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THE NEHARI MANIFOLD FOR FRACTIONAL SYSTEMS INVOLVING CRITICAL NONLINEARITIES

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Abstract. We study the combined effect of concave and convex nonlinearities on the number of positive solutions for a fractional system involving critical Sobolev exponents. With the help of the Nehari manifold, we prove that the system admits at least two positive solutions when the pair of parameters (λ, μ) belongs to a suitable subset of $\mathbb{R}^2$.

1. **Introduction.**

This paper is concerned with the multiplicity of positive solutions for the following elliptic system involving the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)^s v = \mu |v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda, \mu > 0$, $1 < q < 2$ and $\alpha > 1, \beta > 1$ satisfy $\alpha + \beta = 2_s^* = 2N/(N-2s)$, $s \in (0,1)$ and $N > 2s$. When $\alpha = \beta, \alpha + \beta = p \le 2_s^*, \lambda = \mu$ and $u = v$, problem (1) reduces to the semilinear scalar fractional elliptic equation

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{q-2}u + |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Recently, a great attention has been focused on the study of nonlinear problems like (2) which involve the fractional Laplacian. This type of operators naturally arises in physical situations such as thin obstacle problems, optimization, population dynamics, geophysical fluid dynamics, mathematical finance, phases transitions, straitified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flames propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves, see [21]. We refer to [12,23,26,27,29,32]

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for the subcritical case and to [3, 4, 11, 14, 15, 22, 24, 28] for the critical case. In the remarkable paper [9], Caffarelli and Silvestre gave a new formulation of the fractional Laplacian through Dirichlet-Neumann maps. This is extensively used in the recent literature since it allows to transform nonlocal problems to local ones, which permits to use variational methods. For example, Barrios, Colorado, de Pablo and Sánchez [3] used the idea of the s-harmonic extension and studied the effect of lower order perturbations in the existence of positive solutions of (2). Brändle, Colorado and de Pablo [5] investigated the fractional elliptic equation (2) involving concave-convex nonlinearity, and obtained an analogue multiplicity result to the problem considered by Ambrosetti, Brézis and Cerami in [2]. In the case $q = 2$ and $p = 2_s^*$, Servadei and Valdinoci [24] studied (2) and extended the classical Brézis-Nirenberg result [6] to the nonlocal case. In [8], Cabré and Tan defined $(-\Delta)^{1/2}$ through the spectral decomposition of the Laplacian operator on Ω with zero Dirichlet boundary conditions. With classical local techniques, they established existence of positive solutions for problems with subcritical nonlinearities, regularity and L^∞-estimates for weak solutions. In particular, Tan [28] considered

$$\begin{cases} (-\Delta)^{1/2}u = \lambda u + u^{\frac{N+1}{N-1}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

investigating the solvability (see also [32] for a subcritical situation). Very recently, Colorado, de Pablo, and Sánchez [14] studied the following nonhomogeneous fractional equation involving critical Sobolev exponent

$$\begin{cases} (-\Delta)^{s}u = |u|^{2_s^*-2}u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

and proved existence and multiplicity of solutions under appropriate conditions on the size of f. For the same problems, Shang, Zhang and Yang [22] obtained similar results.

The analogue problems to (1) for the Laplacian operator have been studied extensively in recent years, see [1,13,19,20,30,31] and the references therein. In particular, Hu and Lin [20] studied the Laplacian system with critical growth and obtained the existence and multiplicity results of positive solutions by variational methods.

The purpose of this paper is to study system (1) in the critical case $\alpha+\beta = 2_s^*$. Using variational methods and a *Nehari manifold decomposition*, we prove that system (1) admits at least two positive solutions when the pair of parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. To our best knowledge, there are just a few results in the literature on the fractional system (1) with both concave-convex nonlinearities and critical growth terms. We point out that we adopt in the paper the *spectral* (or *regional*) definition of the fractional laplacian in a bounded domain based upon a Caffarelli-Silvestre type extension (see [14]), and not the *integral definition*. We shall refer to [25] for a nice comparison between these two different notions. In [18], a problem like (1) with $q = 2$ is investigated, using the integral notion, from the point of view of existence, nonexistence and regularity.

To formulate the main result, we introduce

$$\Lambda_1 := \left(\frac{2_s^* - q}{2_s^* - 2}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\right)^{-\frac{2}{2-q}} \left[\frac{2-q}{2(2_s^*-q)}(k_s\mathcal{S}_{s,\alpha,\beta})^{\frac{2_s^*}{2}}\right]^{\frac{2}{2_s^*-2}}, \tag{4}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, k_s is a normalization constant and the numbers $\mathcal{S}(s,N), \mathcal{S}_{s,\alpha,\beta}$ are best Sobolev constants that will be introduced later.

For $\gamma > 0$, we also consider

$$\mathscr{C}_\gamma := \left\{(\lambda,\mu) \in \mathbb{R}^2_+ : 0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \gamma\right\}.$$

Then we have the following

Theorem 1.1. *The following facts holds*

(*i*) *system* (1) *has at least one positive solution for all* $(\lambda,\mu) \in \mathscr{C}_{\Lambda_1}$.
(*ii*) *there is* $\Lambda_2 < \Lambda_1$ *such that system* (1) *has at least two positive solutions for all* $(\lambda,\mu) \in \mathscr{C}_{\Lambda_2}$.

Concerning regularity, one can get a priori estimates for the solutions to (1) and hence obtain, as in [3, Proposition 5.2], that $u, v \in C^\infty(\overline{\Omega})$ for $s = 1/2$, $u, v \in C^{0,2s}(\overline{\Omega})$ if $0 < s < 1/2$ and $u, v \in C^{1,2s-1}(\overline{\Omega})$ if $1/2 < s < 1$.

The paper is organized as follows. In Section 2 we introduce the variational setting of the problem and present some preliminary results. In Section 3 we show that the Palais-Smale condition holds for the energy functional associated with (1) at energy levels in a suitable range related to the best Sobolev constants. In Section 4 we give some properties about the Nehari manifold and fibering maps. In Section 5 we investigate the existence of Palais-Smale sequences. In Section 6 we obtain solutions to some related local minimization problems. Finally, the proof of Theorem 1.1 will be given in Section 7.

2. Some preliminary facts.

In this section, we collect some preliminary facts in order to establish the functional setting. First of all, let us introduce the standard notations for future use in this paper. We denote the upper half-space in $\mathbb{R}^{N+1}_+$ by

$$\mathbb{R}^{N+1}_+ := \{z = (x,y) = (x_1, \cdots, x_N, y) \in \mathbb{R}^{N+1} : y > 0\}.$$

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Denote by

$$\mathcal{C}_\Omega := \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+,$$

the cylinder with base Ω and its lateral boundary by $\partial_L \mathcal{C}_\Omega := \partial\Omega \times (0,\infty)$. The powers $(-\Delta)^s$ of the positive Laplace operator $-\Delta$, in Ω, with zero Dirichlet boundary conditions are defined via its spectral decomposition, namely

$$(-\Delta)^s u(x) := \sum_{j=1}^\infty a_j \rho_j^s \varphi_j(x),$$

where (ρ_j, φ_j) is the sequence of eigenvalues and eigenfunctions of the operator $-\Delta$ in Ω under zero Dirichlet boundary data and a_j are the coefficients of u for the base $\{\varphi_j\}_{j=1}^\infty$ in $L^2(\Omega)$. In fact, the fractional Laplacian $(-\Delta)^s$ is well defined in the space of functions

$$H^s_0(\Omega) := \left\{u = \sum_{j=1}^\infty a_j \varphi_j \in L^2(\Omega) : \ \|u\|_{H^s_0} = \Big(\sum_{j=1}^\infty a_j^2 \rho_j^s\Big)^{1/2} < \infty\right\},$$

and $\|u\|_{H^s_0} = \|(-\Delta)^{s/2} u\|_{L^2(\Omega)}$. The dual space $H^{-s}(\Omega)$ is defined in the standard way, as well as the inverse operator $(-\Delta)^{-s}$.

Definition 2.1. We say that $(u,v) \in H_0^s(\Omega) \times H_0^s(\Omega)$ is a solution of (1.1) if the identity

$$\int_\Omega \left((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi_1 + (-\Delta)^{\frac{s}{2}}v(-\Delta)^{\frac{s}{2}}\varphi_2\right) dx - \int_\Omega (\lambda|u|^{q-2}u\varphi_1 + \mu|v|^{q-2}v\varphi_2)dx$$
$$- \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^\beta\varphi_1 dx - \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^{\beta-2}v\varphi_2 dx = 0,$$

holds for all $(\varphi_1,\varphi_2) \in H_0^s(\Omega) \times H_0^s(\Omega)$.

Associated with problem (1), we consider the energy functional

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u,v) :=& \frac{1}{2}\int_\Omega \left(|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2\right) dx - \frac{1}{q}\int_\Omega (\lambda|u|^q + \mu|v|^q)dx \\ &- \frac{2}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta dx.\end{aligned}$$

The functional is well defined in $H_0^s(\Omega) \times H_0^s(\Omega)$, and moreover, the critical points of the functional $\mathcal{J}_{\lambda,\mu}$ correspond to solutions of (1). We now conclude the main ingredients of a recently developed technique used in order to deal with fractional powers of the Laplacian. To treat the nonlocal problem (1), we shall study a corresponding extension problem, which allows us to investigate problem (1) by studying a local problem via classical variational methods. We first define the extension operator and fractional Laplacian for functions in $H_0^s(\Omega) \times H_0^s(\Omega)$. We refer the reader to [3–5, 10] and to the references therein.

Definition 2.2. For a function $u \in H_0^s(\Omega)$, we denote its s-harmonic extension $w = E_s(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution of the problem

$$\begin{cases} \mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega \\ w = 0 & \text{on } \partial_L\mathcal{C}_\Omega \\ w = u & \text{on } \Omega \times \{0\}, \end{cases}$$

and

$$(-\Delta)^s u(x) = -k_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y),$$

where $k_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$ is a normalization constant.

The extension function $w(x,y)$ belongs to the space

$$X_0^s(\mathcal{C}_\Omega) := \overline{C_0^\infty(\Omega\times[0,\infty))}^{\|\cdot\|_{X_0^s(\mathcal{C}_\Omega)}}$$

endowed with the norm

$$\|z\|_{X_0^s(\mathcal{C}_\Omega)} := \left(k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla z|^2 dxdy\right)^{1/2}.$$

The extension operator is an isometry between $H_0^s(\Omega)$ and $X_0^s(\mathcal{C}_\Omega)$, namely

$$\|u\|_{H_0^s(\Omega)} = \|E_s(u)\|_{X_0^s(\mathcal{C}_\Omega)}, \quad \text{for all } u \in H_0^s(\Omega).$$

With this extension we can reformulate (1) as the following local problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_1) = 0, \ -\mathrm{div}(y^{1-2s}\nabla w_2) = 0 & \text{in } \mathcal{C}_\Omega \\ w_1 = w_2 = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ w_1 = u, \ w_2 = v & \text{on } \Omega \times \{0\} \\ \dfrac{\partial w_1}{\partial \nu^s} = \lambda |w_1|^{q-2} w_1 + \dfrac{2\alpha}{\alpha+\beta} |w_1|^{\alpha-2} w_1 |w_2|^\beta & \text{on } \Omega \times \{0\} \\ \dfrac{\partial w_2}{\partial \nu^s} = \mu |w_2|^{q-2} w_2 + \dfrac{2\beta}{\alpha+\beta} |w_1|^{\alpha} |w_2|^{\beta-2} w_2 & \text{on } \Omega \times \{0\}, \end{cases} \tag{5}$$

where

$$\frac{\partial w_i}{\partial \nu^s} := -k_s \lim_{y\to 0^+} y^{1-2s} \frac{\partial w_i}{\partial y}, \quad i = 1, 2,$$

and $w_1, w_2 \in X_0^s(\mathcal{C}_\Omega)$ are the s-harmonic extension of $u, v \in H_0^s(\Omega)$, respectively. Let us set

$$E_0^s(\mathcal{C}_\Omega) := X_0^s(\mathcal{C}_\Omega) \times X_0^s(\mathcal{C}_\Omega).$$

An energy solution to this problem is a function $(w_1, w_2) \in E_0^s(\mathcal{C}_\Omega)$ satisfying

$$\begin{aligned} k_s \int_{\mathcal{C}_\Omega} & y^{1-2s} \nabla w_1 \cdot \nabla \varphi_1 dxdy + k_s \int_{\mathcal{C}_\Omega} y^{1-2s} \nabla w_2 \cdot \nabla \varphi_2 dxdy \\ &= \lambda \int_\Omega |w_1|^{q-2} w_1 \varphi_1 dx + \frac{2\alpha}{\alpha+\beta} \int_\Omega |w_1|^{\alpha-2} w_1 |w_2|^\beta \varphi_1 dx \\ &\quad + \mu \int_\Omega |w_2|^{q-2} w_2 \varphi_2 dx + \frac{2\beta}{\alpha+\beta} \int_\Omega |w_1|^\alpha |w_2|^{\beta-2} w_2 \varphi_2 dx, \end{aligned}$$

for all $(\varphi_1, \varphi_2) \in E_0^s(\mathcal{C}_\Omega)$. If $(w_1, w_2) \in E_0^s(\mathcal{C}_\Omega)$ satisfies (5), then

$$(u, v) = (w_1(\cdot, 0), w_2(\cdot, 0)),$$

defined in the sense of traces, belongs to the space $H_0^s(\Omega) \times H_0^s(\Omega)$ and it is a solution of the original problem (1). The associated energy functional to the problem (5) is denoted by

$$\begin{aligned} \mathcal{I}_{\lambda,\mu}(w) := \mathcal{I}_{\lambda,\mu}(w_1, w_2) &= \frac{k_s}{2} \int_{\mathcal{C}_\Omega} y^{1-2s} (|\nabla w_1|^2 + |\nabla w_2|^2) dxdy \\ &- \frac{1}{q} \int_\Omega (\lambda |w_1|^q + \mu |w_2|^q) dx - \frac{2}{\alpha+\beta} \int_\Omega |w_1|^\alpha |w_2|^\beta dx. \end{aligned}$$

Critical points of $\mathcal{I}_{\lambda,\mu}$ in $E_0^s(\mathcal{C}_\Omega)$ correspond to critical points of $\mathcal{J}_{\lambda,\mu} : H_0^s(\Omega) \times H_0^s(\Omega) \to \mathbb{R}$. In the following lemma we list some relevant inequalities from [5].

Lemma 2.3. *For any $1 \le r \le 2_s^*$ and any $z \in X_0^s(\mathcal{C}_\Omega)$, it holds*

$$\left(\int_\Omega |u(x)|^r dx \right)^{\frac{2}{r}} \le C \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla z(x,y)|^2 dxdy, \qquad u := \mathrm{Tr}(z), \tag{6}$$

for some positive constant $C = C(r, s, N, \Omega)$. Furthermore, the space $X_0^s(\mathcal{C}_\Omega)$ is compactly embedded into $L^r(\Omega)$, for every $r < 2_s^*$.

Remark 1. When $r = 2_s^*$, the best constant in (6) is denoted by $\mathcal{S}(s, N)$, that is

$$\mathcal{S}(s, N) := \inf_{z \in X_0^s(\mathcal{C}_\Omega) \setminus \{0\}} \frac{\int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla z(x,y)|^2 dxdy}{\left(\int_\Omega |z(x,0)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}}. \tag{7}$$

It is not achieved in any bounded domain and, for all $z \in X^s(\mathbb{R}^{N+1}_+)$,

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla z(x,y)|^2 dxdy \geq \mathcal{S}(s,N)\left(\int_{\mathbb{R}^N}|z(x,0)|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{N}}. \tag{8}$$

$\mathcal{S}(s,N)$ is achieved for $\Omega = \mathbb{R}^N$ by functions w_ε which are the s-harmonic extensions of

$$u_\varepsilon(x) := \frac{\varepsilon^{(N-2s)/2}}{(\varepsilon^2+|x|^2)^{(N-2s)/2}}, \qquad \varepsilon > 0,\ \ x \in \mathbb{R}^N. \tag{9}$$

Let $U(x) = (1+|x|^2)^{\frac{2s-N}{2}}$ and let $\mathcal{W}$ be the extension of U (cf. [3,5]). Then

$$\mathcal{W}(x,y) = E_s(U) = c_{N,s}y^{2s}\int_{\mathbb{R}^N}\frac{U(z)dz}{(|x-z|^2+y^2)^{\frac{N+2s}{2}}},$$

is the extreme function for the fractional Sobolev inequality (8). The constant $\mathcal{S}(s,N)$ given in (7) takes the exact value

$$\mathcal{S}(s,N) = \frac{2\pi^s\Gamma(\frac{N+2s}{2})\Gamma(1-s)(\Gamma(\frac{N}{2}))^{\frac{2s}{N}}}{\Gamma(s)\Gamma(\frac{N-2s}{2})(\Gamma(N))^s},$$

and it is achieved for $\Omega = \mathbb{R}^N$ by the functions w_ε.

Now, we consider the following minimization problem

$$\mathcal{S}_{s,\alpha,\beta} := \inf_{(w_1,w_2)\in E_0^s(\mathcal{C}_\Omega)\setminus\{0\}} \frac{\int_{\mathcal{C}_\Omega} y^{1-2s}(|\nabla w_1|^2+|\nabla w_2|^2)dxdy}{\Big(\int_\Omega |w_1|^\alpha|w_2|^\beta dx\Big)^{\frac{2}{2_s^*}}}. \tag{10}$$

Using ideas from [1], we establish a relationship between $\mathcal{S}(s,N)$ and $\mathcal{S}_{s,\alpha,\beta}$ (see also [18]).

Lemma 2.4. *For the constants $\mathcal{S}(s,N)$ and $\mathcal{S}_{s,\alpha,\beta}$ introduced in (7) and (10), it holds*

$$\mathcal{S}_{s,\alpha,\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{2_s^*}}\right]\mathcal{S}(s,N). \tag{11}$$

In particular, the constant $\mathcal{S}_{s,\alpha,\beta}$ is achieved for $\Omega = \mathbb{R}^N$.

Proof. Let $\{z_n\} \subset X_0^s(\mathcal{C}_\Omega)$ be a minimization sequence for $\mathcal{S}(s,N)$. Let $\sigma, t > 0$ to be chosen later and consider the sequences $w_{1,n} := \sigma z_n$ and $w_{2,n} := tz_n$ in $X_0^s(\mathcal{C}_\Omega)$. By means of (10), we have

$$\frac{\sigma^2+t^2}{(\sigma^\alpha t^\beta)^{\frac{2}{2_s^*}}}\frac{\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla z_n(x,y)|^2dxdy}{\big(\int_\Omega |z_n|^{2_s^*}dx\big)^{\frac{2}{2_s^*}}} \geq \mathcal{S}_{s,\alpha,\beta}.$$

Defining $g:\mathbb{R}^+\to\mathbb{R}^+$ by setting $g(x) := x^{\frac{2\beta}{2_s^*}} + x^{\frac{-2\alpha}{2_s^*}}$, we have

$$\frac{\sigma^2+t^2}{(\sigma^\alpha t^\beta)^{\frac{2}{2_s^*}}} = g\Big(\frac{\sigma}{t}\Big), \qquad \min_{\mathbb{R}^+} g = g(x_0) = g(\sqrt{\alpha/\beta}) = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{2_s^*}}.$$

Choosing σ, t in the previous inequality such that $\sigma/t = \sqrt{\alpha/\beta}$ and letting $n\to\infty$ yields

$$\Big[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{2_s^*}}\Big]\mathcal{S}(s,N) \geq \mathcal{S}_{s,\alpha,\beta}.$$

On the other hand, let $\{(w_{1,n}, w_{2,n})\} \subset E_0^s(\mathcal{C}_\Omega) \setminus \{0\}$ be a minimizing sequence for $\mathcal{S}_{s,\alpha,\beta}$. Set $h_n := \sigma_n w_{2,n}$ for $\sigma_n > 0$ with $\int_\Omega |w_{1,n}|^{2_s^*} dx = \int_\Omega |h_n|^{2_s^*} dx$. Then Young's inequality yields

$$\int_\Omega |w_{1,n}|^\alpha |h_n|^\beta dx \leq \frac{\alpha}{2_s^*} \int_\Omega |w_{1,n}|^{2_s^*} dx + \frac{\beta}{2_s^*} \int_\Omega |h_n|^{2_s^*} dx = \int_\Omega |h_n|^{2_s^*} dx = \int_\Omega |w_{1,n}|^{2_s^*} dx.$$

In turn, we can estimate

$$\begin{aligned}
&\frac{\displaystyle\int_{\mathcal{C}_\Omega} y^{1-2s}(|\nabla w_{1,n}(x,y)|^2 + |\nabla w_{2,n}(x,y)|^2) dxdy}{\Big(\displaystyle\int_\Omega |w_{1,n}|^\alpha |w_{2,n}|^\beta dx\Big)^{\frac{2}{\alpha+\beta}}} \\
&= \frac{\sigma_n^{\frac{2\beta}{2_s^*}} \displaystyle\int_{\mathcal{C}_\Omega} y^{1-2s}(|\nabla w_{1,n}(x,y)|^2 + |\nabla w_{2,n}(x,y)|^2) dxdy}{\Big(\displaystyle\int_\Omega |w_{1,n}|^\alpha |h_n|^\beta dx\Big)^{\frac{2}{\alpha+\beta}}} \\
&\geq \frac{\sigma_n^{\frac{2\beta}{2_s^*}} \displaystyle\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w_{1,n}(x,y)|^2 dxdy}{\left(\displaystyle\int_\Omega |w_{1,n}|^{2_s^*} dx\right)^{2/2_s^*}} + \frac{\sigma_n^{\frac{2\beta}{2_s^*}} \sigma_n^{-2} \displaystyle\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla h_n(x,y)|^2 dxdy}{\left(\displaystyle\int_\Omega |h_n|^{2_s^*} dx\right)^{2/2_s^*}} \\
&\geq \mathcal{S}(s,N) g(\sigma_n) \geq \mathcal{S}(s,N) g(\sqrt{\alpha/\beta}).
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the last inequality we obtain

$$\Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{2_s^*}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{2_s^*}}\Big] \mathcal{S}(s,N) \leq \mathcal{S}_{s,\alpha,\beta}.$$

Whence, the conclusion follows by combining the previous inequalities. □

In the end of this section, we fix some notations that will be used in the sequel.

Notations. In this paper we use the following notations:

- $L^p(\Omega)$, $1 \leq p \leq \infty$ denote Lebesgue spaces, with norm $\|\cdot\|_p$. $E = X_0^s(\mathcal{C}_\Omega) \times X_0^s(\mathcal{C}_\Omega)$ is equipped with norm $\|z\|^2 = \|(w_1, w_2)\|^2 = \|w_1\|^2_{X_0^s(\mathcal{C}_\Omega)} + \|w_2\|^2_{X_0^s(\mathcal{C}_\Omega)}$.
- The dual space of a Banach space E will be denoted by E^{-1}. We set $tz = t(w_1, w_2) = (tw_1, tw_2)$ for all $z \in E$ and $t \in \mathbb{R}$. $z = (w_1, w_2)$ is said to be non-negative in $\mathcal{C}_\Omega$ if $w_1(x,y) \geq 0, w_2(x,y) \geq 0$ in $\mathcal{C}_\Omega$ and to be positive if $w_1(x,y) > 0, w_2(x,y) > 0$ in $\mathcal{C}_\Omega$.
- $|\Omega|$ is the Lebesgue measure of Ω. $B(0;r)$ is the ball at the origin of radius r.
- $\mathcal{O}(\varepsilon^t)$ denotes $|\mathcal{O}(\varepsilon^t)|/\varepsilon^t \leq C$ as $\varepsilon \to 0$ for $t \geq 0$. $o_n(1)$ denotes $o_n(1) \to 0$.
- C, C_i, c denote various positive constants which may vary from line to line.

3. **The Palais-Smale condition.** In this section we shall detect the range of values c for which the $(PS)_c$-condition holds for the functional $\mathcal{I}_{\lambda,\mu}$. Let $c \in \mathbb{R}$ and set, for simplicity, $E := E_0^s(\mathcal{C}_\Omega)$. We say $\{z_n\} \subset E$ is a $(PS)_c$-sequence in E for $\mathcal{I}_{\lambda,\mu}$ if $\mathcal{I}_{\lambda,\mu}(z_n) = c + o_n(1)$ and $\mathcal{I}'_{\lambda,\mu}(z_n) = o_n(1)$ strongly in E^{-1}, as $n \to \infty$. If any $(PS)_c$-sequence $\{z_n\}$ in E for $\mathcal{I}_{\lambda,\mu}$ admits a convergent subsequence, we say that $\mathcal{I}_{\lambda,\mu}$ satisfies the $(PS)_c$-condition. We shall need the following preliminary result.

Lemma 3.1. *Let $\{z_n\} \subset E$ be a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$ for some $c \in \mathbb{R}$ with $z_n \rightharpoonup z$ in E. Then $\mathcal{I}'_{\lambda,\mu}(z) = 0$ and there exists a positive constant K_0, depending only on q, N, s and $|\Omega|$, such that*

$$\mathcal{I}_{\lambda,\mu}(z) \geq -K_0\Big(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\Big).$$

Proof. Consider $z_n = (w_{1,n}, w_{2,n}) \subset E$ and $z = (w_1, w_2) \in E$. If $\{z_n\}$ is a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$ with $z_n \rightharpoonup z$ in E, then $w_{1,n} \rightharpoonup w_1$ and $w_{2,n} \rightharpoonup w_2$ in $X_0^s(\mathcal{C}_\Omega)$, as $n \to \infty$. Then, by virtue of Sobolev embedding theorem (Lemma 2.3), we also have $w_{1,n}(\cdot, 0) \to w_1(\cdot, 0)$ and $w_{2,n}(\cdot, 0) \to w_2(\cdot, 0)$ strongly in $L^q(\Omega)$, as $n \to \infty$. Of course, up to a further subsequence, $w_{1,n}(\cdot, 0) \to w_1(\cdot, 0)$ and $w_{2,n}(\cdot, 0) \to w_2(\cdot, 0)$ a.e. in Ω. It is standard to check that $\mathcal{I}'_{\lambda,\mu}(z) = 0$. This implies that $\langle \mathcal{I}'_{\lambda,\mu}(z), z\rangle = 0$, namely

$$k_s \int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla w_1|^2 + |\nabla w_2|^2\right) dxdy = \int_\Omega (\lambda |w_1|^q + \mu |w_2|^q) dx + 2\int_\Omega |w_1|^\alpha |w_2|^\beta dx.$$

Consequently, we get

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z) =& \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) k_s \int_{\mathcal{C}_\Omega} y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2) dxdy \\ &- \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\int_\Omega (\lambda|w_1|^q + \mu|w_2|^q)dx.\end{aligned} \tag{12}$$

By (12), Hölder and Young inequalities and the Sobolev embedding theorem, we obtain

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z) &= \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)\|z\|^2 - \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\int_\Omega (\lambda|w_1|^q + \mu|w_2|^q)dx \\ &\geq \frac{s}{N}\|z\|^2 - \frac{2_s^* - q}{q2_s^*}|\Omega|^{(2_s^*-q)/2_s^*}(\lambda\|w_1\|_{2_s^*}^q + \mu\|w_2\|_{2_s^*}^q) \\ &\geq \frac{s}{N}\|z\|^2 - \frac{2_s^* - q}{q2_s^*}|\Omega|^{(2_s^*-q)/2_s^*}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}(\lambda\|w_1\|^q + \mu\|w_2\|^q) \\ &\geq \frac{s}{N}\|z\|^2 - C\Big(\frac{2-q}{2}\varepsilon\left[\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right] + \frac{q}{2}\varepsilon^{-\frac{2-q}{q}}\left[\|w_1\|^2 + \|w_2\|^2\right]\Big) \\ &= \frac{s}{N}\|z\|^2 - \frac{s}{N}\|z\|^2 - K_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) \\ &= -K_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right),\end{aligned}$$

which yields the assertion, where we have put

$$C := \frac{2_s^* - q}{q2_s^*}|\Omega|^{(2_s^*-q)/2_s^*}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}, \quad \varepsilon := \left(\frac{NqC}{2s}\right)^{\frac{q}{2-q}}, \quad K_0 := \frac{\varepsilon(2-q)}{2}C,$$

the positive constants involving only q, $|\Omega|$, s and N. □

Lemma 3.2. *If $\{z_n\} \subset E$ is a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$, then $\{z_n\}$ is bounded in E.*

Proof. Let $z_n = (w_{1,n}, w_{2,n}) \subset E$ be a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$ and suppose, by contradiction, that $\|z_n\| \to \infty$, as $n \to \infty$. Put

$$\widetilde{z}_n = (\widetilde{w}_{1,n}, \widetilde{w}_{2,n}) := \frac{z_n}{\|z_n\|} = \Big(\frac{w_{1,n}}{\|z_n\|}, \frac{w_{2,n}}{\|z_n\|}\Big).$$

We may assume that $\widetilde{z}_n \rightharpoonup \widetilde{z} = (\widetilde{w}_1, \widetilde{w}_2)$ in E. This implies that $\widetilde{w}_{1,n}(\cdot,0) \to \widetilde{w}_1(\cdot,0)$ and $\widetilde{w}_{2,n}(\cdot,0) \to \widetilde{w}_2(\cdot,0)$ strongly in $L^r(\Omega)$ for all $1 \le r < 2_s^*$ and, thus,

$$\int_\Omega (\lambda|\widetilde{w}_{1,n}|^q + \mu|\widetilde{w}_{2,n}|^q)dx = \int_\Omega (\lambda|\widetilde{w}_1|^q + \mu|\widetilde{w}_2|^q)dx + o_n(1).$$

Since $\{z_n\}$ is a $(PS)_c$ sequence for $\mathcal{I}_{\lambda,\mu}$ and $\|z_n\| \to \infty$, we get

$$\begin{aligned}\frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy - \frac{\|z_n\|^{q-2}}{q}\int_\Omega (\lambda|\widetilde{w}_{1,n}|^q + \mu|\widetilde{w}_{2,n}|^q)dx \\ - \frac{2\|z_n\|^{2_s^*-2}}{2_s^*}\int_\Omega |\widetilde{w}_{1,n}|^\alpha|\widetilde{w}_{2,n}|^\beta dx = o_n(1),\end{aligned} \tag{13}$$

and

$$\begin{aligned}k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy - \|z_n\|^{q-2}\int_\Omega (\lambda|\widetilde{w}_{1,n}|^q + \mu|\widetilde{w}_{2,n}|^q)dx \\ - 2\|z_n\|^{2_s^*-2}\int_\Omega |\widetilde{w}_{1,n}|^\alpha|\widetilde{w}_{2,n}|^\beta dx = o_n(1).\end{aligned} \tag{14}$$

Combining (13) and (14), as $n \to \infty$, we obtain

$$\begin{aligned}&k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy \\ &= \frac{2(2_s^*-q)}{q(2_s^*-2)}\|z_n\|^{q-2}\int_\Omega (\lambda|\widetilde{w}_{1,n}|^q + \mu|\widetilde{w}_{2,n}|^q)dx + o_n(1).\end{aligned} \tag{15}$$

In view of $1 < q < 2$ and $\|z_n\| \to \infty$, (15) implies that

$$k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy \to 0,$$

as $n \to \infty$, which contradicts to the fact that $\|\widetilde{z}_n\| = 1$ for any $n \ge 1$. □

Lemma 3.3. *$\mathcal{I}_{\lambda,\mu}$ satisfies the $(PS)_c$ condition with c satisfying*

$$-\infty < c < c_\infty := \frac{2s}{N}\left(\frac{k_s\mathcal{S}_{s,\alpha,\beta}}{2}\right)^{\frac{N}{2s}} - K_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right),$$

where K_0 is the positive constant introduced in Lemma 3.1

Proof. Let $\{z_n\} \subset E$ be a $(PS)_c$-sequence for $\mathcal{I}_{\lambda,\mu}$ with $c \in (-\infty, c_\infty)$. Write $z_n = (w_{1,n}, w_{2,n})$. By Lemma 3.2, we see that $\{z_n\}$ is bounded in E and $z_n \rightharpoonup z = (w_1, w_2)$ up to a subsequence and z is a critical point of $\mathcal{I}_{\lambda,\mu}$. Furthermore, $w_{1,n} \rightharpoonup w_1$ and $w_{2,n} \rightharpoonup w_2$ weakly in $X_0^s(\mathcal{C}_\Omega)$, $w_{1,n}(\cdot,0) \to w_1(\cdot,0)$ and $w_{2,n}(\cdot,0) \to w_2(\cdot,0)$ strongly in $L^r(\Omega)$ for every $1 \le r < 2_s^*$ and $w_{1,n}(\cdot,0) \to w_1(\cdot,0), w_{2,n}(\cdot,0) \to w_2(\cdot,0)$ a.e. in Ω, up to a subsequence. Hence, we have

$$\int_\Omega (\lambda|w_{1,n}|^q + \mu|w_{2,n}|^q)dx = \int_\Omega (\lambda|w_1|^q + \mu|w_2|^q)dx + o_n(1). \tag{16}$$

Let $\widehat{w}_{1,n} := w_{1,n} - w_1$, $\widehat{w}_{2,n} := w_{2,n} - w_2$ and $\widehat{z}_n := (\widehat{w}_{1,n}, \widehat{w}_{2,n})$. Then, we obtain

$$\|\widehat{z}_n\|^2 = \|z_n\|^2 - \|z\|^2 + o_n(1).$$

In light of [19, Lemma 2.1], we also get

$$\int_\Omega |\widehat{w}_{1,n}|^\alpha |\widehat{w}_{2,n}|^\beta dx = \int_\Omega |w_{1,n}|^\alpha |w_{2,n}|^\beta dx - \int_\Omega |w_1|^\alpha |w_2|^\beta dx + o_n(1). \tag{17}$$

Using $\mathcal{I}_{\lambda,\mu}(z_n) = c + o_n(1)$ and $\mathcal{I}'_{\lambda,\mu}(z_n) = o_n(1)$ and (16)-(17), we conclude

$$\frac{1}{2}\|\widehat{z}_n\|^2 - \frac{2}{2_s^*}\int_\Omega |\widehat{w}_{1,n}|^\alpha |\widehat{w}_{2,n}|^\beta dx = c - \mathcal{I}_{\lambda,\mu}(z) + o_n(1), \tag{18}$$

$$\|\widehat{z}_n\|^2 - 2\int_\Omega |\widehat{w}_{1,n}|^\alpha |\widehat{w}_{2,n}|^\beta dx = \langle \mathcal{I}'_{\lambda,\mu}(z_n), z_n\rangle - \langle \mathcal{I}'_{\lambda,\mu}(z), z\rangle + o_n(1) = o_n(1).$$

Hence, we may assume that

$$\|\widehat{z}_n\|^2 \to \ell, \qquad 2\int_\Omega |\widehat{w}_{1,n}|^\alpha |\widehat{w}_{2,n}|^\beta dx \to \ell. \tag{19}$$

If $\ell = 0$, the proof is complete. If $\ell > 0$ then from (19) and the definition of $\mathcal{S}_{s,\alpha,\beta}$, we have

$$k_s \mathcal{S}_{s,\alpha,\beta}\left(\frac{\ell}{2}\right)^{\frac{2}{2_s^*}} = k_s \mathcal{S}_{s,\alpha,\beta}\lim_{n\to\infty}\left(\int_\Omega |\widehat{w}_{1,n}|^\alpha |\widehat{w}_{2,n}|^\beta dx\right)^{\frac{2}{2_s^*}} \le \lim_{n\to\infty}\|\widehat{z}_n\|^2 = \ell,$$

which implies that $\ell \ge 2(k_s\mathcal{S}_{s,\alpha,\beta}/2)^{\frac{N}{2s}}$. On the other hand, from Lemma 3.1, (18) and (19),

$$c = \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\ell + \mathcal{I}_{\lambda,\mu}(z) \ge \frac{2s}{N}\left(\frac{k_s\mathcal{S}_{s,\alpha,\beta}}{2}\right)^{\frac{N}{2s}} - K_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right),$$

which contradicts $c < c_\infty$. □

4. The Nehari manifold.

Since the energy functional $\mathcal{I}_{\lambda,\mu}$ associated with (5) is not bounded on E, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} := \left\{z \in E\backslash\{0\} : \langle \mathcal{I}'_{\lambda,\mu}(z), z\rangle = 0\right\}.$$

Thus, $z = (w_1, w_2) \in \mathcal{N}_{\lambda,\mu}$ if and only if $z \ne 0$ and

$$\langle \mathcal{I}'_{\lambda,\mu}(z), z\rangle = \|z\|^2 - Q_{\lambda,\mu}(z) - 2\int_\Omega |w_1|^\alpha |w_2|^\beta dx = 0, \tag{20}$$

where

$$Q_{\lambda,\mu}(z) := \int_\Omega (\lambda|w_1|^q + \mu|w_2|^q)dx.$$

It is clear that all critical points of $\mathcal{I}_{\lambda,\mu}$ must lie on $\mathcal{N}_{\lambda,\mu}$ and, as we will see below, local minimizers on $\mathcal{N}_{\lambda,\mu}$ are actually critical points of $\mathcal{I}_{\lambda,\mu}$. We have the following results.

Lemma 4.1. *The energy functional $\mathcal{I}_{\lambda,\mu}$ is bounded below and coercive on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $z = (w_1, w_2) \in \mathcal{N}_{\lambda,\mu}$. Then by (20) and the Hölder and Sobolev inequalities

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z) &= \frac{2_s^* - 2}{22_s^*}\|z\|^2 - \frac{2_s^* - q}{q2_s^*}Q_{\lambda,\mu}(z) \\ &\ge \frac{2_s^* - 2}{22_s^*}\|z\|^2 - \frac{2_s^* - q}{q2_s^*}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\|z\|^q.\end{aligned} \tag{21}$$

Since $1 < q < 2$, the functional $\mathcal{I}_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. □

The Nehari manifold $\mathcal{N}_{\lambda,\mu}$ is closely linked to the fibering map $\Phi_z : t \to \mathcal{I}_{\lambda,\mu}(tz)$ given by

$$\Phi_z(t) := \mathcal{I}_{\lambda,\mu}(tz) = \frac{t^2}{2}\|z\|^2 - \frac{t^q}{q}\int_\Omega (\lambda|w_1|^q + \mu|w_2|^q)dx - \frac{2t^{2_s^*}}{\alpha+\beta}\int_\Omega |w_1|^\alpha|w_2|^\beta dx.$$

Such maps were introduced by Drabek and Pohozaev in [16] and later on used by Brown and Zhang [7]. Notice that we have

$$\Phi_z'(t) = t\|z\|^2 - t^{q-1}\int_\Omega (\lambda|w_1|^q + \mu|w_2|^q)dx - 2t^{2_s^*-1}\int_\Omega |w_1|^\alpha|w_2|^\beta dx,$$
$$\Phi_z''(t) = \|z\|^2 - (q-1)t^{q-2}\int_\Omega (\lambda|w_1|^q + \mu|w_2|^q)dx - 2(2_s^*-1)t^{2_s^*-2}\int_\Omega |w_1|^\alpha|w_2|^\beta dx.$$

It is clear that $\Phi_z'(t) = 0$ if and only if $tz \in \mathcal{N}_{\lambda,\mu}$. Hence, $z \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Phi_z'(1) = 0$. Introduce now the functional

$$\mathcal{R}_{\lambda,\mu}(z) := \langle \mathcal{I}_{\lambda,\mu}'(z), z\rangle.$$

Then, for every $z \in \mathcal{N}_{\lambda,\mu}$, we have

$$\begin{aligned}\langle \mathcal{R}_{\lambda,\mu}'(z), z\rangle &= 2\|z\|^2 - qQ_{\lambda,\mu}(z) - 22_s^*\int_\Omega |w_1|^\alpha|w_2|^\beta dx \\ &= (2-q)\|z\|^2 - 2(2_s^*-q)\int_\Omega |w_1|^\alpha|w_2|^\beta dx \\ &= (2_s^*-q)Q_{\lambda,\mu}(z) - (2_s^*-2)\|z\|^2.\end{aligned} \tag{22}$$

Following the method used in [30], we split $\mathcal{N}_{\lambda,\mu}$ into three parts

$$\begin{aligned}\mathcal{N}_{\lambda,\mu}^+ &:= \{z \in \mathcal{N}_{\lambda,\mu} : \langle \mathcal{R}_{\lambda,\mu}'(z), z\rangle > 0\},\\ \mathcal{N}_{\lambda,\mu}^0 &:= \{z \in \mathcal{N}_{\lambda,\mu} : \langle \mathcal{R}_{\lambda,\mu}'(z), z\rangle = 0\},\\ \mathcal{N}_{\lambda,\mu}^- &:= \{z \in \mathcal{N}_{\lambda,\mu} : \langle \mathcal{R}_{\lambda,\mu}'(z), z\rangle < 0\}.\end{aligned}$$

Then, we have the following lemmas.

Lemma 4.2. *If z_0 is a local minimizer for $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and $z_0 \notin \mathcal{N}_{\lambda,\mu}^0$, then $\mathcal{I}_{\lambda,\mu}'(z_0) = 0$.*

Proof. Let $z_0 = (w_{0,1}, w_{0,2}) \in \mathcal{N}_{\lambda,\mu}$ be a local minimizer for the functional $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$. Hence, there exists a Lagrange multiplier $\gamma \in \mathbb{R}$ such that $\mathcal{I}_{\lambda,\mu}'(z_0) = \gamma\mathcal{R}_{\lambda,\mu}'(z_0)$. Thus,

$$\langle \mathcal{I}_{\lambda,\mu}'(z_0), z_0\rangle = \gamma\langle \mathcal{R}_{\lambda,\mu}'(z_0), z_0\rangle = 0.$$

Since $z_0 \notin \mathcal{N}_{\lambda,\mu}^0$, then $\langle \mathcal{R}_{\lambda,\mu}'(z_0), z_0\rangle \neq 0$, yielding $\gamma = 0$. □

Let Λ_1 be the positive number defined in (4). Then we have the following result.

Lemma 4.3. *Assume that $(\lambda,\mu) \in \mathscr{C}_{\Lambda_1}$. Then $\mathcal{N}_{\lambda,\mu}^0 = \emptyset$.*

Proof. Assume by contradiction that there exist $\lambda > 0$ and $\mu > 0$ with $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$ and such that $\mathcal{N}_{\lambda,\mu}^0 \neq \emptyset$. Let $z \in \mathcal{N}_{\lambda,\mu}^0$. Then, by virtue of (22), we get

$$\|z\|^2 = \frac{2(2_s^*-q)}{2-q}\int_\Omega |w_1|^\alpha|w_2|^\beta dx, \qquad \|z\|^2 = \frac{2_s^*-q}{2_s^*-2}Q_{\lambda,\mu}(z).$$

By Hölder inequality and the Sobolev embedding theorem, we have

$$\|z\| \geq \left[\frac{2-q}{2(2_s^*-q)}(k_s\mathcal{S}_{s,\alpha,\beta})^{\frac{2_s^*}{2}}\right]^{\frac{1}{2_s^*-2}},$$
$$\|z\| \leq \left(\frac{2_s^*-q}{2_s^*-2}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\right)^{\frac{1}{2-q}}\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{1}{2}},$$

which leads to the inequality

$$\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}} \geq \left(\frac{2_s^*-q}{2_s^*-2}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\right)^{-\frac{2}{2-q}}\left[\frac{2-q}{2(2_s^*-q)}(k_s\mathcal{S}_{s,\alpha,\beta})^{\frac{2_s^*}{2}}\right]^{\frac{2}{2_s^*-2}}=\Lambda_1,$$

contradicting the assumption. □

From Lemma 4.3, if $0<\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}<\Lambda_1$, we can write $\mathcal{N}_{\lambda,\mu}=\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$ and define

$$\alpha_{\lambda,\mu}:=\inf_{z\in\mathcal{N}_{\lambda,\mu}}\mathcal{I}_{\lambda,\mu}(z),\qquad \alpha^+_{\lambda,\mu}:=\inf_{z\in\mathcal{N}^+_{\lambda,\mu}}\mathcal{I}_{\lambda,\mu}(z),\qquad \alpha^-_{\lambda,\mu}:=\inf_{z\in\mathcal{N}^-_{\lambda,\mu}}\mathcal{I}_{\lambda,\mu}(z).$$

Moreover, we have the following properties about the Nehari manifold $\mathcal{N}_{\lambda,\mu}$.

Theorem 4.4. *The following facts holds*

- (*i*) *If* $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$, *then we have* $\alpha_{\lambda,\mu}\leq\alpha^+_{\lambda,\mu}<0$;
- (*ii*) *If* $(\lambda,\mu)\in\mathscr{C}_{(q/2)^{2/(2-q)}\Lambda_1}$, *then we have* $\alpha^-_{\lambda,\mu}>c_0$ *for some positive constant* c_0 *depending on* λ,μ,N,s *and* $|\Omega|$.

Proof. (i) Let $z=(w_1,w_2)\in\mathcal{N}^+_{\lambda,\mu}$. By formula (22), we have

$$\frac{2-q}{2(2_s^*-q)}\|z\|^2>\int_\Omega|w_1|^\alpha|w_2|^\beta$$

and so,

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z)&=\left(\frac{1}{2}-\frac{1}{q}\right)\|z\|^2+2\left(\frac{1}{q}-\frac{1}{2_s^*}\right)\int_\Omega|w_1|^\alpha|w_2|^\beta dx\\&\leq\left[\left(\frac{1}{2}-\frac{1}{q}\right)+\left(\frac{1}{q}-\frac{1}{2_s^*}\right)\frac{2-q}{2_s^*-q}\right]\|z\|^2\\&=-\frac{(2-q)s}{Nq}\|z\|^2<0.\end{aligned}$$

Therefore, by the definition of $\alpha_{\lambda,\mu},\alpha^+_{\lambda,\mu}$, we can deduce that $\alpha_{\lambda,\mu}\leq\alpha^+_{\lambda,\mu}<0$.

(ii) Let $z\in\mathcal{N}^-_{\lambda,\mu}$. By equation (22),

$$\frac{2-q}{2(2_s^*-q)}\|z\|^2<\int_\Omega|w_1|^\alpha|w_2|^\beta dx.$$

By the Hölder inequality and the Sobolev embedding theorem, we have

$$\int_\Omega|w_1|^\alpha|w_2|^\beta dx\leq(k_s\mathcal{S}_{s,\alpha,\beta})^{-\frac{2_s^*}{2}}\|z\|^{2_s^*}.$$

Hence, we obtain

$$\|z\|>\left(\frac{2-q}{2(2_s^*-q)}\right)^{\frac{1}{2_s^*-2}}(k_s\mathcal{S}_{s,\alpha,\beta})^{\frac{N}{4s}},\quad\text{for all } z\in\mathcal{N}^-_{\lambda,\mu}.$$

From the last inequality we infer that

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z) &= \frac{2_s^*-2}{22_s^*}\|z\|^2 - \frac{2_s^*-q}{q2_s^*}Q_{\lambda,\mu}(z)\\ &\geq \|z\|^q\left[\frac{2_s^*-2}{22_s^*}\|z\|^{2-q} - \frac{2_s^*-q}{q2_s^*}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\right]\\ &> \left(\frac{2-q}{2(2_s^*-q)}\right)^{\frac{q}{2_s^*-2}}(k_s\mathcal{S}_{s,\alpha,\beta})^{\frac{qN}{4s}}\left[\frac{2_s^*-2}{22_s^*}(k_s\mathcal{S}_{s,\alpha,\beta})^{\frac{(2-q)N}{4s}}\left(\frac{2-q}{2(2_s^*-q)}\right)^{\frac{2-q}{2_s^*-2}}\right.\\ &\quad\left.-\frac{2_s^*-q}{q2_s^*}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\right].\end{aligned}$$

Thus, if $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}} < (q/2)^{\frac{2}{2-q}}\Lambda_1$, then

$$\mathcal{I}_{\lambda,\mu}(z) > c_0, \quad \text{for all } z\in\mathcal{N}^-_{\lambda,\mu},$$

for some positive constant $c_0 = c_0(\lambda,\mu,s,N,|\Omega|)$. □

Lemma 4.5. *Let $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$. Then, for every $z=(w_1,w_2)\in E\backslash\{(0,0)\}$, there exist unique numbers $t^-=t^-(z)>0$ and $t^+=t^+(z)>0$ such that*

$$t^+z\in\mathcal{N}^+_{\lambda,\mu}, \qquad t^-z\in\mathcal{N}^-_{\lambda,\mu}.$$

In particular, we have

$$t^+<\bar t<t^-, \qquad \bar t := \left(\frac{(2_s^*-q)Q_{\lambda,\mu}(z)}{(2_s^*-2)\|z\|^2}\right)^{\frac{1}{2-q}}$$

as well as $t\mapsto\mathcal{I}_{\lambda,\mu}(tz)$ strictly increasing on $[t^+,t^-]$ and

$$\mathcal{I}_{\lambda,\mu}(t^+z) = \min_{0\le t\le t^-}\mathcal{I}_{\lambda,\mu}(tz), \qquad \mathcal{I}_{\lambda,\mu}(t^-z)=\max_{t\ge0}\mathcal{I}_{\lambda,\mu}(tz).$$

Proof. Fix $z=(w_1,w_2)\in E$, so that $Q_{\lambda,\mu}(z)>0$, and let $\mathfrak{m}:(0,\infty)\to\mathbb{R}$ be defined by

$$\mathfrak{m}(t) := t^{2-2_s^*}\|z\|^2 - t^{q-2_s^*}Q_{\lambda,\mu}(z), \quad \text{for } t>0. \tag{23}$$

Obviously, $\mathfrak{m}(t)\to-\infty$ as $t\to0^+$ and $\mathfrak{m}(t)\to0$ and $\mathfrak{m}(t)>0$ as $t\to\infty$. Since

$$\mathfrak{m}'(t) = (2-2_s^*)t^{1-2_s^*}\|z\|^2 - (q-2_s^*)t^{q-2_s^*-1}Q_{\lambda,\mu}(z),$$

we have $\mathfrak{m}'(t)=0$ at $t=\bar t=\bar t_{\max}$, $\mathfrak{m}'(t)>0$ for $t\in(0,\bar t_{\max})$ and $\mathfrak{m}'(t)<0$ for $t\in(\bar t_{\max},\infty)$. Hence, $\mathfrak{m}$ achieves its maximum at $\bar t_{\max}$, is increasing for $t\in(0,\bar t_{\max})$ and decreasing for $t\in(\bar t_{\max},\infty)$. By $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$, (21) and (4), we have

$$\begin{aligned}Q_{\lambda,\mu}(z) &\le (k_s\mathcal{S}(s,N))^{-\frac{q}{2}}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\|z\|^q\\ &< (k_s\mathcal{S}(s,N))^{-\frac{q}{2}}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\Lambda_1^{\frac{2-q}{2}}\|z\|^q\\ &= \frac{2_s^*-2}{2_s^*-q}\|z\|^q\left[\frac{2-q}{2(2_s^*-q)}(k_s\mathcal{S}_{s,\alpha,\beta})^{\frac{2_s^*}{2}}\right]^{\frac{2-q}{2_s^*-2}}.\end{aligned}\tag{24}$$

By (24) and a simple calculation we have

$$
\begin{aligned}
\mathfrak{m}(\bar{t}_{\max}) &= (\bar{t}_{\max})^{2-2_s^*}\|z\|^2 - (\bar{t}_{\max})^{q-2_s^*}Q_{\lambda,\mu}(z) \\
&= \left[\frac{(2_s^*-q)Q_{\lambda,\mu}(z)}{(2_s^*-2)\|z\|^2}\right]^{\frac{2-2_s^*}{2-q}}\|z\|^2 - \left[\frac{(2_s^*-q)Q_{\lambda,\mu}(z)}{(2_s^*-2)\|z\|^2}\right]^{\frac{q-2_s^*}{2-q}}Q_{\lambda,\mu}(z) \\
&= \left(\frac{2_s^*-q}{2_s^*-2}\right)^{\frac{2-2_s^*}{2-q}} \times \frac{2-q}{2_s^*-q}\|z\|^{\frac{2(2_s^*-q)}{2-q}}Q_{\lambda,\mu}(z)^{\frac{2-2_s^*}{2-q}} \qquad (25)\\
&> 2\|z\|^{2_s^*}(k_s\mathcal{S}_{s,\alpha,\beta})^{-\frac{2_s^*}{2}}.
\end{aligned}
$$

Then, taking into account the definition of $\mathcal{S}_{s,\alpha,\beta}$ we have

$$
\mathfrak{m}(0) = -\infty < 0 \le 2\int_\Omega |w_1|^\alpha|w_2|^\beta dx \le 2(k_s\mathcal{S}_{s,\alpha,\beta})^{-\frac{2_s^*}{2}}\|z\|^{2_s^*} < \mathfrak{m}(\bar{t}_{\max}).
$$

In turn, there exist unique t^+ and t^- such that $0 < t^+ < \bar{t}_{\max} < t^-$,

$$
\mathfrak{m}(t^+) = 2\int_\Omega |w_1|^\alpha|w_2|^\beta dx = \mathfrak{m}(t^-),
$$

and $\mathfrak{m}'(t^+) > 0 > \mathfrak{m}'(t^-)$. By the equation for $\Phi_z'(t)$ and (23) we have

$$
\Phi_z'(t) = t^{2_s^*-1}\left[\mathfrak{m}(t) - 2\int_\Omega |w_1|^\alpha|w_2|^\beta dx\right], \qquad (26)
$$

which yields $\Phi_z'(t^\pm) = 0$. By virtue of (26), we also have $\pm\Phi_z''(t^\pm) > 0$. This shows that Φ_z has a local minimum at t^+ and local maximum at t^- with $t^\pm z \in \mathcal{N}_{\lambda,\mu}^\pm$. The function $t \mapsto \mathcal{I}_{\lambda,\mu}(tz)$ is increasing on $[t^+, t^-]$ and decreasing over $[0, t^+] \cup [t^-, \infty)$. Hence, $\mathcal{I}_{\lambda,\mu}(t^+z) = \min_{0\le t\le t^-}\mathcal{I}_{\lambda,\mu}(tz)$ and $\mathcal{I}_{\lambda,\mu}(t^-z) = \max_{t\ge 0}\mathcal{I}_{\lambda,\mu}(tz)$, concluding the proof. □

5. Existence of Palais-Smale sequences.

Lemma 5.1. *Let $(\lambda,\mu) \in \mathscr{C}_{\Lambda_1}$. Then, for any $z \in \mathcal{N}_{\lambda,\mu}$, there exists $r > 0$ and a differentiable map $\xi : B(0;r) \subset E \to \mathbb{R}^+$ such that $\xi(0) = 1$ and $\xi(h)(z-h) \in \mathcal{N}_{\lambda,\mu}$ for every $h \in B(0;r)$. Let us set*

$$
\begin{aligned}
\mathcal{T}_1 &:= 2k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\left(\nabla w_1\cdot\nabla h_1 + \nabla w_2\cdot\nabla h_2\right)dxdy,\\
\mathcal{T}_2 &:= q\int_\Omega \left(\lambda|w_1|^{q-2}w_1h_1 + \mu|w_2|^{q-2}w_2h_2\right)dx,\\
\mathcal{T}_3 &:= 2\int_\Omega \left(\alpha|w_1|^{\alpha-2}w_1h_1|w_2|^\beta + \beta|w_1|^\alpha|w_2|^{\beta-2}w_2h_2\right)dx,
\end{aligned}
$$

for all $(h_1,h_2) \in E$ and $(w_1,w_2) \in E$. Then

$$
\langle\xi'(0),h\rangle = \frac{\mathcal{T}_3 + \mathcal{T}_2 - \mathcal{T}_1}{(2-q)\|z\|^2 - 2(2_s^*-q)\int_\Omega |w_1|^\alpha|w_2|^\beta dx}, \qquad (27)
$$

for all $(h_1,h_2) \in E$.

Proof. For $z=(w_1,w_2)\in\mathcal{N}_{\lambda,\mu}$, define a function $\mathscr{H}_z:\mathbb{R}\times E\to\mathbb{R}$ by

$$\begin{aligned}\mathscr{H}_z(\xi,p):=&\langle\mathcal{I}'_{\lambda,\mu}(\xi(z-p)),\xi(z-p)\rangle\\ =&\xi^2k_s\int_{\mathcal{C}_\Omega}y^{1-2s}\left(|\nabla(w_1-p_1)|^2+|\nabla(w_2-p_2)|^2\right)dxdy\\ &-\xi^q\int_\Omega(\lambda|w_1-p_1|^q+\mu|w_2-p_2|^q)\,dx-2\xi^{2_s^*}\int_\Omega|w_1-p_1|^\alpha|w_2-p_2|^\beta dx.\end{aligned}$$

Then $\mathscr{H}_z(1,0)=\langle\mathcal{I}'_{\lambda,\mu}(z),z\rangle=0$ and, by Lemma 4.3, we have

$$\begin{aligned}\frac{d\mathscr{H}_z(1,(0,0))}{d\xi}&=2\|z\|^2-q\int_\Omega(\lambda|w_1|^q+\mu|w_2|^q)\,dx-22_s^*\int_\Omega|w_1|^\alpha|w_2|^\beta dx\\ &=(2-q)\|z\|^2-2(2_s^*-q)\int_\Omega|w_1|^\alpha|w_2|^\beta dx\neq0.\end{aligned}$$

In turn, by virtue of the Implicit Function Theorem, there exists $r>0$ and a function $\xi:B(0;r)\subset E\to\mathbb{R}$ of class C^1 such that $\xi(0)=1$ and formula (27) holds, via direct computation. Moreover, $\mathscr{H}_z(\xi(h),h)=0$, for all $h\in B(0;r)$, is equivalent to

$$\langle\mathcal{I}'_{\lambda,\mu}(\xi(h)(z-h)),\xi(h)(z-h)\rangle=0,\quad\text{for all }h\in B(0;r),$$

namely $\xi(h)(z-h)\in\mathcal{N}_{\lambda,\mu}$. □

Lemma 5.2. *Let $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$. Then, for each $z\in\mathcal{N}^-_{\lambda,\mu}$, there exists $r>0$ and a differentiable function $\xi^-:B(0;r)\subset E\to\mathbb{R}^+$ such that $\xi^-(0)=1$, $\xi^-(h)(z-h)\in\mathcal{N}^-_{\lambda,\mu}$ for every $h\in B(0;r)$ and formula (27) holds.*

Proof. Arguing as for the proof of Lemma 5.1, there exists $r>0$ and a differentiable function $\xi^-:B(0;r)\subset E\to\mathbb{R}^+$ such that $\xi^-(0)=1$, $\xi^-(h)(z-h)\in\mathcal{N}_{\lambda,\mu}$ for all $h\in B(0;r)$ and formula (27) holds. Since

$$\langle\mathcal{R}'_{\lambda,\mu}(z),z\rangle=(2-q)\|z\|^2-2(2_s^*-q)\int_\Omega|w_1|^\alpha|w_2|^\beta dx<0,$$

by the continuity of the functions $\mathcal{R}'_{\lambda,\mu}$ and ξ^-, up to reducing the size of $r>0$, we get

$$\begin{aligned}\langle\mathcal{R}'_{\lambda,\mu}(\xi^-(h)(z-h)),\xi^-(h)(z-h)\rangle&=(2-q)\|\xi^-(h)(z-h)\|^2\\ &-2(2_s^*-q)\int_\Omega|(\xi^-(h)(z-h))_1|^\alpha|(\xi^-(h)(z-h))_2|^\beta dx<0,\end{aligned}$$

where $(\xi^-(h)(z-h))_i\in X_0^s(\mathcal{C}_\Omega)$ denote the components of $\xi^-(h)(z-h)$. This implies that the functions $\xi^-(h)(z-h)$ belong to $\mathcal{N}^-_{\lambda,\mu}$. □

Proposition 1. *The following facts hold.*

- *(i) Let $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$. Then there is a $(PS)_{\alpha_{\lambda,\mu}}$-sequence $\{z_n\}\subset\mathcal{N}_{\lambda,\mu}$ for $\mathcal{I}_{\lambda,\mu}$.*
- *(ii) Let $(\lambda,\mu)\in\mathscr{C}_{(q/2)^{2/(2-q)}\Lambda_1}$. Then there is a $(PS)_{\alpha^-_{\lambda,\mu}}$-sequence $\{z_n\}\subset\mathcal{N}^-_{\lambda,\mu}$ for $\mathcal{I}_{\lambda,\mu}$.*

Proof. (i) By Lemma 4.1 and Ekeland Variational Principle [17], there exists a minimizing sequence $\{z_n\}\subset\mathcal{N}_{\lambda,\mu}$ such that

$$\begin{aligned}&\mathcal{I}_{\lambda,\mu}(z_n)<\alpha_{\lambda,\mu}+\frac{1}{n},\\ &\mathcal{I}_{\lambda,\mu}(z_n)<\mathcal{I}_{\lambda,\mu}(w)+\frac{1}{n}\|w-z_n\|,\quad\text{for each }w\in\mathcal{N}_{\lambda,\mu}.\end{aligned}\tag{28}$$

Taking n large and using $\alpha_{\lambda,\mu} < 0$, we have

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(z_n) &= \Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)\|z_n\|^2 - \Big(\frac{1}{q}-\frac{1}{2_s^*}\Big)\int_\Omega(\lambda|w_{1,n}|^q+\mu|w_{2,n}|^q)dx \\ &< \alpha_{\lambda,\mu}+\frac{1}{n} < \frac{\alpha_{\lambda,\mu}}{2}.\end{aligned} \tag{29}$$

This yields that

$$\begin{aligned}-\frac{q2_s^*}{2(2_s^*-q)}\alpha_{\lambda,\mu} &< \int_\Omega(\lambda|w_{1,n}|^q+\mu|w_{2,n}|^q)dx \\ &\le |\Omega|^{\frac{2_s^*-q}{2_s^*}}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}})^{\frac{2-q}{2}}\|z_n\|^q.\end{aligned} \tag{30}$$

Consequently, $z_n \neq 0$ and combining with (29) and (30) and using Hölder inequality

$$\|z_n\| > \Big[-\frac{q2_s^*}{2(2_s^*-q)}\alpha_{\lambda,\mu}|\Omega|^{\frac{q-2_s^*}{2_s^*}}(k_s\mathcal{S}(s,N))^{\frac{q}{2}}(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}})^{\frac{q-2}{2}}\Big]^{\frac{1}{q}},$$

and

$$\|z_n\| < \Big[\frac{2(2_s^*-q)}{q(2_s^*-2)}|\Omega|^{\frac{2_s^*-q}{2_s^*}}(k_s\mathcal{S}(s,N))^{-\frac{q}{2}}(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}})^{\frac{2-q}{2}}\Big]^{\frac{1}{2-q}}. \tag{31}$$

Now we prove that

$$\|\mathcal{I}'_{\lambda,\mu}(z_n)\|_{E^{-1}} \to 0, \text{ as } n\to\infty.$$

Fix $n\in\mathbb{N}$. By applying Lemma 5.1 to z_n, we obtain the function $\xi_n : B(0;r_n)\to\mathbb{R}^+$ for some $r_n>0$, such that $\xi_n(h)(z_n-h)\in\mathcal{N}_{\lambda,\mu}$. Take $0<\rho<r_n$. Let $w\in E$ with $w\not\equiv 0$ and put $h^*=\frac{\rho w}{\|w\|}$. We set $h_\rho=\xi_n(h^*)(z_n-h^*)$, then $h_\rho\in\mathcal{N}_{\lambda,\mu}$, and we have from (28) that

$$\mathcal{I}_{\lambda,\mu}(h_\rho)-\mathcal{I}_{\lambda,\mu}(z_n) \ge -\frac{1}{n}\|h_\rho-z_n\|.$$

By the Mean Value Theorem, we get

$$\langle\mathcal{I}'_{\lambda,\mu}(z_n),h_\rho-z_n\rangle+o(\|h_\rho-z_n\|)\ge-\frac{1}{n}\|h_\rho-z_n\|.$$

Thus, we have

$$\langle\mathcal{I}'_{\lambda,\mu}(z_n),-h^*\rangle+(\xi_n(h^*)-1)\langle\mathcal{I}'_{\lambda,\mu}(z_n),z_n-h^*\rangle\ge-\frac{1}{n}\|h_\rho-z_n\|+o(\|h_\rho-z_n\|).$$

Whence, from $\xi_n(h^*)(z_n-h^*)\in\mathcal{N}_{\lambda,\mu}$, it follows that

$$\begin{aligned}-\rho\Big\langle\mathcal{I}'_{\lambda,\mu}(z_n),\frac{w}{\|w\|}\Big\rangle&+(\xi_n(h^*)-1)\langle\mathcal{I}'_{\lambda,\mu}(z_n)-\mathcal{I}'_{\lambda,\mu}(h_\rho),z_n-h^*\rangle\\ &\ge-\frac{1}{n}\|h_\rho-z_n\|+o(\|h_\rho-z_n\|).\end{aligned}$$

So, we get

$$\begin{aligned}\Big\langle\mathcal{I}'_{\lambda,\mu}(z_n),\frac{w}{\|w\|}\Big\rangle\le&\frac{1}{n\rho}\|h_\rho-z_n\|+\frac{o(\|h_\rho-z_n\|)}{\rho}\\ &+\frac{(\xi_n(h^*)-1)}{\rho}\langle\mathcal{I}'_{\lambda,\mu}(z_n)-\mathcal{I}'_{\lambda,\mu}(h_\rho),z_n-h^*\rangle.\end{aligned} \tag{32}$$

Since $\|h_\rho-z_n\|\le\rho|\xi_n(h^*)|+|\xi_n(h^*)-1|\|z_n\|$ and

$$\lim_{\rho\to 0}\frac{|\xi_n(h^*)-1|}{\rho}\le\|\xi'_n(0)\|.$$

For fixed $n \in \mathbb{N}$, if we let $\rho \to 0$ in (32), then by virtue of (31) we can choose a constant $C > 0$ independent of ρ such that

$$\left\langle \mathcal{I}'_{\lambda,\mu}(z_n), \frac{w}{\|w\|} \right\rangle \leq \frac{C}{n}(1 + \|\xi'_n(0)\|).$$

Thus, we are done once we prove that $\|\xi'_n(0)\|$ remains uniformly bounded. By (27), (31) and Hölder inequality, we have

$$\left|\langle \xi'_n(0), h\rangle\right| \leq \frac{C_1\|h\|}{\left|(2-q)\|z_n\|^2 - 2(2_s^* - q)\int_\Omega |w_{1,n}|^\alpha |w_{2,n}|^\beta dx\right|}$$

for some $C_1 > 0$. We only need to prove that

$$\left|(2-q)\|z_n\|^2 - 2(2_s^* - q)\int_\Omega |w_{1,n}|^\alpha |w_{2,n}|^\beta dx\right| \geq C_2,$$

for some $C_2 > 0$ and n large enough. We argue by contradiction. Suppose that there exists a subsequence $\{z_n\}$ such that

$$(2-q)\|z_n\|^2 - 2(2_s^* - q)\int_\Omega |w_{1,n}|^\alpha |w_{2,n}|^\beta dx = o_n(1). \tag{33}$$

By virtue of (33) and the fact that $z_n \in \mathcal{N}_{\lambda,\mu}$, we have

$$\|z_n\|^2 = \frac{2(2_s^* - q)}{2-q}\int_\Omega |w_{1,n}|^\alpha |w_{2,n}|^\beta dx + o_n(1), \qquad \|z_n\|^2 = \frac{2_s^* - q}{2_s^* - 2} Q_{\lambda,\mu}(z_n) + o_n(1).$$

Taking into account that $\mathcal{I}_{\lambda,\mu}(z_n) \to \alpha_{\lambda,\mu} < 0$ as $n \to \infty$, we have $\|z_n\| \not\to 0$ as $n \to \infty$. Then, arguing as in the proof of Lemma 4.3 yields $(\lambda, \mu) \notin \mathscr{C}_{\Lambda_1}$, a contradiction. Then,

$$\left\langle \mathcal{I}'_{\lambda,\mu}(z_n), \frac{w}{\|w\|} \right\rangle \leq \frac{C}{n}.$$

This proves (i). By Lemma 5.2, one can prove (ii), but we shall omit the details here. □

6. **Local minimization problems.** Now, we establish the existence of a local minimizer for $\mathcal{I}_{\lambda,\mu}$ in $\mathcal{N}^+_{\lambda,\mu}$.

Proposition 2. *Let $(\lambda, \mu) \in \mathscr{C}_{\Lambda_1}$. Then $\mathcal{I}_{\lambda,\mu}$ has a local minimizer z^+ in $\mathcal{N}^+_{\lambda,\mu}$ satisfying the following conditions:*

(i) $\mathcal{I}_{\lambda,\mu}(z^+) = \alpha_{\lambda,\mu} = \alpha^+_{\lambda,\mu} < 0$;
(ii) z^+ is a positive solution of (5).

Proof. By (i) of Proposition 1, there exists a minimizing sequence $\{z_n\} = \{(w_{1,n}, w_{2,n})\}$ for $\mathcal{I}_{\lambda,\mu}$ in $\mathcal{N}_{\lambda,\mu}$ such that, as $n \to \infty$,

$$\mathcal{I}_{\lambda,\mu}(z_n) = \alpha_{\lambda,\mu} + o_n(1) \quad \text{and} \quad \mathcal{I}'_{\lambda,\mu}(z_n) = o_n(1) \text{ in } E^{-1}. \tag{34}$$

By Lemma 4.1, we see that $\mathcal{I}_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$, and $\{z_n\}$ is bounded in E. Then there exists a subsequence, still denoted by $\{z_n\}$ and $z^+ = (w_1^+, w_2^+) \in E$ such that, as $n \to \infty$,

$$\begin{cases} w_{1,n} \rightharpoonup w_1^+, \ \ w_{2,n} \rightharpoonup w_2^+, \ \ \text{weakly in } X_0^s(\Omega), \\ w_{1,n} \to w_1^+, \ \ w_{2,n} \to w_2^+, \ \ \text{strongy in } L^r(\Omega) \text{ for all } 1 \leq r < 2_s^*, \\ w_{1,n} \to w_1^+, \ \ w_{2,n} \to w_2^+, \ \ \text{a.e. in } \Omega, \end{cases}$$

up to subsequences. This implies that, as $n \to \infty$,

$$Q_{\lambda,\mu}(z_n) = Q_{\lambda,\mu}(z^+) + o_n(1). \tag{35}$$

We claim that z^+ is a nontrivial solution of (5). It is easy to verify that z^+ is a weak solution of (5). From $z_n \in \mathcal{N}_{\lambda,\mu}$ and (21) we deduce that

$$Q_{\lambda,\mu}(z_n) = \frac{q(2_s^* - 2)}{2(2_s^* - q)}\|z_n\|^2 - \frac{q2_s^*}{2_s^* - q}\mathcal{I}_{\lambda,\mu}(z_n). \tag{36}$$

Let $n \to \infty$ in (36), by (34), (35) and $\alpha_{\lambda,\mu} < 0$, we have

$$Q_{\lambda,\mu}(z^+) \geq -\frac{q2_s^*}{2_s^* - q}\alpha_{\lambda,\mu} > 0.$$

Therefore, $z^+ \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of (5). Now we show that $z_n \to z^+$ strongly in E and $\mathcal{I}_{\lambda,\mu}(z^+) = \alpha_{\lambda,\mu}$. Since $z^+ \in \mathcal{N}_{\lambda,\mu}$, then by (36), we obtain

$$\begin{aligned}
\alpha_{\lambda,\mu} &\leq \mathcal{I}_{\lambda,\mu}(z^+) \\
&= \frac{s}{N}\|z^+\|^2 - \frac{2_s^* - q}{q2_s^*}Q_{\lambda,\mu}(z^+) \\
&\leq \lim_{n\to\infty}\Big(\frac{s}{N}\|z_n\|^2 - \frac{2_s^* - q}{q2_s^*}Q_{\lambda,\mu}(z_n)\Big) \\
&= \lim_{n\to\infty}\mathcal{I}_{\lambda,\mu}(z_n) = \alpha_{\lambda,\mu}.
\end{aligned}$$

This implies that $\mathcal{I}_{\lambda,\mu}(z^+) = \alpha_{\lambda,\mu}$ and $\lim_{n\to\infty}\|z_n\|^2 = \|z^+\|^2$. Since

$$\|z_n - z^+\|^2 = \|z_n\|^2 - \|z^+\|^2 + o_n(1),$$

we conclude that $z_n \to z^+$ in E. We claim that $z^+ \in \mathcal{N}^+_{\lambda,\mu}$. Assume by contradiction that $z^+ \in \mathcal{N}^-_{\lambda,\mu}$. Then, by Lemma 4.5, there exist (unique) t_1^+ and t_1^- with $t_1^+ z^+ \in \mathcal{N}^+_{\lambda,\mu}$ and $t_1^- z^+ \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t_1^+ < t_1^- = 1$. Since

$$\frac{d}{dt}\mathcal{I}_{\lambda,\mu}(tz^+)|_{t=t_1^+} = 0, \quad \text{and} \quad \frac{d^2}{dt^2}\mathcal{I}_{\lambda,\mu}(tz^+)|_{t=t_1^+} > 0,$$

there exists $t_1^+ < t^* \leq t_1^-$ such that $\mathcal{I}_{\lambda,\mu}(t_1^+ z^+) < \mathcal{I}_{\lambda,\mu}(t^* z^+)$. By Lemma 4.5, we have

$$\alpha_{\lambda,\mu} \leq \mathcal{I}_{\lambda,\mu}(t_1^+ z^+) < \mathcal{I}_{\lambda,\mu}(t^* z^+) \leq \mathcal{I}_{\lambda,\mu}(t_1^- z^+) = \mathcal{I}_{\lambda,\mu}(z^+) = \alpha_{\lambda,\mu},$$

a contradiction. Since $\mathcal{I}_{\lambda,\mu}(z^+) = \mathcal{I}_{\lambda,\mu}(|w_1^+|, |w_2^+|)$ and $(|w_1^+|, |w_2^+|) \in \mathcal{N}_{\lambda,\mu}$, by Lemma 4.2 we may assume that z^+ is a nontrivial nonnegative solution of (5). In particular $w_1^+ \not\equiv 0, w_2^+ \not\equiv 0$. In fact, without loss of generality, we assume by contradiction that $w_2^+ \equiv 0$. Then w_1^+ is a nontrivial nonnegative solution of

$$\begin{cases}
-\mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega \\
w = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\
w = u & \text{on } \Omega \times \{0\} \\
\dfrac{\partial w}{\partial \nu^s} = \lambda |w|^{q-2} w & \text{on } \Omega \times \{0\},
\end{cases}$$

By the maximum principle [27] we get $w_1^+ > 0$ in $X_0^s(\mathcal{C}_\Omega)$ and

$$\|(w_1^+, 0)\|^2 = Q_{\lambda,\mu}(w_1^+, 0) > 0.$$

Moreover, we can choose $w_2^* \in X_0^s(\mathcal{C}_\Omega)\backslash\{0\}$ such that

$$\|(0, w_2^*)\|^2 = Q_{\lambda,\mu}(0, w_2^*) > 0.$$

Note that
$$Q_{\lambda,\mu}(w_1^+, w_2^*) = Q_{\lambda,\mu}(w_1^+, 0) + Q_{\lambda,\mu}(0, w_2^*) > 0,$$
and so by Lemma 4.5 there is unique $0 < t^+ < \bar{t} < t^-$ such that $(t^+ w_1^+, t^+ w_2^*) \in \mathcal{N}^+_{\lambda,\mu}$. Moreover,
$$\bar{t} = \left(\frac{(2_s^* - q) Q_{\lambda,\mu}(w_1^+, w_2^*)}{(2_s^* - 2)\|(w_1^+, w_2^*)\|^2} \right)^{\frac{1}{2-q}} = \left(\frac{2_s^* - q}{2_s^* - 2} \right)^{\frac{1}{2-q}} > 1$$
and
$$\mathcal{I}_{\lambda,\mu}(t^+ w_1^+, t^+ w_2^*) = \inf_{0 \le t \le t^-} \mathcal{I}_{\lambda,\mu}(t w_1^+, t w_2^*).$$
This implies that
$$\alpha^+_{\lambda,\mu} \le \mathcal{I}_{\lambda,\mu}(t^+ w_1^+, t^+ w_2^*) \le \mathcal{I}_{\lambda,\mu}(w_1^+, w_2^*) < \mathcal{I}_{\lambda,\mu}(w_1^+, 0) = \alpha^+_{\lambda,\mu}$$
which is a contradiction. Then by the Strong Maximum Principle [10, Lemma 2.4], we have $w_1^+, w_2^+ > 0$ in $\mathcal{C}_\Omega$, hence, z^+ is a positive solution for (5). $\square$

Next we will use $w_\varepsilon = E_s(u_\varepsilon)$, the family of minimizers for the trace inequality (8), where u_ε is given in (9). Without loss of generality, we may assume that $0 \in \Omega$. We then define the cut-off function $\phi \in C_0^\infty(\mathcal{C}_\Omega), 0 \le \phi \le 1$ and for small fixed $\rho > 0$,
$$\phi(x,y) = \begin{cases} 1, & (x,y) \in \overline{B_\rho}, \\ 0, & (x,y) \notin \overline{B_{2\rho}}, \end{cases}$$
where $B_\rho = \{(x,y) : |x|^2 + y^2 < \rho^2, y > 0\}$. We take ρ so small that $\overline{B_{2\rho}} \subset \overline{\mathcal{C}_\Omega}$. Recall $\mathcal{W}$ is the extension of U introduced in Section 2, we have (cf. [3]) $|\nabla \mathcal{W}(x,y)| \le Cy^{-1}\mathcal{W}(x,y)$. Let
$$U_\varepsilon(x) = \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}}, \quad \varepsilon > 0.$$
Then the extension of $U_\varepsilon(x)$ has the form
$$\mathcal{W}_\varepsilon(x,y) = c_{N,s} y^{2s} \int_{\mathbb{R}^N} \frac{U_\varepsilon(z) dz}{(|x-z|^2 + y^2)^{\frac{N+2s}{2}}} = \varepsilon^{2s-N} \mathcal{W}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right).$$
Notice that $\phi \mathcal{W}_\varepsilon \in X_0^s(\mathcal{C}_\Omega)$, for $\varepsilon > 0$ small enough.

Lemma 6.1. *There is $z \in E \backslash \{0\}$ nonnegative and $\Lambda^* > 0$ such that for $(\lambda, \mu) \in \mathscr{C}_{\Lambda^*}$*
$$\sup_{t \ge 0} \mathcal{I}_{\lambda,\mu}(tz) < c_\infty,$$
where c_∞ is given in Lemma 3.3. In particular, $\alpha^-_{\lambda,\mu} < c_\infty$ for all $(\lambda,\mu) \in \mathscr{C}_{\Lambda^*}$.

Proof. By an argument similar to that of the proof of [3, formula (3.26)], we get
$$\|\phi \mathcal{W}_\varepsilon\|^2_{X_0^s} = k_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \mathcal{W}_\varepsilon|^2 dx dy + \mathcal{O}(1) \tag{37}$$
$$= \varepsilon^{2s-N} k_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \mathcal{W}(x,y)|^2 dx dy + \mathcal{O}(1).$$
We notice that
$$\|\phi U_\varepsilon\|^{2_s^*}_{2_s^*} = \int_\Omega |\phi U_\varepsilon|^{2_s^*} dx = \int_\Omega \frac{\phi(x)^{2_s^*}}{(\varepsilon^2 + |x|^2)^N} dx,$$
$$\|U_\varepsilon\|^{2_s^*}_{2_s^*} = \int_{\mathbb{R}^N} \frac{1}{(\varepsilon^2 + |x|^2)^N} dx = \varepsilon^{-N} \|U\|^{2_s^*}_{2_s^*}.$$

Then, one has that

$$\|\phi U_\varepsilon\|_{2_s^*}^{2_s^*} - \varepsilon^{-N}\|U\|_{2_s^*}^{2_s^*} = \int_\Omega \frac{\phi^{2_s^*}(x)-1}{(\varepsilon^2+|x|^2)^N}dx - \int_{\mathbb{R}^N\setminus\Omega}\frac{dx}{(\varepsilon^2+|x|^2)^N},$$

which yields

$$\begin{aligned}\Big|\|\phi U_\varepsilon\|_{2_s^*}^{2_s^*} - \varepsilon^{-N}\|U\|_{2_s^*}^{2_s^*}\Big| &\le \int_{\Omega\setminus B(0;\rho)} \frac{1}{(\varepsilon^2+|x|^2)^N}dx + \int_{\mathbb{R}^N\setminus\Omega}\frac{dx}{(\varepsilon^2+|x|^2)^N} \\ &= \int_{\mathbb{R}^N\setminus B(0;\rho)}\frac{dx}{(\varepsilon^2+|x|^2)^N} \le \int_{\mathbb{R}^N\setminus B(0;\rho)}\frac{dx}{|x|^{2N}} = C_3.\end{aligned}$$

This implies that

$$1 - C_3\varepsilon^N\|U\|_{2_s^*}^{-2_s^*} \le \varepsilon^N\|\phi U_\varepsilon\|_{2_s^*}^{2_s^*}\|U\|_{2_s^*}^{-2_s^*} \le 1 + C_3\varepsilon^N\|U\|_{2_s^*}^{-2_s^*}.$$

Taking ε so small that $C_3\varepsilon^N\|U\|_{2_s^*}^{-2_s^*} < 1$, since $2/2_s^* = (N-2s)/N < 1$, we obtain

$$\begin{aligned}1 - \varepsilon^N C_3\|U\|_{2_s^*}^{-2_s^*} &\le (1-\varepsilon^N C_3\|U\|_{2_s^*}^{-2_s^*})^{2/2_s^*} \le \varepsilon^{N-2s}\|\phi U_\varepsilon\|_{2_s^*}^2\|U\|_{2_s^*}^{-2} \\ &\le (1+\varepsilon^N C_3\|U\|_{2_s^*}^{-2_s^*})^{2/2_s^*} \le 1 + \varepsilon^N C_3\|U\|_{2_s^*}^{-2_s^*}.\end{aligned}$$

Hence $\|\phi U_\varepsilon\|_{2_s^*}^2 = \varepsilon^{2s-N}\|U\|_{2_s^*}^2 + \mathcal{O}(\varepsilon^{2s})$. Since $\mathcal{W} = E_s(U)$ optimizes (8), by (37),

$$\begin{aligned}\frac{\|\phi\mathcal{W}_\varepsilon\|_{X_0^s}^2}{\|\phi U_\varepsilon\|_{2_s^*}^2} &= \frac{\varepsilon^{2s-N}k_s\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\mathcal{W}(x,y)|^2dxdy + \mathcal{O}(1)}{\varepsilon^{2s-N}\|U\|_{2_s^*}^2 + \mathcal{O}(\varepsilon^{2s})} \\ &= \frac{k_s\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\mathcal{W}(x,y)|^2dxdy}{\|U\|_{2_s^*}^2}\Big(1+\mathcal{O}(\varepsilon^{N-2s})\Big) \\ &= k_s\mathcal{S}(s,N) + \mathcal{O}(\varepsilon^{N-2s}).\end{aligned} \tag{38}$$

Now we consider the function $J : E \to \mathbb{R}$ defined by

$$J(z) := 1/2\|z\|^2 - 2/2_s^*\int_\Omega |w_1|^\alpha|w_2|^\beta dx.$$

Set $w_{0,1} := \sqrt{\alpha}\phi\mathcal{W}_\varepsilon$, $w_{0,2} := \sqrt{\beta}\phi\mathcal{W}_\varepsilon$ and $z_0 := (w_{0,1}, w_{0,2}) \in E$. Notice that $J(0) = 0$, $J(tz_0) > 0$ for $t > 0$ small and $J(tz_0) < 0$ for $t > 0$ large. The map $t \mapsto J(tz_0)$ maximizes at

$$t_0 := \left(\frac{\|z_0\|^2}{2\int_\Omega |w_{0,1}|^\alpha|w_{0,2}|^\beta dx}\right)^{\frac{1}{2_s^*-2}}. \tag{39}$$

Then from (11), (38) and (39), we conclude that

$$
\begin{aligned}
\sup_{t\geq 0} J(tz_0) = J(t_0 z_0) &= \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\frac{\|z_0\|^{\frac{22_s^*}{2_s^*-2}}}{\left(2\int_\Omega |w_{0,1}|^\alpha |w_{0,2}|^\beta\right)^{\frac{2}{2_s^*-2}}} \\
&= \frac{s}{N}\left[\frac{(\alpha+\beta)k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla(\phi\mathcal{W}_\varepsilon)|^2 dxdy}{\left(\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}\int_\Omega |\phi U_\varepsilon|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}}\right]^{\frac{2_s^*}{2_s^*-2}}\cdot\frac{1}{2^{\frac{N-2s}{2s}}} \\
&= \frac{s}{N2^{\frac{N-2s}{2s}}}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]^{\frac{N}{2s}}\left[\frac{k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla(\phi\mathcal{W}_\varepsilon)|^2}{\left(\int_\Omega |\phi U_\varepsilon|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}}\right]^{\frac{N}{2s}} \qquad (40) \\
&= \frac{s}{N2^{\frac{N-2s}{2s}}}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]^{\frac{N}{2s}}\left[k_s\mathcal{S}(s,N)+\mathcal{O}(\varepsilon^{N-2s})\right]^{\frac{N}{2s}} \\
&= \frac{s}{N2^{\frac{N-2s}{2s}}}\left[k_s\mathcal{S}_{s,\alpha,\beta}+\mathcal{O}(\varepsilon^{N-2s})\right]^{\frac{N}{2s}} \\
&= \frac{s}{N}\frac{1}{2^{\frac{N-2s}{2s}}}(k_s\mathcal{S}_{s,\alpha,\beta})^{\frac{N}{2s}}+\mathcal{O}(\varepsilon^{N-2s}) = \frac{2s}{N}\left(\frac{k_s\mathcal{S}_{s,\alpha,\beta}}{2}\right)^{\frac{N}{2s}}+\mathcal{O}(\varepsilon^{N-2s}).
\end{aligned}
$$

We now choose $\delta_1>0$ so small that, for all $(\lambda,\mu)\in\mathscr{C}_{\delta_1}$, we get

$$
c_\infty = \frac{2s}{N}\left(\frac{k_s\mathcal{S}_{s,\alpha,\beta}}{2}\right)^{\frac{N}{2s}} - K_0\big(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\big) > 0.
$$

By the definition of $\mathcal{I}_{\lambda,\mu}$ and z_0, we have

$$
\mathcal{I}_{\lambda,\mu}(tz_0) \leq \frac{t^2}{2}\|z_0\|^2, \quad \text{for all } t\geq 0 \text{ and } \lambda,\mu>0,
$$

which implies that there exists $t_0\in(0,1)$ satisfying

$$
\sup_{t\in[0,t_0]}\mathcal{I}_{\lambda,\mu}(tz_0) < c_\infty, \quad \text{for all } (\lambda,\mu)\in\mathscr{C}_{\delta_1}.
$$

Hence, from (40) and $\alpha,\beta>1$ we see that

$$
\begin{aligned}
\sup_{t\geq t_0}\mathcal{I}_{\lambda,\mu}(tz_0) &= \sup_{t\geq t_0}\left(J(tz_0)-\frac{t^q}{q}Q_{\lambda,\mu}(z_0)\right) \qquad (41) \\
&\leq \frac{2s}{N}\Big(\frac{k_s\mathcal{S}_{s,\alpha,\beta}}{2}\Big)^{\frac{N}{2s}}+\mathcal{O}(\varepsilon^{N-2s})-\frac{t_0^q}{q}\left(\lambda\alpha^{\frac{q}{2}}+\mu\beta^{\frac{q}{2}}\right)\int_{B(0;\rho)}|U_\varepsilon|^q dx \\
&\leq \frac{2s}{N}\Big(\frac{k_s\mathcal{S}_{s,\alpha,\beta}}{2}\Big)^{\frac{N}{2s}}+\mathcal{O}(\varepsilon^{N-2s})-\frac{t_0^q}{q}(\lambda+\mu)\int_{B(0;\rho)}|U_\varepsilon|^q dx.
\end{aligned}
$$

Letting $0<\varepsilon\leq\rho$, we have

$$
\int_{B(0;\rho)}|U_\varepsilon|^q dx = \int_{B(0;\rho)}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{q(N-2s)}{2}}}dx \geq \int_{B(0;\rho)}\frac{1}{(2\rho^2)^{\frac{q(N-2s)}{2}}}dx = C_4,
$$

for some $C_4=C_4(N,s,\rho)$. Combining this with (41), for $\varepsilon=\big(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\big)^{\frac{1}{N-2s}}<\rho$,

$$
\sup_{t\geq t_0}\mathcal{I}_{\lambda,\mu}(tz_0) \leq \frac{2s}{N}\Big(\frac{k_s\mathcal{S}_{s,\alpha,\beta}}{2}\Big)^{\frac{N}{2s}}+\mathcal{O}\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)-\frac{t_0^q}{q}(\lambda+\mu)C_4.
$$

Choosing $\delta_2 > 0$ small enough, for all $(\lambda, \mu) \in \mathscr{C}_{\delta_2}$, we have

$$\mathcal{O}\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) - \frac{t_0^q}{q}(\lambda + \mu)C_4 < -K_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right).$$

If we set $\Lambda^* = \min\{\delta_1, \rho^{N-2s}, \delta_2\} > 0$, then for $(\lambda, \mu) \in \mathscr{C}_{\Lambda^*}$,

$$\sup_{t \geq 0} \mathcal{I}_{\lambda,\mu}(tz_0) < c_\infty. \tag{42}$$

Finally, we prove that $\alpha^-_{\lambda,\mu} < c_\infty$ for all $(\lambda, \mu) \in \mathscr{C}_{\Lambda^*}$. Recall that

$$z_0 = (w_{0,1}, w_{0,2}) = (\sqrt{\alpha}\phi\mathcal{W}_\varepsilon, \sqrt{\beta}\phi\mathcal{W}_\varepsilon).$$

By Lemma 4.5 there is $t_0 > 0$ such that $t_0 z_0 \in \mathcal{N}^-_{\lambda,\mu}$. By the definition of $\alpha^-_{\lambda,\mu}$ and (42), we conclude

$$\alpha^-_{\lambda,\mu} \leq \mathcal{I}_{\lambda,\mu}(t_0 z_0) \leq \sup_{t \geq 0} \mathcal{I}_{\lambda,\mu}(tz_0) < c_\infty,$$

for all $(\lambda, \mu) \in \mathscr{C}_{\Lambda^*}$. □

Let Λ^* be as in Lemma 6.1. We prove the existence a local minimizer for $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$.

Proposition 3. *Let $\Lambda^* > 0$ be as in Lemma 6.1 and set*

$$\Lambda_2 := \min\{\Lambda^*, (q/2)^{\frac{2}{2-q}}\Lambda_1\}.$$

For $(\lambda, \mu) \in \mathscr{C}_{\Lambda_2}$, $\mathcal{I}_{\lambda,\mu}$ has a minimizer z^- in $\mathcal{N}^-_{\lambda,\mu}$ with $\mathcal{I}_{\lambda,\mu}(z^-) = \alpha^-_{\lambda,\mu}$. Furthermore, z^- is a positive solution of (5).

Proof. By (ii) of Proposition 1, there is a $(PS)_{\alpha^-_{\lambda,\mu}}$ sequence $\{z_n\} \subset \mathcal{N}^-_{\lambda,\mu}$ for $\mathcal{I}_{\lambda,\mu}$ for all

$$(\lambda, \mu) \in \mathscr{C}_{(q/2)^{2/(2-q)}\Lambda_1}.$$

By Lemmas 3.3 and 6.1 and (ii) of Theorem 4.4, for $(\lambda, \mu) \in \mathscr{C}_{\Lambda^*}$, $\mathcal{I}_{\lambda,\mu}$ satisfies the PS condition at the energy level $\alpha^-_{\lambda,\mu} > 0$. Therefore, there exist a subsequence still denoted by $\{z_n\} = \{(w_{1,n}, w_{2,n})\} \subset \mathcal{N}^-_{\lambda,\mu}$ and $z^- := (w_1^-, w_2^-) \in E$ such that $z_n \to z^-$ strongly in E and

$$\mathcal{I}_{\lambda,\mu}(z^-) = \alpha^-_{\lambda,\mu} > 0 \text{ for all } (\lambda, \mu) \in \mathscr{C}_{\Lambda_2}. \tag{43}$$

Now we prove that $z^- \in \mathcal{N}^-_{\lambda,\mu}$. By virtue of $z_n \in \mathcal{N}^-_{\lambda,\mu}$ we see that

$$\langle \mathcal{R}'_{\lambda,\mu}(z_n), z_n \rangle = (2-q)\|z_n\|^2 - 2(2^*_s - q)\int_\Omega |w_{1,n}|^\alpha |w_{2,n}|^\beta dx < 0, \ \forall n \in \mathbb{N}.$$

Taking the limit in the last inequality and using $z_n \to z^-$ in E, we get

$$\langle \mathcal{R}'_{\lambda,\mu}(z^-), z^- \rangle = (2-q)\|z^-\|^2 - 2(2^*_s - q)\int_\Omega |w_1^-|^\alpha |w_2^-|^\beta dx \leq 0. \tag{44}$$

Observe that we must have the strict inequality in (44). Otherwise, by (43), we have $z^- \neq (0,0)$, and so $z^- \in \mathcal{N}^0_{\lambda,\mu}$, but this contradicts to Lemma 4.3. Thus, $z^- \in \mathcal{N}^-_{\lambda,\mu}$. Since $\mathcal{I}_{\lambda,\mu}(z^-) = \mathcal{I}_{\lambda,\mu}(|z^-|)$ with $|z^-| = (|w_1^-|, |w_2^-|)$, and $|z^-| \in \mathcal{N}^-_{\lambda,\mu}$, by Lemma 4.2 we may assume that z^- is a nontrivial nonnegative solution of (5). Moreover, by $z^- \in \mathcal{N}^-_{\lambda,\mu}$, we get from (44) that

$$\int_\Omega |w_1^-|^\alpha |w_2^-|^\beta dx > \frac{2-q}{2(2^*_s - q)}\|z^-\|^2 > 0.$$

This implies that $w_1^- \not\equiv 0, w_2^- \not\equiv 0$. Using the maximum principle as in the end of the proof of Proposition 2, we have $w_1^-, w_2^- > 0$ in $\mathcal{C}_\Omega$. Hence, $z^- \in \mathcal{N}_{\lambda,\mu}^-$ is a positive solution for (5). $\square$

7. **Proof of Theorem 1.1 concluded.** By Proposition 2, for $(\lambda,\mu) \in \mathscr{C}_{\Lambda_1}$, system (5) has a positive solution $z^+ \in \mathcal{N}_{\lambda,\mu}^+$. By Proposition 3, a positive solution $z^- \in \mathcal{N}_{\lambda,\mu}^-$ exists for $(\lambda,\mu) \in \mathscr{C}_{\Lambda_2}$. Since $\mathcal{N}_{\lambda,\mu}^+ \cap \mathcal{N}_{\lambda,\mu}^- = \emptyset$, then $z^\pm$ are distinct solutions of (5), so that $(u^\pm(x), v^\pm(x)) = (w_1^\pm(x,0), w_2^\pm(x,0))$ are distinct positive solutions of (1). $\square$

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