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NONHOMOGENEOUS POLYHARMONIC ELLIPTIC PROBLEMS AT CRITICAL GROWTH WITH SYMMETRIC DATA*

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Abstract. We show the existence of multiple solutions of a perturbed polyharmonic elliptic problem at critical growth with Dirichlet boundary conditions when the domain and the nonhomogenous term are invariant with respect to some group of symmetries.

1. **Introduction and Main Result.** Let $K \geqslant 1$ and let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $N > 2K$. In this paper we consider the polyharmonic elliptic problem

$$\begin{cases} (-\Delta)^K u = |u|^{K_*-2}u + f & \text{in } \Omega, \\ \left(\frac{\partial}{\partial \nu}\right)^j u\Big|_{\partial\Omega} = 0, & j = 0, \dots, K-1, \end{cases} \qquad (\mathscr{P}_{\Omega,\,f})$$

where $f \in H^{-K}(\Omega)$ and $K_* = \frac{2N}{N-2K}$ denotes the critical exponent for the Sobolev embedding $H_0^K(\Omega) \hookrightarrow L^{K_*}(\Omega)$.

If $f = 0$ this problem is invariant under dilations. Lack of compactness in elliptic problems which are invariant under dilations is known to produce quite interesting phenomena. It often gives rise to solutions of small perturbations of such problems. This behavior has been extensively studied for $K = 1$ (and $1_* = 2^*$), we refer to [5], [27] and [30] for a detailed discussion. For $K > 1$ perturbations of problem $(\mathscr{P}_{\Omega,\,0})$ by adding a subcritical term have been considered by many authors; we refer to the work of Gazzola [14] and the references therein. For $K = 2$ perturbations of the domain giving rise to solutions were also recently considered by Gazzola, Grunau and one of the authors [15].

Adding a nonhomogeneous term produces a similar effect. For $K = 1$ it was shown by Tarantello [29] that, if $f \neq 0$ and $\|f\|_{H^{-1}}$ is small enough, problem $(\mathscr{P}_{\Omega,\,f})$ has at least two nontrivial solutions. This result was extended to the case $K = 2$ by Deng and Wang [11]. One consequence of the main result in this paper is that this is true for every $K \geqslant 1$. We shall show that the following holds.

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COROLLARY 1.1. *There exists a $\kappa > 0$ such that, if $f \neq 0$ and $\|f\|_{H^{-K}} < \kappa$, then problem $(\mathscr{P}_{\Omega,f})$ has at least 2 solutions.*

It is well known that the presence of symmetries gives rise, in many cases, to additional solutions. The impact of symmetries on problem $(\mathscr{P}_{\Omega,f})$ for $K = 1$ has been studied recently by Kavian, Ruf and one of the authors [10]. The main goal of this paper is to study the effect of symmetries on the number and on the type of solutions of problem $(\mathscr{P}_{\Omega,f})$ for arbitrary $K \geqslant 1$. We consider domains Ω which are invariant under the action of some closed subgroup G of the group $\mathscr{O}(N)$ of orthogonal transformations of $\mathbb{R}^N$, that is, $gx \in \Omega$ for every $g \in G$, $x \in \Omega$. We assume that f is G–invariant, that is, $f(gx) = f(x)$ for every $g \in G$, $x \in \Omega$, and look for additional solutions of problem $(\mathscr{P}_{\Omega,f})$ which are also G–invariant. Recall that G is said to act freely on Ω if $g_1 x \neq g_2 x$ for all $g_1 \neq g_2 \in G$, $x \in \Omega$. As a consequence of our main result we obtain the following.

COROLLARY 1.2. *If $G \neq \{1\}$ acts freely on Ω then there exists a $\kappa > 0$ with the property that, for every $f \neq 0$ which is G–invariant and such that $\|f\|_{H^{-K}} < \kappa$, problem $(\mathscr{P}_{\Omega,f})$ has at least 3 solutions one of which is G–invariant and one of which is not.*

For example, if Ω is symmetric with respect to the origin (i.e. $x \in \Omega$ if $-x \in \Omega$) and $0 \notin \Omega$ then, for every even function $f \neq 0$ with $\|f\|_{H^{-K}}$ small enough, problem $(\mathscr{P}_{\Omega,f})$ has at least 3 solutions, one of which is even and one of which is not. We write

$$(u,v)_{K,2} = \begin{cases} \int_\Omega \Delta^q u \, \Delta^q v \, dx & \text{if } K = 2q, \\ \int_\Omega \nabla \Delta^q u \, \nabla \Delta^q v \, dx & \text{if } K = 2q+1 \end{cases} \tag{1.1}$$

for the usual scalar product in the Sobolev space $H_0^K(\Omega)$, and denote by S_K the best Sobolev constant for the embedding $H_0^K(\Omega) \hookrightarrow L^{K_*}(\Omega)$,

$$S_K = \inf \Big\{ \|u\|_{K,2}^2 : \; u \in H_0^K(\Omega), \; \int_\Omega |u|^{K_*} \, dx = 1 \Big\}.$$

We write $\sharp\, G_j$ for the cardinality of G_j. Our main result is the following.

THEOREM 1.3. *Let $\{1\} = G_1, \dots, G_m$ be closed subgroups of $\mathscr{O}(N)$ acting freely on Ω such that $\sharp\, G_1 < \cdots < \sharp\, G_m$ and $G_{m-1} \subset G_m$. Then at least one of the following assertions holds:*

(a) $m > 1$ and for $f = 0$ problem $(\mathscr{P}_{\Omega,0})$ has a nontrivial solution u such that

$$\|u\|_{K,2}^2 \leqslant (\sharp\, G_{m-1})(S_K)^{N/2K} ;$$

(b) $m \geqslant 1$ and there exists a $\kappa > 0$ with the property that, if f is G_i–invariant for each $i = 1, \dots, m$, $f \neq 0$ and $\|f\|_{H^{-K}} < \kappa$, then problem $(\mathscr{P}_{\Omega,f})$ has at least $m+1$ solutions $u_0, u_1, \dots, u_m$ such that u_i is G_i–invariant but not G_{i+1}–invariant for $i = 1, \dots, m-1$, and u_m is G_m–invariant.

We recall that a weak solution of $(\mathscr{P}_{\Omega,f})$ belongs to $C^{2K,\alpha}(\overline{\Omega})$ if $\partial\Omega$ is of class $C^{2K,\alpha}$ and $f \in C^{0,\alpha}(\overline{\Omega})$ [20].

Whether Ω has symmetries or not, Theorem 1.3 (with $m = 1$) asserts the existence of at least two solutions of problem $(\mathscr{P}_{\Omega,f})$ for $f \neq 0$ and $\|f\|_{H^{-K}}$ small enough. This is Corollary 1.1. Moreover, if Ω and f have appropriate symmetries, Theorem 1.3 provides an additional solution. Indeed, since $(\mathscr{P}_{\Omega,0})$ has no ground state solution, Theorem 1.3 includes Corollary 1.2 as a special case (a detailed argument is given in Section 4 below).

Theorem 1.3 asserts the existence of many solutions of problem $(\mathscr{P}_{\Omega,f})$ provided that it has many symmetries and that the unperturbed problem $(\mathscr{P}_{\Omega,0})$ has no nontrivial solution below a certain energy level.

Little is known about nonexistence of solutions of problem $(\mathscr{P}_{\Omega,0})$. For $K = 1$ Pohožaev's identity [23] implies nonexistence of solutions in starshaped domains. But for $K > 1$, even though Oswald did show that there are no positive solutions in domains of this kind [21], as far as we know there is no result excluding sign–changing ones apart from the case $K = 2$ where the existence of radial solutions on a ball has been ruled out [15].

In any case, notice that the condition that G acts freely on Ω is quite strong. It implies that $0 \notin \Omega$, which excludes starshaped domains. It also implies that Ω has nontrivial topology. For $K = 1$ a well known result of Bahri and Coron [1] asserts the existence of a solution of problem $(\mathscr{P}_{\Omega,0})$ if Ω has nontrivial topology. A similar result for any $K \geqslant 1$ was recently obtained by Bartsch, Weth and Willem [2]. Moreover, even in some contractible domains, solutions of $(\mathscr{P}_{\Omega,f})$ are known to exist [22], [15]. Quite recently, however, Ben Ayed, El Mehdi and Hammami [3] obtained a nonexistence result for problem $(\mathscr{P}_{\Omega,0})$ on thin annuli. They showed that, for $K = 1$, problem $(\mathscr{P}_{\Omega,0})$ has no positive solution below a given energy level if the annular domain is thin enough. This fact, together with Theorem 1.3 and some stronger results of this kind, provides multiple solutions of problem $(\mathscr{P}_{\Omega,f})$ for $K = 1$ and small $f \neq 0$ on thin annuli [10].

For $f \geqslant 0$ and $K = 1$ there is an effect of the domain topology [26] together with its symmetries [10] on the number of solutions of $(\mathscr{P}_{\Omega,f})$. Also more general group actions are allowed in this case. This is a consequence of the fact that, for $K = 1$, least energy solutions are positive if $f \geqslant 0$ and small enough. For $K > 1$ this positivity preservation property does not hold in general, due to the lack of maximum principles for $(-\Delta)^K$ [17].

Finally we would like to mention that for nonhomogeneous polyharmonic problems at (small enough) subcritical growth with homogeneous or nonhomogeneous Dirichlet boundary conditions much stronger results hold for arbitrary $K \geqslant 1$ [18].

This paper is organized as follows: in Section 2 we describe the variational setting associated to problem $(\mathscr{P}_{\Omega,f})$ in the presence of symmetries. In Section 3 we give a compactness condition for this problem and obtain a first G–invariant solution. In Section 4 a further G–invariant solution is provided, and Theorem 1.3 is proved. As in the case $K = 1$ [29], [10], the proof of Theorem 1.3 for $K \geqslant 1$ relies, on one hand, on the knowledge of the first G–invariant noncompactness level for the unperturbed problem $(\mathscr{P}_{\Omega,0})$. On the other hand, it requires fine estimates similar to those obtained by Brézis and Nirenberg in [8]. These questions will be handled in Sections 5 and 6 respectively.

2. **The Variational Framework.**

Let G be a closed subgroup of $\mathscr{O}(N)$ and assume that Ω and f are G–invariant. Consider the problem

$$\begin{cases} (-\Delta)^K u = |u|^{K_*-2}u + f & \text{in } \Omega, \\ \left(\frac{\partial}{\partial \nu}\right)^j u\Big|_{\partial\Omega} = 0, & j = 0, \dots, K-1, \\ u(gx) = u(x) & \text{for } g \in G. \end{cases} \tag{$\mathscr{P}^G_{\Omega,f}$}$$

The action of G on Ω induces an orthogonal G–action on $H_0^K(\Omega)$ given by

$$(gu)(x) := u(g^{-1}x),$$

that is, $(u,v)_{K,2} = (gu,gv)_{K,2}$ for all $g \in G$, $u,v \in H_0^K(\Omega)$. We write $|\cdot|_p$ for the L^p–norm. The energy functional

$$E_f(u) = \frac{1}{2}\|u\|_{K,2}^2 - \frac{1}{K_*}|u|_{K_*}^{K_*} - \int_\Omega fu\,dx$$

is G–invariant, that is, $E_f(gu) = E_f(u)$ for all $g \in G$ and $u \in H_0^K(\Omega)$.

Weak solutions of problem $(\mathscr{P}^G_{\Omega,f})$ are critical points of the restriction of E_f to the space of fixed points

$$H_0^K(\Omega)^G = \{u \in H_0^K(\Omega) : u(gx) = u(x) \text{ for all } g \in G\}.$$

They lie on the Nehari set

$$\begin{aligned}\mathcal{N}_f^G &= \Big\{u \in H_0^K(\Omega)^G : \ DE_f(u)u = 0\Big\}\\ &= \Big\{u \in H_0^K(\Omega)^G : \ \|u\|_{K,2}^2 - |u|_{K_*}^{K_*} - \int_\Omega fu\,dx = 0\Big\}\end{aligned}$$

From now on we assume that the following condition holds:

$(\mathscr{H}_1)$ For every $v \in H_0^K(\Omega)^G$ with $|v|_{K_*}^{K_*} = 1$,

$$\Big|\int_\Omega fv\,dx\Big| < b_{N,K}\|v\|_{K,2}^{\frac{N+2K}{2K}}, \qquad b_{N,K} = \frac{4K}{N-2K}\Big(\frac{N-2K}{N+2K}\Big)^{\frac{N+2K}{4K}}.$$

This condition is the same one given by Tarantello [29] for $K=1$ and by Deng and Wang [11] for $K=2$. Observe that condition $(\mathscr{H}_1)$ holds provided that

$$\|f\|_{H^{-K}} < b_{N,K}\,S_K^{N/4K}.$$

Let us recall some properties satisfied by the Nehari set.

Proposition 2.1. *If condition $(\mathscr{H}_1)$ holds then $\mathcal{N}_f^G$ has the following properties:*

(a) if $f \neq 0$ then $\mathcal{N}_f^G$ is a C^1–submanifold of $H_0^K(\Omega)^G$; if $f = 0$ then $\mathcal{N}_0^G \setminus \{0\}$ is a C^1–submanifold of $H_0^K(\Omega)^G$;

(b) for every $0 \neq u \in \mathcal{N}_f^G$ the line $\mathbb{R}u$ is transversal to $\mathcal{N}_f^G$ at u;

(c) $\mathcal{N}_f^G$ has two components

$$\begin{aligned}(\mathcal{N}_f^G)^+ &= \Big\{u \in \mathcal{N}_f^G : \ \|u\|_{K,2}^2 - \Big(\frac{N+2K}{N-2K}\Big)|u|_{K_*}^{K_*} \geqslant 0\Big\},\\ (\mathcal{N}_f^G)^- &= \Big\{u \in \mathcal{N}_f^G : \ \|u\|_{K,2}^2 - \Big(\frac{N+2K}{N-2K}\Big)|u|_{K_*}^{K_*} < 0\Big\};\end{aligned}$$

(d) $(\mathcal{N}_f^G)^-$ is radially diffeomorphic to the unit sphere in $H_0^K(\Omega)^G$;

(e) $E_f(u) < 0$ for $u \in (\mathcal{N}_f^G)^+$ with $u \neq 0$;

(f) E_f is bounded below on $\mathcal{N}_f^G$.

The proof is easy and similar to the one for $K=1$ [29], [10]. Details are left to the reader. We define

$$c_{f,0}^G := \inf_{u \in \mathcal{N}_f^G} E_f = \inf_{u \in (\mathcal{N}_f^G)^+} E_f, \quad c_{f,1}^G := \inf_{u \in (\mathcal{N}_f^G)^-} E_f.$$

These infima are natural candidates for being critical values. We shall show that they are actually achieved. Notice that Proposition 2.1 implies that $-\infty < c^G_{f,0} \leqslant c^G_{f,1} < +\infty$ and that $c^G_{f,0} < 0$ if $f \neq 0$. If $f = 0$ we write

$$\mathcal{N}^G = (\mathcal{N}^G_0)^-, \quad c^G_1 = \inf_{\mathcal{N}^G} E_0 \,.$$

3. **The G–invariant Compactness Range.**

Definition 3.1. *A sequence $(u_k) \subset H^K_0(\Omega)^G$ such that*

$$E_f(u_k) \to c, \qquad \|DE_f(u_k)\|_{H^{-K}} \to 0,$$

will be called a G–PS–sequence for E_f at the level c. E_f will be said to satisfy the G–Palais–Smale condition $(PS)^G_c$ at the level c if every such sequence has a convergent subsequence.

Let us set

$$\mu^G := \Big(\min_{x\in\overline{\Omega}} \sharp\, Gx\Big) \frac{K}{N} \left(S_K\right)^{N/2K},$$

where $\sharp\, Gx$ denotes the cardinality of the G–orbit $Gx = \big\{gx : g \in G\big\}$ of x. The following Proposition, which extends a result of P.L. Lions [19] to any $K \geqslant 1$, will be proved in Section 5.

Proposition 3.2. *E_0 satisfies $(PS)^G_c$ at every $c < \mu^G$. In particular, if every G–orbit in Ω is infinite, then E_0 satisfies $(PS)^G_c$ at every $c \in \mathbb{R}$.*

Corollary 3.3. *E_f satisfies $(PS)^G_c$ at every $c < c^G_{f,0} + \mu^G$. In particular, if every G–orbit in Ω is infinite, then E_f satisfies $(PS)^G_c$ at every $c \in \mathbb{R}$.*

Proof. Let (u_k) be G–PS–sequence for E_f with $E_f(u_k) \to c$. It is readily seen that (u_k) is bounded. Hence a subsequence converges weakly in $H^K_0(\Omega)^G$ to a weak solution u of $(\mathscr{P}^G_{\Omega,\,f})$. Let $v_k := u_k - u$. A standard argument shows that

$$E_f(u_k) = E_f(u) + E_0(v_k) + o(1)$$
$$o(1) = DE_f(u_k) = DE_f(u) + DE_0(v_k) + o(1) = DE_0(v_k) + o(1)$$

as $k \to +\infty$. Therefore (v_k) is a G–PS–sequence for E_0 such that $v_k \rightharpoonup 0$ weakly in $H^K_0(\Omega)^G$ and

$$E_0(v_k) \to c - E_f(u) \leqslant c - c^G_{f,0} < \mu^G.$$

It follows from Proposition 3.2 that, up to a subsequence, $v_k \to 0$ strongly in $H^K_0(\Omega)^G$ and, hence, $u_k \to u$ strongly in $H^K_0(\Omega)^G$. □

Easy consequences of Corollary 3.3 are the following.

Theorem 3.4. *If f satisfies assumption $(\mathscr{H}_1)$ then $c^G_{f,0}$ is achieved at a point $u^G_{f,0} \in (\mathcal{N}^G_f)^+$ which is a critical point of E_f on $H^K_0(\Omega)^G$. Moreover,*

$$\|f\|_{H^{-K}} \to 0 \implies \big\|u^G_{f,0}\big\|_{K,2} \to 0.$$

Proof. If $f = 0$ take $u^G_{f,0} = 0$. For $f \neq 0$ let (u_n) be a minimizing sequence for E_f on $(\mathcal{N}^G_f)^+$. Ekeland's variational principle [30] allows us to assume that (u_n) is a Palais–Smale sequence for E_f on $\mathcal{N}^G_f$. Since $c^G_{f,0} < 0$, we may also assume that $u_n \neq 0$. Hence $\mathbb{R}u_n$ is transversal to $\mathcal{N}^G_f$ at u_n and therefore (u_n) is a G–PS–sequence for E_f at the level $c^G_{f,0}$. Corollary 3.3 above asserts that a subsequence of

(u_n) converges strongly to $u^G_{f,0} \in (\mathcal{N}^G_f)^+$. Hence, $E_f(u^G_{f,0}) = c^G_{f,0}$. The last assertion follows immediately from the inequality

$$\begin{aligned} 0 \geqslant E_f(u^G_{f,0}) &= \frac{K}{N}\,\|u_{f,0}\|^2_{K,2} - \frac{N+2K}{2N}\int_\Omega f u_{f,0}\,dx \\ &\geqslant \frac{K}{N}\,\|u^G_{f,0}\|^2_{K,2} - \frac{N+2K}{2N}\,\|f\|_{H^{-K}}\,\|u^G_{f,0}\|_{K,2} \geqslant -\frac{(N+2K)^2}{16NK}\,\|f\|^2_{H^{-K}}\,. \end{aligned}$$

□

For $K=1$ this result is due to Tarantello [29] if $G=\{1\}$ and for arbitrary G it was proved in [10]. A further consequence of Corollary 3.3 is the following.

PROPOSITION 3.5. *If condition $(\mathscr{H}_1)$ holds then $c^G_{f,0} < c^G_{f,1}$.*

Proof. If $c^G_{f,0} = c^G_{f,1}$ then, arguing as in Theorem 3.4, we obtain a $u^G_{f,1} \in (\mathcal{N}^G_f)^-$ such that $E_f(u^G_{f,1}) = c^G_{f,1}$. Let $t_0 \geqslant 0$ be the largest real number with $E_f(t_0 u^G_{f,1}) \in (\mathcal{N}^G_f)^+$. Assumption $(\mathscr{H}_1)$ implies that

$$c^G_{f,0} \leqslant E_f(t_0 u^G_{f,1}) < E_f(u^G_{f,1}) = c^G_{f,1}.$$

This is a contradiction. □

4. **A Second G–invariant Solution.** We wish to give conditions for $c^G_{f,1}$ to be achieved by E_f on $(\mathcal{N}^G_f)^-$. Corollary 3.3 immediately gives the following.

THEOREM 4.1. *Assume that condition $(\mathscr{H}_1)$ holds. If every G–orbit of Ω is infinite, then $c^G_{f,1}$ is achieved at a point $u^G_{f,1} \in (\mathcal{N}^G_f)^-$ which is a critical point of E_f on $H^K_0(\Omega)^G$.*

Next we consider the case when the domain Ω has a finite G–orbit. We assume throughout that condition $(\mathscr{H}_1)$ holds and also

$(\mathscr{H}_2)$ G is a finite group acting freely on Ω.

For $\varepsilon > 0$ and $y \in \mathbb{R}^N$ we consider the ground state solutions

$$T_{\varepsilon,y}(x) = c_{N,K}\Big(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2}\Big)^{\frac{N-2K}{2}}, \quad c_{N,K} = \Big\{\prod_{j=1-K}^{K}(N-2j)\Big\}^{\frac{N-2K}{4K}}$$

of the problem $(-\Delta)^K u = |u|^{K_*-2}u$ in $\mathbb{R}^N$ (see [28]). It satisfies

$$\|T_{\varepsilon,y}\|^2_{K,2} = (S_K)^{N/2K} = |T_{\varepsilon,y}|^{K_*}_{K_*}\,.$$

For $y \in \Omega$ we consider the multi–bump function

$$w_{\varepsilon,y} = \sum_{g\in G}\varphi_{gy}T_{\varepsilon,gy} \in H^K_0(\Omega)^G$$

where $\varphi \in C^\infty(\mathbb{R}^N)$ is radially symmetric, $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $B(0,1)$ and $\varphi = 0$ outside $B(0,2)$, and $\varphi_{gy}(x) = \varphi(\rho^{-1}(x-gy))$ with

$$0 < \rho < \min\Big\{\frac{1}{2}\mathrm{dist}(y,\partial\Omega), \frac{1}{4}\,|gy - g'y| : \ g,g' \in G,\ g \neq g'\Big\}.$$

Hence $\mathrm{supp}(\varphi_{gy}) \subset \Omega$ and $\mathrm{supp}(\varphi_{gy}) \cap \mathrm{supp}(\varphi_{g'y}) = \emptyset$ if $g \neq g'$.

If $f \neq 0$ then $u^G_{f,0} \neq 0$ and we may assume without loss of generality that $u^G_{f,0} > 0$ in some set $\Sigma \subset \Omega$ of positive measure. The following lemma, which extends a result of Brézis and Nirenberg [8] to any $K \geqslant 1$, will be proved in Section 6.

Lemma 4.2. *For each $K \geqslant 1$ and $t \geqslant 0$ and for a.e. $y \in \Sigma$ the following holds*

$$\begin{aligned}\int_\Omega |u^G_{f,0} + tw_{\varepsilon,y}|^{K_*} = &\int_\Omega |u^G_{f,0}|^{K_*}\,dx + t^{K_*}\int_\Omega |w_{\varepsilon,y}|^{K_*}\,dx \\ &+ K_* t^{K_*-1}\int_\Omega u^G_{f,0}\, w^{K_*-1}_{\varepsilon,y}\,dx \\ &+ K_* t\int_\Omega |u^G_{f,0}|^{K_*-2}u^G_{f,0}\,w_{\varepsilon,y}\,dx + R_\varepsilon + S_\varepsilon\end{aligned}$$

with $R_\varepsilon = o\left(\varepsilon^{(N-2K)/2}\right)$ and $S_\varepsilon = o\left(\varepsilon^{(N-2K)/2}\right)$ as $\varepsilon \to 0$.

Proposition 4.3. *Assume that $f \neq 0$ and that conditions $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$ hold. Then for every $t \geqslant 0$ and a.e. $y \in \Sigma$ it results*

$$E_f(u^G_{f,0} + tw_{\varepsilon,y}) < c^G_{f,0} + \mu^G$$

for each $\varepsilon > 0$ sufficiently small.

Proof. For every $t \geqslant 0$ and each $y \in \Omega$ it results

$$\begin{aligned}E_f(u^G_{f,0} + tw_{\varepsilon,y}) = &\frac{1}{2}\|u^G_{f,0}\|^2_{K,2} + t(u^G_{f,0}, w_{\varepsilon,y})_{K,2} + \frac{t^2}{2}\|w_{\varepsilon,y}\|^2_{K,2} \\ &-\frac{1}{K_*}|u^G_{f,0} + tw_{\varepsilon,y}|^{K_*}_{K_*} - \int_\Omega f u^G_{f,0}\,dx - t\int_\Omega f w_{\varepsilon,y}\,dx.\end{aligned}$$

In view of Lemma 4.2, this equality yields

$$\begin{aligned}E_f(u^G_{f,0} + tw_{\varepsilon,y}) &= E_f(u^G_{f,0}) + tDE_f(u^G_{f,0})w_{\varepsilon,y} + \frac{t^2}{2}\|w_{\varepsilon,y}\|^2_{K,2}\,dx - \frac{t^{K_*}}{K_*}|w_{\varepsilon,y}|^{K_*}_{K_*} \\ &\quad -t^{K_*-1}\int_\Omega u^G_{f,0}\,w^{K_*-1}_{\varepsilon,y}\,dx + o\big(\varepsilon^{(n-2K)/2}\big) \\ &= c^G_{f,0} + \frac{t^2}{2}\|w_{\varepsilon,y}\|^2_{K,2}\,dx - \frac{t^{K_*}}{K_*}|w_{\varepsilon,y}|^{K_*}_{K_*} \\ &\quad -t^{K_*-1}\int_\Omega u^G_{f,0}\,w^{K_*-1}_{\varepsilon,y}\,dx + o\big(\varepsilon^{(n-2K)/2}\big)\end{aligned} \tag{4.1}$$

as $\varepsilon \to 0$ for a.e. $y \in \Sigma$. Arguing as in [14], one finds $C > 0$ such that

$$\|\varphi_y T_{\varepsilon,y}\|^2_{K,2} \leqslant S_K^{N/2K} + C\varepsilon^{N-2K}, \quad |\varphi_y T_{\varepsilon,y}|^{K_*}_{K_*} \geqslant S_K^{N/2K} - C\varepsilon^N, \tag{4.2}$$

as $\varepsilon \to 0$. Since $\frac{t^2}{2} - \frac{t^{K_*}}{K_*} \leqslant \frac{K}{N}$ and since G acts freely on Ω, it follows that

$$\begin{aligned}E_f(u^G_{f,0} + tw_{\varepsilon,y}) \leqslant &c^G_{f,0} + (\sharp\, G)\frac{K}{N}S_K^{N/2K} \\ &- t^{K_*-1}\int_\Omega u^G_{f,0}\,w^{K_*-1}_{\varepsilon,y}\,dx + o\big(\varepsilon^{(N-2K)/2}\big)\end{aligned} \tag{4.3}$$

as $\varepsilon \to 0$ for a.e. $y \in \Sigma$. On the other hand,

$$\begin{aligned}&\int_\Omega u^G_{f,0}\,w^{K_*-1}_{\varepsilon,y}\,dx = (\sharp\, G)\int_\Omega u^G_{f,0}\varphi_y^{K_*-1}T^{K_*-1}_{\varepsilon,y}\,dx \\ &= (\sharp\, G)D_{N,K}\int_{\mathbb{R}^N} u^G_{f,0}(x)\,\varphi_y^{K_*-1}(x)\frac{\varepsilon^{(N+2K)/2}}{(\varepsilon^2 + |x-y|^2)^{(N+2K)/2}}\,dx \\ &= (\sharp\, G)D_{N,K}\varepsilon^{(N-2K)/2}\int_{\mathbb{R}^N} u^G_{f,0}(x)\,\varphi_y^{K_*-1}(x)\frac{1}{\varepsilon^N}\psi\Big(\frac{x-y}{\varepsilon}\Big)\,dx\end{aligned}$$

where $D_{N,K} > 0$ and $\psi(\xi) = (1 + |\xi|^2)^{-(N+2K)/2}$. Since by [13, Theorem 8.15]

$$\int_{\mathbb{R}^N} u^G_{f,0}(x)\, \varphi_y^{K_*-1}(x) \frac{1}{\varepsilon^N} \psi\Big(\frac{x-y}{\varepsilon}\Big)\, dx \to u^G_{f,0}(y) \int_{\mathbb{R}^N} \psi(\xi)\, d\xi$$

as $\varepsilon \to 0$ for a.e. $y \in \Sigma$ it follows that for some $\widetilde{D}_{N,K} > 0$

$$\begin{aligned} E_f(u^G_{f,0} + tw_{\varepsilon,y}) &\leqslant c_{f,0} + \mu^G \\ &\quad - t^{K_*-1} \widetilde{D}_{N,K}(\sharp\, G)\, u^G_{f,0}(y)\, \varepsilon^{(N-2K)/2} + o\big(\varepsilon^{(N-2K)/2}\big). \end{aligned} \tag{4.4}$$

as $\varepsilon \to 0$. Finally, since $u^G_{f,0}(y) > 0$ for a.e. $y \in \Sigma$, the result follows. $\square$

Notice that if we knew that $u^G_{f,0} > 0$ in all of Ω, a similar argument would yield

$$E_f(u^G_{f,0} + tw_{\varepsilon,y}) < c^G_{f,0} + \mu^G = c^G_{f,0} + \Big(\min_{x\in\overline{\Omega}} \sharp\, Gx\Big) \frac{K}{N} (S_K)^{N/2K}$$

even if the action of G on Ω is not free (see [10, Proposition 18]). But, since all we know is that $u^G_{f,0} > 0$ in some set Σ of positive measure it might very well be that, if the action is not free, no G–orbit Gy with $y \in \Sigma$ has minimum cardinality. This is why we consider free actions. Proposition 4.3 and Theorem 4.4 below are still true if instead of $(\mathscr{H}_2)$ we assume that Ω has only one G–orbit type, that is, all G–orbits in Ω are G–isomorphic.

As a consequence of Proposition 4.3 we obtain the following result.

THEOREM 4.4. *Assume that $f \neq 0$ and that conditions $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$ hold. Then $c^G_{f,1}$ is achieved at a point $u^G_{f,1} \in (\mathcal{N}^G_f)^-$ which is a critical point of E_f on $H^K_0(\Omega)^G$.*

Proof. Since the ray $\big\{u^G_{f,0} + tw_{\varepsilon,y} : t \geqslant 0\big\}$ crosses $(\mathcal{N}^G_f)^-$, it follows from Proposition 4.3 above that

$$c^G_{f,1} = \inf_{u\in(\mathcal{N}^G_f)^-} E_f < c^G_{f,0} + \mu^G.$$

By Corollary 3.3 the value $c^G_{f,1}$ is achieved by E_f on $(\mathcal{N}^G_f)^-$. $\square$

For $K = 1$ this result was proved by Tarantello [29] for the trivial group and extended to arbitrary groups in [10].

For the proof of Theorem 1.3 we require the following easy Lemma.

LEMMA 4.5. *For every $\alpha > 0$,*

$$\|f\|_{H^{-K}} \leqslant \Big(\frac{N}{K} c^G_1\Big)^{-1/2} \alpha \quad \Longrightarrow \quad c^G_1 - \alpha \leqslant c^G_{f,1}\,.$$

Proof. Let $\varepsilon > 0$ and let $v \in (\mathcal{N}^G_f)^-$ be such that $E_f(v) < c^G_{f,1} + \varepsilon$. Let $t_0 > 0$ be such that $u = t_0 v \in (\mathcal{N}^G_0)^-$. Then $E_0(u) = \frac{K}{N}\|u\|^2_{K,2} \geqslant c^G_1$ and, therefore,

$$\begin{aligned} c^G_{f,1} + \varepsilon > E_f(u) = E_0(u) - \int_\Omega fu\, dx &\geqslant E_0(u) - \|f\|_{H^{-K}} \Big(\frac{N}{K} E_0(u)\Big)^{1/2} \\ &\geqslant E_0(u) - \big(c^G_1\big)^{-1/2} \alpha\, (E_0(u))^{1/2} \geqslant c^G_1 - \alpha \end{aligned}$$

because the function $\big\{t \mapsto t - (c^G_1)^{-1/2} \alpha\, t^{1/2}\big\}$ is increasing for $t \geqslant c^G_1$. $\square$

Using the results above and arguing as in the case $K = 1$ [10] one can now prove Theorem 1.3. We give the details for the reader's convenience.

Proof of Theorem 1.3. Assume that problem $(\mathscr{P}_{\Omega,0})$ has no solution $u \in \mathcal{N}$ such that $E_0(u) = \frac{K}{N}\|u\|_{K,2}^2 \leqslant (\sharp\, G_{m-1})\frac{K}{N}\,(S_K)^{N/2K}$. Then, by Theorem 4.1 and Proposition 3.2, $c_1^{G_i} = \mu^{G_i} = (\sharp\, G_i)\frac{K}{N}\,(S_K)^{N/2K} < +\infty$ for $i = 1, \dots, m-1$. If $c_1^{G_m}$ is not achieved by E_0 on $\mathcal{N}^{G_m}$ then $c_1^{G_m} = \mu^{G_m}$ too. On the other hand if it is achieved then necessarily $c_1^{G_m} > c_1^{G_{m-1}}$, because $c_1^{G_{m-1}}$ is not achieved by E_0 on $\mathcal{N}^{G_{m-1}} \supset \mathcal{N}^{G_m}$. So, since $\sharp\, G_{i-1} < \sharp\, G_i$, we obtain in any case that $c_1^{G_i} > c_1^{G_{i-1}}$ for all $i = 2, \dots, m$. Let

$$\alpha := \min\left\{c_1^{G_i} - c_1^{G_{i-1}} : \ i = 2, \dots, m\right\} > 0.$$

By Lemma 4.5 above there is a $\kappa > 0$ independent of f, such that, if $\|f\|_{H^{-1}} \leqslant \kappa$, then f satisfies assumption $(\mathscr{H}_1)$ and $c_1^{G_i} - \alpha \leqslant c_{f,1}^{G_i}$ for all $i = 1, \dots, m$. This, together with Proposition 4.3, implies that

$$c_{f,0}^{G_1} < c_{f,1}^{G_1} < \cdots < c_1^{G_{i-1}} \leqslant c_{f,1}^{G_i} < c_1^{G_i} \leqslant \cdots < c_1^{G_{m-1}} \leqslant c_{f,1}^{G_m}.$$

Theorem 3.4 and Theorem 4.4 provide $m+1$ different solutions, $u_{f,0}^{G_1} \in (\mathcal{N}_f^{G_1})^+$, $u_{f,1}^{G_i} \in (\mathcal{N}_f^{G_i})^-$, with $E_f(u_{f,0}^{G_1}) = c_{f,0}^{G_1}$ and $E_f(u_{f,1}^{G_i}) = c_{f,1}^{G_i}$ for $i = 1, \dots, m$. Since $c_{f,1}^{G_{i+1}}$ is the least possible energy of a G_{i+1}–invariant solution on $\mathcal{N}_f^-$, $u_{f,1}^{G_i}$ is not G_{i+1}–invariant for $i = 1, \dots, m-1$. $\square$

Proof of Corollary 1.2. Since problem $(\mathscr{P}_{\Omega,0})$ is invariant under dilations, the best Sobolev constant S_K is independent of the domain. It follows that the infimum

$$c_1^{\{1\}} = \frac{K}{N}(S_K)^{N/2K}$$

is not achieved by E_0 on $\mathcal{N}^{\{1\}} \subset H_0^K(\Omega)$. Indeed, if it were achieved at some point $u_1 \in \mathcal{N}^{\{1\}}$, then extending u_1 by 0 outside of Ω would give a minimum of E_0 on the Nehari manifold in $H^K(\mathbb{R}^N)$, that is, a solution of the problem $(-\Delta)^K u = |u|^{K_*-2}u$ in $\mathbb{R}^N$ which vanishes outside of Ω, contradicting the unique continuation property [24]. As a consequence, assertion (b) of Theorem 1.3 must hold. $\square$

Observe that Theorem 1.3 is still true if we assume that every G_m–orbit of Ω is infinite instead of asking that G_m acts freely on Ω. The proof is similar except that, in this case, both $c_1^{G_m}$ and $c_{f,1}^{G_m}$ are always achieved (cf. Theorem 4.1). In particular, the following holds.

Corollary 4.6. *If every G–orbit of Ω is infinite, then there exists a $\kappa > 0$ with the property that, for every $f \neq 0$ which is G–invariant and such that $\|f\|_{H^{-K}} < \kappa$, problem $(\mathscr{P}_{\Omega,f})$ has at least three solutions one of which is G–invariant and one of which is not.*

Remark 4.7. *These results can be extended to eigenvalue problems*

$$\begin{cases} (-\Delta)^K u = \lambda u + |u|^{K_*-2}u + f \quad \text{in } \Omega, \\ \left(\dfrac{\partial}{\partial\nu}\right)^j u\Big|_{\partial\Omega} = 0, \quad j = 0, \dots, K-1, \end{cases} \qquad (\mathscr{P}_{\Omega,\lambda,f})$$

provided that $0 \leqslant \lambda < \lambda_1$, where λ_1 is the first eigenvalue of $(-\Delta)^K$ with Dirichlet boundary conditions (see [11]). For $f = 0$ and $\Omega = B_R(0)$ this problem has been *studied by many authors (see e.g.* [4, 25]*), mainly in dealing with the so called Pucci–Serrin conjecture which says that the dimensions for which there is* $0 < \Lambda < \lambda_1$ *such that* ($\mathscr{P}_{\Omega,\lambda,0}$) *admits no solution for* $\lambda < \Lambda$ *are precisely*

$$2K+1, \ldots, 2K+j, \ldots, 4K-1.$$

Even though a full proof of the Pucci–Serrin conjecture seems out of reach, a weak version (for positive solutions) has been given by Grunau in [16].

5. **Proof of Proposition 3.2.** In this section we prove Proposition 3.2. Let us first prove the following result.

LEMMA 5.1. *Assume that* $(\overline{v}_k) \subset H^K(\mathbb{R}^N)$ *is such that* $\overline{v}_k \rightharpoonup 0$. *Then*

$$\forall h \in C_c^\infty(B): \quad \|h\overline{v}_k\|_{K,2}^2 = (\overline{v}_k, h^2\overline{v}_k)_{K,2} + o(1)$$

as $k \to +\infty$.

Proof. We consider the case $K = 2q$. The case $K = 2q+1$ can be treated in a similar fashion. Setting for each $k \geqslant 1$

$$A_k = \sum_{\substack{j_1,\ldots,j_q=1 \\ \alpha\in\{0,1,2\}^q,\,\alpha\neq 0}}^{q} c_{j,\alpha} \frac{\partial^{2q-|\alpha|}\overline{v}_k}{\partial x_{j_1}^{2-\alpha_1}\cdots\partial x_{j_q}^{2-\alpha_q}} \frac{\partial^{|\alpha|}h}{\partial x_{j_1}^{\alpha_1}\cdots\partial x_{j_q}^{\alpha_q}},$$

it results

$$|\Delta^q(h\overline{v}_k)|^2 = h^2|\Delta^q\overline{v}_k|^2 + 2hA_k\Delta^q\overline{v}_k + A_k^2\,. \tag{5.1}$$

Moreover, setting

$$B_k = \sum_{\substack{j_1,\ldots,j_q=1 \\ \alpha\in\{0,1,2\}^q,\,\alpha\neq 0}}^{q} c_{j,\alpha} \frac{\partial^{2q-|\alpha|}\overline{v}_k}{\partial x_{j_1}^{2-\alpha_1}\cdots\partial x_{j_q}^{2-\alpha_q}} \frac{\partial^{|\alpha|}h^2}{\partial x_{j_1}^{\alpha_1}\cdots\partial x_{j_q}^{\alpha_q}},$$

it follows that

$$\Delta^q\overline{v}_k\Delta^q(h^2\overline{v}_k) = h^2|\Delta^q\overline{v}_k|^2 + B_k\Delta^q\overline{v}_k\,. \tag{5.2}$$

By combining (5.1) and (5.2) yields

$$|\Delta^q(h\overline{v}_k)|^2 = \Delta^q\overline{v}_k\Delta^q(h^2\overline{v}_k) + A_k^2 - \big(B_k - 2hA_k\big)\Delta^q\overline{v}_k\,. \tag{5.3}$$

Since $D^j\overline{v}_k \to 0$ in $L^2_{loc}(\mathbb{R}^N)$ for $j = 0,\ldots,2q-1$, being $\alpha \neq 0$, it results

$$A_k \to 0, \quad B_k - 2hA_k \to 0 \quad \text{in } L^2(\operatorname{supt} h)$$

as $k \to +\infty$. In particular equation (5.3) yields the assertion. □

Proof of Theorem 3.2. Let $(u_k) \subset H_0^K(\Omega)^G$ be a G–PS–sequence for E_0 such that $E_0(u_k) \to c < \mu^G$. We wish to show that a subsequence of (u_k) converges strongly in $H_0^K(\Omega)$. Since PS–sequences for E_0 are bounded in $H_0^K(\Omega)$,

$$\|u_k\|_{K,2}^2 = \frac{N}{K}E_0(u_k) - \frac{N-2K}{2K}DE_0(u_k)u_k \to \frac{N}{K}c$$

as $k \to +\infty$. Thus $c \geqslant 0$. We may assume that $u_k \rightharpoonup u$ weakly in $H_0^K(\Omega)$ and that $u_k \to u$ a.e. in Ω. It is easy to see that $DE_0(u) = 0$ and that $v_k := u_k - u$ is a PS–sequence for E_0 such that $v_k \rightharpoonup 0$ weakly in $H_0^K(\Omega)$ and

$$E_0(v_k) = E_0(u_k) - E_0(u) + o(1) = c - E_0(u) + o(1).$$

Let $d := c - E_0(u)$. Thus $0 \leqslant d \leqslant c$. If $d = 0$ then $u_k \to u$ strongly in $H_0^K(\Omega)$ and we are done.

So let us assume that $d > 0$. Since (v_k) is a PS–sequence for E_0, it is bounded in $H_0^K(\Omega)$. Hence

$$|v_k|_{K_*}^{K_*} = \frac{N}{K} E_0(v_k) - \frac{N}{2K} DE_0(v_k)v_k \to \frac{N}{K} d > 0.$$

Let $\delta = \min\left\{\frac{Nd}{2K}, \left(\frac{S_K}{2}\right)^{\frac{N}{2K}}\right\}$ and denote by $B(x,\varrho)$ the closed ball in $\mathbb{R}^N$ with center x and radius ϱ. The Levy concentration function

$$\Phi_k(\varrho) = \sup_{x\in\mathbb{R}^N} \int_{B(x,\varrho)} |v_k|^{K_*}\, d\xi$$

satisfies $\Phi_k(0) = 0$ and $\Phi_k(+\infty) > \delta$ for k large enough. Hence we may choose $y_k \in \Omega$ and $\varepsilon_k > 0$ such that

$$\sup_{x\in\mathbb{R}^N} \int_{B(x,\varepsilon_k)} |v_k|^{K_*}\, d\xi = \int_{B(y_k,\varepsilon_k)} |v_k|^{K_*}\, d\xi = \delta > 0\,. \tag{5.4}$$

Observe that, Ω being bounded, the sequence (ε_k) is bounded. We define

$$\overline{v}_k(z) = \varepsilon_k^{\frac{N-2K}{2}} v_k(\varepsilon_k z + y_k) \in H^K(\mathbb{R}^N).$$

It results $\|\overline{v}_k\|_{K,2}^2 = \|v_k\|_{K,2}^2$ and $|\overline{v}_k|_{K_*}^{K_*} = |v_k|_{K_*}^{K_*}$. In particular, $(\overline{v}_k)$ is a bounded sequence in $H^K(\mathbb{R}^N)$ and, up to a subsequence,

$$\begin{aligned}
\overline{v}_k &\rightharpoonup \overline{v} \quad \text{weakly in } H^K(\mathbb{R}^N),\\
D^j\overline{v}_k &\to D^j\overline{v} \quad \text{a.e. in } \mathbb{R}^N, \quad j = 0,\ldots,K-1,\\
D^j\overline{v}_k &\to D^j\overline{v} \quad \text{in } L^2_{loc}(\mathbb{R}^N), \quad j = 0,\ldots,K-1,
\end{aligned}$$

as $k \to +\infty$. We wish to show that $\overline{v} \neq 0$. Assume by contradiction that $\overline{v} = 0$. It follows from Lemma 5.1 and equality (5.4) above, and from Sobolev and Hölder inequalities that, as $k \to +\infty$,

$$\begin{aligned}
S_K\, |h\overline{v}_k|_{K_*}^2 &\leqslant \|h\overline{v}_k\|_{K,2}^2\\
&= \big(\overline{v}_k, h^2\overline{v}_k\big)_{K,2} + o(1)\\
&= \int_{\mathbb{R}^N} h^2\, |\overline{v}_k|^{K_*} dx + DE_0(v_k)\Big(h^2\Big(\frac{\cdot - y_k}{\varepsilon_k}\Big)\ v_k\Big) + o(1)\\
&\leqslant \bigg(\int_{B(z,1)} |\overline{v}_k|^{K_*} dx\bigg)^{2K/N} \bigg(\int_{\mathbb{R}^N} |h\overline{v}_k|^{K_*} dx\bigg)^{2/K_*} + o(1)\\
&\leqslant \delta^{2K/N}\, |h\overline{v}_k|_{K_*}^2 + o(1) \leqslant \frac{S_K}{2}\, |h\overline{v}_k|_{K_*}^2 + o(1)
\end{aligned}$$

for each $z \in \mathbb{R}^N$ and $h \in C_c^\infty(B(z,1))$. Hence $\overline{v}_k \to 0$ in $L^{K_*}_{loc}(\mathbb{R}^N)$. This is a contradiction to (5.4). Therefore, $\overline{v} \neq 0$.

Since Ω is bounded and $v_k \rightharpoonup 0$ in $H_0^K(\Omega)$, up to a subsequence, $y_k \to y \in \overline{\Omega}$ and $\varepsilon_k \to 0$. If $(\varepsilon_k^{-1}\mathrm{dist}(y_k, \partial\Omega))$ is bounded, we may assume that

$$\lim_{k\to+\infty} \varepsilon_k^{-1}\mathrm{dist}(y_k,\partial\Omega) = b.$$

It is then easy to verify that, up to a rotation of $\mathbb{R}^N$, the sets

$$\Omega_k = \big\{z \in \mathbb{R}^N :\ \varepsilon_k z + y_k \in \Omega\big\}$$

satisfy

$$\bigcap_{n=1}^{+\infty}\left(\bigcup_{k=n}^{+\infty}\Omega_k\right)=\mathbb{H}^N:=\Big\{(z_1,\dots,z_N)\in\mathbb{R}^N:z_N\geqslant -b\Big\}.$$

Hence, $\overline{v}$ is a nontrivial solution of the equation

$$(-\Delta)^K u=|u|^{K_*-2}u \quad \text{in } \mathbb{H}^N.$$

On the other hand, if

$$\varepsilon_k^{-1}\mathrm{dist}(y_k,\partial\Omega)\to+\infty$$

then $\overline{v}$ is a nontrivial solution of the equation

$$(-\Delta)^K u=|u|^{K_*-2}u \quad \text{in } \mathbb{R}^N.$$

In both cases we obtain that $E_0(\overline{v})=\frac{K}{N}\|\overline{v}\|_{K,2}^2\geqslant\frac{K}{N}(S_K)^{N/2K}$.

Let $\Gamma=\{g\in G:\ gy=y\}$ be the isotropy subgroup of y. Thus, the G–orbit Gy of y is G–homeomorphic to the homogeneous space of right cosets G/Γ [12]. Let $S=\{g_1,\dots,g_m\}$ be a finite subset of G whose elements represent pairwise distinct cosets $[g_1],\dots,[g_m]$ in G/Γ. Since $y_k\to y$ and $\varepsilon_k\to 0$, it follows that $\varepsilon_k^{-1}|g_iy_k-g_jy_k|\to+\infty$ for each $i\neq j$. Hence, since v_k is G–invariant, it results

$$\begin{aligned}
&\left\|v_k-\sum_{i=1}^m\varepsilon_k^{\frac{2K-N}{2}}\ \overline{v}g_i^{-1}\left(\frac{\cdot\ -g_iy_k}{\varepsilon_k}\right)\right\|_{K,2}^2\\
=&\left\|\varepsilon_k^{\frac{N-2K}{2}}v_k(\varepsilon_k\ \cdot\ +g_1y_k)-\sum_{i=1}^m\overline{v}g_i^{-1}\left(\ \cdot\ +\frac{g_1y_k-g_iy_k}{\varepsilon_k}\right)\right\|_{K,2}^2\\
=&\left\|\overline{v}_kg_1^{-1}-\overline{v}g_1^{-1}-\sum_{i\neq1}\overline{v}g_i^{-1}\left(\ \cdot\ +\frac{g_1y_k-g_iy_k}{\varepsilon_k}\right)\right\|_{K,2}^2\\
=&\left\|\overline{v}_kg_1^{-1}-\sum_{i\neq1}\overline{v}g_i^{-1}\left(\ \cdot\ +\frac{g_1y_k-g_iy_k}{\varepsilon_k}\right)\right\|_{K,2}^2-\left\|\overline{v}g_1^{-1}\right\|_{K,2}^2+o(1)\\
=&\left\|v_k-\sum_{i\neq1}\varepsilon_k^{\frac{2K-N}{2}}\ \overline{v}g_i^{-1}\left(\frac{\cdot\ -g_iy_k}{\varepsilon_k}\right)\right\|_{K,2}^2-\|\overline{v}\|_{K,2}^2+o(1)
\end{aligned}$$

and, inductively,

$$\begin{aligned}
\|v_k\|_{K,2}^2&=\left\|v_k-\sum_{i=1}^m\varepsilon_k^{\frac{2K-N}{2}}\ \overline{v}g_i^{-1}\left(\frac{\cdot\ -g_iy_k}{\varepsilon_k}\right)\right\|_{K,2}^2+m\|\overline{v}\|_{K,2}^2+o(1)\\
&\geqslant m(S_K)^{N/2K}+o(1)
\end{aligned}$$

as $k\to+\infty$ for all $m\leqslant\sharp\, G/\Gamma$. Since $\|v_k\|_{K,2}^2$ is bounded it follows that $Gy\cong G/\Gamma$ is finite. If $c<\mu^G$, then $\|v_k\|_{K,2}^2\to\frac{N}{K}d<\frac{N}{K}\mu^G$. It follows that

$$\sharp\, Gy<\Big(\min_{x\in\overline{\Omega}}\sharp\, Gx\Big).$$

This is a contradiction. Hence E_0 satisfies $(PS)_c^G$ for each $c<\mu^G$.

For a complete description of all G–PS–sequences for $K=1$ we refer to [9].

6. **Proof of Lemma 4.2.** By [8, Lemma 4] there exists $C_{N,K} > 0$ such that for all $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} &\big| |\alpha+\beta|^{K_*} - |\alpha|^{K_*} - |\beta|^{K_*} - K_* \alpha\beta \left(|\alpha|^{K_*-2} + |\beta|^{K_*-2} \right) \big| \\ &\leqslant C_{N,K} \begin{cases} |\alpha|\,|\beta|^{K_*-1} & \text{for } |\alpha| \geqslant |\beta|, \\ |\alpha|^{K_*-1}|\beta| & \text{for } |\alpha| \leqslant |\beta| \end{cases} \end{aligned} \tag{6.1}$$

if $N \geqslant 6K$, and for all $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} &\big| |\alpha+\beta|^{K_*} - |\alpha|^{K_*} - |\beta|^{K_*} - K_* \alpha\beta \left(|\alpha|^{K_*-2} + |\beta|^{K_*-2} \right) \big| \\ &\leqslant C_{N,K} \left(|\alpha|^{K_*-2}|\beta|^2 + |\alpha|^2|\beta|^{K_*-2} \right) \end{aligned}$$

provided that $N \in \{2K+1, \dots, 6K-1\}$.

In the following, it is understood that $u^G_{f,0} = 0$ outside Ω.

• Case $N \geqslant 6K$; by (6.1), to prove the assertion on R_ε it suffices to estimate the right hand side of the inequality

$$|R_\varepsilon| \leqslant C_{N,K} \int_{\{|u^G_{f,0}| \geqslant t w_{\varepsilon,y}\}} |u^G_{f,0}| \, (t w_{\varepsilon,y})^{K_*-1} \, dx \,.$$

Splitting the integration into $|x-y| < \varrho/2$ and $\varrho/2 \leqslant |x-y|$, one gets

$$\begin{aligned} |R_\varepsilon| \leqslant{}& C'_{N,K,t} \, (\sharp\, G) \varepsilon^{\gamma_2(N-2K)/2} \int_{\mathbb{R}^N} \frac{|u^G_{f,0}(x)|^{1+\gamma_1}}{|x-y|^{\gamma_2(N-2K)}} \, dx \\ &+ C'_{N,K,t} \, (\sharp\, G) \, \varepsilon^{(N+2K)/2} \end{aligned}$$

where $\gamma_1, \gamma_2 > 0$ satisfy $\gamma_1 + \gamma_2 = K_* - 1$ and $\gamma_2 < N/(N-2K)$. Note that

$$|u^G_{f,0}|^{1+\gamma_1} \in L^1(\mathbb{R}^N), \quad |x|^{-\gamma_2(N-2K)} \in L^1(\mathbb{R}^N)$$

since $1+\gamma_1 < K_*$ and $\gamma_2(N-2K) < N$. In particular, by [13, p.232],

$$\int_{\mathbb{R}^N} |u^G_{f,0}(x)|^{1+\gamma_1} \frac{1}{|x-y|^{\gamma_2(N-2K)}} \, dx < +\infty$$

for a.e. $y \in \Sigma$ (as convolution of two L^1 functions). It follows that $R_\varepsilon = O\left(\varepsilon^{N\vartheta/2}\right)$ for each $\vartheta < 1$. In a similar fashion, estimating the right hand side of

$$|S_\varepsilon| \leqslant C'_{N,K} \int_{\{|u^G_{f,0}| < t w_{\varepsilon,y}\}} |u^G_{f,0}|^{K_*-1} \, t w_{\varepsilon,y} \, dx$$

yields $S_\varepsilon = O\left(\varepsilon^{N\vartheta/2}\right)$ for each $\vartheta < 1$ as $\varepsilon \to 0$.

• Case $N \in \{2K+1, \dots, 6K-1\}$; it results

$$N = 2K + j, \quad K_* = \frac{4K+2j}{j}, \quad j = 1, \dots, 4K-1.$$

We distinguish three cases:

- $j = 1, \dots, 2K-1$,
- $j = 2K$,
- $j = 2K+1, \dots, 4K-1$.

• Case $j = 1, \dots, 2K-1$; by (6.2) for each j it results

$$|R_\varepsilon| \leqslant C'_{N,K,t}\, \varepsilon^j (\sharp\, G) \int_{\mathbb{R}^N} |u^G_{f,0}(x)|^{4K/j} \, \frac{1}{|x-y|^{2j}}\, dx\,,$$

$$|S_\varepsilon| \leqslant C'_{N,K,t}\, \varepsilon^j (\sharp\, G) \int_{\mathbb{R}^N} u^G_{f,0}(x)^2 \, \frac{\varepsilon^{2K-j}}{(\varepsilon^2+|x-y|^2)^{2K}}\, dx.$$

The first estimate gives $R_\varepsilon = O\left(\varepsilon^j\right)$ since the integral, again by [13, p.232], is finite for a.e. $y \in \Sigma$. The second estimate gives $R_\varepsilon = O\left(\varepsilon^j\right)$ as well, since the kernels

$$\left\{ x \mapsto \frac{\varepsilon^{2K-j}}{(\varepsilon^2+|x-y|^2)^{2K}} = \frac{1}{\varepsilon^N} \frac{1}{\left(1+\left|\frac{x-y}{\varepsilon}\right|^2\right)^{2K}} \right\}$$

correspond to that of a mollifier (up to a constant) and thus

$$\int_{\mathbb{R}^N} u^G_{f,0}(x)^2 \, \frac{\varepsilon^{2K-j}}{(\varepsilon^2+|x-y|^2)^{2K}}\, dx \to c u^G_{f,0}(y)^2$$

as $\varepsilon \to 0$ for some $c > 0$ and a.e. $y \in \Sigma$ (see [13, Theorem 8.15]).

• Case $j = 2K$; by (6.2) it results

$$|R_\varepsilon|, |S_\varepsilon| \leqslant C'_{N,K,t}\, (\sharp\, G) \int_{\mathbb{R}^N} |u^G_{f,0}(x)|^2 \, \frac{\varepsilon^{2K}}{(\varepsilon^2+|x-y|^2)^{2K}}\, dx\,.$$

If $\gamma_1, \gamma_2 > 0$ satisfy $\gamma_1 + \gamma_2 = 2K$, then

$$\frac{1}{(\varepsilon^2+|x-y|^2)^{2K}} \leqslant \frac{1}{\varepsilon^{2\gamma_1}} \frac{1}{|x-y|^{2\gamma_2}},$$

which implies

$$|R_\varepsilon|, |S_\varepsilon| \leqslant C'_{N,K,t}\, \varepsilon^{2K-2\gamma_1} (\sharp\, G) \int_{\mathbb{R}^N} |u^G_{f,0}(x)|^2 \, \frac{1}{|x-y|^{2\gamma_2}}\, dx.$$

This yields $R_\varepsilon = S_\varepsilon = O(\varepsilon^{2K\vartheta})$ for all $\vartheta < 1$.

• Case $j = 2K+1, \dots, 4K-1$; by (6.2), one has

$$|R_\varepsilon| \leqslant C'_{N,K,t}\, \varepsilon^{2K} (\sharp\, G) \int_{\mathbb{R}^N} |u^G_{f,0}(x)|^{4K/j} \, \frac{\varepsilon^{j-2K}}{(\varepsilon^2+|x-y|^2)^j}\, dx\,,$$

$$|S_\varepsilon| \leqslant C'_{N,K,t}\, \varepsilon^{2K} (\sharp\, G) \int_{\mathbb{R}^N} |u^G_{f,0}(x)|^2 \, \frac{1}{|x-y|^{4K}}\, dx.$$

Taking into account that

$$\int_{\mathbb{R}^N} |u^G_{f,0}(x)|^2 \, \frac{1}{|x-y|^{4K}}\, dx < +\infty$$

a.e. $y \in \Sigma$ and that the kernels

$$\left\{ x \mapsto \frac{\varepsilon^{j-2K}}{(\varepsilon^2+|x-y|^2)^j} = \frac{1}{\varepsilon^N} \frac{1}{\left(1+\left|\frac{x-y}{\varepsilon}\right|^2\right)^{j}} \right\}$$

correspond to that of a mollifier (up to a constant), arguing as before one gets

$$R_\varepsilon = S_\varepsilon = O\left(\varepsilon^{2K}\right).$$

Therefore, putting the previous conclusions together, we have

$$R_\varepsilon = S_\varepsilon = \begin{cases} O(\varepsilon) & \text{if } N = 2K+1, \\ \vdots & \vdots \\ O\left(\varepsilon^j\right) & \text{if } N = 2K+j, \\ \vdots & \vdots \\ O\left(\varepsilon^{2K-1}\right) & \text{if } N = 4K-1, \\ O\left(\varepsilon^{2K\vartheta}\right) & \text{if } N = 4K, \\ O\left(\varepsilon^{2K}\right) & \text{if } N \in \left\{4K+1, \dots, 6K-1\right\}, \\ O\left(\varepsilon^{N\vartheta/2}\right) & \text{if } N \geqslant 6K \end{cases}$$

for each $\vartheta < 1$ as $\varepsilon \to 0$. In particular, in any case it results

$$R_\varepsilon = S_\varepsilon = o\Big(\varepsilon^{(N-2K)/2}\Big)$$

as $\varepsilon \to 0$ and the proof is complete. □

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