u \in BV_A(\Omega)$, if*

$$|Du|_A(\Omega) := C_{1,A,u}(\Omega) + C_{2,A,u}(\Omega) < \infty,$$

where we set

$$C_{1,A,u}(\Omega) := \sup \left\{ \int_\Omega \Re u(x)\mathrm{div}\varphi(x) - A(x)\cdot\varphi(x)\Im u(x)dx \mid \varphi \in C_c^\infty(\Omega, \mathbb{R}^N), \ \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\},$$

$$C_{2,A,u}(\Omega) := \sup \left\{ \int_\Omega \Im u(x)\mathrm{div}\varphi(x) + A(x)\cdot\varphi(x)\Re u(x)dx \mid \varphi \in C_c^\infty(\Omega, \mathbb{R}^N), \ \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

A function $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$ is said to be of locally A-bounded variation and we write $u \in BV_{A,\mathrm{loc}}(\Omega)$, if

$$|Du|_A(U) < \infty, \qquad \text{for every open set } U \Subset \Omega.$$

We endow the space $BV_A(\Omega, \mathbb{C})$ with the following norm:

$$\|u\|_{BV_A(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|_A(\Omega).$$

With this choice, the space $(BV_A(\Omega), \|\cdot\|_{BV_A(\Omega)})$ is a real Banach space, see [32, Lemma 3.8].

As for the magnetic Sobolev spaces, in the case $A \equiv 0$, the previous definition is consistent with the classical one of $BV(\Omega)$. We summarize now the basic properties of the space $BV_A(\Omega)$ that has been fully proved in [32]. The coming results can be considered as the natural extension to the magnetic setting of the classical theory, see e.g. [1].

Lemma 2.1 (Lemma 3.2 [32]). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, $A : \mathbb{R}^N \to \mathbb{R}^N$ locally bounded and $u \in BV_A(\Omega)$. Let $E \subset \Omega$ be a Borel set then*

$$|Du|_A(E) := \inf\{C_{1,A,u}(U) \mid E \subset U, \ U \subset \Omega \ open\} + \inf\{C_{2,A,u}(U) \mid E \subset U, \ U \subset \Omega \ open\}$$

extends $|Du|_A(\cdot)$ to a Radon measure in Ω. For any open set $U \subset \Omega$, $C_{1,A,u}(U)$ and $C_{2,A,u}(U)$ are defined requiring the test functions to be supported in U and $|Du|_A(\emptyset) := 0$.

Lemma 2.2 (Lemma 3.3 [32]). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Then*

$$W^{1,1}_{\mathrm{loc}}(\Omega) \subset BV_{A,\mathrm{loc}}(\Omega).$$

Lemma 2.3 (Lemma 3.4 [32]). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Assume that $u \in W^{1,1}_A(\Omega)$. Then $u \in BV_A(\Omega)$ and it holds*

$$|Du|_A(\Omega) = \int_\Omega |\nabla u - iA(x)u|_1 dx.$$

Furthermore, if $u \in BV_A(\Omega) \cap C^\infty(\Omega)$, then $u \in W^{1,1}_A(\Omega)$.

Lemma 2.4 (Lemma 3.5 [32]). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Then $u \in BV_A(\Omega)$ if and only if $u \in BV(\Omega)$. Moreover, for every $u \in BV_A(\Omega)$, there exists a positive constant $K = K(A, \Omega)$ such that*

$$K^{-1}\|u\|_{BV(\Omega)} \le \|u\|_{BV_A(\Omega)} \le K\|u\|_{BV(\Omega)}.$$

Lemma 2.5 (Lemma 3.7 [32]). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Let $\Omega \subset \mathbb{R}^N$ be an open set and $\{u_k\}_{k \in \mathbb{N}} \subset BV_A(\Omega)$ a sequence converging locally in $L^1(\Omega)$ to a function u. Then*

$$\liminf_{k \to \infty} |Du_k|_A(\Omega) \ge |Du|_A(\Omega).$$

Lemma 2.6 (Lemma 3.10 [32]). *Suppose that $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $u \in BV_A(\Omega)$. Then there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset C^\infty(\Omega, \mathbb{C})$ such that*

$$\lim_{k \to \infty} \int_\Omega |u_k - u|_1 \, dx = 0 \quad and \quad \lim_{k \to \infty} |Du_k|_A(\Omega) = |Du|_A(\Omega).$$

Lemma 2.7 (Lemma 3.14 [32]). *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and that $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally bounded. Let $\{u_k\}_{k \in \mathbb{N}}$ be a bounded sequence in $BV_A(\Omega)$. Then, up to a subsequence, it converges strongly in $L^1(\Omega)$ to some function $u \in BV_A(\Omega)$.*

Lemma 2.8 (Lemma 3.12 [32]). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz. Then for any open set $W \supset \overline{\Omega}$, there exists a linear and continuous extension operator $E : BV_A(\Omega) \to BV_A(\mathbb{R}^N)$ such that*

$$Eu = 0, \quad for \ almost \ every \ x \in \mathbb{R}^N \setminus W, \quad and \quad |DEu|_A(\partial\Omega) = 0,$$

for every $u \in BV_A(\Omega)$.

A few words concerning the proofs of the aforementioned results are now in order. Roughly speaking, the strategy of the proofs follow the classical ones as in e.g. [1]. From the technical point of view, once we ask for local boundedness of A we can usually control the extra-terms coming from A. In particular, the *norm equivalence* provided by Lemma 2.4 and the pointwise *Diamagnetic inequality*, see e.g., [14] allow sometimes to get magnetic results from the classical ones. We refer to [32, Section 3] for more details.

3. Magnetic BBM-type formulas

The introduction of the magnetic counterpart of classical Sobolev spaces and BV space leads to the following natural question: do BBM-type formulas still hold in the magnetic setting? The aim of this section is to collect some results that provide a positive answer to the above question.

An useful equality to get BBM-type formulas is

$$(3.1) \qquad \int_{\mathbb{S}^{N-1}} |v \cdot \sigma|_p^p \, d\sigma = p \, Q_{p,N} \, |v|_p^p, \qquad \text{for all } v \in \mathbb{C}^N, p \geq 1.$$

This motivates the introduction of the norm $|\cdot|_p$ on $\mathbb{C}^N$. Indeed, (3.1) does not hold with the classical Euclidean norm for $p \neq 2$. Given $u : \mathbb{R}^N \to \mathbb{C}$ a measurable complex-valued function, we denote

$$\Psi_u(x, y) := e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y), \qquad x, y \in \mathbb{R}^N.$$

The function $\Psi_u(\cdot, \cdot)$ also depends on A but for notational ease, we ignore it.

Theorem 3.1 (Magnetic Bourgain-Brezis-Mironescu type result). *Let $p \geq 1$, $A : \bar{\Omega} \to \mathbb{R}^N$ be of class C^1 and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative radial mollifiers. Then $u \in W_A^{1,p}(\Omega)$ if $p > 1$ and $u \in BV_A(\Omega)$ if $p = 1$ if and only if $u \in L^p(\Omega)$ and*

$$(3.2) \qquad \sup_{n\in\mathbb{N}} \int_\Omega \int_\Omega \frac{|\Psi_u(x,y) - \Psi_u(x,x)|_p^p}{|x-y|^p} \rho_n(|x-y|) \, dx \, dy < +\infty.$$

Moreover,

$$(3.3) \qquad \lim_{n\to+\infty} \int_\Omega \int_\Omega \frac{|\Psi_u(x,y) - \Psi_u(x,x)|_p^p}{|x-y|^p} \rho_n(|x-y|) \, dx dy = Q_{1,N} |Du|_A(\Omega).$$

Statement (3.2) is proved in [39] for $p = 2$, in [32] for $p = 1$ both under the assumption that $A \in C^2(\bar{\Omega})$, in [31] for $p > 1$ and $A \in C^1(\bar{\Omega})$, and for $p \geq 1$ for a more general (anisotropic) setting in [30]. The proof of (3.3) is given in [31] for $p > 1$ and for $p \geq 1$ for a more general setting in [30] [1]. The proof of Theorem 3.1 is essentially based on the works in the case without magnetic field, see [5, 8, 13]. Nevertheless work is required to deal with the presence of the magnetic field A.

4. A magnetic version of the result by Maz'ya and Shaposhnikova

The aim of this section is to describe the generalization proved in [33] of [22] to the magnetic case. For a locally bounded A, let the space of complex valued functions $D_{A,0}^{s,p}(\mathbb{R}^N, \mathbb{C})$ be the completion of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to the norm

$$\|u\|_{D_{A,0}^{s,p}} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\Psi_u(x,y) - \Psi_u(x,x)|^p}{|x-y|^{N+ps}} dx dy \right)^{1/p}.$$

Theorem 4.1 (Magnetic Maz'ya-Shaposhnikova type result). *Let $N \geq 1$ and $p \in [1, \infty)$. Then for every*

$$u \in \bigcup_{0<s<1} D_{A,0}^{s,p}(\mathbb{R}^N, \mathbb{C}),$$

there holds

$$\lim_{s\searrow 0} s \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\Psi_u(x,x) - \Psi_u(x,y)|^p}{|x-y|^{N+ps}} dx dy = \frac{4\pi^{N/2}}{p\Gamma(N/2)} \|u\|_{L^p(\mathbb{R}^N)}^p.$$

In one direction the proof is based on the Diamagnetic inequality to reduce the problem to the non-magnetic case. For the converse inequality, the magnetic effects has to be controlled, and this can be done because the magnetic effect becomes negligible as $s \to 0$.

Remark 4.1. We point out that when $A \equiv 0$ then Theorem 4.1 boils down to the result proved in [22]. It also remains valid for the operator $A_\sharp$ and its proof carries on by trivial modifications of our arguments.

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[1]Some of these works only deal with the whole space setting, nevertheless, one can extend them for a smooth bounded domains as stated.

5. A MAGNETIC VERSION OF THE RESULTS BY BOURGAIN AND NGUYEN

In this section, we present some results in [31]. Set

$$J_\delta(u) := \iint_{\{|\Psi_u(x,y) - \Psi_u(x,x)| > \delta\}} \frac{\delta^2}{|x - y|^{N+2}} \, dx dy, \quad \text{for } u \in L^1_{\text{loc}}(\mathbb{R}^N), \qquad \delta > 0.$$

We prove

Theorem 5.1. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz. Then $u \in H^1_A(\mathbb{R}^N)$ if and only if $u \in L^2(\mathbb{R}^N)$ and*

$$\sup_{0 < \delta < 1} J_\delta(u) < +\infty.$$

Moreover, we have, for $u \in H^1_A(\mathbb{R}^N)$,

$$\lim_{\delta \searrow 0} J_\delta(u) = Q_N \int_{\mathbb{R}^N} |\nabla u - \mathrm{i} A(x) u|^2 \, dx$$

and

$$\sup_{\delta > 0} J_\delta(u) \le C_N \left(\int_{\mathbb{R}^N} |\nabla u - \mathrm{i} A(x) u|^2 \, dx + \left(\|\nabla A\|^2_{L^\infty(\mathbb{R}^N)} + 1 \right) \int_{\mathbb{R}^N} |u|^2 \, dx \right).$$

This provides a new characterization of the H^1_A norm in terms of nonlocal functionals extending to the magnetic setting further results in the spirit of Bourgain, Brézis and Mironescu [5,6] (see also [13,34]).

6. ALMOST EVERYWHERE AND L^1 CONVERGENCE

In this section we collect other results obtained in [31] in the spirit of the works [11] and [36]. We are there interested in other modes of convergence of functionals related to those appearing in Theorems 3.1 and 5.1. We only recall some results for the case $p = 2$. For $u \in L^1_{\text{loc}}(\mathbb{R}^N)$, set

$$D_n(u, x) := \int_{\mathbb{R}^N} \frac{|\Psi_u(x, y) - \Psi_u(x, x)|^2}{|x - y|^2} \rho_n(|x - y|) \, dy, \quad \text{for } x \in \mathbb{R}^N.$$

Concerning Theorem 3.1, we have

Proposition 6.1 (Proposition 4.1 [31]). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz, $u \in H^1_A(\mathbb{R}^N)$, and let (ρ_n) be a sequence of radial mollifiers such that*

$$\sup_{t > 1} \sup_n t^{-2} \rho_n(t) < +\infty.$$

We have

$$\lim_{n \to +\infty} D_n(u, x) = 2Q_N |\nabla u(x) - \mathrm{i} A(x) u(x)|^2, \quad \text{for a.e. } x \in \mathbb{R}^N,$$

and

$$\lim_{n \to +\infty} D_n(u, \cdot) = 2Q_N |\nabla u(\cdot) - \mathrm{i} A(\cdot) u(\cdot)|^2, \quad \text{in } L^1(\mathbb{R}^N).$$

Concerning Theorem 5.1, we set, for $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$,

$$J_\delta(u, x) := \int_{\{|\Psi_u(x,y) - \Psi_u(x,x)| > \delta\}} \frac{\delta^2}{|x - y|^{N+2}} dy.$$

We have

Proposition 6.2 (Proposition 4.2 [31]). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz and let $u \in H^1_A(\mathbb{R}^N)$. Then*

$$(6.1) \qquad \lim_{\delta \searrow 0} J_\delta(u, x) = Q_N |\nabla u(x) - \mathrm{i} A(x) u(x)|^2, \quad \text{for a.e. } x \in \mathbb{R}^N$$

and

$$(6.2) \qquad \lim_{\delta \searrow 0} J_\delta(u, \cdot) = Q_N |\nabla u(\cdot) - \mathrm{i} A(\cdot) u(\cdot)|^2, \quad \text{in } L^1(\mathbb{R}^N).$$

In both cases, we prove the results on smooth functions relying on delicate estimates of maximal-type functions with their roots in [11]. We can then conclude arguing by density. We refer to [31] for detailed proofs of both results.

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