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ON A CLASS OF ELLIPTIC EQUATIONS FOR THE n-LAPLACIAN IN $\mathbb{R}^n$ WITH ONE-SIDED EXPONENTIAL GROWTH*

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Abstract. By means of a suitable nonsmooth critical point theory for lower semicontinuous functionals we prove the existence of infinitely many solutions for a class of quasilinear Dirichlet problems with symmetric nonlinearities having a one-sided growth condition of exponential type.

1. Introduction and main result. The aim of this paper is to get a multiplicity result for the quasilinear elliptic problem

$$(P) \qquad \begin{cases} -\Delta_n u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where Ω is a C^1 bounded domain of $\mathbb{R}^n$ with $n \geq 2$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function with a suitable exponential growth and $\Delta_n u = \operatorname{div}(|Du|^{n-2}Du)$ is the n-Laplacian operator. If, for instance, g is continuous in $\Omega \times \mathbb{R}$ and satisfies the two-sided growth condition

$$|g(x,s)| \leq a(x) + b|s|^{p-1}e^{|s|^p} \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where $a \in L^r(\Omega)$ for some $r > 1$, $b > 0$ and $1 < p < \dfrac{n}{n-1}$, then the functional

$$f(u) = -\int_\Omega G(x,u)\,dx \tag{1.1}$$

is of class C^1 on $W_0^{1,n}(\Omega)$, defined $G : \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$G(x,s) = \int_0^s g(x,t)\,dt.$$

Thus, the given problem is reduced to that of looking for critical points of a smooth functional by means of classical variational tools (see, e.g., [7, 8] and references therein). Here, on the contrary, we want to investigate the existence of solutions of problem (P) when $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is just a Carathéodory function such that

$$\sup_{|s|\leq t} |g(\cdot,s)| \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for every } t > 0 \tag{1.2}$$

and assume that there exists a constant $\kappa > 0$ such that for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ with

$$G(x,s) \leq a_\varepsilon(x) + \kappa e^{\varepsilon|s|^{\frac{n}{n-1}}} \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

The above assumption gives only a one-sided growth condition so that the function f in (1.1) is not necessarily finite and is only lower semicontinuous (see Proposition 3.11); thus, classical variational arguments cannot be applied. However, problem (P) can be solved in a weak sense (see Theorem 3.14) by using a nonsmooth machinery developed by Degiovanni and Zani (see [5, 6] and references therein both for the abstract framework and applications to semilinear problems with a one-sided power type growth).

Before stating the main result of this paper, let us recall, in our setting, the definition of weak solution of the given problem.

Definition 1.1. *A function $u \in W_0^{1,n}(\Omega)$ is a weak solution of (P) if $g(x,u) \in L^1_{\mathrm{loc}}(\Omega)$ and $-\Delta_n u = g(x,u)$ in $\mathcal{D}'(\Omega)$, namely*

$$\int_\Omega |Du|^{n-2} Du \cdot Dv \, dx = \int_\Omega g(x,u) v \, dx \quad \textit{for every } v \in C_c^\infty(\Omega).$$

The next result extends [6, Theorem 6.1] to the case of exponential-type nonlinearities. More precisely, we have the following result.

Theorem 1.2. *Assume that condition (1.2) holds and that there exist $\theta > n$, $R > 0$ and a function $a \in L^1(\Omega)$ such that*

$$(1.3) \qquad |s| \ge R \implies 0 < \theta G(x,s) \le s g(x,s) \quad \textit{for every } s \in \mathbb{R} \textit{ and a.e. } x \in \Omega,$$

$$(1.4) \qquad |s| \le R \implies G(x,s) \le a(x) \quad \textit{for every } s \in \mathbb{R} \textit{ and a.e. } x \in \Omega.$$

Furthermore, there exist $\kappa_1, \kappa_2 > 0$ and $\beta \ge 0$ such that for every $\varepsilon > 0$ there exist two functions $a_\varepsilon, b_\varepsilon \in L^1(\Omega)$ such that

$$(1.5) \qquad s g(x,s) \le a_\varepsilon(x) + \kappa_1 e^{\varepsilon |s|^{\frac{n}{n-1}}} \quad \textit{for every } s \in \mathbb{R} \textit{ and a.e. } x \in \Omega,$$

$$(1.6) \quad G(x,s) - \beta s g(x,s) \ge b_\varepsilon(x) - \kappa_2 e^{\varepsilon |s|^{\frac{n}{n-1}}} \quad \textit{for every } s \in \mathbb{R} \textit{ and a.e. } x \in \Omega.$$

Then, if g is odd with respect to s, i.e.,

$$(1.7) \qquad g(x,-s) = -g(x,s) \quad \textit{for every } s \in \mathbb{R} \textit{ and a.e. } x \in \Omega,$$

problem (P) has infinitely many weak solutions.

Remark 1.3. By virtue of Young's inequality it follows that for every $\varepsilon > 0$

$$e^{\frac{\varepsilon}{2}|s|^{\frac{n}{n-1}}} \le \varepsilon e^{\varepsilon |s|^{\frac{n}{n-1}}} + c_\varepsilon \quad \text{for every } s \in \mathbb{R},$$

for a $c_\varepsilon > 0$. Thus, conditions (1.5) and (1.6) imply that for every $\varepsilon > 0$ there exist two functions a_ε, $b_\varepsilon \in L^1(\Omega)$ such that

$$(1.8) \qquad s g(x,s) \le a_\varepsilon(x) + \varepsilon e^{\varepsilon |s|^{\frac{n}{n-1}}} \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

$$(1.9) \quad G(x,s) - \beta s g(x,s) \ge b_\varepsilon(x) - \varepsilon e^{\varepsilon |s|^{\frac{n}{n-1}}} \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Actually, here the functions a_ε and b_ε differ by a constant from those of (1.5) and (1.6). On the other hand, we avoid to change the notations.

Remark 1.4. By conditions (1.3)–(1.5) and Remark 1.3 it follows that for every $\varepsilon > 0$ there exists a suitable function $a_\varepsilon \in L^1(\Omega)$ such that

$$G(x,s) \le a_\varepsilon(x) + \varepsilon e^{\varepsilon |s|^{\frac{n}{n-1}}} \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega. \tag{1.10}$$

If, furthermore, assumption (1.6) holds with $\beta = 0$, then G satisfies a two-sided growth condition which implies that f is finite and continuous (see Proposition 3.11) but not necessarily of class C^1.

Remark 1.5. It is easy to see that, for instance, the previous setting contains the model exponential nonlinearity $g(x,s) = |s|^{p-2}se^{|s|^p}$ for $p > 1$ any but $p < \dfrac{n}{n-1}$.

The plan of the paper is as follows.

– In Section 2 we recall from [5, 6] some notions of nonsmooth critical point theory.

– In Section 3 we study the variational setting and we prove that the functional associated with (P) satisfies some technical properties (i.e., the $(PS)_c$ and $(\mathrm{epi})_c$ conditions).

– In Section 4 we end up the proof of Theorem 1.2.

2. Preliminaries of nonsmooth analysis.

First of all, we need to introduce some abstract notions of nonsmooth analysis. To this aim, let us just recall the main definitions which extend the classical critical point theory to some classes of lower semicontinuous functions on metric spaces (for more details, see [6] and references therein).

Let (X,d) be a metric space and consider the associated metric, still denoted by d, defined on the product set $X \times \mathbb{R}$ as

$$d((u,\lambda),(v,\mu)) = \sqrt{(d(u,v))^2 + (\lambda - \mu)^2}.$$

Taken $f : X \to \overline{\mathbb{R}}$, we can consider the related set

$$\mathrm{epi}(f) = \{(u,\lambda) \in X \times \mathbb{R} : \ f(u) \le \lambda\}$$

and the corresponding projection $\mathcal{G}_f : (u,\lambda) \in \mathrm{epi}(f) \mapsto \lambda \in \mathbb{R}$, which turns out to be a 1-Lipschitz continuous function.

Definition 2.1. *Let $u \in X$ be such that $f(u) \in \mathbb{R}$. The weak slope of f at u is the extended real number $|df|(u)$ defined as*

$$
\begin{aligned}
|df|(u) = \sup\big\{\sigma \in [0,+\infty[:\ &\exists \delta > 0 \text{ and a continuous map} \\
&H : (B_\delta(u, f(u)) \cap \mathrm{epi}(f)) \times [0,\delta] \to X \text{ s.t.} \\
&d(H((w,\mu),t),w) \le t,\ f(H((w,\mu),t)) \le \mu - \sigma t \\
&\text{for every } ((w,\mu),t) \in (B_\delta(u,f(u)) \cap \mathrm{epi}(f)) \times [0,\delta]\big\},
\end{aligned}
$$

where $B_\delta(u, f(u)) = \big\{(w,\mu) \in X \times \mathbb{R} :\ d((w,\mu),(u,f(u))) < \delta\big\}$.

Definition 2.2. *We say that $u \in X$ is a (lower) critical point of f if $f(u) \in \mathbb{R}$ and $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (lower) critical level of f if there exists a (lower) critical point $u \in X$ such that $f(u) = c$.*

Definition 2.3. *Taken $c \in \mathbb{R}$, we say that f satisfies the Palais-Smale condition at level c, briefly $(PS)_c$, if any sequence $(u_h)_h \subset X$, such that $f(u_h) \to c$ and $|df|(u_h) \to 0$ as $h \to +\infty$, admits a converging subsequence.*

Definition 2.4. *Taken $c \in \mathbb{R}$, we say that f satisfies the $(\mathrm{epi})_c$ condition if*

$$\inf\{|dG_f|(u,\lambda) :\ f(u) < \lambda,\ |\lambda - c| < \gamma_0\} > 0$$

for some $\gamma_0 > 0$.

Furthermore, in order to obtain a multiplicity result for even functionals, the following definition is needed.

Definition 2.5. *Let $f : X \to \overline{\mathbb{R}}$ be an even function such that $f(0) < +\infty$. For any $\lambda \ge f(0)$ the equivariant weak slope in $(0,\lambda)$ is defined as*

$$
\begin{aligned}
|d_{\mathbb{Z}_2} G_f|(0,\lambda) = \sup\big\{\sigma \in [0,+\infty[:\ &\exists \delta > 0 \text{ and a continuous map} \\
&H : (B_\delta(u, f(u)) \cap \mathrm{epi}(f)) \times [0,\delta] \to \mathrm{epi}(f), \\
&\quad H = (H_1, H_2),\ \text{s.t.} \\
&d(H((w,\mu),t),(w,\mu)) \le t,\ H_2((w,\mu),t)) \le \mu - \sigma t \\
&\quad \text{and } H_1((-w,\mu),t) = -H_1((w,\mu),t) \\
&\text{for every } ((w,\mu),t) \in (B_\delta(u,f(u)) \cap \mathrm{epi}(f)) \times [0,\delta]\big\}.
\end{aligned}
$$

From now on, let $(X, \|\cdot\|)$ be a normed space over $\mathbb{R}$ with dual space X^* and dual product $\langle\cdot,\cdot\rangle$. Since the function we will deal with in the next sections is the sum of a smooth term ψ and a lower semicontinuous term f, we need the following results (for more details, see [3], [6, Theorem 2.9]).

Proposition 2.6. *If $\psi \in C^1(X,\mathbb{R})$ then for any $(u,\lambda) \in \operatorname{epi}(f)$ we have*

$$|dG_{f+\psi}|(u, \lambda + \psi(u)) = 1 \iff |dG_f|(u,\lambda) = 1.$$

If, furthermore, both f and ψ are even and $f(0) < +\infty$ then for any $\lambda \geq f(0)$ it is

$$|d_{\mathbb{Z}_2} G_{f+\psi}|(0, \lambda + \psi(0)) = 1 \iff |d_{\mathbb{Z}_2} G_f|(0,\lambda) = 1.$$

Let us remark that some more information about the weak slope of a given function can be obtained by means of a kind of subdifferential (for more details, see [2]).

Definition 2.7. *Taken $u \in X$ such that $f(u) \in \mathbb{R}$, define*

$$\partial f(u) = \{\alpha \in X^* : \langle \alpha, v\rangle \leq f^0(u;v) \text{ for every } v \in X\},$$

where

$$f^0(u;v) = \sup_{j>0} f_j^0(u;v),$$

with

$$\begin{aligned} f_j^0(u;v) = \inf\big\{ r \in \mathbb{R} : \ & \exists \delta > 0 \text{ and a continuous map} \\ & \Phi : (B_\delta(u, f(u)) \cap \operatorname{epi}(f)) \times]0,\delta] \to B_j(v) \text{ s.t.} \\ & f(w + t\Phi((w,\mu),t)) \leq \mu + rt \\ & \text{for every } ((w,\mu),t) \in (B_\delta(u,f(u)) \cap \operatorname{epi}(f)) \times]0,\delta] \big\}. \end{aligned}$$

Proposition 2.8. *If $u \in X$ is such that $f(u) \in \mathbb{R}$, then*

$$|df|(u) < +\infty \iff \partial f(u) \neq \emptyset,$$

$$|df|(u) < +\infty \implies |df|(u) \geq \min\{\|\alpha\| : \alpha \in \partial f(u)\}.$$

Moreover, if $\psi : X \to \mathbb{R}$ is a C^1 functional it is

$$\partial(f+\psi)(u) = \partial f(u) + \partial\psi(u) \text{ and } \partial\psi(u) = \{\psi'(u)\}.$$

At last, we can set the abstract theorem which is a nonsmooth version of the classical Ambrosetti-Rabinowitz theorem (see [1] for smooth functionals and [6, 9] for nonsmooth ones).

Theorem 2.9. *Let $(X, \|\cdot\|)$ be a Banach space and $F : X \to \mathbb{R}\cup\{+\infty\}$ an even lower semicontinuous function. Assume that there exists a sequence $(V_h)_h$ of finite dimensional subspaces of X such that $V_h \subset V_{h+1}$ for every $h \in \mathbb{N}$. Suppose that there exist two constants $\varrho > 0$ and $\eta > 0$ such that*

(a) *there exists a closed subspace Z of X such that $X = V_0 \oplus Z$ and*

$$u \in Z, \quad \|u\| = \varrho \implies F(u) \geq \eta;$$

(b) *there exists a sequence $(R_h)_h \subset]\varrho, +\infty[$ such that*

$$u \in V_h, \quad \|u\| \geq R_h \implies F(u) \leq F(0);$$

(c) *for every $c \geq \eta$ the function F satisfies both $(PS)_c$ and $(\mathrm{epi})_c$;*

(d) *for any $\lambda \geq \eta$ it is $|d_{\mathbb{Z}_2} G_F|(0,\lambda) \neq 0$.*

Then, there exists a sequence $(u_h)_h$ of critical points of F in X such that $F(u_h) \to +\infty$.

3. Variational setting and Palais-Smale condition. In order to introduce the nonsmooth variational setting for our problem, let us consider the Sobolev space $W_0^{1,n}(\Omega)$ equipped with the standard norm $\|u\|_{1,n}^n = \int_\Omega |Du|^n\,dx$.

Definition 3.1. *Fixed $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$, define the set*

$$V_u = \left\{v \in W_0^{1,n}(\Omega) \cap L_c^\infty(\Omega) :\ u \in L^\infty(\{x \in \Omega : v(x) \neq 0\})\right\},$$

where $v \in L_c^\infty(\Omega)$ means $v \in L^\infty(\Omega)$ and $v(x) = 0$ a.e. outside a compact subset of Ω.

Arguing as in [5] the following results can be proved:

Proposition 3.2. *Assume that (1.2) holds. Then, taken $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$, we have:*

(a) *if* $v \in V_u$ *then* $g(x,u)v \in L^1(\Omega)$;

(b) V_u *is a dense linear subspace of* $W_0^{1,n}(\Omega)$;

(c) *if* $v \in W_0^{1,n}(\Omega)$ *and* $(g(x,u)v)^+ \in L^1(\Omega)$ *then there exists a sequence* $(v_h)_h \subset V_u$ *such that* $-v^-(x) \leq v_h(x) \leq v^+(x)$ *for a.e.* $x \in \Omega$, *every* $h \in \mathbb{N}$, *and* $v_h \to v$ *strongly in* $W_0^{1,n}(\Omega)$,

$$\lim_h \int_\Omega g(x,u)v_h\,dx = \int_\Omega g(x,u)v\,dx.$$

Definition 3.3. *Taken* $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$, *we say* $g(x,u) \in W^{-1,n'}(\Omega)$ *if*

$$\sup\left\{\int_\Omega g(x,u)v\,dx:\ v \in W_0^{1,n}(\Omega),\ \|v\|_{1,n} \leq 1,\ g(x,u)v \in L^1(\Omega)\right\} < +\infty.$$

Remark 3.4. If (1.2) holds, then taken $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$ by Proposition 3.2 it follows that

$$\sup\left\{\int_\Omega g(x,u)v\,dx:\ v \in W_0^{1,n}(\Omega),\ \|v\|_{1,n} \leq 1,\ g(x,u)v \in L^1(\Omega)\right\}$$
$$= \sup\left\{\int_\Omega g(x,u)v\,dx:\ v \in V_u,\ \|v\|_{1,n} \leq 1\right\}.$$

Hence, if $g(x,u) \in W^{-1,n'}(\Omega)$, it results that

$$v \in V_u \mapsto \int_\Omega g(x,u)v \in \mathbb{R}$$

is a linear continuous function which has a unique linear continuous extension on $W_0^{1,n}(\Omega)$. Clearly, such a map is in the (classical) dual space of $W_0^{1,n}(\Omega)$ and it can be still named $g(x,u)$. At last, arguing as in [5, Theorem 2.5] it is $g(x,u) = 0$ a.e. in Ω if and only if $g(x,u) = 0$ in $W^{-1,n'}(\Omega)$.

By Remark 3.4 it follows that the set $W^{-1,n'}(\Omega)$ in Definition 3.3 is the classical dual space of $W_0^{1,n}(\Omega)$, so it is endowed by its norm $\|\cdot\|_{-1,n'}$ while $\langle\cdot,\cdot\rangle$ is the scalar product in the duality $W^{-1,n'}(\Omega)$, $W_0^{1,n}(\Omega)$.

As in [5, Theorem 2.8], we have:

Proposition 3.5. *Assume that* (1.2) *holds and let* $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$ *be such that* $g(x,u) \in W^{-1,n'}(\Omega)$. *If* $v \in W_0^{1,n}(\Omega)$ *is such that* $(g(x,u)v)^+ \in L^1(\Omega)$ *then* $g(x,u)v \in L^1(\Omega)$.

In order to apply Propositions 3.2 and 3.5, we need some a priori estimates which allow to prove that $(g(x,u)v)^+ \in L^1(\Omega)$ for a function $v \in W_0^{1,n}(\Omega)$. To this aim, we need hypothesis (1.5) and the following Moser-Trudinger inequality (for more details, cf. [10, 11]).

Theorem 3.6. *For every $\alpha > 0$ we have*

$$e^{\alpha|u|^{\frac{n}{n-1}}} \in L^1(\Omega) \quad \textit{for every } u \in W_0^{1,n}(\Omega).$$

Moreover, there exists a constant $c_{TM} > 0$ such that for every $0 < \alpha \le n\omega_{n-1}^{\frac{1}{n-1}}$ we have

$$u \in W_0^{1,n}(\Omega),\ \int_\Omega |Du|^n\,dx \le 1 \implies \int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}}\,dx \le c_{TM}|\Omega|,$$

where ω_{n-1} is the $(n-1)$-dimensional surface of the unit sphere $\mathbb{R}^n$ and $|\Omega|$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^n$.

Since useful in the following, first of all, we point out an easy consequence of the Trudinger-Moser inequality.

Lemma 3.7. *If $(v_h)_h \subset W_0^{1,n}(\Omega)$ is a bounded sequence, there exists $\varepsilon_0 > 0$ such that*

$$\int_\Omega e^{\varepsilon|v_h|^{\frac{n}{n-1}}}\,dx \le (c_{TM}+1)|\Omega| \quad \textit{for every } h \in \mathbb{N} \textit{ and } \varepsilon \in [0,\varepsilon_0].$$

Proof. Since $(v_h)_h$ is bounded in $W_0^{1,n}(\Omega)$ there exists $\varepsilon_0 > 0$ so small that

$$\varepsilon_0 \|v_h\|_{1,n}^{\frac{n}{n-1}} \le n\omega_{n-1}^{\frac{1}{n-1}} \quad \text{for every } h \in \mathbb{N},$$

with ω_{n-1} as in Theorem 3.6. Then, taken any $0 \le \varepsilon \le \varepsilon_0$, it is either

$$\|v_h\|_{1,n} = 0 \implies \int_\Omega e^{\varepsilon|v_h|^{\frac{n}{n-1}}}\,dx = |\Omega|$$

or $\|v_h\|_{1,n} \neq 0$ which implies by Theorem 3.6 that

$$\begin{aligned}\int_\Omega e^{\varepsilon|v_h|^{\frac{n}{n-1}}}\,dx &= \int_\Omega e^{\varepsilon\|v_h\|_{1,n}^{\frac{n}{n-1}}\left|\frac{v_h}{\|v_h\|_{1,n}}\right|^{\frac{n}{n-1}}}\,dx \le \int_\Omega e^{\varepsilon_0\|v_h\|_{1,n}^{\frac{n}{n-1}}\left|\frac{v_h}{\|v_h\|_{1,n}}\right|^{\frac{n}{n-1}}}\,dx \\ &\le \int_\Omega e^{n\omega_{n-1}^{\frac{1}{n-1}}\left|\frac{v_h}{\|v_h\|_{1,n}}\right|^{\frac{n}{n-1}}}\,dx \le c_{TM}|\Omega|,\end{aligned}$$

which concludes the proof. $\square$

Moreover, by virtue of Proposition 3.5, if hypothesis (1.5) holds, Theorem 3.6 implies the following corollaries.

Corollary 3.8. *Suppose that* (1.2) *and* (1.5) *hold. If* $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$ *is such that* $g(x,u) \in W^{-1,n'}(\Omega)$, *then it is* $g(x,u) \in L^1_{\mathrm{loc}}(\Omega)$ *and* $g(x,u)u \in L^1_{\mathrm{loc}}(\Omega)$.

Corollary 3.9. *Suppose that* (1.2) *and* (1.5) *hold. Assume* $u \in W^{1,n}_0(\Omega)$ *is such that* $g(x,u) \in W^{-1,n'}(\Omega)$. *Then* $g(x,u)u \in L^1(\Omega)$.

Whence, the previous Definition 3.3 allows also to redefine the concept of weak solution (for more details, see [6]).

Lemma 3.10. *If* (1.2) *and* (1.5) *hold, a function* $u \in W^{1,n}_0(\Omega)$ *is a weak solution of problem* (P) *if* $g(x,u) \in W^{-1,n'}(\Omega)$ *and the equation in* (P) *is satisfied in* $W^{-1,n'}(\Omega)$.

Now, let us define the functional $F : W^{1,n}_0(\Omega) \to \overline{\mathbb{R}}$ such that

$$F(u) = \frac{1}{n}\int_\Omega |Du|^n\,dx - \int_\Omega G(x,u)\,dx.$$

Notice that, if (1.10) holds, then Theorem 3.6 implies

$$\int_\Omega G(x,u)^+\,dx < +\infty, \quad \text{whence, } F(u) \in \mathbb{R}\cup\{+\infty\} \text{ for every } u \in W^{1,n}_0(\Omega).$$

Furthermore, let

$$\psi(u) = \frac{1}{n}\int_\Omega |Du|^n\,dx \quad \text{and} \quad f(u) = -\int_\Omega G(x,u)\,dx.$$

It is well known that ψ is a C^1 functional with

$$\psi'(u)[v] = \int_\Omega |Du|^{n-2}Du\cdot Dv\,dx \quad \text{for every } u,v \in W^{1,n}_0(\Omega).$$

On the contrary, in general, f is not smooth, so that also F is not smooth in $W^{1,n}_0(\Omega)$.

Proposition 3.11. *If* (1.10) *holds, then* f *is lower semicontinuous in* $W^{1,n}_0(\Omega)$. *If, furthermore,* (1.9) *holds with* $\beta = 0$ *then* f *is finite and continuous in* $W^{1,n}_0(\Omega)$.

Proof. It is enough to prove that fixed any $u \in W_0^{1,n}(\Omega)$ and $(u_h)_h \subset W_0^{1,n}(\Omega)$ such that $u_h \to u$ in $W_0^{1,n}(\Omega)$ then

$$\limsup_h \int_\Omega G(x,u_h)\,dx \le \int_\Omega G(x,u)\,dx. \tag{3.11}$$

By (1.10), fixed any $\varepsilon > 0$, for a suitable $a_\varepsilon \in L^1(\Omega)$ it is

$$G(x,u_h) - \varepsilon e^{\varepsilon |u_h|^{\frac{n}{n-1}}} \le a_\varepsilon(x) \quad \text{for a.e. } x \in \Omega \text{ and every } h \in \mathbb{N}.$$

Therefore, by Fatou's Lemma, it is

$$\limsup_h \int_\Omega \left(G(x,u_h) - \varepsilon e^{\varepsilon |u_h|^{\frac{n}{n-1}}} \right) dx \le \int_\Omega \left(G(x,u) - \varepsilon e^{\varepsilon |u|^{\frac{n}{n-1}}} \right) dx,$$

which yields

$$\limsup_h \int_\Omega G(x,u_h)\,dx \le \int_\Omega G(x,u)\,dx + \varepsilon \limsup_h \int_\Omega e^{\varepsilon |u_h|^{\frac{n}{n-1}}}\,dx. \tag{3.12}$$

On the other hand, the converging sequence $(u_h)_h$ has to be bounded in $W_0^{1,n}(\Omega)$, so Lemma 3.7 applies and, if $\varepsilon \to 0$, inequality (3.11) follows from (3.12). Assume now that it is $\beta = 0$ in (1.9). Thus, (1.9), (1.10) and Theorem 3.6 imply the finiteness of f in $W_0^{1,n}(\Omega)$. Furthermore, we claim that if $u_h \to u$ in $W_0^{1,n}(\Omega)$ then not only (3.11) holds but also

$$\int_\Omega G(x,u)\,dx \le \liminf_h \int_\Omega G(x,u_h)\,dx.$$

In fact, by (1.9) and Fatou's Lemma for every $\varepsilon > 0$ it is

$$\int_\Omega G(x,u) \le \liminf_h \int_\Omega G(x,u_h)\,dx + \varepsilon \limsup_h \int_\Omega e^{\varepsilon |u_h|^{\frac{n}{n-1}}}\,dx.$$

Hence, the required inequality follows from Lemma 3.7 if $\varepsilon \to 0$. □

Corollary 3.12. *If condition* (1.10) *holds, then the functional F is lower semicontinuous in $W_0^{1,n}(\Omega)$. Furthermore, if also* (1.9) *holds with $\beta = 0$ then F is finite and continuous in $W_0^{1,n}(\Omega)$.*

Now, we want to state a suitable variational principle which allows to reduce our problem to the study of (lower) critical points of functional F in $W_0^{1,n}(\Omega)$. To this aim, we remark that, arguing as in [6, Theorem 3.1] with $-G$ in the place of G, the following results can be proved.

Proposition 3.13. *Assume that* (1.2) *holds. If* $u \in W_0^{1,n}(\Omega)$ *is such that* $f(u) \in \mathbb{R}$ *and* $\partial f(u) \neq \emptyset$*, then* $g(x,u) \in W^{-1,n'}(\Omega)$ *and* $\partial f(u) = \{-g(x,u)\}$.

Whence, by Lemma 3.10 and Propositions 2.8 and 3.13, we have the following Variational Principle.

Theorem 3.14.. *Assume* (1.2) *and* (1.5) *hold. If* $u \in W_0^{1,n}(\Omega)$ *is a (lower) critical point of* F *then it is a weak solution of the given problem* (P).

Thus, we want to look for (lower) critical points of F in $W_0^{1,n}(\Omega)$ by means of the abstract Theorem 2.9. In order to study the Palais-Smale condition let us point out that its direct proof is not easy to manage. So, we introduce the following auxiliary definition.

Definition 3.15. *Taken* $c \in \mathbb{R}$*, we say that* F *satisfies the concrete Palais-Smale condition at level* c*, briefly* $(CPS)_c$*, if every sequence* $(u_h)_h \subset W_0^{1,n}(\Omega)$ *such that*

$$g(x,u_h) \in W^{-1,n'}(\Omega) \quad \text{for every } h \in \mathbb{N}, \tag{3.13}$$

$$F(u_h) \to c, \quad -\Delta_n u_h - g(x,u_h) \to 0 \ \text{ in } W^{-1,n'}(\Omega) \text{ as } h \to +\infty \tag{3.14}$$

admits a converging subsequence in $W_0^{1,n}(\Omega)$.

It is quite easy to see that Propositions 2.8 and 3.13 imply the following result.

Proposition 3.16. *Assume that* (1.2) *holds. Then, fixed any* $c \in \mathbb{R}$*,* F *satisfies* $(PS)_c$ *in* $W_0^{1,n}(\Omega)$ *if it satisfies* $(CPS)_c$.

Proposition 3.17. *If* (1.2), (1.3), (1.4), (1.7) *and* (1.8) *hold, then for every* $c \in \mathbb{R}$ *the functional* F *satisfies the* $(CPS)_c$ *condition.*

Proof. Let $c \in \mathbb{R}$ and let $(u_h)_h \subset W_0^{1,n}(\Omega)$ be a sequence such that (3.13) and (3.14) hold. We set $w_h = -\Delta_n u_h - g(x,u_h)$, so that $w_h \to 0$ in $W^{-1,n'}(\Omega)$. We divide the proof into two steps.

<u>Step I.</u> First of all, let us prove that $(u_h)_h$ is bounded in $W_0^{1,n}(\Omega)$. Being $\theta > n$, let $0 < \delta < \dfrac{\theta - n}{2n}$ and define the continuous cut-function $\gamma : \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(s) = \begin{cases} s & \text{if } |s| \leq R \\ -\delta s + \delta R + R & \text{if } R < s \leq R + \dfrac{R}{\delta} \\ -\delta s - \delta R - R & \text{if } -R - \dfrac{R}{\delta} \leq s < -R \\ 0 & \text{if } |s| > R + \dfrac{R}{\delta}. \end{cases}$$

By applying (1.8) with $\varepsilon = 1$, (1.3) and (1.7) imply

$$g(x, u_h)\gamma(u_h) \le a_1(x) + \bar{b}e^{\left(R+\frac{R}{\delta}\right)^{\frac{n}{n-1}}} \quad \text{for a.e. } x \in \Omega \text{ and every } h \in \mathbb{N}$$

for some $a_1 \in L^1(\Omega)$ and $\bar{b} \in \mathbb{R}$. Then, by Theorem 3.6 and Proposition 3.5, for every $h \in \mathbb{N}$ it follows $g(x, u_h)\gamma(u_h) \in L^1(\Omega)$ and (3.14) and Young's inequality imply the existence of a constant C_1 such that

$$\begin{aligned}
&\int_{\{|u_h|\le R\}} |Du_h|^n \, dx - \delta \int_{\{R\le|u_h|\le R+\frac{R}{\delta}\}} |Du_h|^n \, dx \\
&\qquad = \int_{\{|u_h|\le R\}} g(x, u_h)u_h \, dx + \int_{\{R\le|u_h|\le R+\frac{R}{\delta}\}} g(x, u_h)\gamma(u_h) \, dx + \langle w_h, \gamma(u_h)\rangle \\
&\qquad \le \int_{\{|u_h|\le R\}} g(x, u_h)u_h \, dx + C_1 + \|w_h\|_{-1,n'} \|\gamma(u_h)\|_{1,n} \\
&\qquad \le \int_{\{|u_h|\le R\}} g(x, u_h)u_h \, dx + C_1 + d_\varepsilon \|w_h\|_{-1,n'}^{\frac{n}{n-1}} \\
&\qquad + \varepsilon \left(\int_{\{|u_h|\le R\}} |Du_h|^n \, dx + \delta^n \int_{\{R\le|u_h|\le R+\frac{R}{\delta}\}} |Du_h|^n \, dx \right).
\end{aligned}$$

Hence, fixed ε sufficiently small, there exists a positive constant C_2 (independent of h) such that

$$\int_{\{|u_h|\le R\}} g(x, u_h)u_h \, dx \ge -C_2 - 2\delta \int_{\{|u_h|>R\}} |Du_h|^n \, dx \ge -C_2 - 2\delta \int_\Omega |Du_h|^n \, dx.$$

Notice also that, by virtue of (1.3) and (1.4), it is

$$\begin{aligned}
\int_{\{|u_h|>R\}} g(x, u_h)u_h \, dx &\ge \int_{\{|u_h|>R\}} \theta G(x, u_h) \, dx \\
&\ge \theta \int_\Omega G(x, u_h) \, dx - \theta \int_\Omega |a(x)| \, dx \\
&= \frac{\theta}{n} \int_\Omega |Du_h|^n \, dx - \theta c - \theta \int_\Omega |a(x)| \, dx + o(1)
\end{aligned}$$

where $o(1) \to 0$ as $h \to +\infty$. Thus, by the previous inequalities it follows

$$\int_\Omega g(x,u_h)u_h\,dx \geq \left(\frac{\theta}{n} - 2\delta\right)\|u_h\|_{1,n}^n - C_3 \quad \text{for every } h \in \mathbb{N}$$

for a suitable constant $C_3 > 0$. On the other hand, (3.14) implies

$$\int_\Omega |Du_h|^n\,dx = \int_\Omega g(x,u_h)u_h\,dx + \langle w_h, u_h\rangle; \tag{3.15}$$

whence,

$$\left(\frac{\theta}{n} - 2\delta - 1\right)\|u_h\|_{1,n}^n \leq C_3 + \|w_h\|_{-1,n'}\|u_h\|_{1,n},$$

which implies that $(u_h)_h$ is bounded in $W_0^{1,n}(\Omega)$ for the chosen δ.

STEP II. By Step I, up to a subsequence, $(u_h)_h$ converges to some u weakly in $W_0^{1,n}(\Omega)$. First of all, by (1.8) and arguing as in the proof of Proposition 3.11 it is

$$\limsup_h \int_\Omega g(x,u_h)u_h\,dx \leq \int_\Omega g(x,u)u\,dx, \tag{3.16}$$

where (3.13) and Corollary 3.9 imply $g(x,u_h)u_h \in L^1(\Omega)$. Now, fixed $\rho > 0$, let us consider a test function $\vartheta_\rho \in C_c^\infty(\mathbb{R})$ such that its support is in $[-2\rho, 2\rho]$, $\vartheta_\rho = 1$ in $[-\rho,\rho]$ and $|\vartheta_\rho'(s)| \leq 2/\rho$ for every $s \in \mathbb{R}$. Then, taken any $h \in \mathbb{N}$ for every $v \in V_u$ we have

$$\int_\Omega \vartheta_\rho(u_h)|Du_h|^{n-2}Du_h \cdot Dv\,dx + \int_\Omega v|Du_h|^{n-2}Du_h \cdot D\vartheta_\rho(u_h)\,dx$$
$$- \int_\Omega g(x,u_h)\vartheta_\rho(u_h)v\,dx = \langle w_h, \vartheta_\rho(u_h)v\rangle.$$

Clearly, there exists $C > 0$ (independent of v, h and ρ) such that

$$\left|\int_\Omega v|Du_h|^{n-2}Du_h \cdot D\vartheta_\rho(u_h)\,dx\right| \leq \frac{2}{\rho}\|v\|_\infty\|u_h\|_{1,n}^n \leq \frac{C}{\rho}\|v\|_\infty.$$

Therefore, if $h \to +\infty$ and $\rho \to +\infty$, by the previous equality it follows that

$$\int_\Omega |Du|^{n-2}Du \cdot Dv\,dx = \int_\Omega g(x,u)v\,dx \quad \text{for every } v \in V_u.$$

This last result and Remark 3.4 imply that $g(x,u) \in W^{-1,n'}(\Omega)$; hence, by Corollary 3.9 it follows $g(x,u)u \in L^1(\Omega)$ while (c) of Proposition 3.2 gives

$$\int_\Omega |Du|^n \, dx = \int_\Omega g(x,u)u \, dx. \tag{3.17}$$

By combining (3.15), (3.16) and (3.17), it results

$$\begin{aligned}\limsup_h \int_\Omega |Du_h|^n \, dx &= \limsup_h \int_\Omega g(x,u_h)u_h \, dx + \lim_h \langle w_h, u_h\rangle \\ &\le \int_\Omega g(x,u)u \, dx = \int_\Omega |Du|^n \, dx,\end{aligned}$$

namely $u_h \to u$ strongly in $W_0^{1,n}(\Omega)$. □

At last, some information about the $(\mathrm{epi})_c$ condition and the equivariant weak slope of f can be stated. To this aim, the following lemma needs.

Lemma 3.18. *Suppose* (1.9) *is satisfied with* $\beta > 0$ *and* (1.10) *holds. If* $u \in W_0^{1,n}(\Omega)$ *is such that* $g(x,u)u \in L^1(\Omega)$, *then* $f(u) \in \mathbb{R}$ *and for every* $\lambda > f(u)$ *there exist* δ, $\sigma > 0$ *such that*

$$\int_0^1 \int_\Omega g(x, w+\tau z)(w+\tau z) \, dx d\tau < \int_\Omega g(x,u)u \, dx - \sigma$$

for any w, $z \in W_0^{1,n}(\Omega)$ *such that* $\|w-u\|_{1,n} < \delta$, $\|z\|_{1,n} < \delta$ *and*

$$\int_0^1 \int_\Omega G(x, w+\tau z) \, dx d\tau < -\lambda + \delta.$$

Proof. Since $g(x,u)u \in L^1(\Omega)$, (1.9) and (1.10) imply $G(x,u) \in L^1(\Omega)$; whence, $f(u) \in \mathbb{R}$. On the other hand, arguing by contradiction, there exist two sequences $(w_h)_h$, $(z_h)_h$ in $W_0^{1,n}(\Omega)$ such that $w_h \to u$ and $z_h \to 0$ strongly in $W_0^{1,n}(\Omega)$ while

$$\limsup_h \int_0^1 \int_\Omega G(x, w_h + \tau z_h) \, dx d\tau \le -\lambda, \tag{3.18}$$

$$\liminf_h \int_0^1 \int_\Omega g(x, w_h+\tau z_h)(w_h+\tau z_h) \, dx d\tau \ge \int_\Omega g(x,u)u \, dx, \tag{3.19}$$

where being $\beta > 0$ in (1.9), by (1.10) and Theorem 3.6 it follows

$$\int_0^1 \int_\Omega g(x, w_h + \tau z_h)(w_h + \tau z_h)\, dx d\tau < +\infty \quad \text{for every } h \in \mathbb{N}.$$

Thus, simple calculations, (1.9) and Fatou's Lemma allow to prove that

$$\int_\Omega (G(x,u) - \beta g(x,u)u)\, dx \le \varepsilon \limsup_h \int_0^1 \int_\Omega e^{\varepsilon |w_h + \tau z_h|^{\frac{n}{n-1}}}\, dx d\tau$$
$$+ \liminf_h \int_0^1 \int_\Omega (G(x, w_h + \tau z_h) - \beta g(x, w_h + \tau z_h)(w_h + \tau z_h))\, dx d\tau,$$

where, since $(\|w_h + \tau z_h\|_{1,n})_h$ is bounded uniformly with respect to $\tau \in [0,1]$, by Lemma 3.7 it is

$$\varepsilon \limsup_h \int_0^1 \int_\Omega e^{\varepsilon |w_h + \tau z_h|^{\frac{n}{n-1}}}\, dx d\tau \to 0 \quad \text{if } \varepsilon \to 0.$$

Hence, (3.18) and (3.19) give

$$\int_\Omega G(x,u)\, dx \le \beta \int_\Omega g(x,u)u\, dx$$
$$+ \liminf_h \int_0^1 \int_\Omega (G(x, w_h + \tau z_h) - \beta g(x, w_h + \tau z_h)(w_h + \tau z_h))\, dx d\tau$$
$$\le \beta \int_\Omega g(x,u)u\, dx + \limsup_h \int_0^1 \int_\Omega G(x, w_h + \tau z_h)\, dx d\tau$$
$$- \beta \liminf_h \int_0^1 \int_\Omega g(x, w_h + \tau z_h)(w_h + \tau z_h)\, dx d\tau \le -\lambda$$

in contradiction with $f(u) < \lambda$. □

Proposition 3.19. *Assume that hypotheses* (1.2), (1.9) *and* (1.10) *hold. Then for every* $(u, \lambda) \in \mathrm{epi}(f)$ *such that* $f(u) < \lambda$ *it is* $|dG_f|(u, \lambda) = 1$. *Furthermore, if also (1.7) holds then for every* $\lambda > f(0)$ *it is* $|d_{\mathbb{Z}_2} G_f|(0, \lambda) = 1$.

Proof. The proof can be obtained by arguing as in [6, Theorem 3.4], with $-G$ in place of G. Anyway, for completeness, in the Appendix (Section 5) we outline the main differences between the two proofs. □

4. Proof of the main result. Assume that all the hypotheses of Theorem 1.2 are satisfied. Clearly, Remarks 1.3 and 1.4 imply that (1.8), (1.9), (1.10) hold too. We already know that $F : W_0^{1,n}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is an even lower semicontinuous function (continuous and finite if $\beta = 0$) which satisfies $(PS)_c$ for every $c \in \mathbb{R}$ (see (1.7), Corollary 3.12 and Propositions 3.16, 3.17). Moreover, as (1.6) and (1.10) imply $f(0) \in \mathbb{R}$, by Propositions 2.6, 3.19 it follows that F satisfies $(\mathrm{epi})_c$ for every $c \in \mathbb{R}$ and that $|d_{\mathbb{Z}_2} G_F|(0, \lambda) = 1$ for every $\lambda > f(0)$. Thus, in order to apply Theorem 2.9 we only need to prove the geometric assumptions (a) and (b).

Firstly, let us recall the following useful result (see [9, Lemma 3.8]):

Proposition 4.1. *There exist a strictly increasing sequence $(W_h)_h$ of finite dimensional subspaces of $W_0^{1,n}(\Omega) \cap L^\infty(\Omega)$ and a strictly decreasing sequence $(Z_h)_h$ of closed subspaces of $L^n(\Omega)$ such that*

$$L^n(\Omega) = W_h \oplus Z_h \ \textit{for every } h \in \mathbb{N}, \qquad \bigcap_{h=0}^{\infty} Z_h = \{0\}.$$

Fixed $h \in \mathbb{N}$, let W_h and Z_h be the subspaces given by Proposition 4.1. We have

$$W_0^{1,n}(\Omega) = W_h \oplus \widetilde{Z}_h, \quad \text{where } \widetilde{Z}_h = Z_h \cap W_0^{1,n}(\Omega) \text{ for every } h \in \mathbb{N}.$$

Let $\varepsilon > 0$ and define the maps $G_\varepsilon^{(1)}, G_\varepsilon^{(2)} : \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$G_\varepsilon^{(1)}(x,s) = \min\{G(x,s), a_\varepsilon(x)\} \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

$$G_\varepsilon^{(2)}(x,s) = G(x,s) - G_\varepsilon^{(1)}(x,s) \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where $a_\varepsilon \in L^1(\Omega)$ is the function given by the inequality (1.10) (without loss of generality, we may assume that $a_\varepsilon(x) \geq 0$ for a.e. $x \in \Omega$). Then, it results $G_\varepsilon^{(1)}(x,0) = G_\varepsilon^{(2)}(x,0) = 0$ for a.e. $x \in \Omega$. Moreover, we have

$$G_\varepsilon^{(1)}(x,s) \leq a_\varepsilon(x), \quad G_\varepsilon^{(2)}(x,s) \leq \varepsilon e^{\varepsilon |s|^{\frac{n}{n-1}}} \qquad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Thus, by Theorem 3.6, for every $\varrho > 0$ there exist $\varepsilon > 0$ and $\alpha > 0$ such that

$$u \in W_0^{1,n}(\Omega),\ \|u\|_{1,n} = \varrho \implies \frac{1}{n}\int_\Omega |Du|^n \, dx - \int_\Omega G_\varepsilon^{(2)}(x,u)\, dx > 2\alpha. \tag{4.20}$$

Then, it follows that

$$\liminf_h \{\inf\{F(v) : \ v \in \widetilde{Z}_h,\ \|v\|_{1,n} = \varrho\}\} \geq \alpha.$$

Indeed, assume by contradiction that there exists a sequence $v_j \in \widetilde{Z}_{h_j}$ such that

$$\|v_j\|_{1,n} = \varrho, \qquad \limsup_j F(v_j) \leq \alpha.$$

Up to a subsequence we have $v_j \rightharpoonup z$ in $W_0^{1,n}(\Omega)$ and $v_j \to z$ strongly in $L^n(\Omega)$. Since

$$z \in \bigcap_{h=0}^{\infty} \widetilde{Z}_h \subset \bigcap_{h=0}^{\infty} Z_h,$$

it follows that $z = 0$ and Fatou's Lemma implies that

$$\limsup_j \int_\Omega G_\varepsilon^{(1)}(x, v_j)\, dx \leq 0.$$

Therefore,

$$\limsup_j \left\{ \frac{1}{n} \int_\Omega |Dv_j|^n\, dx - \int_\Omega G_\varepsilon^{(2)}(x, v_j)\, dx \right\} \leq \alpha,$$

which contradicts (4.20). Thus, assumption (a) of Theorem 2.9 is satisfied with $Z = \widetilde{Z}_{\bar{h}}$ for $\bar{h}$ large enough. Now, let $R > 0$ be as in (1.3). It results

$$G(x,s) \geq R^{-\theta} G(x,R)|s|^\theta - G(x,R) - |s| \sup_{|t|\leq R} |g(x,t)| \quad \text{for } s \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Notice also that $G(x,R) \in L^1(\Omega)$; so we can define a norm $\|\cdot\|_G$ by setting

$$\|u\|_G = \left(\int_\Omega G(x,R)|u|^\theta\, dx \right)^{1/\theta}.$$

Fixed any $h \in \mathbb{N}$, since W_h is finite dimensional, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\forall u \in W_h : \quad F(u) \leq C_1 \|u\|_G^n - R^{-\theta} \|u\|_G^\theta + C_2 \|u\|_G.$$

Since $\theta > n$, it results that

$$\lim_{\substack{\|u\|_G \to \infty \\ u \in W_h}} F(u) = -\infty,$$

which implies that condition (b) of Theorem 2.9 holds with $V_h = W_{\bar{h}+h}$.

Taking into account Theorem 3.14, the proof of Theorem 1.2 is complete. □

5. Appendix: proof of Proposition 3.19. Take $(u, \lambda) \in \mathrm{epi}(f)$ such that $f(u) < \lambda$. If $\beta = 0$, by Proposition 3.11 f is finite and continuous in $W_0^{1,n}(\Omega)$, so [4, Proposition 2.3] applies. Now, let $\beta > 0$. Thus, (1.9) and (1.10) imply (1.8). As a first step, let us suppose $g(x,u)u \in L^1(\Omega)$; hence, Lemma 3.18 holds and we can consider the corresponding constants δ, $\sigma > 0$. Fixed $\varepsilon > 0$ with $\varepsilon \le \delta$, by (c) of Proposition 3.2 there exist $v \in V_u$ and $\rho > 0$ large enough (i.e., $\rho > \|u\|_{L^\infty(\{v \neq 0\})}$) such that $|v| \le |u|$ a.e. in Ω, $\|\vartheta_\rho(u)v - u\|_{1,n} < \varepsilon$ and

$$\int_\Omega g(x,u)\vartheta_\rho(u)v\,dx > \int_\Omega g(x,u)u\,dx - \frac{\sigma}{4}, \tag{5.21}$$

where $\vartheta_\rho \in C_c^\infty(\mathbb{R})$ has the support in $[-2\rho, 2\rho]$, $\vartheta_\rho = 1$ in $[-\rho, \rho]$ and $|\vartheta'_\rho(s)| \le 2/\rho$ for $s \in \mathbb{R}$ (whence, $\vartheta_\rho(u)v = v$ a.e. in Ω). Take $\delta_1 \in]0, \delta]$ such that $\delta_1 < 1$ and, for simplicity, if $w \in B_{\delta_1}(u)$ and $t \in [0, \delta_1]$, define $z = t(\vartheta_\rho(w)v - w)$. Since $\|\vartheta_\rho(w)v - \vartheta_\rho(u)v\|_{1,n} \to 0$ if $w \to u$ in $W_0^{1,n}(\Omega)$, δ_1 can be chosen so that $\|\vartheta_\rho(w)v - w\|_{1,n} < \varepsilon$ for every $w \in B_{\delta_1}(u)$. Whence, if $t \in [0, \delta_1]$ it is $\|z\|_{1,n} < \varepsilon$, too. Moreover, let us remark that

$$\begin{aligned}\frac{f(w) - f(w+z)}{t} &= \int_0^1 \int_\Omega \frac{1}{1-\tau t} g(x, w+\tau z)\vartheta_\rho(w)v\,dx d\tau \\ &\quad - \int_0^1 \int_\Omega \frac{\tau t}{1-\tau t} g(x, w+\tau z)(w+\tau z)\,dx d\tau \\ &\quad - \int_0^1 \int_\Omega g(x, w+\tau z)(w+\tau z)\,dx d\tau.\end{aligned}$$

We claim that if δ_1 is small enough then

$$\int_0^1 \int_\Omega \frac{1}{1-\tau t} g(x, w+\tau z)\vartheta_\rho(w)v\,dx d\tau > \int_\Omega g(x,u)u\,dx - \frac{\sigma}{4}, \tag{5.22}$$

$$\int_0^1 \int_\Omega \frac{\tau t}{1-\tau t} g(x, w+\tau z)(w+\tau z)\,dx d\tau < \frac{\sigma}{4}, \tag{5.23}$$

$$\int_0^1 \int_\Omega g(x, w+\tau z)(w+\tau z)\,dx d\tau < \int_\Omega g(x,u)u\,dx + \frac{\sigma}{2}. \tag{5.24}$$

In fact, arguing by contradiction, suppose that at least one of the previous inequalities is not true. Then, there exist $(w_h)_h$ and $(t_h)_h$ such that $w_h \to u$ in $W_0^{1,n}(\Omega)$, $t_h \to 0^+$ and, if (5.22) does not hold, it is

$$\int_0^1 \int_\Omega \frac{1}{1-\tau t_h} g(x, w_h + \tau z_h)\vartheta_\rho(w_h)v \, dx d\tau \leq \int_\Omega g(x,u)u \, dx - \frac{\sigma}{4}$$

for every $h \in \mathbb{N}$, with $z_h = t_h(\vartheta_\rho(w_h)v - w_h)$. Thus, up to subsequences, it is

$$\frac{1}{1-\tau t_h} g(x, w_h + \tau z_h)\vartheta_\rho(w_h)v \to g(x,u)\vartheta_\rho(u)v \quad \text{for every } \tau \in [0,1] \text{ and a.e. } x \in \Omega,$$

so (1.2) and Lebesgue Theorem imply

$$\int_\Omega g(x,u)\vartheta_\rho(u)v \, dx \leq \int_\Omega g(x,u)u \, dx - \frac{\sigma}{4}$$

in contradiction with (5.21). On the contrary, if (5.23) does not hold it is

$$\int_0^1 \int_\Omega \frac{\tau t_h}{1-\tau t_h} g(x, w_h + \tau z_h)(w_h + \tau z_h) \, dx d\tau \geq \frac{\sigma}{4}$$

for every $h \in \mathbb{N}$. But this is impossible since, up to subsequences, it is

$$\frac{\tau t_h}{1-\tau t_h} g(x, w_h + \tau z_h)(w_h + \tau z_h) \to 0 \quad \text{for every } \tau \in [0,1] \text{ and a.e. } x \in \Omega,$$

and (1.8), Fatou's Lemma and Lemma 3.7 imply

$$\limsup_h \int_0^1 \int_\Omega \frac{\tau t_h}{1-\tau t_h} g(x, w_h + \tau z_h)(w_h + \tau z_h) \, dx d\tau = 0.$$

At last, if (5.24) does not hold it is

$$\int_0^1 \int_\Omega g(x, w_h + \tau z_h)(w_h + \tau z_h) \, dx d\tau \geq \int_\Omega g(x,u)u \, dx + \frac{\sigma}{2}$$

for every $h \in \mathbb{N}$. But this cannot be true since, up to subsequences, it is

$$g(x, w_h + \tau z_h)(w_h + \tau z_h) \to g(x,u)u \quad \text{for every } \tau \in [0,1] \text{ and a.e. } x \in \Omega,$$

so by (1.8), Fatou's Lemma and Lemma 3.7 it follows

$$\limsup_h \int_0^1 \int_\Omega g(x, w_h + \tau z_h)(w_h + \tau z_h) \, dx d\tau \leq \int_\Omega g(x,u)u \, dx.$$

Now, if $((w,\mu),t) \in (B_{\delta_1}(u,\lambda) \cap \mathrm{epi}(f)) \times [0,\delta_1]$ let us define

$$H((w,\mu),t) = (H_1(w,t), \mu - \frac{\sigma}{2}t), \quad \text{with } H_1(w,t) = w + t(\vartheta_\rho(w)v - w).$$

Arguing as in the proof of [6, Theorem 3.4], Lemma 3.18 and the previous inequalities imply $H((w,\mu),t) \in \mathrm{epi}(f)$; whence, $|dG_f|(u,\lambda) = 1$. At last, similar arguments and the comments in the proof of [6, Theorem 3.4] allow to complete the proof in the case $g(x,u)u \notin L^1(\Omega)$ and to prove the second assertion when (1.7) holds. $\square$

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