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Two solutions for inhomogeneous nonlinear elliptic equations at critical growth

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Abstract. The existence of two nontrivial solutions for a class of fully nonlinear problems at critical growth with perturbations of lower order is proved. The first solution is obtained via a local minimization argument while the second solution follows by a non-smooth mountain pass theorem.

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1 Introduction and main result

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 < p < n$ and $p < q < p^*$, where p^* denotes the critical Sobolev exponent. In this paper, we are concerned with the existence of solutions $u \in W_0^{1,p}(\Omega)$ of the following problem $(\mathscr{P}_{\varepsilon,\lambda})$

$$\begin{cases} -\operatorname{div}\left(\nabla_\xi \mathscr{L}(x,u,\nabla u)\right) + D_s\mathscr{L}(x,u,\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u + \varepsilon h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $h \in L^{p'}(\Omega)$, $h \neq 0$, provided that $\varepsilon > 0$ is small and $\lambda > 0$ is large.

Motivations for investigating problems as $(\mathscr{P}_{\varepsilon,\lambda})$ come from various situations in geometry and physics which present lack of compactness (see [7]). A typical example is Yamabe's problem, i.e. to find $u > 0$ such that

$$-4\frac{n-1}{n-2}\Delta_M u = R' u^{(n+2)/(n-2)} - R(x)u \quad \text{on } M$$

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for some constant R', where M is an n-dimensional Riemannian manifold, $R(x)$ its scalar curvature and $-\Delta_M$ is the Laplace-Beltrami operator on M. Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ fails to be compact, as known, one encounters serious difficulties in applying variational methods to problems as $(\mathscr{P}_{\varepsilon,\lambda})$.

If $h = 0$ and $\lambda = 0$, to obtain solutions of

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

one has to consider in detail the geometry of Ω (see [5]) or has to replace the critical term u^{p^*-1} with $u^{p^*-1-\varepsilon}$ and then investigate the limits of u_ε as $\varepsilon \to 0$ (nearly critical growth). See [16] and references therein.

Assume instead that $h = 0$ but $\lambda \neq 0$. As we show in Corollary 6.2 by a refined version of the well know Pucci-Serrin identity [18], if

$$p^* \nabla_x \mathscr{L}(x, s, \xi) \cdot x - nD_s \mathscr{L}(x, s, \xi)s \geqslant 0$$

a.e. in Ω and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$, then $(\mathscr{P}_{\varepsilon,\lambda})$ admits no nontrivial smooth (C^1) solution for each $\lambda \leqslant 0$, provided that Ω is star-shaped and $\mathscr{L}$ is sufficiently smooth. Therefore, in general, in this case we are reduced to take λ positive.

Let us briefly recall the historical background of existence results for problems at critical growth having perturbations of lower-order. In 1983, in a pioneering paper [7], Brézis and Nirenberg proved that the problem

$$\begin{cases} -\Delta u = u^{(n+2)/(n-2)} + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least one positive solution $u \in H_0^1(\Omega)$ provided that

- $\lambda \in (0, \lambda_1)$ if $n \geqslant 4$
- $\lambda \in \left(\dfrac{\lambda_1}{4}, \lambda_1\right)$ if $n = 3$, $\Omega = B(0, R)$,

being λ_1 the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

The extension to the p-Laplacian was studied by Garcia Azorero and Peral Alonso [13, 14] (see also [4]). They proved that the problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least one nontrivial solution $u \in W_0^{1,p}(\Omega)$, provided that

- $\lambda \in (0, \lambda_1)$ if $1 < p = q < p^*$ and $p^2 \leqslant n$
- $\lambda \in (\lambda_0, +\infty)$ if $1 < p < q < p^*$ and $p^2 > n$
- $\lambda \in (0, +\infty)$ if $1 < p < q < p^*$ and $p^2 \leqslant n$,

where λ_1 is the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions and λ_0 is a suitable positive real number.

Let us finally assume $h \neq 0$. Then, a natural question is whether inhomogeneous problems like $(\mathscr{P}_{\varepsilon,\lambda})$ have more than one solution. If Ω is bounded, one of the first answers was given in 1992 by Tarantello [21], who showed that the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has two distinct solutions if $h \in H^{-1}(\Omega)$ and $\|h\|_{-1,2} \leqslant \frac{4}{n-2}(\frac{n-2}{n+2})^{(n+2)/4}S^{n/4}$. The existence of two nontrivial solutions for the degenerate problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $1 < p < q < p^*$, $\lambda > 0$ large and $\|h\|_{p'}$ small enough, was proven in 1995 by Chabrowski [9]. Finally, these achievements have been extended by Zhou [22] to the equations

$$-\Delta_p u + c|u|^{p-2}u = |u|^{p^*-2}u + f(x,u) + h(x) \qquad (c > 0)$$

on the entire $\mathbb{R}^n$, being $f(x,u)$ a suitable lower-order perturbation of $|u|^{p^*-2}u$. This latter case involves a double loss of compactness, one due to the unboundedness of the domain and the other due to the Sobolev embedding.

Now, more recently, some results for the more general problems

$$\begin{cases} -\operatorname{div}(\nabla_\xi \mathscr{L}(x,u,\nabla u)) + D_s\mathscr{L}(x,u,\nabla u) = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with g subcritical and superlinear have been considered in [1, 2, 17] and [19]. It is therefore natural to wonder what happens when g reaches the critical growth.

The first answer goes back to a work by Arioli and Gazzola [3], who showed the existence of a solution $u \in H_0^1(\Omega)$ to the quasilinear problem

$$\begin{cases} -\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju & \\ \quad = |u|^{2^*-2}u + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where the coefficients $(a_{ij}(x,s))$ satisfy some suitable assumptions, including a semilinear asymptotic behaviour as s goes to $+\infty$ (see Remark 1.2).

In view of the above mentioned results, it is expected that under natural assumptions on $\mathscr{L}$ problems $(\mathscr{P}_{\varepsilon,\lambda})$ admits at least two nontrivial solutions for λ large and ε small (depending on λ). In order to prove this, we argue on the functional $f_{\varepsilon,\lambda} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$f_{\varepsilon,\lambda}(u) = \int_\Omega \mathscr{L}(x,u,\nabla u)\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \varepsilon\int_\Omega hu\,dx,$$

where $W_0^{1,p}(\Omega)$ is endowed with the standard norm $\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$.

In general, under reasonable assumptions on $\mathscr{L}$, $f_{\varepsilon,\lambda}$ is continuous but fails to be locally Lipschitzian unless $\mathscr{L}$ does not depend on u or it is subjected to some very restrictive growth conditions. Consequently, we will apply techniques of the non-smooth critical point theory developed in [8, 10, 11].

We assume that $\mathscr{L}(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class C^1 in s and of class C^2 in ξ. Additionally the map $\mathscr{L}(x,s,\cdot)$ is strictly convex, p-homogeneous and $\mathscr{L}(x,s,0)=0$. Moreover:

($\mathscr{H}_1$) there exists $\nu > 0$ such that

$$\mathscr{L}(x,s,\xi) \geqslant \frac{\nu}{p}|\xi|^p$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

($\mathscr{H}_2$) there exists $c_1, c_2 > 0$ such that

$$|D_s\mathscr{L}(x,s,\xi)| \leqslant c_1|\xi|^p,$$

$$\left|\nabla^2_{\xi\xi}\mathscr{L}(x,s,\xi)\right| \leqslant c_2|\xi|^{p-2} \tag{2}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

($\mathscr{H}_3$) there exist $R > 0$ and $\gamma \in (0, q-p)$ such that

$$|s| \geqslant R \implies D_s\mathscr{L}(x,s,\xi)s \geqslant 0, \tag{3}$$

$$D_s\mathscr{L}(x,s,\xi)s \leqslant \gamma\mathscr{L}(x,s,\xi) \tag{4}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

Under the previous assumptions, the following is our main result.

Theorem 1.1 *For each $\lambda > 0$ sufficiently large there exists $\varepsilon_0 > 0$ such that $(\mathscr{P}_{\varepsilon,\lambda})$ has at least two nontrivial solutions in $W^{1,p}_0(\Omega)$ for any $0 < \varepsilon < \varepsilon_0$.*

This result extends the achievements of [9, Theorem 6] to a more general class of elliptic boundary value problems. We stress that, unlike in [9], we prove our result without any use of concentration-compactness techniques [15]. Indeed, to prove the existence of the first solution as a local minimum of $f_{\varepsilon,\lambda}$, we merely show that our functional is weakly lower semicontinuous on small balls of $W^{1,p}_0(\Omega)$. From this viewpoint, our approach seems to be simpler and more direct.

Furthermore, we give in Theorem 4.4 a precise range of compactness for $f_{\varepsilon,\lambda}$. This, to the author's knowledge, has not been previously stated for fully nonlinear elliptic problems, not even for the quasilinear problem (1). Infact, in [3] it was only found a "nontrivial energy range" for the functional, inside which weak limits of Palais-Smale sequences are nontrivial and solve (1).

Remark 1.2 No further behaviour is assumed on $\mathscr{L}(x,s,\xi)$ and $D_s\mathscr{L}(x,s,\xi)s$ as s goes to $+\infty$. In [3] it was supposed that

$$\lim_{s\to+\infty} a_{ij}(x,s) = \delta_{ij}, \quad \lim_{s\to+\infty} sD_s a_{ij}(x,s) = 0, \quad (i,j = 1,\dots,n)$$

uniformly inside Ω, i.e. problem (1) converges "in some sense" to the semilinear elliptic equation $-\Delta u = |u|^{2^*-2}u + \lambda u$.

Remark 1.3 We assume (3) for $|s| \geqslant R$ for some $R > 0$. In [3] it was assumed

$$\forall s \in \mathbb{R}: \sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i\xi_j \geqslant 0$$

for a.e. $x \in \Omega$ and each $\xi \in \mathbb{R}^n$.

Remark 1.4 Assumptions (3) and (4) have already been considered in literature (see [1, 17, 19]). For instance, taking $A \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $A' \in L^\infty(\mathbb{R})$, $A(s) \geqslant \nu$ and $\gamma A(s) \geqslant A'(s)s \geqslant 0$ for each $s \in \mathbb{R}$, the class of Lagrangians

$$\mathscr{L}(x,s,\xi) = \frac{1}{p}A(s)|\xi|^p$$

fulfils all the requirements. An example is $A(s) = \frac{1}{\gamma} + \arctan(s^2)$.

2 Recalls of non-smooth critical point theory

We briefly recall from [8] some basic notions of non-smooth critical point theory.

Definition 2.1 Let (X,d) be a metric space, $f : X \to \mathbb{R}$ a continuous function and $u \in X$. We denote by $|df|(u)$ the supremum of $\sigma \in [0,+\infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathscr{H} : B_\delta(u) \times [0,\delta] \to X$$

such that for all $(v,t) \in B_\delta(u) \times [0,\delta]$

$$d(\mathscr{H}(v,t),v) \leqslant t, \quad f(\mathscr{H}(v,t)) \leqslant f(v) - \sigma t.$$

We say that the extended real number $|df|(u)$ is the weak slope of f at u.

Definition 2.2 Let (X,d) be a metric space, $f : X \to \mathbb{R}$ a continuous function and $u \in X$. We say that u is a critical point of f if $|df|(u) = 0$.

Definition 2.3 Let (X,d) be a metric space, $f : X \to \mathbb{R}$ a continuous function and $c \in \mathbb{R}$. We say that f satisfies the Palais-Smale condition at level c if every $(u_h) \subset X$ with $f(u_h) \to c$ and $|df|(u_h) \to 0$ admits a convergent subsequence.

Let us recall the mountain pass theorem without Palais-Smale condition in its non-smooth version (see [8]).

Theorem 2.4 *Assume that X is a Banach space, $f : X \to \mathbb{R}$ is continuous and the following facts hold:*

(a) *there exist $\eta > 0$ and $\varrho > 0$ such that*

$$\forall u \in X : \ \|u\|_X = \varrho \implies f(u) > \eta\,;$$

(b) *$f(0) = 0$ and there exists $w \in X$ such that*

$$f(w) < \eta \ \text{ and } \ \|w\|_X > \varrho.$$

Moreover, let us set

$$\Phi = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \ \gamma(1) = w\}$$

and

$$\eta \leqslant \beta = \inf_{\gamma \in \Phi} \max_{t \in [0,1]} f(\gamma(t)).$$

Then there exists a Palais-Smale sequence for f at level β.

Let us now return to our concrete situation.

Definition 2.5 We say that u is a weak solution to $(\mathscr{P}_{\varepsilon,\lambda})$ if $u \in W_0^{1,p}(\Omega)$ and

$$-\text{div}\ (\nabla_\xi \mathscr{L}(x, u, \nabla u)) + D_s \mathscr{L}(x, u, \nabla u) = |u|^{p^*-2}u + \lambda |u|^{q-2}u + \varepsilon h(x)$$

in $\mathscr{D}'(\Omega)$.

By the growth conditions on $\mathscr{L}$ this definition is well posed.

Definition 2.6 We say that $(u_h) \subset W_0^{1,p}(\Omega)$ is a concrete Palais-Smale sequence at level $c \in \mathbb{R}$ ($(CPS)_c$-sequence, in short) for $f_{\varepsilon,\lambda}$, if $f_{\varepsilon,\lambda}(u_h) \to c$,

$$-\text{div}\ (\nabla_\xi \mathscr{L}(x, u_h, \nabla u_h)) + D_s \mathscr{L}(x, u_h, \nabla u_h) \in W^{-1,p'}(\Omega)$$

eventually as $h \to +\infty$ and

$$\begin{aligned} &-\text{div}\ (\nabla_\xi \mathscr{L}(x, u_h, \nabla u_h)) \\ &\quad + D_s \mathscr{L}(x, u_h, \nabla u_h) - |u_h|^{p^*-2}u_h - \lambda |u_h|^{q-2}u_h - \varepsilon h(x) \to 0 \end{aligned}$$

strongly in $W^{-1,p'}(\Omega)$. We say that $f_{\varepsilon,\lambda}$ satisfies the concrete Palais-Smale condition at level c ($(CPS)_c$ in short), if every $(CPS)_c$-sequence for $f_{\varepsilon,\lambda}$ admits a strongly convergent subsequence.

Proposition 2.7 *Assume that $u \in W_0^{1,p}(\Omega)$ is such that $|df_{\varepsilon,\lambda}|(u) < +\infty$. Then*

$$\begin{aligned} w_u = &-\mathrm{div}\,(\nabla_\xi \mathscr{L}(x,u,\nabla u)) \\ &+ D_s\mathscr{L}(x,u,\nabla u) - |u|^{p^*-2}u - \lambda|u|^{q-2}u - \varepsilon h(x) \in W^{-1,p'}(\Omega) \end{aligned}$$

and $\|w_u\|_{-1,p'} \leqslant |df_{\varepsilon,\lambda}|(u)$.

In particular, if u is a critical point of $f_{\varepsilon,\lambda}$ then u is a weak solution to $(\mathscr{P}_{\varepsilon,\lambda})$.

3 The first solution of $(\mathscr{P}_{\varepsilon,\lambda})$

By combining $\mathscr{L}(x,s,0)=0$ and (2), one finds $b_1, b_2 > 0$ such that

$$\mathscr{L}(x,s,\xi) \leqslant b_1|\xi|^p, \tag{5}$$

a.e. in Ω and for each $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$,

$$|\nabla_\xi \mathscr{L}(x,s,\xi)| \leqslant b_2|\xi|^{p-1} \tag{6}$$

for in Ω and for each $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

We now prove a local weakly lower semicontinuity property for $f_{\varepsilon,\lambda}$.

Theorem 3.1 *There exists $\varrho > 0$ such that $f_{\varepsilon,\lambda}$ is weakly lower semicontinuous on $\overline{B_{W_0^{1,p}(\Omega)}(0,\varrho)}$ for each $\lambda \in \mathbb{R}$ and $\varepsilon > 0$.*

Proof. Let $(u_h) \subset W_0^{1,p}(\Omega)$ and u with $u_h \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\|u_h\|_{1,p} \leqslant \varrho$. Taking into account that, up to a subsequence, we have for $s < p^*$

$$u_h \to u \text{ in } L^s(\Omega)\,, \qquad \nabla u_h \rightharpoonup \nabla u \text{ in } L^p(\Omega) \tag{7}$$

and $u_h(x) \to u(x)$ for a.e. $x \in \Omega$, by (5) it results

$$\int_\Omega \mathscr{L}(x,u_h,\nabla u)\,dx = \int_\Omega \mathscr{L}(x,u,\nabla u)\,dx + o(1)$$

as $h \to +\infty$. Note also that, of course,

$$\int_\Omega |u_h|^q\,dx = \int_\Omega |u|^q\,dx + o(1), \qquad \int_\Omega hu_h\,dx = \int_\Omega hu\,dx + o(1)$$

as $h \to +\infty$. In particular, it suffices to show that

$$\begin{aligned} \liminf_h \Bigg\{ &\int_\Omega \mathscr{L}(x,u_h,\nabla u_h)\,dx - \int_\Omega \mathscr{L}(x,u_h,\nabla u)\,dx \\ &-\frac{1}{p^*}\int_\Omega |u_h|^{p^*}\,dx + \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx \Bigg\} \geqslant 0 \end{aligned} \tag{8}$$

for ϱ sufficiently small. Let $k \geqslant 1$ and consider the function $T_k : \mathbb{R} \to \mathbb{R}$ given by

$$T_k(s) = \begin{cases} -k & \text{if } s \leqslant -k \\ s & \text{if } -k \leqslant s \leqslant k \\ \ k & \text{if } s \geqslant k \end{cases}$$

and let $R_k : \mathbb{R} \to \mathbb{R}$ be the map defined by $R_k = Id - T_k$, namely

$$R_k(s) = \begin{cases} s+k & \text{if } s \leqslant -k \\ 0 & \text{if } -k \leqslant s \leqslant k \\ s-k & \text{if } s \geqslant k. \end{cases}$$

It is readily seen that

$$\begin{aligned} &\int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx \\ &\quad = \int_\Omega \mathscr{L}(x, u_h, \nabla T_k(u_h))\, dx + \int_\Omega \mathscr{L}(x, u_h, \nabla R_k(u_h))\, dx \end{aligned} \tag{9}$$

for each $k \geqslant 1$. Of course, one also has

$$\int_\Omega \mathscr{L}(x, u_h, \nabla u)\, dx = \int_\Omega \mathscr{L}(x, u_h, \nabla T_k(u))\, dx + \int_\Omega \mathscr{L}(x, u_h, \nabla R_k(u))\, dx \tag{10}$$

for each $k \geqslant 1$. Now, taking into account that

$$\int_\Omega |u|^{p^*-1} |u_h - u|\, dx = o(1)$$

as $h \to +\infty$ and that for any $k \geqslant 1$

$$\int_\Omega |T_k(u_h) - T_k(u)|^{p^*}\, dx = o(1)$$

as $h \to +\infty$, there exist $c_1, c_2, c_3 > 0$ such that for any k fixed

$$\begin{aligned} &\frac{1}{p^*} \int_\Omega |u_h|^{p^*}\, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*}\, dx \\ &\quad \leqslant c_1 \int_\Omega \left(|u_h|^{p^*-1} + |u|^{p^*-1} \right) |u_h - u|\, dx \\ &\quad \leqslant c_2 \int_\Omega |u_h - u|^{p^*}\, dx + o(1) \\ &\quad \leqslant c_3 \int_\Omega |T_k(u_h) - T_k(u)|^{p^*}\, dx \\ &\qquad + c_3 \int_\Omega |R_k(u_h) - R_k(u)|^{p^*}\, dx + o(1) \\ &\quad = c_3 \int_\Omega |R_k(u_h) - R_k(u)|^{p^*}\, dx + o(1) \end{aligned} \tag{11}$$

as $h \to +\infty$. For each $h, k \geqslant 1$ we have

$$\int_\Omega \mathscr{L}(x, u_h, \nabla R_k(u_h))\, dx \geqslant \frac{\nu}{p} \int_\Omega |\nabla R_k(u_h)|^p\, dx.$$

On the other hand, by the definition of R_k we obtain

$$\int_\Omega \mathscr{L}(x, u_h, \nabla R_k(u))\, dx \leqslant b_1 \int_\Omega |\nabla R_k(u)|^p\, dx \leqslant \frac{\nu}{p} \int_\Omega |\nabla R_k(u)|^p\, dx + o(1)$$

as $k \to +\infty$, uniformly in $h \in \mathbb{N}$. In particular, since for each $k \geqslant 1$ it holds

$$\liminf_h \left\{ \int_\Omega \mathscr{L}(x, u_h, T_k(\nabla u_h))\, dx - \int_\Omega \mathscr{L}(x, u_h, T_k(\nabla u))\, dx \right\} \geqslant 0,$$

by (9), (10) and (11) there exists $c_p > 0$ such that:

$$\begin{aligned}
&\liminf_h \left\{ \int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx - \int_\Omega \mathscr{L}(x, u_h, \nabla u)\, dx \right.\\
&\quad \left. - \frac{1}{p^*} \int_\Omega |u_h|^{p^*}\, dx + \frac{1}{p^*} \int_\Omega |u|^{p^*}\, dx \right\}\\
&\quad \geqslant \liminf_h \left\{ \int_\Omega \mathscr{L}(x, u_h, \nabla R_k(u_h))\, dx - \int_\Omega \mathscr{L}(x, u_h, \nabla R_k(u))\, dx \right.\\
&\quad \left. - c_3 \int_\Omega |R_k(u_h) - R_k(u)|^{p^*}\, dx \right\}\\
&\quad \geqslant \liminf_h \left\{ \frac{\nu}{p} \int_\Omega |\nabla R_k(u_h)|^p\, dx - \frac{\nu}{p} \int_\Omega |\nabla R_k(u)|^p\, dx \right.\\
&\quad \left. - c_3 \int_\Omega |R_k(u_h) - R_k(u)|^{p^*}\, dx \right\} - o(1)\\
&\quad \geqslant \liminf_h \left\{ c_p \int_\Omega |\nabla R_k(u_h) - \nabla R_k(u)|^p\, dx \right.\\
&\quad \left. - c_3 \int_\Omega |R_k(u_h) - R_k(u)|^{p^*}\, dx \right\} - o(1)
\end{aligned} \tag{12}$$

as $k \to +\infty$. By Sobolev inequality we find $d_1, d_2 > 0$ with

$$\begin{aligned}
&\liminf_h \left\{ c_p \int_\Omega |\nabla R_k(u_h) - \nabla R_k(u)|^p\, dx - c_3 \int_\Omega |R_k(u_h) - R_k(u)|^{p^*}\, dx \right\}\\
&\quad \geqslant \liminf_h \|R_k(u_h) - R_k(u)\|_{p^*}^p \{d_1 - d_2 \|R_k(u_h) - R_k(u)\|_{p^*}^{p^*-p}\} \geqslant 0
\end{aligned}$$

provided that ϱ is sufficiently small (independently of ε and λ). In particular the assertion follows by (12) by the arbitrariness of k. □

Lemma 3.2 *For each $\lambda \in \mathbb{R}$ there exist $\varepsilon > 0$ and $\varrho, \eta > 0$ such that*

$$\forall u \in W_0^{1,p}(\Omega): \ \ \|u\|_{1,p} = \varrho \implies f_{\varepsilon,\lambda}(u) > \eta.$$

Proof. Since

$$f_{\varepsilon,\lambda}(u) \geqslant \frac{\nu}{p}\int_\Omega |\nabla u|^p\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \varepsilon\int_\Omega hu\,dx,$$

by [9, Lemma 2] one gets

$$f_{\varepsilon,\lambda}(u) \geqslant \|u\|_{1,p}\{\|u\|_{1,p}^{p-1}\varphi_\lambda(\|u\|_{1,p}) - \varepsilon\|h\|_{p'}c\mathcal{L}^n(\Omega)^{\frac{p^*-p}{pp^*}}\} \tag{13}$$

where $\varphi_\lambda : [0, +\infty[\to \mathbb{R}$ is given by

$$\varphi_\lambda(\tau) = \frac{\nu}{p} - \frac{S^{-p^*}}{p^*}\tau^{p^*-p} - \frac{\lambda}{q}c^q\mathcal{L}^n(\Omega)^{\frac{p^*-q}{p^*}}\tau^{q-p}$$

for some $c > 0$. By (13) the assertion follows. □

Proposition 3.3 *For each $\lambda \in \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that $(\mathscr{P}_{\varepsilon,\lambda})$ admits at least one solution $u_1 \in W_0^{1,p}(\Omega)$ for each $\varepsilon < \varepsilon_0$. Moreover $f_{\varepsilon,\lambda}(u_1) < 0$.*

Proof. Let us choose $\phi \in W_0^{1,p}(\Omega)$ so that

$$\int_\Omega h\phi\,dx > 0.$$

Since for each $t > 0$ it results

$$\begin{aligned} f_{\varepsilon,\lambda}(t\phi) = t^p &\int_\Omega \mathscr{L}(x, t\phi, \nabla\phi)\,dx \\ &-\frac{t^{p^*}}{p^*}\int_\Omega |\phi|^{p^*}\,dx - \frac{\lambda t^q}{q}\int_\Omega |\phi|^q\,dx - \varepsilon t\int_\Omega h\phi\,dx, \end{aligned}$$

there exists $t_{\varepsilon,\lambda} > 0$ such that $f_{\varepsilon,\lambda}(t\phi) < 0$ for each $t \in]0, t_{\varepsilon,\lambda}[$. In particular,

$$\inf_{\|u\|_{1,p}\leqslant\varrho} f_{\varepsilon,\lambda}(u) < 0\,,$$

for each $\varrho > 0$. By Theorem 3.1 there exists $u_1 \in \overline{B_{W_0^{1,p}(\Omega)}(0, \varrho)}$ such that

$$f_{\varepsilon,\lambda}(u_1) = \min_{\|u\|_{1,p}\leqslant\varrho} f_{\varepsilon,\lambda}(u) < 0$$

for ϱ small enough. Moreover, up to reducing ϱ, it has to be $\|u_1\|_{1,p} < \varrho$ for $\varepsilon > 0$ sufficiently small, otherwise by Lemma 3.2 we get $f_{\varepsilon,\lambda}(u_1) > 0$. In particular u_1 is a weak solution of $(\mathscr{P}_{\varepsilon,\lambda})$. □

Remark 3.4 By (13), one can get a weak solution of $(\mathscr{P}_{\varepsilon,\lambda})$ for each $\varepsilon > 0$ on domains Ω having $\mathcal{L}^n(\Omega)$ sufficiently small.

Remark 3.5 Following Lemmas 3 and 4 of [9], one obtains existence of a weak solution also in the case $p \geqslant q$. On the other hand we remark that if $p \geqslant q$ and $\lambda > 0$ one has to require that $\mathcal{L}^n(\Omega)$ is small enough.

4 The $(CPS)_c$ for $f_{\varepsilon,\lambda}$

In this section we prove that $f_{\varepsilon,\lambda}$ satisfies the concrete Palais-Smale condition inside a suitable range of energies.

Lemma 4.1 *Let $c \in \mathbb{R}$. Then each* $(\mathrm{CPS})_c$*-sequence for $f_{\varepsilon,\lambda}$ is bounded.*

Proof. Let $c \in \mathbb{R}$ and let (u_h) be a $(\mathrm{CPS})_c$-sequence for $f_{\varepsilon,\lambda}$. Set:

$$\langle w_h, \varphi\rangle = \int_\Omega \nabla_\xi \mathscr{L}(x, u_h, \nabla u_h)\cdot \nabla\varphi\, dx + \int_\Omega D_s\mathscr{L}(x, u_h, \nabla u_h)\varphi\, dx$$
$$- \int_\Omega g_{\varepsilon,\lambda}(x, u_h)\varphi\, dx - \int_\Omega |u_h|^{p^*-2}u_h\varphi\, dx$$

for all $\varphi \in C_c^\infty(\Omega)$, where $\|w_h\|_{-1,p'} \to 0$ as $h \to +\infty$ and

$$g_{\varepsilon,\lambda}(x, s) = \lambda|s|^{q-2}s + \varepsilon h(x).$$

It is easily verified that for each $p \leqslant \alpha < p^*$ there exists $b_\alpha \in L^1(\Omega)$ with

$$g_{\varepsilon,\lambda}(x, s)s + |s|^{p^*} \geqslant \alpha\left\{\frac{\lambda}{q}|s|^q + \frac{1}{p^*}|s|^{p^*} + \varepsilon h(x)s\right\} - b_\alpha(x)$$

a.e. in Ω and for each $s \in \mathbb{R}$. Now, from

$$\frac{f'_{\varepsilon,\lambda}(u_h)(u_h)}{\|u_h\|_{1,p}} = o(1)$$

as $h \to +\infty$, one deduces that

$$\int_\Omega p\mathscr{L}(x, u_h, \nabla u_h)\, dx + \int_\Omega D_s\mathscr{L}(x, u_h, \nabla u_h)u_h\, dx$$
$$= \int_\Omega g_{\varepsilon,\lambda}(x, u_h)u_h\, dx + \int_\Omega |u_h|^{p^*}\, dx + \langle w_h, u_h\rangle$$
$$\geqslant \alpha\left\{\frac{\lambda}{q}\int_\Omega |u_h|^q\, dx + \frac{1}{p^*}\int_\Omega |u_h|^{p^*}\, dx + \varepsilon\int_\Omega hu_h\, dx\right\}$$
$$- \int_\Omega b_\alpha(x)\, dx + \langle w_h, u_h\rangle \geqslant \alpha\int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx$$
$$-\alpha f_{\varepsilon,\lambda}(u_h) - \int_\Omega b_\alpha(x)\, dx + \langle w_h, u_h\rangle.$$

On the other hand, by (4) one obtains

$$\frac{\nu}{p}(\alpha-\gamma-p)\int_\Omega |\nabla u_h|^p\,dx \leqslant (\alpha-\gamma-p)\int_\Omega \mathscr{L}(x,u_h,\nabla u_h)\,dx$$
$$\leqslant \alpha f_{\varepsilon,\lambda}(u_h)+\int_\Omega b_\alpha(x)\,dx+\|w_h\|_{-1,p'}\|u_h\|_{1,p}.$$

Choosing $\alpha>p$ such that $\alpha-\gamma-p>0$, the assertion follows. □

Remark 4.2 By exploiting the proof of Lemma 4.1 one notes that

$$\sup\left\{\left|\int_\Omega hu\,dx\right| : u \text{ is critical point of } f_{\varepsilon,\lambda} \text{ at level } c\in\mathbb{R}\right\}\leqslant\sigma$$

for some $\sigma>0$ independent of $\varepsilon>0$ and $\lambda>0$.

Remark 4.3 Let $1\leqslant p<\infty$. It is readily seen that the following fact holds: assume that $u_h\to u$ strongly in $L^p(\Omega)$ and $v_h\to v$ weakly in $L^{p'}(\Omega)$ and a.e. in Ω. Then $u_hv_h\to uv$ strongly in $L^1(\Omega)$.

Let now S denote the best Sobolev constant [20], i.e.

$$S=\inf\{\|\nabla u\|_p^p:\ u\in W_0^{1,p}(\Omega),\ \|u\|_{p^*}=1\}.$$

The next result is the main technical tool of our paper.

Theorem 4.4 *There exist $K>0$ and $\varepsilon_0>0$ such that $f_{\varepsilon,\lambda}$ satisfies $(CPS)_c$ with*

$$0<c<\frac{p^*-\gamma-p}{p^*(\gamma+p)}(\nu S)^{n/p}-K\varepsilon \tag{14}$$

for each $\varepsilon<\varepsilon_0$ and $\lambda>0$.

Proof. Let (u_h) be a concrete Palais-Smale sequence for $f_{\varepsilon,\lambda}$ at level c. Since (u_h) is bounded in $W_0^{1,p}(\Omega)$ by Lemma 4.1, up to a subsequence, we have

$$u_h\to u \quad\text{in } L^p(\Omega), \qquad \nabla u_h\rightharpoonup\nabla u \quad\text{in } L^p(\Omega).$$

Moreover, by the results of [6], we also have

$$\text{for a.e. } x\in\Omega:\quad \nabla u_h(x)\to\nabla u(x).$$

Arguing as in [19, Theorem 3.2] we get

$$\langle w_{\varepsilon,\lambda},u\rangle+\|u\|_{p^*}^{p^*}=\int_\Omega\nabla_\xi\mathscr{L}(x,u,\nabla u)\cdot\nabla u\,dx+\int_\Omega D_s\mathscr{L}(x,u,\nabla u)u\,dx\,,$$

where $w_{\varepsilon,\lambda} \in W^{-1,p'}(\Omega)$ is defined by

$$\langle w_{\varepsilon,\lambda}, \varphi \rangle = \lambda \int_\Omega |u|^{q-2} u\varphi \, dx + \varepsilon \int_\Omega h\varphi \, dx.$$

This, following again [19, Theorem 3.2], yields the existence of $d \in \mathbb{R}$ with

$$\begin{aligned} &\limsup_h \left\{ \int_\Omega \nabla_\xi \mathscr{L}(x, u_h, \nabla u_h) \cdot \nabla u_h - \int_\Omega |u_h|^{p^*} \, dx \right\} \leqslant d \\ &\leqslant \int_\Omega \nabla_\xi \mathscr{L}(x, u, \nabla u) \cdot \nabla u - \int_\Omega |u|^{p^*} \, dx. \end{aligned} \tag{15}$$

Of course, we have:

$$\nabla_\xi \mathscr{L}(x, u_h, \nabla u_h) - \nabla_\xi \mathscr{L}(x, u_h, \nabla(u_h - u)) \rightharpoonup \nabla_\xi \mathscr{L}(x, u, \nabla u)$$

in $L^{p'}(\Omega)$. Let us note that it actually holds the strong limit

$$\nabla_\xi \mathscr{L}(x, u_h, \nabla u_h) - \nabla_\xi \mathscr{L}(x, u_h, \nabla(u_h - u)) \to \nabla_\xi \mathscr{L}(x, u, \nabla u)$$

in $L^{p'}(\Omega)$. Indeed, by (2) there exist $\tau \in]0, 1[$ and $c > 0$ with

$$\begin{aligned} &\left| \nabla_\xi \mathscr{L}(x, u_h, \nabla u_h) - \nabla_\xi \mathscr{L}(x, u_h, \nabla(u_h - u)) \right| \\ &\quad \leqslant \left| \nabla^2_{\xi\xi} \mathscr{L}(x, u_h, \nabla u_h + (\tau - 1)\nabla u) \right| |\nabla u| \\ &\quad \leqslant c|\nabla u_h|^{p-2} |\nabla u| + c|\nabla u|^{p-1}. \end{aligned}$$

Therefore, by Remark 4.3, we have

$$\begin{aligned} &\nabla_\xi \mathscr{L}(x, u_h, \nabla u_h) \cdot \nabla u_h = \nabla_\xi \mathscr{L}(x, u_h, \nabla(u_h - u)) \cdot \nabla u_h \\ &\quad + \nabla_\xi \mathscr{L}(x, u, \nabla u) \cdot \nabla u \to u + o(1) = \nabla_\xi \mathscr{L}(x, u_h, \nabla(u_h - u)) \cdot \nabla(u_h - u) \\ &\quad + \nabla_\xi \mathscr{L}(x, u, \nabla u) \cdot \nabla u + o(1) \quad \text{in } L^1(\Omega) \end{aligned}$$

as $h \to +\infty$, i.e.

$$\begin{aligned} &\nabla_\xi \mathscr{L}(x, u_h, \nabla u_h) \cdot \nabla u_h - \nabla_\xi \mathscr{L}(x, u, \nabla u) \cdot \nabla u \\ &\quad = \nabla_\xi \mathscr{L}(x, u_h, \nabla(u_h - u)) \cdot \nabla(u_h - u) + o(1) \quad \text{in } L^1(\Omega) \end{aligned} \tag{16}$$

as $h \to +\infty$. In a similar way, since there exists $\widetilde{c} > 0$ with

$$\left| |u_h|^{p^*} - |u_h|^{p^*-p} |u_h - u|^p \right| \leqslant \widetilde{c}\, |u_h|^{p^*-p} \big(|u_h|^{p-1} + |u|^{p-1} \big) |u|,$$

one obtains

$$|u_h|^{p^*} - |u_h|^{p^*-p} |u_h - u|^p \to |u|^{p^*} \quad \text{in } L^1(\Omega). \tag{17}$$

In particular, by combining (15), (16) and (17), it results

$$\limsup_h \int_\Omega [\nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) - |u_h|^{p^*-p}|u_h-u|^p]\,dx \leqslant 0. \tag{18}$$

On the other hand, by Hölder and Sobolev inequalities, we get

$$\begin{aligned}&\int_\Omega [\nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) - |u_h|^{p^*-p}|u_h-u|^p]\,dx\\ &\quad\geqslant \nu\|\nabla(u_h-u)\|_p^p - \frac{1}{S}\|u_h\|_{p^*}^{p^*-p}\|\nabla(u_h-u)\|_p^p \\ &\quad= \left\{\nu - \frac{1}{S}\|u_h\|_{p^*}^{p^*-p}\right\}\|\nabla(u_h-u)\|_p^p,\end{aligned} \tag{19}$$

which turns out to be coercive if

$$\limsup_h \|u_h\|_{p^*}^{p^*} < (\nu S)^{n/p}. \tag{20}$$

Now, from $f_{\varepsilon,\lambda}(u_h)\to c$ we deduce

$$\int_\Omega \mathscr{L}(x,u_h,\nabla u_h)\,dx - \frac{1}{p^*}\|u_h\|_{p^*}^{p^*} = \frac{\lambda}{q}\|u\|_q^q + \varepsilon\int_\Omega hu\,dx + c + o(1) \tag{21}$$

as $h\to+\infty$. On the other hand, by using (4), from $f'_{\varepsilon,\lambda}(u_h)(u_h)\to 0$ we obtain

$$\frac{\gamma+p}{p}\int_\Omega \mathscr{L}(x,u_h,\nabla u_h)\,dx - \frac{1}{p}\|u_h\|_{p^*}^{p^*} \geqslant \frac{\lambda}{p}\|u\|_q^q + \frac{\varepsilon}{p}\int_\Omega hu\,dx + o(1) \tag{22}$$

as $h\to+\infty$. Multiplying (21) by $\frac{\gamma+p}{p}$, we obtain

$$\begin{aligned}&\frac{\gamma+p}{p}\int_\Omega \mathscr{L}(x,u_h,\nabla u_h)\,dx - \frac{\gamma+p}{pp^*}\|u_h\|_{p^*}^{p^*}\\ &\quad= \frac{\gamma+p}{pq}\lambda\|u\|_q^q + \frac{\gamma+p}{p}\varepsilon\int_\Omega hu + \frac{\gamma+p}{p}c + o(1)\end{aligned} \tag{23}$$

as $h\to+\infty$. Therefore, by combining (23) with (22), one gets

$$\begin{aligned}\frac{p^*-\gamma-p}{pp^*}\|u_h\|_{p^*}^{p^*} &\leqslant -\frac{q-\gamma-p}{pq}\lambda\|u\|_q^q \\ &\quad + c'\varepsilon\int_\Omega hu\,dx + \frac{\gamma+p}{p}c + o(1)\\ &\leqslant c'\varepsilon\int_\Omega hu\,dx + \frac{\gamma+p}{p}c + o(1)\end{aligned}$$

as $h\to+\infty$. Now, taking into account Remark 4.2, we deduce

$$\|u_h\|_{p^*}^{p^*} \leqslant \frac{p^*(\gamma+p)}{p^*-\gamma-p}c + \widetilde{K}\varepsilon + o(1),$$

as $h \to +\infty$ for some $\widetilde{K} > 0$. In particular, condition (20) is fulfilled if

$$\frac{p^*(\gamma + p)}{p^* - \gamma - p} c + \widetilde{K}\varepsilon < (\nu S)^{n/p}$$

which yields range (14) for ε small and a suitable $K > 0$. By combining (18), (19) and (20) we conclude that u_h goes to u strongly in $W_0^{1,p}(\Omega)$. □

Remark 4.5 for the equation

$$-\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{q-2}u + \varepsilon h(x)$$

being $\gamma = 0$ and $\nu = 1$, the range (14) reduces to

$$0 < c < \frac{S^{n/p}}{n} - K\varepsilon$$

for some $K > 0$, according to the results found in [9].

5 The second solution of $(\mathscr{P}_{\varepsilon,\lambda})$

Finally, we come to the proof of Theorem 1.1.

Proof. Let us choose $\phi \in W_0^{1,p} \cap L^\infty(\Omega)$ such that

$$\|\phi\|_{p^*} = 1 \quad \text{and} \quad \int_\Omega h\phi \, dx < 0.$$

It is readily seen that

$$\lim_{t \to +\infty} f_{\varepsilon,\lambda}(t\phi) = -\infty\,,$$

so that there exists $t_{\lambda,\varepsilon} > 0$ with

$$f_{\varepsilon,\lambda}(t_{\lambda,\varepsilon}\phi) = \sup_{t \geqslant 0} f_{\varepsilon,\lambda}(t\phi) > 0. \tag{24}$$

Taking into account (4), the value $t_{\lambda,\varepsilon}$ must satisfy

$$\begin{aligned}
\varepsilon \int_\Omega h\phi &= t_{\lambda,\varepsilon}^{q-1} \Bigg\{ t_{\lambda,\varepsilon}^{p-q} \Bigg[\int_\Omega p\mathscr{L}(x, t_{\lambda,\varepsilon}\phi, \nabla\phi)\, dx \\
&\quad + \int_\Omega D_s\mathscr{L}(x, t_{\lambda,\varepsilon}\phi, \nabla\phi) t_{\lambda,\varepsilon}\phi \, dx \Bigg] - t_{\lambda,\varepsilon}^{p^*-q} - \lambda \int_\Omega |\phi|^q \, dx \Bigg\} \\
&\leqslant t_{\lambda,\varepsilon}^{q-1} \Bigg\{ t_{\lambda,\varepsilon}^{p-q} M \int_\Omega |\nabla\phi|^p \, dx - t_{\lambda,\varepsilon}^{p^*-q} - \lambda \int_\Omega |\phi|^q \, dx \Bigg\},
\end{aligned}$$

for some $M > 0$. Now, being

$$\lim_{\lambda\to+\infty}\left\{t_{\lambda,\varepsilon}^{p^*-q}+\lambda\int_\Omega|\phi|^q\,dx\right\}=+\infty$$

it has to be $t_{\lambda,\varepsilon}\to 0$ as $\lambda\to+\infty$. In particular, by (24) we obtain

$$\lim_{\lambda\to+\infty}\sup_{t\geqslant 0} f_{\varepsilon,\lambda}(t\phi)=0\,,$$

so that there exists $\lambda_0 > 0$ with

$$0<\sup_{t\geqslant 0} f_{\varepsilon,\lambda}(t\phi)<\frac{p^*-\gamma-p}{p^*(\gamma+p)}\,(\nu S)^{n/p}-K\varepsilon \tag{25}$$

for each $\lambda\geqslant\lambda_0$ and $\varepsilon<\varepsilon_0$. Let $w = t\phi$ with t so large that $f_{\varepsilon,\lambda}(w) < 0$ and set

$$\Phi=\{\gamma\in C([0,1],W_0^{1,p}(\Omega)):\ \gamma(0)=0,\ \gamma(1)=w\}$$

and

$$\beta_{\varepsilon,\lambda}=\inf_{\gamma\in\Phi}\max_{t\in[0,1]} f_{\varepsilon,\lambda}(\gamma(t))$$

Taking into account Lemma 3.2, by Theorem 2.4 one finds $(u_h)\subset W_0^{1,p}(\Omega)$ with

$$f_{\varepsilon,\lambda}(u_h)\to\beta_{\varepsilon,\lambda},\quad |df_{\varepsilon,\lambda}|(u_h)\to 0\,,$$

$$0<\eta\leqslant\beta_{\varepsilon,\lambda}=\inf_{\gamma\in\Phi}\max_{t\in[0,1]} f_{\varepsilon,\lambda}(\gamma(t))\leqslant\sup_{t\geqslant 0} f_{\varepsilon,\lambda}(t\phi). \tag{26}$$

By Theorem 4.4 $f_{\varepsilon,\lambda}$ satisfies $(CPS)_{\beta_{\varepsilon,\lambda}}$, since by (25) and (26)

$$\lambda\geqslant\lambda_0\implies 0<\beta_{\varepsilon,\lambda}<\frac{p^*-\gamma-p}{p^*(\gamma+p)}\,(\nu S)^{n/p}-K\varepsilon$$

for each $\varepsilon<\varepsilon_0$. Therefore there exist a subsequence of $(u_h)\subset W^{1,p}(\Omega)$ strongly convergent to some u_2 which solves $(\mathscr{P}_{\varepsilon,\lambda})$. Since $f_{\varepsilon,\lambda}(u_1) < 0$ and $f_{\varepsilon,\lambda}(u_2) > 0$, of course $u_1\neq u_2$. ☐

Remark 5.1 For the Lagrangian $\mathscr{L}(x,s,\xi)=\frac{1}{p}|\xi|^p$ in [9, Theorem 6] it was also provided a quantitative estimate for the smallness of h (in norm). Given $\lambda > 0$ large enough, one gets two solutions if

$$\max\{\|h\|_{p'},\|h\|_{p^{*\prime}}^{p^{*\prime}}\}\leqslant\ \min\left\{\varepsilon_\lambda,\frac{p^{*\prime}p'^{p^{*\prime}}p^{*\frac{p^{*\prime}}{p^*}}}{n^{\frac{p^{*\prime}}{p^*}+1}}S^{n/p}\right\}$$

being $\varepsilon_\lambda > 0$ such that $f_{\varepsilon,\lambda}(u)\geqslant 0$ if $\|h\|_{p'}\leqslant\varepsilon_\lambda$ and $\|u\|_{1,p}$ is sufficiently small, according to [9, Lemma 2].

Remark 5.2 In the case $1 < q \leqslant p < p^*$, in general, our method is inconclusive since it is not clear whether

$$\sup_{t \geqslant 0} f_{\varepsilon,\lambda}(t\phi) \not\to 0 \quad \text{as} \quad \lambda \to +\infty$$

Furthermore, it may also happen that for each $\lambda > 0$ fixed

$$\sup_{t \geqslant 0} f_{\varepsilon,\lambda}(t\phi) \not\to 0 \quad \text{as} \quad \mathcal{L}^n(\Omega) \to 0.$$

See section 4 of [9] where this is discussed for $\mathscr{L}(x,s,\xi) = \frac{1}{p}|\xi|^p$.

6 Non-existence results for $(\mathscr{P}_{0,\lambda})$

Assume that $\mathscr{L}$ is of class C^1 on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and $\nabla_\xi \mathscr{L}$ is of class C^1. The following results follow by the general variational identity for C^1 solutions recently proven in [12], which relaxes in the regularity assumptions a classical result due to Pucci and Serrin [18].

Theorem 6.1 *Let Ω be star-shaped with respect to the origin and*

$$\nabla_x \mathscr{L}(x,s,\xi) \cdot x - \frac{n}{p^*} D_s \mathscr{L}(x,s,\xi)s - \frac{p^* - q}{p^* q} n\lambda\, |s|^q \geqslant 0\,, \tag{27}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Then problem $(\mathscr{P}_{0,\lambda})$ admits no nontrivial solution $u \in C^1(\overline{\Omega})$.

Proof. If we define $\mathscr{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by setting

$$\forall (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n : \;\; \mathscr{F}(x,s,\xi) = \mathscr{L}(x,s,\xi) - \frac{\lambda}{q}|s|^q - \frac{1}{p^*}|s|^{p^*},$$

the assertion follows by the main result of [12], being the inequality

$$n\mathscr{F} + \nabla_x \mathscr{F} \cdot x - aD_s \mathscr{F} s - (a+1)\nabla_\xi \mathscr{F} \cdot \xi \geqslant 0$$

equivalent to (27) provided that $a = \frac{n-p}{p}$. □

Corollary 6.2 *Let Ω be star-shaped with respect to the origin, $\lambda \leqslant 0$ and*

$$p^* \nabla_x \mathscr{L}(x,s,\xi) \cdot x - nD_s \mathscr{L}(x,s,\xi)s \geqslant 0, \tag{28}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Then problem $(\mathscr{P}_{0,\lambda})$ admits no nontrivial solution $u \in C^1(\overline{\Omega})$.

Proof. Being $q < p^*$ and $\lambda \leqslant 0$, condition (28) implies condition (27). □

Remark 6.3 Assume that $\lambda \leqslant 0$ and $\nabla_x \mathscr{L} = 0$. Then the non-existence condition (28) becomes $D_s\mathscr{L}(s,\xi)s \leqslant 0$. Note that this is the contrary of (3). From this point of view (3) seems to be pretty natural.

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