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Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities

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The semi-classical regime of standing wave solutions of a Schrödinger equation in the presence of non-constant electric and magnetic potentials is studied in the case of non-local nonlinearities of Hartree type. It is shown that there exists a family of solutions having multiple concentration regions which are located around the minimum points of the electric potential.

1. Introduction and main result

1.1. Introduction

Some years ago, Penrose [26] derived a system of nonlinear equations by coupling the linear Schrödinger equation of quantum mechanics with Newton's gravitational law. Roughly speaking, a point mass interacts with a density of matter described by the square of the wave function that solves the Schrödinger equation. If m is the mass of the point, this interaction leads to the following system in $\mathbb{R}^3$:

$$\left.\begin{aligned} \frac{\hbar^2}{2m}\Delta\psi - V(x)\psi + U\psi = 0, \\ \Delta U + 4\pi\gamma|\psi|^2 = 0, \end{aligned}\right\} \tag{1.1}$$

where ψ is the wave function, U is the gravitational potential energy, V is a given Schrödinger potential, $\hbar$ is the Planck constant and $\gamma = Gm^2$, with G being Newton's constant of gravitation. Note that, by means of the scaling

$$\psi(x) = \frac{1}{\hbar}\frac{\hat{\psi}(x)}{\sqrt{8\pi\gamma m}}, \qquad V(x) = \frac{1}{2m}\hat{V}(x), \qquad U(x) = \frac{1}{2m}\hat{U}(x),$$

system (1.1) can be written, maintaining the original notation, as

$$\left.\begin{aligned} \hbar^2\Delta\psi - V(x)\psi + U\psi = 0, \\ \hbar^2\Delta U + |\psi|^2 = 0. \end{aligned}\right\} \tag{1.2}$$

The second equation in (1.2) can be explicitly solved with respect to U so that the system becomes the single non-local equation

$$\hbar^2\Delta\psi - V(x)\psi + \frac{1}{4\pi\hbar^2}\left(\int_{\mathbb{R}^3}\frac{|\psi(\xi)|^2}{|x-\xi|}\,\mathrm{d}\xi\right)\psi = 0 \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

The Coulomb-type convolution potential $W(x) = |x|^{-1}$ in $\mathbb{R}^3$ is also involved in various physical applications, such as electromagnetic waves in a Kerr medium (in nonlinear optics), surface gravity waves (in hydrodynamics) as well as ground-state solutions (in quantum mechanical systems) (see, for instance, [1] for further details and [14] for the derivation of these equations from a many-body Coulomb system).

We will study the semi-classical regime (namely the existence and asymptotic behaviour of solutions as $\hbar \to 0$) for a more general equation having a similar structure. Taking ε in place of $\hbar$, our model will be written as follows:

$$\left(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\right)^2 u + V(x)u = \frac{1}{\varepsilon^2}(W * |u|^2)u \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

where the convolution kernel $W : \mathbb{R}^3 \setminus \{0\} \to (0,\infty)$ is an even smooth kernel, homogeneous of degree -1 and we denote the imaginary unit by i. The choice of $W(x) = |x|^{-1}$ recovers (1.3). Equation (1.4) is equivalent to

$$\left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u = (W * |u|^2)u \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where we have set $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$.

The vector-valued field A represents a given external magnetic potential, and forces the solutions to be, in general, complex valued (see [11] and the references therein). To the best of our knowledge, in this framework, no previous result involving the electromagnetic field can be found in the literature. On the other hand, when $A \equiv 0$, it is known that solutions have a constant phase, so that it is not a restriction to look for real-valued solutions. In this simpler situation, we recall the results contained in [24, 25], stating that, at fixed $\hbar = \varepsilon$, system (1.2) can be uniquely solved by radially symmetric functions. Moreover, these solutions decay exponentially fast at infinity together with their first derivatives. The mere existence of one solution can be traced back to [22].

Wei and Winter [29] proposed a deeper study of the multi-bump solutions to the same system and proved an existence result that can be summarized as follows: if $k \geqslant 1$ and $P_1, \dots, P_k \in \mathbb{R}^3$ are given non-degenerate critical points of V (but local extrema are also included without any further requirements), then multi-bump solutions $\psi_\hbar$ exist that concentrate at these points when $\hbar \to 0$. A similar equation is also studied in [23], where multi-bump solutions are found by some finite-dimensional reduction. The main result about existence leans on some *non-degeneracy* assumption on the solutions of a limiting problem, which was actually proved in [29, theorem 3.1] only in the particular case where $W(x) = |x|^{-1}$ in $\mathbb{R}^3$. Moreover, the equation investigated in [23] cannot be deduced from a singularly perturbed problem like (1.3) because the terms do not scale coherently.

For precise references to some classical works (well-posedness, regularity, long-term behaviour) related to the nonlinear Schrödinger equation with Hartree nonlinearity for Coulomb potential and $A = 0$, we refer the reader to [28, p. 66] (see also [8]).

1.2. Statement of the main result

We shall study equation (1.5) by exploiting a penalization technique which was recently developed in [12] (see also [3–5] for $A = 0$), whose main idea is searching for solutions in a suitable class of functions whose location and shape is the one expected for the solution itself. This approach seems appropriate since it does not need very strong knowledge of the *limiting problem* (2.1) introduced in the next section. In particular, for a general convolution kernel W, we still do not know if its solutions are non-degenerate. In order to state our main result (as well as the technical lemmas contained in §§ 2 and 3), the following conditions will be retained.

(A1) $A : \mathbb{R}^3 \to \mathbb{R}^3$ is of class C^1.

(V1) $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that

$$0 \leqslant V_0 = \inf_{x \in \mathbb{R}^3} V(x), \qquad \liminf_{|x| \to \infty} V(x) > 0.$$

(V2) There exist bounded disjoint open sets $O^1, \dots, O^k$ such that

$$0 < m_i = \inf_{x \in O^i} V(x) < \min_{x \in \partial O^i} V(x), \quad i = 1, \dots, k.$$

(W) $W \colon \mathbb{R}^3 \setminus \{0\} \to (0, \infty)$ is an even function of class C^1 such that $W(\lambda x) = \lambda^{-1} W(x)$ for any $\lambda > 0$ and $x \neq 0$.

Convolution kernels such as $W(x) = x_i^2/|x|^3$ for $x \in \mathbb{R}^3 \setminus \{0\}$ or, more generally, $W(x) = W_1(x)/W_2(x)$ for $x \in \mathbb{R}^3 \setminus \{0\}$, where W_1 and W_2 are positive, even and homogeneous of degree m and $m + 1$, respectively, satisfy (W).

For each $i \in \{1, \dots, k\}$, we define

$$\mathcal{M}^i = \{x \in O^i : V(x) = m_i\},$$

$Z = \{x \in \mathbb{R}^3 : V(x) = 0\}$ and $m = \min_{i \in \{1,\dots,k\}} m_i$. By (V1) we can fix $\tilde{m} > 0$ with

$$\tilde{m} < \min\Big\{m, \liminf_{|x| \to \infty} V(x)\Big\}$$

and define $\tilde{V}_\varepsilon(x) = \max\{\tilde{m}, V_\varepsilon(x)\}$. Let H_ε be the Hilbert space defined by the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{C})$ under the scalar product

$$\langle u, v\rangle_\varepsilon = \mathrm{Re} \int_{\mathbb{R}^3} \left(\frac{1}{\mathrm{i}}\nabla u - A_\varepsilon(x)u\right)\overline{\left(\frac{1}{\mathrm{i}}\nabla v - A_\varepsilon(x)v\right)} + \tilde{V}_\varepsilon(x)u\bar{v}\,\mathrm{d}x$$

and let $\|\cdot\|_\varepsilon$ be the associated norm.

The main result of the paper is the following theorem.

THEOREM 1.1. *Suppose that (A1), (V1), (V2) and (W) hold. Then, for any sufficiently small* $\varepsilon > 0$, *there exists a solution* $u_\varepsilon \in H_\varepsilon$ *of equation (1.5) such that* $|u_\varepsilon|$ *has* k *local maximum points* $x_\varepsilon^i \in O^i$ *satisfying*

$$\lim_{\varepsilon\to 0} \max_{i=1,\dots,k} \mathrm{dist}(\varepsilon x_\varepsilon^i, \mathcal{M}^i) = 0,$$

and for which

$$|u_\varepsilon(x)| \leqslant C_1 \exp\Big\{-C_2 \min_{i=1,\dots,k} |x - x_\varepsilon^i|\Big\},$$

for some positive constants C_1 *and* C_2. *Moreover, for any sequence* $(\varepsilon_n) \subset (0, \varepsilon]$ *with* $\varepsilon_n \to 0$, *there exists a subsequence, still denoted by* (ε_n) *such that, for each* $i \in \{1, \dots, k\}$, *there exist* $x^i \in \mathcal{M}^i$ *with* $\varepsilon_n x_{\varepsilon_n}^i \to x^i$, *a constant* $w_i \in \mathbb{R}$ *and* $U_i \in H^1(\mathbb{R}^3, \mathbb{R})$, *a positive least-energy solution of*

$$-\Delta U_i + m_i U_i - (W * |U_i|^2)U_i = 0, \quad U_i \in H^1(\mathbb{R}^3, \mathbb{R}), \tag{1.6}$$

for which we have

$$u_{\varepsilon_n}(x) = \sum_{i=1}^{k} U_i(x - x_{\varepsilon_n}^i)\exp\{\mathrm{i}(w_i + A(x^i)(x - x_{\varepsilon_n}^i))\} + K_n(x), \tag{1.7}$$

where $K_n \in H_{\varepsilon_n}$ *satisfies* $\|K_n\|_{H_{\varepsilon_n}} = o(1)$ *as* $n \to +\infty$.

The one-dimensional and two-dimensional cases would require a separate analysis in the construction of the penalization argument (see, for example, [7] for a detailed discussion). The study of the cases of dimensions larger than three is less interesting from a physical point of view. Moreover, keeping the soliton dynamics in mind as a possible further development, in dimensions $N \geqslant 4$ the time-dependent Schrödinger equation with kernels, say, of the type $W(x) = |x|^{2-N}$ does not have global existence in time for all H^1 initial data (see, for example, [9, remark 6.8.2] and the heuristic discussion in the next section).

1.3. A heuristic remark: multi-bump dynamics

We can also think of theorem 1.1 as the starting point in order to rigorously justify multi-bump soliton dynamics for the full Schrödinger equation with an external magnetic field

$$\left.\begin{aligned} \mathrm{i}\varepsilon\partial_t u + \frac{1}{2}\left(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\right)^2 u + V(x)u &= \frac{1}{\varepsilon^2}(W * |u|^2)u \quad &&\text{in } \mathbb{R}^3,\\ u(x, 0) &= u_0(x) &&\text{in } \mathbb{R}^3. \end{aligned}\right\} \tag{1.8}$$

In the following, we describe what we expect to hold (the question is open even for $A=0$; see the discussion by Fröhlich *et al.* in [16]). Given $k \geqslant 1$ positive numbers $g_1, \dots, g_k$, if $\mathcal{E} : H^1(\mathbb{R}^3) \to \mathbb{R}$ is defined as

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^6} W(x-y)|u(x)|^2|u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y,$$

let $U_j : \mathbb{R}^3 \to \mathbb{R}$, $j = 1, \dots, k$, be the solutions to the minimum problems

$$\mathcal{E}(U_j) = \min\{\mathcal{E}(u) : u \in H^1(\mathbb{R}^3),\ \|u\|^2_{L^2} = g_j\},$$

which solve the equations

$$-\tfrac{1}{2}\Delta U_j + m_j U_j = (W * |U_j|^2)U_j \quad \text{in } \mathbb{R}^3$$

for some $m_j \in \mathbb{R}$. Consider now in (1.8) an initial datum of the form

$$u_0(x) = \sum_{j=1}^{k} U_j\left(\frac{x - x_0^j}{\varepsilon}\right) \exp\left\{\frac{\mathrm{i}}{\varepsilon}[A(x_0^j)\cdot(x - x_0^j) + x \cdot \xi_0^j]\right\}, \quad x \in \mathbb{R}^3,$$

where $x_0^j \in \mathbb{R}^3$ and $\xi_0^j \in \mathbb{R}^3$, $j = 1, \dots, k$, are the initial position and velocity for the ordinary differential equation

$$\left.\begin{aligned} &\dot{x}_j(t) = \xi_j(t), \\ &\dot{\xi}_j(t) = -\nabla V(x_j(t)) - \varepsilon\sum_{i \neq j}^{k} m_i \nabla W(x_j(t) - x_i(t)) - \xi_j(t) \times B(x_j(t)), \\ &x_j(0) = x_0^j, \quad \xi_j(0) = \xi_0^j, \quad j = 1, \dots, k, \end{aligned}\right\} \tag{1.9}$$

with $B = \nabla \times A$. The systems can be considered as a mechanical system of k interacting particles of mass m_i subjected to an external potential as well as a mutual Newtonian-type interaction. Therefore, the conjecture is that, under suitable assumptions, the following representation formula might hold:

$$u_\varepsilon(x,t) = \sum_{j=1}^{k} U_j\left(\frac{x - x_j(t)}{\varepsilon}\right) \exp\left\{\frac{\mathrm{i}}{\varepsilon}[A(x_j(t))\cdot(x - x_j(t)) + x \cdot \xi_j(t) + \theta_\varepsilon^j(t)]\right\} + \omega_\varepsilon, \tag{1.10}$$

locally in time, for certain phases $\theta_\varepsilon^i : \mathbb{R}^+ \to [0, 2\pi)$, where ω_ε is small (in a suitable sense) as $\varepsilon \to 0$, provided that the centres x_0^j in the initial data are chosen sufficiently far from each other. Now, neglecting, as $\varepsilon \to 0$, the interaction term (ε-dependent)

$$\varepsilon\sum_{i \neq j}^{k} m_i \nabla W(x_j(t) - x_i(t))$$

in the Newtonian system (1.9), and taking

$$x_0^1, \dots, x_0^k \in \mathbb{R}^3 : \nabla V(x_0^j) = 0 \quad \text{and} \quad \xi_0^j = 0 \quad \text{for all } j = 1, \dots, k,$$

then the solution of (1.9) is

$$x_j(t) = x_0^j, \quad \xi_j(t) = 0 \quad \text{for all } t \in [0,\infty) \text{ and } j = 1,\dots,k,$$

so that the representation formula (1.10) reduces, for $\varepsilon = \varepsilon_n \to 0$, to

$$u_{\varepsilon_n}(x,t) = \sum_{j=1}^{k} U_j\left(\frac{x - x_0^j}{\varepsilon_n}\right) \exp\left\{\frac{\mathrm{i}}{\varepsilon_n}[A(x_0^j)\cdot(x - x_0^j) + \theta_{\varepsilon_n}^j(t)]\right\} + \omega_{\varepsilon_n},$$

namely, to formula (1.7) up to a change in the phase terms and up to replacing x with $\varepsilon_n x$ and x_0^j with $\varepsilon_n x_{\varepsilon_n}^j$ for all $j = 1,\dots,k$.

1.4. Plan of the paper

In § 2 we obtain several results about the structure of the solutions of the limiting problem (1.6). In particular, we study the compactness of the set of real ground states solutions and we achieve a result about the orbital stability property of these solutions for the Pekar–Choquard equation. In § 3 we perform the penalization scheme. In particular, we obtain various energy estimates in the semi-classical regime $\varepsilon \to 0$ and we get a Palais–Smale condition for the penalized functional which allows us to find suitable critical points inside the concentration set. Finally, we conclude with the proof of theorem 1.1.

1.5. Main notation

- i is the imaginary unit.
- The complex conjugate of any number $z \in \mathbb{C}$ is denoted by $\bar{z}$.
- The real part of a number $z \in \mathbb{C}$ is denoted by $\operatorname{Re} z$.
- The imaginary part of a number $z \in \mathbb{C}$ is denoted by $\operatorname{Im} z$.
- The symbol $\mathbb{R}^+$ (respectively, $\mathbb{R}^-$) means the positive real line $[0,\infty)$ (respectively, $(-\infty,0]$).
- The ordinary inner product between two vectors $a, b \in \mathbb{R}^3$ is denoted by $\langle a \mid b\rangle$.
- The standard L^p norm of a function u is denoted by $\|u\|_{L^p}$.
- The standard L^∞ norm of a function u is denoted by $\|u\|_{L^\infty}$.
- The symbol Δ means $D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2$.
- The convolution $u * v$ means

$$(u * v)(x) = \int u(x-y)v(y)\,\mathrm{d}y.$$

2. Properties of the set of ground states

For any positive real number a, the limiting equation for the Hartree problem (1.4) is

$$-\Delta u + au = (W * |u|^2)u \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

2.1. A Pohozaev-type identity

We now give the statement of a useful identity satisfied by solutions to problem (2.1).

LEMMA 2.1. *Let $u \in H^1(\mathbb{R}^3, \mathbb{C})$ be a solution to (2.1). Then*

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \tfrac{3}{2}a\int_{\mathbb{R}^3} |u|^2 \,\mathrm{d}x = \frac{5}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u(x)|^2|u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{2.2}$$

Proof. The proof is straightforward, and we include it for the sake of completeness. It is enough to prove it for smooth functions, then use a standard density argument. We multiply equation (2.1) by $\langle x \mid \overline{\nabla u}\rangle$. Note that

$$\mathrm{Re}(\Delta u\langle x \mid \overline{\nabla u}\rangle) = \mathrm{div}(\mathrm{Re}(\langle x \mid \overline{\nabla u}\rangle\nabla u) - \tfrac{1}{2}|\nabla u|^2 x) + \tfrac{1}{2}|\nabla u|^2, \tag{2.3}$$

$$\mathrm{Re}(-au\langle x \mid \overline{\nabla u}\rangle) = -a\,\mathrm{div}(\tfrac{1}{2}|u|^2 x) + \tfrac{3}{2}a|u|^2, \tag{2.4}$$

$$\varphi(x)\,\mathrm{Re}(u\langle x \mid \overline{\nabla u}\rangle) = \mathrm{div}(\tfrac{1}{2}|u|^2\varphi(x)x) - \tfrac{1}{2}|u|^2\,\mathrm{div}(\varphi(x)x), \tag{2.5}$$

where

$$\varphi(x) = \int_{\mathbb{R}^3} W(x-y)|u(y)|^2 \,\mathrm{d}y.$$

We can easily obtain that

$$\begin{aligned}\mathrm{div}(\varphi(x)x) &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(x_i \int_{\mathbb{R}^3} W(x-y)|u(y)|^2 \,\mathrm{d}y\right) \\ &= N\int_{\mathbb{R}^3} W(x-y)|u(y)|^2 \,\mathrm{d}y + \int_{\mathbb{R}^3} \langle\nabla W(x-y) \mid x\rangle|u(y)|^2 \,\mathrm{d}y.\end{aligned}$$

Summing up (2.3)–(2.5) and integrating by parts, we reach the identity

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \tfrac{3}{2}a\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x - \frac{3}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u(x)|^2|u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ - \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \langle\nabla W(x-y) \mid x\rangle|u(x)|^2|u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y = 0.\end{aligned} \tag{2.6}$$

By exchanging x with y, we find that

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3} \langle\nabla W(x-y) \mid x\rangle|u(x)|^2|u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\qquad = -\int_{\mathbb{R}^3\times\mathbb{R}^3} \langle\nabla W(x-y) \mid y\rangle|u(x)|^2|u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3} &\langle \nabla W(x-y) \mid x\rangle |u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \langle \nabla W(x-y) \mid x-y\rangle |u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

via Euler's identity for homogeneous functions. Plugging this into (2.6) yields (2.2). □

2.2. Orbital stability property

In this section, we consider the Schrödinger equation

$$\left.\begin{aligned} \mathrm{i}\frac{\partial u}{\partial t} + \Delta u + (W * |u|)^2 u = 0 \quad & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u_0(x) \quad & \text{in } \mathbb{R}^3, \end{aligned}\right\} \tag{2.7}$$

under assumption, coming from (W), that

$$\frac{C_1}{|x|} \leqslant W(x) \leqslant \frac{C_2}{|x|}, \tag{2.8}$$

for positive constants C_1 and C_2 (cf. (2.27)). This equation is also known as the Pekar–Choquard equation (see, for example, [10, 19, 21]). Consider the functionals

$$\mathcal{E}(u) = \tfrac{1}{2}\|\nabla u\|_{L^2}^2 - \tfrac{1}{4}\mathbb{D}(u), \qquad J(u) = \tfrac{1}{2}\|\nabla u\|_{L^2}^2 + \tfrac{1}{2}a\|u\|_{L^2}^2 - \tfrac{1}{4}\mathbb{D}(u),$$

where

$$\mathbb{D}(u) = \int_{\mathbb{R}^6} W(x-y)|u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y, \tag{2.9}$$

and let us set

$$\begin{aligned}\mathcal{M} &= \{u \in H^1(\mathbb{R}^3,\mathbb{C}) : \|u\|_{L^2}^2 = \rho\}, \\ \mathcal{N} &= \{u \in H^1(\mathbb{R}^3,\mathbb{C}) : u \neq 0 \text{ and } J'(u)(u) = 0\}\end{aligned}$$

for some positive number $\rho > 0$.

Definition 2.2. We denote by $\mathcal{G}$ the set of ground-state solutions of (2.1), i.e. solutions to the minimization problem

$$\Lambda = \min_{u\in\mathcal{N}} J(u). \tag{2.10}$$

Remark 2.3. By corollary 2.7 (see also the correspondence between critical points in the proof of lemma 2.6), the minimization problem in definition 2.2 is equivalent to a constrained minimization problem on a sphere of L^2 with a suitable radius ρ. For the latter problem, one can find a solution by following the arguments of [10, § 1], as minimizing sequences converge strongly in $H^1(\mathbb{R}^3)$. In particular, $\mathcal{G} \neq \emptyset$.

In lemma 2.6 we will prove that a ground-state solution of (2.1) can be obtained as a scaling of a solution to the minimization problem

$$\Lambda = \min_{u\in\mathcal{M}} \mathcal{E}(u), \tag{2.11}$$

which is quite a useful characterization for the stability issue. We now recall two global existence results for problem (2.7) (see, for example, [9, corollary 6.1.2]). We remark that (2.8) holds.

PROPOSITION 2.4. *Let $u_0 \in H^1(\mathbb{R}^3)$. Then problem (2.7) admits a unique global solution $u \in C^1([0,\infty), H^1(\mathbb{R}^3,\mathbb{C}))$. Moreover, the charge and the energy are conserved in time, namely,*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \qquad \mathcal{E}(u(t)) = \mathcal{E}(u_0) \tag{2.12}$$

for all $t \in [0,\infty)$.

DEFINITION 2.5. The set $\mathcal{G}$ of ground-state solutions of (2.1) is said to be orbitally stable for the Pekar–Choquard equation (2.7) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $u_0 \in H^1(\mathbb{R}^3,\mathbb{C})$,

$$\inf_{\phi\in\mathcal{G}} \|u_0 - \phi\|_{H^1} < \delta \quad \text{implies that} \ \sup_{t\geqslant 0} \inf_{\psi\in\mathcal{G}} \|u(t,\cdot) - \psi\|_{H^1} < \varepsilon,$$

where $u(t,\cdot)$ is the solution of (2.7) corresponding to the initial datum u_0.

Roughly speaking, the ground states are orbitally stable if any orbit starting from an initial datum u_0 close to $\mathcal{G}$ remains close to $\mathcal{G}$, uniformly in time.

In the classical orbital stability of Cazenave and Lions (see, for example, [10]) the ground-states set $\mathcal{G}$ is meant as the set of minima of the functional $\mathcal{E}$ constrained to a sphere of $L^2(\mathbb{R}^3)$. In this section we aim to show that orbital stability holds with respect to $\mathcal{G}$ as defined in definition 2.2.

Consider the following sets:

$$\begin{aligned} K_{\mathcal{N}} &= \{m \in \mathbb{R} : \text{there is } w \in \mathcal{N} \text{ with } J'(w) = 0 \text{ and } J(w) = m\}, \\ K_{\mathcal{M}} &= \{c \in \mathbb{R}^- : \text{there is } u \in \mathcal{M} \text{ with } \mathcal{E}'|_{\mathcal{M}}(u) = 0 \text{ and } \mathcal{E}(u) = c\}. \end{aligned}$$

In the next result, we establish the equivalence between minimization problems (2.10) and (2.11), namely, that a suitable scaling of a solution of the first problem corresponds to a solution of the second problem with a mapping between the critical values.

LEMMA 2.6. *The following minimization problems are equivalent*

$$\Lambda = \min_{u\in\mathcal{M}} \mathcal{E}(u), \qquad \Gamma = \min_{u\in\mathcal{N}} J(u) \tag{2.13}$$

for $\Lambda < 0$ and $\Lambda = \Psi(\Gamma)$, where $\Psi : K_{\mathcal{N}} \to K_{\mathcal{M}}$ is defined by

$$\Psi(m) = -\frac{1}{2}\left(\frac{3}{a\rho}\right)^{-3} m^{-2}, \quad m \in K_{\mathcal{N}}.$$

Proof. Observe that if $u \in \mathcal{M}$ is a critical point of $\mathcal{E}|_{\mathcal{M}}$ with $\mathcal{E}(u) = c < 0$, then there exists a Lagrange multiplier $\gamma \in \mathbb{R}$ such that $\mathcal{E}'(u)(u) = -\gamma\rho$, that is $\|\nabla u\|^2_{L^2} - \mathbb{D}(u) = -\gamma\rho$. By combining this identity with $\mathbb{D}(u) = 2\|\nabla u\|^2_{L^2} - 4c$, we obtain $-\|\nabla u\|^2_{L^2} + 4c = -\gamma\rho$, which implies that $\gamma > 0$. The equation satisfied by u is

$$-\Delta u + \gamma u = (W * |u|^2)u \quad \text{in } \mathbb{R}^3.$$

After trivial computations, one shows that the scaling

$$w(x) = T^\lambda u(x) := \lambda^2 u(\lambda x), \quad \lambda := \sqrt{\frac{a}{\gamma}} \tag{2.14}$$

is a solution of equation (2.1). On the contrary, if w is a non-trivial critical point of J, then, choosing

$$\lambda = \rho^{-1}\|w\|^2_{L^2}, \tag{2.15}$$

the function $u = T^{1/\lambda}w$ belongs to $\mathcal{M}$ and is a critical point of $\mathcal{E}|_{\mathcal{M}}$. Now, let m be the value of the free functional J on w, $m = J(w)$. Then

$$\begin{aligned} m &= \tfrac{1}{2}\|\nabla w\|^2_{L^2} + \tfrac{1}{2}a\|w\|^2_{L^2} - \tfrac{1}{4}\mathbb{D}(w) \\ &= \tfrac{1}{2}\lambda^3\|\nabla u\|^2_{L^2} + \tfrac{1}{2}a\lambda\|u\|^2_{L^2} - \tfrac{1}{4}\lambda^3\mathbb{D}(u) \\ &= \lambda^3\mathcal{E}(u) + \tfrac{1}{2}a\lambda\rho \\ &= c\lambda^3 + \tfrac{1}{2}a\lambda\rho. \end{aligned} \tag{2.16}$$

Observe that, since $\mathbb{D}(w) = \|\nabla w\|^2_{L^2} + a\|w\|^2_{L^2}$ and w satisfies the Pohozaev identity (2.2), we have the system

$$\begin{aligned} &\tfrac{1}{2}\|\nabla w\|^2_{L^2} + \tfrac{3}{2}a\|w\|^2_{L^2} = \tfrac{5}{4}(\|\nabla w\|^2_{L^2} + a\|w\|^2_{L^2}), \\ &\tfrac{1}{4}\|\nabla w\|^2_{L^2} + \tfrac{1}{4}a\|w\|^2_{L^2} = m, \end{aligned}$$

namely,

$$\begin{aligned} 3\|\nabla w\|^2_{L^2} - a\|w\|^2_{L^2} &= 0, \\ \|\nabla w\|^2_{L^2} + a\|w\|^2_{L^2} &= 4m, \end{aligned}$$

which, finally, entails

$$\|\nabla w\|^2_{L^2} = m, \qquad \|w\|^2_{L^2} = \frac{3}{a}m. \tag{2.17}$$

As a consequence, a simple rescaling yields the value of λ, that is,

$$\rho\lambda = \|w\|^2_{L^2} = \frac{3}{a}m.$$

Substituting this value of λ in formula (2.16), we obtain

$$m = c\left(\frac{3m}{a\rho}\right)^3 + \frac{3}{2}m,$$

namely,

$$-\tfrac{1}{2}m = c\left(\frac{3m}{a\rho}\right)^3.$$

In conclusion, we obtain

$$c = \Psi(m) \stackrel{\text{def}}{=} -\frac{1}{2}\left(\frac{3}{a\rho}\right)^{-3} m^{-2}, \tag{2.18}$$

where the function $\Psi : \mathbb{R}^+ \to \mathbb{R}^-$ is injective. Of course, formally $m = \Psi^{-1}(c)$, where

$$\Psi^{-1}(c) = \sqrt{-\frac{1}{2c}\left(\frac{3}{a\rho}\right)^{-3}},$$

which is injective. In order to prove that Ψ^{-1} is surjective, let m be a free critical value for J, namely, $m = J(w)$, with w a solution of equation (2.1). Then, if we consider $u = T^{1/\lambda}w(x) = \lambda^{-2}w(\lambda^{-1}x)$ with λ given by (2.15), it follows that $u \in \mathcal{M}$ is a critical point of $\mathcal{E}|_{\mathcal{M}}$ with Lagrange multiplier $\gamma = a\lambda^{-2}$. By using

$$\lambda = \left(\frac{a\rho}{3m}\right)^{-1},$$

in light of (2.17), we have

$$\begin{aligned} 4c &= \|\nabla u\|_{L^2}^2 - \gamma\rho \\ &= \lambda^{-3}\|\nabla w\|_{L^2}^2 - a\rho\lambda^{-2} \\ &= \left(\frac{a\rho}{3m}\right)^3 m - a\rho\left(\frac{3m}{a\rho}\right)^{-2}, \end{aligned}$$

which yields $m = \Psi^{-1}(c)$, after a few computations. We are now ready to prove the assertion. Note that, by formula (2.18), we have

$$\begin{aligned} \Lambda &= \min_{u\in\mathcal{M}} \mathcal{E}(u) = \min_{u\in\mathcal{M}} c_u \\ &= \min_{v\in\mathcal{N}} \Psi(m_v) = -\max_{v\in\mathcal{N}} \frac{1}{2}\left(\frac{3}{a\rho}\right)^{-3} m_v^{-2} \\ &= -\frac{1}{2}\left(\frac{3}{a\rho}\right)^{-3}\left(\min_{v\in\mathcal{N}} m_v\right)^{-2} \\ &= -\frac{1}{2}\left(\frac{3}{a\rho}\right)^{-3}\Gamma^{-2} \\ &= \Psi(\Gamma). \end{aligned}$$

If $\hat{u} \in \mathcal{M}$ is a minimizer for Λ, that is, $\Lambda = \mathcal{E}(\hat{u}) = \min_{\mathcal{M}} \mathcal{E}$, the function $\hat{w} = T^{\lambda}\hat{u}$ is a critical point of J with $J(\hat{w}) = \Psi^{-1}(\Lambda) = \Gamma$, so that w is a minimizer for Γ, that is $J(w) = \min_{\mathcal{N}} J$. This concludes the proof. □

Corollary 2.7. *Any ground-state solution u to equation (2.1) satisfies*

$$\|u\|_{L^2}^2 = \rho, \quad \rho = \frac{3\Gamma}{a}, \tag{2.19}$$

where Γ is defined in (2.13). Moreover, for this precise value of the radius ρ we have

$$\min_{u\in\mathcal{M}} J(u) = \min_{u\in\mathcal{N}} J(u), \tag{2.20}$$

where $\mathcal{M} = \mathcal{M}_\rho$.

Proof. The first conclusion is an immediate consequence of the previous proof. Let us now prove that the second conclusion holds, with ρ as in (2.19). We have

$$\begin{aligned}\min_{u\in\mathcal{M}} J(u) &= \min_{u\in\mathcal{M}} \mathcal{E}(u) + \tfrac{1}{2}a\|u\|_{L^2}^2 \\ &= \Lambda + \frac{a\rho}{2} \\ &= -\frac{1}{2}\left(\frac{3}{a\rho}\right)^{-3}\Gamma^{-2} + \frac{a\rho}{2} \\ &= \Gamma = \min_{u\in\mathcal{N}} J(u),\end{aligned}$$

by the definition of ρ. □

The following theorem is the main result of this section.

Theorem 2.8. *The set $\mathcal{G}$ of the ground-state solutions to (2.1) is orbitally stable for (2.7).*

Proof. Assume by contradiction that the assertion is false. Then we can find $\varepsilon_0 > 0$, a sequence of times $(t_n) \subset (0,\infty)$ and of initial data $(u_0^n) \subset H^1(\mathbb{R}^3, \mathbb{C})$ such that

$$\lim_{n\to\infty} \inf_{\phi\in\mathcal{G}} \|u_0^n - \phi\|_{H^1} = 0 \quad \text{and} \quad \inf_{\psi\in\mathcal{G}} \|u_n(t_n,\cdot) - \psi\|_{H^1} \geqslant \varepsilon_0, \tag{2.21}$$

where $u_n(t,\cdot)$ is the solution of (2.7) corresponding to the initial datum u_0^n. Taking into account (2.19) and (2.20), for any $\phi \in \mathcal{G}$, we have

$$\|\phi\|_{L^2}^2 = \rho_0, \qquad J(\phi) = \min_{u\in\mathcal{M}_{\rho_0}} J(u), \quad \rho_0 = \frac{3\Gamma}{a}.$$

Therefore, considering the sequence $\Upsilon_n(x) := u_n(t_n, x)$ bounded in $H^1(\mathbb{R}^3, \mathbb{C})$, and recalling the conservation of charge, as $n \to \infty$, from (2.21), it follows that

$$\|\Upsilon_n\|_{L^2}^2 = \|u_n(t_n,\cdot)\|_{L^2}^2 = \|u_0^n\|_{L^2}^2 = \rho_0 + o(1).$$

Hence, there exists a sequence $(\omega_n) \subset \mathbb{R}^+$ with $\omega_n \to 1$ as $n \to \infty$ such that

$$\|\omega_n\Upsilon_n\|_{L^2}^2 = \rho_0 \quad \text{for all } n \geqslant 1. \tag{2.22}$$

Moreover, by the conservation of energy (2.12) and the continuity of $\mathcal{E}$, as $n \to \infty$,

$$\begin{aligned} J(\omega_n \Upsilon_n) &= \mathcal{E}(\omega_n \Upsilon_n) + \tfrac{1}{2}a\|\omega_n \Upsilon_n\|_{L^2}^2 \\ &= \mathcal{E}(\Upsilon_n) + \tfrac{1}{2}a\|\Upsilon_n\|_{L^2}^2 + o(1) \\ &= \mathcal{E}(u_n(t_n,\cdot)) + \tfrac{1}{2}a\|u_0^n\|_{L^2}^2 + o(1) \\ &= \mathcal{E}(u_0^n) + \tfrac{1}{2}a\|u_0^n\|_{L^2}^2 + o(1) \\ &= J(u_0^n) + o(1) \\ &= \min_{u \in \mathcal{M}_{\rho_0}} J(u) + o(1). \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23), it follows that $(\omega_n \Upsilon_n) \subset H^1(\mathbb{R}^3, \mathbb{C})$ is a minimizing sequence for the functional J (and also for $\mathcal{E}$) over $\mathcal{M}_{\rho_0}$. By taking into account the homogeneity property of W, following the arguments of [10, § 1], we deduce that, up to a subsequence, $(\omega_n \Upsilon_n)$ converges to some function Υ_0, which thus belongs to the set $\mathcal{G}$, since, by (2.22), (2.23) and equality (2.20),

$$J(\Upsilon_0) = \min_{u \in \mathcal{M}_{\rho_0}} J(u) = \min_{u \in \mathcal{N}} J(u).$$

Evidently, this is a contradiction with (2.21), as we would have

$$\varepsilon_0 \leqslant \lim_{n\to\infty} \inf_{\psi \in \mathcal{G}} \|\Upsilon_n - \psi\|_{H^1} \leqslant \lim_{n\to\infty} \|\Upsilon_n - \Upsilon_0\|_{H^1} = 0.$$

This concludes the proof. □

In the particular case where $W(x) = |x|^{-1}$, due to the uniqueness of ground states up to translations and phase changes [24], theorem 2.8 strengthens as follows.

Corollary 2.9. *Assume that w is the unique real ground state of*

$$-\Delta w + aw = (|x|^{-1} * w^2)w \quad \text{in } \mathbb{R}^3.$$

Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$u_0 \in H^1(\mathbb{R}^3, \mathbb{C}) \quad \text{and} \quad \inf_{\substack{y \in \mathbb{R}^3 \\ \theta \in [0,2\pi)}} \|u_0 - \mathrm{e}^{\mathrm{i}\theta} w(\cdot - y)\|_{H^1} < \delta$$

implies that

$$\sup_{t \geqslant 0} \inf_{\substack{y \in \mathbb{R}^3 \\ \theta \in [0,2\pi)}} \|u(t,\cdot) - \mathrm{e}^{\mathrm{i}\theta} w(\cdot - y)\|_{H^1} < \varepsilon.$$

2.3. Structure of least-energy solutions

We can now state the following.

Lemma 2.10. *Any complex ground-state solution u to (2.1) has the form*

$$u(x) = \mathrm{e}^{\mathrm{i}\theta}|u(x)| \quad \text{for some } \theta \in [0, 2\pi).$$

Proof. In view of lemma 2.6 (see also corollary 2.7), searching for ground-state solutions of (2.1) is equivalent to considering the constrained minimization problem $\min_{u\in\mathcal{M}_\rho}\mathcal{E}(u)$ for a suitable value of $\rho>0$. Then the proof is quite standard; we include a proof here for the sake of self-containedness. Consider

$$\sigma_{\mathbb{C}}=\inf\{\mathcal{E}(u):u\in H^1(\mathbb{R}^3,\mathbb{C}),\ \|u\|_{L^2}^2=\rho\},$$
$$\sigma_{\mathbb{R}}=\inf\{\mathcal{E}(u):u\in H^1(\mathbb{R}^3,\mathbb{R}),\ \|u\|_{L^2}^2=\rho\}.$$

It holds that $\sigma_{\mathbb{C}}=\sigma_{\mathbb{R}}$. Indeed, trivially one has $\sigma_{\mathbb{C}}\leqslant\sigma_{\mathbb{R}}$. Moreover, if $u\in H^1(\mathbb{R}^3,\mathbb{C})$, due to the well-known inequality $|\nabla|u(x)||\leqslant|\nabla u(x)|$ for almost every $x\in\mathbb{R}^3$, it holds that

$$\int|\nabla|u(x)||^2\,\mathrm{d}x\leqslant\int|\nabla u(x)|^2\,\mathrm{d}x,$$

so that $\mathcal{E}(|u|)\leqslant\mathcal{E}(u)$. In particular, $\sigma_{\mathbb{R}}\leqslant\sigma_{\mathbb{C}}$, yielding $\sigma_{\mathbb{C}}=\sigma_{\mathbb{R}}$. Now let u be a solution to $\sigma_{\mathbb{C}}$ and assume by contradiction that $\mu(\{x\in\mathbb{R}^3:|\nabla|u|(x)|<|\nabla u(x)|\})>0$, where μ denotes the Lebesgue measure in $\mathbb{R}^3$. Then $\||u|\|_{L^2}=\|u\|_{L^2}=1$ and

$$\sigma_{\mathbb{R}}\leqslant\frac{1}{2}\int|\nabla|u||^2\,\mathrm{d}x-\tfrac{1}{4}\mathbb{D}(|u|)<\frac{1}{2}\int|\nabla u|^2\,\mathrm{d}x-\tfrac{1}{4}\mathbb{D}(u)=\sigma_{\mathbb{C}},$$

which is a contradiction, being $\sigma_{\mathbb{C}}=\sigma_{\mathbb{R}}$. Hence, we have $|\nabla|u(x)||=|\nabla u(x)|$ for almost every $x\in\mathbb{R}^3$. This is true if and only if $\operatorname{Re}u\nabla(\operatorname{Im}u)=\operatorname{Im}u\nabla(\operatorname{Re}u)$. In turn, if this last condition holds, we obtain

$$\bar{u}\nabla u=\operatorname{Re}u\nabla(\operatorname{Re}u)+\operatorname{Im}u\nabla(\operatorname{Im}u)\quad\text{almost everywhere in }\mathbb{R}^3,$$

which implies that $\operatorname{Re}(\mathrm{i}\bar{u}(x)\nabla u(x))=0$ almost everywhere in $\mathbb{R}^3$. From the last identity we find $\theta\in[0,2\pi)$ such that $u=\mathrm{e}^{\mathrm{i}\theta}|u|$, concluding the proof. □

We then get the following least-energy levels result for the limiting problem (2.1).

Corollary 2.11. *Consider the two problems*

$$-\Delta u+au=(W*|u|^2)u,\quad u\in H^1(\mathbb{R}^3,\mathbb{R}),\tag{2.24}$$
$$-\Delta u+au=(W*|u|^2)u,\quad u\in H^1(\mathbb{R}^3,\mathbb{C}).\tag{2.25}$$

Let E_a and E_a^c denote their least-energy levels. Then

$$E_a=E_a^c.\tag{2.26}$$

Moreover, any least-energy solution of (2.24) has the form $\mathrm{e}^{\mathrm{i}\tau}U$, where U is a positive least-energy solution of (2.25) and $\tau\in\mathbb{R}$.

2.4. Compactness of the ground states set

In light of assumption (W), there exist two positive constants C_1 and C_2 such that

$$\frac{C_1}{|x|}\leqslant W(x)\leqslant\frac{C_2}{|x|}\quad\text{for all }x\in\mathbb{R}^3\setminus\{0\}.\tag{2.27}$$

We recall the following two Hardy–Littlewood–Sobolev-type inequalities (see, for example, [27, theorem 1] and [20, theorem 4.3]) in $\mathbb{R}^3$:

$$\text{for all } u \in L^{6q/(3+2q)}(\mathbb{R}^3), \qquad \||x|^{-1} * u^2\|_{L^q} \leqslant C\|u\|^2_{L^{6q/(3+2q)}}, \tag{2.28}$$

$$\text{for all } u \in L^{12/5}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^6} |x-y|^{-1} u^2(y) u^2(x)\,\mathrm{d}y\,\mathrm{d}x \leqslant C\|u\|^4_{L^{12/5}}. \tag{2.29}$$

We have the following lemma.

Lemma 2.12. *There exists a positive constant C such that, for all $u \in H^1(\mathbb{R}^3)$,*

$$\mathbb{D}(u) \leqslant C\|u\|^3_{L^2}\|u\|_{H^1}.$$

Proof. By combining (2.27), (2.29) and the Gagliardo–Nirenberg inequality, we obtain

$$\mathbb{D}(u) \leqslant C_2 \int_{\mathbb{R}^6} |x-y|^{-1} u^2(x) u^2(y)\,\mathrm{d}x\,\mathrm{d}y \leqslant C\|u\|^4_{L^{12/5}} \leqslant C\|u\|^3_{L^2}\|u\|_{H^1},$$

which proves the assertion. □

More generally, we recall the following facts from [23, § 2].

Lemma 2.13. *Assume that $K \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $s \geqslant \frac{3}{2}$. Then there exists a constant $C > 0$ such that, for any $u, v \in H^1(\mathbb{R}^3)$,*

$$\|K * (uv)\|_{L^\infty} \leqslant C\|u\|_{H^1}\|v\|_{H^1}. \tag{2.30}$$

Moreover, assume that $K \in L^{3-\varepsilon}(\mathbb{R}^3) + L^{3+\varepsilon}(\mathbb{R}^3)$ for some sufficiently small $\varepsilon > 0$. Then

$$\lim_{|x|\to\infty} (K * u^2)(x) = 0 \tag{2.31}$$

for any $u \in H^1(\mathbb{R}^3)$.

Proof. Write $K = K_1 + K_2$ with $K_1 \in L^\infty(\mathbb{R}^3)$ and $K_2 \in L^s(\mathbb{R}^3)$, $s \geqslant \frac{3}{2}$. Then, by Sobolev's embedding theorems, $H^1(\mathbb{R}^3) \subset L^{2s/(s-1)}(\mathbb{R}^3)$ and hence, for some $C > 0$,

$$\begin{aligned}
\|K * (uv)\|_{L^\infty} &= \|(K_1 + K_2) * (uv)\|_{L^\infty} \\
&\leqslant \|K_1\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} + \|K_2\|_{L^s}\|uv\|_{L^{s/(s-1)}} \\
&\leqslant \|K_1\|_{L^\infty}\|u\|_{H^1}\|v\|_{H^1} + \|K_2\|_{L^s}\|u\|_{L^{2s/(s-1)}}\|v\|_{L^{2s/(s-1)}} \\
&\leqslant C\|u\|_{H^1}\|v\|_{H^1}.
\end{aligned}$$

Concerning the second part of the statement,

$$|K * u^2(x)| \leqslant |K_1 * u^2(x)| + |K_2 * u^2(x)|$$

for every $x \in \mathbb{R}^3$ and, for sufficiently small $\varepsilon > 0$, by Sobolev embedding

$$u^2 \in L^{(3-\varepsilon)'}(\mathbb{R}^3) \cap L^{(3+\varepsilon)'}(\mathbb{R}^3),$$

where $(3-\varepsilon)'$ and $(3+\varepsilon)'$ are the conjugate exponents of $3-\varepsilon$ and $3+\varepsilon$, respectively. Then the assertion follows directly from [20, lemma 2.20]. □

Let $\mathcal{S}_a$ denote the set of (complex) least-energy solutions u to equation (2.1) such that

$$|u(0)| = \max_{x\in\mathbb{R}^3} |u(x)|.$$

By lemma 2.10, up to a constant phase change, we can assume that u is real valued. Moreover, $\mathcal{S}_a \neq \emptyset$ (see remark 2.3).

PROPOSITION 2.14. *For any $a > 0$ the set $\mathcal{S}_a$ is compact in $H^1(\mathbb{R}^3, \mathbb{R})$ and there exist positive constants C, σ such that $u(x) \leqslant C\exp\{-\sigma|x|\}$ for any $x \in \mathbb{R}^3$ and all $u \in \mathcal{S}_a$.*

Proof. If $L_a : H^1(\mathbb{R}^3) \to \mathbb{R}$ denotes the functional associated with (2.1),

$$L_a(u) = L(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \tfrac{1}{2}a\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x - \tfrac{1}{4}\mathbb{D}(u),$$

since $\mathbb{D}(u) = \|\nabla u\|_{L^2}^2 + a\|u\|_{L^2}^2$ for all $u \in \mathcal{S}_a$, we have

$$m_a = L_a(u) = \tfrac{1}{4}(\|\nabla u\|_{L^2}^2 + a\|u\|_{L^2}^2),$$

where $m_a = \min\{L_a(u) : u \neq 0 \text{ solves } (2.1)\}$. Hence, it follows that the set $\mathcal{S}_a$ is bounded in $H^1(\mathbb{R}^3)$. Moreover, $\mathcal{S}_a$ is also bounded in $L^\infty(\mathbb{R}^3)$ and $u(x) \to 0$ as $|x| \to \infty$ for any $u \in \mathcal{S}_a$. Indeed, from the Hardy–Littlewood–Sobolev inequality (2.28), for any $q \geqslant 3$,

$$\|W * u^2\|_{L^q} \leqslant C\||x|^{-1} * u^2\|_{L^q} \leqslant C\|u\|_{L^{6q/(3+2q)}}^2 \leqslant C\|u\|_{L^6}^2 \leqslant C, \tag{2.32}$$

so that $|x|^{-1} * u^2 \in L^q(\mathbb{R}^3)$, for any $q \geqslant 3$. Setting

$$f(x) = (W * u^2)(x)u(x) - au(x),$$

for all m with $2 \leqslant m < 6$, by the Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}\|f\|_{L^m} &\leqslant C\|u\|_{L^m} + C\|(|x|^{-1} * u^2)u\|_{L^m} \\ &\leqslant C\|u\|_{L^m} + C\||x|^{-1} * u^2\|_{L^{6m/(6-m)}}\|u\|_{L^6} \\ &\leqslant C.\end{aligned}$$

By virtue of (2.31), we have $(W * u^2)(x) \to 0$ as $|x| \to \infty$, and thus standard arguments show that u is exponentially decaying to zero at infinity (see also the argument just before (2.36)), which readily implies $f(x) \in L^1(\mathbb{R}^3)$. From equation (2.1), namely $-\Delta u = f$, by the Calderón–Zygmund estimate (see [17, theorem 9.9, corollary 9.10 and preceding lines]; note also that $f \in L^1(\mathbb{R}^3)$ so that, by means of [2, lemma A5], it holds that $u = G * f$, where G is the fundamental solution of $-\Delta$ on $\mathbb{R}^N$) and it follows that $u \in W^{2,m}(\mathbb{R}^3)$, for every $2 \leqslant m < 6$, with $\|u\|_{W^{2,m}}$ uniformly bounded in $\mathcal{S}_a$. Hence, it follows that u is a bounded function which vanishes at infinity and $\mathcal{S}_a$ is uniformly bounded in $C^{1,\alpha}(\mathbb{R}^3)$ (take $3 < m < 6$ to obtain this embedding). Actually, u has further regularity as, again using the equation, the boundedness of u, the Calderón–Zygmund estimate and the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}\|u\|_{W^{2,m}} &\leqslant C\|f\|_{L^m}\\ &\leqslant C\|u\|_{L^m}+C\|(|x|^{-1}*u^2)u\|_{L^m}\\ &\leqslant C+C\||x|^{-1}*u^2\|_{L^m}\\ &\leqslant C,\end{aligned}$$

so that $u\in W^{2,m}(\mathbb{R}^3)$ for every $m\geqslant 2$. Let us now show that the limit $u(x)\to 0$ as $|x|\to\infty$ holds uniformly for $u\in\mathcal{S}_a$. Assuming by contradiction that $u_m(x_m)\geqslant\sigma>0$ along some sequences $(u_m)\subset\mathcal{S}_a$ and $(x_m)\subset\mathbb{R}^3$ with $|x_m|\to\infty$, shifting u_m as $v_m(x)=u_m(x+x_m)$, it follows that (v_m) is bounded in $H^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ and it converges, up to a subsequence, to a function v, weakly in $H^1(\mathbb{R}^3)$ and locally uniformly in $C(\mathbb{R}^3)$. If u denotes the weak limit of u_m, we also claim that u and v are both solutions to equation (2.1), which are non-trivial as follows from (local) uniform convergence and $u_m(0)\geqslant u_m(x_m)\geqslant\sigma$ (since 0 is a global maximum for u_m) and $v_m(0)=u_m(x_m)\geqslant\sigma$. To see that u and v are solutions to (2.1), set

$$\varphi_m(x)=\int_{\mathbb{R}^3}W(x-y)u_m^2(y)\,\mathrm{d}y,\qquad \varphi(x)=\int_{\mathbb{R}^3}W(x-y)u^2(y)\,\mathrm{d}y.$$

Let us show that $\varphi_m(x)\to\varphi(x)$, as $m\to\infty$, for any fixed $x\in\mathbb{R}^3$. Indeed, we can write $\varphi_m(x)-\varphi(x)=I_m^1(\rho)+I_m^2(\rho)$ for any $m\geqslant 1$ and any $\rho>0$, where we set

$$\begin{aligned}I_m^1(\rho)&=\int_{B_\rho(0)}W(x-y)(u_m^2(y)-u^2(y))\,\mathrm{d}y,\\ I_m^2(\rho)&=\int_{\mathbb{R}^3\setminus B_\rho(0)}W(x-y)(u_m^2(y)-u^2(y))\,\mathrm{d}y.\end{aligned}$$

Fix $x\in\mathbb{R}^3$ and let $\varepsilon>0$. Choose $\rho_0>0$ sufficiently large that

$$I_m^2(\rho_0)\leqslant\int_{\mathbb{R}^3\setminus B_{\rho_0}(0)}\frac{C}{|y|-|x|}|u_m^2(y)-u^2(y)|\,\mathrm{d}y\leqslant\frac{C}{\rho_0-|x|}<\frac{\varepsilon}{2}.\tag{2.33}$$

On the other hand, by the uniform local convergence of u_m to u as $m\to\infty$ and the Hölder inequality, for some $1<r<3$,

$$\begin{aligned}I_m^1(\rho_0)&\leqslant C\|u_m-u\|_{L^{r'}(B_{\rho_0}(0))}\bigg(\int_{B_{\rho_0}(0)}\frac{1}{|x-y|^r}|u_m(y)+u(y)|^r dy\bigg)^{1/r}\\ &\leqslant C\|u_m-u\|_{L^{r'}(B_{\rho_0}(0))}\bigg(\int_{B_{\rho_0}(x)}\frac{1}{|y|^r}\,\mathrm{d}y\bigg)^{1/r}\\ &\leqslant C\|u_m-u\|_{L^{r'}(B_{\rho_0}(0))}\\ &<\tfrac{1}{2}\varepsilon\end{aligned}\tag{2.34}$$

for all sufficiently large m, where r' denotes the conjugate exponent of r. The bound $r<3$ ensures that the singular integral which appears in the second inequality is finite. Combining (2.33) with (2.34) concludes the proof of the pointwise convergence of φ_m to φ. For all $\eta \in C_0^\infty(\mathbb{R}^3)$ and any measurable set E, we observe that, for any $q \in [2,6)$ and $p^{-1} + q^{-1} = 1$,

$$\begin{aligned}
&\left| \int_E \varphi_m(x) u_m(x) \eta(x)\,\mathrm{d}x \right| \\
&\qquad \leqslant \left(\int_E |\eta(x)|^p\,\mathrm{d}x \right)^{1/p} \left(\int_E |\varphi_m(x) u_m(x)|^q\,\mathrm{d}x \right)^{1/q} \\
&\qquad \leqslant \|\eta\|_{L^p(E)} \left(\int_{\mathbb{R}^3} |u_m(x)|^6\,\mathrm{d}x \right)^{1/6} \left(\int_{\mathbb{R}^3} |\varphi_m(x)|^{6q/(6-q)}\,\mathrm{d}x \right)^{(6-q)/6q},
\end{aligned}$$

and we observe that $6q/(6-q) \geqslant 3$ since $q \geqslant 2$. Since we already know that $\{\varphi_m\}$ is bounded in any L^r with $r \geqslant 3$, we conclude that, for some constant $C > 0$,

$$\left| \int_E \varphi_m(x) u_m(x) \eta(x)\,\mathrm{d}x \right| \leqslant C\|\eta\|_{L^p(E)},$$

and the last term can be made arbitrarily small by taking E of small measure. Since the support of η is a compact set and $\varphi_m u_m \eta \to \varphi u \eta$ almost everywhere, the Vitali convergence theorem implies

$$\lim_{m\to+\infty} \int_K \varphi_m u_m \eta\,\mathrm{d}x = \int_K \varphi u \eta\,\mathrm{d}x$$

for all $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R}^3)$ with compact support K. This concludes the proof that u and v are non-trivial solutions to (2.1). It follows that, for any m and k,

$$\begin{aligned}
J_a(u_m) = J_a(u_k) &= \frac{1}{4}\int_{\mathbb{R}^3} (|\nabla u_m|^2 + a u_m^2)\,\mathrm{d}x, \\
J_a(u) &\geqslant J_a(z) = m_a, \\
J_a(v) &\geqslant J_a(z) = m_a,
\end{aligned}$$

for all $z \in \mathcal{S}_a$. On the other hand, for any $R > 0$ and $m \geqslant 1$ with $2R \leqslant |x_m|$,

$$\begin{aligned}
m_a &= J_a(u_m) \\
&\geqslant \tfrac{1}{4} \liminf_m \int_{B_R(0)} (|\nabla u_m|^2 + a u_m^2)\,\mathrm{d}x \\
&\qquad + \tfrac{1}{4} \liminf_m \int_{B_R(0)} (|\nabla v_m|^2 + a v_m^2)\,\mathrm{d}x \\
&\geqslant J_a(u) + J_a(v) - \varepsilon \\
&\geqslant 2m_a - o(1)
\end{aligned}$$

as $R \to \infty$, which yields a contradiction for sufficiently large R. Hence, the conclusion follows. Let us now prove that

$$\lim_{|x|\to\infty} \varphi(x) = 0, \qquad \varphi(x) = \int_{\mathbb{R}^3} W(x-y) u^2(y)\,\mathrm{d}y. \tag{2.35}$$

Note that, for small $\varepsilon > 0$, $|x|^{-1}$ can be written as the sum of

$$|x|^{-1}\chi_{B(0,1)} \in L^{3-\varepsilon}(\mathbb{R}^3) \quad \text{and} \quad |x|^{-1}\chi_{\mathbb{R}^3\setminus B(0,1)} \in L^{3+\varepsilon}(\mathbb{R}^3).$$

In particular, (2.31) of lemma 2.13 is fulfilled for $u \in H^1(\mathbb{R}^3)$. Then, since $W \leqslant C|x|^{-1}$, the quantity $\sup_{|x|\geqslant R}|W * u^2|$ can be made arbitrarily small by choosing sufficently large R. In light of (2.35), let $R_a > 0$ such that $\varphi(x) \leqslant \frac{1}{2}a$ for any $|x| \geqslant R_a$. As a consequence,

$$-\Delta u(x) + \tfrac{1}{2}au(x) \leqslant 0 \quad \text{for } |x| \geqslant R_a. \tag{2.36}$$

It is thus standard to see that this yields the exponential decay of u, with uniform decay constants in $\mathcal{S}_a$. We can finally conclude the proof. Let (u_n) be any sequence in $\mathcal{S}_a$. Up to a subsequence, it follows that (u_n) converges weakly to a function u, which is also a solution to equation (2.1). If $\mathbb{D}$ is the function defined in (2.9), we immediately obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2 + au_n^2 - \mathbb{D}(u_n) = 0 = \int_{\mathbb{R}^3}|\nabla u|^2 + au^2 - \mathbb{D}(u). \tag{2.37}$$

Hence, the desired strong convergence of (u_n) to u in $H^1(\mathbb{R}^3)$ follows once we prove that $\mathbb{D}(u_n) \to \mathbb{D}(u)$ as $n \to \infty$. In view of the uniform exponential decay of u_n, it follows that $u_n \to u$ strongly in $L^{12/5}(\mathbb{R}^3)$ as $n \to \infty$. Taking the observation that (u_n) is bounded in $H^1(\mathbb{R}^3)$ and that W is even into account, we obtain the inequality

$$|\mathbb{D}(u_n) - \mathbb{D}(u)| \leqslant \sqrt{\tilde{\mathbb{D}}(||u_n|^2 - |u|^2|^{1/2})}\sqrt{\tilde{\mathbb{D}}((|u_n|^2 + |u|^2)^{1/2})}, \quad n \in \mathbb{N},$$

$\tilde{\mathbb{D}}$ defined as the operator $\mathbb{D}$ with the Coulomb kernel $|x|^{-1}$ being in place of $\mathbb{D}$. In fact, taking [20, theorem 9.8] into account, and since W is even, we have

$$\begin{aligned}
&|\mathbb{D}(u_n) - \mathbb{D}(u)| \\
&\quad = \left|\int_{\mathbb{R}^6} W(x-y)|u_n(x)|^2|u_n(y)|^2\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^6} W(x-y)|u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\quad = \left|\int_{\mathbb{R}^6} W(x-y)|u_n(x)|^2|u_n(y)|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^6} W(x-y)|u_n(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y\right. \\
&\qquad \left. - \int_{\mathbb{R}^6} W(x-y)|u(x)|^2|u_n(y)|^2\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^6} W(x-y)|u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\quad = \left|\int_{\mathbb{R}^6} W(x-y)(|u_n(x)|^2 - |u(x)|^2)(|u_n(y)|^2 + |u(y)|^2)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\quad \leqslant \int_{\mathbb{R}^6} W(x-y)||u_n(x)|^2 - |u(x)|^2||\,|u_n(y)|^2 + |u(y)|^2|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant C\int_{\mathbb{R}^6} |x-y|^{-1}||u_n(x)|^2 - |u(x)|^2||\,|u_n(y)|^2 + |u(y)|^2|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant C\sqrt{\tilde{\mathbb{D}}(||u_n|^2 - |u|^2|^{1/2})}\sqrt{\tilde{\mathbb{D}}((|u_n|^2 + |u|^2)^{1/2})}.
\end{aligned}$$

Then, by the Hardy–Littlewood–Sobolev inequality and the Hölder's inequality, it follows that

$$|\tilde{\mathbb{D}}(u_n)-\tilde{\mathbb{D}}(u)|^2 \leqslant C\||\,|u_n|^2-|u|^2|^{1/2}\|^4_{L^{12/5}}\|(|u_n|^2+|u|^2)^{1/2}\|^4_{L^{12/5}} \leqslant C\|u_n-u\|^2_{L^{12/5}}.$$

As a consequence,

$$\mathbb{D}(u_n) = \mathbb{D}(u) + o(1) \quad \text{as } n \to \infty,$$

which concludes the proof in light of formula (2.37). □

3. The penalization argument

Throughout this and the following sections we mainly use the arguments of [12], highlighting the technical steps where the Hartree nonlinearity is involved in place of the local one. For the sake of self-containedness and for the reader's convenience we develop the arguments with all the detail.

For any set $\Omega \subset \mathbb{R}^3$ and $\varepsilon > 0$, let $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \varepsilon x \in \Omega\}$.

3.1. Notation and framework

The following lemmas, taken from [12], show that the norm in H_ε is locally equivalent to the standard H^1 norm.

Lemma 3.1. *Let $K \subset \mathbb{R}^3$ be an arbitrary, fixed, bounded domain. Assume that A is bounded on K and that $0 < \alpha \leqslant V \leqslant \beta$ on K for some α, $\beta > 0$. Then, for any fixed $\varepsilon \in [0, 1]$, the norm*

$$\|u\|^2_{K_\varepsilon} = \int_{K_\varepsilon} \left|\left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon(y)\right)u\right|^2 + V_\varepsilon(y)|u|^2 \,\mathrm{d}y$$

is equivalent to the usual norm on $H^1(K_\varepsilon, \mathbb{C})$. Moreover, these equivalences are uniform; namely, there exist constants $c_1, c_2 > 0$ independent of $\varepsilon \in [0, 1]$ such that

$$c_1\|u\|_{K_\varepsilon} \leqslant \|u\|_{H^1(K_\varepsilon,\mathbb{C})} \leqslant c_2\|u\|_{K_\varepsilon}.$$

Corollary 3.2. *Retain the setting of lemma 3.1. Then the following facts hold.*

(i) *If K is compact, for any $\varepsilon \in (0, 1]$, the norm*

$$\|u\|^2_K := \int_K \left|\left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon(y)\right)u\right|^2 + V_\varepsilon(y)|u|^2 \,\mathrm{d}y$$

is uniformly equivalent to the usual norm on $H^1(K, \mathbb{C})$.

(ii) *For $A_0 \in \mathbb{R}^3$ and fixed $b > 0$, the norm*

$$\|u\|^2 := \int_{\mathbb{R}^3} \left|\left(\frac{1}{\mathrm{i}}\nabla - A_0\right)u\right|^2 + b|u|^2 \,\mathrm{d}y$$

is equivalent to the usual norm on $H^1(\mathbb{R}^3, \mathbb{C})$.

(iii) *If $(u_{\varepsilon_n}) \subset H^1(\mathbb{R}^3, \mathbb{C})$ satisfies $u_{\varepsilon_n} = 0$ on $\mathbb{R}^3 \setminus K_{\varepsilon_n}$ for any $n \in \mathbb{N}$ and $u_{\varepsilon_n} \to u$ in $H^1(\mathbb{R}^3, \mathbb{C})$, then $\|u_{\varepsilon_n} - u\|_{\varepsilon_n} \to 0$ as $n \to \infty$.*

For future reference, we recall the following *diamagnetic inequality*: for every $u \in H_\varepsilon$,

$$\left|\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon\right)u\right| \geqslant |\nabla|u|| \quad \text{almost everywhere in } \mathbb{R}^3. \tag{3.1}$$

See [15] for a proof. As a consequence of (3.1), $|u| \in H^1(\mathbb{R}^3, \mathbb{R})$ for any $u \in H_\varepsilon$.

For any $u \in H_\varepsilon$, let us set

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} |D^\varepsilon u|^2 + V_\varepsilon(x)|u|^2 \,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u(x)|^2|u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{3.2}$$

where we set $D^\varepsilon = (\nabla/\mathrm{i} - A_\varepsilon)$. Define, for all $\varepsilon > 0$,

$$\chi_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in O_\varepsilon, \\ \varepsilon^{-6/\mu} & \text{if } y \notin O_\varepsilon, \end{cases} \qquad \chi_\varepsilon^i(y) = \begin{cases} 0 & \text{if } y \in (O^i)_\varepsilon, \\ \varepsilon^{-6/\mu} & \text{if } y \notin (O^i)_\varepsilon, \end{cases}$$

and

$$Q_\varepsilon(u) = \left(\int_{\mathbb{R}^3} \chi_\varepsilon|u|^2 \,\mathrm{d}x - 1\right)_+^{5/2}, \qquad Q_\varepsilon^i(u) = \left(\int_{\mathbb{R}^3} \chi_\varepsilon^i|u|^2 \,\mathrm{d}x - 1\right)_+^{5/2}. \tag{3.3}$$

The functional Q_ε will act as a penalization to force the concentration phenomena of the solution to occur inside O. In particular, we remark that the penalization terms vanish on elements whose corresponding L^∞-norm is sufficiently small. This device was first introduced in [6]. Finally, we define the functionals $\Gamma_\varepsilon, \Gamma_\varepsilon^1, \dots, \Gamma_\varepsilon^k : H_\varepsilon \to \mathbb{R}$ by setting

$$\Gamma_\varepsilon(u) = \mathcal{F}_\varepsilon(u) + Q_\varepsilon(u), \qquad \Gamma_\varepsilon^i(u) = \mathcal{F}_\varepsilon(u) + Q_\varepsilon^i(u), \quad i = 1, \dots, k. \tag{3.4}$$

It is easy to check, under our assumptions and using the diamagnetic inequality (3.1), that the functionals Γ_ε and Γ_ε^i are of class C^1 over H_ε. Hence, a critical point of $\mathcal{F}_\varepsilon$ corresponds to a solution of (1.5). To find solutions of (1.5) which *concentrate* in O as $\varepsilon \to 0$, we shall look for a critical point of Γ_ε for which Q_ε is zero.

Let

$$\mathcal{M} = \bigcup_{i=1}^k \mathcal{M}^i, \qquad O = \bigcup_{i=1}^k O^i$$

and, for any set $B \subset \mathbb{R}^3$ and $\alpha > 0$, $B^\delta = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, B) \leqslant \delta\}$, set

$$\delta = \tfrac{1}{10}\min\Big\{\operatorname{dist}(\mathcal{M}, \mathbb{R}^3 \setminus O), \min_{i\neq j}\operatorname{dist}(O_i, O_j), \operatorname{dist}(O, Z)\Big\}.$$

We fix $\beta \in (0, \delta)$ and a cut-off $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi(y) = 1$ for $|y| \leqslant \beta$ and $\varphi(y) = 0$ for $|y| \geqslant 2\beta$. Also, setting $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$ for each $x_i \in (\mathcal{M}^i)^\beta$ and $U_i \in \mathcal{S}_{m_i}$, we define

$$U_\varepsilon^{x_1,\dots,x_k}(y) = \sum_{i=1}^k \exp\left\{\mathrm{i}A(x_i)\left(y - \frac{x_i}{\varepsilon}\right)\right\}\varphi_\varepsilon\left(y - \frac{x_i}{\varepsilon}\right)U_i\left(y - \frac{x_i}{\varepsilon}\right).$$

We will find a solution, for sufficiently small $\varepsilon > 0$, near the set

$$X_\varepsilon = \{U_\varepsilon^{x_1\dots,x_k}(y) : x_i \in (\mathcal{M}^i)^\beta \text{ and } U_i \in \mathcal{S}_{m_i} \text{ for each } i = 1,\dots,k\}.$$

For each $i \in \{1,\dots,k\}$ we fix an arbitrary $x_i \in \mathcal{M}^i$ and an arbitrary $U_i \in \mathcal{S}_{m_i}$ and define

$$\mathcal{W}_\varepsilon^i(y) = \exp\left\{\mathrm{i}A(x_i)\left(y - \frac{x_i}{\varepsilon}\right)\right\}\varphi_\varepsilon\left(y - \frac{x_i}{\varepsilon}\right)U_i\left(y - \frac{x_i}{\varepsilon}\right).$$

Setting

$$\mathcal{W}_{\varepsilon,t}^i(y) = \exp\left\{\mathrm{i}A(x_i)\left(y - \frac{x_i}{\varepsilon}\right)\right\}\varphi_\varepsilon\left(y - \frac{x_i}{\varepsilon}\right)U_i\left(\frac{y}{t} - \frac{x_i}{\varepsilon t}\right),$$

we see that

$$\lim_{t\to 0}\|\mathcal{W}_{\varepsilon,t}^i\|_\varepsilon = 0, \quad \Gamma_\varepsilon(\mathcal{W}_{\varepsilon,t}^i) = \mathcal{F}_\varepsilon(\mathcal{W}_{\varepsilon,t}^i), \quad t \geqslant 0.$$

In the next proposition we will show that there exists $T_i > 0$ such that $\Gamma_\varepsilon(\mathcal{W}_{\varepsilon,T_i}^i) < -2$ for any sufficiently small $\varepsilon > 0$. Assuming this holds, let $\gamma_\varepsilon^i(s) = \mathcal{W}_{\varepsilon,s}^i$ for $s > 0$ and $\gamma_\varepsilon^i(0) = 0$. For $s = (s_1,\dots,s_k) \in T = [0,T_1] \times \dots \times [0,T_k]$, we define

$$\gamma_\varepsilon(s) = \sum_{i=1}^k \mathcal{W}_{\varepsilon,s_i}^i \quad \text{and} \quad D_\varepsilon = \max_{s\in T}\Gamma_\varepsilon(\gamma_\varepsilon(s)).$$

Finally, for each $i \in \{1,\dots,k\}$, let $E_{m_i} = L_{m_i}^{\mathrm{c}}(U)$ for $U \in S_{m_i}$. Here L_a^{c} is, for any $a > 0$, the Euler functional associated to (2.1) in which solutions are considered to be complex valued.

3.2. Energy estimates and the Palais–Smale condition

In what follows, we set

$$E_m = \min_{i\in\{1,\dots,k\}} E_{m_i}, \qquad E = \sum_{i=1}^k E_{m_i}.$$

For a set $A \subset H_\varepsilon$ and $\alpha > 0$, we let $A^\alpha = \{u \in H_\varepsilon : \|u - A\|_\varepsilon \leqslant \alpha\}$.

Proposition 3.3. *We have the following results:*

(i) $\lim_{\varepsilon\to 0} D_\varepsilon = E$*;*

(ii) $\limsup_{\varepsilon\to 0}\max_{s\in\partial T}\Gamma_\varepsilon(\gamma_\varepsilon(s)) \leqslant \tilde{E} = \max\{E - E_{m_i} \mid i = 1,\dots,k\} < E$*;*

(iii) *for each $d > 0$, there exists $\alpha > 0$ such that, for sufficiently small $\varepsilon > 0$,*

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geqslant D_\varepsilon - \alpha \text{ implies that } \gamma_\varepsilon(s) \in X_\varepsilon^{d/2}.$$

Proof. Since $\operatorname{supp}(\gamma_\varepsilon(s)) \subset \mathcal{M}_\varepsilon^{2\beta}$ for each $s \in T$, it follows that

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) = \mathcal{F}_\varepsilon(\gamma_\varepsilon(s)) = \sum_{i=1}^k \mathcal{F}_\varepsilon(\gamma_\varepsilon^i(s)).$$

Arguing as in [12, proposition 3.1] we claim that, for each $i \in \{1, \dots, k\}$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon(y)\right)W^i_{\varepsilon,s_i}\right|^2 \mathrm{d}y = s_i\int_{\mathbb{R}^3}|\nabla U_i|^2\,\mathrm{d}y. \tag{3.5}$$

Using the exponential decay of U_i we have that, as $\varepsilon \to 0$,

$$\begin{aligned}\int_{\mathbb{R}^3} V_\varepsilon(y)|W^i_{\varepsilon,s_i}|^2\,\mathrm{d}y &= \int_{\mathbb{R}^3} m_i\left|U_i\left(\frac{y}{s_i}\right)\right|^2\mathrm{d}y + o(1)\\ &= m_i s_i^3\int_{\mathbb{R}^3}|U_i|^2\,\mathrm{d}y + o(1)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^6} W(x-y)|W^i_{\varepsilon,s_i}(x)|^2|W^i_{\varepsilon,s_i}(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= \int_{\mathbb{R}^6} W(x-y)\left|U_i\left(\frac{y}{s_i}\right)\right|^2\left|U_i\left(\frac{x}{s_i}\right)\right|^2\mathrm{d}x\,\mathrm{d}y + o(1)\\ &\quad= s_i^5\int_{\mathbb{R}^6} W(x-y)|U_i(x)|^2|U_i(y)|^2\,\mathrm{d}x\,\mathrm{d}y + o(1).\end{aligned}$$

Thus, from the above limit and from (3.5), (3.6), we derive

$$\begin{aligned}\mathcal{F}_\varepsilon(\gamma^i_\varepsilon(s_i)) &= \frac{1}{2}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon(y)\right)\gamma^i_\varepsilon(s_i)\right|^2\mathrm{d}y + V_\varepsilon(y)|\gamma^i_\varepsilon(s_i)|^2\,\mathrm{d}y\\ &\qquad - \frac{1}{4}\int_{\mathbb{R}^6} W(x-y)|\gamma^i_\varepsilon(s_i)|^2|\gamma^i_\varepsilon(s_i)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &= \tfrac{1}{2}s_i\int_{\mathbb{R}^3}|\nabla U_i|^2\,\mathrm{d}y + \tfrac{1}{2}s_i^3 m_i\int_{\mathbb{R}^3}|U_i|^2\,\mathrm{d}y\\ &\qquad - \tfrac{1}{4}s_i^5\int_{\mathbb{R}^6} W(x-y)|U_i(x)|^2|U^i(y)|^2\,\mathrm{d}x\,\mathrm{d}y + o(1).\end{aligned}$$

Using the Pohozaev identity (2.2) and the relation

$$\int_{\mathbb{R}^3}|\nabla U_i|^2\,\mathrm{d}y + m_i\int_{\mathbb{R}^3}|U_i|^2\,\mathrm{d}y = \int_{\mathbb{R}^6} W(x-y)|U_i(x)|^2|U_i(y)|^2\,\mathrm{d}x\,\mathrm{d}y,$$

we see that

$$\mathcal{F}_\varepsilon(\gamma^i_\varepsilon(s_i)) = \left(\frac{s_i}{6} + \frac{s_i^3}{2} - \frac{2s_i^5}{6}\right)m_i\int_{\mathbb{R}^3}|U_i|^2\,\mathrm{d}y + o(1).$$

Also,

$$\max_{t\in[0,\infty)}\left(\frac{t}{6} + \frac{t^3}{2} - \frac{2t^5}{6}\right)m_i\int_{\mathbb{R}^3}|U_i|^2\,\mathrm{d}y = E_{m_i}.$$

At this point we deduce that (i) and (ii) hold. Clearly, the existence of a $T_i > 0$ such that $\mathit{\Gamma}_\varepsilon(W^i_{\varepsilon,T_i}) < -2$ is also justified. To conclude, we simply observe that, setting

$$g(t) = \frac{t}{6} + \frac{t^3}{2} - \frac{2t^5}{6},$$

the derivative $g'(t)$ of $g(t)$ is positive for $t \in (0,1)$, negative for $t \in (1,+\infty)$ and vanishes at $t=1$. We conclude by observing that $g''(1) < 0$. $\square$

Let us define

$$\Phi^i_\varepsilon = \{\gamma \in C([0,T_i], H_\varepsilon) : \gamma(0) = \gamma^i_\varepsilon(0),\ \gamma(T_i) = \gamma^i_\varepsilon(T_i)\}$$

and

$$C^i_\varepsilon = \inf_{\gamma \in \Phi^i_\varepsilon} \max_{s_i \in [0,T_i]} \Gamma^i_\varepsilon(\gamma(s_i)).$$

Proposition 3.4. *For the level C^i_ε defined earlier, we have*

$$\liminf_{\varepsilon \to 0} C^i_\varepsilon \geqslant E_{m_i}.$$

In particular, $\lim_{\varepsilon \to 0} C^i_\varepsilon = E_{m_i}$.

Proof. The proof of this lemma is analogous to that of [12, proposition 3.2]. $\square$

Next we define, for every $\alpha \in \mathbb{R}$, the sub-level

$$\Gamma^\alpha_\varepsilon = \{u \in H_\varepsilon : \Gamma_\varepsilon(u) \leqslant \alpha\}.$$

Proposition 3.5. *Let (ε_j) be such that* $\lim_{j\to\infty} \varepsilon_j = 0$ *and let $(u_{\varepsilon_j}) \in X^d_{\varepsilon_j}$ be such that*

$$\lim_{j\to\infty} \Gamma_{\varepsilon_j}(u_{\varepsilon_j}) \leqslant E \quad \textit{and} \quad \lim_{j\to\infty} \Gamma'_{\varepsilon_j}(u_{\varepsilon_j}) = 0. \tag{3.7}$$

Then, for sufficiently small $d > 0$, there exist, up to a subsequence, $(y^i_j) \subset \mathbb{R}^3$, $i = 1,\dots,k$, points $x^i \in \mathcal{M}^i$ (not to be confused with the points x_i already introduced), $U_i \in S_{m_i}$ such that

$$\left.\begin{aligned} \lim_{j\to\infty} |\varepsilon_j y^i_j - x^i| = 0,\\ \lim_{j\to\infty} \left\| u_{\varepsilon_j} - \sum_{i=1}^k \mathrm{e}^{\mathrm{i}A_\varepsilon(y^i_j)(\cdot - y^i_j)} \varphi_{\varepsilon_j}(\cdot - y^i_j) U_i(\cdot - y^i_j) \right\|_{\varepsilon_j} = 0. \end{aligned}\right\} \tag{3.8}$$

Proof. For simplicity we write ε instead of ε_j. From proposition 2.14, we know that the S_{m_i} are compact. Then there exist $Z_i \in S_{m_i}$ and $(x^i_\varepsilon) \subset (\mathcal{M}^i)^\beta$, $x^i \in (\mathcal{M}^i)^\beta$ for $i = 1,\dots,k$, with $x^i_\varepsilon \to x^i$ as $\varepsilon \to 0$ such that, passing to a subsequence still denoted by (u_ε),

$$\left\| u_\varepsilon - \sum_{i=1}^k \exp\left\{ \mathrm{i}A(x^i_\varepsilon)\left(\cdot - \frac{x^i_\varepsilon}{\varepsilon}\right)\right\} \varphi_\varepsilon\left(\cdot - \frac{x^i_\varepsilon}{\varepsilon}\right) Z_i\left(\cdot - \frac{x^i_\varepsilon}{\varepsilon}\right)\right\|_\varepsilon \leqslant 2d \tag{3.9}$$

for small $\varepsilon > 0$. We set

$$u_{1,\varepsilon} = \sum_{i=1}^k \varphi_\varepsilon\left(\cdot - \frac{x^i_\varepsilon}{\varepsilon}\right) u_\varepsilon$$

and $u_{2,\varepsilon} = u_\varepsilon - u_{1,\varepsilon}$. As a first step in the proof of the proposition we shall prove that

$$\Gamma_\varepsilon(u_\varepsilon) \geqslant \Gamma_\varepsilon(u_{1,\varepsilon}) + \Gamma_\varepsilon(u_{2,\varepsilon}) + O(\varepsilon). \tag{3.10}$$

Suppose that there exist

$$y_\varepsilon \in \bigcup_{i=1}^{k} B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)$$

and $R > 0$ satisfying

$$\liminf_{\varepsilon \to 0} \int_{B(y_\varepsilon, R)} |u_\varepsilon|^2 \,\mathrm{d}y > 0,$$

which means that

$$\liminf_{\varepsilon \to 0} \int_{B(0,R)} |v_\varepsilon|^2 \,\mathrm{d}y > 0, \tag{3.11}$$

where $v_\varepsilon(y) = u_\varepsilon(y + y_\varepsilon)$. Taking a subsequence, we can assume that $\varepsilon y_\varepsilon \to x_0$ with x_0 in the closure of

$$\bigcup_{i=1}^{k} B(x^i, 2\beta) \backslash B(x^i, \beta).$$

Since (3.9) holds, (v_ε) is bounded in H_ε. Thus, since $\tilde{m} > 0$, (v_ε) is bounded in $L^2(\mathbb{R}^3, \mathbb{C})$, and using the diamagnetic inequality and the Hardy–Sobolev inequality (see also the proof of proposition 2.14) we deduce that (v_ε) is bounded in $L^m(\mathbb{R}^3, \mathbb{C})$ for any $m < 6$. In particular, up to a subsequence, $v_\varepsilon \to \mathcal{W} \in L^m(\mathbb{R}^3, \mathbb{C})$ weakly. Also, by corollary 3.2(i), for any compact $K \subset \mathbb{R}^3$, (v_ε) is bounded in $H^1(K, \mathbb{C})$. Thus, we can assume that $v_\varepsilon \to \mathcal{W}$ in $H^1(K, \mathbb{C})$ weakly for any compact $K \subset \mathbb{R}^3$ strongly in $L^m(K, \mathbb{C})$. Because of (3.11), $\mathcal{W}$ is not the zero function. Now, since $\lim_{\varepsilon \to 0} \Gamma'_\varepsilon(u_\varepsilon) = 0$, $\mathcal{W}$ is a non-trivial solution of

$$-\Delta \mathcal{W} - \frac{2}{\mathrm{i}} A(x_0) \cdot \nabla \mathcal{W} + |A(x_0)|^2 \mathcal{W} + V(x_0)\mathcal{W} = (W * |\mathcal{W}|^2)\mathcal{W}. \tag{3.12}$$

From (3.12) and since $\mathcal{W} \in L^m(\mathbb{R}^3, \mathbb{C})$, we readily deduce, using corollary 3.2(ii), that $\mathcal{W} \in H^1(\mathbb{R}^3, \mathbb{C})$.

Let $\omega(y) = \mathrm{e}^{-\mathrm{i}A(x_0)y}\mathcal{W}(y)$. Then ω is a non-trivial solution of the complex-valued equation

$$-\Delta\omega + V(x_0)\omega = (W * |\omega|^2)\omega.$$

For large $R > 0$ we have

$$\int_{B(0,R)} \left| \left(\frac{\nabla}{\mathrm{i}} - A(x_0) \right) \mathcal{W} \right|^2 \mathrm{d}y \geqslant \frac{1}{2} \int_{\mathbb{R}^3} \left| \left(\frac{\nabla}{\mathrm{i}} - A(x_0) \right) \mathcal{W} \right|^2 \mathrm{d}y \tag{3.13}$$

and thus, by the weak convergence,

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_{B(y_\varepsilon,R)}|D^\varepsilon u_\varepsilon|^2\,\mathrm{d}y &= \liminf_{\varepsilon\to 0}\int_{B(0,R)}\left|\left(\frac{\nabla}{\mathrm{i}}-A_\varepsilon(y+y_\varepsilon)\right)v_\varepsilon\right|^2\mathrm{d}y\\ &\geqslant \int_{B(0,R)}\left|\left(\frac{\nabla}{\mathrm{i}}-A(x_0)\right)\mathcal{W}\right|^2\mathrm{d}y\\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}}-A(x_0)\right)\mathcal{W}\right|^2\mathrm{d}y\\ &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\omega|^2\,\mathrm{d}y. \end{aligned}\tag{3.14}$$

It follows from lemma 2.6 that $E_a > E_b$ if $a > b$ and, using lemma 2.11, we have $L^{\mathrm{c}}_{V(x_0)}(\omega) \geqslant E^{\mathrm{c}}_{V(x_0)} = E_{V(x_0)} \geqslant E_m$ since $V(x_0) \geqslant m$. Thus, from (3.14) and lemma 2.6 we obtain that

$$\liminf_{\varepsilon\to 0}\int_{B(y_\varepsilon,R)}|D^\varepsilon u_\varepsilon|^2\,\mathrm{d}y \geqslant \tfrac{3}{2}L^{\mathrm{c}}_{V(x_0)}(\omega) \geqslant \tfrac{3}{2}E_m > 0,\tag{3.15}$$

which contradicts (3.9), provided $d > 0$ is sufficiently small. Indeed, $x_0 \neq x^i$ for all $i \in \{1,\dots,k\}$ and the Z_i are exponentially decreasing.

Since such a sequence (y_ε) does not exist, we deduce from [22, lemma I.1] that

$$\limsup_{\varepsilon\to 0}\int_{\bigcup_{i=1}^k B(x^i_\varepsilon/\varepsilon,2\beta/\varepsilon)\setminus B(x^i_\varepsilon/\varepsilon,\beta/\varepsilon)}|u_\varepsilon|^5\,\mathrm{d}y = 0.\tag{3.16}$$

As a consequence, using the boundedness of $(\|u_\varepsilon\|_2)$ we can derive that

$$\begin{aligned}\lim_{\varepsilon\to 0}\bigg\{\int_{\mathbb{R}^3\times\mathbb{R}^3}&W(x-y)|u_\varepsilon(x)|^2|u_\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &-\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_{1,\varepsilon}(x)|^2|u_{1,\varepsilon}(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &-\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_{2,\varepsilon}(x)|^2|u_{2,\varepsilon}(y)|^2\,\mathrm{d}x\,\mathrm{d}y\bigg\} = 0.\end{aligned}$$

At this point, writing

$$\begin{aligned}\Gamma_\varepsilon(u_\varepsilon) &= \Gamma_\varepsilon(u_{1,\varepsilon}) + \Gamma_\varepsilon(u_{2,\varepsilon})\\ &\quad+\sum_{i=1}^k\int_{B(x^i_\varepsilon/\varepsilon,2\beta/\varepsilon)\setminus B(x^i_\varepsilon/\varepsilon,\beta/\varepsilon)}\varphi_\varepsilon\left(y-\frac{x^i_\varepsilon}{\varepsilon}\right)\left(1-\varphi_\varepsilon\left(y-\frac{x^i}{\varepsilon}\right)\right)|D^\varepsilon u_\varepsilon|^2\\ &\quad+V_\varepsilon\varphi_\varepsilon\left(y-\frac{x^i_\varepsilon}{\varepsilon}\right)\left(1-\varphi_\varepsilon\left(y-\frac{x^i_\varepsilon}{\varepsilon}\right)\right)|u_\varepsilon|^2\,\mathrm{d}y\\ &\quad-\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_\varepsilon(x)|^2|u_\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

$$- \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} W(x-y)|u_{1,\varepsilon}(x)|^2 |u_{1,\varepsilon}(y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$
$$- \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2 \, \mathrm{d}x \, \mathrm{d}y + o(1)$$

as $\varepsilon \to 0$ shows that the inequality (3.10) holds. We now estimate $\Gamma_\varepsilon(u_{2,\varepsilon})$. We have

$$\begin{aligned}
\Gamma_\varepsilon(u_{2,\varepsilon}) &\geqslant \mathcal{F}_\varepsilon(u_{2,\varepsilon}) \\
&= \frac{1}{2} \int_{\mathbb{R}^3} |D^\varepsilon u_{2,\varepsilon}|^2 + \tilde{V}_\varepsilon |u_{2,\varepsilon}|^2 \, \mathrm{d}y - \frac{1}{2} \int_{\mathbb{R}^3} (\tilde{V}_\varepsilon - V_\varepsilon)|u_{2,\varepsilon}|^2 \, \mathrm{d}y \\
&\quad - \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\geqslant \tfrac{1}{2} \|u_{2,\varepsilon}\|_\varepsilon^2 - \frac{\tilde{m}}{2} \int_{\mathbb{R}^3 \setminus O_\varepsilon^i} |u_{2,\varepsilon}|^2 \, \mathrm{d}y \\
&\quad - \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2 \, \mathrm{d}x \, \mathrm{d}y. \qquad (3.17)
\end{aligned}$$

Here we have used the fact that $\tilde{V}_\varepsilon - V_\varepsilon = 0$ on O_ε^i and $|\tilde{V}_\varepsilon - V_\varepsilon| \leqslant \tilde{m}$ on $\mathbb{R}^3 \setminus O_\varepsilon^i$. For some $C > 0$,

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant C \|u_{2,\varepsilon}\|_{L^2}^3 \|u_{2,\varepsilon}\|_{H^1}.$$

Since (u_ε) is bounded, we see from (3.9) that $\|u_{2,\varepsilon}\|_\varepsilon \leqslant 4d$ for small $\varepsilon > 0$. Thus, taking sufficiently small $d > 0$, we have

$$\tfrac{1}{2} \|u_{2,\varepsilon}\|_\varepsilon^2 - \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2 \, \mathrm{d}x \, \mathrm{d}y \geqslant \tfrac{1}{8} \|u_{2,\varepsilon}\|_\varepsilon^2. \qquad (3.18)$$

Now note that $\mathcal{F}_\varepsilon$ is uniformly bounded in X_ε^d for small $\varepsilon > 0$, and so is Q_ε. This implies that, for some $C > 0$,

$$\int_{\mathbb{R}^3 \setminus O_\varepsilon} |u_{2,\varepsilon}|^2 \, \mathrm{d}y \leqslant C \varepsilon^{6/\mu} \qquad (3.19)$$

and from (3.17)–(3.19) we deduce that $\Gamma_\varepsilon(u_{2,\varepsilon}) \geqslant o(1)$.

Now, for $i = 1, \dots, k$, we define $u_{1,\varepsilon}^i(y) = u_{1,\varepsilon}(y)$ for $y \in O_\varepsilon^i$ and $u_{1,\varepsilon}^i(y) = 0$ for $y \notin O_\varepsilon^i$. Also, we set $\mathcal{W}_\varepsilon^i(y) = u_{1,\varepsilon}^i(y + x_\varepsilon^i/\varepsilon)$. We fix an arbitrary $i \in \{1, \dots, k\}$. Arguing as before, we can assume, up to a subsequence, that $\mathcal{W}_\varepsilon^i$ converges weakly in $L^m(\mathbb{R}^3, \mathbb{C})$, $m < 6$, to a solution $\mathcal{W}^i \in H^1(\mathbb{R}^3, \mathbb{C})$ of

$$-\Delta \mathcal{W}^i - \frac{2}{\mathrm{i}} A(x^i) \cdot \nabla \mathcal{W}^i + |A(x^i)|^2 \mathcal{W}^i + V(x^i) \mathcal{W}^i = (W * \mathcal{W}^i) \mathcal{W}^i, \quad y \in \mathbb{R}^3.$$

We shall prove that $\mathcal{W}_\varepsilon^i$ tends to $\mathcal{W}^i$ strongly in H_ε. Suppose that there exist $R > 0$ and a sequence (z_ε) with $z_\varepsilon \in B(x_\varepsilon^i/\varepsilon, 2\beta/\varepsilon)$ satisfying

$$\liminf_{\varepsilon \to 0} |z_\varepsilon - \varepsilon^{-1} x_\varepsilon^i| = \infty \quad \text{and} \quad \liminf_{\varepsilon \to 0} \int_{B(z_\varepsilon, R)} |u_\varepsilon^{1,i}|^2 \, \mathrm{d}y > 0.$$

We may assume that $\varepsilon z_\varepsilon \to z^i \in O^i$ as $\varepsilon \to 0$. Then $\tilde{\mathcal{W}}^i_\varepsilon(y) = \mathcal{W}^i_\varepsilon(y + z_\varepsilon)$ weakly converges in $L^m(\mathbb{R}^3, \mathbb{C})$ (for any $m < 6$) to $\tilde{\mathcal{W}}^i \in H^1(\mathbb{R}^3, \mathbb{C})$, which satisfies

$$-\Delta\tilde{\mathcal{W}}^i - \frac{2}{\mathrm{i}}A(z^i)\cdot\nabla\tilde{\mathcal{W}} + |A(z^i)|^2\tilde{\mathcal{W}}^i + V(z^i)\tilde{\mathcal{W}}^i = (W * \tilde{\mathcal{W}}^i)\tilde{\mathcal{W}}^i, \quad y \in \mathbb{R}^3$$

and we obtain a contradiction, as before. Then, using [22, lemma I.1], it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\times\mathbb{R}^3} & W(x-y)|\mathcal{W}^i_\varepsilon(x)|^2|\mathcal{W}^i_\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}^i(x)|^2|\mathcal{W}^i(y)|^2\,\mathrm{d}x\,\mathrm{d}y. \qquad (3.20)\end{aligned}$$

Then, from the weak convergence of $\mathcal{W}^i_\varepsilon$ to $\mathcal{W}^i \neq 0$ in $H^1(K, \mathbb{C})$ for any compact $K \subset \mathbb{R}^3$, we obtain, for any $i \in \{1, \dots, k\}$,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Gamma_\varepsilon(u^i_{1,\varepsilon}) &\geqslant \liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u^i_{1,\varepsilon}) \\ &\geqslant \liminf_{\varepsilon\to 0}\frac{1}{2}\int_{B(0,R)}\left|\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon y + x^i_\varepsilon)\right)\mathcal{W}^i_\varepsilon\right|^2 \\ &\quad + V(\varepsilon y + x^i_\varepsilon)|\mathcal{W}^i_\varepsilon|^2\,\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}^i_\varepsilon(x)|^2|\mathcal{W}^i_\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \frac{1}{2}\int_{B(0,R)}\left|\left(\frac{\nabla}{\mathrm{i}} - A(x^i)\right)\mathcal{W}^i\right|^2 + V(x^i)|\mathcal{W}^i|^2\,\mathrm{d}y \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}^i(x)|^2|\mathcal{W}^i(y)|^2\,\mathrm{d}x\,\mathrm{d}y. \qquad (3.21)\end{aligned}$$

Since these inequalities hold for any $R > 0$, we deduce, using lemma 2.11, that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Gamma_\varepsilon(u^i_{1,\varepsilon}) &\geqslant \frac{1}{2}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}} - A(x^i)\right)\mathcal{W}^i\right|^2\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^3} V(x^i)|\mathcal{W}^i|^2\,\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}^i(x)|^2|\mathcal{W}^i(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\omega^i|^2 + V(x^i)|\omega^i|^2\,\mathrm{d}y \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\omega^i(x)|^2|\omega^i(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= L^{\mathrm{c}}_{V(x^i)}(\omega^i) \\ &\geqslant E^{c}_{m_i} \\ &= E_{m_i}, \qquad (3.22)\end{aligned}$$

where we have set $\omega^i(y) = \mathrm{e}^{-\mathrm{i}A(x^i)y}\mathcal{W}^i(y)$. Now, by (3.10),

$$\begin{aligned}
\limsup_{\varepsilon\to 0}\left(\Gamma_\varepsilon(u_{2,\varepsilon}) + \sum_{i=1}^{k}\Gamma_\varepsilon(u^i_{1,\varepsilon})\right) &= \limsup_{\varepsilon\to 0}(\Gamma_\varepsilon(u_{2,\varepsilon}) + \Gamma_\varepsilon(u_{1,\varepsilon})) \\
&\leqslant \limsup_{\varepsilon\to 0}\Gamma_\varepsilon(u_\varepsilon) \\
&\leqslant E \\
&= \sum_{i=1}^{k} E_{m_i}.
\end{aligned} \tag{3.23}$$

Thus, as $\Gamma_\varepsilon(u_{2,\varepsilon}) \geqslant o(1)$ we deduce from (3.22), (3.23) that, for any $i \in \{1,\dots,k\}$,

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon(u^i_{1,\varepsilon}) = E_{m_i}. \tag{3.24}$$

Now (3.22) and (3.24) imply that $L_{V(x^i)}(\omega^i) = E_{m_i}$. Recalling from [18] that $E_a > E_b$ if $a > b$, and using lemma 2.11, we conclude that $x^i \in \mathcal{M}^i$. At this point it is clear that $W^i(y) = \mathrm{e}^{\mathrm{i}A(x^i)y}U_i(y - z_i)$ with $U_i \in S_{m_i}$ and $z_i \in \mathbb{R}^3$.

To establish that $W^i_\varepsilon \to W^i$ strongly in H_ε, we first show that $W^i_\varepsilon \to W^i$ strongly in $L^2(\mathbb{R}^3,\mathbb{C})$. Since (W^i_ε) is bounded in H_ε, the diamagnetic inequality (3.1) immediately yields that $(|W^i_\varepsilon|)$ is bounded in $H^1(\mathbb{R}^3,\mathbb{R})$ and we can assume that $|W^i_\varepsilon| \to |W^i| = |\omega^i|$ weakly in $H^1(\mathbb{R}^3,\mathbb{R})$. Now, since $L_{V(x^i)}(\omega^i) = E_{m_i}$ we obtain, using the diamagnetic inequality, (3.20), (3.24) and the fact that $V \geqslant V(x^i)$ on O^i,

$$\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla\omega^i|^2\,\mathrm{d}y + \int_{\mathbb{R}^3} m_i|\omega^i|^2\,\mathrm{d}y - 2\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\omega^i(x)|^2|\omega^i(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \geqslant \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon y + x^i_\varepsilon)\right)\mathcal{W}^i_\varepsilon\right|^2\mathrm{d}y + \int_{\mathbb{R}^3} V(\varepsilon y + x^i_\varepsilon)|\mathcal{W}^i_\varepsilon|^2\,\mathrm{d}y \\
&\qquad\qquad - 2\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}^i_\varepsilon(x)|^2|\mathcal{W}^i_\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \geqslant \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^3}|\nabla|\mathcal{W}^i_\varepsilon||^2\,\mathrm{d}y + \int_{\mathbb{R}^3} V(x^i)|\mathcal{W}^i_\varepsilon|^2\,\mathrm{d}y \\
&\qquad\qquad - 2\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}^i_\varepsilon(x)|^2|\mathcal{W}^i_\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \geqslant \int_{\mathbb{R}^3}|\nabla|\omega^i||^2\,\mathrm{d}y + \int_{\mathbb{R}^3} m_i|\omega^i|^2\,\mathrm{d}y \\
&\qquad\qquad - 2\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\omega^i(x)|^2|\omega^i(y)|^2\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.25}$$

But from lemma 2.11 we know that, since $L_{V(x^i)}(\omega^i) = E_{m_i}$,

$$\int_{\mathbb{R}^3}|\nabla|\omega^i||^2\,\mathrm{d}y = \int_{\mathbb{R}^3}|\nabla\omega^i|^2\,\mathrm{d}y.$$

Thus, we deduce from (3.25) that

$$\int_{\mathbb{R}^3} V(\varepsilon y + x_\varepsilon^i)|\mathcal{W}_\varepsilon^i|^2\,\mathrm{d}y \to \int_{\mathbb{R}^3} V(x^i)|\mathcal{W}^i|^2\,\mathrm{d}y. \tag{3.26}$$

Thus, since $V \geqslant V(x^i)$ on O^i, we deduce that

$$\mathcal{W}_\varepsilon^i \to \mathcal{W}^i \quad \text{strongly in } L^2(\mathbb{R}^3, \mathbb{C}). \tag{3.27}$$

From (3.27) we easily obtain

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \left|\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon y + x_\varepsilon^i)\right)\mathcal{W}_\varepsilon^i\right|^2 - \left|\left(\frac{\nabla}{\mathrm{i}} - A(x^i)\right)\mathcal{W}_\varepsilon^i\right|^2 \mathrm{d}y = 0. \tag{3.28}$$

Now, using (3.20), (3.25) and (3.26), we see from (3.28) that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left|\left(\frac{\nabla}{\mathrm{i}} - A(x^i)\right)\mathcal{W}^i\right|^2 \mathrm{d}y + \int_{\mathbb{R}^3} V(x^i)|\mathcal{W}^i|^2\,\mathrm{d}y \\
&\qquad \geqslant \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^3} \left|\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon y + x_\varepsilon^i)\right)\mathcal{W}_\varepsilon^i\right|^2 \mathrm{d}y + \int_{\mathbb{R}^3} V(\varepsilon y + x_\varepsilon^i)|\mathcal{W}_\varepsilon^i|^2\,\mathrm{d}y \\
&\qquad \geqslant \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^3} \left|\left(\frac{\nabla}{\mathrm{i}} - A(x^i)\right)\mathcal{W}_\varepsilon^i\right|^2 \mathrm{d}y + \int_{\mathbb{R}^3} V(x^i)|\mathcal{W}_\varepsilon^i|^2\,\mathrm{d}y.
\end{aligned} \tag{3.29}$$

At this point, and using corollary 3.2(ii), we have established the strong convergence $W_\varepsilon^i \to W^i$ in $H^1(\mathbb{R}^3, \mathbb{C})$. Thus, we have

$$u_{1,\varepsilon}^i = \exp\left\{\mathrm{i}A(x^i)\left(\,\cdot - \frac{x_\varepsilon^i}{\varepsilon}\right)\right\} U_i\left(\,\cdot - \frac{x_\varepsilon^i}{\varepsilon} - z_i\right) + o(1)$$

strongly in $H^1(\mathbb{R}^3, \mathbb{C})$. Now setting $y_\varepsilon^i = x_\varepsilon^i/\varepsilon + z_i$ and changing U_i to $\mathrm{e}^{\mathrm{i}A(x^i)z_i}U_i$ we obtain that

$$u_{1,\varepsilon}^i = \mathrm{e}^{\mathrm{i}A(x^i)(\cdot - y_\varepsilon^i)} U_i(\cdot - y_\varepsilon^i) + o(1)$$

strongly in $H^1(\mathbb{R}^3, \mathbb{C})$. Finally, using the exponential decay of U_i and ∇U_i we have

$$u_{1,\varepsilon}^i = \mathrm{e}^{\mathrm{i}A_\varepsilon(y_\varepsilon^i)(\cdot - y_\varepsilon^i)} \varphi_\varepsilon(\cdot - y_\varepsilon^i) U_i(\cdot - y_\varepsilon^i) + o(1).$$

From corollary 3.2(iii) we deduce that this convergence also holds in H_ε, and thus

$$u_{1,\varepsilon} = \sum_{i=1}^k u_{1,\varepsilon}^i = \sum_{i=1}^k \mathrm{e}^{\mathrm{i}A_\varepsilon(y_\varepsilon^i)(\cdot - y_\varepsilon^i)} \varphi_\varepsilon(\cdot - y_\varepsilon^i) U_i(\cdot - y_\varepsilon^i) + o(1)$$

strongly in H_ε. To conclude the proof of the proposition, it suffices to show that $u_{2,\varepsilon} \to 0$ in H_ε. Since $E \geqslant \lim_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon)$ and $\lim_{\varepsilon\to 0} \Gamma_\varepsilon(u_{1,\varepsilon}) = E$, we deduce, using (3.10), that $\lim_{\varepsilon\to 0} \Gamma_\varepsilon(u_{2,\varepsilon}) = 0$. Now, from (3.17)–(3.19), we obtain that $u_{2,\varepsilon} \to 0$ in H_ε. □

3.3. Critical points of the penalized functional

We first state the following proposition.

PROPOSITION 3.6. *For sufficiently small $d > 0$, there exist constants $\omega > 0$ and $\varepsilon_0 > 0$ such that $|\Gamma_\varepsilon'(u)| \geqslant \omega$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{d/2})$ and $\varepsilon \in (0, \varepsilon_0)$.*

Proof. By contradiction, we suppose that, for $d > 0$ sufficiently small such that proposition 3.5 applies, there exist (ε_j) with $\lim_{j\to\infty} \varepsilon_j = 0$ and a sequence (u_{ε_j}) with $u_{\varepsilon_j} \in X_{\varepsilon_j}^d \setminus X_{\varepsilon_j}^{d/2}$ satisfying $\lim_{j\to\infty} \Gamma_{\varepsilon_j}(u_{\varepsilon_j}) \leqslant E$ and $\lim_{j\to\infty} \Gamma_{\varepsilon_j}'(u_{\varepsilon_j}) = 0$. By proposition 3.5, there exist $(y_{\varepsilon_j}^i) \subset \mathbb{R}^3$, $i = 1, \dots, k$, $x^i \in \mathcal{M}^i$ and $U_i \in S_{m_i}$ such that

$$\lim_{\varepsilon_j\to 0} |\varepsilon_j y_{\varepsilon_j}^i - x^i| = 0,$$

$$\lim_{\varepsilon_j\to 0} \left\| u_{\varepsilon_j} - \sum_{i=1}^k \exp\{\mathrm{i} A_{\varepsilon_j}(y_{\varepsilon_j}^i)(\cdot - y_{\varepsilon_j}^i)\} \varphi_{\varepsilon_j}(\cdot - y_{\varepsilon_j}^i) U_i(\cdot - y_{\varepsilon_j}^i) \right\|_{\varepsilon_j} = 0.$$

By the definition of X_{ε_j} we see that $\lim_{\varepsilon_j\to 0} \mathrm{dist}(u_{\varepsilon_j}, X_{\varepsilon_j}) = 0$. This contradicts $u_{\varepsilon_j} \notin X_{\varepsilon_j}^{d/2}$ and completes the proof. □

From now on we fix a $d > 0$ such that proposition 3.6 holds.

PROPOSITION 3.7. *For sufficiently small fixed $\varepsilon > 0$, Γ_ε has a critical point $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$.*

Proof. We can take $R_0 > 0$ sufficiently large that $O \subset B(0, R_0)$ and $\gamma_\varepsilon(s) \in H_0^1(B(0, R/\varepsilon))$ for any $s \in T$, $R > R_0$ and sufficiently small $\varepsilon > 0$.

We note that, by proposition 3.3(iii), there exists $\alpha \in (0, E - \tilde{E})$ such that, for sufficiently small $\varepsilon > 0$,

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geqslant D_\varepsilon - \alpha \Longrightarrow \gamma_\varepsilon(s) \in X_\varepsilon^{d/2} \cap H_0^1\left(B\left(0, \frac{R}{\varepsilon}\right)\right).$$

We begin by showing that, for sufficiently small fixed $\varepsilon > 0$, and $R > R_0$, there exists a sequence $(u_n^R) \subset X_\varepsilon^{d/2} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, R/\varepsilon))$ such that $\Gamma'(u_n^R) \to 0$ in $H_0^1(B(0, R/\varepsilon))$ as $n \to +\infty$.

Arguing by contradiction, we suppose that, for sufficiently small $\varepsilon > 0$, there exists $a_R(\varepsilon) > 0$ such that $|\Gamma_\varepsilon'(u)| \geqslant a_R(\varepsilon)$ on $X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, R/\varepsilon))$. In what follows, any $u \in H_0^1(B(0, R/\varepsilon))$ will be regarded as an element in H_ε by defining $u = 0$ in $\mathbb{R}^3 \setminus B(0, R/\varepsilon)$. Note from proposition 3.6 that there exists $\omega > 0$, independent of $\varepsilon > 0$, such that $|\Gamma_\varepsilon'(u)| \geqslant \omega$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{d/2})$. Thus, by a deformation argument in $H_0^1(B(0, R/\varepsilon))$, starting from γ_ε for sufficiently small $\varepsilon > 0$, there exists a $\mu \in (0, \alpha)$ and a path $\gamma \in C([0, T], H_\varepsilon)$ satisfying

$$\gamma(s) = \gamma_\varepsilon(s) \quad \text{for } \gamma_\varepsilon(s) \in \Gamma_\varepsilon^{D_\varepsilon - \alpha}, \qquad \gamma(s) \in X_\varepsilon^d \quad \text{for } \gamma_\varepsilon(s) \notin \Gamma_\varepsilon^{D_\varepsilon - \alpha}$$

and

$$\Gamma_\varepsilon(\gamma(s)) < D_\varepsilon - \mu, \quad s \in T. \tag{3.30}$$

Let $\psi \in C_0^\infty(\mathbb{R}^3)$ be such that $\psi(y) = 1$ for $y \in O^\delta$, $\psi(y) = 0$ for $y \notin O^{2\delta}$, $\psi(y) \in [0, 1]$ and $|\nabla\psi| \leqslant 2/\delta$. For $\gamma(s) \in X_\varepsilon^d$, we define $\gamma_1(s) = \psi_\varepsilon \gamma(s)$ and

$\gamma_2(s) = (1-\psi_\varepsilon)\gamma(s)$, where $\psi_\varepsilon(y) = \psi(\varepsilon y)$. The dependence on ε will be understood in the notation for γ_1 and γ_2. Note that

$$\begin{aligned}
\Gamma_\varepsilon(\gamma(s)) = \Gamma_\varepsilon(\gamma_1(s)) + \Gamma_\varepsilon(\gamma_2(s)) &+ \int_{\mathbb{R}^3} (\psi_\varepsilon(1-\psi_\varepsilon)|D^\varepsilon\gamma(s)|^2 \\
&+ V_\varepsilon\psi_\varepsilon(1-\psi_\varepsilon)|\gamma(s)|^2)\,\mathrm{d}y \\
&+ Q_\varepsilon(\gamma(s)) - Q_\varepsilon(\gamma_1(s)) - Q_\varepsilon(\gamma_2(s)) \\
&- \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)(|\gamma(s)(x)|^2|\gamma(s)(y)|^2 \\
&\qquad\qquad - |\gamma_1(s)(x)|^2|\gamma_1(s)(y)|^2 \\
&\qquad\qquad - |\gamma_1(s)(x)|^2|\gamma_2(s)(y)|^2)\,\mathrm{d}x\,\mathrm{d}y + o(1).
\end{aligned}$$

Since, for $A, B \geqslant 0$, $(A+B-1)_+ \geqslant (A-1)_+ + (B-1)_+$, it follows that

$$\begin{aligned}
Q_\varepsilon(\gamma(s)) &= \left(\int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_1(s)+\gamma_2(s)|^2\,\mathrm{d}y - 1\right)_+^{5/2} \\
&\geqslant \left(\int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_1(s)|^2\,\mathrm{d}y + \int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_2(s)|^2\,\mathrm{d}y - 1\right)_+^{5/2} \\
&\geqslant \left(\int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_1(s)|^2\,\mathrm{d}y - 1\right)_+^{5/2} + \left(\int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_2(s)|^2\,\mathrm{d}y - 1\right)_+^{5/2} \\
&= Q_\varepsilon(\gamma_1(s)) + Q_\varepsilon(\gamma_2(s)).
\end{aligned}$$

Now, as in the derivation of (3.19), using the fact that $Q_\varepsilon(\gamma(s))$ is uniformly bounded with respect to ε, we have, for some $C > 0$,

$$\int_{\mathbb{R}^3\setminus O_\varepsilon} |\gamma(s)|^2\,\mathrm{d}y \leqslant C\varepsilon^{6/\mu}. \tag{3.31}$$

Since W is even, we have

$$\begin{aligned}
\int_{\mathbb{R}^3\times\mathbb{R}^3} & W(x-y)(|\gamma(s)(x)|^2|\gamma(s)(y)|^2 - |\gamma_1(s)(x)|^2|\gamma_1(s)(y)|^2 \\
&\quad - |\gamma_1(s)(x)|^2|\gamma_2(s)(y)|^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_{O_\varepsilon^\delta} \mathrm{d}y \int_{\mathbb{R}^3\setminus O_\varepsilon^{2\delta}} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2\,\mathrm{d}x \\
&\quad + 2\int_{O_\varepsilon^\delta} \mathrm{d}y \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2\,\mathrm{d}x \\
&\quad + 2\int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} \mathrm{d}y \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2\,\mathrm{d}x \\
&\quad + 2\int_{\mathbb{R}^3\setminus O_\varepsilon^{2\delta}} \mathrm{d}y \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2\,\mathrm{d}x
\end{aligned}$$

$$= 2\int_{O_\varepsilon^\delta} \mathrm{d}y \int_{\mathbb{R}^3\setminus O_\varepsilon^\delta} W(x-y)|\gamma_2(s)(x)|^2|\gamma_1(s)(y)|^2\,\mathrm{d}x$$
$$+ 2\int_{\mathbb{R}^3\setminus O_\varepsilon^\delta} \mathrm{d}y \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2\,\mathrm{d}x.$$

From (3.31) we deduce that

$$\lim_{\varepsilon\to 0}\int_{O_\varepsilon^\delta} \mathrm{d}y \int_{\mathbb{R}^3\setminus O_\varepsilon^\delta} W(x-y)|\gamma_2(s)(x)|^2|\gamma_1(s)(y)|^2\,\mathrm{d}x = 0 \tag{3.32}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus O_\varepsilon^\delta} \mathrm{d}y \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma_2(s)(x)|^2|\gamma_1(s)(y)|^2\,\mathrm{d}x = 0. \tag{3.33}$$

From (3.32) and (3.33) we have (recall that γ_1 and γ_2 depend on ε)

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |W(x-y)(|\gamma(s)(x)|^2|\gamma(s)(y)|^2 - |\gamma_1(s)(x)|^2|\gamma_1(s)(y)|^2 - |\gamma_1(s)(x)|^2|\gamma_2(s)(y)|^2)|\,\mathrm{d}x\,\mathrm{d}y = o(1).$$

Thus, we see that, as $\varepsilon \to 0$,

$$\Gamma_\varepsilon(\gamma(s)) \geqslant \Gamma_\varepsilon(\gamma_1(s)) + \Gamma_\varepsilon(\gamma_2(s)) + o(1).$$

Also,

$$\Gamma_\varepsilon(\gamma_2(s)) \geqslant -\frac{1}{4}\int_{(\mathbb{R}^3\setminus O_\varepsilon)\times(\mathbb{R}^3\setminus O_\varepsilon)} W(x-y)|\gamma_2(s)(x)|^2|\gamma_2(s)(y)|^2\,\mathrm{d}x\,\mathrm{d}y \geqslant o(1).$$

Therefore, it follows that

$$\Gamma_\varepsilon(\gamma(s)) \geqslant \Gamma_\varepsilon(\gamma_1(s)) + o(1). \tag{3.34}$$

For $i = 1, \ldots, k$, we define

$$\gamma_1^i(s)(y) = \begin{cases} \gamma_1(s)(y) & \text{for } y \in (O^i)_\varepsilon^{2\delta}, \\ 0 & \text{for } y \notin (O^i)_\varepsilon^{2\delta}. \end{cases}$$

Note that

$$(A_1 + \cdots + A_n - 1)_+ \geqslant \sum_{i=1}^n (A_i - 1)_+$$

for $A_1, \ldots, A_n \geqslant 0$. Then we see that

$$\Gamma_\varepsilon(\gamma_1(s)) \geqslant \sum_{i=1}^k \Gamma_\varepsilon(\gamma_1^i(s)) = \sum_{i=1}^k \Gamma_\varepsilon^i(\gamma_1^i(s)). \tag{3.35}$$

From proposition 3.3(ii), and since $0 < \alpha < E - \tilde{E}$, we obtain that $\gamma_1^i \in \Phi_\varepsilon^i$ for all $i \in \{1, \ldots, k\}$. Thus, by [13, proposition 3.4], proposition 3.4 and (3.35), we deduce that, as $\varepsilon \to 0$,

$$\max_{s\in T} \Gamma_\varepsilon(\gamma(s)) \geqslant E + o(1).$$

Since $\limsup_{\varepsilon\to 0} D_\varepsilon \leqslant E$, this contradicts (3.30).

Now let (u_n^R) be a Palais–Smale sequence corresponding to a fixed small $\varepsilon > 0$. Since (u_n^R) is bounded in $H_0^1(B(0, R/\varepsilon))$, and by corollary 3.2, we have that, up to a subsequence, u_n^R converges strongly to u^R in $H_0^1(B(0, R/\varepsilon))$. We observe that u^R is a critical point of Γ_ε on $H_0^1(B(0, R/\varepsilon))$, and it solves

$$\begin{aligned}\left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon\right)^2 u^R + V_\varepsilon u^R \\ = (W * |u^R|^2)u^R - 5\left(\int \chi_\varepsilon |u^R|^2 \,\mathrm{d}y - 1\right)_+^{3/2} \chi_\varepsilon u^R \quad \text{in } B\left(0, \frac{R}{\varepsilon}\right).\end{aligned} \tag{3.36}$$

Exploiting Kato's inequality,

$$\Delta |u^R| \geqslant -\,\mathrm{Re}\left(\frac{\overline{u^R}}{|u^R|}\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon\right)^2 u^R\right),$$

we obtain

$$\Delta |u^R| \geqslant V_\varepsilon |u^R| - (W * |u^R|^2)|u^R| + 5\left(\int \chi_\varepsilon |u^R|^2 \,\mathrm{d}y - 1\right)_+^{3/2} \chi_\varepsilon |u^R| \quad \text{in } \mathbb{R}^3. \tag{3.37}$$

Moreover, by Moser iteration, it follows that $\|u^R\|_{L^\infty}$ is bounded. Since $(Q_\varepsilon(u^R))$ is uniformly bounded for small $\varepsilon > 0$, we derive that $(W * |u^R|^2)|u^R| \leqslant \frac{1}{2} V_\varepsilon |u^R(y)|$ if $|y| \geqslant 2R_0$. Applying a comparison principle, we derive that

$$|u^R(y)| \leqslant C \exp\{-(|y| - 2R_0)\} \tag{3.38}$$

for some $C > 0$ independent of $R > R_0$. Therefore, as (u^R) is bounded in H_ε, we may assume that it weakly converges to some u_ε in H_ε as $R \to +\infty$. Since u^R is a solution of (3.36), we see from (3.38) that (u^R) converges strongly to $u_\varepsilon \in X_\varepsilon \cap \Gamma_\varepsilon^{D_\varepsilon}$ and it solves

$$\left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon\right)^2 u_\varepsilon + V_\varepsilon u_\varepsilon = (W * |u_\varepsilon|^2)u_\varepsilon - 5\left(\int \chi_\varepsilon |u_\varepsilon|^2 \,\mathrm{d}y - 1\right)_+^{3/2} \chi_\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^3. \tag{3.39}$$

□

3.4. Proof of the main result

We see from proposition 3.7 that there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, Γ_ε has a critical point $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$. Exploiting Kato's inequality,

$$\Delta |u_\varepsilon| \geqslant -\,\mathrm{Re}\left(\frac{\bar{u_\varepsilon}}{|u_\varepsilon|}\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon\right)^2 u_\varepsilon\right),$$

we obtain

$$\Delta |u_\varepsilon| \geqslant V_\varepsilon |u_\varepsilon| - (W * |u_\varepsilon|^2)|u_\varepsilon| + 5\left(\int \chi_\varepsilon |u_\varepsilon|^2 \,\mathrm{d}y - 1\right)_+^{3/2} \chi_\varepsilon |u_\varepsilon| \quad \text{in } \mathbb{R}^3. \tag{3.40}$$

Moreover, by (2.32) and the subsequent bootstrap arguments, we deduce that $u_\varepsilon \in L^q(\mathbb{R}^3)$ for any $q > 2$. Hence, a Moser iteration scheme shows that $(\|u_\varepsilon\|_{L^\infty})$ is bounded. Now, by proposition 3.5, we see that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus(\mathcal{M}^{2\beta})_\varepsilon}|D^\varepsilon u_\varepsilon|^2+\tilde{V}_\varepsilon|u_\varepsilon|^2\,\mathrm{d}y=0$$

and thus, by elliptic estimates [17], we obtain that

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3\setminus(\mathcal{M}^{2\beta})_\varepsilon)}=0. \tag{3.41}$$

This gives the following decay estimate for u_ε on $\mathbb{R}^3\setminus(\mathcal{M}^{2\beta})_\varepsilon\cup(Z^\beta)_\varepsilon$:

$$|u_\varepsilon(x)|\leqslant C\exp\{-c\,\mathrm{dist}(x,(\mathcal{M}^{2\beta})_\varepsilon\cup(Z^\beta)_\varepsilon)\} \tag{3.42}$$

for some constants $C,c>0$. Indeed, from (3.41) we see that

$$\lim_{\varepsilon\to 0}\|W*|u_\varepsilon|^2\|_{L^\infty(\mathbb{R}^3\setminus(\mathcal{M}^{2\beta})_\varepsilon\cup(Z^\beta)_\varepsilon)}=0.$$

Also,

$$\inf\{V_\varepsilon(y):y\notin(\mathcal{M}^{2\beta})_\varepsilon\cup(Z^\beta)_\varepsilon\}>0.$$

Thus, we obtain the decay estimate (3.42) by applying standard comparison principles to (3.40).

If $Z\neq\emptyset$, we will need, in addition, an estimate for $|u_\varepsilon|$ on $(Z^{2\beta})_\varepsilon$. Let $\{H^i\}_{i\in I}$ be the connected components of $\mathrm{int}(Z^{3\delta})$ for some index set I. Note that $Z\subset\bigcup_{i\in I}H^i$ and Z is compact. Thus, the set I is finite. For each $i\in I$, let (ϕ^i,λ_1^i) be a pair of first positive eigenfunction and eigenvalue of $-\Delta$ on $(H^i)_\varepsilon$ with Dirichlet boundary condition. From now on we fix an arbitrary $i\in I$. By using the fact that $(Q_\varepsilon(u_\varepsilon))$ is bounded we see that, for some constant $C>0$,

$$\|u_\varepsilon\|_{L^3((H^i)_\varepsilon)}\leqslant C\varepsilon^{3/\mu}. \tag{3.43}$$

Thus, from the Hardy–Littlewood–Sobolev inequality we have, for some $C>0$,

$$\|W*|u_\varepsilon|^2\|_{L^\infty((H^i)_\varepsilon)}\leqslant C\|u_\varepsilon\|^2_{L^3((H^i)_\varepsilon)}\leqslant C\varepsilon^6.$$

Denote $\phi^i_\varepsilon(y)=\phi^i(\varepsilon y)$. Then, for sufficiently small $\varepsilon>0$, we deduce that, for $y\in\mathrm{int}((H^i)_\varepsilon)$,

$$\Delta\phi^i_\varepsilon(y)-V_\varepsilon(x)\phi^i_\varepsilon(y)+(W*|u_\varepsilon(y)|^2)\phi^i_\varepsilon(y)\leqslant(C\varepsilon^6-\lambda_1\varepsilon^2)\phi^i_\varepsilon\leqslant 0. \tag{3.44}$$

Now, since $\mathrm{dist}(\partial(Z^{2\beta})_\varepsilon,(Z^\beta)_\varepsilon)=\beta/\varepsilon$, we see from (3.42) that, for some constants $C,c>0$,

$$\|u_\varepsilon\|_{L^\infty(\partial(Z^{2\beta})_\varepsilon)}\leqslant C\exp\left\{-\frac{c}{\varepsilon}\right\}. \tag{3.45}$$

We normalize ϕ^i, requiring that

$$\inf_{y\in(H^i)_\varepsilon\cap\partial(Z^{2\delta})_\varepsilon}\phi^i_\varepsilon(y)=C\exp\left\{-\frac{c}{\varepsilon}\right\} \tag{3.46}$$

for the same C, $c>0$ as in (3.45). Then we see that, for some $\kappa>0$,

$$\phi^i_\varepsilon(y)\leqslant\kappa C\exp\left\{\frac{-c}{\varepsilon}\right\},\quad y\in(H^i)_\varepsilon\cap(Z^{2\beta})_\varepsilon.$$

Now we deduce, using (3.43)–(3.46), that, for each $i \in I$, $|u_\varepsilon| \leqslant \phi^i_\varepsilon$ on $(H^i)_\varepsilon \cap (Z^{2\beta})_\varepsilon$. Therefore,

$$|u_\varepsilon(y)| \leqslant C \exp\left\{\frac{-c}{\varepsilon}\right\} \quad \text{on } (Z^{2\delta})_\varepsilon \tag{3.47}$$

for some $C, c > 0$. Now (3.42) and (3.47) imply that $Q_\varepsilon(u_\varepsilon) = 0$ for sufficiently small $\varepsilon > 0$ and thus u_ε satisfies (1.5). Now using propositions 2.14 and 3.5, we readily deduce that the properties of u_ε given in theorem 1.1 hold. Here, in (1.7) we also use lemma 2.11.

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