Research Article

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A note on the Dancer–Fučík spectra of the fractional p-Laplacian and Laplacian operators

Abstract: We study the Dancer–Fučík spectrum of the fractional p-Laplacian operator. We construct an unbounded sequence of decreasing curves in the spectrum using a suitable minimax scheme. For $p = 2$, we present a very accurate local analysis. We construct the minimal and maximal curves of the spectrum locally near the points where it intersects the main diagonal of the plane. We give a sufficient condition for the region between them to be nonempty and show that it is free of the spectrum in the case of a simple eigenvalue. Finally, we compute the critical groups in various regions separated by these curves. We compute them precisely in certain regions and prove a shifting theorem that gives a finite-dimensional reduction in certain other regions. This allows us to obtain nontrivial solutions of perturbed problems with nonlinearities crossing a curve of the spectrum via a comparison of the critical groups at zero and infinity.

Keywords: Fractional p-Laplacian, Dancer–Fučík spectrum, critical groups

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1 Introduction

For $p \in (1, \infty)$, $s \in (0, 1)$ and $N > sp$, the fractional p-Laplacian $(-\Delta)_p^s$ is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}}\, dy, \quad x \in \mathbb{R}^N.$$

This definition is consistent, up to a normalization constant depending on N and s, with the usual definition of the linear fractional Laplacian operator $(-\Delta)^s$ when $p = 2$. There is currently a rapidly growing literature on problems involving these nonlocal operators. In particular, fractional p-eigenvalue problems have been studied in Lindgren and Lindqvist [29], Iannizzotto and Squassina [25] and Franzina and Palatucci [20], regularity of fractional p-minimizers in Di Castro, Kuusi and Palatucci [15] and existence via Morse theory in Iannizzotto, Liu, Perera and Squassina [24]. We refer to Caffarelli [6] for the motivations that have lead to their study.

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. The Dancer–Fučík spectrum of the operator $(-\Delta)_p^s$ in Ω is the set $\Sigma_p^s(\Omega)$ of all points $(a, b) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} (-\Delta)_p^s\, u = b(u^+)^{p-1} - a(u^-)^{p-1} & \text{in } \Omega, \\ \qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $u^\pm = \max\{\pm u, 0\}$ are the positive and negative parts of u, respectively, has a nontrivial weak solution. Let us recall the weak formulation of (1.1). Let

$$[u]_{s,p} = \Bigg(\int\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dx\, dy \Bigg)^{1/p}$$

be the Gagliardo seminorm of the measurable function $u : \mathbb{R}^N \to \mathbb{R}$ and let

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = (|u|_p^p + [u]_{s,p}^p)^{1/p},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$. We work in the closed linear subspace

$$X_p^s(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$ (see Di Nezza, Palatucci and Valdinoci [16, Theorem 7.1]). A function $u \in X_p^s(\Omega)$ is a weak solution of problem (1.1) if

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega (b(u^+)^{p-1} - a(u^-)^{p-1})v\,dx \quad \text{for all } v \in X_p^s(\Omega). \tag{1.2}$$

This notion of spectrum for linear local elliptic partial differential operators has been introduced by Dancer [10, 11] and Fučík [21], who recognized its significance for the solvability of related semilinear boundary value problems. In particular, the Dancer–Fučík spectrum of the Laplacian in Ω with the Dirichlet boundary condition is the set $\Sigma(\Omega)$ of all points $(a, b) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -\Delta u = bu^+ - au^- & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

has a nontrivial solution. Denoting by $\lambda_k \nearrow +\infty$ the Dirichlet eigenvalues of $-\Delta$ in Ω, the spectrum $\Sigma(\Omega)$ clearly contains the sequence of points (λ_k, λ_k). For $N = 1$, where Ω is an interval, Fučík [21] showed that $\Sigma(\Omega)$ with the periodic boundary condition consists of a sequence of hyperbolic-like curves passing through the points (λ_k, λ_k), with one or two curves going through each point. For $N \geq 2$, the spectrum $\Sigma(\Omega)$ consists locally of curves emanating from the points (λ_k, λ_k) (see Gallouët and Kavian [22], Ruf [42], Lazer and McKenna [27], Lazer [26], Các [5], Magalhães [32], Cuesta and Gossez [9], de Figueiredo and Gossez [14] and Margulies and Margulies [33]). Schechter [43] showed that in the square $(\lambda_{k-1}, \lambda_{k+1}) \times (\lambda_{k-1}, \lambda_{k+1})$, the spectrum $\Sigma(\Omega)$ contains two strictly decreasing curves, which may coincide, such that the points in the square that are either below the lower curve or above the upper curve are not in $\Sigma(\Omega)$, while the points between them may or may not belong to $\Sigma(\Omega)$ when they do not coincide.

The Dancer–Fučík spectrum of the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the set $\Sigma_p(\Omega)$ of all $(a, b) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -\Delta_p u = b(u^+)^{p-1} - a(u^-)^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a nontrivial solution. For $N = 1$, the Dirichlet spectrum $\sigma(-\Delta_p)$ of $-\Delta_p$ in Ω consists of a sequence of simple eigenvalues $\lambda_k \nearrow +\infty$ and $\Sigma_p(\Omega)$ has the same general shape as $\Sigma(\Omega)$ (see Drábek [17]). For $N \geq 2$, the first eigenvalue λ_1 of $-\Delta_p$ is positive, simple and has an associated eigenfunction that is positive in Ω (see Anane [2] and Lindqvist [30, 31]), so $\Sigma_p(\Omega)$ contains the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. Moreover, λ_1 is isolated in the spectrum, so the second eigenvalue $\lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty)$ is well-defined (see Anane and Tsouli [3]), and a first nontrivial curve in $\Sigma_p(\Omega)$ passing through (λ_2, λ_2) and asymptotic to $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ at infinity was constructed using the mountain pass theorem by Cuesta, de Figueiredo and Gossez [8]. Although a complete description of $\sigma(-\Delta_p)$ is not yet available, an increasing and unbounded sequence of eigenvalues can be constructed via a standard minimax scheme based on the Krasnosel'skiĭ genus, or via nonstandard schemes based on the cogenus as in Drábek and Robinson [18] and the cohomological index as in Perera [35]. Unbounded sequences of decreasing curves in $\Sigma_p(\Omega)$, analogous to the lower and upper curves of Schechter [43] in the semilinear case, have been constructed using various minimax schemes by Cuesta [7], Micheletti and Pistoia [34], and Perera [36].

Goyal and Sreenadh [23] recently studied the Dancer–Fučík spectrum for a class of linear nonlocal elliptic operators that includes the fractional Laplacian $(-\Delta)^s$. As in Cuesta, de Figueiredo and Gossez [8], they constructed a first nontrivial curve in the Dancer–Fučík spectrum that passes through (λ_2, λ_2) and is asymptotic to $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ at infinity. Very recently, in [4], the authors proved, among other things, that the second variational eigenvalue λ_2 is larger than λ_1 and (λ_1, λ_2) does not contain any other eigenvalues.

The purpose of this note is to point out that the general theories developed in Perera, Agarwal and O'Regan [37] and Perera and Schechter [41] apply to the fractional p-Laplacian and Laplacian operators, respectively, and draw some conclusions about their Dancer–Fučík spectra. We construct an unbounded sequence of decreasing curves in $\Sigma_p^s(\Omega)$ using a suitable minimax scheme. For $p = 2$, we present a very accurate local analysis. We construct the minimal and maximal curves of the spectrum locally near the points where it intersects the main diagonal of the plane. We give a sufficient condition for the region between them to be nonempty and show that it is free of the spectrum in the case of a simple eigenvalue. Finally, we compute the critical groups in various regions separated by these curves. We compute them precisely in certain regions and prove a shifting theorem that gives a finite-dimensional reduction in certain other regions. This allows us to obtain nontrivial solutions of perturbed problems with nonlinearities crossing a curve of the spectrum via a comparison of the critical groups at zero and infinity.

2 The Dancer–Fučík spectrum of the fractional p-Laplacian

The general theory developed in Perera, Agarwal and O'Regan [37] applies to problem (1.1). Indeed, the odd $(p-1)$-homogeneous operator $A_p^s \in C(X_p^s(\Omega), X_p^s(\Omega)^*)$, where $X_p^s(\Omega)^*$ is the dual of $X_p^s(\Omega)$, defined by

$$A_p^s(u)\,v = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy, \quad u, v \in X_p^s(\Omega),$$

that is associated with the left-hand side of equation (1.2) satisfies

$$A_p^s(u)\,u = \|u\|^p, \quad |A_p^s(u)\,v| \le \|u\|^{p-1}\|v\| \quad \text{for all } u, v \in X_p^s(\Omega) \tag{2.1}$$

and is the Fréchet derivative of the C^1-functional

$$I_p^s(u) = \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy, \quad u \in X_p^s(\Omega).$$

Moreover, since $X_p^s(\Omega)$ is uniformly convex, it follows from (2.1) that A_p^s is of type (S), i.e., every sequence $(u_j) \subset X_p^s(\Omega)$ such that

$$u_j \rightharpoonup u, \quad A_p^s(u_j)(u_j - u) \to 0$$

has a subsequence that converges strongly to u (see [37, Proposition 1.3]). Hence, the operator A_p^s satisfies the structural assumptions of [37, Chapter 1].

When $a = b = \lambda$, problem (1.1) reduces to the nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)_p^s\,u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.2}$$

Eigenvalues of this problem coincide with critical values of the functional

$$\Psi(u) = \left(\int_\Omega |u|^p\,dx\right)^{-1}$$

on the manifold

$$\mathcal{M} = \{u \in X_p^s(\Omega) : \|u\| = 1\}.$$

The first eigenvalue

$$\lambda_1 = \inf_{u \in \mathcal{M}} \Psi(u)$$

is positive, simple, isolated and has an associated eigenfunction that is positive in Ω (see Lindgren and Lindqvist [29] and Franzina and Palatucci [20]), so $\Sigma_p^s(\Omega)$ contains the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, let $i(M)$ denote the $\mathbb{Z}_2$-cohomological index of $M \in \mathcal{F}$ (see Fadell and Rabinowitz [19]) and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \geq 2.$$

Then, $\lambda_k \nearrow +\infty$ is a sequence of eigenvalues of problem (2.2) (see [37, Theorem 4.6]), so $\Sigma_p^s(\Omega)$ contains the sequence of points (λ_k, λ_k).

Following [37, Chapter 8], we now construct an unbounded sequence of decreasing curves in $\Sigma_p^s(\Omega)$. For $t > 0$, let

$$\Psi_t(u) = \left(\int_\Omega ((u^+)^p + t(u^-)^p)\, dx \right)^{-1}, \quad u \in \mathcal{M}.$$

Then, the point $(c, ct) \in \Sigma_p^s(\Omega)$ if and only if c is a critical value of Ψ_t (see [37, Lemma 8.3]). For each $k \geq 2$ such that $\lambda_k > \lambda_{k-1}$, let

$$C\Psi_t^{\lambda_{k-1}} = (\Psi_t^{\lambda_{k-1}} \times [0, 1])/(\Psi_t^{\lambda_{k-1}} \times \{1\})$$

be the cone on the sublevel set $\Psi_t^{\lambda_{k-1}} = \{u \in \mathcal{M} : \Psi_t(u) \leq \lambda_{k-1}\}$, let Γ_k denote the class of maps $\gamma \in C(C\Psi_t^{\lambda_{k-1}}, \mathcal{M})$ such that $\gamma|_{\Psi_t{}^{\lambda_{k-1}}}$ is the identity and set

$$c_k^s(t) = \inf_{\gamma \in \Gamma_k} \sup_{u \in \gamma(C\Psi_t^{\lambda_{k-1}})} \Psi_t(u).$$

We have the following theorem as a special case of [37, Theorem 8.8].

Theorem 2.1. *Let*

$$\mathcal{C}_k^s = \left\{ (c_k^s(t), c_k^s(t)t) : \frac{\lambda_{k-1}}{\lambda_k} < t < \frac{\lambda_k}{\lambda_{k-1}} \right\}.$$

Then, $\mathcal{C}_k$ is a decreasing continuous curve in $\Sigma_p^s(\Omega)$ and $c_k^s(1) \geq \lambda_k$.

3 The Dancer–Fučík spectrum of the fractional Laplacian

The Dancer–Fučík spectrum of the operator $(-\Delta)^s$ in Ω is the set $\Sigma^s(\Omega)$ of all points $(a, b) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} (-\Delta)^s u = bu^+ - au^- & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

has a nontrivial weak solution. The general theory developed in Perera and Schechter [41] applies to problem (3.1). Indeed, set $X^s(\Omega) = X_2^s(\Omega)$ and let A^s be the inverse of the solution operator

$$S : L^2(\Omega) \to S(L^2(\Omega)) \subset X^s(\Omega), \quad f \mapsto u,$$

of the problem

$$\begin{cases} (-\Delta)^s u = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then, A^s is a self-adjoint operator on $L^2(\Omega)$ and we have

$$(u, v) = (A^{s/2} u, A^{s/2} v)_2 = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}}\, dx\, dy \quad \text{for all } u, v \in X^s(\Omega)$$

and

$$\|u\| = \|A^{s/2} u\|_2 = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy \right)^{1/2} \quad \text{for all } u \in X^s(\Omega),$$

where $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$ are the inner products in $X^s(\Omega)$ and $L^2(\Omega)$, respectively. Moreover, its spectrum $\sigma(A^s) \subset (0,\infty)$ and $(A^s)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a compact operator since the embedding $X^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Thus, $\sigma(A^s)$ consists of isolated eigenvalues λ_k, $k \geq 1$, of finite multiplicities satisfying $0 < \lambda_1 < \lambda_2 < \cdots$. The first eigenvalue λ_1 is simple and has an associated eigenfunction $\varphi_1 > 0$ and if $w \in ((\mathbb{R}\varphi_1)^\perp \cap X^s(\Omega)) \setminus \{0\}$, then

$$0 = (w, \varphi_1) = (A^s\, w, \varphi_1)_2 = (w, A^s\, \varphi_1)_2 = \lambda_1 (w, \varphi_1)_2,$$

so $w^\pm \neq 0$. Hence, the operator A^s satisfies all the assumptions of [41, Chapter 4].

Now, we describe the minimal and maximal curves of $\Sigma^s(\Omega)$ in the square

$$Q_k = (\lambda_{k-1}, \lambda_{k+1})^2, \quad k \geq 2,$$

constructed in [41]. Weak solutions of problem (3.1) coincide with critical points of the C^1-functional

$$I(u,a,b) = \frac{1}{2} \int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy - \frac{1}{2} \int\limits_{\Omega} (b(u^+)^2 + a(u^-)^2) dx, \quad u \in X^s(\Omega).$$

Denote by E_k the eigenspace of λ_k and set

$$N_k = \bigoplus_{j=1}^{k} E_j, \quad M_k = N_k^\perp \cap X^s(\Omega).$$

Then, $X^s(\Omega) = N_k \oplus M_k$ is an orthogonal decomposition with respect to both $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$. When $(a,b) \in Q_k$, $I(v+y+w)$, $v+y+w \in N_{k-1} \oplus E_k \oplus M_k$ is strictly concave in v and strictly convex in w, i.e., if $v_1 \neq v_2 \in N_{k-1}$, $w \in M_{k-1}$, then

$$I((1-t)v_1 + tv_2 + w) > (1-t)I(v_1+w) + tI(v_2+w) \quad \text{for all } t \in (0,1)$$

and if $v \in N_k$, $w_1 \neq w_2 \in M_k$, then

$$I(v + (1-t)w_1 + tw_2) < (1-t)I(v+w_1) + tI(v+w_2) \quad \text{for all } t \in (0,1)$$

(see [41, Proposition 4.6.1]).

Proposition 3.1 ([41, Proposition 4.7.1, Corollary 4.7.3, Proposition 4.7.4]). *Let $(a,b) \in Q_k$.*

(i) *There is a positive homogeneous map $\theta(\cdot,a,b) \in C(M_{k-1}, N_{k-1})$ such that $v = \theta(w)$ is the unique solution of*

$$I(v+w) = \sup_{v' \in N_{k-1}} I(v'+w), \quad w \in M_{k-1}.$$

Moreover, $I'(v+w) \perp N_{k-1}$ if and only if $v = \theta(w)$. Furthermore, the map θ is continuous on $M_{k-1} \times Q_k$ and satisfies $\theta(w, \lambda_k, \lambda_k) = 0$ for all $w \in M_{k-1}$.

(ii) *There is a positive homogeneous map $\tau(\cdot,a,b) \in C(N_k, M_k)$ such that $w = \tau(v)$ is the unique solution of*

$$I(v+w) = \inf_{w' \in M_k} I(v+w'), \quad v \in N_k.$$

Moreover, $I'(v+w) \perp M_k$ if and only if $w = \tau(v)$. Furthermore, the map τ is continuous on $N_k \times Q_k$ and satisfies $\tau(v, \lambda_k, \lambda_k) = 0$ for all $v \in N_k$.

For $(a,b) \in Q_k$, let

$$\begin{aligned} \sigma(w,a,b) &= \theta(w,a,b) + w, \quad w \in M_{k-1}, \quad S_k(a,b) = \sigma(M_{k-1},a,b),\\ \zeta(v,a,b) &= v + \tau(v,a,b), \quad v \in N_k, \quad S^k(a,b) = \zeta(N_k,a,b). \end{aligned}$$

Then, S_k and S^k are topological manifolds modeled on M_{k-1} and N_k, respectively. Thus, S_k is infinite-dimensional, while S^k is d_k-dimensional, where $d_k = \dim N_k$. For $B \subset X^s(\Omega)$, set $\tilde{B} = \{u \in B : \|u\| = 1\}$. We say that B is a radial set if $B = \{tu : u \in \tilde{B},\ t \geq 0\}$. Since θ and τ are positive homogeneous, so are σ and ζ and hence S_k and S^k are radial manifolds.

Let

$$K(a,b) = \{u \in X^s(\Omega) : I'(u,a,b) = 0\}$$

be the set of critical points of $I(\cdot,a,b)$. Since I' is positive homogeneous, it follows that K is a radial set. As $I(u) = (I'(u),u)/2$, we have

$$I(u) = 0 \quad \text{for all } u \in K. \tag{3.2}$$

Since $X^s(\Omega) = N_{k-1} \oplus E_k \oplus M_k$, Proposition 3.1 implies

$$K = \{u \in S_k \cap S^k : I'(u) \perp E_k\}. \tag{3.3}$$

Together with (3.2), it also implies

$$K \subset \{u \in S_k \cap S^k : I(u) = 0\}. \tag{3.4}$$

Set

$$n_{k-1}(a,b) = \inf_{w \in \tilde{M}_{k-1}} \sup_{v \in N_{k-1}} I(v+w,a,b), \quad m_k(a,b) = \sup_{v \in \tilde{N}_k} \inf_{w \in M_k} I(v+w,a,b).$$

Since $I(u,a,b)$ is nonincreasing in a for fixed u and b and in b for fixed u and a, it follows that $n_{k-1}(a,b)$ and $m_k(a,b)$ are nonincreasing in a for fixed b and in b for fixed a. By Proposition 3.1,

$$n_{k-1}(a,b) = \inf_{w \in \tilde{M}_{k-1}} I(\sigma(w,a,b),a,b), \quad m_k(a,b) = \sup_{v \in \tilde{N}_k} I(\zeta(v,a,b),a,b).$$

Proposition 3.2 ([41, Proposition 4.7.5, Lemma 4.7.6, Proposition 4.7.7]). *Let $(a,b),(a',b') \in Q_k$.*

(i) *Assume that $n_{k-1}(a,b) = 0$. Then,*

$$I(u,a,b) \geq 0 \quad \textit{for all } u \in S_k(a,b), \quad K(a,b) = \{u \in S_k(a,b) : I(u,a,b) = 0\}$$

and $(a,b) \in \Sigma^s(\Omega)$.

(a) *If $a' \leq a$, $b' \leq b$ and $(a',b') \neq (a,b)$, then $n_{k-1}(a',b') > 0$,*

$$I(u,a',b') > 0 \quad \textit{for all } u \in S_k(a',b') \setminus \{0\}$$

and $(a',b') \notin \Sigma^s(\Omega)$.

(b) *If $a' \geq a$, $b' \geq b$ and $(a',b') \neq (a,b)$, then $n_{k-1}(a',b') < 0$ and there is some $u \in S_k(a',b') \setminus \{0\}$ such that*

$$I(u,a',b') < 0.$$

Furthermore, n_{k-1} is continuous on Q_k and $n_{k-1}(\lambda_k,\lambda_k) = 0$.

(ii) *Assume that $m_k(a,b) = 0$. Then,*

$$I(u,a,b) \leq 0 \quad \textit{for all } u \in S^k(a,b), \quad K(a,b) = \{u \in S^k(a,b) : I(u,a,b) = 0\}$$

and $(a,b) \in \Sigma^s(\Omega)$.

(a) *If $a' \geq a$, $b' \geq b$ and $(a',b') \neq (a,b)$, then $m_k(a',b') < 0$,*

$$I(u,a',b') < 0 \quad \textit{for all } u \in S^k(a',b') \setminus \{0\}$$

and $(a',b') \notin \Sigma^s(\Omega)$.

(b) *If $a' \leq a$, $b' \leq b$ and $(a',b') \neq (a,b)$, then $m_k(a',b') > 0$ and there is some $u \in S^k(a',b') \setminus \{0\}$ such that*

$$I(u,a',b') > 0.$$

Furthermore, m_k is continuous on Q_k and $m_k(\lambda_k,\lambda_k) = 0$.

For $a \in (\lambda_{k-1},\lambda_{k+1})$, set

$$\nu_{k-1}(a) = \sup\{b \in (\lambda_{k-1},\lambda_{k+1}) : n_{k-1}(a,b) \geq 0\}, \quad \mu_k(a) = \inf\{b \in (\lambda_{k-1},\lambda_{k+1}) : m_k(a,b) \leq 0\}.$$

Then,

$$b = \nu_{k-1}(a) \iff n_{k-1}(a,b) = 0, \quad b = \mu_k(a) \iff m_k(a,b) = 0$$

(see [41, Lemma 4.7.8]).

Theorem 3.3 ([41, Theorem 4.7.9]). *Let $(a,b) \in Q_k$.*
(i) *The function ν_{k-1} is continuous, strictly decreasing and satisfies*
  (a) $\nu_{k-1}(\lambda_k) = \lambda_k$,
  (b) $b = \nu_{k-1}(a) \implies (a,b) \in \Sigma^s(\Omega)$,
  (c) $b < \nu_{k-1}(a) \implies (a,b) \notin \Sigma^s(\Omega)$.
(ii) *The function μ_k is continuous, strictly decreasing and satisfies*
  (a) $\mu_k(\lambda_k) = \lambda_k$,
  (b) $b = \mu_k(a) \implies (a,b) \in \Sigma^s(\Omega)$,
  (c) $b > \mu_k(a) \implies (a,b) \notin \Sigma^s(\Omega)$.
(iii) $\nu_{k-1}(a) \le \mu_k(a)$.

Thus,

$$C_k : b = \nu_{k-1}(a), \quad C^k : b = \mu_k(a)$$

are strictly decreasing curves in Q_k that belong to $\Sigma^s(\Omega)$. They both pass through the point (λ_k, λ_k) and may coincide. The region

$$\mathrm{I}_k = \{(a,b) \in Q_k : b < \nu_{k-1}(a)\}$$

below the lower curve C_k and the region

$$\mathrm{I}^k = \{(a,b) \in Q_k : b > \mu_k(a)\}$$

above the upper curve C^k are free of $\Sigma^s(\Omega)$. They are the minimal and maximal curves of $\Sigma^s(\Omega)$ in Q_k in this sense. Points in the region

$$\mathrm{II}_k = \{(a,b) \in Q_k : \nu_{k-1}(a) < b < \mu_k(a)\}$$

between C_k and C^k, when it is nonempty, may or may not belong to $\Sigma^s(\Omega)$.

For $(a,b) \in Q_k$, let

$$\mathcal{N}_k(a,b) = S_k(a,b) \cap S^k(a,b).$$

Since S_k and S^k are radial sets, so is $\mathcal{N}_k$. The next two propositions show that $\mathcal{N}_k$ is a topological manifold modeled on E_k and hence

$$\dim \mathcal{N}_k = d_k - d_{k-1}.$$

We will call it the null manifold of I.

Proposition 3.4 ([41, Proposition 4.8.1, Lemma 4.8.3, Proposition 4.8.4]). *Let $(a,b) \in Q_k$.*
(i) *There is a positive homogeneous map $\eta(\cdot, a, b) \in C(E_k, N_{k-1})$ such that $v = \eta(y)$ is the unique solution of*

$$I(\zeta(v+y)) = \sup_{v' \in N_{k-1}} I(\zeta(v'+y)), \quad y \in E_k.$$

Moreover, $I'(\zeta(v+y)) \perp N_{k-1}$ if and only if $v = \eta(y)$. Furthermore, the map η is continuous on $E_k \times Q_k$ and satisfies $\eta(y, \lambda_k, \lambda_k) = 0$ for all $y \in E_k$.
(ii) *There is a positive homogeneous map $\xi(\cdot, a, b) \in C(E_k, M_k)$ such that $w = \xi(y)$ is the unique solution of*

$$I(\sigma(y+w)) = \inf_{w' \in M_k} I(\sigma(y+w')), \quad y \in E_k.$$

Moreover, $I'(\sigma(y+w)) \perp M_k$ if and only if $w = \xi(y)$. Furthermore, the map ξ is continuous on $E_k \times Q_k$ and satisfies $\xi(y, \lambda_k, \lambda_k) = 0$ for all $y \in E_k$.
(iii) *For all $y \in E_k$,*

$$\zeta(\eta(y) + y) = \sigma(y + \xi(y)),$$

i.e., $\eta(y) = \theta(y + \xi(y))$ and $\xi(y) = \tau(\eta(y) + y)$.

Let

$$\varphi(y) = \zeta(\eta(y) + y) = \sigma(y + \xi(y)), \quad y \in E_k.$$

Proposition 3.5 ([41, Proposition 4.8.5]). *Let $(a,b) \in Q_k$.*
(i) *$\varphi(\cdot, a, b) \in C(E_k, X^s(\Omega))$ is a positive homogeneous map such that*

$$I(\varphi(y)) = \inf_{w \in M_k} \sup_{v \in N_{k-1}} I(v + y + w) = \sup_{v \in N_{k-1}} \inf_{w \in M_k} I(v + y + w), \quad y \in E_k,$$

and $I'(\varphi(y)) \in E_k$ for all $y \in E_k$.
(ii) *If $(a', b') \in Q_k$ with $a' \geq a$ and $b' \geq b$, then*

$$I(\varphi(y, a', b'), a', b') \leq I(\varphi(y, a, b), a, b) \quad \text{for all } y \in E_k.$$

(iii) *φ is continuous on $E_k \times Q_k$.*
(iv) *$\varphi(y, \lambda_k, \lambda_k) = y$ for all $y \in E_k$.*
(v) *$\mathcal{N}_k(a,b) = \{\varphi(y, a, b) : y \in E_k\}$.*
(vi) *$\mathcal{N}_k(\lambda_k, \lambda_k) = E_k$.*

By (3.3) and (3.4),

$$K = \{u \in \mathcal{N}_k : I'(u) \perp E_k\} \subset \{u \in \mathcal{N}_k : I(u) = 0\}. \tag{3.5}$$

The following theorem shows that the curves C_k and C^k are closely related to $\tilde{I} = I|_{\mathcal{N}_k}$.

Theorem 3.6 ([41, Theorem 4.8.6]). *Let $(a,b) \in Q_k$.*
(i) *If $b < \nu_{k-1}(a)$, then*

$$\tilde{I}(u, a, b) > 0 \quad \text{for all } u \in \mathcal{N}_k(a,b) \setminus \{0\}.$$

(ii) *If $b = \nu_{k-1}(a)$, then*

$$\tilde{I}(u, a, b) \geq 0 \quad \text{for all } u \in \mathcal{N}_k(a,b), \quad K(a,b) = \{u \in \mathcal{N}_k(a,b) : \tilde{I}(u, a, b) = 0\}.$$

(iii) *If $\nu_{k-1}(a) < b < \mu_k(a)$, then there are $u_i \in \mathcal{N}_k(a,b) \setminus \{0\}$, $i = 1, 2$, such that*

$$\tilde{I}(u_1, a, b) < 0 < \tilde{I}(u_2, a, b).$$

(iv) *If $b = \mu_k(a)$, then*

$$\tilde{I}(u, a, b) \leq 0 \quad \text{for all } u \in \mathcal{N}_k(a,b), \quad K(a,b) = \{u \in \mathcal{N}_k(a,b) : \tilde{I}(u, a, b) = 0\}.$$

(v) *If $b > \mu_k(a)$, then*

$$\tilde{I}(u, a, b) < 0 \quad \text{for all } u \in \mathcal{N}_k(a,b) \setminus \{0\}.$$

By (3.5), solutions of (3.1) are in $\mathcal{N}_k$. The set $K(a,b)$ of solutions is all of $\mathcal{N}_k(a,b)$ exactly when $(a,b) \in Q_k$ is on both C_k and C^k (see [41, Theorem 4.8.7]). When λ_k is a simple eigenvalue, $\mathcal{N}_k$ is 1-dimensional and hence this implies that (a,b) is on exactly one of those curves if and only if

$$K(a,b) = \{t\varphi(y_0, a, b) : t \geq 0\}$$

for some $y_0 \in E_k \setminus \{0\}$ (see [41, Corollary 4.8.8]).

The following theorem gives a sufficient condition for the region II_k to be nonempty.

Theorem 3.7 ([41, Theorem 4.9.1]). *If there is a function $y \in E_k$ such that $|y^+|_2 \neq |y^-|_2$, then there is a neighborhood $N \subset Q_k$ of (λ_k, λ_k) such that every point $(a,b) \in N \setminus \{(\lambda_k, \lambda_k)\}$ with $a + b = 2\lambda_k$ is in II_k.*

For the local problem (1.3), this result is due to Li, Li and Liu [28]. When λ_k is a simple eigenvalue, the region II_k is free of $\Sigma^s(\Omega)$ (see [41, Theorem 4.10.1]). For problem (1.3), this is due to Gallouët and Kavian [22].

When $(a,b) \notin \Sigma^s(\Omega)$, 0 is the only critical point of I and its critical groups are given by

$$C_q(I, 0) = H_q(I^0, I^0 \setminus \{0\}), \quad q \geq 0,$$

where $I^0 = \{u \in X^s(\Omega) : I(u) \leq 0\}$ and H denotes singular homology. We take the coefficient group to be the field $\mathbb{Z}_2$. The following theorem gives our main results on the critical groups.

Theorem 3.8 ([41, Theorem 4.11.2]). *Let* $(a,b) \in Q_k \setminus \Sigma^s(\Omega)$.

(i) *If* $(a,b) \in \mathrm{I}_k$, *then*
$$C_q(I,0) \approx \delta_{qd_{k-1}} \mathbb{Z}_2.$$

(ii) *If* $(a,b) \in \mathrm{I}^k$, *then*
$$C_q(I,0) \approx \delta_{qd_k} \mathbb{Z}_2.$$

(iii) *If* $(a,b) \in \mathrm{II}_k$, *then*
$$C_q(I,0) = 0, \quad q \le d_{k-1} \text{ or } q \ge d_k$$

and
$$C_q(I,0) \approx \tilde{H}_{q-d_{k-1}-1}(\{u \in \mathcal{N}_k : I(u) < 0\}), \quad d_{k-1} < q < d_k,$$

where $\tilde{H}$ *denotes the reduced homology groups. In particular,* $C_q(I,0) = 0$ *for all* q *when* λ_k *is simple.*

For the local problem (1.3), this result is due to Dancer [12, 13] and Perera and Schechter [38–40]. It can be used, for example, to obtain nontrivial solutions of perturbed problems with nonlinearities that cross a curve of the Dancer–Fučík spectrum, via a comparison of the critical groups at zero and infinity. Consider the problem
$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.6}$$
where f is a Carathéodory function on $\Omega \times \mathbb{R}$.

Theorem 3.9 ([41, Theorem 5.6.1]). *If*
$$f(x,t) = \begin{cases} b_0 t^+ - a_0 t^- + o(t) & \text{as } t \to 0, \\ bt^+ - at^- + o(t) & \text{as } |t| \to \infty, \end{cases}$$
uniformly a.e. in Ω *for some* (a_0,b_0) *and* (a,b) *in* $Q_k \setminus \Sigma^s(\Omega)$ *that are on opposite sides of* C_k *or* C^k, *then problem* (3.6) *has a nontrivial weak solution.*

For problem (1.3), this was proved in Perera and Schechter [39]. It generalizes a well-known result of Amann and Zehnder [1] on the existence of nontrivial solutions for problems crossing an eigenvalue.

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