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Multiple solutions for semilinear elliptic systems with non-homogeneous boundary conditions

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1. Introduction

Since the early seventies, many authors have widely investigated existence and multiplicity of solutions for semilinear elliptic problems with Dirichlet boundary conditions, especially by means of variational methods (see [22] and references therein). In particular, if φ is a real L^2-function on a bounded domain $\Omega \subset \mathbb{R}^n$, $p > 2$ and $p < 2^*$ if $n \geqslant 3$ (here, $2^* = \frac{2n}{n-2}$), the following model problem

$$\begin{cases} -\Delta u = |u|^{p-2}u + \varphi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{P}_{0,\varphi,1}$}$$

has been extensively studied, even when the nonlinear term is more general.

If $\varphi \equiv 0$, the problem is symmetric, so multiplicity results have been achieved via the equivariant Lusternik-Schnirelman theory and the notion of genus for $\mathbb{Z}_2$-symmetric sets (see [19] and references therein).

On the contrary, if $\varphi \not\equiv 0$ the problem loses its $\mathbb{Z}_2$-symmetry and a natural question is whether the infinite number of solutions persists under perturbation of the odd equation.

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In this case, a detailed analysis was carried on by Rabinowitz [18], Struwe [21], Bahri and Berestycki [2], Dong and Li [12] and Tanaka [23]: the existence of infinitely many solutions was obtained via techniques of classical critical point theory provided that a suitable restriction on the growth of the exponent p is assumed.

Furthermore, Bahri and Lions have improved some such results via a technique based on Morse theory (see [3,4]); while, more recently, Paleari and Squassina have extended some of the above mentioned achievements to the quasilinear case by means of techniques of non-smooth critical point theory (see [17]).

Other perturbation results were obtained by Ambrosetti [1] and by Bahri and Berestycki [2] when $p>2$ is any but subcritical: in particular, they proved that for each $v\in\mathbb{N}$ there exists $\varepsilon>0$ such that $(\mathscr{P}_{0,\varphi,1})$ has at least v distinct solutions provided that $\|\varphi\|_2<\varepsilon$.

The success in looking for solutions of a non-symmetric problem as $(\mathscr{P}_{0,\varphi,1})$ made quite interesting to study the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u+\varphi & \text{in } \Omega,\\ u=\chi & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{P}_{\chi,\varphi,1}$}$$

where, in general, the boundary condition χ is different from zero. Some multiplicity results for $(\mathscr{P}_{\chi,\varphi,1})$ have been proved by some of the authors in [7,8] provided that

$$2<p<2\frac{n+1}{n},\qquad \chi\in C(\partial\Omega,\mathbb{R})\cap H^{1/2}(\partial\Omega,\mathbb{R}),\qquad \varphi\in L^2(\Omega,\mathbb{R}).$$

The upper bound to p seems to be a natural extension of the assumption $2<p<4$ considered by Ekeland et al. [13] in order to solve such a problem when $n=1$ (in this case, the range $p<2$ was covered by Clarke and Ekeland in the previous paper [10]).

We stress that an improvement of the results in [7,13] has been reached with a different technique by Bolle [5] and Bolle et al. [6]. From one hand, they prove that if $\Omega\subset\mathbb{R}^n$ is a C^2 bounded domain and

$$2<p<\frac{2n}{n-1},\qquad \chi\in C^2(\partial\Omega,\mathbb{R}),\qquad \varphi\in C(\bar{\Omega},\mathbb{R}),$$

then $(\mathscr{P}_{\chi,\varphi,1})$ has infinitely many classical solutions. On the other hand, they show that in the case $n=1$ it suffices to assume $p>2$, namely the result becomes optimal.

It remains open, even for $\chi\equiv 0$, the problem of whether $(\mathscr{P}_{\chi,\varphi,1})$ has an infinite number of solutions for p all the way up to 2^*. For $\chi\equiv 0$, the most satisfactory result remains the one contained in the celebrated paper [4] of Bahri and Lions where they prove that this fact is true for a subset of φ dense in $L^2(\Omega,\mathbb{R})$.

Let us fix $N\geqslant 1$. The purpose of this paper is to show the multiplicity of solutions for the following semilinear elliptic system

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_j(a_{ij}^{hk}(x)D_i u_h) = b(x)|u|^{p-2}u_k+\varphi_k(x) & \text{in } \Omega,\\ u=\chi \quad \text{on } \partial\Omega, & \\ k=1,\dots,N & \end{cases} \tag{$\mathscr{P}_{\chi,\varphi,N}$}$$

taken any $\chi \in H^{1/2}(\partial\Omega, \mathbb{R}^N)$. Clearly, problem $(\mathscr{P}_{\chi,\varphi,N})$ reduces to the problem $(\mathscr{P}_{\chi,\varphi,1})$ if $N = 1$, $a_{ij}^{hk} = \delta_{ij}^{hk}$ and $b(x) \equiv 1$.

To the authors knowledge no other result can be found in the literature about multiplicity for systems of semilinear elliptic equations with non-homogeneous boundary conditions; on the contrary, some multiplicity results are known in the case of Dirichlet boundary conditions (see [9] for the semilinear case and [17,20] for some extensions to the quasilinear case).

It is well known that the functional $f : \mathscr{M}_\chi \to \mathbb{R}$ associated with $(\mathscr{P}_{\chi,\varphi,N})$ is given by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \,\mathrm{d}x - \frac{1}{p}\int_\Omega b(x)|u|^p \,\mathrm{d}x - \int_\Omega \varphi u \,\mathrm{d}x,$$

where $\mathscr{M}_\chi = \{u \in H^1(\Omega, \mathbb{R}^N)\colon u = \chi \text{ a.e. on } \partial\Omega\}$.

In the next, Ω will denote a Lipschitz bounded domain of $\mathbb{R}^n$ with $n \geqslant 3$ while throughout the paper we shall assume that the coefficients a_{ij}^{hk} and b belong to $C(\bar{\Omega}, \mathbb{R})$ with $a_{ij}^{hk} = a_{ji}^{kh}$ and $b > 0$. Moreover, there exists $v > 0$ such that

$$\sum_{i,j=1}^{n}\sum_{h,k=1}^{N} a_{ij}^{hk}(x)\xi_i\xi_j\eta^h\eta^k \geqslant v|\xi|^2|\eta|^2 \tag{1}$$

for all $x \in \Omega$ and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^N$ (Legendre–Hadamard condition).

Here, we state our main results.

Theorem 1.1. *Assume that* $p \in]2, 2(\frac{n+1}{n})[$. *Then for each* $\varphi \in L^2(\Omega, \mathbb{R}^N)$ *and* $\chi \in H^{1/2}(\partial\Omega, \mathbb{R}^N)$ *the system* $(\mathscr{P}_{\chi,\varphi,N})$ *has a sequence* $(u^m)_m$ *of solutions in* $\mathscr{M}_\chi$ *such that* $f(u^m) \to +\infty$.

In order to prove Theorem 1.1, we use some perturbation arguments developed in [2,18,21]; so the condition $p < 2(n + 1/n)$ is quite natural.

An improvement of such a "control" can be obtained by means of Bolle's techniques, but more assumptions are needed. In fact, all the weak solutions must be regular and the system has to be diagonal, i.e., $a_{ij}^{hk} = \delta_{ij}^{hk}$.

More precisely, we can prove the following theorem.

Theorem 1.2. *Assume that* $p \in]2, \frac{2n}{n-1}[$, $\partial\Omega$ *is of class* C^2, $\chi \in C^2(\partial\Omega, \mathbb{R}^N)$, $\varphi \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^N)$ *for some* $\alpha \in]0, 1[$ *and* $a_{ij}^{hk} = \delta_{ij}^{hk}$. *Then* $(\mathscr{P}_{\chi,\varphi,N})$ *has a sequence* $(u^m)_m$ *of classical solutions such that* $f(u^m) \to +\infty$.

Clearly, Theorems 1.1 and 1.2 extend the results of [6–8] to semilinear elliptic systems. We underline that (1) is weaker than the strong ellipticity condition.

In this paper we use two different approaches. In Sections 2–5 we prove Theorem 1.1 by the classical perturbation arguments of Bahri and Berestycki, Rabinowitz, Struwe, while in Sections 6 and 7 we provide a much simpler proof by means of the technique recently introduced by Bolle. Such tools are also used in Section 8 in order to prove Theorem 1.2.

Let us point out that, in general, whereas De Giorgi's famous example of an unbounded weak solution of a linear elliptic system shows (cf. [11]), we cannot hope to find everywhere regular solutions for coefficients $a_{ij}^{hk} \in L^\infty(\Omega,\mathbb{R})$. Anyway, if $a_{ij}^{hk} \in C(\bar{\Omega},\mathbb{R})$ and (1) holds we have that if u solves $(\mathscr{P}_{\chi,\varphi,N})$ then

$$u \in C^{0,\alpha}(\Omega,\mathbb{R}^N)$$

for each $\alpha \in]0,1[$ (see [14]); but if we look for classical solutions, namely u of class C^2 on $\bar{\Omega}$, the coefficients a_{ij}^{hk} have to be sufficiently smooth while we have to assume $\varphi \in C^{0,\alpha}(\bar{\Omega},\mathbb{R}^N)$ for some $\alpha \in]0,1[$ and $\chi \in C^2(\partial\Omega,\mathbb{R}^N)$ (see [15] and references therein).

2. The "standard" perturbation argument

In the next theorem we recall a generalization of the classical Mountain Pass Theorem due to Rabinowitz (cf. [18]) which allows to deal with non-symmetric problems.

Theorem 2.1. *Let $\mathscr{X}$ be a Hilbert space and $f : \mathscr{X} \to \mathbb{R}$ a C^1-functional. Assume that there exists $M \in \mathbb{R}$ such that f satisfies the Palais–Smale condition at each level $c \in \mathbb{R}$ with $c \geqslant M$. Let $\mathscr{Y}$ be a finite dimensional subspace of $\mathscr{X}$ and, fixed $u^* \in \mathscr{X} \setminus \mathscr{Y}$, set*

$$\mathscr{Y}^* = \mathscr{Y} \oplus \langle u^* \rangle, \qquad \mathscr{Y}^*_+ = \{u + \lambda u^* \colon u \in \mathscr{Y}^*,\ \lambda \geqslant 0\}.$$

Suppose that:

(a) $f(0) \leqslant 0$,

(b) *there exists $R > 0$ such that*

$$u \in \mathscr{Y}, \quad \|u\| \geqslant R \Rightarrow f(u) \leqslant f(0),$$

(c) *there exists $R^* > 0$ such that*

$$u \in \mathscr{Y}^*, \quad \|u\| \geqslant R^* \Rightarrow f(u) \leqslant f(0)$$

and define

$$\Gamma = \{\gamma \in C(\mathscr{X},\mathscr{X})\colon \gamma \textit{ odd and } \gamma(u) = u \text{ if } \max\{f(u), f(-u)\} \leqslant 0\}.$$

Then, if

$$c^* = \inf_{\gamma\in\Gamma} \sup_{u\in\mathscr{Y}^*_+} f(\gamma(u)) > \inf_{\gamma\in\Gamma} \sup_{u\in\mathscr{Y}} f(\gamma(u)) \geqslant M,$$

f admits at least one critical value $\bar{c} \geqslant c^*$.

Proof. See [18] or [22]. □

3. Reduction to homogeneous boundary conditions

As a first step, let us reduce $(\mathscr{P}_{\chi,\varphi,N})$ to a Dirichlet type problem. To this aim, let us denote by $\phi \in \mathscr{M}_\chi$ the only solution of the linear system

$$-\sum_{i,j=1}^{n} \sum_{h=1}^{N} D_j(a_{ij}^{hk}(x) D_i \phi_h) = 0 \quad \text{in } \Omega,$$

$$\phi = \chi \quad \text{on } \partial\Omega.$$

$$k = 1, \ldots, N \tag{2}$$

Since $p < 2^*$, it results $\phi \in L^p(\Omega, \mathbb{R}^N)$.

From now on, we shall assume that $b \equiv 1$. Otherwise, taking into account that there exist two positive constants m_b and M_b such that

$$m_b \leqslant b(x) \leqslant M_b \quad \text{for all } x \in \bar{\Omega},$$

the general case can be covered by slight modifications of some lemmas proved in the next sections.

It is easy to show that the following fact holds:

Proposition 3.1. *$u \in \mathscr{M}_\chi$ solves $(\mathscr{P}_{\chi,\varphi,N})$ if and only if $z \in H_0^1(\Omega, \mathbb{R}^N)$ solves*

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \sum_{h=1}^{N} D_j(a_{ij}^{hk}(x) D_i z_h) = |z + \phi|^{p-2}(z_k + \phi_k) + \varphi_k(x) \quad \text{in } \Omega, \\ z = 0 \quad \text{on } \partial\Omega, \\ k = 1, \ldots, N \end{cases}$$

where $u(x) = z(x) + \phi(x)$ for a.e. $x \in \bar{\Omega}$.

Therefore, in order to find solutions of our problem it is enough looking for critical points of the C^1-functional $f_\chi : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ given by

$$f_\chi(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p} \int_\Omega |u + \phi|^p \, dx - \int_\Omega \varphi u \, dx$$

(we refer the reader to [19,22] for some recalls of classical critical point theory).

Lemma 3.2. *There exists $A > 0$ such that if $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a critical point of f_χ, then*

$$\int_\Omega |u + \phi|^p \, dx \leqslant pA(f_\chi^2(u) + 1)^{1/2}.$$

Proof. By Young's inequality, for each $\varepsilon > 0$ there exist $\alpha_\varepsilon, \beta_\varepsilon > 0$ such that

$$|u + \phi|^{p-1}|\phi| \leqslant \varepsilon|u + \phi|^p + \alpha_\varepsilon|\phi|^p, \qquad |u + \phi||\varphi| \leqslant \varepsilon|u + \phi|^p + \beta_\varepsilon|\varphi|^{p'}, \tag{3}$$

with $1/p + 1/p' = 1$.

Therefore, if u is a critical point of f_χ, we get

$$f_\chi(u) = f_\chi(u) - \tfrac{1}{2} f'_\chi(u)[u]$$

$$= \left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega |u+\phi|^p\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u+\phi|^{p-2}(u+\phi)\phi\,\mathrm{d}x$$

$$-\frac{1}{2}\int_\Omega \varphi u\,\mathrm{d}x$$

$$\geqslant \frac{p-2}{2p}\int_\Omega |u+\phi|^p\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u+\phi|^{p-1}|\phi|\,\mathrm{d}x$$

$$-\frac{1}{2}\int_\Omega (|u+\phi||\varphi| + |\varphi||\phi|)\,\mathrm{d}x$$

$$\geqslant \left(\frac{p-2}{2p}-\varepsilon\right)\int_\Omega |u+\phi|^p\,\mathrm{d}x - \frac{1}{2}(\alpha_\varepsilon\|\phi\|_p^p$$

$$+\beta_\varepsilon\|\varphi\|_{p'}^{p'} + \|\varphi\|_2\|\phi\|_2).$$

Choosing ε such that $p-2-2p\varepsilon>0$, i.e., $\varepsilon\in\,]0,\frac{1}{2}-\frac{1}{p}[$, we get

$$pM_\varepsilon f_\chi(u) \geqslant \int_\Omega |u+\phi|^p\,\mathrm{d}x - pM_\varepsilon\gamma_\varepsilon(p,\phi,\varphi),$$

where $M_\varepsilon = \frac{2}{p-2-2p\varepsilon}$ and

$$\gamma_\varepsilon(p,\phi,\varphi) = \tfrac{1}{2}(\alpha_\varepsilon\|\phi\|_p^p + \beta_\varepsilon\|\varphi\|_{p'}^{p'} + \|\varphi\|_2\|\phi\|_2).$$

At this point, the assertion follows by $A\geqslant\sqrt{2}M_\varepsilon\max\{1,\gamma_\varepsilon(p,\phi,\varphi)\}$. □

Now, let $\eta\in C^\infty(\mathbb{R},\mathbb{R})$ be a cut function such that $\eta(s)=1$ for $s\leqslant 1$, $\eta(s)=0$ for $s\geqslant 2$ while $-2<\eta'(s)<0$ when $1<s<2$. For each $u\in H_0^1(\Omega,\mathbb{R}^N)$ let us define

$$\zeta(u) = 2pA(f_\chi^2(u)+1)^{1/2}, \qquad \psi(u) = \eta\left(\zeta(u)^{-1}\int_\Omega |u+\phi|^p\,\mathrm{d}x\right), \tag{4}$$

where A is as in Lemma 3.2. Finally, let us introduce the modified functional $\tilde{f}_\chi : H_0^1(\Omega,\,\mathbb{R}^N)\to\mathbb{R}$ in order to apply the techniques used in [7]:

$$\tilde{f}_\chi(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n\sum_{h,k=1}^N a_{ij}^{hk}(x)D_iu_hD_ju_k\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x - \psi(u)\int_\Omega \Theta(x,u)\,\mathrm{d}x,$$

with

$$\Theta(x,u) = \frac{|u+\phi|^p}{p} - \frac{|u|^p}{p} + \varphi u.$$

Let us provide an estimate for the loss of symmetry of $\tilde{f}_\chi$.

Lemma 3.3. *There exists $\beta>0$ such that*

$$|\tilde{f}_\chi(u) - \tilde{f}_\chi(-u)| \leqslant \beta(|\tilde{f}_\chi(u)|^{\frac{p-1}{p}}+1) \quad \textit{for all } u\in\operatorname{supp}(\psi)$$

(*here*, $\operatorname{supp}(\psi)$ *is the support of* ψ).

Proof. First of all, let us show that there exist $c_1, c_2 > 0$ such that there results

$$\left| \int_\Omega (|u+\phi|^p - |u|^p)\,\mathrm{d}x \right| \leqslant c_1 |f_\chi(u)|^{\frac{p-1}{p}} + c_2, \tag{5}$$

$$\left| \int_\Omega (|u-\phi|^p - |u|^p)\,\mathrm{d}x \right| \leqslant c_1 |f_\chi(u)|^{\frac{p-1}{p}} + c_2, \tag{6}$$

$$\left| \int_\Omega \varphi u\,\mathrm{d}x \right| \leqslant c_1 |f_\chi(u)|^{\frac{p-1}{p}} + c_2 \tag{7}$$

for all $u \in \mathrm{supp}(\psi)$. In fact, taken any $u \in H_0^1(\Omega, \mathbb{R})$ it is easy to see that

$$||u+\phi|^p - |u|^p| \leqslant p2^{p-2}|u+\phi|^{p-1}|\phi| + p2^{p-2}|\phi|^p, \tag{8}$$

$$||u-\phi|^p - |u|^p| \leqslant p2^{p-2}|u+\phi|^{p-1}|\phi| + p2^{2p-3}|\phi|^p. \tag{9}$$

Hence, by (8) we get

$$\left| \int_\Omega (|u+\phi|^p - |u|^p)\,\mathrm{d}x \right| \leqslant p2^{p-2}\|\phi\|_p \left(\int_\Omega |u+\phi|^p\,\mathrm{d}x \right)^{\frac{p-1}{p}} + p2^{p-2}\|\phi\|_p^p,$$

while (9) implies

$$\left| \int_\Omega (|u-\phi|^p - |u|^p)\,\mathrm{d}x \right| \leqslant p2^{p-2}\|\phi\|_p \left(\int_\Omega |u+\phi|^p\,\mathrm{d}x \right)^{\frac{p-1}{p}} + p2^{2p-3}\|\phi\|_p^p.$$

Moreover, by Hölder and Young's inequalities it results

$$\left| \int_\Omega \varphi u\,\mathrm{d}x \right| \leqslant \left(\int_\Omega |u+\phi|^p\,\mathrm{d}x \right)^{p-1/p} + (p-2)\left(\frac{\|\varphi\|_{p'}}{p-1} \right)^{\frac{p-1}{p-2}} + \|\varphi\|_2\|\phi\|_2.$$

If, furthermore, we assume $u \in \mathrm{supp}(\psi)$, it follows

$$\int_\Omega |u+\phi|^p\,\mathrm{d}x \leqslant 4pA(|f_\chi(u)|+1)$$

which implies (5)–(7).

Then, again by Young's inequality, simple calculations and (5), (7) give

$$|f_\chi(u)| \leqslant a_1|\tilde{f}_\chi(u)| + a_2, \tag{10}$$

for suitable $a_1, a_2 > 0$. The assertion follows by combining inequalities (5)–(7) and (10). $\square$

Now, we want to link the critical points of $\tilde{f}_\chi$ to those ones of f_χ. To this aim we need more information about $\tilde{f}'_\chi$.

Taken $u \in H_0^1(\Omega, \mathbb{R}^N)$, by direct computations we get

$$\tilde{f}'_\chi(u)[u] = (1+T_1(u)) \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k\,\mathrm{d}x$$

$$-(1-\psi(u))\int_\Omega |u|^p\,\mathrm{d}x-(\psi(u)+T_1(u))\int_\Omega \varphi u\,\mathrm{d}x$$
$$-(\psi(u)+T_2(u))\int_\Omega |u+\phi|^{p-2}(u+\phi)u\,\mathrm{d}x, \tag{11}$$

where $T_1, T_2 : H_0^1(\Omega,\mathbb{R}^N)\to\mathbb{R}$ are defined by setting

$$T_1(u)=4p^2A^2\eta'(\delta(u))\delta(u)\zeta(u)^{-2}f_\chi(u)\int_\Omega \Theta(x,u)\,\mathrm{d}x,$$

$$T_2(u)=p\eta'(\delta(u))\zeta(u)^{-1}\int_\Omega \Theta(x,u)\,\mathrm{d}x+T_1(u),$$

with $\delta(u)=\zeta(u)^{-1}\int_\Omega |u+\phi|^p\,\mathrm{d}x$.

Remark 3.4. In order to point out some properties of the maps T_1 and T_2 defined above, let us remark that by (5) and (7) there exist $b_1, b_2>0$ such that for all $u\in\operatorname{supp}(\psi)$ it is

$$|T_i(u)|\leqslant b_1|f_\chi(u)|^{-1/p}+b_2|f_\chi(u)|^{-1}\quad\text{for both } i=1,2.$$

Therefore, arguing as in [18] (see also [7, Lemma 2.9]), there exist $\alpha_0, M_0>0$ such that if $M\geqslant M_0$ then

$$\tilde f_\chi(u)\geqslant M,\quad u\in\operatorname{supp}(\psi)\Rightarrow f_\chi(u)\geqslant \alpha_0 M,$$

whence, it results $|T_i(u)|\to 0$ as $M\to+\infty$ for $i=1,2$ (trivially, it is $T_1(u)=T_2(u)=0$ if $u\notin\operatorname{supp}(\psi)$).

Theorem 3.5. *There exists $M_1>0$ such that if u is a critical point of $\tilde f_\chi$ and $\tilde f_\chi(u)\geqslant M_1$ then u is a critical point of f_χ and $f_\chi(u)=\tilde f_\chi(u)$.*

Proof. Let $u\in H_0^1(\Omega,\mathbb{R}^N)$ be a critical point of $\tilde f_\chi$. By the definition of ψ it suffices to show that, if $\tilde f_\chi(u)\geqslant M_1$ for a large enough M_1, then $\delta(u)<1$, i.e.,

$$\zeta(u)^{-1}\int_\Omega |u+\phi|^p\,\mathrm{d}x<1.$$

By (11) we have

$$\begin{aligned} f_\chi(u)&=f_\chi(u)-\frac{1}{2(1+T_1(u))}\tilde f'_\chi(u)[u]\\ &=-\frac{1}{p}\int_\Omega |u+\phi|^p\,\mathrm{d}x-\int_\Omega \varphi u\,\mathrm{d}x+\frac{1-\psi(u)}{2(1+T_1(u))}\int_\Omega |u|^p\,\mathrm{d}x\\ &\quad+\frac{\psi(u)+T_1(u)}{2(1+T_1(u))}\int_\Omega \varphi u\,\mathrm{d}x+\frac{\psi(u)+T_2(u)}{2(1+T_1(u))}\int_\Omega |u+\phi|^{p-2}(u+\phi)u\,\mathrm{d}x\\ &=\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega |u+\phi|^p\,\mathrm{d}x-\frac{T_1(u)-T_2(u)}{2(1+T_1(u))}\int_\Omega |u|^p\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\frac{\psi(u)+T_2(u)}{1+T_1(u)}-1\right)\int_\Omega(|u+\phi|^p-|u|^p)\,\mathrm{d}x\\
&-\frac{\psi(u)+T_2(u)}{2(1+T_1(u))}\int_\Omega|u+\phi|^{p-2}(u+\phi)\phi\,\mathrm{d}x\\
&-\left(1-\frac{\psi(u)+T_1(u)}{2(1+T_1(u))}\right)\int_\Omega\varphi u\,\mathrm{d}x.
\end{aligned}
$$

Then, by Remark 3.4 it is possible to choose $M_1>0$ so large that

$$
\left|\frac{1-\psi(u)}{1+T_1(u)}\right|\leqslant 2,\qquad \left|\frac{\psi(u)+T_1(u)}{1+T_1(u)}\right|\leqslant 2,
$$

$$
\left|\frac{\psi(u)+T_2(u)}{1+T_1(u)}-1\right|\leqslant 2,\qquad \left|\frac{\psi(u)+T_2(u)}{1+T_1(u)}\right|\leqslant 2,
$$

so, working as in the proof of [8, Proposition 2.6], we deduce that for each $\varepsilon>0$ there exist h_ε, $\tilde{\gamma}_\varepsilon(p,\phi,\varphi)>0$ such that

$$
f_\chi(u)\geqslant\left(\frac{p-2}{2p}-2^{p-2}\left|\frac{T_2(u)-T_1(u)}{1+T_1(u)}\right|-h_\varepsilon\right)\int_\Omega|u+\phi|^p\,\mathrm{d}x-\tilde{\gamma}_\varepsilon(p,\phi,\varphi),
$$

where $h_\varepsilon\to 0$ as $\varepsilon\to 0$. At this point, choosing a priori ε and M_1 in such a way that

$$
2^{p-2}\left|\frac{T_2(u)-T_1(u)}{1+T_1(u)}\right|+h_\varepsilon\leqslant\frac{p-2}{4p},
$$

we obtain

$$
f_\chi(u)\geqslant\frac{p-2}{4p}\int_\Omega|u+\phi|^p\,\mathrm{d}x-\tilde{\gamma}_\varepsilon(p,\phi,\varphi),
$$

which completes the proof if, as in Lemma 3.2, the constant A taken in definition (4) is large enough. □

4. The Palais–Smale condition

Let us point out that, in the check of the Palais–Smale condition for semilinear elliptic systems under the assumption (1), an important role is played by the so called Gårding's inequality.

Lemma 4.1. *Let $(u^m)_m$ be a bounded sequence in $H_0^1(\Omega,\mathbb{R}^N)$ and let $(w^m)_m$ be a strongly convergent sequence in $H^{-1}(\Omega,\mathbb{R}^N)$ such that*

$$
\int_\Omega\sum_{i,j=1}^{n}\sum_{h,k=1}^{N}a_{ij}^{hk}(x)D_iu_h^mD_jv_k\,\mathrm{d}x=\langle w^m,v\rangle\quad \textit{for all } v\in H_0^1(\Omega,\mathbb{R}^N).
$$

Then $(u^m)_m$ has a subsequence $(u^{m_k})_k$ strongly convergent in $H_0^1(\Omega,\mathbb{R}^N)$.

Proof. First of all, in our setting the following Gårding-type inequality holds: taken ν as in (1) for each $\varepsilon \in\ ]0,\nu[$ there exists $c_\varepsilon \geqslant 0$ such that

$$\int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \,\mathrm{d}x \geqslant (\nu - \varepsilon)\|Du\|_2^2 - c_\varepsilon \|u\|_2^2$$

for all $u \in H_0^1(\Omega, \mathbb{R}^N)$ (see [16, Theorem 6.5.1]). Therefore, fixed $\varepsilon > 0$, we have

$$\begin{aligned}\langle w^l - w^m, u^l - u^m \rangle &= \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i(u_h^l - u_h^m) D_j(u_k^l - u_k^m) \,\mathrm{d}x \\ &\geqslant (\nu - \varepsilon)\|Du^l - Du^m\|_2^2 - c_\varepsilon \|u^l - u^m\|_2^2\end{aligned}$$

for all $m, l \in \mathbb{N}$. Since $u^m \to u$ in $L^2(\Omega, \mathbb{R}^N)$, up to subsequences, we can conclude that $Du^m \to Du$ in $L^2(\Omega, \mathbb{R}^N)$. $\square$

Now, let $d \geqslant 0$ be such that

$$\int_\Omega \left(\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k + d|u|^2 \right) \mathrm{d}x \geqslant \frac{\nu}{2}\|Du\|_2^2 \tag{12}$$

for all $u \in H_0^1(\Omega, \mathbb{R}^N)$.

Lemma 4.2. *There exists $M_2 > 0$ such that if $(u^m)_m$ is a* $(\mathrm{PS})_c$*-sequence of $\tilde{f}_\chi$ with $c \geqslant M_2$, then $(u^m)_m$ is bounded in $H_0^1(\Omega, \mathbb{R}^N)$.*

Proof. Let $M_2 > 0$ be fixed and consider $(u^m)_m$, a $(\mathrm{PS})_c$-sequence of $\tilde{f}_\chi$, with $c \geqslant M_2$, such that

$$M_2 \leqslant \tilde{f}_\chi(u^m) \leqslant K,$$

for a certain $K > M_2$.

First of all, let us remark that if there exists a subsequence $(u^{m_k})_k$ such that $u^{m_k} \notin \mathrm{supp}(\psi)$ for all $k \in \mathbb{N}$, then it is a Palais–Smale sequence for the symmetric functional

$$f_0(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \,\mathrm{d}x - \frac{1}{p} \int_\Omega |u|^p \,\mathrm{d}x$$

in $H_0^1(\Omega, \mathbb{R}^N)$. Whence, it is easier to prove that such a subsequence is bounded.

So, we can assume $u^m \in \mathrm{supp}(\psi)$ for all $m \in \mathbb{N}$.

For $m \in \mathbb{N}$ large enough and any $\varrho > 0$, taken d as in (12) by (11) it results

$$\begin{aligned}K + \varrho\|Du^m\|_2 &\geqslant \tilde{f}_\chi(u^m) - \varrho \tilde{f}_\chi'(u^m)[u^m] \\ &= \frac{1}{2}(1 - 2\varrho(1 + T_1(u^m))) \int_\Omega \left(\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h^m D_j u_k^m + d|u^m|^2 \right) \mathrm{d}x \\ &\quad - \frac{d}{2}(1 - 2\varrho(1 + T_1(u^m)))\,\|u^m\|_2^2 + \left(\varrho(1 - \psi(u^m)) - \frac{1}{p} \right) \int_\Omega |u^m|^p \,\mathrm{d}x\end{aligned}$$

$$+\varrho(\psi(u^m)+T_2(u^m))\int_\Omega |u^m+\phi|^{p-2}(u^m+\phi)u^m\,\mathrm{d}x$$

$$+\varrho(\psi(u^m)+T_1(u^m))\int_\Omega \varphi u^m\,\mathrm{d}x-\psi(u^m)\int_\Omega \Theta(x,u^m)\,\mathrm{d}x.$$

Since it is $p>2$, we can fix, a priori, a constant $\hbar\in\,]1,p/2[$ such that, taken $\mu\in\,]0,1-2\hbar/p[$, $\varrho\in\,]\hbar/p,\frac{1-\mu}{2}[$ and $\bar{\mu}\in\,]0,\varrho(1-1/\hbar)[$, by Remark 3.4 if M_2 is large enough for all $m\in\mathbb{N}$ we have

$$|T_1(u^m)|<\min\left\{1,\frac{1-\mu}{2\varrho}-1\right\},\qquad |T_2(u^m)|<1-\frac{1}{\hbar}-\frac{\bar{\mu}}{\varrho}$$

and then

$$\mu<1-2\varrho(1+T_1(u^m))\leqslant 1, \tag{13}$$

$$\bar{\mu}\leqslant\varrho(1+T_2(u^m))-\frac{1}{p}. \tag{14}$$

So, by (12) and (13) we obtain

$$\begin{aligned}K+\varrho\ \|Du^m\|_2 \geqslant\ & \frac{\nu\mu}{4}\|Du^m\|_2^2-\frac{d}{2}\|u^m\|_2^2+\left(\varrho(1-T_2(u^m))-\frac{1}{p}\right)\int_\Omega |u^m|^p\,\mathrm{d}x\\ & -(\varrho(1+|T_1(u^m)|)+1)\int_\Omega |\varphi||u^m|\,\mathrm{d}x-\varrho(1+|T_2(u^m)|)\\ & \int_\Omega |u^m+\phi|^{p-1}|\phi|\,\mathrm{d}x+\left(\varrho(\psi(u^m)+T_2(u^m))-\frac{\psi(u^m)}{p}\right)\\ & \int_\Omega (|u^m+\phi|^p-|u^m|^p)\,\mathrm{d}x.\end{aligned}$$

Hence, fixed any $\varepsilon>0$, by (3), (14) and a suitable choice of the positive constants a_1 and a_2^ε there results

$$\begin{aligned}&K+\varrho\|Du^m\|_2+\frac{d}{2}\|u^m\|_2^2\\ &\quad\geqslant\frac{\nu\mu}{4}\|Du^m\|_2^2+(\bar{\mu}-\varepsilon a_1)\|u^m\|_p^p+\left(\varrho(\psi(u^m)+T_2(u^m))-\frac{\psi(u^m)}{p}\right)\\ &\quad\int_\Omega (|u^m+\phi|^p-|u^m|^p)\,\mathrm{d}x-a_2^\varepsilon.\end{aligned}$$

Let us point out that, as $u^m\in\operatorname{supp}(\psi)$, (5) and (10) imply

$$\left(\int_\Omega (|u^m+\phi|^p-|u^m|^p)\,\mathrm{d}x\right)_{m\in\mathbb{N}}\qquad\text{is bounded.}$$

Whence, $p>2$ and a suitable choice of ε small enough allow to complete the proof. $\square$

Lemma 4.3. *Let M_2 be as in Lemma* 4.2 *and $c \geqslant M_2$. Then, taken any* $(\mathrm{PS})_c$*-sequence* $(u^m)_m$ *for $\tilde{f}_\chi$, the sequence*

$$\hat{g}(x,u^m) = |u^m|^{p-2}u^m + \psi(u^m)\Theta'(x,u^m) + \psi'(u^m)\int_\Omega \Theta(x,u^m)\,\mathrm{d}x$$

admits a convergent subsequence in $H^{-1}(\Omega,\mathbb{R}^N)$.

Proof. Follow the steps of [17, Lemma 3.3]. □

The next is one of the main tools of this paper, the $(\mathrm{PS})_c$ condition for $\tilde{f}_\chi$.

Theorem 4.4. *The functional $\tilde{f}_\chi$ satisfies the Palais–Smale condition at each level $c \in \mathbb{R}$ with $c \geqslant M_2$, where M_2 is as in Lemma* 4.2.

Proof. Let $(u^m)_m$ be a Palais–Smale sequence for $\tilde{f}_\chi$ at level $c \geqslant M_2$. Therefore, $(u^m)_m$ is bounded in $H_0^1(\Omega,\mathbb{R}^N)$ and by Lemma 4.3, up to a subsequence, $(\hat{g}(x,u^m))_m$ is strongly convergent in $H^{-1}(\Omega,\mathbb{R}^N)$. Hence, the assertion follows by Lemma 4.1 applied to $w^m = \hat{g}(x,u^m) + \tilde{f}'_\chi(u^m)$ where, by assumption, $\tilde{f}'_\chi(u^m) \to 0$ in $H^{-1}(\Omega,\mathbb{R}^N)$. □

5. Comparison of growths for min–max values

In this section we shall build two min–max classes for $\tilde{f}_\chi$ and then we compare the growth of the associated min–max values.

Let $(\lambda^l,u^l)_l$ be a sequence in $\mathbb{R}\times H_0^1(\Omega,\mathbb{R}^N)$ such that

$$\begin{cases} -\Delta u_k^l = \lambda^l u_k^l & \text{in } \Omega,\\ u^l = 0 & \text{on } \partial\Omega,\\ k = 1,\dots,N, \end{cases}$$

with $(u^l)_l$ orthonormalized. Let us consider the finite dimensional subspaces

$$V_0 := \langle u^0\rangle, \quad V_{l+1} := V_l \oplus \mathbb{R}u^{l+1} \ \text{ for any } l \in \mathbb{N}.$$

Fixed $l \in \mathbb{N}$ it is easy to check that some constants $\beta_1,\beta_2,\beta_3,\beta_4 > 0$ exist such that

$$\tilde{f}_\chi(u) \leqslant \beta_1\|u\|_{1,2}^2 - \beta_2\|u\|_{1,2}^p - \beta_3\|u\|_{1,2} - \beta_4 \quad \text{for all } u \in V_l.$$

Then, there exists $R_l > 0$ such that

$$u \in V_l, \quad \|u\|_{1,2} \geqslant R_l \Rightarrow \tilde{f}_\chi(u) \leqslant \tilde{f}_\chi(0) \leqslant 0.$$

Definition 5.1. For any $l \geqslant 1$ we set $D_l = V_l \cap B(0,R_l)$,

$$\Gamma_l = \left\{\gamma \in C(D_l, H_0^1(\Omega,\mathbb{R}^N)) \colon \gamma \text{ odd and } \gamma_{|\partial B(0,R_l)} = Id\right\}$$

and

$$b_l = \inf_{\gamma\in\Gamma_l}\max_{u\in D_l} \tilde{f}_\chi(\gamma(u)).$$

In order to prove some estimates on the growth of the levels b_l, a result due to Tanaka (cf. [23]) implies the following lemma.

Lemma 5.2. *There exist* $\beta > 0$ *and* $l_0 \in \mathbb{N}$ *such that*

$$b_l \geqslant \beta l^{\frac{2p}{n(p-2)}} \quad \textit{for all } l \geqslant l_0.$$

Proof. By (12) and simple calculations $a_1, a_2 > 0$ exist such that

$$\tilde{f}_\chi(u) \geqslant \frac{\nu}{4}\|Du\|_2^2 - a_1\|u\|_p^p - a_2 \quad \text{for all } u \in \partial B(0, R_l) \cap V_{l-1}^{\perp}.$$

Then, it is enough to follow the proof of [23, Theorem 1]. □

Now, let us introduce a second class of min–max values to be compared with b_l.

Definition 5.3. Taken $l \in \mathbb{N}$, define

$$U_l = \{\xi = tu^{l+1} + w\colon 0 \leqslant t \leqslant R_{l+1},\ \ w \in B(0, R_{l+1}) \cap V_l,\ \ \|\xi\|_{1,2} \leqslant R_{l+1}\}$$

and

$$\Lambda_l = \{\lambda \in C(U_l, H_0^1(\Omega, \mathbb{R}^N))\colon \lambda_{|_{D_l}} \in \Gamma_l \text{ and}$$
$$\lambda|_{\partial B(0,R_{l+1}) \cup ((B(0,R_{l+1}) \setminus B(0,R_l)) \cap V_l)} = Id\}.$$

Assume

$$c_l = \inf_{\lambda \in \Lambda_l} \max_{u \in U_l} \tilde{f}_\chi(\lambda(u)).$$

The following result is the concrete version of Theorem 2.1.

Lemma 5.4. *Assume* $c_l > b_l \geqslant \max\{M_1, M_2\}$. *Taken* $\delta \in\ ]0, c_l - b_l[$, *let us set*

$$\Lambda_l(\delta) = \{\lambda \in \Lambda_l\colon\ \tilde{f}_\chi(\lambda(u)) \leqslant b_l + \delta \textit{ for all } u \in D_l\},$$

$$c_l(\delta) = \inf_{\lambda \in \Lambda_l(\delta)} \max_{u \in U_l} \tilde{f}_\chi(\lambda(u)).$$

Then, $c_l(\delta)$ *is a critical value for* $\tilde{f}_\chi$.

Proof. The proof can be obtained by arguing as in [18, Lemma 1.57]. □

Now, we prove that the situation $c_l = b_l$ cannot occur for all large l.

Lemma 5.5. *Assume that* $c_l = b_l$ *for all* $l \geqslant l_1$. *Then there exists* $\gamma > 0$ *with*

$$b_l \leqslant \gamma l^p.$$

Proof. Working as in [18, Lemma 1.64] it is possible to prove that

$$b_{l+1} \leqslant b_l + \beta(|b_l|^{\frac{p-1}{p}} + 1) \quad \text{for all } l \geqslant l_1.$$

The assertion follows by [2, Lemma 5.3]. □

Proof of Theorem 1.1. Observe that the inequality $2 < p < 2(n+1)/n$ implies

$$p < \frac{2p}{n(p-2)}.$$

Therefore, by Lemmas 5.2 and 5.5 it follows that there exists a diverging sequence $(l_n)_n \subset \mathbb{N}$ such that $c_{l_n} > b_{l_n}$ for all $n \in \mathbb{N}$, then Lemma 5.4 implies that $(c_{l_n}(\delta))_n$ is a sequence of critical values for $\tilde{f}_\chi$. Whence, by Theorem 3.5 the functional f_χ has a diverging sequence of critical values. □

Remark 5.6. When p goes all the way up to 2^*, in a similar fashion, one can prove that for each $v \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $(\mathscr{P}_{\varepsilon\chi,\varepsilon\varphi,N})$ has at least v distinct solutions in $\mathscr{M}_{\varepsilon\chi}$. This is possible since there exists $\beta > 0$ such that

$$|\tilde{f}_\chi^\varepsilon(u) - \tilde{f}_\chi^\varepsilon(-u)| \leqslant \varepsilon\beta(|\tilde{f}_\chi^\varepsilon(u)|^{\frac{p-1}{p}} + 1),$$

for each $\varepsilon > 0$ and $u \in \operatorname{supp}(\psi)$, where $\tilde{f}_\chi^\varepsilon : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ is defined by:

$$\tilde{f}_\chi^\varepsilon(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x - \psi_\varepsilon(u)\int_\Omega \Theta_\varepsilon(x,u)\,\mathrm{d}x$$

with

$$\Theta_\varepsilon(x,u) = \frac{|u+\varepsilon\phi|^p}{p} - \frac{|u|^p}{p} + \varepsilon\varphi u, \qquad \psi_\varepsilon(u) = \eta\left(\zeta(u)^{-1}\int_\Omega |u+\varepsilon\phi|^p \,\mathrm{d}x\right)$$

(for more details in the scalar case, see [1,2,8]).

6. Bolle's method for non-symmetric problems

In this section we briefly recall from [5] the theory devised by Bolle for dealing with problems with broken symmetry.

The idea is to consider a continuous path of functionals starting from the symmetric functional f_0 and to prove a preservation result for min–max critical levels in order to get critical points also for the end-point functional f_1.

Let $\mathscr{X}$ be a Hilbert space equipped with the norm $\|\cdot\|$ and $f:[0,1]\times\mathscr{X}\to\mathbb{R}$ a C^2-functional. Set $f_\theta = f(\theta,\cdot)$ if $\theta\in[0,1]$.

Assume that $\mathscr{X} = \mathscr{X}_- \oplus \mathscr{X}_+$ and let $(e_l)_{l\geqslant 1}$ be an orthonormal base of $\mathscr{X}_+$ such that we can define an increasing sequence of subspaces as follows:

$$\mathscr{X}_0 := \mathscr{X}_-, \quad \mathscr{X}_{l+1} := \mathscr{X}_l \oplus \mathbb{R}e_{l+1} \quad \text{if } l \in \mathbb{N}.$$

Provided that $\dim(\mathscr{X}_-) < +\infty$, let us set

$$\mathscr{K} = \{\zeta \in C(\mathscr{X}, \mathscr{X}) \colon \zeta \text{ is odd and for a fixed } R > 0 \ \zeta(u) = u \text{ if } \|u\| \geqslant R\}$$

and

$$c_l = \inf_{\zeta \in \mathscr{K}} \sup_{u \in \mathscr{X}_l} f_0(\zeta(u)).$$

Assume that

($\mathscr{H}_1$) f satisfies a kind of Palais–Smale condition in $[0,1] \times \mathscr{X}$: any $((\theta^m, u^m))_m$ such that

$$(f(\theta^m, u^m))_m \quad \text{is bounded and } f'_{\theta^m}(u^m) \to 0 \quad \text{as } m \to +\infty \tag{15}$$

converges up to subsequences;

($\mathscr{H}_2$) for any $b > 0$ there exists $C_b > 0$ such that

$$|f_\theta(u)| \leqslant b \Rightarrow \left| \frac{\partial}{\partial\theta} f(\theta, u) \right| \leqslant C_b(\|f'_\theta(u)\| + 1)(\|u\| + 1)$$

for all $(\theta, u) \in [0,1] \times \mathscr{X}$;

($\mathscr{H}_3$) there exist two continuous maps $\eta_1, \eta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ which are Lipschitz continuous with respect to the second variable and such that $\eta_1 \leqslant \eta_2$. Suppose

$$\eta_1(\theta, f_\theta(u)) \leqslant \frac{\partial}{\partial\theta} f(\theta, u) \leqslant \eta_2(\theta, f_\theta(u)) \tag{16}$$

at each critical point u of f_θ;

($\mathscr{H}_4$) f_0 is even and for each finite dimensional subspace $\mathscr{W}$ of $\mathscr{X}$ it results

$$\lim_{\substack{u \in \mathscr{W} \\ \|u\| \to +\infty}} \sup_{\theta \in [0,1]} f(\theta, u) = -\infty.$$

Taken $i = 1, 2$, let us denote by $\psi_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ the solutions of the problem

$$\begin{cases} \dfrac{\partial}{\partial\theta} \psi_i(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)), \\ \psi_i(0, s) = s. \end{cases}$$

Note that $\psi_i(\theta, \cdot)$ are continuous, non-decreasing on $\mathbb{R}$ and $\psi_1 \leqslant \psi_2$. Set

$$\bar{\eta}_1(s) = \sup_{\theta \in [0,1]} \eta_1(\theta, s), \qquad \bar{\eta}_2(s) = \sup_{\theta \in [0,1]} \eta_2(\theta, s).$$

In this framework, the following abstract result can be proved.

Theorem 6.1. *There exists $C \in \mathbb{R}$ such that if $l \in \mathbb{N}$ then*
(a) *either f_1 has a critical level $\tilde{c}_l$ with $\psi_2(1, c_l) < \psi_1(1, c_{l+1}) \leqslant \tilde{c}_l$,*
(b) *or we have $c_{l+1} - c_l \leqslant C(\bar{\eta}_1(c_{l+1}) + \bar{\eta}_2(c_l) + 1)$.*

Proof. See [5, Theorem 3] and [6, Theorem 2.2]. □

7. Application to semilinear elliptic systems

In this section we want to prove Theorem 1.1 in a simpler fashion by means of the arguments introduced in Section 6.

For $\theta \in [0,1]$, let us consider the functional $f_\theta : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ as

$$f_\theta(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \, \mathrm{d}x - \frac{1}{p} \int_\Omega |u + \theta\phi|^p \, \mathrm{d}x - \theta \int_\Omega \varphi u \, \mathrm{d}x.$$

It can be proved that the assumption $(\mathscr{H}_1)$ is satisfied.

Lemma 7.1. *Let* $((\theta^m, u^m))_m \subset [0,1] \times H_0^1(\Omega, \mathbb{R}^N)$ *be such that* (15) *holds. Then* $((\theta^m, u^m))_m$ *converges up to subsequences.*

Proof. Let $((\theta^m, u^m))_m$ be such that (15) holds. For a suitable $K > 0$ and any $\varrho > 0$ it is

$$\begin{aligned} K + \varrho \|Du^m\|_2 &\geqslant f_{\theta^m}(u^m) - \varrho f'_{\theta^m}(u^m)[u^m] \\ &= \left(\frac{1}{2} - \varrho\right) \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h^m D_j u_k^m \, \mathrm{d}x \\ &\quad + \left(\varrho - \frac{1}{p}\right) \int_\Omega |u^m + \theta^m \phi|^p \, \mathrm{d}x \\ &\quad - \theta^m \varrho \int_\Omega |u^m + \theta^m \phi|^{p-2}(u^m + \theta^m \phi)\phi \, \mathrm{d}x \end{aligned}$$

for all m large enough. Then, fixed any $\varepsilon > 0$ and taken d as in (12), (3) and simple computations imply

$$\begin{aligned} \varrho \|Du^m\|_2 + \left(\frac{1}{2} - \varrho\right) d \|u^m\|_2^2 &\geqslant \left(\frac{1}{2} - \varrho\right) \frac{\nu}{2} \|Du^m\|_2^2 \\ &\quad + \frac{1}{2^{p-1}} \left(\varrho(1-\varepsilon) - \frac{1}{p}\right) \|u^m\|_p^p - a_\varepsilon \end{aligned}$$

for a certain $a_\varepsilon > 0$. Hence, if we fix $\varrho \in]\frac{1}{p}, \frac{1}{2}[$ and $\varepsilon \in]0, 1 - \frac{1}{\varrho p}[$, by this last inequality it follows that $(u^m)_m$ has to be bounded in $H_0^1(\Omega, \mathbb{R}^N)$.

So, if we assume $w^m = f'_{\theta^m}(u^m) + |u^m + \theta^m \phi|^{p-2}(u^m + \theta^m \phi) + \theta^m \varphi$ it is easy to prove that $(w^m)_m$ strongly converges in $H^{-1}(\Omega, \mathbb{R}^N)$, up to subsequences. Whence, Lemma 4.1 implies that $(u^m)_m$ has a converging subsequence in $H_0^1(\Omega, \mathbb{R}^N)$. $\square$

In the following result we see that the assumption $(\mathscr{H}_2)$ is also fulfilled.

Lemma 7.2. *For each* $b > 0$ *there exists* $C_b > 0$ *such that*

$$|f_\theta(u)| \leqslant b \Rightarrow \left| \frac{\partial}{\partial \theta} f(\theta, u) \right| \leqslant C_b(\|f'_\theta(u)\| + 1)(\|u\|_{1,2} + 1)$$

for all $(\theta, u) \in [0,1] \times H_0^1(\Omega, \mathbb{R}^N)$.

Proof. Fix $b > 0$. The condition $|f_\theta(u)| \leqslant b$ is equivalent to

$$\left| \int_\Omega \left(\frac{1}{2} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k - \frac{1}{p} |u + \theta\phi|^p - \theta\varphi u \right) \mathrm{d}x \right| \leqslant b \tag{17}$$

which implies that

$$\theta \int_\Omega \varphi u \,\mathrm{d}x \geqslant \frac{p}{2} \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{h,k}(x) D_i u_h D_j u_k \,\mathrm{d}x$$

$$- \int_\Omega |u + \theta\phi|^p \,\mathrm{d}x - (p-1)\theta \int_\Omega \varphi u \,\mathrm{d}x - pb. \tag{18}$$

So, taken d as in (12), we have

$$\begin{aligned} -f_\theta'(u)[u] &= - \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \,\mathrm{d}x \\ &\quad + \int_\Omega |u + \theta\phi|^{p-2}(u + \theta\phi) u \,\mathrm{d}x + \theta \int_\Omega \varphi u \,\mathrm{d}x \\ &\geqslant \left(\frac{p}{2} - 1 \right) \int_\Omega \left(\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k + d|u|^2 \right) \mathrm{d}x \\ &\quad - \left(\frac{p}{2} - 1 \right) d \|u\|_2^2 - \int_\Omega |u + \theta\phi|^{p-2}(u + \theta\phi)\theta\phi \,\mathrm{d}x \\ &\quad - (p-1)\theta \int_\Omega \varphi u \,\mathrm{d}x - pb \\ &\geqslant (p-2)\frac{v}{4} \|Du\|_2^2 - \left(\frac{p}{2} - 1 \right) d \|u\|_2^2 \\ &\quad - \int_\Omega |u + \theta\phi|^{p-2}(u + \theta\phi)\theta\phi \,\mathrm{d}x - (p-1)\theta \int_\Omega \varphi u \,\mathrm{d}x - pb. \end{aligned}$$

By Hölder inequality there exist $c_1, c_2, c_3 > 0$ such that

$$\left| \int_\Omega |u + \theta\phi|^{p-2}(u + \theta\phi)\theta\phi \,\mathrm{d}x \right| \leqslant c_1 \|u + \theta\phi\|_p^{p-1}, \tag{19}$$

$$\left| \int_\Omega \varphi u \,\mathrm{d}x \right| \leqslant c_2 \|u + \theta\phi\|_p + c_3, \tag{20}$$

while (17) implies

$$\|u + \theta\phi\|_p^p \leqslant c_4 \|Du\|_2^2 + c_5(b) \tag{21}$$

for suitable $c_4, c_5(b) > 0$. Then, since Young's inequality yields

$$c_1\|u+\theta\phi\|_p^{p-1} \leqslant \varepsilon\|u+\theta\phi\|_p^p + \tilde{c}_1(\varepsilon),$$
$$c_2\|u+\theta\phi\|_p \leqslant \varepsilon\|u+\theta\phi\|_p^p + \tilde{c}_2(\varepsilon), \tag{22}$$

for all $\varepsilon > 0$ and certain $\tilde{c}_1(\varepsilon), \tilde{c}_2(\varepsilon) > 0$, it can be proved that $c_6, c_7(\varepsilon, b) > 0$ exist such that

$$-f'_\theta(u)[u] \geqslant \left((p-2)\frac{\nu}{4} - \varepsilon c_6\right)\|Du\|_2^2 - c_7(\varepsilon, b).$$

So, if ε is small enough, some $\tilde{c}_6, \tilde{c}_7(b) > 0$ can be find such that

$$\tilde{c}_6\|Du\|_2^2 - \tilde{c}_7(b) \leqslant -f'_\theta(u)[u]. \tag{23}$$

On the other hand, since

$$\frac{\partial}{\partial\theta} f(\theta, u) = -\int_\Omega |u+\theta\phi|^{p-2}(u+\theta\phi)\phi\,\mathrm{d}x - \int_\Omega \varphi u\,\mathrm{d}x$$

by (19) and (20) it follows

$$\left|\frac{\partial}{\partial\theta} f(\theta, u)\right| \leqslant c_8\|u+\theta\phi\|_p^{p-1} + c_9 \tag{24}$$

and then by (22)

$$\left|\frac{\partial}{\partial\theta} f(\theta, u)\right| \leqslant \varepsilon\|u+\theta\phi\|_p^p + c_{10}(\varepsilon)$$

for any $\varepsilon > 0$ and $c_8, c_9, c_{10}(\varepsilon) > 0$ suitable constants. So, for all $\varepsilon > 0$ and a certain $c_{11}(\varepsilon, b) > 0$, (21) implies

$$\left|\frac{\partial}{\partial\theta} f(\theta, u)\right| \leqslant \varepsilon c_4\|Du\|_2^2 + c_{11}(\varepsilon, b). \tag{25}$$

Hence, the proof follows by (23), (25) and a suitable choice of ε. □

Lemma 7.3. *If $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a critical point of f_θ, there exists $\sigma > 0$ such that*

$$\int_\Omega |u+\theta\phi|^p\,\mathrm{d}x \leqslant \sigma(f_\theta^2(u)+1)^{1/2}.$$

Proof. It suffices to argue as in Lemma 3.2. □

Finally, we check that also the assumption $(\mathscr{H}_3)$ is fulfilled.

Lemma 7.4. *At each critical point u of f_θ inequality* (16) *holds if η_1, η_2 are defined in $(\theta, s) \in [0,1] \times \mathbb{R}$ as*

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = C(s^2+1)^{\frac{p-1}{2p}} \tag{26}$$

for a suitable constant $C > 0$.

Proof. It is sufficient to combine (24) with Lemma 7.3. □

New proof of Theorem 1.1. Clearly, f_0 is an even functional. Moreover, by Lemmas 7.1, 7.2 and 7.4 the hypotheses $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ and $(\mathscr{H}_3)$ hold.

Now, consider $(V_l)_l$, the sequence of subspaces of $H_0^1(\Omega,\mathbb{R}^N)$ introduced in Section 5. Defined the set of maps $\mathscr{K}$ as in Section 6 with $\mathscr{X}=H_0^1(\Omega,\mathbb{R}^N)$, assume

$$c_l = \inf_{\zeta\in \mathrm{K}} \sup_{u\in V_l} f_0(\zeta(u)).$$

Simple computations allow to prove that, taken any finite dimensional subspace $\mathscr{W}$ of $H_0^1(\Omega,\mathbb{R}^N)$, some constants $\beta_1,\beta_2,\beta_3>0$ exist such that

$$f_\theta(u) \leqslant \beta_1\|u\|_{1,2}^2 - \beta_2\|u\|_{1,2}^p - \beta_3 \quad \text{for all } u\in\mathscr{W}.$$

Then,

$$\lim_{\substack{u\in\mathscr{W}\\ \|u\|_{1,2}\to+\infty}} \sup_{\theta\in[0,1]} f_\theta(u) = -\infty$$

and also $(\mathscr{H}_4)$ has been proved. Hence, Theorem 6.1 applies and, by the choice made in (26), condition (b) implies that there exists $\tilde{C}>0$ such that

$$|c_{l+1}-c_l| \leqslant \tilde{C}((c_l)^{\frac{p-1}{p}} + (c_{l+1})^{\frac{p-1}{p}} + 1), \tag{27}$$

which implies $c_l \leqslant \tilde{\gamma}\, l^p$ for some $\tilde{\gamma}>0$ in view of [2, Lemma 5.3]. Taking into account Lemma 5.2 we conclude that (27) cannot hold provided that

$$\frac{2p}{n(p-2)} > p,$$

namely $p\in\,]2, 2(\frac{n+1}{n})[$. Whence, the assertion follows by (a) of Theorem 6.1. □

8. The diagonal case

Now, we want to prove Theorem 1.2. To this aim let us point out that we deal with the problem

$$\begin{cases} -\Delta u_k = |u|^{p-2}u_k + \varphi_k(x) & \text{in } \Omega,\\ u=\chi \quad \text{on } \partial\Omega, \\ k=1,\ldots,N \end{cases} \tag{28}$$

and want to prove that (28) has an infinite number of solutions if $p\in\,]2,\frac{2n}{n-1}[$. In this case, the functional f_θ defined in the previous section becomes

$$f_\theta(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u+\theta\phi|^p\,\mathrm{d}x - \theta\int_\Omega \varphi u\,\mathrm{d}x,$$

where ϕ solves the system (2) with $a_{ij}^{hk}=\delta_{ij}^{hk}$.

By the regularity assumptions we made on $\partial\Omega$, χ and φ the following lemma can be proved.

Lemma 8.1. *There exists $c>0$ such that if u is a critical point of f_θ, then*

$$\left|\int_{\partial\Omega}\left(\frac{1}{2}|\nabla w|^2-\left|\frac{\partial w}{\partial n}\right|^2\right)\mathrm{d}\sigma\right|\leqslant c\int_\Omega(|\nabla w|^2+|w|^p+1)\,\mathrm{d}x,$$

where $w=u+\theta\phi$.

Proof. If $u\in H_0^1(\Omega,\mathbb{R}^N)$ is such that $f'_\theta(u)=0$, then some regularity theorems imply that u is a classical solution of the problem

$$\begin{cases}-\Delta u_k=|u+\theta\phi|^{p-2}(u_k+\theta\phi)+\theta\varphi_k & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\\ k=1,\dots,N,\end{cases}$$

then, $w=u+\theta\phi\in C^2(\Omega,\mathbb{R}^N)$ solves the elliptic system

$$\begin{cases}-\Delta w_k=|w|^{p-2}w_k+\theta\varphi_k & \text{in } \Omega,\\ w_k=\theta\phi_k & \text{on } \partial\Omega,\\ k=1,\dots,N.\end{cases}\tag{29}$$

Taken $\delta>0$, let us consider a cut function $\tilde\eta\in C^\infty(\mathbb{R},\mathbb{R})$ such that $\tilde\eta(s)=1$ for $s\leqslant 0$ and $\tilde\eta(s)=0$ for $s\geqslant\delta$. Moreover, taken any $x\in\mathbb{R}^N$, let $d(x,\partial\Omega)$ be the distance of x from the boundary of Ω. Let us point out that, since Ω is smooth enough, δ can be choosen in such a way that $d(\cdot,\partial\Omega)$ is of class C^2 on

$$\bar\Omega\cap\{x\in\mathbb{R}^n\colon d(x,\partial\Omega)<\delta\},$$

and $\hat n(x)=\nabla d(x,\partial\Omega)$ coincides on $\partial\Omega$ with the inner normal.

So, defined $g:\mathbb{R}^N\to\mathbb{R}$ as $g(x)=\tilde\eta(d(x,\partial\Omega))$, for each $k=1,\dots,N$ let us multiply the kth equation in (29) by $g(x)\nabla w_k\cdot\hat n(x)$. Hence, working as in [6, Lemma 4.2] and summing up with respect to k, we get

$$\sum_{k=1}^N\int_\Omega-\Delta w_k g(x)\nabla w_k\cdot\hat n\,\mathrm{d}x=\int_{\partial\Omega}\left(\frac{1}{2}|\nabla w|^2-\left|\frac{\partial w}{\partial n}\right|^2\right)\mathrm{d}\sigma+O(\|\nabla w\|_2^2),$$

$$\sum_{k=1}^N\int_\Omega|w|^{p-2}w_k g(x)\nabla w_k\cdot\hat n\,\mathrm{d}x=\frac{\theta^p}{p}\int_{\partial\Omega}|\phi|^p\,\mathrm{d}\sigma+O(\|w\|_p^p),$$

$$\sum_{k=1}^N\int_\Omega\theta\varphi_k(x)g(x)\nabla w_k\cdot\hat n\,\mathrm{d}x=\theta^2\int_{\partial\Omega}\varphi\phi\,\mathrm{d}\sigma+O(\|w\|_p).$$

Whence, the proof follows by putting together these identities. □

With the stronger assumptions we made in this section, the estimates in Lemma 7.4 can be improved.

Lemma 8.2. *At each critical point u of f_θ inequality* (16) *holds if η_1, η_2 are defined in* $(\theta, s) \in [0,1] \times \mathbb{R}$ *as*

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = C(s^2+1)^{1/4}$$

for a suitable constant $C > 0$.

Proof. Let u be a critical point of f_θ. Then,

$$\frac{\partial}{\partial \theta} f(\theta, u) = \int_{\partial\Omega} \frac{\partial u}{\partial n} \phi \, \mathrm{d}\sigma + \int_{\Omega} \varphi(\theta\phi - u) \, \mathrm{d}x,$$

so, taking into account Lemma 8.1, it is enough to argue as in [6, Lemma 4.3]. □

Proof of Theorem 1.2. Arguing as in the proof of Theorem 1.1 given in Section 7, we have that the proof of Theorem 1.2 follows by Theorem 6.1 since also in this case condition (b) cannot occur. Let us point out that, by Lemma 8.2, the incompatibility condition is $\frac{2p}{n(p-2)} > 2$, i.e., $p \in]2, \frac{2n}{n-1}[$. □

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