Asymptotics of ground states for fractional Hénon systems

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Dedicated to Professor Djairo G. de Figueiredo on the occasion of his 80th birthday

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1 Introduction and main results

Let $s \in (0,1)$, $N > 2s$ and $B = \{x \in \mathbb{R}^N : |x| < 1\}$. Consider the fractional system of Hénon type

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$$\begin{cases} (-\Delta)^s u = \dfrac{2p}{p+q}|x|^{\alpha} u^{p-1} v^q & \text{in } B, \\ (-\Delta)^s v = \dfrac{2q}{p+q}|x|^{\alpha} u^{p} v^{q-1} & \text{in } B, \\ u > 0, \ v > 0 & \text{in } B, \\ u = v = 0 & \text{in } \partial B, \end{cases} \tag{1}$$

where $(-\Delta)^s$ stands for the fractional Laplacian. Recently, a great attention has been focused on the study of nonlinear problems involving the fractional Laplacian, in view of concrete real-world applications. For instance, this type of operators arises in the thin obstacle problem, optimization, finance, phase transitions, stratified materials, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves, see, e.g., [1, 3, 11, 26, 28, 30]. See also [20] and the references therein. In a smooth bounded domain $B \subset \mathbb{R}^N$, the operator $(-\Delta)^s$ can be defined by using the eigenvalues $\{\lambda_k\}$ and corresponding eigenfunctions $\{\varphi_k\}$ of the Laplace operator $-\Delta$ in B with zero Dirichlet boundary values, normalized by $\|\varphi_k\|_{L^2(B)} = 1$, for all $k \in \mathbb{N}$, that is,

$$-\Delta \varphi_k = \lambda_k \varphi_k \text{ in } B, \qquad \varphi_k = 0 \text{ on } \partial B.$$

We define the space $H_0^s(B)$ by

$$H_0^s(B) := \Big\{ u = \sum_{k=1}^{\infty} u_k \varphi_k \text{ in } L^2(B) : \sum_{k=1}^{\infty} u_k^2 \lambda_k^s < \infty \Big\},$$

equipped with the norm

$$\|u\|_{H_0^s(B)} := \Big(\sum_{k=1}^{\infty} u_k^2 \lambda_k^s \Big)^{1/2}.$$

Thus, for all $u \in H_0^s(B)$, the fractional Laplacian $(-\Delta)^s$ can be defined as

$$(-\Delta)^s u(x) := \sum_{k=1}^{\infty} u_k \lambda_k^s \varphi_k(x), \quad x \in B.$$

We wish to point out that a different notion of fractional Laplacian, available in the literature, is given by $(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F}(u)(\xi))$, where $\mathscr{F}$ denotes the Fourier transform. This is also called the integral fractional Laplacian. This definition, in bounded domains, is really different from the spectral one. In the case of the integral notion, due to the strong nonlocal character of the operator, the Dirichlet

datum is given in $\mathbb{R}^N \setminus B$ and not simply on ∂B. Recently, Caffarelli and Silvestre [12] developed a local interpretation of the fractional Laplacian given in $\mathbb{R}^N$ by considering a Dirichlet to Neumann type operator in the domain $\{(x,t) \in \mathbb{R}^{N+1} : t > 0\}$. A similar extension, in a bounded domain with zero Dirichlet boundary condition, was established, for instance, by Cabrè and Tan in [10], Tan [32], Capella, Dàvila, Dupaigne, and Sire [14], and by Brändle, Colorado, de Pablo, and Sànchez [7]. For any $u \in H_0^s(B)$, the solution $w \in H^1_{0,L}(C_B)$ of

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 \text{ in } C_B := B \times (0,\infty), \\ \qquad\qquad w = 0 \text{ on } \partial_L C_B := \partial B \times (0,\infty), \\ \qquad\qquad w = u \text{ on } B \times \{0\}, \end{cases} \tag{2}$$

is called the s-harmonic extension $w = E_s(u)$ of u, and it belongs to the space

$$H^1_{0,L}(C_B) = \left\{ w \in L^2(C_B) : w = 0 \text{ on } \partial_L C_B : \int_{C_B} y^{1-2s}|\nabla w|^2 dxdy < \infty \right\}.$$

It is proved (see [7, Section 4.1–4.2]) that

$$-k_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y} = (-\Delta)^s u,$$

where $k_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$. Here $H^1_{0,L}(C_B)$ is a Hilbert space endowed with the norm

$$\|u\|_{H^1_{0,L}(C_B)} = \left(k_s \int_{C_B} y^{1-2s}|\nabla w|^2 dxdy\right)^{1/2}.$$

In the local case, the so-called Hénon problem

$$(HP) \qquad \begin{cases} -\Delta u = |x|^\alpha u^{p-1} & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

was first studied in [29] after being introduced by Hénon in [24] in connection with the research of rotating stellar structures. This problem has been studied by several authors, e.g. [4, 13, 31] and the references therein. For this class of problems, moving plane methods [22] cannot be applied, and numerical calculations [15] suggest that the existence of non-radial solutions is in fact possible. In [13] the authors have shown that the maximum point x_p of a ground state solution for the Hénon equation (HP) approaches a point $x_0 \in \partial B$ as $p \to 2^*$, where $2^* = 2N/(N-2)$. This result was extended to local Hénon type variational systems in [33], as well as for scalar nonlocal Hénon type equations in [18]. The main goal of this paper is to get a similar result for the nonlocal Hénon system (1).

We reformulate the nonlocal systems (1) into a local system, by using the local reduction, that is, we set

$$(LS)\qquad \begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_1) = 0 \text{ in } C_B = B\times(0,\infty), \\ -\mathrm{div}(y^{1-2s}\nabla w_2) = 0 \text{ in } C_B = B\times(0,\infty), \\ w_1 = w_2 = 0 \text{ on } \partial_L C_B = \partial B\times(0,\infty), \\ w_1 = u \geq 0 \text{ on } B\times\{0\}, \\ w_2 = v \geq 0 \text{ on } B\times\{0\}, \\ k_s y^{1-2s}\frac{\partial w_1}{\partial v} = \frac{2p}{p+q}|x|^{\alpha}u^{p-1}v^{q} \text{ on } B\times\{0\}, \\ k_s y^{1-2s}\frac{\partial w_2}{\partial v} = \frac{2q}{p+q}|x|^{\alpha}u^{p}v^{q-1} \text{ on } B\times\{0\}. \end{cases}$$

Here $u(x) = w_1(x,0)$, $v(x) = w_2(x,0)$, and the outward normal derivative should be understood as

$$y^{1-2s}\frac{\partial w}{\partial v} = -\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}.$$

Let us define the space $H := H^1_{0,L}(C_B)\times H^1_{0,L}(C_B)$ and the functional $I : H \to \mathbb{R}$,

$$I(w_1,w_2) = \frac{k_s}{2}\int_{C_B} y^{1-2s}(|\nabla w_1|^2+|\nabla w_2|^2)dxdy - \frac{2}{p+q}\int_B |x|^{\alpha}w_1(x,0)^p w_2(x,0)^q dx.$$

A weak solution to system (LS) is a vector $(w_1,w_2)\in H$ verifying $I'(w_1,w_2)(h,k) = 0$ for all $(h,k)\in H$,

$$I'(w_1,w_2)(h,k) = k_s\int_{C_B} y^{1-2s}(\nabla w_1\cdot\nabla h + \nabla w_2\cdot\nabla k)\,dxdy$$
$$-\frac{2p}{p+q}\int_B |x|^{\alpha}w_1(x,0)^{p-1}w_2(x,0)^q h dx - \frac{2q}{p+q}\int_B |x|^{\alpha}w_1(x,0)^p w_2(x,0)^{q-1}k dx.$$

For the nonlocal scalar problem

$$\begin{cases} (-\Delta)^s u = |x|^{\alpha}u^p & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{in } \partial B, \end{cases} \tag{3}$$

we have

$$(LE)\qquad \begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 \text{ in } C_B = B\times(0,\infty), \\ w = 0 \text{ on } \partial_L C_B = \partial B\times(0,\infty), \\ k_s y^{1-2s}\frac{\partial w}{\partial v} = |x|^{\alpha}u^{p-1} \text{ on } B\times\{0\}. \end{cases}$$

For this problem consider the associated minimization problem

$$\mathscr{S}^{\alpha}_{s,p}(C_B) = \inf_{w\in H^1_{0,L}(C_B)} \frac{k_s \displaystyle\int_{C_B} y^{1-2s}|\nabla w|^2 dxdy}{\Big(\displaystyle\int_B |x|^{\alpha}|w(x,0)|^p dx\Big)^{2/p}}.$$

Then $\mathscr{S}^{0}_{s,2^*_s}(C_B)$, where $2^*_s := 2N/(N-2s)$, is never achieved [7] and $\mathscr{S}^{0}_{s,2^*_s}(\mathbb{R}^{N+1}_+)$ is attained by the w which are the s-harmonic extensions of

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{4}}}{(\varepsilon+|x|^2)^{\frac{N-2s}{2}}}, \quad \varepsilon > 0,\ x \in \mathbb{R}^N.$$

Let $U(x) = (1+|x|^2)^{\frac{2s-N}{2}}$ and let W be the extension of U, namely

$$\mathscr{W}(x,y) = E_s(U) = cy^{2s}\int_{\mathbb{R}^N} \frac{U(z)}{(|x-z|^2+y^2)^{\frac{N+2s}{2}}}dz.$$

For the system (LS) consider the following minimization problem

$$\mathscr{S}^{\alpha}_{s,p,q}(C_B) = \inf_{w_1,w_2\in H^1_{0,L}(C_B)} \frac{k_s \displaystyle\int_{C_B} y^{1-2s}(|\nabla w_1|^2+|\nabla w_2|^2)dxdy}{\Big(\displaystyle\int_B |x|^{\alpha}|w_1(x,0)|^p|w_2(x,0)|^q dx\Big)^{\frac{2}{p+q}}} \tag{4}$$

Theorem 1.1. *For any $\alpha > 0$, $\mathscr{S}^{\alpha}_{s,p,q}(C_B)$ is achieved if $2 < p+q < 2^*_s$.*

Proof. Since B is bounded and $\alpha > 0$ we have $|x|^{\alpha}|u|^r \le C|u|^r$. The trace operator from $H^1_{0,L}(C_B)$ to $L^r(B)$ is continuous if $1/r \ge 1/2 - s/N$, and compact if strict inequality holds, see [7, Theorem 4.4] see also [5, 10]. Then the trace operator $t_r : H^1_{0,L}(C_B) \to L^r(|x|^{\alpha}, B)$ is compact for $r < 2N/(N-2s)$. Taking a minimizing sequence $(w_{1,n}, w_{2,n})$, there is $(w_1, w_2) \in H$ with $w_{i,n} \rightharpoonup w_i$, as $n \to \infty$. Then

$$w_{1,n} \to w_1 \text{ in } L^{p+q}(|x|^{\alpha}, B), \quad p+q < 2^*_s,$$
$$w_{2,n} \to w_2 \text{ in } L^{p+q}(|x|^{\alpha}, B), \quad p+q < 2^*_s.$$

By Young inequality we conclude that

$$\int_B |x|^{\alpha}|w_{1,n}(x,0)|^p|w_{2,n}(x,0)|^q dx \to \int_B |x|^{\alpha}|w_1(x,0)|^p|w_2(x,0)|^q dx, \text{ as } n \to \infty.$$

This implies that $\mathscr{S}^{\alpha}_{s,p,q}(C_B)$ is achieved if $2 < p+q < 2^*_s$. □

Remark 1.2. If $(w_{1,n}, w_{2,n})$ is a minimizing sequence to $S^{\alpha}_{s,p,q}(C_B)$, then it is readily seen that the sequence $(|w_{1,n}|, |w_{2,n}|)$ is minimizing too. Thus, we can assume that the minimizer (w_1, w_2) is nonnegative, that is, $w_{1,n}, w_{2,n} \geq 0$. By maximum principle we have $w_{1,n}, w_{2,n} > 0$. Finally, invoking the regularity theory we infer that $w_{1,n}, w_{2,n} \in C^{\gamma}(C_B)$, for some $\gamma \in (0,1)$. Notice that (w_1, w_2) is a weak solution for (LS). Indeed, by Lagrange multiplier theorem, considering the constraint

$$\mathscr{M} := \Big\{(w_1, w_2) \in H : \int_B |x|^{\alpha} w_1(x,0)^p w_2(x,0)^q dx = 1\Big\},$$

there exists $\lambda \in \mathbb{R}$ such

$$F'(w_1, w_2)(h,k) = \lambda G'(w_1, w_2)(h,k), \quad \forall (h,k) \in H,$$

where

$$F(w_1, w_2) = \frac{k_s}{2} \int_{C_B} y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2) dxdy,$$

$$G(w_1, w_2) = \int_B |x|^{\alpha} w_1(x,0)^p w_2(x,0)^q dx - 1.$$

Then, for all $(h,k) \in H$, we have

$$k_s \int_{C_B} y^{1-2s}(\nabla w_1 \cdot \nabla h + \nabla w_2 \cdot \nabla k) dxdy$$
$$= \lambda p \int_B |x|^{\alpha} w_1(x,0)^{p-1} w_2(x,0)^q h dx + \lambda q \int_B |x|^{\alpha} w_1(x,0)^p w_2(x,0)^{q-1} k dx.$$

By choosing $(h,k) = (w_1, w_2)$, we get

$$k_s \int_{C_B} y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2) dxdy = \lambda(p+q) \int_B |x|^{\alpha} w_1(x,0)^p w_2(x,0)^q h dx$$
$$= \lambda(p+q).$$

Therefore $\lambda > 0$ and $(\hat{w}_1, \hat{w}_2) = (\beta w_1, \beta w_2)$ with $\beta = \big(\frac{\lambda(p+q)}{2}\big)^{\frac{1}{p+q-2}}$ is a weak solution of (LS).

Now, we state the asymptotic behavior of ground states when $p + q \to 2^*_s$.

Theorem 1.3. *Let $\alpha > 0$, p_ε, $q > 1$ with $p_\varepsilon + q < 2^*_s$, $p_\varepsilon \to p$ as $\varepsilon \to 0$ and $p + q = 2^*_s$. Let $(w_{1,\varepsilon}, w_{2,\varepsilon}) \in H$ be a solution to the minimization problem (4). Then there exists $x_0 \in \partial B$ such that*

i) $k_s y^{1-2s}(|\nabla w_{1,\varepsilon}|^2 + |\nabla w_{2,\varepsilon}|^2) \rightharpoonup \mu \delta_{(x_0,0)}$ in the sense of measure,

ii) $|u_{1,\varepsilon}|^{p_\varepsilon}|u_{2,\varepsilon}|^{q} \rightharpoonup \gamma\delta_{x_0}$ *in the sense of measure,*

where $\mu > 0, \gamma > 0$ *satisfy* $\mu \geq S\gamma^{2/2_s^*}$ *and* δ_{x_0} *is the Dirac mass at* x_0.

Let $(w_{1,\varepsilon}, w_{2,\varepsilon})$ be a minimizer of $\mathscr{S}^{\alpha}_{s,p_\varepsilon,q}(C_B)$ which exists because $2 < p_\varepsilon + q < 2_s^*$. By regularity results (see, e.g., [7, 9, 14]), $(w_{1,\varepsilon}, w_{2,\varepsilon})$ is Hölder continuous. We will show that there exists $x_\varepsilon,\ y_\varepsilon \in \overline{B}$ with

$$M_{i,\varepsilon} = w_{i,\varepsilon}(x_\varepsilon, 0) = \max_{(x,y)\in\overline{B}\times(0,\infty)} w_{i,\varepsilon}(x, y).$$

Let $\lambda_\varepsilon > 0$ and $\bar{\lambda}_\varepsilon > 0$ be such that $\lambda_\varepsilon^{\frac{N-2s}{2}} M_{1,\varepsilon} = 1$ and $\bar{\lambda}_\varepsilon^{\frac{N-2s}{2}} M_{2,\varepsilon} = 1$, where

$$\lambda_\varepsilon, \bar{\lambda}_\varepsilon \to 0, \quad \text{as } p_\varepsilon + q \to 2_s^*.$$

We state another description of the phenomenon exhibited in Theorem 1.3.

Theorem 1.4. *There hold*

i) $M_{1,\varepsilon} = \mathscr{O}_\varepsilon(1)M_{2,\varepsilon}$ *as* $\varepsilon \to 0$, *hence,* $\lambda_\varepsilon = \mathscr{O}_\varepsilon(1)\bar{\lambda}_\varepsilon$ *as* $\varepsilon \to 0$.

ii) $\operatorname{dist}(x_\varepsilon, \partial B) \to 0$ *and* $\dfrac{\operatorname{dist}(x_\varepsilon, \partial B)}{\lambda_\varepsilon} \to \infty$ *as* $p_\varepsilon + q \to 2_s^*$;

iii) $\lim\limits_{p_\varepsilon+q\to 2_s^*} k_s \displaystyle\int_{C_B} y^{1-2s}(|\nabla \mathscr{T}_{1,\varepsilon}|^2 + |\nabla \mathscr{T}_{2,\varepsilon}|^2)dxdy = 0,$

where we have set $\mathscr{T}_{i,\varepsilon}(x, y) := w_{1,\varepsilon}(x, y) - \lambda_\varepsilon^{\frac{2s-N}{2}} \mathscr{W}\left(\frac{x-x_\varepsilon}{\lambda_\varepsilon}, \frac{y}{\lambda_\varepsilon}\right)$, for $i = 1, 2$.

2 Preliminaries

For any $u_i \in H_0^s(B)$, there is a unique extension $w_i = E_s(u_i) \in H^1_{0,L}(C_B)$ of u_i. The extension operator is an isometry between $H_0^s(B)$ and $H^1_{0,L}(C_B)$, that is (see [5, 7, 18])

$$\|E_s(u_i)\|_{H^1_{0,L}(C_B)} = \|u_i\|_{H_0^s(B)}, \quad i = 1, 2.$$

Let us set

$$x_0 := \left(1 - \frac{1}{|\ln\varepsilon|}, 0, \ldots, 0\right) \in \mathbb{R}^N, \qquad z_0 := (x_0, 0) \in \mathbb{R}^{N+1}.$$

Let us denote $B_\rho := \{x \in \mathbb{R}^N : |x - x_0| < \rho\}$ and

$$\mathbb{A}_\rho := \{(x, y) \in \mathbb{R}^{N+1} : |(x, y) - z_0| < \rho\}, \quad \mathbb{B}_\rho := \{(x, y) \in \mathbb{R}^{N+1} : |(x, y)| < \rho\}.$$

Let $\varphi \in C_0^\infty(C_B)$ be a cut-off function satisfying

$$\varphi(x,y) := \begin{cases} 1 \text{ if } (x,y) \in \mathbb{A}_{\frac{1}{2|\ln\varepsilon|}} \\ 0 \text{ if } (x,y) \notin \mathbb{A}_{\frac{1}{|\ln\varepsilon|}}, \end{cases}$$

with $0 \le \varphi(x,y) \le 1$ and $|\nabla\varphi(x,y)| \le C|\ln\varepsilon|$, for $(x,y) \in C_B$. If $\mathscr{W}$ is the extension of the function U previously introduced, we have (see [5]) $|\nabla\mathscr{W}(x,y)| \le Cy^{-1}\mathscr{W}(x,y)$, for $(x,y) \in \mathbb{R}^{N+1}_+$. The extension of $U_\varepsilon(x) = (\varepsilon + |x|^2)^{(2s-N)/2}$ has the form

$$\mathscr{W}_\varepsilon(x,y) = \varepsilon^{\frac{2s-N}{2}}\mathscr{W}\Big(\frac{x-x_0}{\sqrt{\varepsilon}}, \frac{y}{\sqrt{\varepsilon}}\Big), \qquad \varepsilon > 0.$$

Notice that $\varphi\mathscr{W}_\varepsilon \in H^1_{0,L}(C_B)$ for ε small enough. The following lemma is proved in [18, Lemma 3.1]

Lemma 2.1. *There holds*

$$\frac{\displaystyle\int_{C_B} k_s y^{1-2s}|\nabla(\varphi\mathscr{W}_\varepsilon)|^2 dxdy}{\Big(\displaystyle\int_B |x|^\alpha |\varphi(x,0)\mathscr{W}_\varepsilon(x,0)|^p dx\Big)^{2/p}} = \mathscr{S}^0_{s,2^*_s}(C_B) + o_\varepsilon(1),$$

as $p \to 2^*_s$, *and* $\varepsilon \to 0$.

A minimizer of $\mathscr{S}^\alpha_{s,p,q}(C_B)$ exists as $2 < p+q < 2^*_s$ and arguing as in [2, Theorem 5] we have

$$\mathscr{S}^\alpha_{s,p,q}(C_B) = C_{p,q}\,\mathscr{S}^\alpha_{s,p+q}(C_B), \qquad C_{p,q} := \Big[\Big(\frac{p}{q}\Big)^{\frac{q}{p+q}} + \Big(\frac{p}{q}\Big)^{-\frac{p}{p+q}}\Big], \tag{5}$$

where we have set

$$\mathscr{S}^\alpha_{s,p+q}(C_B) := \inf_{w \in H^1_{0,L}(C_B)} \frac{\displaystyle\int_{C_B} k_s y^{1-2s}|\nabla w|^2 dxdy}{\Big(\displaystyle\int_B |x|^\alpha |w(x,0)|^{p+q} dx\Big)^{2/(p+q)}}.$$

In particular

$$\mathscr{S}_{s,p,q}(C_B) := \mathscr{S}^0_{s,p,q}(C_B) = C_{p,q}\mathscr{S}^0_{s,p+q}(C_B) = C_{p,q}\mathscr{S}_{p+q}(C_B).$$

Furthermore, if w_0 realizes $\mathscr{S}^\alpha_{s,p+q}(C_B)$, then $(u_0, v_0) = (Bw_0, Cw_0)$ realizes $\mathscr{S}^\alpha_{s,p,q}(C_B)$, for

$$B, C > 0, \quad B = \sqrt{p/q}\,C.$$

Setting $\hat{u}_\varepsilon = \sqrt{p_\varepsilon}\varphi\mathscr{W}_\varepsilon$ and $\hat{v}_\varepsilon = \sqrt{q}\varphi\mathscr{W}_\varepsilon$ and applying identity (5), we have

$$\frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla\hat{u}_\varepsilon|^2 + |\nabla\hat{v}_\varepsilon|^2)dxdy}{\Big(\displaystyle\int_B |x|^\alpha |\hat{u}_\varepsilon(x,0)|^{p_\varepsilon}|\hat{v}_\varepsilon(x,0)|^q dx\Big)^{2/(p_\varepsilon+q)}}$$

$$= C_{p_\varepsilon,q}\frac{\displaystyle\int_{C_B} k_s y^{1-2s}|\nabla(\varphi\mathscr{W}_\varepsilon)|^2 dxdy}{\Big(\displaystyle\int_B |x|^\alpha|\varphi(x,0)\mathscr{W}_\varepsilon(x,0)|^{p_\varepsilon+q}dx\Big)^{2/(p_\varepsilon+q)}} = C_{p_\varepsilon,q}\mathscr{S}^0_{s,2^*_s}(C_B) + o_\varepsilon(1),$$

as $p_\varepsilon + q \to 2^*_s$ for $\varepsilon \to 0$. Following [18, Lemma 3.2], we have

Lemma 2.2. *Let $(u_\varepsilon, v_\varepsilon)$ be a minimizer of $\mathscr{S}^\alpha_{s,p_\varepsilon,q}(C_B)$ and $p_\varepsilon + q \to 2^*_s$ for $\varepsilon \to 0$. Then we have*

$$\lim_{\varepsilon\to 0}\frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dxdy}{\Big(\displaystyle\int_B |x|^\alpha |u_\varepsilon(x,0)|^{p_\varepsilon}|v_\varepsilon(x,0)|^q dx\Big)^{2/(p_\varepsilon+q)}} = C_{p,q}\mathscr{S}^0_{s,2^*_s}(C_B) = C_{p,q}\mathscr{S}^0_{s,2^*_s}(\mathbb{R}^{N+1}_+),$$

$$\lim_{\varepsilon\to 0}\frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dxdy}{\Big(\displaystyle\int_B |u_\varepsilon(x,0)|^{p_\varepsilon}|v_\varepsilon(x,0)|^q dx\Big)^{2/(p_\varepsilon+q)}} = C_{p,q}\mathscr{S}^0_{s,2^*_s}(C_B) = C_{p,q}\mathscr{S}^0_{s,2^*_s}(\mathbb{R}^{N+1}_+).$$

Proof. We already know that $\mathscr{S}^0_{s,2^*_s}(C_B) = \mathscr{S}^0_{s,2^*_s}(\mathbb{R}^{N+1}_+)$. Notice that, by (5), we get by Lemma 2.1

$$\frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dxdy}{\Big(\displaystyle\int_B |u_\varepsilon(x,0)|^{p_\varepsilon}|v_\varepsilon(x,0)|^q dx\Big)^{2/(p_\varepsilon+q)}} \le \frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dxdy}{\Big(\displaystyle\int_B |x|^\alpha |u_\varepsilon(x,0)|^{p_\varepsilon}|v_\varepsilon(x,0)|^q dx\Big)^{2/(p_\varepsilon+q)}}$$

$$\le C_{p_\varepsilon,q}\frac{\displaystyle\int_{C_B} k_s y^{1-2s}|\nabla(\varphi W_\varepsilon)|^2 dxdy}{\Big(\displaystyle\int_B |x|^\alpha|\varphi(x,0)W_\varepsilon(x,0)|^{p_\varepsilon+q}dx\Big)^{2/(p_\varepsilon+q)}} = C_{p_\varepsilon,q}\mathscr{S}^0_{s,2^*_s}(C_B) + o(\varepsilon),$$

as $p_\varepsilon + q \to 2_s^*$, for $\varepsilon \to 0$. On the other hand, we infer that

$$\frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dxdy}{\Big(\displaystyle\int_B |u_\varepsilon(x,0)|^{p_\varepsilon}|v_\varepsilon(x,0)|^q dx\Big)^{2/(p_\varepsilon+q)}} \geq \mathscr{S}^0_{s,p_\varepsilon,q}(C_B)$$

$$= C_{p_\varepsilon,q}\mathscr{S}^0_{s,p_\varepsilon+q}(C_B) \geq C_{p_\varepsilon,q}\mathscr{S}^0_{s,2_s^*}(C_B).$$

The last inequality is due to Hölder inequality. This concludes the proof. □

Corollary 2.3. *Let $p + q = 2_s^*$. Then the infimum $\mathscr{S}^\alpha_{s,p,q}(C_B)$ cannot be achieved.*

Proof. Observe that, for all $\alpha \geq 0$, there holds $\mathscr{S}^\alpha_{s,p,q}(C_B) = C_{p,q}\,\mathscr{S}^\alpha_{s,2_s^*}(C_B)$. Suppose, by contradiction, that $\mathscr{S}^\alpha_{s,p,q}(C_B)$ is achieved by a function $(w_1, w_2) \in H$. Without loss of generality, we may assume that $w_1 \geq 0$ and $w_2 \geq 0$. By Lemma 2.2, we get

$$C_{p,q}\mathscr{S}^0_{s,2_s^*}(C_B) = \mathscr{S}^\alpha_{s,p,q}(C_B) = \frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)dxdy}{\Big(\displaystyle\int_B |x|^\alpha w_1(x,0)^p w_2(x,0)^q dx\Big)^{2/(p+q)}}$$

$$\geq \frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)dxdy}{\Big(\displaystyle\int_B w_1(x,0)^p w_2(x,0)^q dx\Big)^{2/(p+q)}} \geq \mathscr{S}^0_{s,p,q}(C_B) = C_{p,q}\mathscr{S}^0_{s,2_s^*}(C_B),$$

so that $\mathscr{S}^0_{s,2_s^*}(C_B)$ is achieved at $(w_1, w_2) \in H$, being

$$C_{p,q}\mathscr{S}^0_{s,2_s^*}(C_B) = \frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)dxdy}{\Big(\displaystyle\int_B w_1(x,0)^p w_2(x,0)^q dx\Big)^{2/(p+q)}}.$$

By setting $\tilde{w}_i(x,t) := w_i(x,t)$ for $(x,t) \in B \times (0,\infty)$ and $\tilde{w}_i(x,t) := 0$ for $(x,t) \in \mathbb{R}^N \setminus B \times (0,\infty)$ we get the minimizer $(\tilde{w}_1, \tilde{w}_2) \in \mathscr{S}^0_{2_s^*}(\mathbb{R}^{N+1}_+)$. A contradiction, since $w_i > 0$, by the maximum principle. □

Definition 2.4. A sequence $(w_{1,n}, w_{2,n}) \subset H$ is said to be tight if, for all $\eta > 0$, there is $\rho_0 > 0$ with

$$\sup_{n\in\mathbb{N}} \int_{\{y>\rho_0\}} \int_B k_s y^{1-2s}(|\nabla w_{1,n}|^2 + |\nabla w_{2,n}|^2)dxdy \leq \eta.$$

The following concentration compactness principle [27] can be adapted from [5, Theorem 5.1]

Proposition 2.5. *Let $(w_{1,n}, w_{2,n}) \subset H$ be tight and weakly convergent to (w_1, w_2) in H. Let us denote $u_{i,n} = \mathrm{Tr}(w_{i,n})$ and $u_i = \mathrm{Tr}(w_i)$, $p + q = 2_s^*$. Let μ, ν be two nonnegative measures such that*

i) $k_s y^{1-2s}(|\nabla w_{1,n}|^2 + |\nabla w_{2,n}|^2) \rightharpoonup \mu$ in the sense of measure,
ii) $|u_{1,n}|^p |u_{2,n}|^q \rightharpoonup \nu$ in the sense of measure.

Then there exist an at most countable set I and points $\{x_i\}_{i\in I} \subset B$ such that

$$\mu \geq k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2) + \sum_{k\in I} \mu_k \delta_{(x_k,0)}, \qquad \nu = |u_1|^p |u_2|^q + \sum_{k\in I} \nu_k \delta_{x_k}, \tag{6}$$

with $\mu_k > 0, \nu_k > 0$ and $\mu_k \geq C_{p,q} \mathscr{S}^0_{s,2_s^*} \nu_k^{2/2_s^*}$.

Finally, we give an explicit form to the solutions of the problem

$$\begin{cases} (-\Delta)^s u = \dfrac{2p}{p+q} u^{p-1} v^q & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \dfrac{2q}{p+q} u^p v^{q-1} & \text{in } \mathbb{R}^N, \\ u > 0, \ v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{7}$$

where $p + q = 2_s^*$. Let $u, v \in L^{2_s^*}(\mathbb{R}^N)$ be solutions of the following problem

$$\begin{cases} u = \dfrac{2p}{p+q} \displaystyle\int_{\mathbb{R}^N} \frac{u^{p-1}(y) v^q(y)}{|x-y|^{N-2s}}\, dy, \\ v = \dfrac{2q}{p+q} \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y) v^{q-1}(y)}{|x-y|^{N-2s}}\, dy. \\ u > 0, \ v > 0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{8}$$

Denote by

$$\tilde{u}(x) := \frac{1}{|x|^{N-2s}} u\Big(\frac{x}{|x|^2}\Big), \qquad \tilde{v}(x) := \frac{1}{|x|^{N-2s}} v\Big(\frac{x}{|x|^2}\Big),$$

the Kelvin transform of u and v, respectively. Hence, $(\tilde{u}, \tilde{v})$ is also a solution of (8). We may prove as in [16, Theorem 4.5] that problems (7) and (8) are equivalent, that is if (u, v) with $u, v \in H^s(\mathbb{R}^N)$ is a weak solution of (7), then (u, v) is a solution of (8), while if (u, v) with $u, v \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$ solves (8), then (u, v) is a solution of (7).

Now we show that $L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$ solution (u,v) of the following problem is radially symmetric.

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^{p-1}(y)v^q(y)}{|x-y|^{N-2s}}\,dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)v^{q-1}(y)}{|x-y|^{N-2s}}\,dy. \\ \quad u>0,\ v>0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{9}$$

Let $\Sigma_\lambda = \{x = (x_1, \cdots, x_N) : x_1 > \lambda\}$, $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_N)$ and $u_\lambda(x) = u(x^\lambda)$.

Lemma 2.6. *Let (u,v) be a solution of (9). Then (u,v) is radially symmetric with respect to some point.*

Proof. The result is proved by the moving plane methods developed for integral equations, see [17]. The argument is now standard, we sketch the proof. For details, we refer to similar arguments in [35]. We have

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^{N-2s}} - \frac{1}{|x^\lambda - y|^{N-2s}}\Big)\Big(u_\lambda^{p-1}(y)v_\lambda^q(y) - u^{p-1}(y)v^q(y)\Big)\,dy$$

and

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^{N-2s}} - \frac{1}{|x^\lambda - y|^{N-2s}}\Big)\Big(u_\lambda^{p}(y)v_\lambda^{q-1}(y) - u^{p}(y)v^{q-1}(y)\Big)\,dy.$$

Next, we claim that there exist $K \geq 0$, such that if $\lambda < -K$, there holds

$$u(x) \geq u_\lambda(x) \quad \text{and} \quad v(x) \geq v_\lambda(x).$$

Indeed, define

$$\Sigma_\lambda^u = \{x \in \Sigma_\lambda : u(x) \leq u_\lambda(x)\}, \quad \Sigma_\lambda^v = \{x \in \Sigma_\lambda : v(x) \leq v_\lambda(x)\}$$

and $\Sigma_\lambda^- = \Sigma_\lambda \setminus \big(\Sigma_\lambda^u \cup \Sigma_\lambda^v\big)$, we can deduce as [35] that

$$\begin{aligned} u_\lambda(x) - u(x) \leq & \int_{\Sigma_\lambda^v} \frac{1}{|x-y|^{N-2s}} u_\lambda^{p-1}(y)\Big(v_\lambda^q(y) - v^q(y)\Big)\,dy \\ & + \int_{\Sigma_\lambda^u} \frac{1}{|x-y|^{N-2s}} v_\lambda^q(y)\Big(u_\lambda^{p-1}(y) - u^{p-1}(y)\Big)\,dy. \end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality,

$$\|u_\lambda(x)-u(x)\|_{L^{2_s^*}(\Sigma_\lambda^u)} \le C\|u_\lambda^{p-1}v_\lambda^{q-1}(v_\lambda-v)\|_{L^{\frac{2_s^*N}{N+2s2_s^*}}(\Sigma_\lambda^v)} + C\|u_\lambda^{p-2}v_\lambda^q(u_\lambda-u)\|_{L^{\frac{2_s^*N}{N+2s2_s^*}}(\Sigma_\lambda^u)}.$$

By Hölder's inequality,

$$\|u_\lambda(x)-u(x)\|_{L^{2_s^*}(\Sigma_\lambda^u)} \le C\|u_\lambda\|^{p-1}_{L^{2_s^*}(\Sigma_\lambda^v)}\|v_\lambda\|^{q-1}_{L^{2_s^*}(\Sigma_\lambda^v)}\|(v_\lambda-v)\|_{L^{2_s^*}(\Sigma_\lambda^v)} + C\|u_\lambda\|^{p-2}_{L^{2_s^*}(\Sigma_\lambda^u)}\|v_\lambda\|^q_{L^{2_s^*}(\Sigma_\lambda^u)}\|(u_\lambda-u)\|_{L^{2_s^*}(\Sigma_\lambda^u)}.$$

Choose $K>0$ large and for $\lambda<-K$, we have

$$\|u_\lambda(x)-u(x)\|_{L^{2_s^*}(\Sigma_\lambda^u)} \le \frac{1}{4}\|u_\lambda(x)-u(x)\|_{L^{2_s^*}(\Sigma_\lambda^u)} + \frac{1}{4}\|v_\lambda(x)-v(x)\|_{L^{2_s^*}(\Sigma_\lambda^v)}.$$

Similarly,

$$\|v_\lambda(x)-v(x)\|_{L^{2_s^*}(\Sigma_\lambda^u)} \le \frac{1}{4}\|u_\lambda(x)-u(x)\|_{L^{2_s^*}(\Sigma_\lambda^u)} + \frac{1}{4}\|v_\lambda(x)-v(x)\|_{L^{2_s^*}(\Sigma_\lambda^v)}.$$

The claim follows easily. Now, we may proceed as the proof of [35, Theorem 1.1]. □

It is known [16] that a positive solution $U\in L^{2_s^*}(\mathbb{R}^N)$ of the problem

$$(-\Delta)^s u = u^{\frac{N+2s}{N-2s}} \quad \text{in } \mathbb{R}^N, \tag{10}$$

is given by

$$U(x)=C\Big(\frac{t}{t^2+|x-x_0|^2}\Big)^{\frac{N-2s}{2}},$$

for some constant $C=C(N,s)>0$, some $t>0$ and $x_0\in\mathbb{R}^N$.

Lemma 2.7. *Let (u,v) be a nontrivial weak solution of problem (7). There exist $A,B>0$ such that $u=AU$ and $v=BU$.*

Proof. We known that the solutions (u,v) of (7) are solutions of (8). By Lemma 2.6, any solution (u,v) of (8) is radially symmetric and monotone decreasing about some point. Let $(\tilde{u},\tilde{v})$ be the Kelvin transform of (u,v) with the pole $p\neq 0$

$$\tilde{u}(x) = \frac{1}{|x-p|^{N-2s}} u\Big(\frac{x-p}{|x-p|^2} + p\Big), \qquad \tilde{v}(x) = \frac{1}{|x-p|^{N-2s}} v\Big(\frac{x-p}{|x-p|^2} + p\Big).$$

We remark that $(\tilde{u}, \tilde{v})$ is a solution of (8) too, and then $(\tilde{u}, \tilde{v})$ is radially symmetric with respect to some point q. Following the argument on page 280 in [23], we can see that if $p = q$, then (u, v) is constant, which is not true in our case. Hence, $p \neq q$. Now, using the Kelvin transform

$$K(f)(x) = \frac{1}{|x|^{N-2s}} f\Big(\frac{x}{|x|^2}\Big),$$

we deduce as in [6, proof of Lemma 7] that $u = AU$ and $v = BU$. □

3 Proof of Theorem 1.3

Choose p_k such that $p_k + q \to 2_s^*$, as $k \to \infty$. Let $(w_{1,k}, w_{2,k}) \in H$ be a nonnegative solution to

$$\mathcal{S}^{\alpha}_{s,p_k,q}(C_B) = \frac{k_s \displaystyle\int_{C_B} y^{1-2s}(|\nabla w_{1,k}|^2 + |\nabla w_{2,k}|^2)dxdy}{\Big(\displaystyle\int_B |x|^{\alpha} |w_{1,k}(x,0)|^{p_k} |w_{2,k}(x,0)|^q dx\Big)^{\frac{2}{p_k+q}}}. \tag{11}$$

Up to the factor $((p_k + q)\lambda_k/2)^{1/(p_k+q-2)}$ depending upon the Lagrange multiplier λ_k, $(w_{1,k}, w_{2,k})$ solves

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_{1,k}) = 0, \quad -\mathrm{div}(y^{1-2s}\nabla w_{2,k}) = 0, & \text{in } C_B, \\ k_s y^{1-2s} \frac{\partial w_{1,k}}{\partial \nu} = \frac{2p_k}{p_k+q} |x|^{\alpha} w_{1,k}(x,0)^{p_k-1} w_{2,k}(x,0)^q, & \text{on } B \times \{0\}, \\ k_s y^{1-2s} \frac{\partial w_{2,k}}{\partial \nu} = \frac{2q}{p_\varepsilon+q} |x|^{\alpha} w_{1,k}(x,0)^{p_k} w_{2,k}(x,0)^{q-1}, & \text{on } B \times \{0\}, \\ w_{1,k} = w_{2,k} = 0, & \text{on } \partial_L C_B. \end{cases} \tag{12}$$

In particular, we get

$$\int_{C_B} k_s y^{1-2s}(|\nabla w_{1,k}|^2 + |\nabla w_{2,k}|^2)dxdy = 2\int_B |x|^{\alpha} w_{1,k}(x,0)^{p_k} w_{2,k}(x,0)^q dx. \tag{13}$$

One may now set, for every $x \in B$ and $y > 0$,

$$\tilde{w}_{i,k}(x,y) := C_k w_{i,k}(x,y), \qquad C_k = \Big(\int_B w_{1,k}(x,0)^{p_k} w_{2,k}(x,0)^q dx\Big)^{-\frac{1}{p_k+q}}, \quad i = 1, 2. \tag{14}$$

We have

$$\int_B \tilde{w}_{1,k}(x,0)^{p_k}\tilde{w}_{2,k}(x,0)^q dx = 1, \qquad \text{for all } k \in \mathbb{N},$$

and by (11) and Lemma 2.2, we have

$$\int_{C_B} k_s y^{1-2s}(|\nabla \tilde{w}_{1,k}|^2 + |\nabla \tilde{w}_{2,k}|^2)dxdy = C_{p,q}\mathscr{S}^0_{s,2^*_s} + o_k(1), \quad \text{as } k \to \infty.$$

The sequence C_k converges to some $C > 0$, whenever $k \to \infty$. This can be proved by comparison with the term $\int_B |x|^\alpha w_{1,k}(x,0)^{p_k} w_{2,k}(x,0)^q dx$, which converges to a constant in view of formulas (11), (13) and Lemma 2.2. In fact, taking into account the Sobolev trace inequality, we have

$$0 < \sigma \le \int_B |x|^\alpha w_{1,k}(x,0)^{p_k} w_{2,k}(x,0)^q dx \le C_k^{-p_k-q} \le C\|w_{1,k}\|^{p_k}_{L^{2^*_s}(B)}\|w_{2,k}\|^q_{L^{2^*_s}(B)} \le C.$$

The sequence $(\tilde{w}_{1,k}, \tilde{w}_{2,k})$ is bounded in H. Furthermore, it is *tight*. This fact can be proved by arguing as in [5, Lemma 3.6]. By Proposition 2.5, there exist nonnegative measures μ, ν, a pair of functions $(w_1, w_2) \in H$, an at most countable set J and points with $\{x_i\}_{i\in J} \subset B$ such that

i) $\tilde{w}_{i,k} \rightharpoonup w_i, \quad i = 1, 2,$
ii) $k_s y^{1-2s}(|\nabla \tilde{w}_{1,k}|^2 + |\nabla \tilde{w}_{2,k}|^2) \rightharpoonup \mu$ in the sense of measure,
iii) $|\tilde{w}_{1,k}(x,0)|^p |\tilde{w}_{2,k}(x,0)|^q \rightharpoonup \nu$ in the sense of measure,

and (6) holds with $\nu_k > 0$ and $\mu_k \ge C_{p,q}\mathscr{S}^0_{s,2^*_s}\nu_k^{2/2^*_s}$. It follows that

$$\lim_k \int_{\mathbb{R}^{N+1}_+} k_s y^{1-2s}(|\nabla \tilde{w}_{1,k}|^2 + |\nabla \tilde{w}_{2,k}|^2)\varphi dxdy = \int_{\mathbb{R}^{N+1}_+} \varphi \, d\mu, \quad \forall \varphi \in L^\infty \cap C(\mathbb{R}^{N+1}_+),$$

$$\lim_k \int_{\mathbb{R}^N} |\tilde{w}_{1,k}(x,0)|^p |\tilde{w}_{2,k}(x,0)|^q \varphi dx = \int_{\mathbb{R}^N} \varphi \, d\nu, \quad \forall \varphi \in L^\infty \cap C(\mathbb{R}^N).$$

In particular, we infer that

$$\lim_k \int_{\mathbb{R}^{N+1}_+} k_s y^{1-2s}(|\nabla \tilde{w}_{1,k}|^2 + |\nabla \tilde{w}_{2,k}|^2)dxdy = \mu(\mathbb{R}^{N+1}_+),$$

$$\lim_k \int_{\mathbb{R}^N} |\tilde{w}_{1,k}(x,0)|^p |\tilde{w}_{2,k}(x,0)|^q dx = \nu(\mathbb{R}^N).$$

Claim: $I \neq \emptyset$.

Verification: If $I = \emptyset$, we would have $\int_B |w_1(x,0)|^p |w_2(x,0)|^q dx = 1$ and

$$C_{p,q,}\mathscr{S}^0_{s,2^*_s} = \lim_k \int_{C_B} k_s y^{1-2s}(|\nabla \tilde{w}_{1,k}|^2 + |\nabla \tilde{w}_{2,k}|^2)dxdy$$
$$\geq \int_{C_B} (k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2))dxdy,$$

yielding $C_{p,q,}\mathscr{S}^0_{s,2^*_s} = \int_{C_B} k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)dxdy$, namely a contradiction to Corollary 2.3.

Claim: I contains only one point and $w_1 = w_2 = 0$.

Verification: We argue by contradiction and consider the following three cases:

i) $w_1 \neq 0$ and $w_2 \neq 0$;
ii) $w_1 \neq 0$ and $w_2 = 0$;
iii) $w_1 = 0$ and $w_2 \neq 0$.

In the case i), we have $\sum_{j\in J} \nu_j \in (0,1)$. Notice that

$$C_{p,q,}\mathscr{S}^0_{s,2^*_s} = \lim_k \int_{C_B} k_s y^{1-2s}(|\nabla \tilde{w}_{1,k}|^2 + |\nabla \tilde{w}_{2,k}|^2)dxdy$$
$$\geq \int_{C_B} (k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2))dxdy + C_{p,q,}\mathscr{S}^0_{s,2^*_s} \sum_{j\in I} \nu_j^{2/2^*_s},$$

as well as

$$1 = \nu(\mathbb{R}^N) = \int_B |w_1(x,0)|^p |w_2(x,0)|^q dx + \sum_{j\in I} \nu_j.$$

These facts imply that

$$\int_{C_B} k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)dxdy \leq C_{p,q,}\mathscr{S}^0_{s,2^*_s}\Big(1 - \sum_{j\in I} \nu_j^{2/2^*_s}\Big)$$
$$\leq C_{p,q,}\mathscr{S}^0_{s,2^*_s}\Big(1 - \sum_{j\in I} \nu_j\Big)^{2/2^*_s}$$
$$= \mathscr{S}^0_{s,p,q}\Big(\int_B |w_1(x,0)|^p |w_2(x,0)|^q dx\Big)^{2/2^*_s},$$

which is a contradiction. In the case ii) or iii), we have $\sum_{j\in J} \nu_j = 1$. Notice that

$$C_{p,q,}\mathscr{S}^0_{s,2^*_s} \geq \int_{C_B} (k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2))dxdy + C_{p,q,}\mathscr{S}^0_{s,2^*_s} \sum_{j\in I} \nu_j^{2/2^*_s}.$$

This implies, as above, that

$$\int_{C_B} k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)dxdy \le C_{p,q}\mathscr{S}^0_{s,2^*_s}\Big(1 - \sum_{j\in I}\nu_j\Big)^{2/2^*_s} = 0,$$

which is a contradiction. Then $w_1 = w_2 = 0$. We claim that J is singleton. Notice again

$$1 \ge \sum_{j\in I}\nu_j^{2/2^*_s} \ge \Big(\sum_{j\in I}\nu_j\Big)^{2/2^*_s} = 1,$$

so there is at most one $j^* \in I$ such that $\nu_{j^*} \ne 0$, proving the claim. Hence there exists $x_0 \in \overline{B}$ with

$$k_s y^{1-2s}(|\nabla \tilde{w}_{1,k}|^2 + |\nabla \tilde{w}_{2,k}|^2) \rightharpoonup \mu_0\delta_{x_0}, \quad |\tilde{w}_{1,k}(x,0)|^p|\tilde{w}_{2,k}(x,0)|^q \rightharpoonup \nu_0\delta_{x_0}, \tag{15}$$

in the sense of measure with $\mu_0 \ge C_{p,q}\mathscr{S}^0_{s,2^*_s}\nu_0^{2/2^*_s}$. Taking into account the relation (14) between $\tilde{w}_{i,k}$ and $w_{i,k}$ the same conclusion follows for the $w_{i,k}$.

Assume by contradiction that $x_0 \in B$. Then it follows $\mathrm{dist}(x_0, \partial B) = \sigma$, for $\sigma \in (0,1)$. Notice that $|w_{1,k}(x,0)|^{p_k} \le |w_{1,k}(x,0)|^p + o_k(1)$. By the concentration behavior of the sequence $|w_{1,k}(x,0)|^p|w_{2,k}(x,0)|^q$ stated in (15), there exists $\varphi \in L^\infty \cap C(\mathbb{R}^N)$ with $\varphi(x_0) = 0$ and

$$\int_{\mathbb{R}^N} |w_{1,k}(x,0)|^p|w_{2,k}(x,0)|^q \chi_{B\setminus B(x_0,\sigma/2)}(x)dx \le \int_{\mathbb{R}^N} |w_{1,k}(x,0)|^p|w_{2,k}(x,0)|^q\varphi(x)dx = o_k(1).$$

Since $\int_B |w_{2,k}(x,0)|^q dx \le C$ by the Sobolev inequality [5, formula (2.11)], then we conclude

$$\begin{aligned}\int_B |x|^\alpha |w_{1,k}(x,0)|^{p_k}|w_{2,k}(x,0)|^q dx &= \int_{B(x_0,\sigma/2)} |x|^\alpha |w_{1,k}(x,0)|^{p_k}|w_{2,k}(x,0)|^q dx \\ &+ \int_{\mathbb{R}^N} |x|^\alpha |w_{1,k}(x,0)|^{p_k}|w_{2,k}(x,0)|^q \chi_{B\setminus B(x_0,\sigma/2)}(x)dx \\ &\le (1-\sigma/2)^\alpha \int_{B(x_0,\sigma/2)} |w_{1,k}(x,0)|^{p_k}|w_{2,k}(x,0)|^q dx \\ &+ \int_{\mathbb{R}^N} |w_{1,k}(x,0)|^p|w_{2,k}(x,0)|^q \chi_{B\setminus B(x_0,\sigma/2)}(x)dx + o_k(1) \\ &\le \Lambda(\sigma)^{2^*_s/2}\Big(\int_B |w_{1,k}(x,0)|^{p_k}|w_{2,k}(x,0)|^q dx + o_k(1)\Big),\end{aligned} \tag{16}$$

where $\Lambda(\sigma) := (1-\sigma/2)^{2\alpha/2_s^*} \in (0,1)$. By formula (16), on account of by Lemma 2.2, it follows that

$$C_{p,q,\mathscr{S}^0_{s,2_s^*}} = \lim_k \frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla w_{1,k}|^2 + |\nabla w_{2,k}|^2)dxdy}{\Big(\displaystyle\int_B |x|^\alpha |w_{1,k}(x,0)|^{p_k}|w_{2,k}(x,0)|^q dx\Big)^{2/(p_k+q)}}$$

$$\geq \frac{1}{\Lambda(\sigma)} \lim_k \frac{\displaystyle\int_{C_B} k_s y^{1-2s}(|\nabla w_{1,k}|^2 + |\nabla w_{2,k}|^2)dxdy}{\Big(\displaystyle\int_B |w_{1,k}(x,0)|^{p_k}|w_{2,k}(x,0)|^q dx\Big)^{2/(p_k+q)}} > C_{p,q,\mathscr{S}^0_{s,2_s^*}},$$

which is a contradiction, since $\Lambda^{-1}(\sigma) > 1$. The proof of Theorem 1.3 is complete.

4 Proof of Theorem 1.4

Let $(w_{1,\varepsilon}, w_{2,\varepsilon}) \in H$ be a nonnegative solution to (11). Then, up to a multiplicative constant depending upon the Lagrange multiplier, we may assume that $(w_{1,\varepsilon}, w_{2,\varepsilon})$ solves the system (12). In particular, identity (13) follows. Hence, from Lemma 2.2, we infer

$$\lim_{\varepsilon\to 0}\int_{C_B} k_s y^{1-2s}(|\nabla w_{1,\varepsilon}|^2 + |\nabla w_{2,\varepsilon}|^2)dxdy = \frac{(\mathscr{S}^0_{s,p,q}(C_B))^{\frac{N}{2s}}}{2^{\frac{N-2s}{2s}}} = \frac{(\mathscr{S}^0_{s,p,q}(\mathbb{R}^{N+1}_+))^{\frac{N}{2s}}}{2^{\frac{N-2s}{2s}}}. \tag{17}$$

We know that $(w_{1,\varepsilon}, w_{2,\varepsilon})$ is a solution of the system

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_{1,\varepsilon}) = 0, \quad -\mathrm{div}(y^{1-2s}\nabla w_{2,\varepsilon}) = 0, & x \in C_B, \\ k_s y^{1-2s}\frac{\partial w_{1,\varepsilon}}{\partial \nu} = \frac{2p_\varepsilon}{p_\varepsilon+q}|x|^\alpha w_{1,\varepsilon}(x,0)^{p_\varepsilon-1}w_{2,\varepsilon}(x,0)^q, & x \in B, \\ k_s y^{1-2s}\frac{\partial w_{2,\varepsilon}}{\partial \nu} = \frac{2q}{p_\varepsilon+q}|x|^\alpha w_{1,\varepsilon}(x,0)^{p_\varepsilon}w_{2,\varepsilon}(x,0)^{q-1}, & x \in B, \\ w_{1,\varepsilon} = w_{2,\varepsilon} = 0, & x \in \partial_L C_B. \end{cases}$$

Then, we can assume $w_{i,\varepsilon} \in C^\tau(B)$, for some $\tau \in (0,1)$. There exist $x_{1,\varepsilon}, x_{2,\varepsilon} \in \overline{B}$ such that

$$M_{i,\varepsilon} = w_{i,\varepsilon}(x_{i,\varepsilon},0) = \sup_{(x,y)\in\overline{B}\times(0,\infty)} w_{i,\varepsilon}(x,y), \quad i=1,2. \tag{18}$$

In fact, let $x_{1,\varepsilon}, x_{2,\varepsilon} \in \overline{B}$ be such that

$$M_{i,\varepsilon} := \sup_{x\in\overline{B}} w_{i,\varepsilon}(x,0) = w_{i,\varepsilon}(x_{i,\varepsilon},0), \quad i=1,2.$$

Then the second equality in (18) holds, since we have the following maximum principle

Lemma 4.1. *$w_{i,\varepsilon}(x,y) \le M_{i,\varepsilon}$ for a.e. $(x,y) \in B \times (0,\infty)$, for $i = 1,2$.*

Proof. Define $\tau_i(x,y) := (w_{i,\varepsilon}(x,y) - M_{i,\varepsilon})^+$ for $(x,y) \in B \times (0,\infty)$. Then, testing in (LS), we obtain

$$k_s \int_{C_B} y^{1-2s} |\nabla \tau_1(x,y)|^2 dxdy = \frac{2p_\varepsilon}{p_\varepsilon + q} \int_B |x|^\alpha w_{1,\varepsilon}(x,0)^{p_\varepsilon - 1} w_{2,\varepsilon}(x,0)^q \tau_1(x,0) dx = 0,$$

$$k_s \int_{C_B} y^{1-2s} |\nabla \tau_2(x,y)|^2 dxdy = \frac{2q}{p_\varepsilon + q} \int_B |x|^\alpha w_{1,\varepsilon}(x,0)^{p_\varepsilon} w_{2,\varepsilon}(x,0)^{q-1} \tau_2(x,0) dx = 0.$$

Then $\tau_i \equiv 0$ for $i = 1,2$, yielding he conclusion. □

Lemma 4.2. *For all $i = 1,2$, we have $M_{i,\varepsilon} \to +\infty$ as $p_\varepsilon + q \to 2_s^*$.*

Proof. Suppose by contradiction that there exist $C > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ such that $p_{\varepsilon_n} + q \to 2_s^*$ and $M_{2,\varepsilon_n} \le C$, for all $n \in \mathbb{N}$. Since (w_{i,ε_n}) is bounded in $H^1_{0,L}(C_B)$, up to a subsequence, by the conclusions of Theorem 1.3 we get $w_{i,\varepsilon_n} \rightharpoonup 0$ weakly in $H^1_{0,L}(C_B)$ and $w_{i,\varepsilon_n} \to 0$ in $L^r(B)$, for every $r < 2_s^*$. Then, from identity (13) and formula (17), there exists a positive constant σ independent of ε_n such that

$$\begin{aligned} 0 < \sigma \le \int_B |x|^\alpha |w_{1,\varepsilon_n}(x,0)|^{p_\varepsilon} |w_{2,\varepsilon_n}(x,0)|^q dx \\ \le \|w_{2,\varepsilon_n}(x,0)\|_\infty^q \|w_{1,\varepsilon_n}(x,0)\|_{L^{p_{\varepsilon_n}}(B)}^{p_{\varepsilon_n}} \le Co_n(1), \end{aligned}$$

which yields a contradiction. □

Now we want to recall some general Pohožaev type identity. Consider the following system

$$(LG) \qquad \begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_1) = 0, \text{ in } C_B = B \times (0,\infty), \\ -\mathrm{div}(y^{1-2s}\nabla w_2) = 0, \text{ in } C_B = B \times (0,\infty), \\ w_1 = w_2 = 0, \text{ on } \partial_L C_B = \partial B \times (0,\infty), \\ k_s y^{1-2s} \frac{\partial w_1}{\partial \nu} = C_1 w_1(x,0)^{p-1} w_2(x,0)^q, \text{ on } B \times \{0\}, \\ k_s y^{1-2s} \frac{\partial w_2}{\partial \nu} = C_2 w_1(x,0)^p w_2(x,0)^{q-1}, \text{ on } B \times \{0\}, \end{cases}$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative, and ν is exterior normal vector to ∂B. For the scalar case, the next result was obtained in [7], while for the system we refer to [19].

Theorem 4.3. *Let $p + q = 2_s^*$. Then system (LG) does not admit any nontrivial nonnegative solution.*

The following nonexistence result is crucial for our argument. Consider the following problem

$$(LS) \qquad \begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_1) = 0, \text{ in } \mathbb{R}^{N+1}_{++}, \\ -\mathrm{div}(y^{1-2s}\nabla w_2) = 0, \text{ in } \mathbb{R}^{N+1}_{++}, \\ w_1 = w_2 = 0, \text{ on } \{x_N = 0, y > 0\}, \\ k_s y^{1-2s}\frac{\partial w_1}{\partial \nu} = C_1 w_1(x,0)^{p-1} w_2(x,0)^q, \text{ on } \{x_N > 0, y = 0\}, \\ k_s y^{1-2s}\frac{\partial w_2}{\partial \nu} = C_2 w_1(x,0)^{p} w_2(x,0)^{q-1}, \text{ on } \{x_N > 0, y = 0\}, \end{cases}$$

where $C_1, C_2 > 0$, $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative, $p + q = 2^*_s$ and

$$\mathbb{R}^{N+1}_{++} := \{(x_1, x_2, \ldots, x_{N-1}, x_N, y) \in \mathbb{R}^{N+1} : x_N > 0, y > 0\}.$$

Proposition 4.4. *Let $w_1, w_2 \in H^1_{0,L}(\mathbb{R}^{N+1}_{++})$ be a bounded solution of (LS). Then $(w_1, w_2) = (0, 0)$.*

Proof. Since $(x, y)\cdot \nu = 0$ on $\partial\mathbb{R}^{N+1}_{++}$, one cannot apply directly Pohožaev identities. Whence, we use the Kelvin transformation as in [19, 21] to study a new system set in a ball. Let $w_i \in H^1_{0,L}(\mathbb{R}^{N+1}_{++})$ be a solution to system (*LS*). Then, the Kelvin transformation of w_i is defined by

$$\widetilde{w_i}(z) = |z|^{2s-N} w_i\Big(\frac{z}{|z|^2}\Big), \quad z \in \mathbb{R}^{N+1}_{++}.$$

and from [21, Proposition 2.6] we infer that $\widetilde{w_i}$ is also a solution to (*LS*). By [25, Corollary 2.1, Proposition 2.4], there exists $\gamma \in (0, 1)$ with $\widetilde{w_i}(z) \le C|z|^\gamma$, for $z \in B_1(0)$. Then there exists $C > 0$ such that

$$|w_i(z)| \le C(1 + |z|^2)^{-\frac{N-2s+\gamma}{2}}, \quad \text{for all } z \in \mathbb{R}^{N+1}_{++}. \tag{19}$$

Arguing as in [18], denote by $B_{\frac{1}{2}}(\frac{e_N}{2}) \subset \mathbb{R}^N$ the ball centered at $\frac{e_N}{2}$ with radius $\frac{1}{2}$. Define

$$v_i(z) := |z|^{2s-N} w_i\Big(-(e_N, 0) + \frac{z}{|z|^2}\Big), \quad \text{for all } z \in \overline{C_{B_{\frac{1}{2}}(\frac{e_N}{2})}} \setminus \{0\}.$$

By means of (19), for a positive constant C and for $|z|$ small enough, we have

$$v_i(z) \le C|z|^\gamma, \quad \text{for all } z \in \mathbb{R}^{N+1}_{++} \setminus \{0\}.$$

Therefore, we may extend v_i by 0 at 0. Then, as above, (v_1, v_2) is a weak solution of the system

$$(LSB)\qquad \begin{cases} -\mathrm{div}(y^{1-2s}\nabla v_1) = 0, \text{ in } C_{B_{\frac{1}{2}}(\frac{e_N}{2})}, \\ -\mathrm{div}(y^{1-2s}\nabla v_2) = 0, \text{ in } C_{B_{\frac{1}{2}}(\frac{e_N}{2})}, \\ v_1 = v_2 = 0, \text{ on } \partial_L C_{B_{\frac{1}{2}}(\frac{e_N}{2})}, \\ k_s y^{1-2s}\frac{\partial v_1}{\partial \nu} = C_1 v_1(x,0)^{p-1} v_2(x,0)^q, \text{ on } B_{\frac{1}{2}}(\frac{e_N}{2}) \times \{0\}, \\ k_s y^{1-2s}\frac{\partial v_2}{\partial \nu} = C_2 v_1(x,0)^{p} v_2(x,0)^{q-1}, \text{ on } B_{\frac{1}{2}}(\frac{e_N}{2}) \times \{0\}. \end{cases}$$

Now, applying Theorem 4.3 to system (*LSB*) we infer that $v_i = 0, i = 1, 2$, that is, $w_i = 0, i = 1, 2$. □

We are now ready to complete the proof. By Lemma 4.2, we may assume

$$M_{1,\varepsilon} = w_{1,\varepsilon}(x_{1,\varepsilon}, 0) = \sup_{(x,y)\in \overline{B}\times(0,\infty)} w_{1,\varepsilon}(x,y) \to +\infty,$$

We may assume $M_{1,\varepsilon} \geq M_{2,\varepsilon}$. Let $\lambda_\varepsilon > 0$ be such that

$$\lambda_\varepsilon^{\frac{N-2s}{2}} M_{1,\varepsilon} = 1, \qquad 0 \leq \lambda_\varepsilon^{\frac{N-2s}{2}} M_{2,\varepsilon} \leq 1,$$

where $\lambda_\varepsilon \to 0$, as $p_\varepsilon + q \to 2_s^*$. Define the scaled functions

$$\tilde{w}_{1,\varepsilon}(x,y) := \lambda_\varepsilon^{\frac{N-2s}{2}} w_{1,\varepsilon}(\lambda_\varepsilon x + x_{1,\varepsilon}, \lambda_\varepsilon y),$$

$$\tilde{w}_{2,\varepsilon}(x,y) := \lambda_\varepsilon^{\frac{N-2s}{2}} w_{2,\varepsilon}(\lambda_\varepsilon x + x_{1,\varepsilon}, \lambda_\varepsilon y),$$

$B_\varepsilon := \{x \in \mathbb{R}^N : \lambda_\varepsilon x + x_{1,\varepsilon} \in B_1(0)\}$ and $C_{B_\varepsilon} := B_\varepsilon \times (0,\infty)$. Then $(\tilde{w}_{1,\varepsilon}(x,y), \tilde{w}_{1,\varepsilon}(x,y))$ satisfies

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla \tilde{w}_{1,\varepsilon}) = 0, \quad -\mathrm{div}(y^{1-2s}\nabla \tilde{w}_{2,\varepsilon}) = 0, \ x \in C_{B_\varepsilon} \\ 0 < \tilde{w}_{1,\varepsilon} \leq 1, \ 0 < \tilde{w}_{2,\varepsilon} \leq 1, \ \tilde{w}_{1,\varepsilon}(0,0) = 1, \ x \in C_{B_\varepsilon} \\ k_s y^{1-2s}\frac{\partial \tilde{w}_{1,\varepsilon}}{\partial \nu} = \frac{2p_\varepsilon}{p_\varepsilon+q}|\lambda_\varepsilon x + x_{1,\varepsilon}|^\alpha \lambda_\varepsilon^{N-\frac{N-2s}{2}(p_\varepsilon+q)} \tilde{w}_{1,\varepsilon}(x,0)^{p_\varepsilon-1}\tilde{w}_{2,\varepsilon}(x,0)^q, \ x \in B_\varepsilon, \\ k_s y^{1-2s}\frac{\partial \tilde{w}_{2,\varepsilon}}{\partial \nu} = \frac{2q}{p_\varepsilon+q}|\lambda_\varepsilon x + x_{1,\varepsilon}|^\alpha \lambda_\varepsilon^{N-\frac{N-2s}{2}(p_\varepsilon+q)} \tilde{w}_{1,\varepsilon}(x,0)^{p_\varepsilon}\tilde{w}_{2,\varepsilon}(x,0)^{q-1}, \ x \in B_\varepsilon, \\ \tilde{w}_{1,\varepsilon} = 0, \quad \tilde{w}_{2,\varepsilon} = 0, \ x \in \partial B_\varepsilon \cap \partial_L C_{B_\varepsilon}. \end{cases}$$

Suppose $x_{1,\varepsilon} \to x_0$ for some $x_0 \in \bar{B}_1(0)$. We claim that $x_0 \in \partial B_1(0)$. By contradiction, assume that $x_0 \in B_1(0)$ and let $d := \frac{1}{2}\mathrm{dist}(x_0, \partial B_1(0))$. Denote $\mathscr{B}(0,r) = \{z \in \mathbb{R}^{N+1} : |z| < r\}$. For $\varepsilon > 0$ small, both $\tilde{w}_{1,\varepsilon}$ and $\tilde{w}_{2,\varepsilon}$ are well defined in $\mathscr{B}(0, d/\lambda_\varepsilon) \cap \mathbb{R}^{N+1}_+$, and

$$\sup_{(x,y)\in\mathscr{B}(0,\frac{d}{\lambda_\varepsilon})\cap\mathbb{R}^{N+1}_+} \tilde{w}_{1,\varepsilon}(x,y) = \tilde{w}_{1,\varepsilon}(0,0) = 1, \qquad \sup_{(x,y)\in\mathscr{B}(0,\frac{d}{\lambda_\varepsilon})\cap\mathbb{R}^{N+1}_+} \tilde{w}_{2,\varepsilon}(x,y) \in (0,1].$$

Since $M_{1,\varepsilon} \to +\infty$, we have $0 \le \lambda_\varepsilon \le 1$, for $\varepsilon > 0$ small. Let

$$h(\varepsilon) := \lambda_\varepsilon^{N-\frac{N-2s}{2}(p_\varepsilon+q)} \quad \text{and} \quad h(0) := \lim_{p_\varepsilon+q\to 2_s^*} h(\varepsilon).$$

Hence, $0 \le h(\varepsilon) \le 1$. Three possibilities may occur, namely

(1) $h(0) = 0$,
(2) $h(0) = \beta \in (0,1)$,
(3) $h(0) = 1$.

We show that any of these cases yields a contradiction. We observe that, for any $R > 0$, $B_R(0) \subset B_{d/\lambda_\varepsilon}(0)$ for $\varepsilon > 0$ small enough. By Schauder estimates [7, 12, 14, 25] there are $C > 0$ and $0 < \vartheta < 1$ with

$$\|\tilde{w}_{1,\varepsilon}\|_{C^{0,\vartheta}(\mathscr{B}(0,2R)\cap\mathbb{R}^{N+1}_+)} \le C, \quad \|\tilde{w}_{2,\varepsilon}\|_{C^{0,\vartheta}(\mathscr{B}(0,2R)\cap\mathbb{R}^{N+1}_+)} \le C$$

for ε small enough. By Arzelà-Ascoli's Theorem, there exist subsequences $\{\tilde{w}_{i,\varepsilon_k}\}$ such that $\tilde{w}_{i,\varepsilon_k} \to w_i$ as $k \to \infty$, for $i = 1, 2$, in $C^{0,\vartheta_0}_{\text{loc}}$ for some $\vartheta_0 \in (0, \vartheta)$. Then, we derive that (w_1, w_2) satisfies

$$\begin{cases} -\text{div}(y^{1-2s}\nabla w_1) = 0, \quad -\text{div}(y^{1-2s}\nabla w_2) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ k_s y^{1-2s}\frac{\partial w_1}{\partial \nu} = \frac{2p}{2_s^*}|x_0|^\alpha h(0) w_1^{p-1}(x,0) w_2^q(x,0), & \text{on } \mathbb{R}^N \times \{0\}, \\ k_s y^{1-2s}\frac{\partial w_2}{\partial \nu} = \frac{2q}{2_s^*}|x_0|^\alpha h(0) w_1^{p}(x,0) w_2^{q-1}(x,0), & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{20}$$

and $w_1(0,0) = 1$, $0 \le w_2 \le 1$. Moreover, $w_i \in H^1_{0,L}(\mathbb{R}^{N+1}_+)$. If $x_0 = 0$ or $h(0) = 0$ or $w_2 = 0$, we have $w_1 \equiv 0$, which is impossible since $w_1(0,0) = 1$. Suppose $x_0 \neq 0$, $w_2 \not\equiv 0$ and $h(0) = \beta \in (0,1]$. Then

$$\begin{cases} -\text{div}(y^{1-2s}\nabla w_1) = 0, \quad -\text{div}(y^{1-2s}\nabla w_2) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ k_s y^{1-2s}\frac{\partial w_1}{\partial \nu} = \frac{2p}{2_s^*}\Lambda w_1^{p-1}(x,0) w_2^q(x,0), & \text{on } \mathbb{R}^N \times \{0\}, \\ k_s y^{1-2s}\frac{\partial w_2}{\partial \nu} = \frac{2q}{2_s^*}\Lambda w_1^{p}(x,0) w_2^{q-1}(x,0), & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{21}$$

where $\Lambda = |x_0|^\alpha \beta \in (0,1)$. Setting

$$\bar{w}_1 := \Lambda^{\frac{1}{2_s^*-2}} w_1, \qquad \bar{w}_2 := \Lambda^{\frac{1}{2_s^*-2}} w_2,$$

we have $0 < \bar{w}_1 \le \Lambda^{\frac{1}{2_s^*-2}}$, $0 < \bar{w}_2 \le \Lambda^{\frac{1}{2_s^*-2}}$, $\bar{w}_1(0,0) = \Lambda^{\frac{1}{2_s^*-2}}$ and $(\bar{w}_1, \bar{w}_2)$ satisfies

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla \bar{w}_1) = 0, \quad -\mathrm{div}(y^{1-2s}\nabla \bar{w}_2) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ k_s y^{1-2s}\frac{\partial \bar{w}_1}{\partial \nu} = \frac{2p}{2^*_s}\bar{w}_1^{p-1}(x,0)\bar{w}_2^q(x,0), & \text{on } \mathbb{R}^N \times \{0\}, \\ k_s y^{1-2s}\frac{\partial \bar{w}_2}{\partial \nu} = \frac{2q}{2^*_s}\bar{w}_1^{p}(x,0)\bar{w}_2^{q-1}(x,0), & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{22}$$

Define $\mathscr{S} := \mathscr{S}^0_{s,p,q}(\mathbb{R}^{N+1}_+)$ and observe that

$$\int_{\mathbb{R}^{N+1}_+} k_s y^{1-2s}(|\nabla \bar{w}_1|^2 + |\nabla \bar{w}_2|^2)dxdy = 2\int_{\mathbb{R}^N} \bar{w}_1^p(x,0)\bar{w}_2^q(x,0)dx.$$

Then, by formula (17), we have

$$\begin{aligned} \mathscr{S}^{\frac{N}{2s}} &\le 2^{\frac{N-2s}{2s}} \int_{\mathbb{R}^{N+1}_+} k_s y^{1-2s}(|\nabla \bar{w}_1|^2 + |\nabla \bar{w}_2|^2)dxdy \\ &= \Lambda^{\frac{N-2s}{2s}} 2^{\frac{N-2s}{2s}} \int_{\mathbb{R}^{N+1}_+} k_s y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)dxdy \\ &\le \Lambda^{\frac{N-2s}{2s}} \liminf_{\varepsilon\to 0} 2^{\frac{N-2s}{2s}} \int_{C_{B_\varepsilon}} k_s y^{1-2s}(|\nabla \tilde{w}_{1,\varepsilon}|^2 + |\nabla \tilde{w}_{2,\varepsilon}|^2)dxdy \\ &= \Lambda^{\frac{N-2s}{2s}} \liminf_{\varepsilon\to 0} 2^{\frac{N-2s}{2s}} \int_{C_B} k_s y^{1-2s}(|\nabla w_{1,\varepsilon}|^2 + |\nabla w_{2,\varepsilon}|^2)dxdy \\ &= \Lambda^{\frac{N-2s}{2s}} \mathscr{S}^{\frac{N}{2s}} < \mathscr{S}^{\frac{N}{2s}}, \end{aligned} \tag{23}$$

a contradiction. Then $x_0 \in \partial B_1(0)$. We can straighten ∂B in a neighborhood of x_0 by a non-singular C^1 change of coordinates. Let $x_N = \psi(x')$ be the equation of ∂B, where $x' = (x_1, x_2, \ldots, x_{N-1})$, $\psi \in C^1$. Define new coordinate system given by $z_i = x_i$ for $i = 1, \ldots, N-1$, $z_N = x_N - \psi(x')$ and $z_{N+1} = y$. Let $d_\varepsilon = \mathrm{dist}(x_\varepsilon, \partial B)$. For $p_\varepsilon + q \to 2^*_s$ as $\varepsilon \to 0$, $\tilde{w}_{i,\varepsilon}$ are well defined in $B(0, \frac{\delta}{\lambda_\varepsilon}) \cap \mathbb{R}^{N+1}_+ \cap \{z_N > -\frac{d_\varepsilon}{\lambda_\varepsilon}\}$ for some $\delta > 0$ small enough. Moreover

$$\sup_{B(0,\frac{\delta}{\lambda_\varepsilon})\cap\mathbb{R}^{N+1}_+\cap\{z_N>-\frac{d_\varepsilon}{\lambda_\varepsilon}\}} \tilde{w}_{1,\varepsilon}(x,y) = \tilde{w}_{1,\varepsilon}(0,0) = 1,$$

$$\sup_{B(0,\frac{\delta}{\lambda_\varepsilon})\cap\mathbb{R}^{N+1}_+\cap\{z_N>-\frac{d_\varepsilon}{\lambda_\varepsilon}\}} \tilde{w}_{2,\varepsilon}(x,y) \in (0,1].$$

We now have the following

Claim: $d_\varepsilon/\lambda_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

Verification: Suppose by contradiction that $d_\varepsilon/\lambda_\varepsilon$ remains bounded and $d_\varepsilon/\lambda_\varepsilon \to s$ for some $s \ge 0$. By the previous argument, since $|x_0| = 1$, we get $\tilde{w}_{i,\varepsilon} \to \tilde{w}_i$ in $C^{0,\gamma}_{\mathrm{loc}}$, $\tilde{w}_1(0,0) = 1$ and

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla\tilde{w}_1) = 0, \text{ in } \{(z_1,\dots,z_N,z_{N+1}) : z_N > -s, z_{N+1} > 0\}, \\ \operatorname{div}(y^{1-2s}\nabla\tilde{w}_2) = 0, \text{ in } \{(z_1,\dots,z_N,z_{N+1}) : z_N > -s, z_{N+1} > 0\}, \\ k_s y^{1-2s}\frac{\partial\tilde{w}_1}{\partial\nu} = \Lambda\frac{2p}{2_s^*}\tilde{w}_1^{p-1}(x,0)\tilde{w}_2^q(x,0), \text{ on } \{(z_1,\dots,z_{N+1}) : z_N > -s, z_{N+1} = 0\}, \\ k_s y^{1-2s}\frac{\partial\tilde{w}_2}{\partial\nu} = \Lambda\frac{2p}{2_s^*}\tilde{w}_1^{p}(x,0)\tilde{w}_2^{q-1}(x,0), \text{ on } \{(z_1,\dots,z_{N+1}) : z_N > -s, z_{N+1} = 0\}, \\ \tilde{w}_1(z) = \tilde{w}_2(z) = 0, \text{ on } \{(z_1,\dots,z_N,z_{N+1}) : z_N = -s, z_{N+1} > 0\}, \\ \tilde{w}_1(z),\tilde{w}_2(z) \in (0,1], \text{ in } \{(z_1,\dots,z_N,z_{N+1}) : z_N > -s, z_{N+1} > 0\}. \end{cases} \tag{24}$$

By a translation, $(\tilde{w}_1,\tilde{w}_2)$ verifies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla\tilde{w}_1) = 0, \quad \operatorname{div}(y^{1-2s}\nabla\tilde{w}_2) = 0, \text{ in } \mathbb{R}^{N+1}_{++}, \\ k_s y^{1-2s}\frac{\partial\tilde{w}_1}{\partial\nu} = \Lambda\frac{2p}{2_s^*}\tilde{w}_1^{p-1}(x,0)\tilde{w}_2^q(x,0), \text{ on } \{(z_1,\dots,z_N,z_{N+1}) : z_N > 0, z_{N+1} = 0\}, \\ k_s y^{1-2s}\frac{\partial\tilde{w}_2}{\partial\nu} = \Lambda\frac{2p}{2_s^*}\tilde{w}_1^{p}(x,0)\tilde{w}_2^{q-1}(x,0), \text{ on } \{(z_1,\dots,z_N,z_{N+1}) : z_N > 0, z_{N+1} = 0\}, \\ \tilde{w}_1(z) = \tilde{w}_2(z) = 0, \text{ on}\{(z_1,\dots,z_N,z_{N+1}) : z_N = 0, z_{N+1} > 0\}, \\ \tilde{w}_2(z) \in (0,1], \tilde{w}_1(0,\dots,s,0) = 1, \text{ in}\mathbb{R}^{N+1}_{++}. \end{cases} \tag{25}$$

Since $\tilde{w}_i \in H^1_{0,L}(\mathbb{R}^{N+1}_{++})$, by Proposition 4.4, $(\tilde{w}_1,\tilde{w}_2) = (0,0)$, which violates $\tilde{w}_1(0,\dots,s,0) = 1$. Then the claim follows and C_{B_ε} converges to the entire $\mathbb{R}^{N+1}_{++}$ as $\varepsilon \to 0$.

Claim. $\Lambda = |x_0|h(0) = h(0) = 1$. We can assume $(\tilde{w}_{1,\varepsilon},\tilde{w}_{2,\varepsilon}) \to (\tilde{w}_1,\tilde{w}_2)$, as $\varepsilon \to 0$, and $(\tilde{w}_1,\tilde{w}_2)$ satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla\tilde{w}_1) = 0, \quad \operatorname{div}(y^{1-2s}\nabla\tilde{w}_2) = 0, & \text{in } \mathbb{R}^{N+1}_{++}, \\ k_s y^{1-2s}\frac{\partial\tilde{w}_1}{\partial\nu} = \Lambda\frac{2p}{2_s^*}\tilde{w}_1^{p-1}(x,0)\tilde{w}_2^q(x,0), & \text{on } \mathbb{R}^N\times\{0\}, \\ k_s y^{1-2s}\frac{\partial\tilde{w}_2}{\partial\nu} = \Lambda\frac{2p}{2_s^*}\tilde{w}_1^{p}(x,0)\tilde{w}_2^{q-1}(x,0), & \text{on } \mathbb{R}^N\times\{0\}, \\ \tilde{w}_1(z) = \tilde{w}_2(z) = 0, & \text{on } \{0\}\times(0,\infty), \\ \tilde{w}_i(z) \in (0,1], \tilde{w}_i(0,0) = 1, & \text{in } \mathbb{R}^{N+1}_{++}. \end{cases} \tag{26}$$

If $\tilde{w}_2 \equiv 0$ or $0 \le \Lambda < 1$, we reach the contradiction either as in (23) or by Proposition 4.4. Hence, $\Lambda = 1$ and $\tilde{w}_2 \not\equiv 0$. This implies $M_{1,\varepsilon}^{-1}\tilde{w}_{2,\varepsilon}(\lambda_\varepsilon x + x_\varepsilon) \to v(x) \not\equiv 0$, and then $1 \ge M_{1,\varepsilon}^{-1}M_{2,\varepsilon} \to \sigma > 0$ as $\varepsilon \to 0$, that is $M_{1,\varepsilon} = \mathcal{O}(1)M_{2,\varepsilon}$. This is (i) of Theorem 1.4.

Let $y_\varepsilon \in B_1(0)$ be such that $w_{2,\varepsilon}(y_\varepsilon) = \max_{B_1(0)} w_{2,\varepsilon}(y)$. We define $\tilde{w}_{2,\varepsilon}(x) = (\bar{\lambda}_\varepsilon)^{(N-2s)/2}w_{2,\varepsilon}(\bar{\lambda}_\varepsilon x + y_\varepsilon)$, where $\bar{\lambda}_\varepsilon^{(N-2s)/2}w_{2,\varepsilon}(y_\varepsilon) = 1$. Suppose $y_\varepsilon \to y_0$. Again, using a blow-up argument, we get $y_0 \in \partial B_1(0)$. Then, in light of Lemma 2.7, we have

$$\tilde{w}_1(x,y) = a\mathscr{W}_1(x,y), \qquad \tilde{w}_2(x,y) = b\mathscr{W}_1(x,y)$$

for some positive numbers a, b such that $a/b = \sqrt{p/q}$. Let $\tilde{v}_{i,\varepsilon} = \tilde{w}_{i,\varepsilon} - \tilde{w}_i$. Then $\tilde{v}_{i,\varepsilon} \rightharpoonup 0$ weakly in $H^1_{0,L}(C_\omega)$ for any $\omega \subset \mathbb{R}^{N+1}_+$ and

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla \tilde{v}_{1,\varepsilon}) = 0, \quad \operatorname{div}(y^{1-2s}\nabla \tilde{v}_{2,\varepsilon}) = 0, & \text{in } C_{B_\varepsilon}, \\ \kappa_{2s} y^{1-2s}\frac{\partial \tilde{v}_{1,\varepsilon}}{\partial \nu} = \frac{2p_\varepsilon}{p_\varepsilon+q} Q_\varepsilon \tilde{w}^{p_\varepsilon-1}_{1,\varepsilon}(x,0)\tilde{w}^q_{2,\varepsilon}(x,0) - \frac{p(N-2s)}{N} w_1^{p-1} w_2^q & \text{on } B_\varepsilon \times \{0\} \\ \kappa_{2s} y^{1-2s}\frac{\partial \tilde{v}_{2,\varepsilon}}{\partial \nu} = \frac{2q}{p_\varepsilon+q} Q_\varepsilon \tilde{w}^{p_\varepsilon}_{1,\varepsilon}(x,0)\tilde{w}^{q-1}_{2,\varepsilon}(x,0) - \frac{q(N-2s)}{N} w_1^{p} w_2^{q-1} & \text{on } B_\varepsilon \times \{0\} \\ \tilde{v}_{i,\varepsilon} = -\tilde{w}_i, & \text{on } \partial_L C_{B_\varepsilon}, \end{cases} \tag{27}$$

where we have set $Q_\varepsilon := |\lambda_\varepsilon x + x_{1,\varepsilon}|^\alpha h(\varepsilon)$. Multiplying the first equation in (27) by $\tilde{v}_{1,\varepsilon}$ and $\tilde{v}_{2,\varepsilon}$, respectively, integrating by parts, and applying Brézis-Lieb Lemma, as $p_\varepsilon + q \to 2^*_s$, we have

$$\begin{aligned} & k_s \int_{C_{B_\varepsilon}} y^{1-2s}(|\nabla \tilde{v}_{1,\varepsilon}|^2 + |\nabla \tilde{v}_{2,\varepsilon}|^2)dxdy \\ & = \int_{B_\varepsilon} \Big(\frac{2p_\varepsilon}{p_\varepsilon+q} Q_\varepsilon \tilde{w}^{p_\varepsilon-1}_{1,\varepsilon}(x,0)\tilde{w}^q_{2,\varepsilon}(x,0) - \frac{p(N-2s)}{N}\tilde{w}_1^{p-1}(x,0)\tilde{w}_2^q(x,0)\Big)\tilde{v}_{1,\varepsilon}(x,0)dx \\ & - k_s \int_{\partial_L B_\varepsilon} y^{1-2s}\frac{\partial \tilde{v}_{1,\varepsilon}}{\partial \nu} w_1 dS \\ & + \int_{B_\varepsilon} \Big(\frac{2q}{p_\varepsilon+q} Q_\varepsilon \tilde{w}^{p_\varepsilon}_{1,\varepsilon}(x,0)\tilde{w}^{q-1}_{2,\varepsilon}(x,0) - \frac{q(N-2s)}{N}\tilde{w}_1^{p}(x,0)\tilde{w}_2^{q-1}(x,0)\Big)\tilde{v}_{2,\varepsilon}(x,0)dx \\ & - k_s \int_{\partial_L B_\varepsilon} y^{1-2s}\frac{\partial \tilde{v}_{2,\varepsilon}}{\partial \nu} w_2 dS \\ & = \frac{p(N-2s)}{N}\int_{B_\varepsilon} Q_\varepsilon \Big(\tilde{w}^{p_\varepsilon-1}_{1,\varepsilon}(x,0)\tilde{w}^q_{2,\varepsilon}(x,0) - \tilde{w}_1^{p-1}(x,0)\tilde{w}_2^q(x,0)\Big)\tilde{v}_{1,\varepsilon}(x,0)dx \\ & - k_s \int_{\partial_L B_\varepsilon} y^{1-2s}\frac{\partial \tilde{v}_{1,\varepsilon}}{\partial \nu}\tilde{w}_1 dS + \frac{p(N-2s)}{N}\int_{B_\varepsilon}(Q_\varepsilon - 1)\tilde{w}_1^{p-1}(x,0)\tilde{w}_2^q(x,0)dx \\ & + \frac{q(N-2s)}{N}\int_{B_\varepsilon} Q_\varepsilon \Big(\tilde{w}^{p_\varepsilon}_{1,\varepsilon}(x,0)\tilde{w}^{q-1}_{2,\varepsilon}(x,0) - \tilde{w}_1^{p}(x,0)\tilde{w}_2^{q-1}(x,0)\Big)\tilde{v}_{2,\varepsilon}(x,0)dx \\ & - k_s \int_{\partial_L B_\varepsilon} y^{1-2s}\frac{\partial \tilde{v}_{2,\varepsilon}}{\partial \nu}\tilde{w}_2 dS + \frac{q(N-2s)}{N}\int_{B_\varepsilon}(Q_\varepsilon - 1)\tilde{w}_1^{p}(x,0)\tilde{w}_2^{q-1}(x,0)dx + o_\varepsilon(1), \end{aligned}$$

since $Q_\varepsilon \to 1$ as $\varepsilon \to 0$. Recalling that $\mathscr{W}_\varepsilon$ decay at infinity, we obtain

$$\begin{aligned} & k_s \int_{C_{B_\varepsilon}} y^{1-2s}(|\nabla \tilde{v}_{1,\varepsilon}|^2 + |\nabla \tilde{v}_{2,\varepsilon}|^2)dxdy \\ & \qquad = \frac{p(N-2s)}{N}\int_{B_\varepsilon} Q_\varepsilon \Big((\tilde{v}_{1,\varepsilon} + \tilde{w}_1)^{p_\varepsilon-1}(x,0)(\tilde{v}_{2,\varepsilon} + \tilde{w}_2)^q(x,0) \end{aligned}$$

$$- \tilde{w}_1^{p-1}(x,0)\tilde{w}_2^q(x,0)\Big)\tilde{v}_{1,\varepsilon}(x,0)dx$$
$$+ \frac{q(N-2s)}{N}\int_{B_\varepsilon} Q_\varepsilon\Big((\tilde{v}_{1,\varepsilon}+\tilde{w}_1)^{p_\varepsilon}(x,0)(\tilde{v}_{2,\varepsilon}+\tilde{w}_2)^{q-1}(x,0)$$
$$- \tilde{w}_1^p(x,0)\tilde{w}_2^{q-1}(x,0)\Big)\tilde{v}_{2,\varepsilon}(x,0)dx + o_\varepsilon(1).$$

Inserting $\tilde{w}_{i,\varepsilon} = \tilde{v}_{i,\varepsilon} = \tilde{w}_i$, and using the following inequalities (cf. [8, 34])

$$||a+b|^p - |a|^p - |b|^p - pab(|a|^{p-2}+|b|^{p-2})| \le C\begin{cases} |a||b|^{p-1} \text{ if } |a| \ge |b|, \\ |a|^{p-1}|b| \text{ if } |a| \le |b|, \\ 1 \le p \le 3, \end{cases}$$
$$||a+b|^p - |a|^p - |b|^p - pab(|a|^{p-2}+|b|^{p-2})| \le C(|a|^{p-2}|b|^2 + |a|^2|b|^{p-2}), \quad p \ge 3,$$

we infer that

$$k_s\int_{C_{B_\varepsilon}} y^{1-2s}(|\nabla\tilde{v}_{1,\varepsilon}|^2 + |\nabla\tilde{v}_{2,\varepsilon}|^2)dxdy = \frac{p(N-2s)}{N}\int_{B_\varepsilon} Q_\varepsilon\tilde{v}_{1,\varepsilon}^p(x,0)\tilde{v}_{2,\varepsilon}^q(x,0)dx$$
$$+\frac{q(N-2s)}{N}\int_{B_\varepsilon} Q_\varepsilon\tilde{v}_{1,\varepsilon}^p(x,0)\tilde{v}_{2,\varepsilon}^q(x,0)dx + o_\varepsilon(1) = 2\int_{B_\varepsilon} Q_\varepsilon\tilde{v}_{1,\varepsilon}^p(x,0)\tilde{v}_{2,\varepsilon}^q(x,0)dx + o_\varepsilon(1). \tag{28}$$

By definition of $\mathscr{S}^\alpha_{s,p_\varepsilon,q}$ and recalling that $\mathscr{S}^\alpha_{s,p_\varepsilon,q} = \mathscr{S} + o_\varepsilon(1)$, we get

$$k_s\int_{C_{B_\varepsilon}} y^{1-2s}(|\nabla\tilde{v}_{1,\varepsilon}|^2 + |\nabla\tilde{v}_{2,\varepsilon}|^2)dxdy$$
$$\ge \mathscr{S}\Big(\int_{B_\varepsilon} Q_\varepsilon\tilde{v}_{1,\varepsilon}^p(x,0)\tilde{v}_{2,\varepsilon}^q(x,0)dx\Big)^{\frac{2}{p_\varepsilon+q}} + o_\varepsilon(1).$$

Assume by contradiction that

$$\lim_{\varepsilon\to 0} k_s\int_{C_{B_\varepsilon}} y^{1-2s}(|\nabla\tilde{v}_{1,\varepsilon}|^2 + |\nabla\tilde{v}_{2,\varepsilon}|^2)dxdy = \rho > 0.$$

Then, we have

$$k_s\int_{C_{B_\varepsilon}} y^{1-2s}(|\nabla\tilde{v}_{1,\varepsilon}|^2 + |\nabla\tilde{v}_{2,\varepsilon}|^2)dxdy$$
$$= 2\int_{B_\varepsilon} Q_\varepsilon\tilde{v}_{1,\varepsilon}^p(x,0)\tilde{v}_{2,\varepsilon}^q(x,0)dx + o(1) \ge \frac{\mathscr{S}^{\frac{N}{2s}}}{2^{\frac{N-2s}{2s}}} + o_\varepsilon(1).$$

By using a Brézis-Lieb type Lemma and arguments similar to the ones above, we get

$$I(\tilde{w}_{1,\varepsilon},\tilde{w}_{2,\varepsilon}) = \int_{C_{B_\varepsilon}} \frac{k_s}{2} y^{1-2s}(|\nabla \tilde{v}_{1,\varepsilon}|^2 + |\nabla \tilde{v}_{2,\varepsilon}|^2)dxdy$$
$$- \frac{2}{p_\varepsilon + q}\int_{B_\varepsilon} Q_\varepsilon \tilde{v}_{1,\varepsilon}(x,0)^p \tilde{v}_{2,\varepsilon}(x,0)^q dx + \int_{\mathbb{R}^{N+1}_+} \frac{k_s}{2} y^{1-2s}(a^2+b^2)|\nabla \mathscr{W}_1|^2 dxdy$$
$$- \frac{2}{p_\varepsilon + q}\int_{\mathbb{R}^N} |a\mathscr{W}_1(x,0)|^{p_\varepsilon}|b\mathscr{W}_1(x,0)|^q dx + o_\varepsilon(1) \geq \frac{2s}{N}\frac{\mathscr{S}^{\frac{N}{2s}}}{2^{\frac{N-2s}{2s}}} + o_\varepsilon(1).$$

On the other hand,

$$I(\tilde{w}_{1,\varepsilon},\tilde{w}_{2,\varepsilon}) := \frac{k_s}{2}\int_{C_{B_\varepsilon}} y^{1-2s}(|\nabla \tilde{w}_{1,\varepsilon}|^2 + |\nabla \tilde{w}_{2,\varepsilon}|^2)dxdy$$
$$- \frac{2}{p+q}\int_{B_\varepsilon} Q_\varepsilon \tilde{w}^p_{1,\varepsilon}(x,0)\tilde{w}^q_{2,\varepsilon}(x,0)dx = \frac{s}{N}\frac{\mathscr{S}^{\frac{N}{2s}}}{2^{\frac{N-2s}{2s}}} + o_\varepsilon(1),$$

a contradiction. Hence $\rho = 0$, proving also Theorem 1.4(ii).

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