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On the long term spatial segregation for a competition–diffusion system

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Abstract. We investigate the long term behavior for a class of competition–diffusion systems of Lotka–Volterra type for two competing species in the case of low regularity assumptions on the data. Due to the coupling that we consider the system cannot be reduced to a single equation yielding uniform estimates with respect to the inter-specific competition rate parameter. Moreover, in the particular but meaningful case of initial data with disjoint support and Dirichlet boundary data which are time-independent, we prove that as the competition rate goes to infinity the solution converges, along with suitable sequences, to a spatially segregated state satisfying some variational inequalities.

Keywords: competition–diffusion systems, Lotka–Volterra model, spatial segregation, population dynamics, asymptotic behaviour, stationary solution, dissipative systems

1. Introduction

Let Ω be a bounded, open, connected subset of $\mathbb{R}^N$ with smooth boundary and let κ be a positive parameter. The aim of this paper is to investigate the asymptotic behavior of a competition–diffusion system of Lotka–Volterra type for two competing species of population of densities u and v, with Dirichlet boundary conditions,

$$\begin{cases} u_t - \Delta u = f(u) - \kappa u v^2, & \text{in } \Omega \times (0,\infty), \\ v_t - \Delta v = g(v) - \kappa v u^2, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = \psi(x,t), & \text{on } \partial\Omega \times [0,\infty), \\ v(x,t) = \zeta(x,t), & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ v(x,0) = v_0(x), & \text{in } \Omega. \end{cases} \tag{P_κ}$$

A relevant problem in population ecology is the understanding of the interactions between different species, in particular in the case when the interactions are large and of competitive type. As the inter-specific parameter κ ruling the mutual interaction of the species gets large, competitive reaction–diffusion systems are expected to approach a limiting configuration where the populations survive but exhibit disjoint habitats (cf. [7,8,10,19,22,24]). For population dynamics models which require Dirichlet boundary conditions we refer to [8,22], while for the more ecologically natural Neumann boundary

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conditions we refer to [10,15] and references therein. As pointed out in [8], the Dirichlet case presents further difficulties compared with the Neumann case, as the boundary terms which pop up after integration by parts cannot be estimated independently of κ. The classical stationary Lotka–Volterra model for two populations

$$\begin{cases} -\Delta u = f(u) - \kappa uv, & \text{in } \Omega, \\ -\Delta v = g(v) - \kappa vu, & \text{in } \Omega, \\ u = \psi, & \text{on } \partial\Omega, \\ v = \zeta, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has been intensively studied with respect to the spatial segregation limit as $\kappa \to \infty$. If, for instance, ψ and ζ belong to $W^{1,\infty}(\partial\Omega)$, then there exists a sequence of solution (u_κ, v_κ) to (1.1), bounded in $W^{1,\infty}(\overline{\Omega})$, and a limiting positive state (u, v) with $uv = 0$, satisfying suitable variational inequalities and such that, up to a subsequence, $u_\kappa \to u$ and $v_\kappa \to v$ in $H^1(\Omega)$ with a precise rate of convergence (see [5]). Concerning the parabolic system associated with (1.1), in [8] Crooks, Dancer, Hilhorst, Mimura and Ninomiya proved (also in the case of possibly different diffusion coefficients) that, for any $T > 0$, there exists subsequences u_{κ_m} and v_{κ_m} of the solutions converging in $L^2(\Omega \times (0,T))$ to a bounded state with disjoint support and solving a limiting free boundary problem. Beside this convergence results on finite time intervals, in [7], in the case of equal diffusion coefficients and stationary boundary conditions, Crooks, Dancer and Hilhorst recently studied the long term segregation for large interactions, by reducing the system to a single equation whose solutions admit uniform estimates in κ. Typically, stabilization is based upon a variational structure yielding an energy functional, bounded and decreasing along the trajectories (see e.g. [13,27]). Unfortunately, as far as we know, due to the coupling term $-\kappa uv$, the parabolic system associated with (1.1) does not admit a natural Lyapunov functional and a direct analysis is therefore not possible. Now, system (P_κ) can be regarded as a variant of the standard Lotka–Volterra model, with different inter-specific competition coupling terms. In addition, if one considers homogeneous boundary data, then (P_κ) admits a natural non-increasing energy functional $\Lambda_\kappa : [0,\infty) \to \mathbb{R}$

$$\Lambda_\kappa(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\nabla v(t)\|_2^2 - \int_\Omega \int_0^{u(t)} f(\sigma)\,\mathrm{d}\sigma - \int_\Omega \int_0^{v(t)} g(\sigma)\,\mathrm{d}\sigma + \frac{\kappa}{2}\int_\Omega u^2(t)v^2(t).$$

As we will see, a non-increasing energy functional can be constructed also for general boundary conditions (see the Proof of Theorem 2.11). We shall tackle the problem with techniques from the theory of dissipative dynamical systems to show the convergence towards the solutions to the stationary system, formally written as

$$\begin{cases} -\Delta u = f(u) - \kappa uv^2, & \text{in } \Omega, \\ -\Delta v = g(v) - \kappa vu^2, & \text{in } \Omega, \\ u = \psi_\infty, & \text{on } \partial\Omega, \\ v = \zeta_\infty, & \text{on } \partial\Omega. \end{cases} \tag{S_κ}$$

A question which naturally arises is whether the solutions stabilize towards a segregated state along some $t_j \to \infty$ and $\kappa_j \to \infty$, for instance in the natural case when the initial data have disjoint support and the boundary data are stationary in time (see problem (1.2) in the next section). Some numerical computations in a square domain in $\mathbb{R}^2$ have been performed in [8] (see Sections 1 and 4) for the Lotka–Volterra model under these assumptions on the initial and boundary conditions (see also [17], where an algorithm for parallel computing was implemented in order to efficiently track the interfaces). In [26] we arranged a complete set of numerical experiments both for (1.2) (i.e. system (P_κ) with time-independent boundary data) and the corresponding model with the standard Lotka–Volterra coupling. Although on one hand working with (P_κ) gives some advantages in the study of the long term dynamics for κ fixed as it directly admits a Lyapunov functional, on the other hand the asymptotic analysis for the solutions of (S_κ) is far more complicated than the study of (1.1) (subtracting the equations of (1.1) one reduces to the single equation $\Delta u = \kappa u(u - \Phi)$ where Φ is an harmonic function, while this is not the case working with (S_κ)). For instance, the global boundedness in κ of the solutions in H^1 will be derived from the corresponding boundedness for the solution flow of the parabolic system uniformly with respect to κ. To show the boundedness directly on the elliptic systems seems out of reach. In addition, the blow up analysis based on Lipschitz rescalings performed in [5] does not seem to work.

Concerning some physical motivations to consider coupling terms between the equations which are different from the standard one uv, we refer the reader, e.g., to Section 3.3 of classical Murray's book [21] (looking at formula (3.14) at p. 87, our system corresponds to the choice $F(N,P) = 1 - N - \kappa P^2$ and $G(N,P) = 1 - P - \kappa N^2$ with respect to the book's notations). It is also useful to think about systems of two Schrödinger [1] or Gross–Pitaevskii [9] equations modelling particle interaction (and populations can also be thought as discrete collections of interacting particles), intensively investigated in recent time (nonlinear optics, Bose–Einstein binary condensates, etc.), which present all the coupling of (P_κ), yielding a variational structure. We refer the reader to [20] for the case $\kappa < 0$, with physical motivations e.g. from [11], and to [23] for the case where $\kappa > 0$, with physical motivations e.g. from [4]. Both [20,23] deal with the semi-classical regime analysis.

1.1. The main result

The main result of the paper concerns with the long-term behaviour in large-competition regime for the system with time-independent boundary data, that is

$$\begin{cases} u_t - \Delta u = f(u) - \kappa u v^2, & \text{in } \Omega \times (0,\infty), \\ v_t - \Delta v = g(v) - \kappa v u^2, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = \psi(x), & \text{on } \partial\Omega \times [0,\infty), \\ v(x,t) = \zeta(x), & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ v(x,0) = v_0(x), & \text{in } \Omega. \end{cases} \tag{1.2}$$

Concerning the functions $f, g : \mathbb{R} \to \mathbb{R}$, let:

$$\begin{aligned} &f, g \in C^1\big([0,\infty)\big), \qquad f(s) = g(s) = 0, \quad \text{for all } s \leqslant 0, \\ &f(s) < 0, \qquad g(s) < 0, \qquad\qquad\qquad \text{for all } s > 1, \end{aligned} \tag{1.3}$$

and we set

$$F(t) = \int_0^t f(\sigma)\,\mathrm{d}\sigma, \qquad G(t) = \int_0^t g(\sigma)\,\mathrm{d}\sigma.$$

The initial and boundary data are required to satisfy:

$$u_0, v_0 \in H^1(\Omega), \quad 0 \leqslant u_0(x) \leqslant 1, \qquad 0 \leqslant v_0(x) \leqslant 1, \quad \text{a.e. in } \Omega, \tag{1.4}$$

$$\psi, \zeta \in H^{1/2}(\partial\Omega), \quad \psi = u_0|_{\partial\Omega}, \qquad \zeta = v_0|_{\partial\Omega}, \tag{1.5}$$

$$0 \leqslant \psi(x) \leqslant 1, \qquad 0 \leqslant \zeta(x) \leqslant 1, \quad \text{on } \partial\Omega. \tag{1.6}$$

Under these assumptions, as well as those of Section 2, for all $\kappa > 0$, system (P_κ) admits a unique global solution $u_\kappa, v_\kappa \in C^0([0,\infty), H^1(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$. For the local existence, we refer the reader to a paper by Hoshino–Yamada [16] (see e.g. Theorems 1 and 2, having in mind to choose $\theta = \alpha = \gamma = \frac{1}{2}$ in Theorem 1(i) and $\gamma = 0$ in Theorem 2(ii), with respect to the notations therein). For smoothing effects we also wish to refer to the classical book of Henry [14]. The global existence result can be deduced by the comparison principle for parabolic equations (see, for example, the book of Smoller [25]). For $u_t - \Delta u = f(u) - \kappa uv^2$, $v_t - \Delta v = g(v) - \kappa vu^2$ with positive initial data, one can show $0 \leqslant u(t) \leqslant U(t)$ and $0 \leqslant v(t) \leqslant V(t)$, where U, V are solutions of $U_t - \Delta U = f(U), V_t - \Delta V = g(V)$ with the same initial and boundary conditions. Since U and V exist globally in time due to assumptions (1.3), (1.4) and (2.2) (a priori uniform-in-time L^∞-estimates for the solutions hold, see Lemma 2.3), one also recovers the global existence result (for the sake of completeness, we also mention Theorem 3 in Hoshino–Yamada [16] for small initial data and part (iv) of Proposition 7.3.2 in [18] for smooth initial data). In the following we set $\mathbb{H} = H^1(\Omega) \times H^1(\Omega)$, endowed with the standard Dirichlet norm, and

$$\mathbb{H}_0 = \big\{(u,v) \in \mathbb{H}\colon\ uv = 0 \text{ a.e. in } \Omega\big\}.$$

The following theorem is the main result of the paper, regarding system (1.2).

Theorem 1.1. *Assume* (1.3)–(1.6) *and* $(u_0, v_0) \in \mathbb{H}_0$. *Let* (u_κ, v_κ) *be the solution to system* (1.2). *Then there exist two diverging sequences* $(\kappa_m), (t_m) \subset \mathbb{R}^+$ *and* $(u_\infty, v_\infty) \in \mathbb{H}_0$ *such that*

$$\big(u_{\kappa_m}(t_m), v_{\kappa_m}(t_m)\big) \to (u_\infty, v_\infty) \quad \textit{in the } L^p \times L^p \textit{ norm, for any } p \in [2,\infty),$$

as $m \to \infty$, *where*

$$u_\infty, v_\infty \geqslant 0, \quad -\Delta u_\infty \leqslant f(u_\infty), \qquad -\Delta v_\infty \leqslant g(v_\infty), \quad u_\infty|_{\partial\Omega} = \psi, v_\infty|_{\partial\Omega} = \zeta.$$

Moreover, in the one-dimensional case, we have

$$\big\|\big(u_{\kappa_m}(t_m), v_{\kappa_m}(t_m)\big) - (u_\infty, v_\infty)\big\|_{L^\infty \times L^\infty} \to 0, \quad \textit{as } m \to \infty.$$

Hence, starting with segregated data, the system evolves towards a limiting segregated state satisfying suitable variational inequalities. As we have previously pointed out, in Sections 1, 4 of [8], the reader can find very nice pictures reproducing (for the classical model) these kind of separation phenomena. Notice that, due to the nonstandard coupling in system (S_κ) the H^1 convergence seems pretty hard to obtain either working directly on the system (which would require precise quantitative estimate of the rate of convergence of the solutions to u_∞ and v_∞) or using indirect arguments such combining blow up analysis with Liouville theorems (which, however, would naturally require stronger regularity assumptions on the boundary conditions). In Section 2, we will obtain, for κ fixed, the asymptotic behaviour of the system in the case of almost stationary boundary data. The author is not aware of any other result of this type in the literature (see also [3]).

2. Long term behaviour for κ fixed

The goal of this section is the study of the long term behaviour of the parabolic system (P_κ), for any $\kappa > 0$ fixed. We cover the general case of boundary data depending on time. Finally, in the particular case of segregated initial data and time independent boundary conditions, we will prove a stronger global boundedness result.

2.1. Assumptions and main result

Concerning f and g we will assume condition (1.3). Moreover, the initial and boundary data are required to satisfy (1.4) and

$$\psi, \zeta \in C^0([0,\infty), H^{1/2}(\partial\Omega)), \quad \psi(0) = u_0|_{\partial\Omega}, \qquad \zeta(0) = v_0|_{\partial\Omega}, \tag{2.1}$$

$$0 \leqslant \psi(x,t) \leqslant 1, \qquad 0 \leqslant \zeta(x,t) \leqslant 1, \quad \text{on } \partial\Omega \times [0,\infty). \tag{2.2}$$

We will assume that:

$$\psi(\cdot,t) \to \psi_\infty \quad \text{and} \quad \zeta(\cdot,t) \to \zeta_\infty \quad \text{in } H^{1/2}(\partial\Omega) \text{ as } t \to \infty, \tag{2.3}$$

$$\psi_t, \zeta_t \in L^1(0,\infty; H^{1/2}(\partial\Omega)) \cap L^2(0,\infty; H^{-1/2}(\partial\Omega)), \quad \psi_t(\cdot,t), \zeta_t(\cdot,t) \to 0 \quad \text{as } t \to \infty, \tag{2.4}$$

$$\psi_t(\cdot,0) = \zeta_t(\cdot,0) = 0, \tag{2.5}$$

$$\psi_{tt}, \zeta_{tt} \in L^1(0,\infty; H^{-1/2}(\partial\Omega)). \tag{2.6}$$

Under the previous assumptions we have the following result.

Theorem 2.1. *Let $(u_0, v_0) \in \mathbb{H}$ and $\kappa > 0$. Then for every diverging sequence $(t_h) \subset \mathbb{R}^+$ there exist a subsequence $(t_j) \subset \mathbb{R}^+$ and a solution $(\hat{u}_\kappa, \hat{v}_\kappa) \in \mathbb{H}$ to system (S_κ) such that*

$$\left\| \left(u_\kappa(t_j), v_\kappa(t_j)\right) - (\hat{u}_\kappa, \hat{v}_\kappa) \right\|_{\mathbb{H}} \to 0, \quad \textit{as } j \to \infty.$$

Moreover, the convergence holds in the $L^p \times L^p$ norm for any $p \in [2,\infty)$.

Strenghtening the assumptions we obtain the global boundedness uniformly in κ.

Theorem 2.2. *Assume that $(u_0, v_0) \in \mathbb{H}_0$ and the boundary conditions are time-independent. Then, in addition to the conclusion of Theorem 2.1, we have*

$$\sup_{t \geqslant 0} \sup_{\kappa > 0} \left\| (u_\kappa(t), v_\kappa(t)) \right\|_{\mathbb{H}} < \infty,$$

namely (u_κ, v_κ) is bounded in $\mathbb{H}$ (and in any $L^p \times L^p$ space), uniformly with respect to κ.

This second achievement will be of course an important step in order to prove the main result of the paper.

2.2. Some preliminary results

From a direct computation, we have positivity and a priori bounds for the solutions to (P_κ), uniformly with respect to κ.

Lemma 2.3. $\Sigma = [0,1] \times [0,1]$ *is a globally positively invariant region for system* (P_κ)*, uniformly with respect to* κ*, namely*

$$0 \leqslant u_\kappa(x,t) \leqslant 1, \qquad 0 \leqslant v_\kappa(x,t) \leqslant 1, \quad \textit{a.e. } x \in \Omega, t \geqslant 0.$$

Proof. Testing the first equation of (P_κ) with $-u_\kappa^-$ and using (1.3), (1.4), (2.1) and (2.2), we easily obtain that $u_\kappa \geqslant 0$, while testing the same equation with $(u_\kappa - 1)^+$ we deduce similarly that $u_\kappa \leqslant 1$. An analogous manipulation of the second equation in (P_κ) yields the corresponding bounds for the component v_κ. $\square$

Let $A = -\Delta$ be the Laplace operator on $L^2(\Omega)$ with domain $\mathscr{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and consider the hierarchy of Hilbert spaces $H^\alpha = \mathscr{D}(A^{\alpha/2})$, $\alpha \in \mathbb{R}$, with $\|u\|_{H^\alpha} = \|A^{\alpha/2}u\|_2$. We recall an exponential decay property of the heat kernel operator $\mathrm{e}^{t\Delta}$.

Lemma 2.4. *Let* $\alpha > 0$*. Then there exist* $\omega > 0$ *and* $C_\alpha > 0$ *such that*

$$\|\mathrm{e}^{t\Delta}\|_{\mathcal{L}(L^2,H^{2\alpha})} \leqslant C_\alpha \mathrm{e}^{-\omega t} t^{-\alpha}, \quad t > 0. \tag{2.7}$$

In particular,

$$\int_0^\infty \|\mathrm{e}^{\sigma\Delta}\|_{\mathcal{L}(L^2,H^{2\alpha})}\,\mathrm{d}\sigma < \infty$$

provided that $\alpha \in (0,1)$.

Proof. As the real part of the spectrum of A is bounded away from zero by a positive constant ω, by [14, Theorem 1.4.3, p. 26], for $\alpha > 0$ there exists $C_\alpha > 0$ such that $\|A^\alpha \mathrm{e}^{-tA}\|_{\mathcal{L}(L^2,L^2)} \leqslant C_\alpha \mathrm{e}^{-\omega t} t^{-\alpha}$, for all $t > 0$. Hence, $\|\mathrm{e}^{t\Delta}\|_{\mathcal{L}(L^2,H^{2\alpha})} = \|(-\Delta)^\alpha \mathrm{e}^{t\Delta}\|_{\mathcal{L}(L^2,L^2)} \leqslant C_\alpha \mathrm{e}^{-\omega t} t^{-\alpha}$, for all $t > 0$. The second assertion follows by (2.7). $\square$

Next we provide a compactness result for the trajectories of (P_κ).

Lemma 2.5. *For any* $(u_0, v_0) \in \mathbb{H}$*,* $\kappa > 0$ *and* $\tau > 0$ *the set* $\{(u_\kappa(t), v_\kappa(t))\colon t \geqslant \tau\}$ *is relatively compact in* $\mathbb{H}$.

Proof. Let U and V denote the solutions to the linear problems

$$\begin{cases} U_t - \Delta U = 0, & \text{in } \Omega \times (0,\infty), \\ U(x,t) = \psi(x,t), & \text{on } \partial\Omega \times (0,\infty), \\ U(x,0) = U_0(x), & \text{in } \Omega, \end{cases} \tag{2.8}$$

and

$$\begin{cases} V_t - \Delta V = 0, & \text{in } \Omega \times (0,\infty), \\ V(x,t) = \zeta(x,t), & \text{on } \partial\Omega \times (0,\infty), \\ V(x,0) = V_0(x), & \text{in } \Omega, \end{cases} \tag{2.9}$$

where $U_0, V_0 \in H^1(\Omega)$ satisfy

$$\begin{cases} -\Delta U_0 = 0, & \text{in } \Omega, \\ U_0(x) = \psi(x,0), & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta V_0 = 0, & \text{in } \Omega, \\ V_0(x) = \zeta(x,0), & \text{on } \partial\Omega. \end{cases}$$

By assumption (2.2) and the maximum principle for harmonic functions, $0 \leqslant U_0(x) \leqslant 1$ and $0 \leqslant V_0(x) \leqslant 1$ for a.e. $x \in \Omega$. Hence, arguing as in the proof of Lemma 2.3, we have $0 \leqslant U(x,t) \leqslant 1$ and $0 \leqslant V(x,t) \leqslant 1$ for a.e. $x \in \Omega$ and $t \geqslant 0$. Now, the functions

$$\tilde{u}_\kappa(x,t) = u_\kappa(x,t) - U(x,t), \qquad \tilde{v}_\kappa(x,t) = v_\kappa(x,t) - V(x,t) \tag{2.10}$$

solve the system with homogeneous boundary conditions

$$\begin{cases} (\tilde{u}_\kappa)_t - \Delta\tilde{u}_\kappa = f(\tilde{u}_\kappa + U) - \kappa(\tilde{u}_\kappa + U)(\tilde{v}_\kappa + V)^2, & \text{in } \Omega \times (0,\infty), \\ (\tilde{v}_\kappa)_t - \Delta\tilde{v}_\kappa = g(\tilde{v}_\kappa + V) - \kappa(\tilde{v}_\kappa + V)(\tilde{u}_\kappa + U)^2, & \text{in } \Omega \times (0,\infty), \\ \tilde{u}_\kappa(x,t) = \tilde{v}_\kappa(x,t) = 0, & \text{on } \partial\Omega \times [0,\infty), \\ \tilde{u}_\kappa(x,0) = u_0(x) - U_0(x), & \text{in } \Omega, \\ \tilde{v}_\kappa(x,0) = v_0(x) - V_0(x), & \text{in } \Omega. \end{cases} \tag{$\widetilde{P}_\kappa$}$$

Denote now by $\Psi = \Psi(x;t) \in C^0([0,\infty), H^1(\Omega))$ the family of harmonic extensions to Ω of ψ

$$\begin{cases} -\Delta\Psi(x;t) = 0, & \text{in } \Omega, \\ \Psi(x;t) = \psi(x,t), & \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

and set $\bar{U}(x,t) = U(x,t) - \Psi(x;t)$. Then $\bar{U}$ solves the nonautonomous problem with homogeneous boundary and initial conditions

$$\begin{cases} \bar{U}_t - \Delta\bar{U} = -\Psi_t, & \text{in } \Omega \times (0,\infty), \\ \bar{U}(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ \bar{U}(x,0) = 0, & \text{in } \Omega. \end{cases} \tag{2.12}$$

Notice that $\bar{U}(x,0) = 0$ since $U_0(x)$ and $\Psi(x;0)$ are both harmonic functions with the same boundary conditions. From (2.5)–(2.6) and classical regularity theory for harmonic functions,

$$\|\Psi_t\|_{L^\infty(0,\infty;L^2(\Omega))} \leqslant c\|\psi_t\|_{L^\infty(0,\infty;H^{-1/2}(\partial\Omega))} \leqslant \|\psi_{tt}\|_{L^1(0,\infty;H^{-1/2}(\partial\Omega))}.$$

By Duhamel's formula $\bar{U}$ is given by

$$\bar{U}(t) = -\int_0^t \mathrm{e}^{(t-\sigma)\Delta}\Psi_t(\sigma)\,\mathrm{d}\sigma.$$

If $\alpha \in (1/2, 1)$, in light of (2.7) of Lemma 2.4, since Ψ_t is in $L^\infty(0,\infty; L^2(\Omega))$,

$$\sup_{t\geqslant 0}\|\bar U(t)\|_{H^{2\alpha}} < \infty. \tag{2.13}$$

Of course the same control holds for $\bar V(t)$. Let now Ψ_∞ denote the harmonic extension of ψ_∞, the limit of $\psi(t)$ in $H^{1/2}(\partial\Omega)$ as $t \to \infty$ according to (2.3) . By standard regularity estimates, $\|\Psi(t) - \Psi_\infty\|_{H^1(\Omega)} \leqslant c\|\psi(t) - \psi_\infty\|_{H^{1/2}(\partial\Omega)}$, so that $\Psi(t) \to \Psi_\infty$ in $H^1(\Omega)$ as $t \to \infty$. Of course the same control holds for the boundary extensions of ζ. Also, by Duhamel's formula we have

$$\tilde u_\kappa(t) = \mathrm{e}^{t\Delta}(u_0 - U_0) + \int_0^t \mathrm{e}^{(t-\sigma)\Delta}\Phi^1_\kappa(\sigma)\,\mathrm{d}\sigma,$$
$$\tilde v_\kappa(t) = \mathrm{e}^{t\Delta}(v_0 - V_0) + \int_0^t \mathrm{e}^{(t-\sigma)\Delta}\Phi^2_\kappa(\sigma)\,\mathrm{d}\sigma,$$

where

$$\Phi^1_\kappa(\sigma) = f\big(u_\kappa(\sigma)\big) - \kappa u_\kappa(\sigma)v^2_\kappa(\sigma), \qquad \Phi^2_\kappa(\sigma) = g\big(v_\kappa(\sigma)\big) - \kappa v_\kappa(\sigma)u^2_\kappa(\sigma).$$

By means of Lemma 2.3, we have $\Phi^1_\kappa,\ \Phi^2_\kappa \in L^\infty(0,\infty; L^\infty(\Omega))$. If $\alpha \in (1/2, 1)$, then again by (2.7) one obtains for any $\tau > 0$

$$\sup_{t\geqslant\tau}\|\tilde u_\kappa(t)\|_{H^{2\alpha}} < \infty. \tag{2.14}$$

As $H^{2\alpha}$ is compactly embedded in $H^1(\Omega)$ and $u_\kappa(t) = \tilde u_\kappa(t) + \bar U(t) + \Psi(t)$ the assertion follows by (2.13)–(2.14) for the component u_κ. The same arguments works for $\tilde v_\kappa$. □

Remark 2.6. By strengthening the regularity assumptions on the boundary data, say $W^{1,\infty}(\partial\Omega)$ in place of $H^{1/2}(\partial\Omega)$ in the assumptions at the beginning of the section, and defining $-\Delta$ over $L^q(\Omega)$ for any $q \geqslant 2$, the previous result can of course be improved, yielding compactness of the trajectories in $W^{2\alpha,q}(\Omega)$ for any $q \geqslant 2$, and hence into spaces of Hölder continuous functions. Unfortunately the estimates are not independent of κ and in order to have H^1 bounds uniformly in κ we shall need to exploit energy arguments.

For every $\tau > 0$ and every function $h : (0,\infty) \to H^1(\Omega)$, let us set

$$h^\tau(t) = h(t+\tau), \quad t > 0.$$

The following result gives a stabilization property for the solutions of the linear parabolic equation with nonhomogeneous time-dependent boundary conditions.

Lemma 2.7. *Let U be the solution to the problem* (2.8). *Then $U(t) \to U_\infty$ in $H^1(\Omega)$ as $t \to \infty$, where $U_\infty \in H^1(\Omega)$ is the solution to*

$$\begin{cases} -\Delta U_\infty = 0, & \textit{in } \Omega,\\ U_\infty = \psi_\infty, & \textit{on } \partial\Omega. \end{cases} \tag{2.15}$$

Proof. With the notations introduced in the proof of Lemma 2.5, we consider, for $\tau > 0$, the functions $W(t) = \bar{U}^\tau(t) - \bar{U}(t)$ and $\varrho(t) = \Psi_t(t) - \Psi_t^\tau(t)$, which satisfy

$$\begin{cases} W_t - \Delta W = \varrho(t), & \text{in } \Omega \times (0,\infty), \\ W(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ W(x,0) = U(\tau) - U_0 + \Psi(0) - \Psi(\tau), & \text{in } \Omega. \end{cases} \tag{2.16}$$

By multiplying the equation by $-\Delta W$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla W(t)\|_2^2 + \|\Delta W(t)\|_2^2 = -\int_\Omega \varrho(t)\Delta W(t).$$

By applying Hölder and then Young inequalities on the right-hand side, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla W(t)\|_2^2 + \frac{1}{2}\|\Delta W(t)\|_2^2 \leqslant \frac{1}{2}\|\varrho(t)\|_2^2.$$

Let A be the positive operator on $L^2(\Omega)$ defined by $A = -\Delta$, with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Due to the (compact and dense) injection $H^2(\Omega) \cap H_0^1(\Omega) = \mathcal{D}(A) \hookrightarrow \mathcal{D}(A^{1/2}) = H_0^1(\Omega)$, we have $\|\nabla W\|_2 \leqslant \alpha_1^{-1/2}\|\Delta W\|_2$ for some $\alpha_1 > 0$ (see e.g. Henry [14]), so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla W(t)\|_2^2 + \frac{\alpha_1}{2}\|\nabla W(t)\|_2^2 \leqslant \frac{1}{2}\|\varrho(t)\|_2^2.$$

Finally, Gronwall inequality entails

$$\|W(t)\|_{H_0^1}^2 \leqslant \|W(0)\|_{H_0^1}^2 \mathrm{e}^{-\sigma t} + c\mathrm{e}^{-\sigma t}\int_0^t \mathrm{e}^{\sigma s}\|\varrho(s)\|_2^2\,\mathrm{d}s,$$

for some $\sigma > 0$ and $c > 0$. In turn, we readily obtain

$$\lim_{t\to\infty}\|U^\tau(t) - U(t)\|_{H^1} \leqslant \frac{c}{\sqrt{\sigma}}\lim_{t\to\infty}\|\Psi_t^\tau(t) - \Psi_t(t)\|_2 + \lim_{t\to\infty}\|\Psi^\tau(t) - \Psi(t)\|_{H^1}.$$

In view of (2.3) and standard elliptic equations, we deduce

$$\|U^\tau(t) - U(t)\|_{H^1} \to 0, \quad \text{as } t \to \infty. \tag{2.17}$$

The same argument shows that $\{U(t)\}_{t\geqslant 0}$ is bounded in $H^1(\Omega)$. Let now $(t_h) \subset \mathbb{R}^+$ be any diverging sequence. Since $\{U(t)\}_{t\geqslant 1}$ is relatively compact in $H^1(\Omega)$, there exists a subsequence, that we still denote by (t_h), such that $U(t_h) \to U_\infty$ in $H^1(\Omega)$. Let $\eta \in C_c^\infty(\Omega)$. By integrating the equation for U on $(t_h, t_h+1) \times \Omega$, yields

$$\lim_h \left[\int_{t_h}^{t_h+1}\int_\Omega U_t\eta + \int_{t_h}^{t_h+1}\int_\Omega \nabla U \cdot \nabla\eta\right] = 0.$$

On one hand, we have

$$\lim_h \left| \int_{t_h}^{t_h+1} \int_\Omega U_t \eta \right| \leqslant \lim_h \int_\Omega |U(t_h+1) - U(t_h)||\eta| \leqslant c \lim_h \|U^1(t_h) - U(t_h)\|_2 = 0.$$

Moreover, there exists $(s_h) \subset \mathbb{R}^+$ with $s_h = t_h + \xi_h$, $0 \leqslant \xi_h \leqslant 1$, such that by (2.17)

$$\int_{t_h}^{t_h+1} \int_\Omega \nabla U \cdot \nabla \eta = \int_\Omega \nabla U(s_h) \cdot \nabla \eta = \int_\Omega \nabla U(t_h) \cdot \nabla \eta + \mathrm{o}(1), \quad \text{as } h \to \infty.$$

Hence, taking the limit as $h \to \infty$, we get $\int_\Omega \nabla U_\infty \cdot \nabla \eta = 0$. Moreover, from the convergence of $U(t_h)$ to U_∞ in $H^1(\Omega)$ we deduce that $U(t_h)|_{\partial\Omega} \to U_\infty|_{\partial\Omega}$ in $H^{1/2}(\partial\Omega)$. From (2.3) we deduce that $U_\infty = \psi_\infty$ on $\partial\Omega$. Therefore U_∞ solves (2.15). Since (2.15) has a unique solution, we actually deduce the convergence of the whole flow $U(t)$. □

Next, we obtain a summability result for the solutions to (2.8).

Lemma 2.8. *Let U be the solution to* (2.8). *Then* $U_t \in L^1(0, \infty; H^1(\Omega))$.

Proof. As in the proof of Lemma 2.5, $\bar{U}$ is the solution to (2.12). Hence, taking into account (2.5), it turns out that $\widetilde{U}(x,t) = \bar{U}_t(x,t)$ is a solution to

$$\begin{cases} \widetilde{U}_t - \Delta \widetilde{U} = -\Psi_{tt}, & \text{in } \Omega \times (0,\infty), \\ \widetilde{U}(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ \widetilde{U}(x,0) = 0, & \text{in } \Omega. \end{cases} \tag{2.18}$$

By assumption (2.6) it follows $\Psi_{tt} \in L^1(0,\infty; L^2(\Omega))$. In addition, we have $\Psi_t \in L^1(0,\infty; H^1(\Omega))$. By Lemma 2.4 we have $\|\mathrm{e}^{t\Delta}\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))} \leqslant C\mathrm{e}^{-\omega t} t^{-1/2}$, for some $C, \omega > 0$. Hence,

$$\widetilde{U}(t) = -\int_0^t \mathrm{e}^{(t-\sigma)\Delta} \Psi_{tt}(\sigma)\, \mathrm{d}\sigma,$$

and we obtain

$$\begin{aligned} \|\widetilde{U}\|_{L^1(0,\infty;H_0^1(\Omega))} &\leqslant C \int_0^\infty \left[\int_0^t \mathrm{e}^{-\omega(t-\sigma)} (t-\sigma)^{-1/2} \|\Psi_{tt}(\sigma)\|_2 \, \mathrm{d}\sigma \right] \mathrm{d}t \\ &= C \int_0^\infty \|\Psi_{tt}(\sigma)\|_2 \left[\int_\sigma^\infty \mathrm{e}^{-\omega(t-\sigma)} (t-\sigma)^{-1/2} \, \mathrm{d}t \right] \mathrm{d}\sigma \\ &= C \left(\int_0^\infty \mathrm{e}^{-\omega\sigma} \sigma^{-1/2} \, \mathrm{d}\sigma \right) \|\Psi_{tt}\|_{L^1(0,\infty;L^2(\Omega))}. \end{aligned}$$

Hence $\widetilde{U} \in L^1(0,\infty; H_0^1(\Omega))$, which yields $\bar{U}_t \in L^1(0,\infty; H_0^1(\Omega))$ and, in turn, taking into account (2.4), also $U_t \in L^1(0,\infty; H^1(\Omega))$, concluding the proof. □

Lemma 2.9. *Let $\tilde{u}_\kappa$ and $\tilde{v}_\kappa$ be as in system $\widetilde{P}_\kappa$. Then*

$$\int_0^T \|\partial_t \tilde{u}_\kappa(\sigma)\|_2^2 \, \mathrm{d}\sigma < \infty, \qquad \int_0^T \|\partial_t \tilde{v}_\kappa(\sigma)\|_2^2 \, \mathrm{d}\sigma < \infty,$$

for any $T > 0$.

Proof. Setting $\Upsilon(x,t) = f(\tilde{u}_\kappa(x,t)+U(x,t)) - \kappa(\tilde{u}_\kappa(x,t)+U(x,t))(\tilde{v}_\kappa(x,t)+V(x,t))^2$ for any $x \in \Omega$ and $t > 0$ and $m(x) = u_0(x) - U_0(x)$, it follows that $\tilde{u}_\kappa$ is the solution to

$$\begin{cases} \partial_t \tilde{u}_\kappa - \Delta \tilde{u}_\kappa = \Upsilon, & \text{in } \Omega \times (0,\infty), \\ \tilde{u}_\kappa(x,t) = 0, & \text{on } \partial\Omega \times [0,\infty), \\ \tilde{u}_\kappa(x,0) = m(x), & \text{in } \Omega. \end{cases}$$

Hence, since $m \in H_0^1(\Omega)$ and $\Upsilon \in L^2(0,T,L^2(\Omega))$ for any $T > 0$ (as $0 \leqslant u_\kappa, v_\kappa \leqslant 1$ and f is continuous), the desired summability for $\partial_t \tilde{u}_\kappa$ follows, e.g., by [12, Theorem 5, p. 360]. The proof for $\partial_t \tilde{v}_\kappa$ is similar. □

Let us recall a useful elementary Gronwall-type inequality.

Lemma 2.10. *Let $g \in L^1([0,\infty),[0,\infty))$. Assume that $\Upsilon : [0,\infty) \to [0,\infty)$ is an absolutely continuous function such that*

$$\Upsilon(t) \leqslant c_1 + c_2 \int_0^t g(\sigma)\sqrt{\Upsilon(\sigma)} \, \mathrm{d}\sigma, \quad t \geqslant 0,$$

for some $c_1, c_2 > 0$. Then

$$\Upsilon(t) \leqslant 2c_1 + c_2^2 \|g\|_{L^1(0,\infty)}^2, \quad t \geqslant 0.$$

Proof. Let $t > 0$ and consider $\bar{t} \in [0,t]$ such that $\Upsilon(\bar{t}) = \max\{\Upsilon(\sigma)\colon \sigma \in [0,t]\}$. Hence

$$\Upsilon(\bar{t}) \leqslant c_1 + c_2 \int_0^{\bar{t}} g(\sigma)\sqrt{\Upsilon(\sigma)} \, \mathrm{d}\sigma \leqslant c_1 + c_2\sqrt{\Upsilon(\bar{t})}\|g\|_{L^1(0,t)} \leqslant c_1 + c_2\sqrt{\Upsilon(\bar{t})}\|g\|_{L^1(0,\infty)},$$

so the assertion immediately follows by Young inequality and $\Upsilon(t) \leqslant \Upsilon(\bar{t})$. □

Next we obtain an H^1 stabilization result for the solutions (u_κ, v_κ) to (P_κ).

Theorem 2.11. *Assume that $(u_0, v_0) \in \mathbb{H}$ and set*

$$\mu = \|u_0 v_0\|_2^2 + \|\Psi_t\|_{L^1(0,\infty;L^2(\Omega))} + \|Z_t\|_{L^1(0,\infty;L^2(\Omega))}. \tag{2.19}$$

Then there exists a positive constant $R = R(u_0, v_0, \psi, \zeta)$ independent of κ such that

$$\big\|\big(u_\kappa(t), v_\kappa(t)\big)\big\|_{\mathbb{H}} \leqslant R + \kappa\mu, \quad \text{for all } t \geqslant 0. \tag{2.20}$$

Moreover, for any $\tau_0 > 0$ and $\kappa > 0$,

$$\lim_{t\to\infty}\sup_{\tau\in[0,\tau_0]}\|u_\kappa(t+\tau)-u_\kappa(t)\|_{H^1}=0,\qquad \lim_{t\to\infty}\sup_{\tau\in[0,\tau_0]}\|v_\kappa(t+\tau)-v_\kappa(t)\|_{H^1}=0.$$

Proof. Let $\tau_0 > 0$ and $\kappa > 0$. Let us first prove that

$$\lim_{t\to\infty}\sup_{\tau\in[0,\tau_0]}\|u_\kappa(t+\tau)-u_\kappa(t)\|_2=0,\qquad \lim_{t\to\infty}\sup_{\tau\in[0,\tau_0]}\|v_\kappa(t+\tau)-v_\kappa(t)\|_2=0. \tag{2.21}$$

According to the proof of Lemma 2.5, let again U (resp. V) be the solution of the linear problems (2.8) (resp. (2.9)), where U_0 (resp. V_0) is the harmonic extensions of $\psi(0)$ (resp. $\zeta(0)$). Then $\tilde u_\kappa(x,t) = u_\kappa(x,t) - U(x,t)$ and $\tilde v_\kappa(x,t) = v_\kappa(x,t) - V(x,t)$ are solutions to system $(\widetilde P_\kappa)$ having homogeneous boundary conditions. Let now $\varepsilon \in (0,1)$ and, taking into account Lemma 2.9, introduce the auxiliary energy functional $\Lambda_\kappa : [0,\infty) \to \mathbb{R}$ defined by setting:

$$\begin{aligned}
\Lambda_\kappa(t) ={}& \frac{1}{2}\|\nabla\tilde u_\kappa(t)\|_2^2+\frac{1}{2}\|\nabla\tilde v_\kappa(t)\|_2^2-\int_\Omega F\big(\tilde u_\kappa(t)+U(t)\big)-\int_\Omega G\big(\tilde v_\kappa(t)+V(t)\big)\\
&+\frac{\kappa}{2}\int_\Omega\big(\tilde u_\kappa(t)+U(t)\big)^2\big(\tilde v_\kappa(t)+V(t)\big)^2+2\int_0^t\left[\int_\Omega\nabla\tilde u_\kappa(\sigma)\cdot\nabla U_t(\sigma)\right]\mathrm{d}\sigma\\
&-\int_\Omega\nabla U(t)\cdot\nabla\tilde u_\kappa(t)-\int_0^t{}_{H^{-1/2}}\left\langle\frac{\partial\tilde u_\kappa(\sigma)}{\partial\nu},\psi_t(\sigma)\right\rangle_{H^{1/2}}\mathrm{d}\sigma\\
&+2\int_0^t\left[\int_\Omega\nabla\tilde v_\kappa(\sigma)\cdot\nabla V_t(\sigma)\right]\mathrm{d}\sigma\\
&-\int_\Omega\nabla V(t)\cdot\nabla\tilde v_\kappa(t)-\int_0^t{}_{H^{-1/2}}\left\langle\frac{\partial\tilde v_\kappa(\sigma)}{\partial\nu},\zeta_t(\sigma)\right\rangle_{H^{1/2}}\mathrm{d}\sigma\\
&+\varepsilon\int_0^t\|\partial_t\tilde u_\kappa(\sigma)\|_2^2\,\mathrm{d}\sigma+\varepsilon\int_0^t\|\partial_t\tilde v_\kappa(\sigma)\|_2^2\,\mathrm{d}\sigma.
\end{aligned}$$

We prove that Λ_κ is nonincreasing and there exist two constants $\alpha_\kappa \in \mathbb{R}$ and $\beta_\kappa \in \mathbb{R}$ (which we will write down explicitely) such that $\alpha_\kappa \leqslant \Lambda_\kappa(t) \leqslant \beta_\kappa$, for all $t \geqslant 0$. By multiplying the first equation of $(\widetilde P_\kappa)$ by $\partial_t u_\kappa$ and the second one by $\partial_t v_\kappa$, using the fact that U and V solve problems (2.8)–(2.9), and adding the resulting identities, we reaches

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\kappa(t)=-(1-\varepsilon)\|\partial_t\tilde u_\kappa(t)\|_2^2-(1-\varepsilon)\|\partial_t\tilde v_\kappa(t)\|_2^2\leqslant 0. \tag{2.22}$$

In particular, $\{t \mapsto \Lambda_\kappa(t)\}$ is a nonincreasing function. Hence,

$$\begin{aligned}
\Lambda_\kappa(t)\leqslant\Lambda_\kappa(0)={}&\frac{1}{2}\|\nabla(u_0-U_0)\|_2^2+\frac{1}{2}\|\nabla(v_0-V_0)\|_2^2-\int_\Omega F(u_0)-\int_\Omega G(v_0)\\
&-\int_\Omega\nabla U_0\cdot\nabla(u_0-U_0)-\int_\Omega\nabla V_0\cdot\nabla(v_0-V_0)+\frac{\kappa}{2}\int_\Omega u_0^2v_0^2,
\end{aligned}$$

for all $t \geqslant 0$, namely Λ_κ is bounded from above, uniformly in time and β_κ is of the form

$$\beta_\kappa = P + \kappa \|u_0 v_0\|_2^2, \quad P = P(u_0, v_0, \psi, \zeta). \tag{2.23}$$

Now, using the trace inequality, the first equation of $(\widetilde{P}_\kappa)$, the L^∞-boundedness of the solutions and the Young inequality, we find $c > 0$ and $c_\varepsilon > 0$ such that

$$\begin{aligned}
&\left| \int_0^t {}_{H^{-1/2}}\left\langle \frac{\partial \tilde{u}_\kappa(\sigma)}{\partial \nu}, \psi_t(\sigma) \right\rangle_{H^{1/2}} \mathrm{d}\sigma \right| \\
&\quad \leqslant \int_0^t \|\nabla \tilde{u}_\kappa(\sigma)\|_2 \|\nabla \Psi_t(\sigma)\|_2 \,\mathrm{d}\sigma + \int_0^t \|\Delta \tilde{u}_\kappa(\sigma)\|_2 \|\Psi_t(\sigma)\|_2 \,\mathrm{d}\sigma \\
&\quad \leqslant \int_0^t \|\nabla \tilde{u}_\kappa(\sigma)\|_2 \|\nabla \Psi_t(\sigma)\|_2 \,\mathrm{d}\sigma + \int_0^t \|\partial_t \tilde{u}_\kappa(\sigma)\|_2 \|\Psi_t(\sigma)\|_2 \,\mathrm{d}\sigma + c\kappa \int_0^t \|\Psi_t(\sigma)\|_2 \,\mathrm{d}\sigma \\
&\quad \leqslant \int_0^t \|\nabla \tilde{u}_\kappa(\sigma)\|_2 \|\nabla \Psi_t(\sigma)\|_2 \,\mathrm{d}\sigma + \varepsilon \int_0^t \|\partial_t \tilde{u}_\kappa(\sigma)\|_2^2 \,\mathrm{d}\sigma \\
&\qquad + c_\varepsilon \int_0^t \|\Psi_t(\sigma)\|_2^2 \,\mathrm{d}\sigma + c\kappa \int_0^t \|\Psi_t(\sigma)\|_2 \,\mathrm{d}\sigma,
\end{aligned}$$

where Ψ_t is the harmonic extension of ψ_t to Ω (see formula (2.11)). Analogously, we reach

$$\begin{aligned}
&\left| \int_0^t {}_{H^{-1/2}}\left\langle \frac{\partial \tilde{v}_\kappa(\sigma)}{\partial \nu}, \zeta_t(\sigma) \right\rangle_{H^{1/2}} \mathrm{d}\sigma \right| \\
&\quad \leqslant \int_0^t \|\nabla \tilde{v}_\kappa(\sigma)\|_2 \|\nabla Z_t(\sigma)\|_2 \,\mathrm{d}\sigma + \varepsilon \int_0^t \|\partial_t \tilde{v}_\kappa(\sigma)\|_2^2 \,\mathrm{d}\sigma \\
&\qquad + c_\varepsilon \int_0^t \|Z_t(\sigma)\|_2^2 \,\mathrm{d}\sigma + c\kappa \int_0^t \|Z_t(\sigma)\|_2 \,\mathrm{d}\sigma,
\end{aligned}$$

where, instead, Z_t denotes the harmonic extension of ζ_t to Ω, namely

$$\begin{cases} -\Delta Z_t(x;t) = 0, & \text{in } \Omega, \\ Z_t(x;t) = \zeta_t(x,t), & \text{on } \partial\Omega. \end{cases}$$

From the above estimates, the definition of Λ_κ, (1.3), Lemma 2.7 and assumptions (2.4) we obtain that

$$\begin{aligned}
\|\nabla \tilde{u}_\kappa(t)\|_2^2 + \|\nabla \tilde{v}_\kappa(t)\|_2^2 \leqslant{}& C_1 + C_2 \int_0^t \|\nabla \tilde{u}_\kappa(\sigma)\|_2 \left[\|\nabla U_t(\sigma)\|_2 + \|\nabla \Psi_t(\sigma)\|_2\right] \mathrm{d}\sigma \\
&+ C_3 \int_0^t \|\nabla \tilde{v}_\kappa(\sigma)\|_2 \left[\|\nabla V_t(\sigma)\|_2 + \|\nabla Z_t(\sigma)\|_2\right] \mathrm{d}\sigma,
\end{aligned}$$

for some positive constant $C_1 = C_1(\kappa)$ independent of t,

$$C_1(\kappa) = Q + \kappa\mu, \quad Q = Q(u_0, v_0, \psi, \zeta), \tag{2.24}$$

for C_2, C_3 independent of t and κ, where μ has been defined in (2.19). Hence, by the Cauchy–Schwarz inequality

$$\|\nabla\tilde{u}_\kappa(t)\|_2^2+\|\nabla\tilde{v}_\kappa(t)\|_2^2\leqslant C_1+C_4\int_0^t\sqrt{\|\nabla\tilde{u}_\kappa(\sigma)\|_2^2+\|\nabla\tilde{v}_\kappa(\sigma)\|_2^2}\\ \times\left[\|\nabla U_t(\sigma)\|_2+\|\nabla\Psi_t(\sigma)\|_2+\|\nabla V_t(\sigma)\|_2+\|\nabla Z_t(\sigma)\|_2\right]\mathrm{d}\sigma$$

for all $t\geqslant 0$, for some positive constant C_4 independent of t and κ. From Lemma 2.10 it follows that, for all $t\geqslant 0$,

$$\|\nabla\tilde{u}_\kappa(t)\|_2^2+\|\nabla\tilde{v}_\kappa(t)\|_2^2\leqslant 2C_1+C_4^2\big[\|\nabla U_t\|_{L^1(0,\infty;L^2(\Omega))}+\|\nabla\Psi_t\|_{L^1(0,\infty;L^2(\Omega))}\\ +\|\nabla V_t\|_{L^1(0,\infty;L^2(\Omega))}+\|\nabla Z_t\|_{L^1(0,\infty;L^2(\Omega))}\big]^2,$$

which, by Lemma 2.8 and assumption (2.4), yields boundedness of $(\tilde{u}_\kappa(t),\tilde{v}_\kappa(t))$ and, consequently of the sequence $(u_\kappa(t),v_\kappa(t))$ in $\mathbb{H}$, with the estimate appearing in (2.20).

In particular, from the $\mathbb{H}$ boundedness of $(\tilde{u}_\kappa(t),\tilde{v}_\kappa(t))$ we deduce that Λ_κ is bounded from below uniformly with respect to t, with a constant α_κ of the same form as the one appearing in inequality (2.20) (say, $\Lambda_\kappa(t)\geqslant -M-N\kappa\mu$, for some constants $M,N\geqslant 0$). To prove this it suffices to repeat the estimates that we have obtained above (see the inequalities following formula (2.23)) on the term which appear in the functional as time integrals, using the H^1 bound of $\tilde{u}_\kappa$ and $\tilde{v}_\kappa$, uniform in time. Notice that the time integrals $\varepsilon\int_0^t\|\partial_t\tilde{u}_\kappa(\sigma)\|_2^2\,\mathrm{d}\sigma$ and $\varepsilon\int_0^t\|\partial_t\tilde{v}_\kappa(\sigma)\|_2^2\,\mathrm{d}\sigma$ which appear in the estimate of the boundary term are balanced by the corresponding term in the definition of Λ_κ. More precisely, we obtain

$$2\left|\int_0^t\int_\Omega\nabla\tilde{u}_\kappa(\sigma)\cdot\nabla U_t(\sigma)\,\mathrm{d}\sigma\right|\leqslant 2(R+\kappa\mu)\|\nabla U_t\|_{L^1(0,\infty;L^2(\Omega))}\leqslant A+B\kappa\mu,$$

$$\left|\int_\Omega\nabla U(t)\cdot\nabla\tilde{u}_\kappa(t)\right|\leqslant(R+\kappa\mu)\sup_{t\geqslant 1}\|\nabla U(t)\|_2\leqslant C+D\kappa\mu,$$

as well as

$$-\int_0^t {}_{H^{-1/2}}\left\langle\frac{\partial\tilde{u}_\kappa(\sigma)}{\partial\nu},\psi_t(\sigma)\right\rangle_{H^{1/2}}\mathrm{d}\sigma-\int_0^t {}_{H^{-1/2}}\left\langle\frac{\partial\tilde{v}_\kappa(\sigma)}{\partial\nu},\zeta_t(\sigma)\right\rangle_{H^{1/2}}\mathrm{d}\sigma\\ +\varepsilon\int_0^t\|\partial_t\tilde{u}_\kappa(\sigma)\|_2^2\,\mathrm{d}\sigma+\varepsilon\int_0^t\|\partial_t\tilde{u}_\kappa(\sigma)\|_2^2\,\mathrm{d}\sigma\\ \geqslant-(R+\kappa\mu)\|\nabla\Psi_t\|_{L^1(0,\infty,L^2(\Omega))}-c_\varepsilon\|\Psi_t\|^2_{L^2(0,\infty,L^2(\Omega))}-c\kappa\|\Psi_t\|_{L^1(0,\infty,L^2(\Omega))}\\ -(R+\kappa\mu)\|\nabla Z_t\|_{L^1(0,\infty,L^2(\Omega))}-c_\varepsilon\|Z_t\|^2_{L^2(0,\infty,L^2(\Omega))}-c\kappa\|Z_t\|_{L^1(0,\infty,L^2(\Omega))}\geqslant -E-F\kappa\mu,$$

for some constants $A,B,C,D,E,F\geqslant 0$ independent of κ and t. Now, for all $\tau\in[0,\tau_0]$,

$$\|\tilde{u}_\kappa^\tau(t)-\tilde{u}_\kappa(t)\|_2^2+\|\tilde{v}_\kappa^\tau(t)-\tilde{v}_\kappa(t)\|_2^2\\ =\int_\Omega|\tilde{u}_\kappa^\tau(t)-\tilde{u}_\kappa(t)|^2+\int_\Omega|\tilde{v}_\kappa^\tau(t)-\tilde{v}_\kappa(t)|^2\leqslant\tau\int_t^{t+\tau}\|\partial_t\tilde{u}_\kappa(\sigma)\|_2^2\,\mathrm{d}\sigma+\tau\int_t^{t+\tau}\|\partial_t\tilde{v}_\kappa(\sigma)\|_2^2\,\mathrm{d}\sigma\\ =\frac{\tau}{1-\varepsilon}\int_t^{t+\tau}\left(-\frac{\mathrm{d}}{\mathrm{d}\sigma}\Lambda_\kappa(\sigma)\right)\mathrm{d}\sigma\leqslant\frac{\tau_0}{1-\varepsilon}\left[\Lambda_\kappa(t)-\Lambda_\kappa(t+\tau_0)\right],$$

where we exploited Hölder inequality, Fubini's theorem and identity (2.22) (in the spirit of [6]). Hence, we obtain

$$
\begin{aligned}
&\|u_\kappa^\tau(t) - u_\kappa(t)\|_2^2 + \|v_\kappa^\tau(t) - v_\kappa(t)\|_2^2 \\
&\quad \leqslant 2\big(\|\tilde{u}_\kappa^\tau(t) - \tilde{u}_\kappa(t)\|_2^2 + \|\tilde{v}_\kappa^\tau(t) - \tilde{v}_\kappa(t)\|_2^2 + \|U^\tau(t) - U(t)\|_2^2 + \|V^\tau(t) - V(t)\|_2^2\big) \\
&\quad \leqslant \frac{2\tau_0}{1-\varepsilon}\big[\Lambda_\kappa(t) - \Lambda_\kappa(t+\tau_0)\big] + 2\|U^\tau(t) - U(t)\|_2^2 + 2\|V^\tau(t) - V(t)\|_2^2.
\end{aligned}
\tag{2.25}
$$

Since Λ_κ is nonincreasing and bounded from below at fixed κ, it follows that $\Lambda_\kappa(t)$ admits a finite limit as $t \to \infty$. Therefore, letting $t \to \infty$ in (2.25), and taking into account Lemma 2.7, we obtain (2.21). Now, assume by contradiction that, for some $\varepsilon_0 > 0$,

$$
\|u_\kappa(t_h + \tau_h) - u_\kappa(t_h)\|_{H^1} \geqslant \varepsilon_0 > 0,
$$

along a diverging sequence $(t_h) \subset \mathbb{R}^+$ and for $(\tau_h) \subset \mathbb{R}^+$ bounded. In light of Lemma 2.5, there exist $\hat{u}$ and $\check{u} \in H^1(\Omega)$ such that, up to a subsequence that we still denote by (t_h), $u_\kappa(t_h + \tau_h) \to \hat{u}$ in $H^1(\Omega)$ as $h \to \infty$, and $u_\kappa(t_h) \to \check{u}$ in $H^1(\Omega)$ as $h \to \infty$. In particular, $\|\hat{u} - \check{u}\|_{H^1} \geqslant \varepsilon_0 > 0$, while (2.21) yields $\|\hat{u} - \check{u}\|_{L^2} = 0$, thus giving rise to a contradiction. One argues similarly for v_κ. This concludes the proof of the theorem. □

Next we have an important consequence of the previous lemma, proving Theorem 2.2.

Corollary 2.12. *Assume that $(u_0, v_0) \in \mathbb{H}_0$ and that the boundary data are stationary. Then the sequence $(u_\kappa(t), v_\kappa(t))$ is uniformly bounded in H^1 with respect to t and κ. Moreover, the energy functional which appears in the proof of Theorem* 2.11 *is bounded below and above by constants which are independent of κ.*

Proof. If $(u_0, v_0) \in \mathbb{H}_0$, since $u_0 v_0 = 0$ and $\psi_t = \zeta_t = 0$ by (2.19) we have that $\mu = 0$. In turn, by (2.20), the sequence $(u_\kappa(t), v_\kappa(t))$ is uniformly bounded with respect to t and κ. By inspecting the proof of Theorem 2.11 it is easy to check that the auxiliary energy functional satisfies

$$
-M - N\kappa\mu \leqslant \Lambda_\kappa(t) \leqslant O + P\kappa\mu, \quad t \geqslant 0,
$$

for some constants $M, N, O, P \geqslant 0$ independent of κ. Hence, being $\mu = 0$ it follows that Λ_κ has bounds uniform in time and in k. □

2.3. *Proof of Theorem* 2.1 *concluded*

Let $\kappa > 0$ and let $(t_h) \subset \mathbb{R}^+$ be any diverging sequence. Then, by virtue of Theorem 2.11, we have

$$
\lim_{h\to\infty} \|u_\kappa(t_h + \tau_h) - u_\kappa(t_h)\|_{H^1} = 0, \qquad \lim_{h\to\infty} \|v_\kappa(t_h + \tau_h) - v_\kappa(t_h)\|_{H^1} = 0,
\tag{2.26}
$$

for every sequence $(\tau_h) \subset [0,1]$. Let us fix $\eta, \xi \in C_c^\infty(\Omega)$. By integrating over $(t_h, t_h+1) \times \Omega$ the equations of (P_κ) multiplied by η and ξ respectively, we reach

$$\lim_h \Big[\int_{t_h}^{t_h+1}\int_\Omega \partial_t u_\kappa \eta + \int_{t_h}^{t_h+1}\int_\Omega \nabla u_\kappa \cdot \nabla \eta - \int_{t_h}^{t_h+1}\int_\Omega f(u_\kappa)\eta + \kappa \int_{t_h}^{t_h+1}\int_\Omega u_\kappa v_\kappa^2 \eta\Big] = 0,$$
$$\lim_h \Big[\int_{t_h}^{t_h+1}\int_\Omega \partial_t v_\kappa \xi + \int_{t_h}^{t_h+1}\int_\Omega \nabla v_\kappa \cdot \nabla \xi - \int_{t_h}^{t_h+1}\int_\Omega g(v_\kappa)\xi + \kappa \int_{t_h}^{t_h+1}\int_\Omega v_\kappa u_\kappa^2 \xi\Big] = 0.$$

Regarding the first terms in the previous identities, we obtain

$$\lim_h \Big|\int_{t_h}^{t_h+1}\int_\Omega \partial_t u_\kappa \eta\Big| \leqslant \lim_h \int_\Omega \big|u_\kappa(t_h) - u_\kappa(t_h+1)\big||\eta| \leqslant \lim_h c\big\|u_\kappa(t_h) - u_\kappa(t_h+1)\big\|_2 = 0,$$
$$\lim_h \Big|\int_{t_h}^{t_h+1}\int_\Omega \partial_t v_\kappa \xi\Big| \leqslant \lim_h \int_\Omega \big|v_\kappa(t_h) - v_\kappa(t_h+1)\big||\xi| \leqslant \lim_h c\big\|v_\kappa(t_h) - v_\kappa(t_h+1)\big\|_2 = 0.$$

Moreover, there exist two sequences $(s_h), (r_h) \subset \mathbb{R}^+$ such that

$$t_h \leqslant s_h \leqslant t_h + 1, \qquad t_h \leqslant r_h \leqslant t_h + 1, \quad s_h = t_h + \rho_h^1, r_h = t_h + \rho_h^2,$$

with $(\rho_h^1), (\rho_h^2) \subset [0,1]$, and

$$\int_{t_h}^{t_h+1}\int_\Omega \nabla u_\kappa \cdot \nabla \eta - f(u_\kappa)\eta + \kappa u_\kappa v_\kappa^2 \eta = \int_\Omega \nabla u_\kappa(s_h) \cdot \nabla \eta - f\big(u_\kappa(s_h)\big)\eta + \kappa u_\kappa(s_h) v_\kappa^2(s_h)\eta,$$
$$\int_{t_h}^{t_h+1}\int_\Omega \big[\nabla v_\kappa \cdot \nabla \xi - g(v_\kappa)\xi + \kappa v_\kappa u_\kappa^2 \xi\big] = \int_\Omega \nabla v_\kappa(r_h) \cdot \nabla \xi - g\big(v_\kappa(r_h)\big)\xi + \kappa v_\kappa(r_h) u_\kappa^2(r_h)\xi.$$

In turn, we get

$$\lim_h \Big[\int_\Omega \nabla u_\kappa(s_h) \cdot \nabla \eta - \int_\Omega f\big(u_\kappa(s_h)\big)\eta + \kappa \int_\Omega u_\kappa(s_h) v_\kappa^2(s_h)\eta\Big] = 0,$$
$$\lim_h \Big[\int_\Omega \nabla v_\kappa(r_h) \cdot \nabla \xi - \int_\Omega g\big(v_\kappa(r_h)\big)\xi + \kappa \int_\Omega v_\kappa(r_h) u_\kappa^2(r_h)\xi\Big] = 0.$$

On the other hand, in light of (2.26), there holds

$$\lim_h \big\|\nabla u_\kappa(s_h) - \nabla u_\kappa(t_h)\big\|_2 = \lim_h \big\|\nabla u_\kappa(t_h + \rho_h^1) - \nabla u_\kappa(t_h)\big\|_2 = 0,$$
$$\lim_h \big\|\nabla v_\kappa(r_h) - \nabla v_\kappa(t_h)\big\|_2 = \lim_h \big\|\nabla v_\kappa\big(t_h + \rho_h^2\big) - \nabla v_\kappa(t_h)\big\|_2 = 0.$$

Hence $\int_\Omega (\nabla u_\kappa(s_h) - \nabla u_\kappa(t_h)) \cdot \nabla \eta \to 0$ and, as f, g are C^1 on $[0,1]$ and $0 \leqslant u_\kappa, v_\kappa \leqslant 1$,

$$\Big|\int_\Omega \big(f\big(u_\kappa(s_h)\big) - f\big(u_\kappa(t_h)\big)\big)\eta\Big| \leqslant c \sup_{[0,1]} |f'| \big\|u_\kappa(s_h) - u_\kappa(t_h)\big\|_2 \to 0,$$

as $h \to \infty$, and finally,

$$\left| \int_\Omega \big(u_\kappa(s_h) v_\kappa^2(s_h) - u_\kappa(t_h) v_\kappa^2(t_h)\big)\eta \right| \leqslant c\big\|u_\kappa(s_h) - u_\kappa(t_h)\big\|_2 + c\big\|v_\kappa(s_h) - v_\kappa(t_h)\big\|_2 \to 0,$$

as $h \to \infty$, the positive constant c varying from line to line. Of course, the same conclusions hold for the limit involving the sequence $v_\kappa(r_h)$. In conclusion, we reach

$$\lim_h \left[\int_\Omega \nabla u_\kappa(t_h) \cdot \nabla\eta - \int_\Omega f\big(u_\kappa(t_h)\big)\eta + \kappa \int_\Omega u_\kappa(t_h) v_\kappa^2(t_h)\eta \right] = 0,$$
$$\lim_h \left[\int_\Omega \nabla v_\kappa(t_h) \cdot \nabla\xi - \int_\Omega g\big(v_\kappa(t_h)\big)\xi + \kappa \int_\Omega v_\kappa(t_h) u_\kappa^2(t_h)\xi \right] = 0.$$

Again in view of Theorem 2.11, we can assume that, up to a subsequence, which we shall denote again by t_h, we have that $u_\kappa(t_h) \rightharpoonup \hat{u}_\kappa$ and $v_\kappa(t_h) \rightharpoonup \hat{v}_\kappa$ weakly in $H^1(\Omega)$. Up to a subsequence, in light of Lemma 2.5, this convergence is actually strong. Notice also that

$$\hat{u}_\kappa|_{\partial\Omega} = \lim_h u_\kappa(t_h)|_{\partial\Omega} = \lim_h \psi(t_h)|_{\partial\Omega} = \psi_\infty,$$
$$\hat{v}_\kappa|_{\partial\Omega} = \lim_h v_\kappa(t_h)|_{\partial\Omega} = \lim_h \zeta(t_h)|_{\partial\Omega} = \zeta_\infty,$$

where we exploited the compact embedding $H^1(\Omega) \hookrightarrow H^{1/2}(\partial\Omega)$. Moreover, by Lemma 2.3 and the Dominated Convergence Theorem, as $h \to \infty$, we get

$$\int_\Omega \nabla\hat{u}_\kappa \cdot \nabla\eta - \int_\Omega f(\hat{u}_\kappa)\eta + \kappa \int_\Omega \hat{u}_\kappa \hat{v}_\kappa^2 \eta = 0, \quad \forall \eta \in H_0^1(\Omega),$$
$$\int_\Omega \nabla\hat{v}_\kappa \cdot \nabla\xi - \int_\Omega g(\hat{v}_\kappa)\xi + \kappa \int_\Omega \hat{v}_\kappa \hat{u}_\kappa^2 \xi = 0, \quad \forall \xi \in H_0^1(\Omega).$$

Hence $(\hat{u}_\kappa, \hat{v}_\kappa) \in \mathbb{H}$ is a solution to (S_κ). The convergence occurs of course in $L^p(\Omega)$ for any $p \in [2, 2^*)$. For $p \geqslant 2^*$, taking $\varepsilon > 0$ and using the bounds $0 \leqslant u_\kappa(t_h) \leqslant 1$ and $0 \leqslant \hat{u}_\kappa \leqslant 1$, we have

$$\int_\Omega \big|u_\kappa(t_h) - \hat{u}_\kappa\big|^p \leqslant 2^{p+\varepsilon-2^*} \big\|u_\kappa(t_h) - \hat{u}_\kappa\big\|_{2^*-\varepsilon}^{2^*-\varepsilon},$$

concluding the proof.

3. Proof of Theorem 1.1

Before concluding the proof of Theorem 1.1, we provide the convergence of the sequences $(\hat{u}_\kappa, \hat{v}_\kappa)$ in any L^p space with $p \geqslant 2$ towards a segregated state. Notice that the solutions $(\hat{u}_\kappa, \hat{v}_\kappa)$ to (S_κ) pop up as H^1 limits of the solutions to (1.2), and the boundedness of $(\hat{u}_\kappa, \hat{v}_\kappa)$ in H^1 is inherited by the boundedness of $(u_\kappa(t_h), v_\kappa(t_h))$ in H^1 uniform in t and κ (in the case $(u_0, v_0) \in \mathbb{H}_0$). Without this information it would not have been clear how to show the boundedness of $(\hat{u}_\kappa, \hat{v}_\kappa)$ working directly on the elliptic system (instead, for system (1.1), this is an easy task, cf. [5], Lemma 2.1).

Lemma 3.1. *Assume that* $(u_0, v_0) \in \mathbb{H}_0$. *Let* $(\hat{u}_\kappa, \hat{v}_\kappa) \in \mathbb{H}$ *be the solution to system* (S_κ) *as obtained in Theorem* 2.1 *for* $\kappa > 0$. *Then there exists* $(u_\infty, v_\infty) \in \mathbb{H}_0$ *with*

$$u_\infty, v_\infty \geqslant 0, \quad -\Delta u_\infty \leqslant f(u_\infty), \qquad -\Delta v_\infty \leqslant g(v_\infty)$$

and $u_\infty|_{\partial\Omega} = \psi$, $v_\infty|_{\partial\Omega} = \zeta$ *such that, up to a subsequence, as* $\kappa \to \infty$,

$$(\hat{u}_\kappa, \hat{v}_\kappa) \to (u_\infty, v_\infty) \quad \text{in the } L^p \times L^p \text{ norm for any } p \in [2, \infty).$$

Proof. By virtue of Corollary 2.12 the sequence $(u_\kappa(t_h), v_\kappa(t_h))$ is bounded in $\mathbb{H}$, uniformly with respect to κ. Hence, since $(\hat{u}_\kappa, \hat{v}_\kappa)$ is the H^1-limit of $(u_\kappa(t_h), v_\kappa(t_h))$ as $h \to \infty$, we deduce that $(\hat{u}_\kappa, \hat{v}_\kappa)$ is bounded in $\mathbb{H}$ and $0 \leqslant \hat{u}_\kappa(x) \leqslant 1$, $0 \leqslant \hat{v}_\kappa(x) \leqslant 1$, for a.e. $x \in \Omega$. Taking into account that some terms in the functional Λ_κ introduced within the proof of Theorem 2.11 vanish under the current assumptions (stationary boundary conditions) and that the terms $\varepsilon \int_0^t \|\partial_t \tilde{u}_\kappa(\sigma)\|_2^2$ and $\varepsilon \int_0^t \|\partial_t \tilde{v}_\kappa(\sigma)\|_2^2$ were artificially attached to make things work (notice that the original Λ_κ is decreasing also in the case $\varepsilon = 0$, see formula (2.22)), we now just consider the natural energy functional (for the sake of simplicity we do not change the name)

$$\Lambda_\kappa(t) = \frac{1}{2}\|\nabla \tilde{u}_\kappa(t)\|_2^2 + \frac{1}{2}\|\nabla \tilde{v}_\kappa(t)\|_2^2 - \int_\Omega F\big(\tilde{u}_\kappa(t) + U(t)\big)$$
$$- \int_\Omega G\big(\tilde{v}_\kappa(t) + V(t)\big) + \frac{\kappa}{2}\int_\Omega \big(\tilde{u}_\kappa(t) + U(t)\big)^2\big(\tilde{v}_\kappa(t) + V(t)\big)^2.$$

Then, we have

$$\kappa \int_\Omega u_\kappa^2(t_h) v_\kappa^2(t_h) = 2\Lambda_\kappa(t_h) - \|\nabla \tilde{u}_\kappa(t_h)\|_2^2 - \|\nabla \tilde{v}_\kappa(t_h)\|_2^2 + 2\int_\Omega F\big(u_\kappa(t_h)\big) + G\big(v_\kappa(t_h)\big).$$

Since by Corollary 2.12 the right-hand side is uniformly bounded with respect to κ, we have

$$\kappa \int_\Omega \hat{u}_\kappa^2 \hat{v}_\kappa^2 \leqslant C, \tag{3.1}$$

for some positive constant C independent of κ. Let $u_\infty \in H^1(\Omega)$ and $v_\infty \in H^1(\Omega)$ be the weak limits, as $\kappa \to \infty$, of $\hat{u}_\kappa$ and $\hat{v}_\kappa$ in $H^1(\Omega)$ respectively. Or course, by the compact embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, up to a further subsequence, $\hat{u}_\kappa \to u_\infty$ and $\hat{v}_\kappa \to v_\infty$ in $L^p(\Omega)$ for any $p \in [2, 2^*)$ and $0 \leqslant u_\infty(x) \leqslant 1$, $0 \leqslant v_\infty(x) \leqslant 1$, for a.e. $x \in \Omega$. In the case $p \geqslant 2^*$, let $\varepsilon > 0$, so that

$$\int_\Omega |\hat{u}_\kappa - u_\infty|^p \leqslant 2^{p+\varepsilon-2^*} \|\hat{u}_\kappa - u_\infty\|_{2^*-\varepsilon}^{2^*-\varepsilon},$$

yielding again the convergence. Due to inequality (3.1), we get

$$\lim_{\kappa \to \infty} \int_\Omega \hat{u}_\kappa^2 \hat{v}_\kappa^2 = \int_\Omega u_\infty^2 v_\infty^2 = 0,$$

which yields $u_\infty v_\infty = 0$ a.e. in Ω, namely $(u_\infty, v_\infty) \in \mathbb{H}_0$. Moreover, for each $\kappa > 0$,

$$-\Delta \hat{u}_\kappa \leqslant f(\hat{u}_\kappa), \qquad -\Delta \hat{v}_\kappa \leqslant g(\hat{v}_\kappa),$$

which pass to the weak the limit, yielding $-\Delta u_\infty \leqslant f(u_\infty)$ and $-\Delta v_\infty \leqslant g(v_\infty)$. By the compact embedding $H^1(\Omega) \hookrightarrow H^{1/2}(\partial\Omega)$, also the boundary conditions are conserved. $\square$

3.1. Proof of Theorem 1.1 *concluded*

We can now conclude the proof of Theorem 1.1. Let $(u_0, v_0) \in \mathbb{H}_0$, $p \in [2, \infty)$ and let $(t_h) \subset \mathbb{R}^+$ be any diverging sequence. In light of Theorem 2.1, for every $\kappa \geqslant 1$, there exist a solution $(\hat{u}_\kappa, \hat{v}_\kappa)$ of (S_κ) and a subsequence $(t_h^\kappa) \subset \mathbb{R}^+$ such that,

$$\big\|\big(u_\kappa(t_h^\kappa), v_\kappa(t_h^\kappa)\big) - (\hat{u}_\kappa, \hat{v}_\kappa)\big\|_{\mathbb{H}} \to 0, \quad \text{as } h \to \infty.$$

Moreover, by Lemma 3.1, there exists $(u_\infty, v_\infty) \in \mathbb{H}_0$ with the required properties, such that, up to a subsequence,

$$\big\|(\hat{u}_\kappa, \hat{v}_\kappa) - (u_\infty, v_\infty)\big\|_{L^p \times L^p} \to 0, \quad \text{as } \kappa \to \infty.$$

Now, let $m \geqslant 1$ and let $\kappa_m \geqslant 1$ be such that

$$\big\|(\hat{u}_{\kappa_m}, \hat{v}_{\kappa_m}) - (u_\infty, v_\infty)\big\|_{L^p \times L^p} < \frac{1}{2m}.$$

Then, there exists $t_{h_m}^{\kappa_m} \geqslant 1$ such that

$$\big\|\big(u_{\kappa_m}(t_{h_m}^{\kappa_m}), v_{\kappa_m}(t_{h_m}^{\kappa_m})\big) - (\hat{u}_{\kappa_m}, \hat{v}_{\kappa_m})\big\|_{L^p \times L^p} < \frac{1}{2m}.$$

In turn, setting $t_m = t_{h_m}^{\kappa_m}$, and combining the previous inequalities, we get

$$\big\|\big(u_{\kappa_m}(t_m), v_{\kappa_m}(t_m)\big) - (u_\infty, v_\infty)\big\|_{L^p \times L^p} < \frac{1}{m},$$

which concludes the proof of the first assertion. In the one dimensional case, by means of Morrey theorem, for every $x, y \in \Omega$, we have

$$\big|u_{\kappa_m}(t_m)(x) - u_{\kappa_m}(t_m)(y)\big| \leqslant 4\big\|\nabla u_{\kappa_m}(t_m)\big\|_2 \sqrt{|x-y|} \leqslant C\sqrt{|x-y|},$$

together with $|u_{\kappa_m}(t_m)(x)| \leqslant 1$, yielding the convergence to (u_∞, v_∞) in the $L^\infty \times L^\infty$ norm via Ascoli's theorem.

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