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EXISTENCE, MULTIPLICITY, PERTURBATION, AND CONCENTRATION RESULTS FOR A CLASS OF QUASI-LINEAR ELLIPTIC PROBLEMS

MARCO SQUASSINA

To my parents and to Maria and Giulia

ABSTRACT. The aim of this monograph is to present a comprehensive survey of results about existence, multiplicity, perturbation from symmetry and concentration phenomena for the quasi-linear elliptic equation

$$-\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x,u)D_{x_i}u)+\frac{1}{2}\sum_{i,j=1}^{n} D_u a_{ij}(x,u)D_{x_i}uD_{x_j}u = g(x,u) \quad \text{in } \Omega,$$

where Ω is a smooth domain of $\mathbb{R}^n$, $n \geq 1$. Under natural assumptions on the coefficients a_{ij}, the above problem admits a standard variational structure, but the associated functional $f : H_0^1(\Omega) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_{x_i}uD_{x_j}u\,dx - \int_\Omega G(x,u)\,dx,$$

turns out to be merely continuous. Therefore, some tools of non-smooth critical point theory will be employed throughout the various sections.

CONTENTS

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Key words and phrases. Critical points; non-smooth functionals; weak slope; deformation theorem; invariant functionals, mountain pass theorem; relative category; calculus of variations, Euler equations; quasi-linear elliptic equations; multiple solutions; quasi-linear variational inequalities; Palais-Smale condition; lack of compactness; singularly perturbed equations; spiked solutions; variational identities; nonexistence results.

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Preface

This monograph is an updated, expanded and restyled elaboration of the Ph.D. thesis that the author defended at the University of Milan on January 2002. It contains some of the author's researches undertaken from 1997 to 2003 in the field of variational quasi-linear elliptic partial differential equations, under the supervision of Marco Degiovanni. The author thanks him for his teaching, encouragement and advice. The author is grateful to Lucio Boccardo and Filomena Pacella for supporting a couple of stay at Rome University La Sapienza in 2002 and 2005. Further thanks are due to the Managing Editors of the Electronic Journal of Differential Equations, in particular to Professor Alfonso Castro for his kindness. The author was supported by the MIUR research project "Variational and Topological Methods in the Study of Nonlinear Phenomena" and by the Istituto Nazionale di Alta Matematica "F.Severi".

The presentation of the material is essentially self-contained. It only requires some basic knowledge in functional analysis as well as in the theory of linear elliptic problems. The work is arranged into nine paragraphs, and each of these is divided into various numbered subsections. All results are formally stated as Theorems, Propositions, Lemmas or Corollaries which are numbered by their section number and order within that section. Throughout the manuscript formulae have double indexing in each section, the first digit being the section number. When formulae from another section are referred to, the number corresponding to the section is placed first.

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NOTATION

(1) $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ denote the set of natural, integer, rational, real numbers;
(2) $\mathbb{R}^n$ (or $\mathbb{R}^N$) is the usual real Euclidean space;
(3) Ω is an open set (often implicitly assumed smooth) in $\mathbb{R}^n$;
(4) $\partial\Omega$ is the boundary of Ω;
(5) a.e. stands for almost everywhere;
(6) p' is the conjugate exponent of p;
(7) $L^p(\Omega)$ is the space of u measurable with $\int_\Omega |u|^p dx < \infty$, $1 \leq p < \infty$;
(8) $L^\infty(\Omega)$ is the space of u measurable with $|u(x)| \leq C$ for a.e. $x \in \Omega$;
(9) $\|\cdot\|_p$ and $\|\cdot\|_\infty$ norms of the spaces L^p and L^∞;
(10) $D_{x_i}u(x)$ is the i-th partial derivative of u at x;
(11) $\nabla u(x)$ stands for $(D_{x_1}u(x), \dots, D_{x_n}u(x))$;
(12) $\Delta u(x)$ stands for $\sum_{i=1}^n D^2_{x_i^2}u(x)$;
(13) $H^1(\Omega)$, $H^1(\mathbb{R}^n)$, $H^1_0(\Omega)$, $W^{1,p}(\Omega)$, $W^{1,p}(\mathbb{R}^n)$, $W^{1,p}_0(\Omega)$ are Sobolev spaces;
(14) $H^{-1}(\Omega)$, $W^{-1,p'}_0(\Omega)$ are the first duals of Sobolev spaces;
(15) $W^{k,p}(\mathbb{R}^n)$, $W^{k,p}_0(\Omega)$ denotes higher order Sobolev spaces;
(16) $\|\cdot\|_{1,p}$, $\|\cdot\|_{k,p}$, $\|\cdot\|_{-1,p}$ norms of the Sobolev spaces;
(17) $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ indicate the space locally Lipschitz functions;
(18) $C^\infty_c(\Omega)$ functions differentiable at any order with compact support;
(19) $\mathcal{L}^n(E)$ denotes Lebesgue measure of E;
(20) $\mathcal{H}^{n-1}(A)$ denotes the Hausdorff measure of A;
(21) $\mathcal{H}$ usually stands for a suitable deformation;
(22) $|df|(u)$ stands for the weak slope of f at u;
(23) (u_m) denotes a sequence of scalar functions;
(24) (u^m) denotes a sequence of vector valued functions;
(25) u^+ (resp. u^-) is the positive (resp. negative) part of u;
(26) $\rightharpoonup$ (resp. $\to$) stands for the weak (resp. strong) convergence;
(27) $\lim_n$ means the limit as $n \to +\infty$;
(28) $B_r(x)$ or $B(x,r)$ is the ball of center x and radius r;
(29) $d(x, E)$ is the distance of x from E.
(30) $\langle \varphi, x \rangle$ evaluation of the linear functional φ at x;
(31) $x \cdot y$ scalar product between elements $x, y \in \mathbb{R}^n$;
(32) δ_{ij} is 1 for $i = j$ and 0 for $i \neq j$;
(33) χ_E (or 1_E) is the characteristic function of the set E;
(34) $A \oplus B$ is the direct sum between A and B.

1. INTRODUCTION

The recent years have been marked out by an evergrowing interest in the research of solutions (and, besides, of their various qualitative behaviors) of semi-linear elliptic problems via techniques of classical critical point theory. Readers which are interested in these aspects may look at the following books: Aubin-Ekeland [13], Chabrowski [39, 40], Ghoussoub [74], Mawhin-Willem [104], Rabinowitz [121], Struwe [137], Willem [148] and Zeidler [149].

The present work aims to show how various achievements, well-established in the semilinear case, can be extended to a more general class of problems. More precisely, let Ω be an open bounded subset in $\mathbb{R}^n$ ($n \geq 2$) and $f : H_0^1(\Omega) \to \mathbb{R}$ a functional of the form

$$f(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_{x_i} u D_{x_j} u \, dx - \int_\Omega G(x,u)\, dx.$$

Since the pioneering paper of Ambrosetti-Rabinowitz [5], critical point theory has been successfully applied to the functional f, yielding several important results (see e.g. [42, 104, 121, 137]). However, the assumption that $f : H_0^1(\Omega) \to \mathbb{R}$ is of class C^1 turns out to be very restrictive for more general functionals of calculus of variations, like

$$f(u) = \int_\Omega \mathcal{L}(x,u,\nabla u)\, dx - \int_\Omega G(x,u)\, dx,$$

(see e.g. [53]). In particular, if f has the form

$$f(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_{x_i} u D_{x_j} u \, dx - \int_\Omega G(x,u)\, dx,$$

we may expect f to be of class C^1 only when the a_{ij}'s are independent of u or when $n = 1$. In fact, if f was locally Lipschitz continuous, for $u \in H_0^1(\Omega)$, we would have

$$\sup \left\{ f'(u)(v) : v \in C_c^\infty(\Omega),\ \|v\|_{H_0^1(\Omega)} \leq 1 \right\} < \infty,$$

that is to say

$$\sum_{i,j=1}^n D_s a_{ij}(x,u) D_{x_i} u D_{x_j} u \in H^{-1}(\Omega).$$

The above term naturally belongs to $L^1(\Omega)$, which is not included in $H^{-1}(\Omega)$ for $n \geq 2$. On the other hand, since the papers of Chang [43] and Marino-Scolozzi [102], techniques of critical point theory have been extended to some classes of non-smooth functionals. In our setting, in which f is naturally continuous but not locally Lipschitz, it turns out to be convenient to apply the theory developed in [50, 58, 87, 88] according to the approach started by Canino [33]. Let us point out that a different approach has been also considered in the literature. If we consider the space $H_0^1(\Omega) \cap L^\infty(\Omega)$ endowed with the family of norms

$$\|u\|_\varepsilon = \|u\|_{H_0^1} + \varepsilon \|u\|_{L^\infty}, \quad \varepsilon > 0,$$

then, under suitable assumptions, f is of class C^1 in $(H_0^1(\Omega) \cap L^\infty(\Omega), \|\cdot\|_\varepsilon)$ for each $\varepsilon > 0$. This allows an approximation procedure by smooth problems (the original one is obtained as a limit when $\varepsilon \to 0$). The papers of Struwe [138] and Arcoya-Boccardo [6, 7] follow, with some variants, this kind of approach. However, in view of multiplicity results, it is hard to keep the multiplicity of solutions at the limit. In particular, when f is even and satisfies assumptions of Ambrosetti-Rabinowitz type, the existence of infinitely many solutions has been so far proved only by the former approach. The aim of this manuscript is to present some results concerning existence, nonexistence, multiplicity, perturbation from symmetry, and concentration for quasi-linear problems such as

$$\begin{gathered} -\sum_{i,j=1}^n D_{x_j}(a_{ij}(x,u) D_{x_i} u) + \frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x,u) D_{x_i} u D_{x_j} u = g(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

and even for the more general class of elliptic problems

$$-\operatorname{div}\left(\nabla_\xi \mathcal{L}(x,u,\nabla u)\right) + D_s \mathcal{L}(x,u,\nabla u) = g(x,u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega,$$

including the case when g reaches the critical growth with respect to the Sobolev embedding. New results have been obtained in the following situations:

Section 3: infinitely many solutions for quasi-linear problems with odd nonlinearities; existence of a weak solution for a general class of Euler's equations of multiple integrals of calculus of variations; existence and multiplicity for quasi-linear elliptic equations having unbounded coefficients (cf. [133, 134, 115]).

Section 4: multiplicity of solutions for perturbed symmetric quasi-linear elliptic problems; multiplicity results for semi-linear systems with broken symmetry and non-homogeneous boundary data (cf. [129, 130, 30, 111]).

Section 5: problems of jumping type for a general class of Euler's equations of multiple integrals of calculus of variations; problems of jumping type for a general class of nonlinear variational inequalities (cf. [80, 81]).

Section 6: positive entire solutions for fully nonlinear elliptic equations; existence of two solutions for fully nonlinear problems at critical growth with perturbations of lower order; asymptotics of solutions for a class of nonlinear problems at nearly critical growth (cf. [128, 135, 131, 108]).

Section 7: concentration phenomena for singularly perturbed quasi-linear elliptic equations. Existence of families of solutions with a spike-like shape around a suitable point (cf. [132]).

Section 8: multi-peak solutions for degenerate singularly perturbed elliptic equations. Existence of families of solutions with multi spike-like profile around suitable points (cf. [75]).

Section 9: Pucci-Serrin type identities for C^1 solutions of Euler's equations and related non-existence results (cf. [59]).

For the sake of completeness, we wish to mention a quite recent paper [35] dealing with the variational bifurcation for quasi-linear elliptic equations (extending some early results due to Rabinowitz in the semi-linear case [122]) and the paper [92] regarding improved Morse index type estimates for the functional f.

2. Review of Critical Point Theory

In this section, we shall recall some results of abstract critical point theory [36, 50, 58, 87, 88]. For the proofs, we refer to [36] or [50].

2.1. **Notions of non-smooth analysis.** Let X be a metric space endowed with the metric d and let $f : X \to \overline{\mathbb{R}}$ be a function. We denote by $B_r(u)$ the open ball of center u and radius r and we set

$$\operatorname{epi}(f) = \{(u, \lambda) \in X \times \mathbb{R} : f(u) \le \lambda\}.$$

In the following, $X \times \mathbb{R}$ will be endowed with the metric

$$d((u,\lambda),(v,\mu)) = \Big(d(u,v))^2 + (\lambda - \mu)^2\Big)^{1/2}$$

and $\operatorname{epi}(f)$ with the induced metric.

Definition 2.1. For every $u \in X$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : (B_\delta(u, f(u)) \cap \operatorname{epi}(f)) \times [0, \delta] \to X$$

satisfying
$$d(\mathcal{H}((v,\mu),t),v) \le t, \quad f(\mathcal{H}((v,\mu),t)) \le \mu - \sigma t,$$
whenever $(v,\mu) \in B_\delta(u, f(u)) \cap \operatorname{epi}(f)$ and $t \in [0,\delta]$. The extended real number $|df|(u)$ is called the weak slope of f at u.

Proposition 2.2. *Let $u \in X$ with $f(u) \in \mathbb{R}$. If (u_h) is a sequence in X with $u_h \to u$ and $f(u_h) \to f(u)$, then we have $|df|(u) \le \liminf_h |df|(u_h)$.*

Remark 2.3. If the restriction of f to $\{u \in X : f(u) \in \mathbb{R}\}$ is continuous, then
$$|df| : \{u \in X : f(u) \in \mathbb{R}\} \to [0,+\infty]$$
is lower semi-continuous.

Proposition 2.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Set*
$$\mathcal{D}(f) := \{u \in X : f(u) < +\infty\}$$
and assume that $f|_{\mathcal{D}(f)}$ is continuous. Then for every $u \in \mathcal{D}(f)$ we have
$$|df|(u) = \big|df|_{\mathcal{D}(f)}\big|(u)$$
and this value is in turn equal to the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map
$$\mathcal{H} : (B_\delta(u) \cap \mathcal{D}(f)) \times [0,\delta] \to X$$
satisfying
$$d(\mathcal{H}(v,t),v) \le t, \quad f(\mathcal{H}(v,t)) \le f(v) - \sigma t,$$
whenever $v \in B_\delta(u) \cap \mathcal{D}(f)$ and $t \in [0,\delta]$.

Definition 2.5. An element $u \in X$ is said to be a (lower) critical point of f if $|df|(u) = 0$. A real number c is said to be a (lower) critical value of f if there exists a critical point $u \in X$ of f such that $f(u) = c$. Otherwise c is said to be a regular value of f.

Definition 2.6. Let c be a real number. The function f is said to satisfy the Palais-Smale condition at level c ($(CPS)_c$ for short), if every sequence (u_h) in X with $|df|(u_h) \to 0$ and $f(u_h) \to c$ admits a subsequence (u_{h_k}) converging in X to some u.

Let us also introduce some usual notations. For every $b \in \mathbb{R} \cup \{+\infty\}$ and $c \in \mathbb{R}$ we set
$$f^b = \{u \in X : f(u) \le b\}, \quad K_c = \{u \in X : |df|(u) = 0, f(u) = c\}.$$

Theorem 2.7 (Deformation Theorem)**.** *Let $c \in \mathbb{R}$. Assume that X is complete, $f : X \to \mathbb{R}$ is a continuous function which satisfies $(CPS)_c$. Then, given $\bar\varepsilon > 0$, a neighborhood U of K_c (if $K_c = \emptyset$, we allow $U = \emptyset$) and $\lambda > 0$, there exist $\varepsilon > 0$ and a continuous map $\eta : X \times [0,1] \to X$ such that for every $u \in X$ and $t \in [0,1]$ we have:*

(a) $d(\eta(u,t),u) \le \lambda t$;
(b) $f(\eta(u,t)) \le f(u)$;
(c) $f(u) \notin]c - \bar\varepsilon, c + \bar\varepsilon[\Rightarrow \eta(u,t) = u$;
(d) $\eta(f^{c+\varepsilon} \setminus U, 1) \subset f^{c-\varepsilon}$.

Theorem 2.8 (Noncritical Interval Theorem)**.** *Let $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ ($a < b$). Assume that $f : X \to \mathbb{R}$ is a continuous function which has no critical points u with $a \le f(u) \le b$, that $(CPS)_c$ holds and f^c is complete whenever $c \in [a,b[$. Then there exists a continuous map $\eta : X \times [0,1] \to X$ such that for every $u \in X$ and $t \in [0,1]$ we have:*

(a) $\eta(u,0) = u$;

(b) $f(\eta(u,t)) \leq f(u)$;
(c) $f(u) \leq a \Rightarrow \eta(u,t) = u$;
(d) $f(u) \leq b \Rightarrow f(\eta(u,1)) \leq a$.

Theorem 2.9. *Let X be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a function such that $\mathcal{D}(f)$ is closed in X and $f|_{\mathcal{D}(f)}$ is continuous. Let u_0, v_0, v_1 be in X and suppose that there exists $r > 0$ such that $\|v_0 - u_0\|_X < r$, $\|v_1 - u_0\|_X > r$, $\inf f(\overline{B_r(u_0)}) > -\infty$, and*

$$a' = \inf\{f(u) : u \in X, \|u - u_0\|_X = r\} > \max\{f(v_0), f(v_1)\}.$$

Let

$$\Gamma = \{\gamma : [0,1] \to \mathcal{D}(f) \quad \text{continuous with} \quad \gamma(0) = v_0, \gamma(1) = v_1\}$$

and assume that $\Gamma \neq \emptyset$ and that f satisfies the Palais-Smale condition at the two levels

$$c_1 = \inf f(\overline{B_r(u_0)}), \quad c_2 = \inf_{\gamma \in \Gamma} \max_{[0,1]} (f \circ \gamma).$$

Then $-\infty < c_1 < c_2 < +\infty$ and there exist at least two critical points u_1, u_2 of f such that $f(u_i) = c_i$ $(i = 1, 2)$.

We now recall the mountain pass theorem without Palais-Smale.

Theorem 2.10. *Let X is a Banach space and $f : X \to \mathbb{R}$ is a continuous functional. Assume that the following facts hold:*

(a) *There exist $\eta > 0$ and $\varrho > 0$ such that*

$$\forall u \in X : \|u\|_X = \varrho \Rightarrow f(u) > \eta;$$

(b) *$f(0) = 0$ and there exists $w \in X$ such that:*

$$f(w) < \eta \quad \text{and} \quad \|w\|_X > \varrho.$$

Moreover, let us set

$$\Phi = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \quad \gamma(1) = w\}$$

and

$$\eta \leq \beta = \inf_{\gamma \in \Phi} \max_{t \in [0,1]} f(\gamma(t)).$$

Then there exists a Palais-Smale sequence for f at level β.

In the next theorem, we recall a generalization of the classical perturbation argument of Bahri, Berestycki, Rabinowitz and Struwe devised around 1980 for dealing with problems with broken symmetry adapted to our non-smooth framework (See [119]).

Theorem 2.11. *Let X be a Hilbert space endowed with a norm $\|\cdot\|_X$ and let $f : X \to \mathbb{R}$ be a continuous functional. Assume that there exists $M > 0$ such that f satisfies the concrete Palais-Smale condition at each level $c \geq M$. Let Y be a finite dimensional subspace of X and $u^* \in X \setminus Y$ and set*

$$Y^* = Y \oplus \langle u^* \rangle, \quad Y^*_+ = \{u + \lambda u^* \in Y^* : u \in Y, \quad \lambda \geq 0\}.$$

Assume now that $f(0) \leq 0$ and that

(a) *There exists $R > 0$ such that*

$$\forall u \in Y : \|u\|_X \geq R \Rightarrow f(u) \leq f(0);$$

(b) *there exists $R^* \geq R$ such that:*

$$\forall u \in Y^* : \|u\|_X \geq R^* \Rightarrow f(u) \leq f(0).$$

Let us set

$$\mathcal{P} = \{\gamma \in C(X, X) : \ \gamma \ \text{odd}, \ \gamma(u) = u \ \textit{if} \ \max\{f(u), f(-u)\} \leq 0\}.$$

Then, if

$$c^* = \inf_{\gamma \in \mathcal{P}} \sup_{u \in Y_+^*} f(\gamma(u)) > c = \inf_{\gamma \in \mathcal{P}} \sup_{u \in Y} f(\gamma(u)) \geq M,$$

f admits at least one critical value $\bar{c} \geq c^*$.

2.2. **The case of lower semi-continuous functionals.** Let X be a metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. We set

$$\operatorname{dom}(f) = \{u \in X : \ f(u) < +\infty\} \quad \text{and} \quad \operatorname{epi} f = \{(u, \eta) \in X \times \mathbb{R} : \ f(u) \leq \eta\}.$$

The set $\operatorname{epi} f$ is endowed with the metric

$$d\left((u, \eta), (v, \mu)\right) = \left(d(u, v)^2 + (\eta - \mu)^2\right)^{1/2}.$$

Let us define the function $\mathcal{G}_f : \operatorname{epi} f \to \mathbb{R}$ by setting

$$\mathcal{G}_f(u, \eta) = \eta. \tag{2.1}$$

Note that $\mathcal{G}_f$ is Lipschitz continuous of constant 1.
From now on we denote with $B(u, \delta)$ the open ball of center u and of radius δ. We recall the definition of the weak slope for a continuous function introduced in [50, 58, 87, 88].

Definition 2.12. Let X be a complete metric space, $g : X \to \mathbb{R}$ a continuous function, and $u \in X$. We denote by $|dg|(u)$ the supremum of the real numbers σ in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B(u, \delta) \times [0, \delta] \to X,$$

such that, for every v in $B(u, \delta)$, and for every t in $[0, \delta]$ it results

$$d(\mathcal{H}(v, t), v) \leq t,$$
$$g(\mathcal{H}(v, t)) \leq g(v) - \sigma t.$$

The extended real number $|dg|(u)$ is called the weak slope of g at u.

According to the previous definition, for every lower semi-continuous function f we can consider the metric space $\operatorname{epi} f$ so that the weak slope of $\mathcal{G}_f$ is well defined. Therefore, we can define the weak slope of a lower semi-continuous function f by using $|d\mathcal{G}_f|(u, f(u))$.

More precisely, we have the following

Definition 2.13. For every $u \in \operatorname{dom}(f)$ let

$$|df|(u) = \begin{cases} \dfrac{\left|\mathcal{G}_f\right|(u, f(u))}{\sqrt{1 - \left|\mathcal{G}_f\right|(u, f(u))^2}}, & \text{if } \left|\mathcal{G}_f\right|(u, f(u)) < 1, \\ +\infty, & \text{if } \left|\mathcal{G}_f\right|(u, f(u)) = 1. \end{cases}$$

The previous notion allow us to give the following concepts.

Definition 2.14. Let X be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function. We say that $u \in \operatorname{dom}(f)$ is a (lower) critical point of f if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (lower) critical value of f if there exists a (lower) critical point $u \in \operatorname{dom}(f)$ of f with $f(u) = c$.

Definition 2.15. Let X be a complete metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function and let $c \in \mathbb{R}$. We say that f satisfies the Palais-Smale condition at level c ($(PS)_c$ in short), if every sequence $\{u_n\}$ in $\mathrm{dom}(f)$ such that

$$|df|(u_n) \to 0,$$
$$f(u_n) \to c,$$

admits a subsequence $\{u_{n_k}\}$ converging in X.

For every $\eta \in \mathbb{R}$, let us define the set

$$f^\eta = \{u \in X : f(u) < \eta\}. \tag{2.2}$$

The next result gives a criterion to obtain an estimate from below of $|df|(u)$ (cf. [58]).

Proposition 2.16. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function defined on the complete metric space X, and let $u \in \mathrm{dom}(f)$. Assume that there exist $\delta > 0$, $\eta > f(u)$, $\sigma > 0$ and a continuous function $\mathcal{H} : B(u,\delta) \cap f^\eta \times [0,\delta] \to X$ such that*

$$d(\mathcal{H}(v,t), v) \le t, \quad \forall\, v \in B(u,\delta) \cap f^\eta,$$
$$f(\mathcal{H}(v,t)) \le f(v) - \sigma t, \quad \forall\, v \in B(u,\delta) \cap f^\eta.$$

Then $|df|(u) \ge \sigma$.

We will also use the notion of equivariant weak slope (see [36]).

Definition 2.17. Let X be a normed linear space and $f : X \to \mathbb{R} \cup \{+\infty\}$ an even lower semi-continuous function with $f(0) < +\infty$. For every $(0,\eta) \in \mathrm{epi}\, f$ we denote by $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\eta)$ the supremum of the numbers σ in $[0,\infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : (B((0,\eta),\delta) \cap \mathrm{epi}\, f) \times [0,\delta] \to \mathrm{epi}\, f$$

satisfying

$$d(\mathcal{H}((w,\mu),t), (w,\mu)) \le t, \quad \mathcal{H}_2((w,\mu),t) \le \mu - \sigma t,$$
$$\mathcal{H}_1((-w,\mu),t) = -\mathcal{H}_1((w,\mu),t),$$

for every $(w,\mu) \in B((0,\eta),\delta) \cap \mathrm{epi}\, f$ and $t \in [0,\delta]$.

To compute $|d\mathcal{G}_f|(u,\eta)$, the next result will be useful (cf. [58]).

Proposition 2.18. *Let X be a normed linear space, $J : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous functional, $I : X \to \mathbb{R}$ a C^1 functional and let $f = J + I$. Then the following facts hold:*

(a) *For every $(u,\eta) \in \mathrm{epi}(f)$ we have*

$$|d\mathcal{G}_f|(u,\eta) = 1 \iff |d\mathcal{G}_J|(u,\eta - I(u)) = 1\,;$$

(b) *if J and I are even, for every $\eta \ge f(0)$, we have*

$$|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\eta) = 1 \iff |d_{\mathbb{Z}_2}\mathcal{G}_J|(0,\eta - I(0)) = 1\,;$$

(c) *if $u \in \mathrm{dom}(f)$ and $I'(u) = 0$, then $|df|(u) = |dJ|(u)$.*

Proof. Assertions (a) and (c) follow by arguing as in [58]. Assertion (b) can be reduced to (a) after observing that, since I is even, it results $I'(0) = 0$. □

In [50, 58] variational methods for lower semi-continuous functionals are developed. Moreover, it is shown that the following condition is fundamental in order to apply the abstract theory to the study of lower semi-continuous functions

$$\forall (u,\eta) \in \operatorname{epi} f : f(u) < \eta \quad \Rightarrow \quad \big|\mathcal{G}_f\big|(u,\eta) = 1. \tag{2.3}$$

In the next section we will prove that the functional f satisfies (2.3). The next result gives a criterion to verify condition (2.3) (cf. [60, Corollary 2.11]).

Theorem 2.19. *Let $(u,\eta) \in \operatorname{epi}(f)$ with $f(u) < \eta$. Assume that, for every $\varrho > 0$, there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : \{w \in B(u,\delta) : f(w) < \eta + \delta\} \times [0,\delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t), w) \le \varrho t\,, \quad f(\mathcal{H}(w,t)) \le (1-t) f(w) + t(f(u) + \varrho)$$

whenever $w \in B(u,\delta)$, $f(w) < \eta + \delta$ and $t \in [0,\delta]$. Then $|d\mathcal{G}_f|(u,\eta) = 1$. In addition, if f is even, $u = 0$ and $\mathcal{H}(-w,t) = -\mathcal{H}(w,t)$, then we have $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\eta) = 1$.

Let us now recall from [50] the following result.

Theorem 2.20. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function satisfying (2.3). Assume that there exist $v_0, v_1 \in X$ and $r > 0$ such that $\|v_1 - v_0\| > r$ and*

$$\inf\{f(u) : u \in X,\ \|u - v_0\| = r\} > \max\{f(v_0), f(v_1)\}. \tag{2.4}$$

Let us set

$$\Gamma = \{\gamma : [0,1] \to \operatorname{dom}(f), \quad \gamma \text{ continuous, } \gamma(0) = v_0 \text{ and } \gamma(1) = v_1\}\,,$$

and assume that

$$c_1 = \inf_{\gamma \in \Gamma} \sup_{[0,1]} f \circ \gamma < +\infty$$

and that f satisfies the Palais-Smale condition at the level c_1. Then, there exists a critical point u_1 of f such that $f(u_1) = c_1$. If, moreover,

$$c_0 = \inf f(\overline{B_r(v_0)}) > -\infty,$$

and f satisfies the Palais-Smale condition at the level c_0, then there exists another critical point u_0 of f with $f(u_0) = c_0$.

In the equivariant case we shall apply the following result (see [103]).

Theorem 2.21. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous even function. Let us assume that there exists a strictly increasing sequence (W_h) of finite dimensional subspaces of X with the following properties:*

(a) *There exist $\rho > 0$, $\gamma > f(0)$ and a subspace $V \subset X$ of finite codimension with*
$$\forall u \in V : \|u\| = \rho \quad \Rightarrow \quad f(u) \ge \gamma;$$
(b) *there exists a sequence (R_h) in (ρ, ∞) such that*
$$\forall u \in W_h : \|u\| \ge R_h \quad \Rightarrow \quad f(u) \le f(0);$$
(c) *f satisfies $(PS)_c$ for any $c \ge \gamma$ and f satisfies (2.3);*
(d) *$|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\eta) \ne 0$ for every $\eta > f(0)$.*

Then there exists a sequence $\{u_h\}$ of critical points of f such that $f(u_h) \to +\infty$.

2.3. **Functionals of the calculus of variations.** Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$ and let $f : W_0^{1,p}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ ($N \geq 1$) be a functional of the form

$$f(u) = \int_\Omega \mathcal{L}(x,u,\nabla u)dx. \tag{2.5}$$

The associated Euler's equation is formally given by the quasi-linear problem

$$\begin{gathered} -\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + \nabla_s \mathcal{L}(x,u,\nabla u) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.6}$$

Assume that $\mathcal{L} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ and of class C^1 in (s,ξ) for a.e. $x \in \Omega$. Moreover, assume that there exist $a_0 \in L^1(\Omega)$, $b_0 \in \mathbb{R}$, $a_1 \in L^1_{loc}(\Omega)$ and $b_1 \in L^\infty_{loc}(\Omega)$ such that for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ we have

$$|\mathcal{L}(x,s,\xi)| \leq a_0(x) + b_0|s|^{np/(n-p)} + b_0|\xi|^p, \tag{2.7}$$

$$|\nabla_s \mathcal{L}(x,s,\xi)| \leq a_1(x) + b_1(x)|s|^{np/(n-p)} + b_1(x)|\xi|^p, \tag{2.8}$$

$$\big|\nabla_\xi \mathcal{L}(x,s,\xi)\big| \leq a_1(x) + b_1(x)|s|^{np/(n-p)} + b_1(x)|\xi|^p. \tag{2.9}$$

Conditions (2.8) and (2.9) imply that for every $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ we have

$$\begin{gathered} \nabla_\xi \mathcal{L}(x,u,\nabla u) \in L^1_{loc}(\Omega;\mathbb{R}^{nN}), \\ \nabla_s \mathcal{L}(x,u,\nabla u) \in L^1_{loc}(\Omega;\mathbb{R}^N). \end{gathered}$$

Therefore, for every $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ we have

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + \nabla_s \mathcal{L}(x,u,\nabla u) \in \mathcal{D}'(\Omega;\mathbb{R}^N).$$

Definition 2.22. We say that u is a weak solution of *(2.6)*, if $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ and

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + \nabla_s \mathcal{L}(x,u,\nabla u) = 0$$

in $\mathcal{D}'(\Omega;\mathbb{R}^N)$.

If the integrand $\mathcal{L}$ is subjected to suitable restrictive conditions, it turns out that f is of class C^1 and

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + \nabla_s \mathcal{L}(x,u,\nabla u) \in W^{-1,p'}(\Omega,\mathbb{R}^N)$$

for every $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$. In this regular setting, we have that f satisfies condition $(PS)_c$, if and only of every sequence (u_h) in $W_0^{1,p}(\Omega,\mathbb{R}^N)$ with $f(u_h) \to c$ and

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\big) + \nabla_s \mathcal{L}(x,u_h,\nabla u_h) \to 0$$

strongly in $W^{-1,p'}(\Omega,\mathbb{R}^N)$ has a strongly convergent subsequence in $W_0^{1,p}(\Omega,\mathbb{R}^N)$.

Now, a condition of this kind can be formulated also in our general context, without any reference to the differentiability of the functional f.

Definition 2.23. Let $c \in \mathbb{R}$. A sequence (u_h) in $W_0^{1,p}(\Omega,\mathbb{R}^N)$ is said to be a concrete Palais-Smale sequence at level c ($(CPS)_c-$sequence, in short) for f, if $f(u_h) \to c$,

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\big) + \nabla_s \mathcal{L}(x,u_h,\nabla u_h) \in W^{-1,p'}(\Omega,\mathbb{R}^N)$$

eventually as $h \to \infty$ and

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\big) + \nabla_s \mathcal{L}(x,u_h,\nabla u_h) \to 0$$

strongly in $W^{-1,p'}(\Omega,\mathbb{R}^N)$.

We say that f satisfies the concrete Palais-Smale condition at level c ($(CPS)_c$ in short), if every $(CPS)_c-$sequence for f admits a strongly convergent subsequence in $W_0^{1,p}(\Omega,\mathbb{R}^N)$.

The next result allow us to connect these "concrete" notions with the abstract critical point theory.

Theorem 2.24. *The functional f is continuous and for all $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$,*

$$|df|(u) \geq \sup\Big\{\int_\Omega \big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla v + \nabla_s\mathcal{L}(x,u,\nabla u)\cdot v\big)dx : v \in C_c^\infty,\ \|v\|_{1,p}\leq 1\Big\}.$$

Therefore, if $|df|(u) < +\infty$ it follows

$$-\operatorname{div}\big(\nabla_\xi\mathcal{L}(x,u,\nabla u)\big) + \nabla_s\mathcal{L}(x,u,\nabla u) \in W^{-1,p'}(\Omega,\mathbb{R}^N)$$

and

$$\big\| -\operatorname{div}\big(\nabla_\xi\mathcal{L}(x,u,\nabla u)\big) + \nabla_s\mathcal{L}(x,u,\nabla u)\big\|_{1,p'} \leq |df|(u).$$

Corollary 2.25. *Let $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$, $c \in \mathbb{R}$ and let (u_h) be a sequence in $W_0^{1,p}(\Omega,\mathbb{R}^N)$. Then the following facts hold:*

(a) *If u is a (lower) critical point of f, then u is a weak solution of (2.6));*
(b) *if (u_h) is a $(PS)_c-$sequence for f, then (u_h) is a $(CPS)_c-$sequence for f;*
(c) *if f satisfies $(CPS)_c$, then f satisfies $(PS)_c$.*

By means of the previous result, it is easy to deduce some versions of the Mountain Pass Theorem adapted to the functional f.

Theorem 2.26. *Let (D,S) be a compact pair, let $\psi : S \to W_0^{1,p}(\Omega,\mathbb{R}^N)$ be a continuous map and let*

$$\Phi = \Big\{\varphi \in \mathcal{C}(D, W_0^{1,p}(\Omega,\mathbb{R}^N)) : \varphi_{|S} = \psi\Big\}.$$

Assume that there exists a closed subset A of $W_0^{1,p}(\Omega,\mathbb{R}^N)$ such that

$$\inf_A f \geq \max_{\psi(S)} f, \quad A\cap\psi(S) = \emptyset, \quad A\cap\varphi(D)\neq\emptyset\ \forall\varphi\in\Phi.$$

If f satisfies the concrete Palais-Smale condition at level $c = \inf_{\varphi\in\Phi}\max_{\varphi(D)} f$, then there exists a weak solution u of (2.6) with $f(u) = c$. Furthermore, if $\inf_A f \geq c$, then there exists a weak solution u of (2.6) with $f(u) = c$ and $u \in A$.

Theorem 2.27. *Suppose that*

$$\mathcal{L}(x,-s,-\xi) = \mathcal{L}(x,s,\xi)$$

for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Assume also that

(a) *There exist $\rho > 0$, $\alpha > f(0)$ and a subspace $V \subset W_0^{1,p}(\Omega,\mathbb{R}^N)$ of finite codimension with*
$$\forall u \in V : \|u\| = \rho \Rightarrow f(u)\geq\alpha;$$
(b) *for every finite dimensional subspace $W \subset W_0^{1,p}(\Omega,\mathbb{R}^N)$, there exists $R > 0$ with*
$$\forall u\in W : \|u\|\geq R \Rightarrow f(u)\leq f(0);$$
(c) *f satisfies $(CPS)_c$ for any $c\geq\alpha$.*

Then there exists a sequence $(u_h)\subset W_0^{1,p}(\Omega,\mathbb{R}^N)$ of weak solutions of (2.6) with $\lim_h f(u_h) = +\infty$.

3. Super-linear Elliptic Problems

We refer the reader to [133, 134]. Some parts of these publications have been slightly modified to give the monograph a more uniform appearance.

3.1. **Quasi-linear elliptic systems.**

Many papers have been published on the study of multiplicity of solutions for quasi-linear elliptic equations via non-smooth critical point theory; see e.g. [6, 8, 9, 33, 32, 36, 49, 113, 138]. However, for the vectorial case only a few multiplicity results have been proven: [138, Theorem 3.2] and recently [9, Theorem 3.2], where systems with multiple identity coefficients are treated. In this section, we consider the following diagonal quasi-linear elliptic system, in an open bounded set $\Omega \subset \mathbb{R}^n$ with $n \geq 3$,

$$-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u)D_i u_k) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h = D_{s_k} G(x,u) \quad \text{in } \Omega\,, \tag{3.1}$$

for $k = 1, \dots, N$, where $u : \Omega \to \mathbb{R}^N$ and $u = 0$ on $\partial\Omega$. To prove the existence of weak solutions, we look for critical points of the functional $f : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - \int_\Omega G(x,u)\, dx. \tag{3.2}$$

This functional is not locally Lipschitz if the coefficients a_{ij}^h depend on u; however, as pointed out in [6, 33], it is possible to evaluate f',

$$\begin{aligned} f'(u)(v) &= \int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,u) D_i u_h D_j v_h \, dx \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} D_s a_{ij}^h(x,u)\cdot v D_i u_h D_j u_h \, dx - \int_\Omega D_s G(x,u)\cdot v\, dx \end{aligned}$$

for all $v \in H_0^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$.

To prove our main result and to provide some regularity of solutions, we consider the following assumptions.

- $\big(a_{ij}^h(\cdot, s)\big)$ is measurable in x for every $s \in \mathbb{R}^N$, and of class C^1 in s for a.e. $x \in \Omega$ with $a_{ij}^h = a_{ji}^h$. Furthermore, we assume that there exist $\nu > 0$ and $C > 0$ such that for a.e. $x \in \Omega$, all $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$

$$\sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,s)\xi_i^h \xi_j^h \geq \nu|\xi|^2, \quad \left|a_{ij}^h(x,s)\right| \leq C, \quad \left|D_s a_{ij}^h(x,s)\right| \leq C \tag{3.3}$$

and

$$\sum_{i,j=1}^{n}\sum_{h=1}^{N} s\cdot D_s a_{ij}^h(x,s)\xi_i^h \xi_j^h \geq 0. \tag{3.4}$$

- there exists a bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$, such that for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$, $\sigma \in \{-1, 1\}^N$ and $r, s \in \mathbb{R}^N$

$$\sum_{i,j=1}^{n}\sum_{h=1}^{N}\Big(\frac{1}{2} D_s a_{ij}^h(x,s)\cdot \exp_\sigma(r,s) + a_{ij}^h(x,s) D_{s_h}(\exp_\sigma(r,s))_h\Big)\xi_i^h\xi_j^h \leq 0 \tag{3.5}$$

where $(\exp_\sigma(r,s))_i := \sigma_i \exp[\sigma_i(\psi(r_i) - \psi(s_i))]$ for each $i = 1, \dots, N$.

- the function $G(x,s)$ is measurable in x for all $s \in \mathbb{R}^N$ and of class C^1 in s for a.e. $x \in \Omega$, with $G(x,0) = 0$. Moreover for a.e. $x \in \Omega$ we will denote with $g(x,\cdot)$ the gradient of G with respect to s.
- for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{2n/(n+2)}(\Omega)$ such that

$$|g(x,s)| \le a_\varepsilon(x) + \varepsilon |s|^{(n+2)/(n-2)} \tag{3.6}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^N$ and that there exist $q > 2$, $R > 0$ such that for all $s \in \mathbb{R}^N$ and for a.e. $x \in \Omega$

$$|s| \ge R \Rightarrow 0 < qG(x,s) \le s \cdot g(x,s). \tag{3.7}$$

- there exists $\gamma \in (0, q-2)$ such that for all $\xi \in \mathbb{R}^{nN}$, $s \in \mathbb{R}^N$ and a.e. in Ω

$$\sum_{i,j=1}^n \sum_{h=1}^N s \cdot D_s a_{ij}^h(x,s) \xi_i^h \xi_j^h \le \gamma \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,s) \xi_i^h \xi_j^h. \tag{3.8}$$

Under these assumptions we will prove the following result.

Theorem 3.1. *Assume that for a.e. $x \in \Omega$ and for each $s \in \mathbb{R}^N$*

$$a_{ij}^h(x,-s) = a_{ij}^h(x,s), \quad g(x,-s) = -g(x,s).$$

Then there exists a sequence $(u^m) \subset H_0^1(\Omega, \mathbb{R}^N)$ of weak solutions to (3.1) such that $f(u^m) \to +\infty$ as $m \to \infty$.

The above result is well known for the semi-linear scalar problem

$$-\sum_{i,j=1}^n D_j(a_{ij}(x) D_i u) = g(x,u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Ambrosetti and Rabinowitz in [5, 121] studied this problem using techniques of classical critical point theory. The quasi-linear scalar problem

$$-\sum_{i,j=1}^n D_j(a_{ij}(x,u) D_i u) + \frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u = g(x,u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

was studied in [32, 33, 36] and in [113] in a more general setting. In this case the functional

$$f(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u \, dx - \int_\Omega G(x,u) \, dx$$

is continuous under appropriate conditions, but it is not locally Lipschitz. Consequently, techniques of non-smooth critical point theory have to be applied. In the vectorial case, to my knowledge, problem (3.1) has only been considered in [138, Theorem 3.2] and recently in [9, Theorem 3.2] for coefficients of the type $a_{ij}^{hk}(x,s) = \delta^{hk} \alpha_{ij}(x,s)$.

3.2. **The concrete Palais-Smale condition.** The first step for the $(CPS)_c$ to hold is the boundedness of $(CPS)_c$ sequences.

Lemma 3.2. *For all $c \in \mathbb{R}$ each $(CPS)_c$ sequence of f is bounded in $H_0^1(\Omega, \mathbb{R}^N)$.*

Proof. Let $a_0 \in L^1(\Omega)$ be such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^N$

$$qG(x, s) \leq s \cdot g(x, s) + a_0(x).$$

Now let (u^m) be a $(CPS)_c$ sequence for f and let $w^m \to 0$ in $H^{-1}(\Omega, \mathbb{R}^N)$ such that for all $v \in C_c^\infty(\Omega, \mathbb{R}^N)$,

$$\begin{aligned}\langle w^m, v\rangle &= \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j v_h \, dx \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x, u^m) \cdot v D_i u_h^m D_j u_h^m \, dx - \int_\Omega g(x, u^m) \cdot v.\end{aligned}$$

Taking into account the previous Lemma, for every $m \in \mathbb{N}$ we obtain

$$\begin{aligned}&-\|w^m\|_{H^{-1}(\Omega,\mathbb{R}^N)}\|u^m\|_{H_0^1(\Omega,\mathbb{R}^N)} \\ &\leq \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x, u^m) \cdot u^m D_i u_h^m D_j u_h^m \, dx - \int_\Omega g(x, u^m) \cdot u^m \, dx \\ &\leq \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx + \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x, u^m) \cdot u^m D_i u_h^m D_j u_h^m \, dx - q\int_\Omega G(x, u^m)\, dx + \int_\Omega a_0 \, dx.\end{aligned}$$

Taking into account the expression of f and assumption (3.8), we have that for each $m \in \mathbb{N}$,

$$\begin{aligned}&-\|w^m\|_{H^{-1}(\Omega,\mathbb{R}^N)}\|u^m\|_{H_0^1(\Omega,\mathbb{R}^N)} \\ &\leq -\Big(\frac{q}{2} - 1\Big)\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x, u^m) \cdot u^m D_i u_h^m D_j u_h^m \, dx + qf(u^m) + \int_\Omega a_0 \, dx \\ &\leq -\Big(\frac{q}{2} - 1 - \frac{\gamma}{2}\Big)\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\ &\quad + qf(u^m) + \int_\Omega a_0 \, dx.\end{aligned}$$

Because of (3.3), for each $m \in \mathbb{N}$,

$$\begin{aligned}\nu(q-2-\gamma)\|Du^m\|_2^2 &\le (q-2-\gamma)\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m dx \\ &\le 2\|w^m\|_{H^{-1}(\Omega,\mathbb{R}^N)} \|u^m\|_{H_0^1(\Omega,\mathbb{R}^N)} + 2qf(u^m) + 2\int_\Omega a_0 dx.\end{aligned}$$

Since $w^m \to 0$ in $H^{-1}(\Omega,\mathbb{R}^N)$, (u^m) is a bounded sequence in $H_0^1(\Omega,\mathbb{R}^N)$. □

Lemma 3.3. *If condition (3.6) holds, then the map*

$$\begin{aligned} H_0^1(\Omega,\mathbb{R}^N) &\longrightarrow L^{2n/(n+2)}(\Omega,\mathbb{R}^N) \\ u &\longmapsto g(x,u) \end{aligned}$$

is completely continuous.

The statement of the above lemma is a direct consequence of [36, Theorem 2.2.7]. The next result is crucial for the $(CPS)_c$ condition to hold for our elliptic system.

Lemma 3.4. *Let (u^m) be a bounded sequence in $H_0^1(\Omega,\mathbb{R}^N)$, and set*

$$\begin{aligned}\langle w^m, v\rangle = &\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j v_h\, dx \\ &+ \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u^m)\cdot v D_i u_h^m D_j u_h^m\, dx\end{aligned}$$

for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$. If (w^m) is strongly convergent to some w in $H^{-1}(\Omega,\mathbb{R}^N)$, then (u^m) admits a strongly convergent subsequence in $H_0^1(\Omega,\mathbb{R}^N)$.

Proof. Since (u^m) is bounded, we have $u^m \rightharpoonup u$ for some u up to a subsequence. Each component u_k^m satisfies (2.5) in [22], so we may suppose that $D_i u_k^m \to D_i u_k$ a.e. in Ω for all $k = 1,\dots,N$ (see also [54]). We first prove that

$$\begin{aligned}&\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h\, dx \\ &+ \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u)\cdot u D_i u_h D_j u_h\, dx = \langle w,u\rangle.\end{aligned} \tag{3.9}$$

Let ψ be as in assumption (3.5) and consider the following test functions

$$v^m = \varphi(\sigma_1 \exp[\sigma_1(\psi(u_1)-\psi(u_1^m))],\dots,\sigma_N \exp[\sigma_N(\psi(u_N)-\psi(u_N^m))]),$$

where $\varphi \in C_c^\infty(\Omega)$, $\varphi \ge 0$ and $\sigma_l = \pm 1$ for all l. Therefore, since we have

$$D_j v_k^m = \big(\sigma_k D_j\varphi + (\psi'(u_k) D_j u_k - \psi'(u_k^m) D_j u_k^m)\varphi\big) \exp[\sigma_k(\psi(u_k)-\psi(u_k^m))],$$

we deduce that for all $m \in \mathbb{N}$,

$$\begin{aligned}&\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m (\sigma_h D_j\varphi + \psi'(u_h) D_j u_h \varphi) \exp[\sigma_h(\psi(u_h)-\psi(u_h^m))]\, dx \\ &+ \int_\Omega \sum_{i,j=1}^n \sum_{h,l=1}^N \frac{\sigma_l}{2} D_{s_l} a_{ij}^h(x,u^m) \exp[\sigma_l(\psi(u_l)-\psi(u_l^m))] D_i u_h^m D_j u_h^m \varphi\, dx\end{aligned}$$

$$
- \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \psi'(u_h^m) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))]\varphi \, dx
$$
$$
= \langle w^m, v^m \rangle.
$$

Let us study the behavior of each term of the previous equality as $m \to \infty$. First of all, if $v = (\sigma_1\varphi, \dots, \sigma_N\varphi)$, we have that $v^m \rightharpoonup v$ implies

$$
\lim_m \langle w^m, v^m \rangle = \langle w, v \rangle. \tag{3.10}
$$

Since $u^m \rightharpoonup u$, by Lebesgue's Theorem we obtain

$$
\lim_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m (D_j(\sigma_h\varphi) \tag{3.11}
$$
$$
+ \varphi\psi'(u_h) D_j u_h) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))] \, dx \tag{3.12}
$$
$$
= \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h (D_j v_h + \varphi\psi'(u_h) D_j u_h) \, dx. \tag{3.13}
$$

Finally, note that by assumption (3.5) we have

$$
\sum_{i,j=1}^n \sum_{h=1}^N \Big(\sum_{l=1}^N \frac{\sigma_l}{2} D_{s_l} a_{ij}^h(x,u^m) \exp[\sigma_l(\psi(u_l) - \psi(u_l^m))]
$$
$$
- a_{ij}^h(x,u^m)\psi'(u_h^m) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))] \Big) D_i u_h^m D_j u_h^m \le 0.
$$

Hence, we can apply Fatou's Lemma to obtain

$$
\limsup_m \Big\{ \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,l=1}^N D_{s_l} a_{ij}^h(x,u^m) \exp[\sigma_l(\psi(u_l) - \psi(u_l^m))] D_i u_h^m D_j u_h^m (\sigma_l\varphi) \, dx
$$
$$
- \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \psi'(u_h^m) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))]\varphi \, dx \Big\}
$$
$$
\le \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,l=1}^N D_{s_l} a_{ij}^h(x,u) D_i u_h D_j u_h (\sigma_l\varphi) \, dx
$$
$$
- \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \psi'(u_h)\varphi \, dx ,
$$

which, together with (3.10) and (3.12), yields

$$
\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j v_h \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u) \cdot v D_i u_h D_j u_h \, dx
$$
$$
\ge \langle w, v \rangle
$$

for all test functions $v = (\sigma_1\varphi, \dots, \sigma_N\varphi)$ with $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$, $\varphi \ge 0$. Since we may exchange v with $-v$ we get

$$
\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j v_h \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u) \cdot v D_i u_h D_j u_h \, dx
$$

$$= \langle w, v \rangle$$

for all test functions $v = (\sigma_1 \varphi, \dots, \sigma_N \varphi)$, and since every function $v \in C_c^\infty(\Omega, \mathbb{R}^N)$ can be written as a linear combination of such functions, we infer (3.9). Now, let us prove that

$$\limsup_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \le \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u) D_i u_h D_j u_h \, dx. \tag{3.14}$$

Because of (3.4), Fatou's Lemma implies that

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u \cdot D_s a_{ij}^h(x, u) D_i u_h D_j u_h \, dx \\ &\le \liminf_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u^m \cdot D_s a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx. \end{aligned}$$

Combining this fact with (3.9), we deduce that

$$\begin{aligned} &\limsup_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\ &= \limsup_m \Big[-\frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u^m \cdot D_s a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx + \langle w^m, u^m \rangle \Big] \\ &\le -\frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u \cdot D_s a_{ij}^h(x, u) D_i u_h D_j u_h \, dx + \langle w, u \rangle \\ &= \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u) D_i u_h D_j u_h \, dx, \end{aligned}$$

so that (3.14) is proved. Finally, by (3.3) we have

$$\begin{aligned} &\nu \| Du^m - Du \|_2^2 \\ &\le \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) \big(D_i u_h^m D_j u_h^m - 2 D_i u_h^m D_j u_h + D_i u_h D_j u_h \big) \, dx. \end{aligned}$$

Hence, by (3.14) we obtain

$$\limsup_m \| Du^m - Du \|_2 \le 0$$

which proves that $u^m \to u$ in $H_0^1(\Omega, \mathbb{R}^N)$. □

We now come to the $(CPS)_c$ condition for system (3.1).

Theorem 3.5. *f satisfies $(CPS)_c$ condition for each $c \in \mathbb{R}$.*

Proof. Let (u^m) be a $(CPS)_c$ sequence for f. Since (u^m) is bounded in $H_0^1(\Omega, \mathbb{R}^N)$, from Lemma 3.3 we deduce that, up to a subsequence, $(g(x, u^m))$ is strongly convergent in $H^{-1}(\Omega, \mathbb{R}^N)$. Applying Lemma 3.4, we conclude the present proof. □

3.3. **Existence of multiple solutions for elliptic systems.** We now prove the main result, which is an extension of theorems of [33, 36] and a generalization of [9, Theorem 3.2] to systems in diagonal form.

Proof of Theorem 3.1. We want to apply [36, Theorem 2.1.6]. First of all, because of Theorem 3.5, f satisfies $(CPS)_c$ for all $c \in \mathbb{R}$. Whence, (c) of [36, Theorem 2.1.6] is satisfied. Moreover we have

$$\frac{\nu}{2}\int_\Omega |Du|^2\,dx - \int_\Omega G(x,u)\,dx \le f(u) \le \frac{1}{2}nNC\int_\Omega |Du|^2\,dx - \int_\Omega G(x,u)\,dx.$$

We want to prove that assumptions (a) and (b) of [36, Theorem 2.1.6] are also satisfied. Let us observe that, instead of (b) of [36, Theorem 2.1.6], it is enough to find a sequence (W_n) of finite dimensional subspaces with $\dim(W_n) \to +\infty$ satisfying the inequality of (b) (see also [103, Theorem 1.2]). Let W be a finite dimensional subspace of $H_0^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$. From (3.7) we deduce that for all $s \in \mathbb{R}^N$ with $|s| \ge R$

$$G(x,s) \ge \frac{G\left(x, R\frac{s}{|s|}\right)}{R^q}|s|^q \ge b_0(x)|s|^q,$$

where

$$b_0(x) = R^{-q}\inf\{G(x,s) : |s| = R\} > 0$$

a.e. $x \in \Omega$. Therefore, there exists $a_0 \in L^1(\Omega)$ such that

$$G(x,s) \ge b_0(x)|s|^q - a_0(x) \tag{3.15}$$

a.e. $x \in \Omega$ and for all $s \in \mathbb{R}^N$. Since $b_0 \in L^1(\Omega)$, we may define a norm $\|\cdot\|_G$ on W by

$$\|u\|_G = \left(\int_\Omega b_0|u|^q\,dx\right)^{1/q}.$$

Since W is finite dimensional and $q > 2$, from (3.15) it follows

$$\lim_{\|u\|_G \to +\infty,\, u\in W} f(u) = -\infty$$

and condition (b) of [36, Theorem 2.1.6] is clearly fulfilled too for a sufficiently large $R > 0$. Let now (λ_h, u_h) be the sequence of eigenvalues and eigenvectors for the problem

$$\begin{aligned} \Delta u &= -\lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let us prove that there exist $h_0, \alpha > 0$ such that

$$\forall u \in V^+ : \|Du\|_2 = 1 \Rightarrow f(u) \ge \alpha,$$

where $V^+ = \overline{\operatorname{span}}\left\{u_h \in H_0^1(\Omega,\mathbb{R}^N) : h \ge h_0\right\}$. In fact, given $u \in V^+$ and $\varepsilon > 0$, we find

$$a_\varepsilon^{(1)} \in C_c^\infty(\Omega), \quad a_\varepsilon^{(2)} \in L^{2n/(n+2)}(\Omega),$$

such that $\|a_\varepsilon^{(2)}\|_{2n/(n+2)} \le \varepsilon$ and

$$|g(x,s)| \le a_\varepsilon^{(1)}(x) + a_\varepsilon^{(2)}(x) + \varepsilon|s|^{(n+2)/(n-2)}.$$

If $u \in V^+$, it follows that

$$\begin{aligned} f(u) &\ge \frac{\nu}{2}\|Du\|_2^2 - \int_\Omega G(x,u)\,dx \\ &\ge \frac{\nu}{2}\|Du\|_2^2 - \int_\Omega \left(\left(a_\varepsilon^{(1)} + a_\varepsilon^{(2)}\right)|u| + \frac{n-2}{2n}\varepsilon|u|^{2n/(n-2)}\right)dx \end{aligned}$$

$$\geq \frac{\nu}{2}\|Du\|_2^2 - \|a_\varepsilon^{(1)}\|_2\|u\|_2 - c_1\|a_\varepsilon^{(2)}\|_{2n/(n+2)}\|Du\|_2 - \varepsilon c_2\|Du\|_2^{2n/(n-2)}$$
$$\geq \frac{\nu}{2}\|Du\|_2^2 - \|a_\varepsilon^{(1)}\|_2\|u\|_2 - c_1\varepsilon\|Du\|_2 - \varepsilon c_2\|Du\|_2^{2n/(n-2)}.$$

Then if h_0 is sufficiently large, from the fact that (λ_h) diverges, for all $u \in V^+$, $\|Du\|_2 = 1$ implies

$$\|a_\varepsilon^{(1)}\|_2\|u\|_2 \leq \frac{\nu}{6}.$$

Hence, for $\varepsilon > 0$ small enough, $\|Du\|_2 = 1$ implies that $f(u) \geq \nu/6$.

Finally, set $V^- = \overline{\operatorname{span}}\left\{u_h \in H_0^1(\Omega, \mathbb{R}^N) : h < h_0\right\}$, we have the decomposition

$$H_0^1(\Omega; \mathbb{R}^N) = V^+ \oplus V^-.$$

Therefore, since the hypotheses for [36, Theorem 2.1.6] are fulfilled, we can find a sequence (u^m) of weak solution of system (3.1) such that $\lim_m f(u^m) = +\infty$. The proof is complete. □

3.4. **Regularity of weak solutions for elliptic systems.** Consider the nonlinear elliptic system

$$\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x,u) D_i u_h D_j v_k \, dx = \int_\Omega b(x,u,Du) \cdot v \, dx \tag{3.16}$$

for all $v \in H_0^1(\Omega; \mathbb{R}^N)$. For $l = 1, \dots, N$, we choose

$$b_l(x,u,Du) = \Big\{ -\sum_{i,j=1}^n \sum_{h,k=1}^N D_{s_l} a_{ij}^{hk}(x,u) D_i u_h D_j u_k + g_l(x,u) \Big\}.$$

Assume that there exist $c > 0$ and $q < \frac{n+2}{n-2}$ such that for all $s \in \mathbb{R}^N$ and a.e. in Ω

$$|g(x,s)| \leq c\,(1 + |s|^q)\,. \tag{3.17}$$

Then it follows that for every $M > 0$, there exists $C(M) > 0$ such that for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$ and $s \in \mathbb{R}^N$ with $|s| \leq M$

$$|b(x,s,\xi)| \leq c(M)\left(1 + |\xi|^2\right). \tag{3.18}$$

A nontrivial regularity theory for quasi-linear systems (see, [76, Chapter VI]) yields the following

Theorem 3.6. *For every weak solution $u \in H^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ of the system (3.1) there exist an open subset $\Omega_0 \subseteq \Omega$ and $s > 0$ such that*

$$\forall p \in (n, +\infty) : u \in C^{0,1-\frac{n}{p}}(\Omega_0; \mathbb{R}^N),$$
$$\mathcal{H}^{n-s}(\Omega \setminus \Omega_0) = 0.$$

For the proof of the above theorem, see [76, Chapter VI]. We now consider the particular case when $a_{ij}^{hk}(x,s) = \alpha_{ij}(x,s)\delta^{hk}$, and provide an almost everywhere regularity result.

Lemma 3.7. *Assume that (3.18) holds. Then the weak solutions $u \in H_0^1(\Omega, \mathbb{R}^N)$ of the system*

$$\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}(x,u) D_i u_h D_j v_h \, dx +$$

$$+\frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}(x,u)\cdot v D_i u_h D_j u_h \, dx = \int_\Omega g(x,u)\cdot v\, dx \tag{3.19}$$

for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$, belong to $L^\infty(\Omega,\mathbb{R}^N)$.

Proof. By [138, Lemma 3.3], for each $(CPS)_c$ sequence (u^m) there exist $u \in H_0^1 \cap L^\infty$ and a subsequence (u^{m_k}) with $u^{m_k} \rightharpoonup u$. Then, given a weak solution u, consider the sequence (u^m) such that each element is equal to u and the assertion follows. □

We can finally state a partial regularity result for our system.

Theorem 3.8. *Assume condition (3.18) and let $u \in H_0^1(\Omega,\mathbb{R}^N)$ be a weak solution of the system*

$$\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}(x,u) D_i u_h D_j v_h \, dx + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}(x,u)\cdot v D_i u_h D_j u_h \, dx$$
$$= \int_\Omega g(x,u)\cdot v\, dx$$

for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$. Then there exist an open subset $\Omega_0 \subseteq \Omega$ and $s > 0$ such that

$$\forall p \in (n,+\infty) : u \in C^{0,1-\frac{n}{p}}(\Omega_0;\mathbb{R}^N), \quad \mathcal{H}^{n-s}(\Omega\setminus\Omega_0) = 0.$$

To prove the above theorem, it suffices to combine the previous Lemma with Theorem 3.6.

3.5. Fully nonlinear scalar problems.

Recently, some results for the quite general problem

$$\begin{gathered} -\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) = g(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{3.20}$$

have been considered in [6, 7] and [113]. The goal of this section is to extend some of the results of [6, 113]. To solve (3.20), we shall look for critical points of functionals $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$f(u) = \int_\Omega \mathcal{L}(x,u,\nabla u)\, dx - \int_\Omega G(x,u)\, dx. \tag{3.21}$$

In general, f is continuous but not even locally Lipschitz unless $\mathcal{L}$ does not depend on u or $\mathcal{L}$ is subjected to some very restrictive growth conditions. Then, again we shall refer to non-smooth critical point theory.

We assume that $\mathcal{L} : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$, of class C^1 in (s,ξ) for a.e. $x \in \Omega$, the function $\mathcal{L}(x,s,\cdot)$ is strictly convex and $\mathcal{L}(x,s,0) = 0$ for a.e. $x \in \Omega$. Furthermore, we will assume that:

- there exist $a \in L^1(\Omega)$ and $b_0, \nu > 0$ such that
$$\nu|\xi|^p \le \mathcal{L}(x,s,\xi) \le a(x) + b_0|s|^p + b_0|\xi|^p\,, \tag{3.22}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$;
- for each $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that
$$|D_s\mathcal{L}(x,s,\xi)| \le a_\varepsilon(x) + \varepsilon|s|^{p^*} + b_1|\xi|^p\,, \tag{3.23}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, with $b_1 \in \mathbb{R}$ independent of ε. Furthermore, there exists $a_1 \in L^{p'}(\Omega)$ such that

$$\left|\nabla_\xi \mathcal{L}(x,s,\xi)\right| \le a_1(x) + b_1|s|^{\frac{p^*}{p'}} + b_1|\xi|^{p-1}, \tag{3.24}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;
- there exists $R > 0$ such that

$$|s| \ge R \Rightarrow D_s\mathcal{L}(x,s,\xi)s \ge 0\,, \tag{3.25}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;
- $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$G(x,s) \le d(x)|s|^p + b|s|^{p^*} \tag{3.26}$$

$$\lim_{s\to 0} \frac{G(x,s)}{|s|^p} = 0 \tag{3.27}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, where $d \in L^{\frac{n}{p}}(\Omega)$ and $b \in \mathbb{R}$. Moreover,

$$G(x,s) = \int_0^s g(x,\tau)\,d\tau$$

and there exist $c_1, c_2 > 0$ such that

$$|g(x,s)| \le c_1 + c_2|s|^\sigma$$

for a.e. $x \in \Omega$ and each $s \in \mathbb{R}$, where $\sigma < p^* - 1$.
- there is $q > p$ and $R' > 0$ such that for each $\varepsilon > 0$ there is $a_\varepsilon \in L^1(\Omega)$ with

$$0 < qG(x,s) \le g(x,s)s\,, \tag{3.28}$$

$$q\mathcal{L}(x,s,\xi) - \nabla_\xi\mathcal{L}(x,s,\xi)\cdot\xi - D_s\mathcal{L}(x,s,\xi)s \ge \nu|\xi|^p - a_\varepsilon(x) - \varepsilon|s|^p \tag{3.29}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ with $|s| \ge R'$.

Under the previous assumptions, the following is our main result.

Theorem 3.9. *The boundary value problem*

$$\begin{gathered} -\operatorname{div}\left(\nabla_\xi\mathcal{L}(x,u,\nabla u)\right) + D_s\mathcal{L}(x,u,\nabla u) = g(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

has at least one nontrivial weak solution $u \in W_0^{1,p}(\Omega)$.

This result is an extension of [6, Theorem 3.3], since instead of assuming that

$$\forall s \in \mathbb{R} :\quad q\mathcal{L}(x,s,\xi) - \nabla_\xi\mathcal{L}(x,s,\xi)\cdot\xi - D_s\mathcal{L}(x,s,\xi)s \ge \nu|\xi|^p\,,$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, we only request condition (3.29). In this way the proof of Lemma (3.12) becomes more difficult. The key-point, to deal with the more general assumption, is constituted by Lemma (3.11).

Similarly, in [113, Theorem 1], a multiplicity result for (3.20) is proved, assuming that

$$\forall s \in \mathbb{R} :\quad D_s\mathcal{L}(x,s,\xi)s \ge 0\,,$$
$$\forall s \in \mathbb{R} :\quad q\mathcal{L}(x,s,\xi) - \nabla_\xi\mathcal{L}(x,s,\xi)\cdot\xi - D_s\mathcal{L}(x,s,\xi)s \ge \nu|\xi|^p\,,$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, which are both stronger than (3.25) and (3.29). In particular, the first inequality above and the more general condition (3.25) are involved in Theorem 3.10.

Finally, let us point out that the growth conditions (3.22) - (3.24) are a relaxation of those of [6, 113], where it is assumed that

$$\nu|\xi|^p \leq \mathcal{L}(x,s,\xi) \leq \beta|\xi|^p, \quad |D_s\mathcal{L}(x,s,\xi)| \leq \gamma|\xi|^p,$$
$$\left|\nabla_\xi \mathcal{L}(x,s,\xi)\right| \leq a_1(x) + b_1|s|^{p-1} + b_1|\xi|^{p-1},$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

3.6. **The concrete Palais-Smale condition.** Let us point out that as a consequence of assumption (3.22) and convexity of $\mathcal{L}(x,s,\cdot)$, we can find $M > 0$ such that for each $\varepsilon > 0$ there is $a_\varepsilon \in L^1(\Omega)$ with

$$\nabla_\xi \mathcal{L}(x,s,\xi)\cdot\xi \geq \nu|\xi|^p - a(x) - b_0|s|^p, \tag{3.30}$$
$$|D_s\mathcal{L}(x,s,\xi)| \leq M\nabla_\xi \mathcal{L}(x,s,\xi)\cdot\xi + a_\varepsilon(x) + \varepsilon|s|^{p^*}, \tag{3.31}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

We now come to a local compactness property, which is crucial for the $(CPS)_c$ condition to hold. This result improves [113, Lemma 2], since (3.29) relaxes condition (8) in [113].

Theorem 3.10. *Let (u_h) be a bounded sequence in $W_0^{1,p}(\Omega)$ and set*

$$\langle w_h, v\rangle = \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla v\,dx + \int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)v\,dx\,, \tag{3.32}$$

for all $v \in C_c^\infty(\Omega)$. If (w_h) is strongly convergent to some w in $W^{-1,p'}(\Omega)$, then (u_h) admits a strongly convergent subsequence in $W_0^{1,p}(\Omega)$.

Proof. Since (u_h) is bounded in $W_0^{1,p}(\Omega)$, we find a u in $W_0^{1,p}(\Omega)$ such that, up to a subsequence,

$$\nabla u_h \rightharpoonup \nabla u \quad \text{in } L^p(\Omega), \quad u_h \to u \quad \text{in } L^p(\Omega), \quad u_h(x) \to u(x) \text{ for a.e. } x \in \Omega.$$

By [22, Theorem 2.1], up to a subsequence, we have

$$\nabla u_h(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega. \tag{3.33}$$

Therefore, by (3.24) we deduce that

$$\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h) \rightharpoonup \nabla_\xi \mathcal{L}(x,u,\nabla u) \quad \text{in } L^{p'}(\Omega,\mathbb{R}^n).$$

We now want to prove that u solves the equation

$$\forall v \in C_c^\infty(\Omega): \ \langle w, v\rangle = \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla v\,dx + \int_\Omega D_s\mathcal{L}(x,u,\nabla u)v\,dx. \tag{3.34}$$

To this aim, let us test equation (3.32) with the functions

$$v_h = \varphi\exp\{-M(u_h+R)^+\}, \quad \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \quad \varphi \geq 0.$$

It results that for each $h \in \mathbb{N}$,

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\varphi\exp\{-M(u_h+R)^+\}\,dx - \left\langle w_h, \varphi\exp\{-M(u_h+R)^+\}\right\rangle \\
&+ \int_\Omega \left[D_s\mathcal{L}(x,u_h,\nabla u_h) - M\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h+R)^+\right] \\
&\varphi\exp\{-M(u_h+R)^+\}\,dx = 0.
\end{aligned}$$

Of course, for a.e. $x \in \Omega$, we obtain

$$\varphi \exp\{-M(u_h + R)^+\} \to \varphi \exp\{-M(u + R)^+\}.$$

Since by inequality (3.31) and (3.25) for each $\varepsilon > 0$ and $h \in \mathbb{N}$ we have

$$\begin{aligned}&\big[D_s\mathcal{L}(x, u_h, \nabla u_h) - M\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\big] \\ &\qquad \times\varphi \exp\{-M(u_h + R)^+\} - \varepsilon|u_h|^{p^*}\varphi \le a_\varepsilon(x)\varphi\,,\end{aligned}$$

Fatou's Lemma implies that for each $\varepsilon > 0$,

$$\begin{aligned}&\limsup_h \int_\Omega \big[D_s\mathcal{L}(x, u_h, \nabla u_h) - M\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\big]\varphi e^{-M(u_h+R)^+} \\ &- \varepsilon|u_h|^{p^*}\varphi\, dx \\ &\le \int_\Omega \big[D_s\mathcal{L}(x, u, \nabla u) - M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+\big]\varphi e^{-M(u+R)^+} - \varepsilon|u|^{p^*}\varphi\, dx.\end{aligned}$$

Since (u_h) is bounded in $L^{p^*}(\Omega)$, we find $c > 0$ such that for each $\varepsilon > 0$

$$\begin{aligned}&\limsup_h \int_\Omega \big[D_s\mathcal{L}(x, u_h, \nabla u_h) - M\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\big]\varphi e^{-M(u_h+R)^+} dx \\ &\le \int_\Omega \big[D_s\mathcal{L}(x, u, \nabla u) - M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+\big]\varphi e^{-M(u+R)^+} dx - c\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, the previous inequality yields

$$\begin{aligned}&\limsup_h \int_\Omega \big[D_s\mathcal{L}(x, u_h, \nabla u_h) - M\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\big] \\ &\times \varphi e^{-M(u_h+R)^+} dx \\ &\le \int_\Omega \big[D_s\mathcal{L}(x, u, \nabla u) - M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+\big]\varphi e^{-M(u+R)^+} dx.\end{aligned}$$

Note that we have also

$$\varphi \exp\{-M(u_h + R)^+\} \rightharpoonup \varphi \exp\{-M(u + R)^+\} \quad \text{in } W_0^{1,p}(\Omega).$$

Moreover,

$$\nabla\varphi \exp\{-M(u_h + R)^+\} \to \nabla\varphi \exp\{-M(u + R)^+\} \quad \text{in } L^p(\Omega, \mathbb{R}^n),$$

so that

$$\begin{aligned}&\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla\varphi \exp\{-M(u_h + R)^+\}dx \\ &\to \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\varphi \exp\{-M(u + R)^+\}dx.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}&\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\varphi \exp\{-M(u + R)^+\}\, dx - \langle w, \varphi \exp\{-M(u + R)^+\}\rangle \\ &+ \int_\Omega \big[D_s\mathcal{L}(x, u, \nabla u) - M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+\big]\varphi \exp\{-M(u + R)^+\}dx \\ &\ge 0.\end{aligned}$$

Consider now the test functions

$$\varphi_k := \varphi H(\frac{u}{k}) \exp\{M(u + R)^+\}\,, \quad \varphi \in C_c^\infty(\Omega)\,, \quad \varphi \ge 0\,,$$

where $H \in C^1(\mathbb{R})$, $H = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $H = 0$ in $]-\infty, -1] \cup [1, +\infty[$. It follows that

$$\begin{aligned}&\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi_k \exp\{-M(u + R)^+\}\, dx - \langle w, \varphi H(\frac{u}{k})\rangle \\ &+ \int_\Omega \big[D_s\mathcal{L}(x, u, \nabla u) - M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+\big]\varphi H(\frac{u}{k})\, dx \geq 0.\end{aligned}$$

Furthermore, standard computations yield

$$\nabla \varphi_k = \exp\{M(u + R)^+\}\Big[\nabla\varphi H(\frac{u}{k}) + H'(\frac{u}{k})\frac{\varphi}{k}\nabla u + M\nabla(u + R)^+\varphi H(\frac{u}{k})\Big].$$

Since $\varphi H(\frac{u}{k})$ goes to φ in $W_0^{1,p}(\Omega)$, as $k \to +\infty$ we have

$$\Big\langle w, \varphi H(\frac{u}{k})\Big\rangle \to \langle w, \varphi\rangle\,.$$

By the properties of H and the growth conditions on $\nabla_\xi \mathcal{L}$, letting $k \to +\infty$ yields

$$\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\varphi H(\frac{u}{k}) \to \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\varphi\, dx\,,$$

$$\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u\, H'(\frac{u}{k})\frac{\varphi}{k}\, dx \to 0\,,$$

$$\int_\Omega M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+\varphi H(\frac{u}{k}) \to \int_\Omega M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+\varphi.$$

Whence, we conclude that for all $\varphi \in C_c^\infty(\Omega)$,

$$\varphi \geq 0 \Rightarrow \langle w, \varphi\rangle \leq \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\varphi\, dx + \int_\Omega D_s\mathcal{L}(x, u, \nabla u)\varphi\, dx.$$

Choosing now as test functions

$$v_h := \varphi \exp\{-M(u_h - R)^-\}\,,$$

where as before $\varphi \geq 0$, we obtain the opposite inequality so that (3.34) is proven.

In particular, taking into account the Brezis-Browder type results, we immediately obtain

$$\langle w, u\rangle = \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u\, dx + \int_\Omega D_s\mathcal{L}(x, u, \nabla u)u\, dx. \tag{3.35}$$

The final step is to show that (u_h) goes to u in $W_0^{1,p}(\Omega)$. Consider the function $\zeta : \mathbb{R} \to \mathbb{R}$ defined by

$$\zeta(s) = \begin{cases} Ms & \text{if } 0 < s < R \\ MR & \text{if } s \geq R \\ -Ms & \text{if } -R < s < 0 \\ MR & \text{if } s \leq -R\,, \end{cases} \tag{3.36}$$

and let us prove that

$$\begin{aligned}&\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{\zeta(u_h)\}\, dx \\ &\leq \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp\{\zeta(u)\}\, dx.\end{aligned}$$

Since $u_h \exp\{\zeta(u_h)\}$ are admissible test functions for (3.32), we have

$$\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{\zeta(u_h)\}\, dx - \langle w_h, u_h \exp\{\zeta(u_h)\}\rangle +$$

$$+ \int_\Omega \big[D_s\mathcal{L}(x,u_h,\nabla u_h) + \zeta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h\big]\, u_h \exp\{\zeta(u_h)\}\, dx = 0.$$

Let us observe that (3.33) implies that

$$\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h \to \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u \quad \text{for a.e. } x\in\Omega.$$

Since by inequality (3.31) for each $\varepsilon > 0$ and $h\in\mathbb{N}$ we have

$$\begin{aligned}
&\big[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \zeta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\big]\\
&\times u_h \exp\{\zeta(u_h)\} - R\exp\{MR\}\varepsilon|u_h|^{p^*}\\
&\le R\exp\{MR\}a_\varepsilon(x)\,,
\end{aligned}$$

Fatou's Lemma yields

$$\begin{aligned}
&\limsup_h \int_\Omega \big[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \zeta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\big]\\
&\times u_h e^{\zeta(u_h)} - Re^{MR}\varepsilon|u_h|^{p^*}\,dx\\
&\le \int_\Omega \big[-D_s\mathcal{L}(x,u,\nabla u) - \zeta'(u)\nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\big]u\exp\{\zeta(u)\}\\
&\quad - Re^{MR}\varepsilon|u|^{p^*}\,dx.
\end{aligned}$$

Therefore, since (u_h) is bounded in $L^{p^*}(\Omega)$, we find $c>0$ such that for all $\varepsilon>0$

$$\begin{aligned}
&\limsup_h \int_\Omega \big[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \zeta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\big]\,u_h e^{\zeta(u_h)}dx\\
&\le \int_\Omega \big[-D_s\mathcal{L}(x,u,\nabla u) - \zeta'(u)\nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\big]\,u e^{\zeta(u)}dx - c\varepsilon.
\end{aligned}$$

Taking into account that ε is arbitrary, we conclude that

$$\begin{aligned}
&\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h \exp\{\zeta(u_h)\}\,dx\\
&= \limsup_h \Big\{\int_\Omega \big[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \zeta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\big]\,u_h e^{\zeta(u_h)}dx\\
&+ \langle w_h, u_h\exp\{\zeta(u_h)\}\rangle\Big\}\\
&\le \int_\Omega \big[-D_s\mathcal{L}(x,u,\nabla u) - \zeta'(u)\nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\big]\,u e^{\zeta(u)}dx + \langle w, u\exp\{\zeta(u)\}\rangle\\
&= \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u e^{\zeta(u)}\,dx.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u e^{\zeta(u)}\,dx &\le \liminf_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h e^{\zeta(u_h)}dx\\
&\le \limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h e^{\zeta(u_h)}\,dx\\
&\le \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u e^{\zeta(u)}dx,
\end{aligned}$$

; namely,

$$\lim_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h \exp\{\zeta(u_h)\}dx = \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\exp\{\zeta(u)\}dx.$$

Therefore, by (3.30), generalized Lebesgue's theorem yields

$$\limsup_h \int_\Omega |\nabla u_h|^p \, dx \le \int_\Omega |\nabla u|^p \, dx,$$

that implies the strong convergence of (u_h) to u in $W_0^{1,p}(\Omega)$. □

Lemma 3.11. *Let $c \in \mathbb{R}$ and let (u_h) be a $(CPS)_c$-sequence in $W_0^{1,p}(\Omega)$. Then for each $\varepsilon > 0$ and $\varrho > 0$ there exists $K_{\varrho,\varepsilon} > 0$ such that, for all $h \in \mathbb{N}$,*

$$\begin{aligned}
&\int_{\{|u_h|\le\varrho\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h \, dx \\
&\le \varepsilon \int_{\{\varrho<|u_h|<K_{\varrho,\varepsilon}\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h \, dx + K_{\varrho,\varepsilon}
\end{aligned}$$

and

$$\int_{\{|u_h|\le\varrho\}} |\nabla u_h|^p \, dx \le \varepsilon \int_{\{\varrho<|u_h|<K_{\varrho,\varepsilon}\}} |\nabla u_h|^p \, dx + K_{\varrho,\varepsilon}.$$

Proof. Let $\sigma, \varepsilon > 0$ and $\varrho > 0$. For all $v \in W_0^{1,p}(\Omega)$, we set

$$\begin{aligned}
\langle w_h, v\rangle &= \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla v \, dx \\
&\quad + \int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h) v \, dx - \int_\Omega g(x,u_h) v \, dx.
\end{aligned} \tag{3.37}$$

Let us now consider $\vartheta_1 : \mathbb{R} \to \mathbb{R}$ given by

$$\vartheta_1(s) = \begin{cases} s & \text{if } |s| < \sigma \\ -s + 2\sigma & \text{if } \sigma \le s < 2\sigma \\ -s - 2\sigma & \text{if } -2\sigma < s \le -\sigma \\ 0 & \text{if } |s| \ge 2\sigma. \end{cases} \tag{3.38}$$

Then, testing (3.37) with $\vartheta_1(u_h) \in L^\infty([-2\sigma, 2\sigma])$, we obtain

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\vartheta_1(u_h) \, dx + \int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)\vartheta_1(u_h) \, dx \\
&\le \int_\Omega g(x,u_h)\vartheta_1(u_h) \, dx + \|w_h\|_{-1,p'}\|\vartheta_1(u_h)\|_{1,p}.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
&\int_{\{|u_h|\le\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h \, dx - \int_{\{\sigma<|u_h|\le2\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h \, dx \\
&+ \int_{\{|u_h|\le\sigma\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h \, dx + \int_{\{\sigma<|u_h|\le2\sigma\}} D_s\mathcal{L}(x,u_h,\nabla u_h)\vartheta_1(u_h) \, dx \\
&\le \int_\Omega \Big(c_1 + c_2|2\sigma|^{\frac{n(p-1)+p}{n-p}}\Big)\sigma \, dx + \frac{4^{\frac{p'}{p}}}{p' p^{\frac{p'}{p}} \nu^{\frac{p'}{p}}}\|w_h\|_{-1,p'}^{p'} + \frac{\nu}{4}\|\vartheta_1(u_h)\|_{1,p}^p.
\end{aligned}$$

Let $K_0 > 0$ be such that $\|w_h\|_{-1,p'} \le K_0$. Then, since by (3.30) we have

$$\begin{aligned}
&\nu\|\vartheta_1(u_h)\|_{1,p}^p \\
&\le \int_{\{|u_h|\le\sigma\}} \nu|\nabla u_h|^p \, dx + \int_{\{\sigma<|u_h|\le2\sigma\}} \nu|\nabla u_h|^p \, dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\{|u_h|\le\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + \int_{\{\sigma<|u_h|\le 2\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\
&\quad + \int_{\{|u_h|\le\sigma\}} a(x)\,dx + b_0\int_{\{|u_h|\le\sigma\}} |u_h|^p\,dx + \int_{\{\sigma<|u_h|\le 2\sigma\}} a(x)\,dx\\
&\quad + b_0\int_{\{\sigma<|u_h|\le 2\sigma\}} |u_h|^p\,dx\,,
\end{aligned}
$$

taking into account (3.31), we get for a sufficiently small value of $\sigma > 0$,

$$
\begin{aligned}
&\big(1-\sigma M-\frac{1}{4}\big)\int_{\{|u_h|\le\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\
&\le \big(1+\sigma M+\frac{1}{4}\big)\int_{\{\sigma<|u_h|\le 2\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\
&\quad + \int_\Omega \Big(c_1 + c_2|2\sigma|^{\frac{n(p-1)+p}{n-p}}\Big)\sigma\,dx + \frac{4^{\frac{p'}{p}}}{p' p^{\frac{p'}{p}}\nu^{\frac{p'}{p}}} K_0^{p'}\\
&\quad + \int_\Omega a_\varepsilon(x)\,dx + \Big[b_0(2^p+1)\sigma^p + \varepsilon\sigma^{p^*+1}\Big]\mathcal{L}^n(\Omega).
\end{aligned}
$$

Whence, we have shown an inequality of the type

$$
\begin{aligned}
&\int_{\{|u_h|\le\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\
&\le K_1\int_{\{\sigma<|u_h|\le 2\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + K_2.
\end{aligned}
$$

Let us now define for each $k \ge 1$ the functions ϑ_{2k} , $\vartheta_{2k-1} : \mathbb{R} \to \mathbb{R}$ by setting

$$
\vartheta_{2k}(s) = \begin{cases}
0 & \text{if } |s| \le k\sigma\\
s-k\sigma & \text{if } k\sigma < s < (k+1)\sigma\\
s+k\sigma & \text{if } -(k+1)\sigma < s < -k\sigma\\
-s+(k+2)\sigma & \text{if } (k+1)\sigma \le s < (k+2)\sigma\\
-s-(k+2)\sigma & \text{if } -(k+2)\sigma < s \le -(k+1)\sigma\\
0 & \text{if } |s| \ge (k+1)\sigma\,,
\end{cases}
$$

and

$$
\vartheta_{2k-1}(s) = \begin{cases}
\frac{s}{k} & \text{if } |s| \le k\sigma\\
-s+(k+1)\sigma & \text{if } k\sigma < s < (k+1)\sigma\\
s-(k+1)\sigma & \text{if } (k+1)\sigma < s < (k+2)\sigma\\
-s-(k+1)\sigma & \text{if } -(k+1)\sigma \le s < -k\sigma\\
s+(k+1)\sigma & \text{if } -(k+2)\sigma < s \le -(k+1)\sigma\\
0 & \text{if } |s| \ge (k+1)\sigma.
\end{cases}
$$

Therefore, by iterating on k, we obtain the k-th inequality

$$
\begin{aligned}
&\int_{\{|u_h|\le k\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\
&\le K_1(k)\int_{\{k\sigma<|u_h|\le (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + K_2(k).
\end{aligned}
\tag{3.39}
$$

Let now choose $k \geq 1$ such that $k\sigma \geq \varrho$ and $k\sigma \geq R$. Take $0 < \delta < 1$ and let $\vartheta_\delta : \mathbb{R} \to \mathbb{R}$ be the function defined by setting

$$\vartheta_\delta(s) = \begin{cases} 0 & \text{if } |s| \leq k\sigma \\ s - k\sigma & \text{if } k\sigma < s < (k+1)\sigma \\ s + k\sigma & \text{if } -(k+1)\sigma < s < -k\sigma \\ -\delta s + \sigma + \delta(k+1)\sigma & \text{if } (k+1)\sigma \leq s < (k+1)\sigma + \frac{\sigma}{\delta} \\ -\delta s - \sigma - \delta(k+1)\sigma & \text{if } -(k+1)\sigma - \frac{\sigma}{\delta} < s \leq -(k+1)\sigma \\ 0 & \text{if } |s| \geq (k+1)\sigma + \frac{\sigma}{\delta}. \end{cases}$$

As before, we get

$$\begin{aligned} &\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h)\, dx + \int_\Omega D_s \mathcal{L}(x, u_h, \nabla u_h) \vartheta_\delta(u_h)\, dx \\ &\leq \int_\Omega g(x, u_h) \vartheta_\delta(u_h)\, dx + \frac{1}{p' p^{\frac{p'}{p}} \delta^{\frac{p'}{p}}} \|w_h\|_{-1,p'}^{p'} + \delta \|\vartheta_\delta(u_h)\|_{1,p}^p. \end{aligned}$$

Taking into account (3.25), by computations, we deduce that

$$\int_\Omega D_s \mathcal{L}(x, u_h, \nabla u_h) \vartheta_\delta(u_h)\, dx \geq 0.$$

Moreover, we have as before

$$\begin{aligned} &\|\vartheta_\delta(u_h)\|_{1,p}^p \\ &\leq \int_{\{k\sigma < |u_h| \leq (k+1)\sigma\}} |\nabla u_h|^p\, dx + \int_{\{|u_h| \geq (k+1)\sigma\}} |\nabla u_h|^p\, dx \\ &\leq \frac{1}{\nu} \int_{\{k\sigma < |u_h| \leq (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h\, dx \\ &+ \frac{1}{\nu} \int_{\{|u_h| \geq (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h\, dx \\ &+ \frac{1}{\nu} \int_{\{k\sigma < |u_h| \leq (k+1)\sigma\}} a(x)\, dx + \frac{b_0}{\nu} \int_{\{k\sigma < |u_h| \leq (k+1)\sigma\}} |u_h|^p\, dx \\ &+ \frac{1}{\nu} \int_{\{|u_h| \geq (k+1)\sigma\}} a(x)\, dx + \frac{b_0}{\nu} \int_{\{(k+1)\sigma + \frac{\sigma}{\delta} \geq |u_h| \geq (k+1)\sigma\}} |u_h|^p\, dx\,, \end{aligned}$$

so that

$$\begin{aligned} &\Big(1 - \frac{\delta}{\nu}\Big) \int_{\{k\sigma < |u_h| \leq (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h\, dx \\ &\leq \Big(\delta + \frac{\delta}{\nu}\Big) \int_{\{|u_h| > (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h\, dx \\ &+ \int_\Omega \Big(c_1 + c_2 \Big|(k+1)\sigma + \frac{\sigma}{\delta}\Big|^{\frac{n(p-1)+p}{n-p}}\Big) \sigma\, dx + \frac{1}{p' p^{\frac{p'}{p}} \delta^{\frac{p'}{p}}} K_0^{p'} \\ &\quad + \frac{2}{\nu} \int_\Omega a(x)\, dx + \frac{b_0}{\nu} \Big[(k+1)^p + \Big((k+1) + \frac{1}{\delta}\Big)^p\Big] \sigma^p \mathcal{L}^n(\Omega). \end{aligned}$$

Therefore, we get

$$\begin{aligned}&\int_{\{k\sigma<|u_h|\le(k+1)\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\ &\le\le \frac{\nu\delta+\delta}{\nu-\delta}\int_{\{|u_h|\ge(k+1)\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + K_3(k,\delta).\end{aligned}$$

Combining this inequality with (3.39) we conclude that

$$\begin{aligned}&\int_{\{|u_h|\le\varrho\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\ &\le \int_{\{|u_h|\le k\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\ &\le K_1(k)\int_{\{k\sigma<|u_h|\le(k+1)\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + K_2(k)\\ &\le K_1(k)\frac{\nu\delta+\delta}{\nu-\delta}\int_{\{|u_h|>(k+1)\sigma\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\ &\quad + K_1(k)K_3(k,\delta)+K_2(k)\le\\ &\le \varepsilon\int_{\{|u_h|>\varrho\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + K_{\varrho,\varepsilon},\end{aligned}$$

where we have fixed $\delta>0$ in such a way that $K_1(k)\frac{\nu\delta+\delta}{\nu-\delta}\le\varepsilon$. □

The next result is an extension of [113, Lemma 1], since (3.29) relaxes [113, condition (9)].

Lemma 3.12. *Let $c\in\mathbb{R}$. Then each $(CPS)_c$-sequence for f is bounded in $W_0^{1,p}(\Omega)$.*

Proof. First of all, we can find $a_0\in L^1(\Omega)$ such that for a.e. $x\in\Omega$ and all $s\in\mathbb{R}$

$$qG(x,s)\le sg(x,s)+a_0(x).$$

Now, let (u_h) be a $(CPS)_c$-sequence for f and let for all $v\in C_c^\infty(\Omega)$

$$\langle w,v\rangle=\int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla v\,dx+\int_\Omega D_s\mathcal{L}(x,u,\nabla u)v\,dx-\int_\Omega g(x,u_h)v\,dx.$$

According to Lemma 3.11, for each $\varepsilon>0$ we have

$$\begin{aligned}&-\|w_h\|_{-1,p'}\|u_h\|_{1,p}\\ &\le \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+\int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx-\int_\Omega g(x,u_h)u_h\,dx\\ &\le \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+\int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx\\ &\quad -q\int_\Omega G(x,u_h)\,dx+\int_\Omega a_0\,dx\\ &\le (1+\varepsilon)\int_{\{|u_h|>R'\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+\int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx\\ &\quad -q\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx+qf(u_h)+\int_\Omega a_0\,dx+K_{R',\varepsilon}.\end{aligned}$$

On the other hand, from Lemma 3.11 and (3.29), for each $\varepsilon > 0$ we obtain

$$\begin{aligned}
&\int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx \\
&= \int_{\{|u_h|\le R'\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx + \int_{\{|u_h|>R'\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx \\
&\le \varepsilon M R' \int_{\{K_{R',\varepsilon}>|u_h|>R'\}} \nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx \\
&\quad - \int_{\{|u_h|>R'\}} \nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + q\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx + \int_\Omega a_\varepsilon(x)\,dx \\
&\quad + \varepsilon\int_{\{|u_h|>R'\}} |u_h|^p\,dx - \nu\int_{\{|u_h|>R'\}} |\nabla u_h|^p\,dx + K_{R',\varepsilon}.
\end{aligned}$$

Taking into account Poincaré and Young's inequalities, by (3.24) we find $c > 0$ and $C_{R',\varepsilon} > 0$ with

$$\begin{aligned}
&\int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx \\
&\le \varepsilon c\int_{\{|u_h|>R'\}} |\nabla u_h|^p\,dx - \int_{\{|u_h|>R'\}} \nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx \\
&\quad + q\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx + \int_\Omega a_\varepsilon(x)\,dx - \nu\int_{\{|u_h|>R'\}} |\nabla u_h|^p\,dx + C_{R',\varepsilon}.
\end{aligned}$$

Therefore, for a sufficiently small $\varepsilon > 0$, there exists $\vartheta_\varepsilon > 0$ with

$$\begin{aligned}
&\vartheta_\varepsilon\int_{\{|u_h|>R'\}} |\nabla u_h|^p\,dx \\
&\le \|w_h\|_{-1,p'}\|u_h\|_{1,p} + qf(u_h) + \int_\Omega a_0\,dx + \int_\Omega a_\varepsilon\,dx + K_{R',\varepsilon} + C_{R',\varepsilon}.
\end{aligned}$$

Moreover, it satisfies

$$\int_\Omega |\nabla u_h|^p\,dx \le (1+\varepsilon)\int_{\{|u_h|>R'\}} |\nabla u_h|^p\,dx + K_{R',\varepsilon}.$$

Since $w_h \to 0$ in $W^{-1,p'}(\Omega)$, the assertion follows. □

3.7. **Existence of a weak solution.**

Lemma 3.13. *Under assumptions (3.27) we have*

$$\frac{\int_\Omega G(x,u_h)\,dx}{\|u_h\|_{1,p}^p} \to 0 \quad \text{as } h \to +\infty,$$

for each (u_h) that goes to 0 in $W_0^{1,p}(\Omega)$.

Proof. Let $(u_h) \subseteq W_0^{1,p}(\Omega)$ with $u_h \to 0$ in $W_0^{1,p}(\Omega)$. We can find $(\varrho_h) \subseteq \mathbb{R}$ and a sequence $(w_h) \subseteq W_0^{1,p}(\Omega)$ such that $u_h = \varrho_h w_h$, $\varrho_h \to 0$ and $\|w_h\|_{1,p} = 1$. Taking into account (3.27), it follows

$$\lim_h \frac{G(x,u_h(x))}{\|u_h\|_{1,p}^p} = 0 \quad \text{for a.e. } x \in \Omega.$$

Moreover, for a.e. $x \in \Omega$,

$$\frac{G(x, u_h(x))}{\|u_h\|_{1,p}^p} \leq d|w_h|^p + b\varrho_h^{p^2/(n-p)}|w_h|^{p^*}.$$

If w is the weak limit of (w_h), since $d|w_h|^p \to d|w|^p$ in $L^1(\Omega)$ and $b\varrho_h^{p^2/(n-p)}|w_h|^{p^*} \to 0$ in $L^1(\Omega)$, (a variant of) Lebesgue's Theorem concludes the proof. □

We conclude with the proof of the main result of this section.

Proof of Theorem 3.9. From Lemma (3.12) and Theorem (3.10) it follows that f satisfies the $(CPS)_c$ condition for each $c \in \mathbb{R}$. By (3.22) and (3.28) it easily follows that

$$\forall u \in W_0^{1,p}(\Omega)\backslash\{0\} : \lim_{t\to+\infty} f(tu) = -\infty.$$

From Lemma (3.13) and (3.22), we deduce that 0 is a strict local minimum for f. From Theorem (2.26) the assertion follows. □

Remark 3.14. As proved by Arcoya and Boccardo in [6], each weak solution of (3.20) belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ provided that $\mathcal{L}$ and g satisfy suitable conditions. Then, some nice regularity results hold for various classes of integrands $\mathcal{L}$ (see [91]).

3.8. **Super-linear problems with unbounded coefficients.** The aim of this section is to prove existence and multiplicity results of unbounded critical points for a class of lower semi-continuous functionals (cf. [115]). Let us consider a bounded open set $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ and define the functional $f : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$f(u) = \int_\Omega j(x, u, \nabla u) - \int_\Omega G(x, u),$$

where $j(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a measurable function with respect to x for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and of class C^1 with respect to (s, ξ) for a.e. $x \in \Omega$. We also assume that for almost every x in Ω and every s in $\mathbb{R}$

$$\text{the mapping } \{\xi \mapsto j(x, s, \xi)\} \text{ is strictly convex .} \tag{3.40}$$

Moreover, we suppose that there exist a constant $\alpha_0 > 0$ and a positive increasing function $\alpha \in C(\mathbb{R})$ such that the following hypothesis is satisfied for almost every $x \in \Omega$ and for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$

$$\alpha_0|\xi|^2 \leq j(x, s, \xi) \leq \alpha(|s|)|\xi|^2. \tag{3.41}$$

The functions $j_s(x, s, \xi)$ and $j_\xi(x, s, \xi)$ denote the derivatives of $j(x, s, \xi)$ with respect of the variables s and ξ respectively. Regarding the function $j_s(x, s, \xi)$, we assume that there exist a positive increasing function $\beta \in C(\mathbb{R})$ and a positive constant R such that the following conditions are satisfied almost everywhere in Ω and for every $\xi \in \mathbb{R}^N$:

$$\big|j_s(x, s, \xi)\big| \leq \beta(|s|)|\xi|^2, \quad \text{for every } s \text{ in } \mathbb{R}, \tag{3.42}$$

$$j_s(x, s, \xi)s \geq 0, \quad \text{for every } s \text{ in } \mathbb{R} \text{ with } |s| \geq R. \tag{3.43}$$

Note that, from (3.40) and (3.41), it follows that $j_\xi(x, s, \xi)$ satisfies the following growth condition (see Remark 3.24 for more details)

$$\big|j_\xi(x, s, \xi)\big| \leq 4\alpha(|s|)|\xi|. \tag{3.44}$$

The function $G(x, s)$ is the primitive with respect to s such that $G(x, 0) = 0$ of a Carathéodory (i.e. measurable with respect to x and continuous with respect to s) function $g(x, s)$. We will study two different kinds of problems, according to different nonlinearities $g(x, s)$, that have a main common feature. Indeed, in both cases we cannot expect to find critical points in $L^\infty(\Omega)$. To be more precise, let us consider a first model example of nonlinearity and suppose that there exists p such that

$$g_1(x,s) = a(x)\mathrm{arctg}s + |s|^{p-2}s, \quad 2 < p < \frac{2N}{N-2}, \tag{3.45}$$

where $a(x) \in L^{\frac{2N}{N+2}}(\Omega)$ and $a(x) > 0$. Notice that from hypotheses (3.41) and (3.45) it follows that f is lower semi-continuous on $H_0^1(\Omega)$. We will also assume that

$$\lim_{|s|\to\infty} \frac{\alpha(|s|)}{|s|^{p-2}} = 0. \tag{3.46}$$

Condition (3.46), together with (3.41), allows f to be unbounded from below, so that we cannot look for a global minimum. Moreover, notice that $g(x,s)$ is odd with respect to s, so that it would be natural to expect, if $j(x,-s,-\xi) = j(x,s,\xi)$, the existence of infinitely many solutions as in the semi-linear case (see [6]). Unfortunately, we cannot apply any of the classical results of critical point theory, because our functional f is not of class C^1 on $H_0^1(\Omega)$. Indeed, notice that $\int_\Omega j(x,v,\nabla v)$ is not differentiable. More precisely, since $j_\xi(x,s,\xi)$ and $j_s(x,s,\xi)$ are not supposed to be bounded with respect to s, the terms $j_\xi(x,u,\nabla u)\cdot\nabla v$ and $j_s(x,u,\nabla u)v$ may not be $L^1(\Omega)$ even if $v \in C_0^\infty(\Omega)$. Notice that if $j_s(x,s,\xi)$ and $j_\xi(x,s,\xi)$ were supposed to be bounded with respect to s, f would be Gateaux derivable for every u in $H_0^1(\Omega)$ and along any direction $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ (see [6, 33, 36, 113, 133] for the study of this class of functionals). On the contrary, in our case, for every $u \in H_0^1(\Omega)$, $f'(u)(v)$ does not even exist along directions $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$.

To deal with the Euler equation of f let us define the following subspace of $H_0^1(\Omega)$ for a fixed u in $H_0^1(\Omega)$

$$W_u = \big\{v \in H_0^1(\Omega) : j_\xi(x,u,\nabla u)\cdot\nabla v \in L^1(\Omega) \text{ and } j_s(x,u,\nabla u)v \in L^1(\Omega)\big\}. \tag{3.47}$$

We will see that W_u is dense in $H_0^1(\Omega)$. We give the definition of generalized solution

Definition 3.15. Let $\Lambda \in H^{-1}(\Omega)$ and assume (3.40), (3.41), (3.42). We say that u is a generalized solution to

$$\begin{gathered} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = \Lambda, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

if $u \in H_0^1(\Omega)$ and it results

$$\begin{gathered} j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega), \quad j_s(x,u,\nabla u)u \in L^1(\Omega), \\ \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle\Lambda, v\rangle, \quad \forall\, v \in W_u. \end{gathered}$$

Theorem 3.16. *Assume conditions (3.40), (3.41), (3.42), (3.43), (3.45), (3.46). Moreover, suppose that there exist $R' > 0$ and $\delta > 0$ such that*

$$|s| \geq R' \quad\Rightarrow\quad pj(x,s,\xi) - j_s(x,s,\xi)s - j_\xi(x,s,\xi)\cdot\xi \geq \delta|\xi|^2, \tag{3.48}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$. If $j(x,-s,-\xi) = j(x,s,\xi)$, then there exists a sequence $\{u_h\} \subset H_0^1(\Omega)$ of generalized solutions of

$$\begin{gathered} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g_1(x,u), \quad \textit{in } \Omega, \\ u = 0, \quad \textit{on } \partial\Omega \end{gathered} \tag{3.49}$$

such that $f(u_h) \to +\infty$.

In the nonsymmetric case we consider a different class of nonlinearities $g(x,s)$. A simple model example can be the following

$$g_2(x,s) = d(x)\mathrm{arctg}(s^2) + |s|^{p-2}s, \quad 2 < p < \frac{2N}{N-2}, \tag{3.50}$$

where $d(x) \in L^{\frac{N}{2}}(\Omega)$ and $d(x) > 0$. We will prove the following result.

Theorem 3.17. *Assume conditions (3.40), (3.41), (3.42), (3.43), (3.46), (3.48), (3.50). Then there exists a nontrivial generalized solution of the problem*

$$\begin{gathered} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g_2(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.51}$$

Since the functions $\alpha(|s|)$ and $\beta(|s|)$ in (3.41) and (3.42) are not supposed to be bounded, we are dealing with integrands $j(x,s,\xi)$ which may be unbounded with respect to s. This class of functionals has also been treated in [7], [21] and [23]. In these papers the existence of a nontrivial solution $u \in L^\infty(\Omega)$ is proved when $g(x,s) = |s|^{p-2}s$. Note that, in this case it is natural to expect solutions in $L^\infty(\Omega)$. To prove the existence result, in [21] and [23], a fundamental step is to prove that every cluster point of a Palais-Smale sequence belongs to $L^\infty(\Omega)$. That is, to prove that u is bounded before knowing that it is a solution. Notice that if u is in $L^\infty(\Omega)$ and $v \in C_0^\infty(\Omega)$ then $j_\xi(x,u,\nabla u)\cdot\nabla v$ and $j_s(x,u,\nabla u)v$ are in $L^1(\Omega)$. Therefore, if $g(x,s) = |s|^{p-2}s$, it would be possible to define a solution as a function $u \in L^\infty(\Omega)$ that satisfies the equation associated to (P_1) (or (P_2)) in the distributional sense. In our case the function $a(x)$ in (3.45) belongs to $L^{2N/(N+2)}(\Omega)$, so that we can only expect to find solutions in $H_0^1(\Omega)$. In the same way, the function $d(x)$ in (3.50) is in $L^{N/2}(\Omega)$ and also in this case the solutions are not expected to be in $L^\infty(\Omega)$. For these reasons, we have given a definition of solution weaker than the distributional one and we have considered the subspace W_u as the space of the admissible test functions. Notice that if $u \in H_0^1(\Omega)$ is a generalized solution of problem (P_1) (resp. (P_2)) and $u \in L^\infty(\Omega)$, then u is a distributional solution of (P_1) (resp. (P_2)).

We want to stress that we have considered here particular nonlinearities (i.e. g_1 and g_2) just to present - in a simple case - the main difficulties we are going to tackle. Indeed, Theorems 3.16 and 3.17 will be proved as consequences of two general results (Theorems 3.18 and 3.20). To prove these general results we will use an abstract critical point theory for lower semi-continuous functionals developed in [50, 58, 60]. So, firstly, we will show that the functional f can be studied by means of this theory (see Theorem 3.23). Then, we will give a definition of a Palais-Smale sequence $\{u_n\}$ suitable to this situation (Definition 3.37), and we will prove that u_n is compact in $H_0^1(\Omega)$ (Theorems 3.34 and 3.43). To do this we will follow the arguments of [33, 36, 113, 133] where the case in which $\alpha(s)$ and $\beta(s)$ are bounded is studied. In our case we will have to modify the test functions used in these papers in order to get the compactness result. Indeed, here the main difficulty is to find suitable approximations of u_n that belong to W_{u_n}, in order to choose them as test functions. For this reason a large amount of work (Theorems 3.30, 3.31, 3.32 and 3.33) is devoted to find possible improvements of the class of allowed test functions.

3.9. **General setting and main results.** Let us consider Ω a bounded open set in $\mathbb{R}^N$ $(N \geq 3)$. Let us define the functional $J : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$J(v) = \int_\Omega j(x,v,\nabla v), \tag{3.52}$$

where $j(x,s,\xi)$ satisfies hypotheses (3.40), (3.41), (3.42), (3.43). We will prove existence and multiplicity results of generalized solutions (see Definition 3.15) of the problem

$$\begin{gathered} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.53}$$

To do this, we will use variational methods, so that we will study the functional $f : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$f(v) = J(v) - \int_\Omega G(x,v),$$

where $G(x,s) = \int_0^s g(x,t)dt$ is the primitive of the function $g(x,s)$ with $G(x,0) = 0$.

To state our multiplicity result let us suppose that $g(x,s)$ satisfies the following conditions. Assume that for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{2N/(N+2)}(\Omega)$ such that

$$|g(x,s)| \le a_\varepsilon(x) + \varepsilon|s|^{\frac{N+2}{N-2}}, \tag{3.54}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. Moreover, there exist $p > 2$ and functions $a_0(x)$, $\overline{a}(x) \in L^1(\Omega)$, $b_0(x)$, $\overline{b}(x) \in L^{\frac{2N}{N+2}}(\Omega)$ and $k(x) \in L^\infty(\Omega)$ with $k(x) > 0$ almost everywhere, such that

$$pG(x,s) \le g(x,s)s + a_0(x) + b_0(x)|s|, \tag{3.55}$$

$$G(x,s) \ge k(x)|s|^p - \overline{a}(x) - \overline{b}(x)|s|, \tag{3.56}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ (the constant p is the same as the one in (3.48)).

In this case we will prove the following result.

Theorem 3.18. *Assume conditions (3.40), (3.41), (3.42), (3.43), (3.46), (3.48), (3.54), (3.55), (3.56). Moreover, let*

$$j(x,-s,-\xi) = j(x,s,\xi) \quad \textit{and} \quad g(x,-s) = -g(x,s), \tag{3.57}$$

for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Then there exists a sequence $\{u_h\} \subset H_0^1(\Omega)$ of generalized solutions of problem (3.53) with $f(u_h) \to +\infty$.

Remark 3.19. In the classical results of critical point theory different conditions from (3.54), (3.55) and (3.56) are usually supposed. Indeed, as a growth condition on $g(x,s)$, it is assumed that

$$|g(x,s)| \le a(x) + b|s|^{\sigma-1}, \quad 2 < \sigma < \frac{2N}{N-2}, \quad b \in \mathbb{R}^+, \quad a(x) \in L^{\frac{2N}{N+2}}(\Omega). \tag{3.58}$$

Note that (3.58) implies (3.54). Indeed, suppose that $g(x,s)$ satisfies (3.58), then Young inequality implies that (3.54) is satisfied with $a_\varepsilon(x) = a(x) + C(b,\varepsilon)$. Moreover, as a superlinearity condition, it is usually assumed that there exist $p > 2$ and $R > 0$ with

$$0 < pG(x,s) \le g(x,s)s, \quad \text{for every } s \text{ in } \mathbb{R} \text{ with } |s| \ge R. \tag{3.59}$$

Note that this condition is stronger than conditions (3.55), (3.56). Indeed, suppose that $g(x,s)$ satisfies (3.59) and notice that this implies that there exists $a_0 \in L^1(\Omega)$ such that

$$pG(x,s) \le g(x,s)s + a_0(x), \quad \text{for every } s \text{ in } \mathbb{R}.$$

Then (3.55) is satisfied with $b_0(x) \equiv 0$. Moreover, from (3.59) we deduce that there exists $\overline{a}(x) \in L^1(\Omega)$ such that

$$G(x,s) \ge \frac{1}{R^p} \min\{G(x,R), G(x,-R), 1\}|s|^p - \overline{a}(x),$$

so that also (3.56) is satisfied.

To state our existence result in the nonsymmetric case, assume that the function g satisfies the following condition

$$|g(x,s)| \le a_1(x)|s| + b|s|^{\sigma-1}, \qquad 2 < \sigma < \frac{2N}{N-2}, \quad a_1(x) \in L^{\frac{N}{2}}(\Omega), \quad b \in \mathbb{R}^+. \tag{3.60}$$

We will prove the following

Theorem 3.20. *Assume conditions (3.40), (3.41), (3.42), (3.43), (3.46), (3.48), (3.55), (3.56), (3.60). Also, let*

$$\lim_{s\to 0} \frac{g(x,s)}{s} = 0, \quad \text{a.e. in } \Omega. \tag{3.61}$$

Then there exists a nontrivial generalized solution of the problem (3.53). In addition, there exist $\varepsilon > 0$ such that for every $\Lambda \in H^{-1}(\Omega)$ with $\|\Lambda\|_{-1,2} < \varepsilon$ the problem

$$\begin{gathered} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) + \Lambda, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.62}$$

has at least two generalized solutions u_1, u_2 with $f(u_1) \le 0 < f(u_2)$.

Remark 3.21. Notice that, in order to have $g(x,v)v \in L^1(\Omega)$ for every $v \in H^1_0(\Omega)$, the function $a_1(x)$ has to be in $L^{\frac{N}{2}}(\Omega)$. Nevertheless, also in this case we cannot expect to find bounded solution of problem (3.53). The situation is even worse in problem (3.62), indeed in this case we can only expect to find solutions that belong to $H^1_0(\Omega) \cap \operatorname{dom}(J)$.

Remark 3.22. Notice that condition (3.60) implies (3.54). Indeed, suppose that $g(x,s)$ satisfies (3.60). Then Young inequality implies that, for every $\varepsilon > 0$, we have

$$|g(x,s)| \le \beta(\varepsilon)(a_1(x))^{\frac{N+2}{4}} + \varepsilon|s|^{\frac{N+2}{N-2}} + \gamma(\varepsilon,b),$$

where $\beta(\varepsilon)$ and $\gamma(\varepsilon,b)$ are positive constants depending on ε and b. Now, since we have $a_1(x) \in L^{\frac{N}{2}}(\Omega)$, there holds

$$a_\varepsilon(x) = \Big(\beta(\varepsilon)(a_1(x))^{\frac{N+2}{4}} + \gamma(\varepsilon,b)\Big) \in L^{\frac{2N}{N+2}}(\Omega),$$

which yields (3.54).

3.10. **Verification of the key condition.** Let us now set $X = H^1_0(\Omega)$ and consider the functional $J : H^1_0(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined in (3.52). From hypothesis (3.41), we immediately obtain that J is lower semicontinuous. We will now prove that J satisfies (2.3). To this aim, for every $k \ge 1$, we define the truncation $T_k : \mathbb{R} \to \mathbb{R}$ at height k, defined as

$$T_k(s) = s \quad \text{if } |s| \le k, \qquad T_k(s) = k\frac{s}{|s|} \quad \text{if } |s| \ge k. \tag{3.63}$$

Theorem 3.23. *Assume conditions (3.40), (3.41), (3.43). Then, for every $(u,\eta) \in \operatorname{epi} J$ with $J(u) < \eta$, there holds*

$$|d\mathcal{G}_J|(u,\eta) = 1.$$

Moreover, if $j(x,-s,-\xi) = j(x,s,\xi)$, $\forall\, \eta > J(0)(=0)$ it results $|d_{\mathbb{Z}_2}\mathcal{G}_J|(0,\eta) = 1$.

Proof. Let $(u,\eta) \in \operatorname{epi} J$ with $J(u) < \eta$ and let $\varrho > 0$. Then, there exists $\delta \in (0,1]$, $\delta = \delta(\varrho)$, and $k \ge 1$, $k = k(\varrho)$, such that $k \ge R$ (where R is as in (3.43)) and

$$\|T_k(v) - v\|_{1,2} < \varrho, \quad \text{for every } v \in B(u,\delta). \tag{3.64}$$

From (3.41) we have

$$j(x,v,\nabla T_k(v)) \leq \alpha(k)|\nabla v|^2.$$

Then, up to reducing δ, we get the following inequalities

$$\int_\Omega j(x,v,\nabla T_k(v)) < \int_\Omega j(x,u,\nabla T_k(u)) + \varrho \leq \int_\Omega j(x,u,\nabla u) + \varrho, \tag{3.65}$$

for each $v \in B(u,\delta)$. We now prove that, for every $t \in [0,\delta]$ and $v \in B(u,\delta)$, there holds

$$J((1-t)v + tT_k(v)) \leq (1-t)J(v) + t(J(u)+\varrho). \tag{3.66}$$

From (3.40) and since $j(x,s,\xi)$ is of class C^1 with respect to the variable s, there exists $\theta \in [0,1]$ such that

$$\begin{aligned}
&j(x,(1-t)v + tT_k(v),(1-t)\nabla v + t\nabla T_k(v)) - j(x,v,\nabla v)\\
&= j(x,(1-t)v + tT_k(v),(1-t)\nabla v + t\nabla T_k(v)) - j(x,v,(1-t)\nabla v + t\nabla T_k(v))\\
&\quad + j(x,v,(1-t)\nabla v + t\nabla T_k(v)) - j(x,v,\nabla v)\\
&\leq tj_s(x,v+\theta t(T_k(v)-v),(1-t)\nabla v + t\nabla T_k(v))(T_k(v)-v)\\
&\quad + t\left(j(x,v,\nabla T_k(v)) - j(x,v,\nabla v)\right).
\end{aligned}$$

Notice that

$$\begin{aligned}
v(x) \geq k \quad &\Rightarrow \quad v(x) + \theta t(T_k(v(x)) - v(x)) \geq k \geq R,\\
v(x) \leq -k \quad &\Rightarrow \quad v(x) + \theta t(T_k(v(x)) - v(x)) \leq -k \leq -R.
\end{aligned}$$

Then, in light of (3.43) one has

$$j_s(x,v+\theta t(T_k(v)-v),(1-t)\nabla v + t\nabla T_k(v))(T_k(v)-v) \leq 0.$$

It follows that

$$\begin{aligned}
&j(x,(1-t)v + tT_k(v),(1-t)\nabla v + t\nabla T_k(v))\\
&\leq (1-t)j(x,v,\nabla v) + tj(x,v,\nabla T_k(v)).
\end{aligned}$$

Therefore, from (3.65) one gets (3.66). To apply Theorem 2.19 we define

$$\mathcal{H} : \{v \in B(u,\delta) : J(v) < \eta + \delta\} \times [0,\delta] \to H_0^1(\Omega)$$

by setting

$$\mathcal{H}(v,t) = (1-t)v + tT_k(v).$$

Hence, taking into account (3.64) and (3.66), it results

$$d(\mathcal{H}(v,t),v) \leq \varrho t \quad \text{and} \quad J(\mathcal{H}(v,t)) \leq (1-t)J(v) + t(J(u)+\varrho),$$

for $v \in B(u,\delta)$, $J(v) < \eta + \delta$ and $t \in [0,\delta]$. The first assertion now follows from Theorem 2.19. Finally, since $\mathcal{H}(-v,t) = \mathcal{H}(v,t)$ one also has $|d_{\mathbb{Z}_2}\mathcal{G}_J|(0,\eta) = 1$, whenever $j(x,-s,-\xi) = j(x,s,\xi)$. □

3.11. **The variational setting.** This section concerns the relationship between $|dJ|(u)$ and the directional derivatives of the functional J. Moreover, we will obtain some Brezis-Browder (see [27]) type results.

First of all, we make a few observations.

Remark 3.24. It is readily seen that hypothesis (3.40) and the right inequality of (3.41) imply that there exists a positive increasing function $\overline{\alpha}(|s|)$ such that

$$\big|j_\xi(x,s,\xi)\big| \le \overline{\alpha}(|s|)|\xi|, \tag{3.67}$$

for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$. Indeed, from (3.40) one has

$$\forall v \in \mathbb{R}^N : |v| \le 1 \quad \Rightarrow \quad j(x,s,\xi+|\xi|v) \ge j(x,s,\xi) + j_\xi(x,s,\xi)\cdot v|\xi|.$$

This, and (3.41) yield

$$j_\xi(x,s,\xi)\cdot v|\xi| \le 4\alpha(|s|)|\xi|^2.$$

From the arbitrariness of v, (3.67) follows. On the other hand, if (3.67) holds we have

$$|j(x,s,\xi)| \le \int_0^1 |j_\xi(x,s,t\xi)\cdot\xi|dt \le \frac{1}{2}\overline{\alpha}(|s|)|\xi|^2.$$

As a consequence, it is not restrictive to suppose that the functions in the right hand side of (3.41) and (3.67) are the same. Notice that, in particular, there holds $j_\xi(x,s,0) = 0$.

Remark 3.25. It is not restrictive to suppose that the functions $\alpha(s)$ and $\beta(s)$ are both increasing. Indeed, if this is not the case, we can consider the functions

$$A_r(|s|) = \sup_{|s|\le r} \alpha(|s|) \quad \text{and} \quad B_r(|s|) = \sup_{|s|\le r} \beta(|s|),$$

which are increasing.

Remark 3.26. The assumption of strict convexity on the function $\{\xi \to j(x,s,\xi)\}$ implies that, for almost every x in Ω and for every s in $\mathbb{R}$, we have

$$[j_\xi(x,s,\xi) - j_\xi(x,s,\xi^*)]\cdot(\xi-\xi^*) > 0, \tag{3.68}$$

for every ξ, $\xi^* \in \mathbb{R}^N$, with $\xi \ne \xi^*$. Moreover, hypotheses (3.40) and (3.41) imply that,

$$j_\xi(x,s,\xi)\cdot\xi \ge \alpha_0|\xi|^2. \tag{3.69}$$

Indeed, we have

$$0 = j(x,s,0) \ge j(x,s,\xi) + j_\xi(x,s,\xi)\cdot(0-\xi),$$

so that inequality (3.69) follows by virtue of (3.41).

Now, for every $u \in H_0^1(\Omega)$, we define the subspace

$$V_u = \Big\{v \in H_0^1(\Omega)\cap L^\infty(\Omega) : u \in L^\infty(\{x\in\Omega : v(x)\ne 0\})\Big\}. \tag{3.70}$$

As proved in [61], V_u is a vector space dense in $H_0^1(\Omega)$. Since $V_u \subset W_u$, also W_u (see the introduction) is dense in $H_0^1(\Omega)$. In the following proposition we study the conditions under which we can compute the directional derivatives of J.

Proposition 3.27. *Assume conditions (3.41), (3.42), (3.44). Then there exists $J'(u)(v)$ for every $u \in \mathrm{dom}(J)$ and $v \in V_u$. Furthermore, we have*

$$j_s(x,u,\nabla u)v \in L^1(\Omega) \quad \textit{and} \quad j_\xi(x,u,\nabla u)\cdot\nabla v \in L^1(\Omega),$$

and

$$J'(u)(v) = \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v + \int_\Omega j_s(x,u,\nabla u)v.$$

Proof. Let $u \in \mathrm{dom}(J)$ and $v \in V_u$. For every $t \in \mathbb{R}$ and a.e. $x \in \Omega$, we set

$$F(x,t) = j(x, u(x) + tv(x), \nabla u(x) + t\nabla v(x)).$$

Since $v \in V_u$ and by using (3.41), it follows that $F(x,t) \in L^1(\Omega)$. Moreover, it results

$$\frac{\partial F}{\partial t}(x,t) = j_s(x, u + tv, \nabla u + t\nabla v)v + j_\xi(x, u + tv, \nabla u + t\nabla v) \cdot \nabla v.$$

From hypotheses (3.42) and (3.44) we get that for every $x \in \Omega$ with $v(x) \neq 0$, it results

$$\begin{aligned}\Big|\frac{\partial F}{\partial t}(x,t)\Big| &\le \|v\|_\infty \beta(\|u\|_\infty + \|v\|_\infty)(|\nabla u| + |\nabla v|)^2 \\ &\quad + \alpha(\|u\|_\infty + \|v\|_\infty)(|\nabla u| + |\nabla v|)|\nabla v|.\end{aligned}$$

Since the function in the right hand side of the previous inequality belongs to $L^1(\Omega)$, the assertion follows. $\square$

In the sequel we will often use the cut-off function $H \in C^\infty(\mathbb{R})$ given by

$$H(s) = 1 \quad \text{on } [-1,1], \qquad H(s) = 0 \quad \text{outside } [-2,2], \qquad |H'(s)| \le 2. \tag{3.71}$$

Now we can prove a fundamental inequality regarding the weak slope of J.

Proposition 3.28. *Assume conditions* (3.41)*,* (3.42)*,* (3.44)*. Then*

$$\begin{aligned}|d(J - w)|(u) \ge \sup\Big\{ \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla v + \int_\Omega j_s(x, u, \nabla u)v - \langle w, v\rangle : \\ v \in V_u,\ \|v\|_{1,2} \le 1\Big\}\end{aligned}$$

for every $u \in \mathrm{dom}(J)$ *and every* $w \in H^{-1}(\Omega)$.

Proof. If $|d(J - w)|(u) = \infty$, or if

$$\sup\Big\{ \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla v + \int_\Omega j_s(x, u, \nabla u)v - \langle w, v\rangle : v \in V_u,\ \|v\|_{1,2} \le 1\Big\} = 0,$$

then the inequality holds. Otherwise, let $u \in \mathrm{dom}(J)$ and let $\eta \in \mathbb{R}^+$ be such that $J(u) < \eta$. Moreover, let us consider $\overline{\sigma} > 0$ and $\overline{v} \in V_u$ such that $\|\overline{v}\|_{1,2} \le 1$ and

$$\int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla \overline{v} + \int_\Omega j_s(x, u, \nabla u)\overline{v} - \langle w, \overline{v}\rangle < -\overline{\sigma}. \tag{3.72}$$

Let us fix $\varepsilon > 0$ and let us prove that there exists $k_0 \ge 1$ such that

$$\Big\| H\Big(\frac{u}{k_0}\Big)\overline{v}\Big\|_{1,2} < 1 + \varepsilon \tag{3.73}$$

and

$$\int_\Omega j_s(x, u, \nabla u)H\Big(\frac{u}{k_0}\Big)\overline{v} + \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla\Big(H\Big(\frac{u}{k_0}\Big)\overline{v}\Big) - \Big\langle w, H\Big(\frac{u}{k_0}\Big)\overline{v}\Big\rangle < -\overline{\sigma}. \tag{3.74}$$

Let us set $v_k = H(u/k)\overline{v}$, where $H(s)$ is defined as in (3.71). Since $\overline{v} \in V_u$ we deduce that $v_k \in V_u$ for every $k \ge 1$ and v_k converges to $\overline{v}$ in $H^1_0(\Omega)$. This, together with the fact that $\|\overline{v}\|_{1,2} \le 1$, implies (3.73). Moreover, Proposition 3.27 implies that we can consider $J'(u)(v_k)$. In addition, as k goes to infinity, we have

$$\begin{aligned} j_s(x, u(x), \nabla u(x))v_k(x) &\to j_s(x, u(x), \nabla u(x))\overline{v}(x), \quad \text{for a.e. } x \in \Omega, \\ j_\xi(x, u(x), \nabla u(x)) \cdot \nabla v_k(x) &\to j_\xi(x, u(x), \nabla u(x)) \cdot \nabla\overline{v}(x), \quad \text{for a.e. } x \in \Omega.\end{aligned}$$

Moreover, we get

$$\Big|j_s(x,u,\nabla u)H(\frac{u}{k})\overline{v}\Big| \le |j_s(x,u,\nabla u)\overline{v}|\,,$$
$$\big|j_\xi(x,u,\nabla u)\cdot\nabla v_k\big| \le \big|j_\xi(x,u,\nabla u)\big|\big|\nabla\overline{v}\big| + 2|\overline{v}||j_\xi(x,u,\nabla u)\cdot\nabla u|.$$

Since $\overline{v}\in V_u$ and by using (3.42) and (3.44), we can apply Lebesgue Dominated Convergence Theorem to obtain

$$\lim_{k\to\infty}\int_\Omega j_s(x,u,\nabla u)v_k = \int_\Omega j_s(x,u,\nabla u)\overline{v}\,,$$
$$\lim_{k\to\infty}\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v_k = \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\overline{v}\,,$$

which, together with (3.72), implies (3.74). Since we want to apply Proposition 2.16, let us consider J^η as defined in (2.2). Let us now show that there exists $\delta_1>0$ such that

$$\Big\|H\Big(\frac{z}{k_0}\Big)\overline{v}\Big\| \le 1+\varepsilon, \tag{3.75}$$

as well as

$$\int_\Omega j_\xi(x,z,\nabla z)\cdot\nabla\Big(H\Big(\frac{z}{k_0}\Big)\overline{v}\Big) + \int_\Omega j_s(x,z,\nabla z)H\Big(\frac{z}{k_0}\Big)\overline{v} - \Big\langle w, H\Big(\frac{z}{k_0}\Big)\overline{v}\Big\rangle < -\overline{\sigma}, \tag{3.76}$$

for every $z\in B(u,\delta_1)\cap J^\eta$. Indeed, take $u_n\in J^\eta$ such that $u_n\to u$ in $H_0^1(\Omega)$ and set

$$v_n = H\Big(\frac{u_n}{k_0}\Big)\overline{v}.$$

We have that $v_n \to H(u/k_0)\overline{v}$ in $H_0^1(\Omega)$, so that (3.75) follows from (3.73). Moreover, note that $v_n\in V_{u_n}$, so that from Proposition 3.27 we deduce that we can consider $J'(u_n)(v_n)$. From (3.42) and (3.44) it follows

$$|j_s(x,u_n,\nabla u_n)v_n| \le \beta(2k_0)\|\overline{v}\|_\infty|\nabla u_n|^2,$$
$$\big|j_\xi(x,u_n,\nabla u_n)\cdot\nabla v_n\big| \le \alpha(2k_0)|\nabla u_n|\Big[\frac{2}{k_0}\|\overline{v}\|_\infty|\nabla u_n| + |\nabla\overline{v}|\Big].$$

Then we obtain

$$\lim_{n\to\infty}\int_\Omega j_s(x,u_n,\nabla u_n)v_n = \int_\Omega j_s(x,u,\nabla u)H(\frac{u}{k_0})\overline{v}\,,$$
$$\lim_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla v_n = \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\Big[H\Big(\frac{u}{k_0}\Big)\overline{v}\Big],$$

which, combined with (3.74), immediately implies (3.76). Now, observe that (3.76) is equivalent to say that $J'(z)\left(H\left(\frac{z}{k}\right)\overline{v}\right) - \langle w, H\left(\frac{z}{k}\right)\overline{v}\rangle < -\overline{\sigma}$. Thus, there exists $\delta<\delta_1$ with

$$J\Big(z+\frac{t}{1+\varepsilon}H\Big(\frac{z}{k_0}\Big)\overline{v}\Big) - J(z) - \Big\langle w, \frac{t}{1+\varepsilon}H\Big(\frac{z}{k}\Big)\overline{v}\Big\rangle \le -\frac{\overline{\sigma}}{1+\varepsilon}t, \tag{3.77}$$

for every $t\in[0,\delta]$ and $z\in B(u,\delta)\cap J^\eta$. Finally, let us define the continuous function $\mathcal{H}: B(u,\delta)\cap J^\eta\times[0,\delta]\to H_0^1(\Omega)$ given by

$$\mathcal{H}(z,t) = z + \frac{t}{1+\varepsilon}H(\frac{z}{k_0})\overline{v}.$$

From (3.75) and (3.77) we deduce that $\mathcal{H}$ satisfies all the hypotheses of Proposition 2.16. Then, $|d(J-w)|(u) > \frac{\overline{\sigma}}{1+\varepsilon}$, and the conclusion follows from the arbitrariness of ε. $\square$

The next Lemma will be useful in proving two Brezis-Browder type results for J.

Lemma 3.29. *Assume conditions* (3.40), (3.41), (3.42), (3.43) *and let* $u \in \mathrm{dom}(J)$. *Then*

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u \le |dJ|(u)\|u\|_{1,2}. \tag{3.78}$$

In particular, if $|dJ|(u) < \infty$, *then*

$$j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega) \quad \textit{and} \quad j_s(x,u,\nabla u)u \in L^1(\Omega).$$

Proof. First, notice that if u is such that $|dJ|(u) = \infty$, or

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u \le 0,$$

then the conclusion holds. Otherwise, let $k \ge 1$, $u \in \mathrm{dom}(J)$ with $|dJ|(u) < \infty$, and $\sigma > 0$ be such that

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla T_k(u) + \int_\Omega j_s(x,u,\nabla u)T_k(u) > \sigma\|T_k(u)\|_{1,2},$$

where $T_k(s)$ is defined in (3.63). We will prove that $|dJ|(u) \ge \sigma$. Fixed $\varepsilon > 0$, we first want to show that there exists $\delta_1 > 0$ such that

$$\|T_k(w)\|_{1,2} \le (1+\varepsilon)\|T_k(u)\|_{1,2}, \tag{3.79}$$

$$\int_\Omega j_\xi(x,w,\nabla w)\cdot\nabla T_k(w) + \int_\Omega j_s(x,w,\nabla w)T_k(w) > \sigma\|T_k(u)\|_{1,2}, \tag{3.80}$$

for every $w \in H^1_0(\Omega)$ with $\|w-u\|_{1,2} < \delta_1$. Indeed, take $w_n \in H^1_0(\Omega)$ such that $w_n \to u$ in $H^1_0(\Omega)$. Then, (3.79) follows directly. Moreover, notice that from (3.42) and (3.43) there holds

$$j_s(x,w_n(x),\nabla w_n(x))w_n(x) \ge -R\beta(R)|\nabla w_n(x)|^2.$$

Since $w_n \to u$ in $H^1_0(\Omega)$, from (3.69) and by applying Fatou Lemma we get

$$\begin{aligned}
&\liminf_{n\to\infty}\Big[\int_\Omega j_\xi(x,w_n,\nabla w_n)\cdot\nabla T_k(w_n) + \int_\Omega j_s(x,w_n,\nabla w_n)T_k(w_n)\Big] \\
&\ge \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla T_k(u) + \int_\Omega j_s(x,u,\nabla u)T_k(u) > \sigma\|T_k(u)\|_{1,2},
\end{aligned}$$

which yields (3.80). Consider now the continuous map $\mathcal{H} : B(u,\delta_1)\times[0,\delta_1] \to H^1_0(\Omega)$ defined as

$$\mathcal{H}(w,t) = w - \frac{t}{\|T_k(u)\|_{1,2}(1+\varepsilon)}T_k(w).$$

From (3.79) and (3.80) we deduce that there exists $\delta < \delta_1$ such that

$$d(\mathcal{H}(w,t),w) \le t,$$
$$J(\mathcal{H}(w,t)) - J(w) \le -\frac{\sigma}{1+\varepsilon},$$

for every $t \in [0,\delta]$ and $w \in H^1_0(\Omega)$ with $\|w-u\|_{1,2} < \delta$ and $J(w) < J(u)+\delta$. Then, the arbitrariness of ε yields $|dJ|(u) \ge \sigma$. Therefore, for every $k \ge 1$ we get

$$\int_\Omega j_s(x,u,\nabla u)T_k(u) + \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla T_k(u) \le |dJ|(u)\|T_k(u)\|_{1,2}.$$

Taking the limit as $k \to \infty$, the Monotone Convergence Theorem yields (3.78). □

Notice that a generalized solution u (see Definition 3.15) is not, in general, a distributional solution. This, because a test function $v \in W_u$ may not belong to C_0^∞. Thus, it is natural to study the conditions under which it is possible to enlarge the class of admissible test functions. This kind of argument was introduced in [27]. More precisely, suppose we have a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla z + \int_\Omega j_s(x,u,\nabla u)z = \langle w, z\rangle, \quad \forall z \in V_u, \tag{3.81}$$

where V_u is defined in (3.70) and $w \in H^{-1}(\Omega)$. A natural question is whether or not we can take as test function $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$. The next result gives an answer to this question.

Theorem 3.30. *Assume that conditions (3.40), (3.41), (3.42) hold. Let $w \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ that satisfies (3.81). Moreover, suppose that $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ and there exist $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and $\eta \in L^1(\Omega)$ such that*

$$j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v \geq \eta. \tag{3.82}$$

Then $j_\xi(x,u,\nabla u)\cdot\nabla v + j_s(x,u,\nabla u)v \in L^1(\Omega)$ and

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle w, v\rangle.$$

Proof. Since $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$, then $H(\frac{u}{k})v \in V_u$. From (3.81) we have

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\Big[H(\frac{u}{k})v\Big] + \int_\Omega j_s(x,u,\nabla u)H(\frac{u}{k})v = \Big\langle w, H(\frac{u}{k})v\Big\rangle, \tag{3.83}$$

for every $k \geq 1$. Note that

$$\int_\Omega \Big|j_\xi(x,u,\nabla u)\cdot\nabla u H'(\frac{u}{k})\frac{v}{k}\Big| \leq \frac{2}{k}\|v\|_\infty \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u.$$

Since $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$, the Lebesgue Dominated Convergence Theorem yields

$$\lim_{k\to\infty}\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u H'(\frac{u}{k})\frac{v}{k} = 0,$$
$$\lim_{k\to\infty}\Big\langle w, H(\frac{u}{k})v\Big\rangle = \langle w, v\rangle.$$

As far as the remaining terms in (3.83) concerns, note that from (3.82) it follows

$$[j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v]H(\frac{u}{k}) \geq H(\frac{u}{k})\eta \geq -\eta^- \in L^1(\Omega).$$

Thus, we can apply Fatou Lemma and obtain

$$\int_\Omega j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v \leq \langle w, v\rangle\,.$$

The previous inequality and (3.82) imply that

$$j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v \in L^1(\Omega). \tag{3.84}$$

Now, notice that

$$\Big|\big[j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v\big]H(\frac{u}{k})\Big| \leq \big|j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v\big|\,.$$

From (3.84) we deduce that we can use Lebesgue Dominated Convergence Theorem to pass to the limit in (3.83) and obtain the conclusion. $\square$

In the next result we find the conditions under which we can use $v \in H_0^1(\Omega)$ in (3.81). Moreover, we prove, under suitable hypotheses, that if u satisfies (3.81) then u is a generalized solution (see Definition 3.15) of the corresponding problem.

Theorem 3.31. *Assume that conditions (3.40), (3.41), (3.42), (3.43) hold. Let $w \in H^{-1}(\Omega)$, and let $u \in H_0^1(\Omega)$ be such that (3.81) is satisfied. Moreover, suppose that $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$, and that there exist $v \in H_0^1(\Omega)$ and $\eta \in L^1(\Omega)$ such that*

$$j_s(x,u,\nabla u)v \geq \eta \quad \text{and} \quad j_\xi(x,u,\nabla u)\cdot\nabla v \geq \eta. \tag{3.85}$$

Then $j_s(x,u,\nabla u)v \in L^1(\Omega)$, $j_\xi(x,u,\nabla u)\cdot\nabla v \in L^1(\Omega)$ and

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle w, v\rangle. \tag{3.86}$$

In particular, it results $j_s(x,u,\nabla u)u,\ j_s(x,u,\nabla u) \in L^1(\Omega)$ and

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u = \langle w, u\rangle.$$

Moreover, u is a generalized solution of the problem

$$\begin{gathered} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = w, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.87}$$

Proof. Let $k \geq 1$ be fixed. For every $v \in H_0^1(\Omega)$ we have that $T_k(v) \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and $-v^- \leq T_k(v) \leq v^+$. Then, from (3.85), we get

$$j_s(x,u,\nabla u)T_k(v) \geq -\eta^- \in L^1(\Omega). \tag{3.88}$$

Moreover,

$$j_\xi(x,u,\nabla u)\cdot\nabla T_k(v) \geq -[j_\xi(x,u,\nabla u)\cdot\nabla T_k(v)]^- \geq -\eta^- \in L^1(\Omega). \tag{3.89}$$

Then, we can apply Theorem 3.30 and obtain

$$\int_\Omega j_s(x,u,\nabla u)T_k(v) + \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla T_k(v) = \langle w, T_k(v)\rangle \tag{3.90}$$

for every $k \geq 1$. By using again (3.88) and (3.89) and by arguing as in Theorem 3.30 we obtain

$$j_s(x,u,\nabla u)v \in L^1(\Omega) \quad \text{and} \quad j_\xi(x,u,\nabla u)\cdot\nabla v \in L^1(\Omega).$$

Thus, we can use Lebesgue Dominated Convergence Theorem to pass to the limit in (3.90) and get (3.86). In particular, by (3.42), (3.43) and (3.69) we can choose $v = u$. Finally, since

$$j_s(x,u,\nabla u) = j_s(x,u,\nabla u)\chi_{\{|u|<1\}} + j_s(x,u,\nabla u)\chi_{\{|u|\geq 1\}}$$

and

$$\left|j_s(x,u,\nabla u)\chi_{\{|u|\geq 1\}}\right| \leq |j_s(x,u,\nabla u)u|,$$

by (3.42) it results also $j_s(x,u,\nabla u) \in L^1(\Omega)$. Finally, notice that if $v \in W_u$ we can take $\eta = j_\xi(x,u,\nabla u)\cdot\nabla v$ and $\eta = j_s(x,u,\nabla u)v$, so that (3.86) is satisfied. Thus, u is a generalized solution to Problem (3.87). □

We point out that the previous result readily implies that, if $u \in H_0^1(\Omega)$ satisfies (3.81) and $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$, it results that $j_s(x,u,\nabla u) \in L^1(\Omega)$, then $j_s(x,u,\nabla u)v \in L^1(\Omega)$ for every $v \in C_0^\infty(\Omega)$. Instead, the term which has not a distributional interpretation in (3.81) is $j_\xi(x,u,\nabla u)$. In the next result we show that if we multiply $j_\xi(x,u,\nabla u)$ by a suitable sequence of C_c^1 functions, we obtain, passing to the limit, a distributional interpretation of (3.81).

Theorem 3.32. *Assume conditions* (3.40), (3.41), (3.42), (3.43). *Let* $w \in H^{-1}(\Omega)$ *and* $u \in H_0^1(\Omega)$ *be such that* (3.81) *is satisfied. Let* (ϑ_h) *be a sequence in* $C_c^1(\mathbb{R})$ *with*

$$\sup_{h\geq 1} \|\vartheta_h\|_\infty < \infty, \quad \sup_{h\geq 1} \|\vartheta_h'\|_\infty < \infty,$$
$$\lim_{h\to\infty} \vartheta_h(s) = 1, \quad \lim_{h\to\infty} \vartheta_h'(s) = 0.$$

If $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$, *the sequence*

$$\operatorname{div}\left[\vartheta_h(u) j_\xi(x,u,\nabla u)\right]$$

is strongly convergent in $W^{-1,q}(\Omega)$ *for every* $1 < q < \frac{N}{N-1}$, *and*

$$\lim_{h\to\infty} \left\{-\operatorname{div}\left[\vartheta_h(u) j_\xi(x,u,\nabla u)\right]\right\} + j_s(x,u,\nabla u) = w \quad in \quad W^{-1,q}(\Omega).$$

Proof. Let $w = -\operatorname{div} F$ with $F \in L^2(\Omega,\mathbb{R}^N)$ and $v \in C_c^\infty(\Omega)$. Then $\vartheta_h(u)v \in V_u$ and we can take v as test function in (3.81). It results

$$\begin{aligned}\int_\Omega j_\xi(x,u,\nabla u)\vartheta_h(u)\cdot\nabla v &= -\int_\Omega j_\xi(x,u,\nabla u)\vartheta_h'(u)\cdot\nabla u\, v - \int_\Omega j_s(x,u,\nabla u)\vartheta_h(u)v \\ &\quad + \int_\Omega F\vartheta_h'(u)\nabla u\, v + \int_\Omega F\vartheta_h(u)\nabla v.\end{aligned}$$

Then u is a solution of the following equation

$$-\operatorname{div}\Big[\vartheta_h(u) j_\xi(x,u,\nabla u)\Big] = \xi_h \quad \text{in } \mathcal{D}'(\Omega),$$

where

$$\xi_h = -\Big[\vartheta_h'(u)(j_\xi(x,u,\nabla u) - F)\cdot\nabla u + \vartheta_h(u) j_s(x,u,\nabla u)\Big] - \operatorname{div}(\vartheta_h(u)F).$$

Now, notice that

$$\vartheta_h(u)F \to F, \quad \text{strongly in } L^2(\Omega).$$

Then, $\operatorname{div}(\vartheta_h(u)F)$ is a convergent sequence in $H^{-1}(\Omega)$. Since the embedding of $H^{-1}(\Omega)$ in $W^{-1,q}(\Omega)$ is continuous, we get the desired convergence. Moreover, Theorem 3.31 implies that $j_s(x,u,\nabla u) \in L^1(\Omega)$. Then, the remaining terms in ξ_h converge strongly in $L^1(\Omega)$. Thus, we get the conclusion by observing that the embedding of $L^1(\Omega)$ in $W^{-1,q}(\Omega)$ is continuous. □

Consider the case $j(x,s,\xi) = a(x,s)|\xi|^2$ with $a(x,s)$ measurable with respect to x, continuous with respect to s and such that hypotheses (3.40), (3.41), (3.42), (3.43), (3.46) are satisfied. The next result proves that, in particular, if there exists $u \in H_0^1(\Omega)$ that satisfies (3.81) and if $a(x,u)|\nabla u|^2 \in L^1(\Omega)$, then u satisfies (3.81) in the sense of distribution.

Theorem 3.33. *Assume conditions* (3.40), (3.41), (3.42), (3.43), (3.46). *Let* $w \in H^{-1}(\Omega)$ *and* $u \in H_0^1(\Omega)$ *that satisfies* (3.81). *Moreover, suppose that* $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ *and that*

$$j(x,s,\xi) = \widehat{j}(x,s,|\xi|). \tag{3.91}$$

Then $j_\xi(x,u,\nabla u) \in L^1(\Omega)$ *and* u *is a distributional solution to*

$$\begin{gathered}-\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = w, \quad in\ \Omega,\\ u = 0, \quad on\ \partial\Omega.\end{gathered}$$

Proof. It is readily seen that, in view of (3.40) and (3.91), it results
$$|\xi||j_\xi(x,s,\xi)| \leq \sqrt{2}\, j_\xi(x,s,\xi)\cdot\xi.$$
for a.e. $x \in \Omega$, every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. Then
$$j_\xi(x,u,\nabla u)\chi_{\{|\nabla u|>1\}} \in L^1(\Omega).$$
Moreover, we take into account (3.46), and we observe that (3.44) implies that there exists a positive constant C such that
$$|\xi| \leq 1 \;\Rightarrow\; |j_\xi(x,s,\xi)| \leq 4\alpha(|s|) \leq C(|s|^{p-2}+1),$$
which, by the Sobolev embedding, implies also that $j_\xi(x,u,\nabla u)\chi_{\{|\nabla u|\leq 1\}} \in L^1(\Omega)$. Then $j_\xi(x,u,\nabla u) \in L^1(\Omega)$. Moreover, from (3.42) and (3.43) we have
$$j_s(x,u,\nabla u)u \geq j_s(x,u,\nabla u)u\chi_{\{x:\,|u(x)|<R\}} \in L^1(\Omega).$$
Then Theorem 3.31 implies that $j_s(x,u,\nabla u)u \in L^1(\Omega)$. Finally, again Theorem 3.31 yields the conclusion. □

3.12. **A compactness result for J.** In this section we will prove the following compactness result for J. We will follow an argument similar to the one used in [36] and in [133].

Theorem 3.34. *Assume conditions* (3.40), (3.41), (3.42), (3.43)*. Let $\{u_n\} \subset H_0^1(\Omega)$ be a bounded sequence with $j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n \in L^1(\Omega)$ and let $\{w_n\} \subset H^{-1}(\Omega)$ be such that*
$$\forall v \in V_{u_n}: \int_\Omega j_s(x,u_n,\nabla u_n)v + j_\xi(x,u_n,\nabla u_n)\cdot\nabla v = \langle w_n, v\rangle. \tag{3.92}$$
If w_n is strongly convergent in $H^{-1}(\Omega)$, then, up to a subsequence, u_n is strongly convergent in $H_0^1(\Omega)$.

Proof. Let w be the limit of $\{w_n\}$ and let $L > 0$ be such that
$$\|u_n\|_{1,2} \leq L, \quad \text{for every } n \geq 1. \tag{3.93}$$
From (3.93) we deduce that there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence,
$$u_n \rightharpoonup u, \quad \text{weakly in } H_0^1(\Omega). \tag{3.94}$$
Step 1. Let us first prove that u is such that
$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\psi + \int_\Omega j_s(x,u,\nabla u)\psi = \langle w,\psi\rangle, \quad \forall \psi \in V_u. \tag{3.95}$$
First of all, from Rellich Compact Embedding Theorem, up to a subsequence,
$$\begin{gathered} u_n \to u, \quad \text{in } L^q(\Omega), \quad \forall\, q \in [1, 2N/(N-2)), \\ u_n(x) \to u(x), \quad \text{for a.e. } x \in \Omega. \end{gathered} \tag{3.96}$$
We now want to prove that, up to a subsequence,
$$\nabla u_n(x) \to \nabla u(x), \quad \text{for a.e. } x \in \Omega. \tag{3.97}$$
Let $h \geq 1$. For every $v \in C_c^\infty(\Omega)$ we have that $H\left(\frac{u_n}{h}\right)v \in V_{u_n}$ (where H is again the function defined in (3.71)), then
$$\begin{aligned} &\int_\Omega H\Big(\frac{u_n}{h}\Big) j_\xi(x,u_n,\nabla u_n)\cdot\nabla v \\ &= -\int_\Omega \Big[H\Big(\frac{u_n}{h}\Big) j_s(x,u_n,\nabla u_n) + H'\Big(\frac{u_n}{h}\Big) j_\xi(x,u_n,\nabla u_n)\cdot\frac{\nabla u_n}{h}\Big]v \end{aligned}$$

$$+\left\langle w_n, H\left(\frac{u_n}{h}\right) v\right\rangle.$$

Let $w_n = -\operatorname{div}(F_n)$, with (F_n) strongly convergent in $L^2(\Omega, \mathbb{R}^N)$. Then it follows that

$$\begin{aligned}
&\int_\Omega H\left(\frac{u_n}{h}\right) j_\xi(x, u_n, \nabla u_n) \cdot \nabla v \\
&= \int_\Omega \Big[H'\left(\frac{u_n}{h}\right)(F_n - j_\xi(x, u_n, \nabla u_n)) \cdot \frac{\nabla u_n}{h} - H\left(\frac{u_n}{h}\right) j_s(x, u_n, \nabla u_n)\Big] v \\
&\quad + \int_\Omega H\left(\frac{u_n}{h}\right) F_n \cdot \nabla v.
\end{aligned}$$

For the square bracket is bounded in $L^1(\Omega)$ and $(H\left(\frac{u_n}{h}\right) F_n)$ is strongly convergent in $L^2(\Omega, \mathbb{R}^N)$ we can apply [54, Theorem 5] with

$$b_n(x, \xi) = H\left(\frac{u_n(x)}{h}\right) j_\xi(x, u_n(x), \xi) \quad \text{and} \quad E = E_h = \{x \in \Omega : |u(x)| \le h\}$$

and deduce (3.97) by the arbitrariness of $h \ge 1$. Notice that, by virtue of Theorem 3.31, for every n we have

$$\int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \nabla u_n + \int_\Omega j_s(x, u_n, \nabla u_n) u_n = \langle w_n, u_n\rangle.$$

Then, in view of (3.43), one has

$$\sup_{n\ge 1} \int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \nabla u_n < \infty. \tag{3.98}$$

Let now $k \ge 1$, $\varphi \in C_c^\infty(\Omega)$, $\varphi \ge 0$ and consider

$$v = \varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right), \quad \text{where } M_k = \frac{\beta(2k)}{\alpha_0}. \tag{3.99}$$

Note that $v \in V_{u_n}$ and

$$\begin{aligned}
\nabla v &= \nabla\varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) - M_k \varphi e^{-M_k(u_n+R)^+} \nabla(u_n + R)^+ H\left(\frac{u_n}{k}\right) \\
&\quad + \varphi e^{-M_k(u_n+R)^+} H'\left(\frac{u_n}{k}\right) \frac{\nabla u_n}{k}.
\end{aligned}$$

Taking v as test function in (3.92), we obtain

$$\begin{aligned}
&\int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \nabla\varphi \\
&+ \int_\Omega \Big[j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla(u_n + R)^+\Big] \varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \\
&= \int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \varphi e^{-M_k(u_n+R)^+} H'\left(\frac{u_n}{k}\right) \frac{\nabla u_n}{k} \\
&\quad + \left\langle w_n, \varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right)\right\rangle.
\end{aligned} \tag{3.100}$$

Observe that

$$\Big[j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla(u_n + R)^+\Big] \varphi e^{-M_k(u_n+R)^+} H\left(\frac{u_n}{k}\right) \le 0.$$

Indeed, the assertion follows from (3.43), for almost every x such that $u_n(x) \le -R$ while, for almost every x in $\{x : -R \le u_n(x) \le 2k\}$ from (3.44), (3.69) and (3.99) we get

$$\Big[j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla (u_n + R)^+ \Big] \le (\beta(2k) - \alpha_0 M_k)|\nabla u_n|^2 \le 0.$$

Moreover, from (3.44), (3.93), (3.96) and (3.97) we have

$$\begin{aligned} &\int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big) \nabla \varphi \\ &\to \int_\Omega j_\xi(x, u, \nabla u) \cdot e^{-M_k(u+R)^+} H(\frac{u}{k}) \nabla \varphi, \end{aligned}$$

and

$$\Big\langle w_n, \varphi e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big)\Big\rangle \to \Big\langle w, \varphi e^{-M_k(u+R)^+} H\Big(\frac{u}{k}\Big)\Big\rangle,$$

as $n \to \infty$. We take into account (3.98) and deduce that there exists a positive constant C such that

$$\Big| \int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \varphi e^{-M_k(u_n+R)^+} H'\Big(\frac{u_n}{k}\Big) \frac{\nabla u_n}{k} \Big| \le \frac{C}{k}.$$

We take the limit superior in (3.100) and we apply Fatou Lemma to obtain

$$\begin{aligned} &\int_\Omega j_\xi(x, u, \nabla u) \cdot e^{-M_k(u+R)^+} H(\frac{u}{k}) \nabla \varphi + \int_\Omega j_s(x, u, \nabla u) \varphi e^{-M_k(u+R)^+} H(\frac{u}{k}) \\ &- M_k \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla u^+ \varphi e^{-M_k(u+R)^+} H(\frac{u}{k}) \\ &\ge -\frac{C}{k} + \Big\langle w, \varphi e^{-M_k(u+R)^+} H(\frac{u}{k}) \Big\rangle \end{aligned} \tag{3.101}$$

for every $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$. Then, the previous inequality holds for every $\varphi \in H_0^1 \cap L^\infty(\Omega)$ with $\varphi \ge 0$. We now choose in (3.101) the admissible test function

$$\varphi = e^{M_k(u+R)^+} \psi, \quad \psi \in V_u, \quad \psi \ge 0.$$

It results

$$\int_\Omega j_\xi(x, u, \nabla u) \cdot H(\frac{u}{k}) \nabla \psi + \int_\Omega j_s(x, u, \nabla u) H(\frac{u}{k}) \psi \ \ge -\frac{C}{k} + \Big\langle w, H(\frac{u}{k}) \psi \Big\rangle . \tag{3.102}$$

Notice that

$$\begin{aligned} \Big| j_\xi(x, u, \nabla u) \cdot H(\frac{u}{k}) \nabla \psi \Big| &\le \big| j_\xi(x, u, \nabla u) \big| \, |\nabla \psi|, \\ \Big| j_s(x, u, \nabla u) H(\frac{u}{k}) \psi \Big| &\le | j_s(x, u, \nabla u) \psi | . \end{aligned}$$

Since $\psi \in V_u$ and from (3.42) and (3.44) we deduce that we can pass to the limit in (3.102) as $k \to \infty$, and we obtain

$$\int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla \psi + \int_\Omega j_s(x, u, \nabla u) \psi \ \ge \langle w, \psi \rangle, \quad \forall \psi \in V_u, \ \psi \ge 0.$$

To show the opposite inequality, we can take $v = \varphi e^{-M_k(u_n-R)^-} H\big(\frac{u_n}{k}\big)$ as test function in (3.92) and we can repeat the same argument as before. Thus, (3.95) follows.

Step 2. In this step we will prove that $u_n \to u$ strongly in $H_0^1(\Omega)$. From (3.69), (3.98) and Fatou Lemma, we have

$$0 \le \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla u \le \liminf_n \int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \nabla u_n < \infty$$

so that $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$. Therefore, by Theorem 3.31 we deduce

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u = \langle w,u\rangle. \tag{3.103}$$

To prove that u_n converges to u strongly in $H_0^1(\Omega)$ we follow the argument of [133, Theorem 3.2] and we consider the function $\zeta:\mathbb{R}\to\mathbb{R}$ defined by

$$\zeta(s) = \begin{cases} Ms, & \text{if } 0 < s < R,\ M = \frac{\beta(R)}{\alpha_0}, \\ MR, & \text{if } s \geq R, \\ -Ms, & \text{if } -R < s < 0, \\ MR, & \text{if } s \leq -R. \end{cases} \tag{3.104}$$

We have that $v_n = u_n e^{\zeta(u_n)}$ belongs to $H_0^1(\Omega)$, and conditions (3.42), (3.43) and (3.44) imply that hypotheses of Theorem 3.31 are satisfied. Then, we can use v_n as test function in (3.92). It results

$$\begin{aligned} &\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)} \\ &= \langle w_n, v_n\rangle - \int_\Omega \big[j_s(x,u_n,\nabla u_n) + j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n \zeta'(u_n)\big] v_n \end{aligned}$$

Note that v_n converges to $ue^{\zeta(u)}$ weakly in $H_0^1(\Omega)$ and almost everywhere in Ω. Moreover, conditions (3.42), (3.43) and (3.104) allow us to apply Fatou Lemma and get that

$$\begin{aligned} &\limsup_h \int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)} \\ &\leq \langle w, ue^{\zeta(u)}\rangle - \int_\Omega \big[j_s(x,u,\nabla u) + j_\xi(x,u,\nabla u)\cdot\nabla u\zeta'(u)\big] ue^{\zeta(u)}. \end{aligned} \tag{3.105}$$

On the other hand (3.103) and (3.104) imply that

$$\begin{gathered} j_\xi(x,u,\nabla u)\cdot\nabla\big[ue^{\zeta(u)}\big] + j_s(x,u,\nabla u)ue^{\zeta(u)} \in L^1(\Omega), \\ j_\xi(x,u,\nabla u)\cdot\nabla\big[ue^{\zeta(u)}\big] \in L^1(\Omega). \end{gathered} \tag{3.106}$$

Therefore, from Theorem 3.31,

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\Big[ue^{\zeta(u)}\Big] + \int_\Omega j_s(x,u,\nabla u)ue^{\zeta(u)} = \langle w, ue^{\zeta(u)}\rangle. \tag{3.107}$$

Thus, (3.105) and (3.107) imply

$$\begin{aligned} \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u\, e^{\zeta(u)} &\leq \liminf_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)} \\ &\leq \limsup_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)} \\ &\leq \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u e^{\zeta(u)}. \end{aligned}$$

Then (3.69) implies that $u_n \to u$ strongly in $H_0^1(\Omega)$. □

3.13. **Proofs of the main Theorems.** In this section we give the definition of a Concrete Palais-Smale sequence, we study the relation between a Palais-Smale sequence and a Concrete Palais-Smale sequence, and we prove that f satisfies the $(PS)_c$ for every $c \in \mathbb{R}$. Finally, we conclude by giving the proofs of Theorems 3.18 and 3.20.

Let us consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(v) = -\int_\Omega G(x, v) - \langle \Lambda, v\rangle,$$

where $\Lambda \in H^{-1}(\Omega)$, $G(x, s) = \int_0^s g(x, t)\,dt$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying assumption (3.54). Then (3.41) implies that the functional $f : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by $f(v) = J(v) + I(v)$ is lower semi-continuous. To apply the abstract theory, it is crucial to have the following result.

Theorem 3.35. *Assume conditions (3.40), (3.41), (3.43), (3.54). Then, for every $(u, \eta) \in \operatorname{epi} f$ with $f(u) < \eta$, it results*

$$|d\mathcal{G}_f|(u, \eta) = 1.$$

Moreover, if $j(x, -s, -\xi) = j(x, s, \xi)$, $g(x, -s) = -g(x, s)$ and $\Lambda = 0$, for every $\eta > f(0)$ one has $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0, \eta) = 1$.

Proof. Since G is of class C^1, Theorem 3.23 and Proposition 2.18 imply the result. □

Furthermore, since G a C^1 functional, as a consequence of Proposition 3.28 one has the following

Proposition 3.36. *Assume conditions (3.41), (3.42), (3.44), (3.54) and consider $u \in \operatorname{dom}(f)$ with $|df|(u) < \infty$. Then there exists $w \in H^{-1}(\Omega)$ such that $\|w\|_{-1,2} \le |df|(u)$ and*

$$\forall v \in V_u : \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla v + \int_\Omega j_s(x, u, \nabla u)v - \int_\Omega g(x, u)v - \langle \Lambda, v\rangle = \langle w, v\rangle.$$

Proof. Given $u \in \operatorname{dom}(f)$ with $|df|(u) < \infty$, let

$$\widehat{J}(v) = J(v) - \int_\Omega g(x, u)v - \langle \Lambda, v\rangle,$$
$$\widehat{I}(v) = I(v) + \int_\Omega g(x, u)v + \langle \Lambda, v\rangle.$$

Then, since $\widehat{I}$ is of class C^1 with $\widehat{I}'(u) = 0$, by (c) of Proposition 2.18 we get $|df|(u) = |d\widehat{J}|(u)$. By Proposition 3.28, there exists $w \in H^{-1}(\Omega)$ with $\|w\|_{-1,2} \le |df|(u)$ and

$$\forall v \in V_u : \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla v + \int_\Omega j_s(x, u, \nabla u)v - \int_\Omega g(x, u)v - \langle \Lambda, v\rangle = \langle w, v\rangle,$$

and the assertion is proved. □

We can now give the definition of the Concrete Palais-Smale condition.

Definition 3.37. Let $c \in \mathbb{R}$. We say that $\{u_n\}$ is a Concrete Palais-Smale sequence for f at level c ($(CPS)_c$-sequence for short) if there exists $w_n \in H^{-1}(\Omega)$ with $w_n \to 0$ such that $j_\xi(x, u_n, \nabla u_n) \cdot \nabla u_n \in L^1(\Omega)$ for every $n \ge 1$, and

$$f(u_n) \to c, \tag{3.108}$$

$$\int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \nabla v + \int_\Omega j_s(x, u_n, \nabla u_n)v - \int_\Omega g(x, u_n)v - \langle \Lambda, v\rangle \tag{3.109}$$

$$= \langle w_n, v\rangle, \quad \forall\, v \in V_{u_n}.$$

We say that f satisfies the Concrete Palais-Smale condition at level c ($(CPS)_c$ for short) if every $(CPS)_c$-sequence for f admits a strongly convergent subsequence in $H^1_0(\Omega)$.

Proposition 3.38. *Assume conditions (3.41), (3.42), (3.43), (3.44), (3.54). If $u \in \mathrm{dom}(f)$ satisfies $|df|(u) = 0$, then u is a generalized solution to*

$$-\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) + \Lambda, \textit{in } \Omega,$$
$$u = 0, \quad \textit{on } \partial\Omega.$$

Proof. It is sufficient to combine Lemma 3.29, Proposition 3.36, and Theorem 3.31. □

The following result concerns the relation between the $(PS)_c$ condition and the $(CPS)_c$ condition.

Proposition 3.39. *Assume conditions (3.41), (3.42), (3.43), (3.44), (3.54). Then if f satisfies the $(CPS)_c$ condition, it satisfies the $(PS)_c$ condition.*

Proof. Let $\{u_n\} \subset \mathrm{dom}(f)$ that satisfies the Definition 2.15. From Lemma 3.29 and Proposition 3.36 we get that u_n satisfies the conditions in Definition 3.37. Thus, there exists a subsequence, which converges in $H^1_0(\Omega)$. □

We now want to prove that f satisfies the $(CPS)_c$ condition at every level c. To do this, let us consider a $(CPS)_c$-sequence $\{u_n\} \in \mathrm{dom}(f)$.

From Theorem 3.34 we deduce the following result.

Proposition 3.40. *Assume that conditions (3.40), (3.41), (3.42), (3.43), (3.54) are satisfied. Let $\{u_n\}$ be a $(CPS)_c$-sequence for f, bounded in $H^1_0(\Omega)$. Then $\{u_n\}$ admits a strongly convergent subsequence in $H^1_0(\Omega)$.*

Proof. Let $\{u_n\} \subset \mathrm{dom}(f)$ be a concrete Palais-Smale sequence for f at level c. Taking into account that, as known, by (3.54) the map $\{u \mapsto g(x,u)\}$ is compact from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$, it suffices to apply Theorem 3.92 to see that $\{u_n\}$ is strongly compact in $H^1_0(\Omega)$. □

Proposition 3.41. *Assume conditions (3.40), (3.41), (3.42), (3.43), (3.48), (3.54), (3.55). Then every $(CPS)_c$-sequence $\{u_n\}$ for f is bounded in $H^1_0(\Omega)$.*

Proof. Condition (3.43) and (3.69) allow us to apply Theorem 3.31 to deduce that we may choose $v = u_n$ as test functions in (3.109). Taking into account conditions (3.48), (3.54), (3.59), (3.108), the boundedness of $\{u_n\}$ in $H^1_0(\Omega)$ follows by arguing as in [133, Lemma 4.3]. □

Remark 3.42. Note that we use condition (3.48) only in Proposition 3.41.

We can now state the following result.

Theorem 3.43. *Assume conditions (3.40), (3.41), (3.42), (3.43), (3.48), (3.54), (3.55). Then the functional f satisfies the $(PS)_c$ condition at every level $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset \mathrm{dom}(f)$ be a concrete Palais-Smale sequence for f at level c. From Proposition 3.41 it follows that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. By Proposition 3.40 f satisfies the Concrete Palais-Smale condition. Finally Proposition 3.39 implies that f satisfies the $(PS)_c$ condition. □

Proof of Theorem 3.18. This theorem will as a consequence of Theorem 2.21. First, note that (3.41) and (3.54) imply that f is lower semi-continuous. Moreover, from (3.57) we deduce that f is an even functional, and from Theorem 3.23 we deduce that (2.3) and condition (d) of Theorem 2.21 are satisfied. Hypotheses (3.56) implies that condition (b) of Theorem 2.21 is verified (see the subsequent proof of Theorem 3.20). Let now (λ_h, φ_h) be the sequence of solutions of $-\Delta u = \lambda u$ with homogeneous Dirichlet boundary conditions. Moreover, let us consider $V^+ = \overline{\text{span}}\{\varphi_h \in H_0^1(\Omega) : h \geq h_0\}$ and note that V^+ has finite codimension. To prove (a) of Theorem 2.21 it is enough to show that there exist $h_0, \gamma > 0$ such that for all $u \in V^+$ with $\|\nabla u\|_2 = 1$ there holds $f(u) \geq \gamma$. First, note that condition (3.54) implies that, for every $\varepsilon > 0$, we find $a_\varepsilon^{(1)} \in C_c^\infty(\Omega)$ and $a_\varepsilon^{(2)} \in L^{2N/(N+2)}(\Omega)$ with $\|a_\varepsilon^{(2)}\|_{2N/(N+2)} \leq \varepsilon$ and

$$|g(x,s)| \leq a_\varepsilon^{(1)}(x) + a_\varepsilon^{(2)}(x) + \varepsilon|s|^{\frac{N+2}{N-2}}.$$

Now, let $u \in V^+$ and notice that there exist two positive constants c_1, c_2 such that

$$\begin{aligned}
f(u) &\geq \alpha_0\|\nabla u\|_2^2 - \int_\Omega G(x,u) \\
&\geq \alpha_0\|\nabla u\|_2^2 - \int_\Omega \Big(\Big(a_\varepsilon^{(1)} + a_\varepsilon^{(2)}\Big)|u| + \frac{N-2}{2N}\varepsilon|u|^{\frac{2N}{N-2}}\Big) \\
&\geq \alpha_0\|\nabla u\|_2^2 - \|a_\varepsilon^{(1)}\|_2\|u\|_2 - c_1\|a_\varepsilon^{(2)}\|_{\frac{2N}{N+2}}\|\nabla u\|_2 - \varepsilon c_2\|\nabla u\|_2^{\frac{2N}{N-2}} \\
&\geq \alpha_0\|\nabla u\|_2^2 - \|a_\varepsilon^{(1)}\|_2\|u\|_2 - c_1\varepsilon\|\nabla u\|_2 - \varepsilon c_2\|\nabla u\|_2^{\frac{2N}{N-2}}.
\end{aligned}$$

Then if h_0 is sufficiently large, since $\lambda_h \to +\infty$, for all $u \in V^+$, $\|\nabla u\|_2 = 1$ implies $\|a_\varepsilon^{(1)}\|_2\|u\|_2 \leq \alpha_0/2$. Thus, for $\varepsilon > 0$ small enough, $\|\nabla u\|_2 = 1$ implies $f(u) \geq \gamma$ for some $\gamma > 0$. Then also (a) of Theorem 2.21 is satisfied. Theorem 3.43 implies that f satisfies $(PS)_c$ condition at every level c, so that we get the existence of a sequence of critical points $\{u_h\} \subset H_0^1(\Omega)$ with $f(u_h) \to +\infty$. Proposition 3.38 yields the assertion. $\square$

Let us conclude this section with the following proof.

Proof of Theorem 3.20. We will prove this theorem as a consequence of Theorem 2.20. To do this, let us notice that, from (3.41) and (3.60), f is lower semi-continuous on $H_0^1(\Omega)$. Moreover, Theorem 3.23 implies that condition (2.3) is satisfied. From Theorem 3.43 we deduce that f satisfies $(PS)_c$ condition at every level c. It is left to show that f satisfies the geometrical assumptions of Theorem 2.20.

Let us first consider the case in which $\Lambda = 0$. Notice that conditions (3.41), (3.60) and (3.61) imply that there exist $\gamma > 0$ and $r > 0$ such that for $\|u\|_{1,2} = r$ there holds $f(u) \geq \gamma$. Conditions (3.41) and (3.56) imply that there holds

$$f(v) \leq \int_\Omega \alpha(|v|)|\nabla v|^2 - \int_\Omega k(x)|v|^p + \|\bar{a}\|_1 + C_0\|\bar{b}\|_{\frac{2N}{N+2}}\|v\|_{1,2}. \tag{3.110}$$

Now, let us consider a finite dimensional subspace W of $H_0^1(\Omega)$ such that $W \subset L^\infty(\Omega)$. Condition (3.46) implies that, for every $\varepsilon > 0$, there exists $R > r$, $w \in W$, with $\|w\|_\infty > R$ and a positive constant C_ε such that

$$\int_\Omega \alpha(|w|)|\nabla w|^2 \leq \varepsilon C_W\|w\|_{1,2}^p + C_\varepsilon\|w\|_{1,2}^2, \tag{3.111}$$

where C_W is a positive constant depending on W. Then, by suitably choosing ε, (3.110) and (3.111) yield condition (2.4) for a suitable $v_1 \in H_0^1(\Omega)$ and for $v_0 = 0$. Thus, we can apply Theorem 2.20 and deduce the existence of a nontrivial critical point u of f. From Proposition 3.38, u is a generalized solution of Problem (3.53).

Now, let us consider the case in which $\Lambda \neq 0$. Let φ_1 be the first eigenfunction of the Laplace operator with homogeneous Dirichlet boundary conditions and set $v_0 = t_0\varphi_1$ for $t_0 > 0$. Then, if t_0 sufficiently small, thanks to (3.41) and (3.60), we get $f(v_0) < 0$. As before, (3.41), (3.60) and (3.61) imply that there exist $\varepsilon > 0$, $r = r(\varepsilon) > 0$ and $\gamma > 0$ such that, for every $\Lambda \in H^{-1}(\Omega)$ with $\|\Lambda\|_{-1} < \varepsilon$, there holds

$$f(u) \geq \gamma, \quad \text{for every } u \text{ with } \|u - v_0\|_{1,2} = r.$$

Moreover, we use condition (3.41), (3.46) and (3.56) and we argue as before to deduce the existence of $v_1 \in H_0^1(\Omega)$ with $\|v_1 - v_0\| > r$ and $f(v_1) < 0$. Condition (2.4) is thus fulfilled. Then, we can apply Theorem 2.20 getting the existence of two distinct nontrivial critical points of f. Finally, Proposition 3.38 yields the conclusion. □

Remark 3.44. Notice that Theorems 3.16 and 3.17 are an easy consequence of Theorems 3.18 and 3.20 respectively. Indeed, consider for example $g_1(x,s) = a(x)\mathrm{arctg}s + |s|^{p-2}s$. To prove Theorem 3.16, it is left to show that $g_1(x,s)$ satisfies conditions (3.54), (3.55) and (3.56). First, notice that Young inequality implies that, for every $\varepsilon > 0$, there exists a positive constant $\beta(\varepsilon)$ such that (3.54) holds with $a_\varepsilon(x) = \beta(\varepsilon) + a(x)$. Moreover, (3.55) is satisfied with $a_0(x) = 0$ and $b_0(x) = \pi/2(p-1)$. Finally, (3.56) is verified with $k(x) = 1/p$, $\overline{a}(x) = 0$ and $\overline{b}(x) = (\pi/2 + C)\,a(x)$ where $C \in \mathbb{R}^+$ is sufficiently large. Theorem 3.17 can be obtained as a consequence of Theorem 3.20 in a similar fashion.

3.14. **Summability results.** In this section we suppose that $g(x,s)$ satisfies the following growth condition

$$|g(x,s)| \leq a(x) + b|s|^{\frac{N+2}{N-2}}, \quad a(x) \in L^r(\Omega), \quad b \in \mathbb{R}^+. \tag{3.112}$$

Note that (3.54) implies (3.112). Let us set $2^* = 2N/(N-2)$. We prove the following

Theorem 3.45. *Assume conditions* (3.40), (3.41), (3.42), (3.43), (3.112). *Let $u \in H_0^1(\Omega)$ be a generalized solution of problem (P). Then the following conclusions hold:*

(a) *If $r \in (2N/(N+2), N/2)$, then u belongs to $L^{r^{**}}(\Omega)$, where $r^{**} = Nr/(N-2r)$;*
(b) *if $r > N/2$, then u belongs to $L^\infty(\Omega)$.*

The above theorem will be proved as a consequence of the following result.

Lemma 3.46. *Let us assume that conditions* (3.40), (3.41), (3.42), (3.43) *are satisfied. Let $u \in H_0^1(\Omega)$ be a generalized solution of the problem*

$$\begin{gathered} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) + c(x)u = f(x), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.113}$$

Then the following conclusions hold:

(i) *If $c \in L^{\frac{N}{2}}(\Omega)$ and $f \in L^r(\Omega)$, with $r \in (2N/(N+2), N/2)$, then u belongs to $L^{r^{**}}(\Omega)$, where $r^{**} = Nr/(N-2r)$;*
(ii) *if $c \in L^t(\Omega)$ with $t > N/2$ and $f \in L^q(\Omega)$, with $q > N/2$, then u belongs to $L^\infty(\Omega)$.*

Proof. Let us first prove conclusion (i). For every $k > R$ (where R is defined in (3.43)), let us define the function $\eta_k(s) : \mathbb{R} \to \mathbb{R}$ such that $\eta_k \in C^1$, η_k is odd and

$$\eta_k(s) = \begin{cases} 0, & \text{if } 0 < s < R, \\ (s-R)^{2\gamma+1}, & \text{if } R < s < k, \\ b_k s + c_k, & \text{if } s > k, \end{cases} \tag{3.114}$$

where b_k and c_k are constant such that η_k is C^1. Since u is a generalized solution of (3.113), $v = \eta_k(u)$ belongs to W_u. Then we can take it as test function, moreover, $j_s(x, u, \nabla u)\eta_k(u) \geq 0$. Then from (3.43) and (3.69) we get

$$\alpha_0 \int_\Omega \eta_k'(u)|\nabla u|^2 \leq \int_\Omega f(x)\eta_k(u) - \int_\Omega c(x)u\eta_k(u). \tag{3.115}$$

Now, let us consider the odd function $\psi_k(s) : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_k(s) = \int_0^s \sqrt{\eta_k'(t)}\, dt. \tag{3.116}$$

The following properties of the functions ψ_k and η_k can be deduced from (3.114) and (3.116) by easy calculations

$$\big[\psi_k'(s)\big]^2 = \eta_k'(s), \tag{3.117}$$

$$0 \leq \eta_k(s)(s-R) \leq C_0\psi_k(s)^2, \tag{3.118}$$

$$|\eta_k(s)| \leq C_0|\psi_k(s)|^{\frac{2\gamma+1}{\gamma+1}}, \tag{3.119}$$

where C_0 is a positive constant. Notice that for every $\varepsilon > 0$ there exist $c_1(x) \in L^{\frac{N}{2}}(\Omega)$, with $\|c_1\|_{\frac{N}{2}} \leq \varepsilon$ and $c_2 \in L^\infty(\Omega)$ such that $c(x) = c_1(x) + c_2(x)$. From (3.115), (3.117), (3.118) and Hölder inequality, we deduce

$$\begin{aligned} &\alpha_0 \int_\Omega |\nabla(\psi_k(u))|^2 \\ &\leq C_0\|c_1(x)\|_{\frac{N}{2}}\Big[\int_\Omega |\psi_k(u)|^{2^*}\Big]^{2/2^*} + \int_\Omega |f(x) - Rc_1(x) - c_2(x)u||\eta_k(u)|. \end{aligned}$$

We fix $\varepsilon = (\alpha_0\mathcal{S})/(2C_0)$, where $\mathcal{S}$ is the Sobolev constant. We obtain

$$\Big[\int_\Omega |\psi_k(u)|^{2^*}\Big]^{2/2^*} \leq C\int_\Omega |f(x) - Rc_1(x) - c_2(x)u||\eta_k(u)|. \tag{3.120}$$

Now, let us define the function

$$h(x) = |f(x) - Rc_1(x) - c_2(x)u(x)|, \tag{3.121}$$

and note that $h(x)$ belongs to $L^t(\Omega)$ with

$$t = \min\{r, 2^*\}. \tag{3.122}$$

Let us consider first the case in which $t = r$, then from (3.119) and (3.120), we get

$$\Big[\int_\Omega |\psi_k(u)|^{2^*}\Big]^{2/2^*} \leq C\|h\|_r\Big[\int_\Omega |\psi_k(u)|^{r'\frac{2\gamma+1}{\gamma+1}}\Big]^{1/r'}.$$

Since $2N/(N+2) < r < N/2$ we can define $\gamma \in \mathbb{R}^+$ by

$$\gamma = \frac{r(N+2) - 2N}{2(N-2r)} \ \Rightarrow\ 2^*(\gamma+1) = r'(2\gamma+1) = r^{**}. \tag{3.123}$$

Moreover, since $r < N/2$ we have that $2/2^* > 1/r'$, then

$$\Big[\int_\Omega |\psi_k(u)|^{2^*}\Big]^{\frac{2}{2^*}-\frac{1}{r'}} \le C\|h\|_r. \tag{3.124}$$

Note that $|\psi_k(u)| \to C(\gamma)|u-R|^{\gamma+1}\chi_{\{x:|u(x)|>R\}}$ almost everywhere in Ω. Then Fatou Lemma implies that $|u-R|^{\gamma+1}\chi_{\{x:|u(x)|>R\}}$ belongs to $L^{2^*}(\Omega)$. Thus, u belongs to $L^{2^*(\gamma+1)}(\Omega) = L^{r^{**}}(\Omega)$ and the conclusion follows. Consider now the case in which $t = 2^*$ and note that this implies that $N > 6$. In this case we get

$$\Big[\int_\Omega |\psi_k(u)|^{2^*}\Big]^{2/2^*} \le C\|h\|_{2^*}\Big[\int_\Omega s|\psi_k(u)|^{(2^*)'\frac{2\gamma+1}{\gamma+1}}\Big]^{\frac{1}{(2^*)'}}.$$

Since $N > 6$ it results $2/2^* > 1/(2^*)'$. Moreover, we can choose γ such that

$$2^*(\gamma+1) = (2^*)'(2\gamma+1).$$

Thus, we follow the same argument as in the previous case and we deduce that u belongs to $L^{s_1}(\Omega)$ where

$$s_1 = \frac{2^*N}{N-2\,2^*}.$$

If it still holds $s_1 < r$ we can repeat the same argument to gain more summability on u. In this way for every $s \in [2^*, r)$ we can define the increasing sequence

$$s_0 = 2^*, \quad s_{n+1} = \frac{Ns_n}{N-2s_n},$$

and we deduce that there exists $\overline{n}$ such that $s_{\overline{n}-1} < r$ and $s_{\overline{n}} \ge r$. At this step from (3.122) we get that $t = r$ and then $u \in L^{r^{**}}(\Omega)$, that is the maximal summability we can achieve.

Now, let us prove conclusion (ii). First, note that since $f \in L^q(\Omega)$, with $q > N/2$, f belongs to $L^r(\Omega)$ for every $r > (2N)/(N+2)$. Then, conclusion (i) implies that $u \in L^\sigma(\Omega)$ for every $\sigma > 1$. Now, take $\delta > 0$ such that $t-\delta > N/2$, since $u \in L^{\frac{t}{\delta}}(\Omega)$ it results

$$\int_\Omega |c(x)u(x)|^{t-\delta} \le \|c(x)\|_t^{t-\delta}\Big[\int_\Omega |u(x)|^{t/\delta}\Big]^{\frac{\delta}{t}} < \infty.$$

Then, the function $d(x) = f(x) - c(x)u(x)$ belongs to $L^r(\Omega)$ with $r = \min\{q, t-\delta\} > N/2$. Let us take $k > R$ (R is defined in (3.43)) and consider the function $v = G_k(u) = u - T_k(u)$ (where $T_k(s)$ is defined in (3.63)). Since u is a generalized solution of (3.113) we can take v as test function. From (3.43) and (3.69) it results

$$\alpha_0\int_\Omega |\nabla G_k(u)|^2 \le \int_\Omega |d(x)||G_k(u)|.$$

The conclusion follows from Theorem 4.2 of [136]. □

Remark 3.47. In classical results of this type (see e.g. [105] or [26]) it is usually considered as test function $v = |u|^{2\gamma}u$. Note that this type of function cannot be used here for it does not belong to the space W_u. Moreover, the classical truncation T_u seems not to be useful because of the presence of $c(x)u$. Then, we have chosen a suitable truncation of u in order to manage also the term $c(x)u$.

Proof of Theorem 3.45. This theorem will be proved as a consequence of Lemma 3.46. So, consider u a generalized solution of Problem (3.53), we have to prove that u is a generalized solution of Problem 3.113 for suitable $f(x)$ and $c(x)$. This is shown in Theorem 2.2.5 of [36], then we will give here a sketch of the proof of [36] just for clearness. We set

$$g_0(x,s) = \min\{\max\{g(x,s), -a(x)\}, a(x)\},$$

$$g_1(x,s) = g(x,s) - g_0(x,s).$$

It follows that $g(x,s) = g_0(x,s) + g_1(x,s)$ and $|g_0(x,s)| \le a(x)$ so that we can set $f(x) = g_0(x,u(x))$. Moreover, we define

$$c(x) = \begin{cases} -\frac{g_1(x,u(x))}{u(x)}, & \text{if } u(x) \neq 0, \\ 0, & \text{if } u(x) = 0. \end{cases}$$

Then $|c(x)| \le b|u(x)|^{\frac{4}{N-2}}$, so that $c(x) \in L^{\frac{N}{2}}(\Omega)$. Lemma 3.46 implies that conclusion (a) holds. Now, if $r > N/2$ we have that $f(x) \in L^r(\Omega)$ with $r > N/2$. Moreover, conclusion (a) implies that $u \in L^t(\Omega)$ for every $t < \infty$, so that $c(x) \in L^t(\Omega)$ with $t > N/2$. Then Lemma 3.46 implies that $u \in L^\infty(\Omega)$. □

Remark 3.48. When dealing with quasi-linear equations (i.e. $j(x,s,\xi) = a(x,s)\xi \cdot \xi$), a standard technique, to prove summability results, is to reduce the problem to the linear one and to apply the classical result (see e.g. [136]). Note that here this is not possible due to the general form of j.

4. Perturbation from Symmetry

We refer the reader to [129, 130, 111, 93, 30]. Some parts of these publications have been slightly modified to give this collection a more uniform appearance.

4.1. **Quasi-linear elliptic systems.**

In critical point theory, an open problem concerning existence, is the role of symmetry in obtaining multiple critical points for even functionals. Around 1980, the semi-linear scalar problem

$$
\begin{gathered}
-\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) = g(x,u) + \varphi \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega\,,
\end{gathered}
$$

with g super-linear and odd in u and $\varphi \in L^2(\Omega)$, has been object of a very careful analysis by A. Bahri and H. Berestycki in [15], M. Struwe in [138], G-C. Dong and S. Li in [66] and by P.H. Rabinowitz in [119] via techniques of classical critical point theory. Around 1990, A. Bahri and P.L. Lions in [17, 18] improved the previous results via a Morse-Index type technique. Later on, since 1994, several efforts have been devoted to study existence for quasi-linear scalar problems of the type

$$
\begin{gathered}
-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u = g(x,u) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega\,.
\end{gathered}
$$

We refer the reader to [9, 33, 32, 36, 138] and to [6, 113, 133] for a more general setting. In this case the associated functional $f : H_0^1(\Omega,\mathbb{R}^N) \to \mathbb{R}$ given by

$$
f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j u \, dx - \int_\Omega G(x,u)\, dx\,,
$$

is not even locally Lipschitz unless the a_{ij}'s do not depend on u or $n = 1$. Consequently, techniques of non-smooth critical point theory have to be applied. It seems now natural to ask whether some existence results for perturbed even functionals still hold in a quasi-linear setting, both scalar ($N = 1$) and vectorial ($N \geq 2$). In [134] it has recently been proved that diagonal quasi-linear elliptic systems of the type ($k = 1, \dots, N$)

$$
-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u)D_i u_k) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h = D_{s_k} G(x,u) \quad \text{in } \Omega\,, \tag{4.1}
$$

possess a sequence (u^m) of weak solutions in $H_0^1(\Omega,\mathbb{R}^N)$ under suitable assumptions, including symmetry, on coefficients a_{ij}^h and G. To prove this result, we looked for critical points of the functional $f_0 : H_0^1(\Omega,\mathbb{R}^N) \to \mathbb{R}$ defined by

$$
f_0(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - \int_\Omega G(x,u)\, dx. \tag{4.2}
$$

In this section we want to investigate the effects of destroying the symmetry of system (4.1) and show that for each $\varphi \in L^2(\Omega, \mathbb{R}^N)$ the perturbed problem

$$\begin{aligned} &-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u)D_i u_k) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h \\ &= D_{s_k} G(x,u) + \varphi_k \qquad \text{in } \Omega\,, \end{aligned} \tag{4.3}$$

still has infinitely many weak solutions. Of course, to this aim, we shall study the associated functional

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - \int_\Omega G(x,u)\, dx - \int_\Omega \varphi \cdot u \, dx. \tag{4.4}$$

In the next, Ω will denote an open and bounded subset of $\mathbb{R}^n$. To adapt the perturbation argument of [119], we shall consider the following assumptions: - the matrix $\left(a_{ij}^h(x,s)\right)$ is measurable in x for each $s \in \mathbb{R}^N$ and of class C^1 in s for a.e. $x \in \Omega$ with

$$a_{ij}^h(x,s) = a_{ji}^h(x,s).$$

Moreover, there exist $\nu > 0$ and $C > 0$ such that

$$\begin{gathered} \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,s)\xi_i^h\xi_j^h \geq \nu|\xi|^2, \quad \left|a_{ij}^h(x,s)\right| \leq C, \\ \left|D_s a_{ij}^h(x,s)\right| \leq C\,, \quad \sum_{i,j=1}^{n}\sum_{h=1}^{N} s\cdot D_s a_{ij}^h(x,s)\xi_i^h\xi_j^h \geq 0\,, \end{gathered} \tag{4.5}$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$; - (if $N \geq 2$) there exists a bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\sum_{i,j=1}^{n}\sum_{h=1}^{N}\Big(\frac{1}{2} D_s a_{ij}^h(x,s)\cdot \exp_\sigma(r,s) + a_{ij}^h(x,s) D_{s_h}(\exp_\sigma(r,s))_h\Big)\xi_i^h\xi_j^h \leq 0\,, \tag{4.6}$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$, $\sigma \in \{-1,1\}^N$ and $r, s \in \mathbb{R}^N$, where

$$(\exp_\sigma(r,s))_i := \sigma_i \exp[\sigma_i(\psi(r_i) - \psi(s_i))]\,,$$

for each $i = 1, \dots, N$. - the function $G(x,s)$ is measurable in x for all $s \in \mathbb{R}^N$, of class C^1 in s for a.e. $x \in \Omega$ with $G(x,0) = 0$ and $g(x,\cdot)$ denotes the gradient of G with respect of s. - there exist $q > 2$ and $R > 0$ such that

$$|s| \geq R \Rightarrow 0 < qG(x,s) \leq s\cdot g(x,s)\,, \tag{4.7}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^N$; - there exists $\gamma \in]0, q-2[$ such that

$$\sum_{i,j=1}^{n}\sum_{h=1}^{N} s\cdot D_s a_{ij}^h(x,s)\xi_i^h\xi_j^h \leq \gamma \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,s)\xi_i^h\xi_j^h\,, \tag{4.8}$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$. Under the previous assumptions, the following is our main result.

Theorem 4.1. *Assume that there exists σ in $]1, \frac{qn+(q-1)(n+2)}{qn+(q-1)(n-2)}[$ such that*

$$|g(x,s)| \leq a + b|s|^{\sigma}, \tag{4.9}$$

with $a, b \in \mathbb{R}$ and that for a.e. $x \in \Omega$ and for each $s \in \mathbb{R}^N$

$$a_{ij}^h(x,-s) = a_{ij}^h(x,s), \quad g(x,-s) = -g(x,s).$$

Then there exists a sequence $(u^m) \subseteq H_0^1(\Omega,\mathbb{R}^N)$ of solutions to the system

$$\begin{aligned} &-\sum_{i,j=1}^n D_j(a_{ij}^k(x,u)D_i u_k) + \frac{1}{2}\sum_{i,j=1}^n \sum_{h=1}^N D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h \\ &= D_{s_k} G(x,u) + \varphi_k \quad \text{in } \Omega \end{aligned}$$

such that $f(u^m) \to +\infty$ as $m \to \infty$.

This is clearly an extension of the results of [15, 66, 119, 138] to the quasi-linear case, both scalar ($N = 1$) and vectorial ($N \geq 2$).

Let us point out that in the case $N = 1$ a stronger version of the previous result can be proven. Indeed, we may completely drop assumption (b) and replace Lemma 3.4 with [36, Lemma 2.2.4]. To the best of our knowledge, in the case $N > 1$ only very few multiplicity results have been obtained so far via non-smooth critical point theory (see [9, 134, 138]).

4.2. **Symmetry perturbed functionals.** Given $\varphi \in L^2(\Omega,\mathbb{R}^N)$, we shall now consider the functional $f : H_0^1(\Omega,\mathbb{R}^N) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - \int_\Omega G(x,u)\, dx - \int_\Omega \varphi \cdot u \, dx.$$

If $\varphi \not\equiv 0$, clearly f is not even. Note that by (4.7) we find $c_1, c_2, c_3 > 0$ such that

$$\frac{1}{q}(s \cdot g(x,s) + c_1) \geq G(x,s) + c_2 \geq c_3|s|^q. \tag{4.10}$$

Lemma 4.2. *Assume that $u \in H_0^1(\Omega,\mathbb{R}^N)$ is a weak solution to (4.3). Then there exists $\sigma > 0$ such that*

$$\int_\Omega (G(x,u) + c_2)\, dx \leq \sigma \left(f(u)^2 + 1\right)^{1/2}.$$

Proof. If $u \in H_0^1(\Omega,\mathbb{R}^N)$ is a weak solution to (4.3), taking into account (4.8), we deduce that

$$\begin{aligned} f(u) &= f(u) - \frac{1}{2} f'(u)(u) \\ &= \int_\Omega \Big[\frac{1}{2} g(x,u)\cdot u - G(x,u) - \frac{1}{2}\varphi \cdot u\Big]\, dx \\ &\quad - \frac{1}{4}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u)\cdot u D_i u_h D_j u_h \, dx \\ &\geq \big(\frac{1}{2} - \frac{1}{q}\big)\int_\Omega (g(x,u)\cdot u + c_1)\, dx - \frac{1}{2}\|\varphi\|_2 \|u\|_2 \\ &\quad - \frac{\gamma}{4}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - c_4 \end{aligned}$$

$$\geq \left(\frac{q}{2} - 1 - \frac{\gamma}{2}\right) \int_\Omega (G(x,u) + c_2)\, dx - \frac{\gamma}{2} f(u) - \varepsilon \|u\|_q^q - \beta(\varepsilon)\|\varphi\|_2^{q'} - c_5$$

with $\varepsilon \to 0$ and $\beta(\varepsilon) \to +\infty$. Choosing $\varepsilon > 0$ small enough, by (4.10) we have

$$\sigma f(u) \geq \int_\Omega (G(x,u) + c_2)\, dx - c_6,$$

where $\sigma = \frac{2+\gamma}{q-2-\gamma}$, and the assertion follows as in [119, Lemma 1.8]. □

We now want to introduce the modified functional, which is the main tool in order to obtain our result. Let us define $\chi \in C^\infty(\mathbb{R})$ by setting $\chi = 1$ for $s \leq 1$, $\chi = 0$ for $s \geq 2$ and $-2 < \chi' < 0$ when $1 < s < 2$, and let for each $u \in H_0^1(\Omega, \mathbb{R}^N)$

$$\phi(u) = 2\sigma \left(f(u)^2 + 1\right)^{1/2}, \quad \psi(u) = \chi\Big(\phi(u)^{-1} \int_\Omega (G(x,u) + c_2)\, dx\Big).$$

Finally, we define the modified functional by

$$\begin{aligned} \tilde{f}(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx + \\ - \int_\Omega G(x,u)\, dx - \psi(u) \int_\Omega \varphi \cdot u \, dx. \end{aligned} \tag{4.11}$$

The Euler's equation associated to the previous functional is given by

$$-\sum_{i,j=1}^n D_j(a_{ij}^k(x,u) D_i u_k) + \frac{1}{2} \sum_{i,j=1}^n \sum_{h=1}^N D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h = \tilde{g}(x,u) \quad \text{in } \Omega, \tag{4.12}$$

where we set

$$\tilde{g}(x,u) = g(x,u) + \psi(u)\varphi + \psi'(u) \int_\Omega \varphi \cdot u \, dx.$$

Note that taking into account the previous Lemma, if $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a weak solution to (4.3), we have that $\psi(u) = 1$ and therefore $\tilde{f}(u) = f(u)$. In the next result, we measure the defect of symmetry of $\tilde{f}$, which turns out to be crucial in the final comparison argument.

Lemma 4.3. *There exists $\beta > 0$ such that for all $u \in H_0^1(\Omega, \mathbb{R}^N)$*

$$|\tilde{f}(u) - \tilde{f}(-u)| \leq \beta \left(|\tilde{f}(u)|^{1/q} + 1\right).$$

Proof. Note first that if $u \in \mathrm{supp}(\psi)$ then

$$\Big|\int_\Omega \varphi \cdot u \, dx\Big| \leq \alpha\big(|f(u)|^{1/q} + 1\big), \tag{4.13}$$

where $\alpha > 0$ depends on $\|\varphi\|_2$. Indeed, by (4.10) we have

$$\Big|\int_\Omega \varphi \cdot u \, dx\Big| \leq \|u\|_2 \|\varphi\|_2 \leq c\|u\|_q \leq \hat{c}\Big(\int_\Omega (G(x,u) + c_2)\, dx\Big)^{1/q},$$

and since $u \in \mathrm{supp}(\psi)$,

$$\int_\Omega (G(x,u) + c_2)\, dx \leq 4\sigma \left(f(u)^2 + 1\right)^{1/2} \leq \tilde{c}(|f(u)| + 1),$$

inequality (4.13) easily follows. Now, since of course

$$|f(u)| \leq |\widetilde{f}(u)| + 2\Big|\int_\Omega \varphi \cdot u\, dx\Big|,$$

by (4.13) we immediately get for some $b > 0$

$$\psi(u)\Big|\int_\Omega \varphi \cdot u\, dx\Big| \leq b\psi(u)\Big(|\widetilde{f}(u)|^{1/q} + \Big|\int_\Omega \varphi \cdot u\, dx\Big|^{1/q} + 1\Big).$$

Using Young's inequality, for some $b_1, b_2 > 0$ we have that

$$\psi(u)\Big|\int_\Omega \varphi \cdot u\, dx\Big| \leq b_1\left(|\widetilde{f}(u)|^{1/q} + 1\right),$$

and

$$\psi(-u)\left|\int_\Omega \varphi \cdot u\, dx\right| \leq b_2\left(|\widetilde{f}(u)|^{1/q} + 1\right),$$

and since

$$|\widetilde{f}(u) - \widetilde{f}(-u)| = (\psi(u) + \psi(-u))\Big|\int_\Omega \varphi \cdot u\, dx\Big|,$$

the assertion follows. □

Theorem 4.4. *There exists $M > 0$ such that if $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a weak solution to (4.12) with $\widetilde{f}(u) \geq M$ then u is a weak solution to (4.3) and $\widetilde{f}(u) = f(u)$.*

Proof. Let us first prove that there exist $\widetilde{M} > 0$ and $\widetilde{\alpha} > 0$ such that

$$\forall M \in [\widetilde{M}, +\infty[: \quad \widetilde{f}(u) \geq M, \quad u \in \operatorname{supp}(\psi) \Rightarrow f(u) \geq \widetilde{\alpha} M. \tag{4.14}$$

Since we have

$$f(u) \geq \widetilde{f}(u) - \Big|\int_\Omega \varphi \cdot u\Big|,$$

by 4.13 we deduce that

$$f(u) + \alpha\, |f(u)|^{1/q} \geq \widetilde{f}(u) - \alpha \geq \frac{M}{2}$$

for $M \geq \widetilde{M}$, with $\widetilde{M}$ large enough. Now, if it was $f(u) \leq 0$, we would obtain

$$\frac{\alpha^{q'}}{q'} + \frac{1}{q}\,|f(u)| \geq \alpha\, |f(u)|^{1/q} \geq \frac{M}{2} + |f(u)|,$$

which is not possible if we take $\widetilde{M} > 2\alpha^{q'}(q')^{-1}$. Therefore it is $f(u) > 0$ and

$$f(u) > \frac{M}{4} \quad \text{or} \quad f(u) \geq \left(\frac{M}{4\alpha}\right)^q,$$

and (4.14) is proven. Of course, taking into account the definition of ψ, to prove the Lemma it suffices to show that if $M > 0$ is sufficiently large and $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a weak solution to (4.12) with $\widetilde{f}(u) \geq M$, then

$$\phi(u)^{-1}\int_\Omega (G(x, u) + c_2)\, dx \leq 1.$$

If we set

$$\vartheta(u) = \phi(u)^{-1}\int_\Omega (G(x, u) + c_2)\, dx,$$

it follows that

$$\psi'(u)(u) = \chi'(\vartheta(u))\phi(u)^{-2}\Big[\phi(u)\int_\Omega g(x, u) \cdot u\, dx - (2\sigma)^2\vartheta(u) f(u) f'(u)(u)\Big].$$

Define now $T_1, T_2 : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ by setting

$$T_1(u) = \chi'(\vartheta(u))(2\sigma)^2 \vartheta(u)\phi(u)^{-2} f(u) \int_\Omega \varphi \cdot u \, dx,$$

$$T_2(u) = \chi'(\vartheta(u))\phi(u)^{-1} \int_\Omega \varphi \cdot u \, dx + T_1(u).$$

Then we obtain

$$\begin{aligned}
\tilde{f}'(u)(u) &= (1 + T_1(u)) \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx \\
&\quad + \frac{1}{2}(1 + T_1(u)) \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u) \cdot u D_i u_h D_j u_h \, dx \\
&\quad - (1 + T_2(u)) \int_\Omega g(x,u) \cdot u \, dx - (\psi(u) + T_1(u)) \int_\Omega \varphi \cdot u \, dx.
\end{aligned}$$

Consider now the term

$$\tilde{f}(u) - \frac{1}{2(1 + T_1(u))} \tilde{f}'(u)(u).$$

If $\psi(u) = 1$ and $T_1(u) = 0 = T_2(u)$, the assertion follows from Lemma 4.2. Otherwise, since $0 \le \psi(u) \le 1$, if $T_1(u)$ and $T_2(u)$ are both small enough the computations we have made in Lemma 4.2 still hold true with σ replaced by $(2-\varepsilon)\sigma$, for a small $\varepsilon > 0$, and again assertion follows as in Lemma 4.2.

It then remains to show that if $M \to \infty$, then $T_1(u), T_2(u) \to 0$. We may assume that $u \in \operatorname{supp}(\psi)$, otherwise $T_i(u) = 0$, for $i = 1, 2$. Therefore, taking into account (4.13), there exists $c > 0$ with

$$|T_1(u)| \le c \frac{|f(u)|^{1/q} + 1}{|f(u)|}.$$

Finally, by (4.14) we deduce $|T_1(u)| \to 0$ as $M \to \infty$. Similarly, $|T_2(u)| \to 0$. □

4.3. **Boundedness of concrete Palais-Smale sequences.**

Definition 4.5. Let $c \in \mathbb{R}$. A sequence $(u^m) \subseteq H_0^1(\Omega, \mathbb{R}^N)$ is said to be *a concrete Palais-Smale sequence at level c* (*$(CPS)_c$-sequence*, in short) for $\tilde{f}$, if $\tilde{f}(u^m) \to c$,

$$\sum_{i,j=1}^n \sum_{h=1}^N D_{s_k} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \in H^{-1}(\Omega, \mathbb{R}^N)$$

eventually as $m \to \infty$ and

$$-\sum_{i,j=1}^n D_j(a_{ij}^k(x,u^m) D_i u_k^m) + \frac{1}{2} \sum_{i,j=1}^n \sum_{h=1}^N D_{s_k} a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m - \tilde{g}_k(x,u^m),$$

approaches zero strongly in $H^{-1}(\Omega, \mathbb{R}^N)$, where

$$\tilde{g}(x,u) = g(x,u) + \psi(u)\varphi + \psi'(u) \int_\Omega \varphi \cdot u \, dx.$$

We say that $\tilde{f}$ satisfies *the concrete Palais-Smale condition at level c*, if every $(CPS)_c$ sequence for $\tilde{f}$ admits a strongly convergent subsequence in $H_0^1(\Omega, \mathbb{R}^N)$.

Lemma 4.6. *There exists $M > 0$ such that each $(CPS)_c$-sequence (u^m) for $\widetilde{f}$ with $c \geq M$ is bounded in $H^1_0(\Omega, \mathbb{R}^N)$.*

Proof. Let $M > 0$ and (u^m) be a $(CPS)_c$-sequence for $\widetilde{f}$ with $c \geq M$ in $H^1_0(\Omega, \mathbb{R}^N)$ such that, eventually as $m \to +\infty$

$$M \leq \widetilde{f}(u^m) \leq K.$$

for some $K > 0$. Taking into account [134, Lemma 3], we have $\widetilde{f}'(u^m)(u^m) \to 0$ as $m \to +\infty$. Therefore, for large $m \in \mathbb{N}$ and any $\varrho > 0$, it follows

$$\begin{aligned}
&\varrho\|u^m\|_{1,2} + K \geq \widetilde{f}(u^m) - \varrho\widetilde{f}'(u^m)(u^m) \\
&= \big(\frac{1}{2} - \varrho(1 + T_1(u^m))\big) \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\
&\quad - \frac{\varrho}{2}(1 + T_1(u^m)) \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x, u^m) \cdot u^m D_i u_h^m D_j u_h^m \, dx \\
&\quad + \varrho(1 + T_2(u^m)) \int_\Omega g(x, u^m) \cdot u^m \, dx \\
&\quad - \int_\Omega G(x, u^m) \, dx + [\varrho(\psi(u^m) + T_1(u^m)) - \psi(u^m)] \int_\Omega \varphi \cdot u^m \, dx \geq \\
&\geq \Big(\frac{1}{2} - \varrho(1 + T_1(u^m)) - \frac{\varrho\gamma}{2}(1 + T_1(u^m))\Big) \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\
&\quad + \varrho(1 + T_2(u^m)) \int_\Omega g(x, u^m) \cdot u^m \, dx \\
&\quad - \int_\Omega G(x, u^m) \, dx + [\varrho(\psi(u^m) + T_1(u^m)) - \psi(u^m)] \int_\Omega \varphi \cdot u^m \, dx \\
&\geq \frac{\nu}{2} \left(1 - \varrho\,(2 + \gamma)\,(1 + T_1(u^m))\right) \|u^m\|_{1,2}^2 + (q\varrho(1 + T_2(u^m)) - 1) \int_\Omega G(x, u^m) \, dx \\
&\quad - [\varrho(1 + T_1(u^m)) + 1]\|\varphi\|_2 \|u^m\|_2.
\end{aligned}$$

If we choose M sufficiently large, we find $\varepsilon > 0$, $\eta > 0$ and $\varrho \in \left]\frac{1+\eta}{q}, \frac{1-\varepsilon}{\gamma+2}\right[$ such that uniformly in $m \in \mathbb{N}$

$$(1 - \varrho\,(2 + \gamma)\,(1 + T_1(u^m))) > \varepsilon, \quad (q\varrho(1 + T_2(u^m)) - 1) > \eta.$$

Hence we obtain

$$\varrho\|u^m\|_{1,2} + K \geq \frac{\nu\varepsilon}{2}\|u^m\|_{1,2}^2 + b\eta\|u^m\|_q^q - c\|u^m\|_{1,2},$$

which implies that the sequence (u^m) is bounded in $H^1_0(\Omega, \mathbb{R}^N)$. □

Lemma 4.7. *Let $c \in \mathbb{R}$. Then there exists $M > 0$ such that for each bounded $(CPS)_c$ sequence (u^m) for $\widetilde{f}$ with $c \geq M$, the sequence $(\widetilde{g}(x, u^m))$ admits a convergent subsequence in $H^{-1}(\Omega, \mathbb{R}^N)$.*

Proof. Let (u^m) be a bounded $(CPS)_c$-sequence for $\widetilde{f}$ with $c \geq M$. We may assume that $(u^m) \subseteq \operatorname{supp}(\psi)$, otherwise $\psi(u^m) = 0$ and $\psi'(u^m) = 0$. Recall that

$$\widetilde{g}(x, u^m) = g(x, u^m) + \psi(u^m)\varphi + \psi'(u^m) \int_\Omega \varphi \cdot u^m \, dx.$$

Since by [36, Theorem 2.2.7] the maps

$$\begin{array}{ccc} H_0^1(\Omega,\mathbb{R}^N) & \longrightarrow & H^{-1}(\Omega,\mathbb{R}^N) \\ u & \longmapsto & g(x,u) \end{array}$$

and

$$\begin{array}{ccc} H_0^1(\Omega,\mathbb{R}^N) & \longrightarrow & H^{-1}(\Omega,\mathbb{R}^N) \\ u & \longmapsto & \psi(u)\varphi, \end{array}$$

are completely continuous, the sequences $(g(x,u^m))$ and $(\psi(u^m)\varphi)$ admit a convergent subsequence in $H^{-1}(\Omega,\mathbb{R}^N)$. Now, we have

$$\begin{aligned} \psi'(u^m) = &\Big[\chi'(\vartheta(u^m))\phi(u^m)^{-1}\Big] g(x,u^m)+ \\ &- \Big[4\sigma^2\chi'(\vartheta(u^m))\phi(u^m)^{-2}\vartheta(u^m)f(u^m)\Big] f'(u^m). \end{aligned}$$

On the other hand,

$$f'(u^m) = \tilde{f}'(u^m) + \Big[\int_\Omega \varphi\cdot u^m\,dx\Big]\psi'(u^m) + [\psi(u^m)-1]\varphi.$$

Therefore,

$$\begin{aligned} &\Big[1+\Big[4\sigma^2\chi'(\vartheta(u^m))\phi(u^m)^{-2}\vartheta(u^m)f(u^m)\int_\Omega \varphi\cdot u^m\,dx\Big]\Big]\psi'(u^m) \\ &= \Big[\chi'(\vartheta(u^m))\phi(u^m)^{-1}\Big] g(x,u^m) \\ &\quad - \Big[4\sigma^2\chi'(\vartheta(u^m))\phi(u^m)^{-2}\vartheta(u^m)f(u^m)\Big]\tilde{f}'(u^m) \\ &\quad - \Big[4\sigma^2\chi'(\vartheta(u^m))\phi(u^m)^{-2}\vartheta(u^m)f(u^m)(\psi(u^m)-1)\Big]\varphi. \end{aligned} \tag{4.15}$$

By assumption we have $\tilde{f}'(u^m)\to 0$ in $H^{-1}(\Omega,\mathbb{R}^N)$. Taking into account the definition of χ, ϕ and ϑ, all of the square brackets in equation (4.15) are bounded in $\mathbb{R}$ for some $M>0$ and we conclude that also $(\psi'(u^m))$ admits a convergent subsequence in $H^{-1}(\Omega,\mathbb{R}^N)$. The assertion is now proven. □

4.4. **Compactness of concrete Palais-Smale sequences.** The next result is the crucial property for Palais-Smale condition to hold.

Lemma 4.8. *Let (u^m) be a bounded sequence in $H_0^1(\Omega,\mathbb{R}^N)$ and set*

$$\begin{aligned} \langle w^m, v\rangle = &\int_\Omega \sum_{i,j=1}^n\sum_{h=1}^N a_{ij}^h(x,u^m)D_iu_h^mD_jv_h\,dx \\ &+\frac{1}{2}\int_\Omega \sum_{i,j=1}^n\sum_{h=1}^N D_sa_{ij}^h(x,u^m)\cdot vD_iu_h^mD_ju_h^m\,dx \end{aligned}$$

for all $v\in C_c^\infty(\Omega,\mathbb{R}^N)$. Then, if (w^m) is strongly convergent to some w in $H^{-1}(\Omega,\mathbb{R}^N)$, (u^m) admits a strongly convergent subsequence in $H_0^1(\Omega,\mathbb{R}^N)$.

For the proof of the above lemma, sew [134, Lemma 6].

Theorem 4.9. *There exists $M>0$ such that $\tilde{f}$ satisfies $(CPS)_c$-condition for $c\geq M$.*

Proof. Let (u^m) be a $(CPS)_c$ sequence for f with $c \geq M$, where $M > 0$ is as in Lemma 4.6. Therefore, (u^m) is bounded in $H_0^1(\Omega, \mathbb{R}^N)$ and from Lemma 4.7 we deduce that, up to subsequences, $(\tilde{g}(x, u^m))$ is strongly convergent in $H^{-1}(\Omega, \mathbb{R}^N)$. Therefore, the assertion follows from Lemma 4.8. □

4.5. **Existence of multiple solutions.** Let (λ_h, u_h) be the sequence of eigenvalues and eigenvectors for the problem

$$\Delta u = -\lambda u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

and set

$$V_k = \overline{\text{span}} \left\{ u_1, \dots, u_k \in H_0^1(\Omega, \mathbb{R}^N) \right\}.$$

We deduce that for all $s \in \mathbb{R}^N$

$$|s| \geq R \Rightarrow G(x, s) \geq \frac{G\left(x, R\frac{s}{|s|}\right)}{R^q}|s|^q \geq b_0(x)|s|^q,$$

where

$$b_0(x) = R^{-q} \inf\{G(x, s) : \ |s| = R\} > 0.$$

Then it follows that for each $k \in \mathbb{N}$ there exists $R_k > 0$ such that for all $u \in V_k$

$$\|u\|_{1,2} \geq R_k \Rightarrow \tilde{f}(u) \leq 0.$$

Definition 4.10. For each $k \in \mathbb{N}$ set

$$D_k = V_k \cap B(0, R_k),$$
$$\Gamma_k = \left\{ \gamma \in C(D_k, H_0^1) : \gamma \text{is odd and } \gamma_{|\partial B(0, R_k)} = Id \right\},$$
$$b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in D_k} \tilde{f}(\gamma(u)).$$

Lemma 4.11. *For each $k \in \mathbb{N}$, $\varrho \in]0, R_k[$ and $\gamma \in \Gamma_k$*

$$\gamma(D_k) \cap \partial B(0, \varrho) \cap V_{k-1}^{\perp} \neq \emptyset.$$

For the proof of the above lemma, wee, [119, Lemma 1.44].

Lemma 4.12. *There exist $\beta > 0$ and $k_0 \in \mathbb{N}$ such that*

$$\forall k \geq k_0 : \ \ b_k \geq \beta k^{\frac{(n+2)-(n-2)\sigma}{n(\sigma-1)}}.$$

Proof. Let $\gamma \in \Gamma_k$ and $\varrho \in]0, R_k[$. By previous Lemma there exists

$$w \in \gamma(D_k) \cap \partial B(0, \varrho) \cap V_{k-1}^{\perp},$$

and therefore

$$\max_{u \in D_k} \tilde{f}(\gamma(u)) \geq \tilde{f}(w) \geq \inf_{u \in \partial B(0,\varrho) \cap V_{k-1}^{\perp}} \tilde{f}(u). \tag{4.16}$$

Given $u \in \partial B(0, \varrho) \cap V_{k-1}^{\perp}$, by (4.9) we find $\alpha_1, \alpha_2, \alpha_3 >$ with

$$\tilde{f}(u) \geq \frac{1}{2}\varrho^2 - \alpha_1 \|u\|_{\sigma+1}^{\sigma+1} - \alpha_2 \|\varphi\|_2 \|u\|_2 - \alpha_3.$$

Now, By Gagliardo-Nirenberg inequality, there is $\alpha_4 > 0$ such that

$$\|u\|_{\sigma+1} \leq \alpha_4 \|u\|_{1,2}^{\vartheta} \|u\|_2^{1-\vartheta},$$

where $\vartheta = \frac{n(\sigma-1)}{2(\sigma+1)}$. As is well known, it is

$$\|u\|_2 \leq \frac{1}{\sqrt{\lambda_{k-1}}}\|u\|_{1,2},$$

so that we obtain

$$\tilde{f}(u) \geq \frac{1}{2}\varrho^2 - \alpha_1 \lambda_k^{-\frac{(1-\vartheta)(\sigma+1)}{2}} \varrho^{\sigma+1} - \alpha_2 \|\varphi\|_2 \lambda_k^{-\frac{1}{2}} \varrho - \alpha_3.$$

Choosing now

$$\varrho = c\lambda_k^{-\frac{(1-\vartheta)}{2}\frac{(\sigma+1)}{(\sigma-1)}},$$

yields

$$\tilde{f}(u) \geq \frac{1}{4}\varrho_k^2 - \alpha_2 \|\varphi\|_2 \lambda_k^{-\frac{1}{2}} \varrho_k - \alpha_3.$$

Now, as is shown in [52], there exists $\alpha_5 > 0$ such that for large k, $\lambda_k \geq \alpha_5 k^{\frac{2}{n}}$. Therefore. we find $\beta > 0$ with

$$\tilde{f}(u) \geq \beta k^{\frac{(n+2)-(n-2)\sigma}{n(\sigma-1)}},$$

and by (4.16) the Lemma is proved. □

Definition 4.13. For each $k \in \mathbb{N}$ set

$$U_k = \Big\{\xi = tu_{k+1} + w : \ t \in [0, R_{k+1}], \ w \in B(0, R_{k+1}) \cap V_k, \ \|\xi\|_{1,2} \leq R_{k+1}\Big\},$$

$$\Lambda_k = \Big\{\lambda \in C(U_k, H_0^1) : \lambda_{|D_k} \in \Gamma_{k+1} \text{ and}$$

$$\lambda|_{\partial B(0,R_{k+1}) \cup ((B(0,R_{k+1}) \setminus B(0,R_k)) \cap V_k)} = Id\Big\}$$

$$c_k = \inf_{\lambda \in \Lambda_k} \max_{u \in U_k} \tilde{f}(\lambda(u)).$$

We now come to the our main existence tool. Of course, differently from the proof of [119, Lemma 1.57], in this non-smooth framework, we shall apply [36, Theorem 1.1.13] instead of the classical Deformation Lemma [119, Lemma 1.60].

Lemma 4.14. *Assume that $c_k > b_k \geq M$, where M is as in Theorem 4.9. If $\delta \in]0, c_k - b_k[$ and*

$$\Lambda_k(\delta) = \Big\{\lambda \in \Lambda_k : \ \tilde{f}(\lambda(u)) \leq b_k + \delta \ \text{ for } \ u \in D_k\Big\},$$

set

$$c_k(\delta) = \inf_{\lambda \in \Lambda_k(\delta)} \max_{u \in U_k} \tilde{f}(\lambda(u)).$$

Then $c_k(\delta)$ is a critical value for $\tilde{f}$.

Proof. Let $\bar{\varepsilon} = \frac{1}{2}(c_k - b_k - \delta) > 0$ and assume by contradiction that $c_k(\delta)$ is not a critical value for $\tilde{f}$. Therefore, taking into account Lemma 4.9, by [36, Theorem 1.1.13], there exists $\varepsilon > 0$ and a continuous map

$$\eta : H_0^1(\Omega, \mathbb{R}^N) \times [0,1] \to H_0^1(\Omega, \mathbb{R}^N)$$

such that for each $u \in H_0^1(\Omega, \mathbb{R}^N)$ and $t \in [0,1]$,

$$\tilde{f}(u) \notin]c_k(\delta) - \bar{\varepsilon}, c_k(\delta) + \bar{\varepsilon}[\ \Rightarrow \ \eta(u,t) = u, \tag{4.17}$$

$$\eta(\tilde{f}^{c_k(\delta)+\varepsilon}, 1) \subseteq \tilde{f}^{c_k(\delta)-\varepsilon}. \tag{4.18}$$

Choose $\lambda \in \Lambda_k(\delta)$ so that

$$\max_{u\in U_k} \widetilde{f}(\lambda(u)) \le c_k(\delta) + \varepsilon \tag{4.19}$$

and consider $\eta(\lambda(\cdot), 1) : U_k \to H_0^1(\Omega, \mathbb{R}^N)$.

Observe that if $u \in \partial B(0, R_{k+1})$ or $u \in (B(0, R_{k+1})\setminus B(0, R_k)) \cap V_k$, by definition $\widetilde{f}(\lambda(u)) = \widetilde{f}(u)$. Hence, by (4.17), it is $\eta(\lambda(u), 1) = u$. We conclude that $\eta(\lambda(\cdot), 1) \in \Lambda_k$. Moreover, by our choice of $\overline{\varepsilon} > 0$ and $\delta > 0$ we obtain

$$\forall u \in D_k : \quad \widetilde{f}(\lambda(u)) \le b_k + \delta \le c_k - \overline{\varepsilon} \le c_k(\delta) - \overline{\varepsilon}.$$

Therefore (4.17) implies that $\eta(\lambda(\cdot), 1) \in \Lambda_k(\delta)$. On the other hand, again by (4.18) and (4.19)

$$\max_{u\in U_k} \widetilde{f}(\eta(\lambda(u), 1)) \le c_k(\delta) - \varepsilon, \tag{4.20}$$

which is not possible, by definition of $c_k(\delta)$. □

It only remains to prove that we cannot have $c_k = b_k$ for k sufficiently large.

Lemma 4.15. *Assume that $c_k = b_k$ for all $k \ge k_1$. Then, there exist $\gamma > 0$ and $\widetilde{k} \ge k_1$ with*

$$b_{\widetilde{k}} \le \gamma \widetilde{k}^{\frac{q}{q-1}}.$$

Proof. Choose $k \ge k_1$, $\varepsilon > 0$ and a $\lambda \in \Lambda_k$ such that

$$\max_{u\in U_k} \widetilde{f}(\lambda(u)) \le b_k + \varepsilon.$$

Define now $\widetilde{\lambda} : D_{k+1} \to H_0^1$ such that

$$\widetilde{\lambda}(u) = \begin{cases} \lambda(u) & \text{if } u \in U_k \\ -\lambda(-u) & \text{if } u \in -U_k. \end{cases}$$

Since $\widetilde{\lambda}_{|B(0,R_{k+1})\cap V_k}$ is continuous and odd, it follows $\widetilde{\lambda} \in \Gamma_{k+1}$. Then

$$b_{k+1} \le \max_{u\in D_{k+1}} \widetilde{f}(\widetilde{\lambda}(u)).$$

By Lemma 4.3 we have

$$\max_{u\in -U_k} \widetilde{f}(\widetilde{\lambda}(u)) \le b_k + \varepsilon + \beta\left(|b_k + \varepsilon|^{1/q} + 1\right),$$

and since $D_{k+1} = U_k \cup (-U_k)$, we get

$$\forall \varepsilon > 0 : \quad b_{k+1} \le b_k + \varepsilon + \beta\left(|b_k + \varepsilon|^{1/q} + 1\right),$$

that yields

$$\forall k \ge k_1 : \quad b_{k+1} \le b_k + \beta\left(|b_k|^{1/q} + 1\right).$$

The assertion now follows recursively as in [121, Proposition 10.46]. □

We finally come to the proof of the main result, which extends the theorems of [15, 66, 119, 138] to the quasi-linear case, both scalar and vectorial.

Proof of Theorem 4.1. Observe that the inequality

$$1 < \sigma < \frac{qn + (q-1)(n+2)}{qn + (q-1)(n-2)},$$

implies

$$\frac{q}{q-1} < \frac{(n+2) - \sigma(n-2)}{n(\sigma - 1)}.$$

Therefore, combining Lemma 4.12 and Lemma 4.15 we deduce $c_k > b_k$ so that we may apply Lemma 4.14 and obtain that $(c_k(\delta))$ is a sequence of critical values for $\widetilde{f}$. By Theorem 4.4 we finally conclude that f has a diverging sequence of critical values. □

4.6. **Semi-linear systems with nonhomogeneous data.** Since the early seventies, many authors have widely investigated existence and multiplicity of solutions for semi-linear elliptic problems with Dirichlet boundary conditions, especially by means of variational methods (see [137] and references therein). In particular, if φ is a real L^2 function on a bounded domain $\Omega \subset \mathbb{R}^n$, $p > 2$ and $p < 2^*$ if $n \geq 3$ (here, $2^* = \frac{2n}{n-2}$), the following model problem $(\mathcal{P}_{0,\varphi,1})$

$$\begin{gathered} -\Delta u = |u|^{p-2}u + \varphi \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.21}$$

has been extensively studied, even when the nonlinear term is more general.

If $\varphi \equiv 0$, the problem is symmetric, so multiplicity results have been achieved via the equivariant Lusternik-Schnirelman theory and the notion of genus for $\mathbb{Z}_2$-symmetric sets (see [121] and references therein).

On the contrary, if $\varphi \not\equiv 0$, the problem loses its $\mathbb{Z}_2$-symmetry and a natural question is whether the infinite number of solutions persists under perturbation of the odd equation. In this case, a detailed analysis was carried on by Rabinowitz in [119], Struwe in [138], Bahri and Berestycki in [15], Dong and Li in [66] and Tanaka in [140]: the existence of infinitely many solutions was obtained via techniques of classical critical point theory provided that a suitable restriction on the growth of the exponent p is assumed.

Furthermore, Bahri and Lions have improved some of such results via a technique based on Morse theory (see [17, 18]); while, more recently, Paleari and Squassina have extended some of the above mentioned achievements to the quasi-linear case by means of techniques of non-smooth critical point theory (see [111]).

Other perturbation results were obtained by Bahri and Berestycki in [15] and by Ambrosetti in [2] when $p > 2$ is any but subcritical: in particular, they proved that for each $\nu \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $(\mathcal{P}_{0,\varphi,1})$ has at least ν distinct solutions provided that $\|\varphi\|_2 < \varepsilon$.

The success in looking for solutions of a non-symmetric problem as $(\mathcal{P}_{0,\varphi,1})$ made quite interesting to study the problem $(\mathcal{P}_{\chi,\varphi,1})$

$$\begin{gathered} -\Delta u = |u|^{p-2}u + \varphi \quad \text{in } \Omega \\ u = \chi \quad \text{on } \partial\Omega \end{gathered} \tag{4.22}$$

where, in general, the boundary condition χ is different from zero. Some multiplicity results for (4.22) have been proved in [29] provided that

$$2 < p < 2\,\frac{n+1}{n}, \qquad \chi \in C(\partial\Omega, \mathbb{R}) \cap H^{1/2}(\partial\Omega, \mathbb{R}), \qquad \varphi \in L^2(\Omega, \mathbb{R}).$$

The upper bound to p seems to be a natural extension of the assumption $2 < p < 4$ considered by Ekeland, Ghoussoub and Tehrani in [67] in order to solve such a problem when $n = 1$ (in this case, the range $p < 2$ was covered by Clarke and Ekeland in a previous paper [47]).

We stress that an improvement of the results in [29, 67] has been reached with a different technique by Bolle in [24] and Bolle, Ghoussoub and Tehrani in [25]. From one hand, they prove that if $\Omega \subset \mathbb{R}^n$ is a C^2 bounded domain and

$$2 < p < \frac{2n}{n-1}, \quad \chi \in C^2(\partial\Omega, \mathbb{R}), \quad \varphi \in C(\overline{\Omega}, \mathbb{R}),$$

then $(\mathcal{P}_{\chi,\varphi,1})$ has infinitely many classical solutions. On the other hand, they show that in the case $n = 1$ it suffices to assume $p > 2$, namely the result becomes optimal.

It remains open, even for $\chi \equiv 0$, the problem of whether $(\mathcal{P}_{\chi,\varphi,1})$ has an infinite number of solutions for p all the way up to 2^*. For $\chi \equiv 0$, the most satisfactory result remains the one contained in the celebrated paper [18] of Bahri and Lions where they prove that this fact is true for a subset of φ dense in $L^2(\Omega, \mathbb{R})$.

Let us fix $N \geq 1$. The purpose of this section is to show the multiplicity of solutions for the following semi-linear elliptic system $(\mathcal{P}_{\chi,\varphi,N})$

$$\begin{gathered} -\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_j(a_{ij}^{hk}(x) D_i u_h) = b(x)|u|^{p-2}u_k + \varphi_k(x) \quad \text{in } \Omega \\ u = \chi \quad \text{on } \partial\Omega \\ k = 1, \dots, N \end{gathered} \tag{4.23}$$

taken any $\chi \in H^{1/2}(\partial\Omega, \mathbb{R}^N)$. Clearly, (4.23) reduces to the problem (4.22) if $N = 1$, $a_{ij}^{hk} = \delta_{ij}^{hk}$ and $b(x) \equiv 1$.

To the best of our knowledge no other result can be found in the literature about multiplicity for systems of semi-linear elliptic equations with non-homogeneous boundary conditions; on the contrary, some multiplicity results are known in the case of Dirichlet boundary conditions (see [46] for the semi-linear case and [111, 134] for some extensions to the quasi-linear case).

It is well known that the functional $f : \mathcal{M}_\chi \to \mathbb{R}$ associated with (4.23) is given by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p}\int_\Omega b(x)|u|^p \, dx - \int_\Omega \varphi \cdot u \, dx$$

where $\mathcal{M}_\chi = \{u \in H^1(\Omega, \mathbb{R}^N) : u = \chi \quad \text{a.e. on } \partial\Omega\}$.

In the next, Ω will denote a Lipschitz bounded domain of $\mathbb{R}^n$ with $n \geq 3$ while we shall always assume that the coefficients a_{ij}^{hk} and b belong to $C(\overline{\Omega}, \mathbb{R})$ with $a_{ij}^{hk} = a_{ji}^{kh}$ and $b > 0$. Moreover, there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{n}\sum_{h,k=1}^{N} a_{ij}^{hk}(x)\xi_i\xi_j\eta^h\eta^k \geq \nu|\xi|^2|\eta|^2 \tag{4.24}$$

for all $x \in \Omega$ and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^N$ (Legendre-Hadamard condition).

Here, we state our main results.

Theorem 4.16. *Let $p \in]2, 2\frac{n+1}{n}[$. Then for each φ in $L^2(\Omega, \mathbb{R}^N)$ and χ in the space $H^{1/2}(\partial\Omega, \mathbb{R}^N)$ the system (4.23) admits a sequence $(u^m)_m$ of solutions in $\mathcal{M}_\chi$ such that $f(u^m) \to +\infty$.*

To prove Theorem 4.16, we use some perturbation arguments developed in [15, 119, 138]; so the condition $p < 2\,\frac{n+1}{n}$ is quite natural.

An improvement of such a "control" can be obtained by means of the Bolle's techniques, but more assumptions need. In fact, all the weak solutions must be regular and the system has to be diagonal, i.e. $a_{ij}^{hk} = \delta_{ij}^{hk}$.

More precisely, we can prove the following theorem.

Theorem 4.17. *Let $p \in]2, \frac{2n}{n-1}[$, $\partial\Omega$ is of class C^2, χ in $C^2(\partial\Omega, \mathbb{R}^N)$, φ in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ for some $\alpha \in]0,1[$ and $a_{ij}^{hk} = \delta_{ij}^{hk}$. Then (4.23) has a sequence $(u^m)_m$ of classical solutions such that $f(u^m) \to +\infty$.*

Clearly, Theorems 4.16 and 4.17 extend the results of [29] and [25] to semilinear elliptic systems. We underline that (4.24) is weaker than the strong ellipticity condition.

Let us point out that, in general, whereas De Giorgi's famous example of an unbounded weak solution of a linear elliptic system shows (cf. [57]), we can not hope to find everywhere regular solutions for coefficients $a_{ij}^{hk} \in L^\infty(\Omega, \mathbb{R})$. Anyway, if $a_{ij}^{hk} \in C(\overline{\Omega}, \mathbb{R})$ and (4.24) holds we have that if u solves $(\mathcal{P}_{\chi,\varphi,N})$ then

$$u \in C^{0,\alpha}(\Omega, \mathbb{R}^N)$$

for each $\alpha \in]0,1[$ (see [76]); but if we look for classical solutions, namely u of class C^2 on $\overline{\Omega}$, the coefficients a_{ij}^{hk} have to be sufficiently smooth while $\varphi \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ for some $\alpha \in]0,1[$ and $\chi \in C^2(\partial\Omega, \mathbb{R}^N)$ (see [91] and references therein).

4.7. **Reduction to homogeneous boundary conditions.** As a first step, let us reduce (4.23) to a Dirichlet type problem. To this aim, let us denote by $\phi \in \mathcal{M}_\chi$ the only solution of the linear system

$$\begin{gathered} -\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_j(a_{ij}^{hk}(x) D_i \phi_h) = 0 \quad \text{in } \Omega \\ \phi = \chi \quad \text{on } \partial\Omega \\ k = 1, \dots, N \end{gathered} \tag{4.25}$$

Since $p < 2^*$, it results $\phi \in L^p(\Omega, \mathbb{R}^N)$.

From now on, we shall assume that $b \equiv 1$. Taking into account that there exist two positive constants m_b and M_b such that

$$m_b \le b(x) \le M_b \quad \text{for all } x \in \overline{\Omega},$$

the general case can be covered by slight modifies of some lemmas proved in the next sections.

It is easy to show that the following fact holds.

Proposition 4.18. *$u \in \mathcal{M}_\chi$ is a solution of $(\mathcal{P}_{\chi,\varphi,N})$ if and only if $z \in H_0^1(\Omega, \mathbb{R}^N)$ solves*

$$\begin{gathered} -\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_j(a_{ij}^{hk}(x) D_i z_h) = |z+\phi|^{p-2}(z_k+\phi_k) + \varphi_k(x) \quad \text{in } \Omega \\ z = 0 \quad \text{on } \partial\Omega \\ k = 1, \dots, N, \end{gathered}$$

where $u(x) = z(x) + \phi(x)$ for a.e. $x \in \overline{\Omega}$.

Therefore, in order to find solutions of our problem it is enough looking for critical points of the C^1-functional $f_\chi : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ given by

$$f_\chi(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p}\int_\Omega |u+\phi|^p \, dx - \int_\Omega \varphi \cdot u \, dx$$

(we refer the reader to [121, 137] for some recalls of classical critical point theory).

Lemma 4.19. *There exists $A > 0$ such that if $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a critical point of f_χ, then*

$$\int_\Omega |u+\phi|^p \, dx \le pA\left(f_\chi^2(u) + 1\right)^{1/2}.$$

Proof. By Young's inequality, for each $\varepsilon > 0$ there exist $\alpha_\varepsilon, \beta_\varepsilon > 0$ such that

$$|u+\phi|^{p-1}|\phi| \le \varepsilon|u+\phi|^p + \alpha_\varepsilon|\phi|^p, \quad |u+\phi||\varphi| \le \varepsilon|u+\phi|^p + \beta_\varepsilon|\varphi|^{p'}, \tag{4.26}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, if u is a critical point of f_χ, we get

$$\begin{aligned}
&f_\chi(u)\\
&= f_\chi(u) - \frac{1}{2} f_\chi'(u)[u]\\
&= \big(\frac{1}{2} - \frac{1}{p}\big)\int_\Omega |u+\phi|^p \, dx - \frac{1}{2}\int_\Omega |u+\phi|^{p-2}(u+\phi)\cdot\phi \, dx - \frac{1}{2}\int_\Omega \varphi\cdot u \, dx\\
&\ge \frac{p-2}{2p}\int_\Omega |u+\phi|^p \, dx - \frac{1}{2}\int_\Omega |u+\phi|^{p-1}|\phi| \, dx - \frac{1}{2}\int_\Omega (|u+\phi||\varphi| + |\varphi||\phi|) \, dx\\
&\ge \big(\frac{p-2}{2p} - \varepsilon\big)\int_\Omega |u+\phi|^p \, dx - \frac{1}{2}\left(\alpha_\varepsilon\|\phi\|_p^p + \beta_\varepsilon\|\varphi\|_{p'}^{p'} + \|\varphi\|_2\|\phi\|_2\right).
\end{aligned}$$

Choosing ε such that $p - 2 - 2p\varepsilon > 0$, i.e., $\varepsilon \in]0, \frac{1}{2} - \frac{1}{p}[$, we get

$$pM_\varepsilon f_\chi(u) \ge \int_\Omega |u+\phi|^p \, dx - pM_\varepsilon\gamma_\varepsilon(p,\phi,\varphi),$$

where $M_\varepsilon = \frac{2}{p-2-2p\varepsilon}$ and

$$\gamma_\varepsilon(p,\phi,\varphi) = \frac{1}{2}\left(\alpha_\varepsilon\|\phi\|_p^p + \beta_\varepsilon\|\varphi\|_{p'}^{p'} + \|\varphi\|_2\|\phi\|_2\right).$$

At this point, the assertion follows by $A \ge \sqrt{2}M_\varepsilon \max\{1, \gamma_\varepsilon(p,\phi,\varphi)\}$. □

Now, let $\eta \in C^\infty(\mathbb{R},\mathbb{R})$ be a cut function such that $\eta(s) = 1$ for $s \le 1$, $\eta(s) = 0$ for $s \ge 2$ while $-2 < \eta'(s) < 0$ when $1 < s < 2$. For each $u \in H_0^1(\Omega,\mathbb{R}^N)$ let us define

$$\zeta(u) = 2pA\left(f_\chi^2(u) + 1\right)^{1/2}, \quad \psi(u) = \eta\big(\zeta(u)^{-1}\int_\Omega |u+\phi|^p \, dx\big), \tag{4.27}$$

where A is as in Lemma 4.19. Finally, we introduce the modified functional $\widetilde{f}_\chi : H_0^1(\Omega,\mathbb{R}^N) \to \mathbb{R}$ in order to apply the techniques used in [29]:

$$\widetilde{f}_\chi(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p}\int_\Omega |u|^p \, dx - \psi(u)\int_\Omega \Theta(x,u) \, dx,$$

with

$$\Theta(x,u) = \frac{|u+\phi|^p}{p} - \frac{|u|^p}{p} + \varphi\cdot u.$$

Let us provide an estimate for the loss of symmetry of $\widetilde{f}_\chi$.

Lemma 4.20. *There exists $\beta > 0$ such that*

$$\left|\widetilde{f}_\chi(u) - \widetilde{f}_\chi(-u)\right| \le \beta\left(|\widetilde{f}_\chi(u)|^{\frac{p-1}{p}} + 1\right) \quad \text{for all } u \in \operatorname{supp}(\psi)$$

(here, $\operatorname{supp}(\psi)$ *is the support of* ψ*).*

Proof. First of all, let us show that there exist $c_1, c_2 > 0$ such that there results

$$\Big|\int_\Omega (|u+\phi|^p - |u|^p)\,dx\Big| \le c_1|f_\chi(u)|^{\frac{p-1}{p}} + c_2, \tag{4.28}$$

$$\Big|\int_\Omega (|u-\phi|^p - |u|^p)\,dx\Big| \le c_1|f_\chi(u)|^{\frac{p-1}{p}} + c_2, \tag{4.29}$$

$$\Big|\int_\Omega \varphi\cdot u\,dx\Big| \le c_1|f_\chi(u)|^{\frac{p-1}{p}} + c_2 \tag{4.30}$$

for all $u \in \operatorname{supp}(\psi)$. In fact, taken any $u \in H_0^1(\Omega,\mathbb{R})$ it is easy to see that

$$\big||u+\phi|^p - |u|^p\big| \le p2^{p-2}|u+\phi|^{p-1}|\phi| + p2^{p-2}|\phi|^p, \tag{4.31}$$

$$\big||u-\phi|^p - |u|^p\big| \le p2^{p-2}|u+\phi|^{p-1}|\phi| + p2^{2p-3}|\phi|^p. \tag{4.32}$$

Hence, by (4.31) we get

$$\Big|\int_\Omega (|u+\phi|^p - |u|^p)\,dx\Big| \le p2^{p-2}\|\phi\|_p\Big(\int_\Omega |u+\phi|^p\,dx\Big)^{\frac{p-1}{p}} + p2^{p-2}\|\phi\|_p^p,$$

while (4.32) implies

$$\Big|\int_\Omega (|u-\phi|^p - |u|^p)\,dx\Big| \le p2^{p-2}\|\phi\|_p\Big(\int_\Omega |u+\phi|^p\,dx\Big)^{\frac{p-1}{p}} + p2^{2p-3}\|\phi\|_p^p.$$

Moreover, by Hölder and Young's inequalities it results

$$\Big|\int_\Omega \varphi\cdot u\,dx\Big| \le \Big(\int_\Omega |u+\phi|^p\,dx\Big)^{\frac{p-1}{p}} + (p-2)\Big(\frac{\|\varphi\|_{p'}}{p-1}\Big)^{\frac{p-1}{p-2}} + \|\varphi\|_2\|\phi\|_2.$$

If, furthermore, we assume $u \in \operatorname{supp}(\psi)$, it follows

$$\int_\Omega |u+\phi|^p\,dx \le 4pA(|f_\chi(u)| + 1)$$

which implies (4.28), (4.29) and (4.30). Then, again by Young's inequality, simple calculations and (4.28), (4.30) give

$$|f_\chi(u)| \le a_1|\widetilde{f}_\chi(u)| + a_2, \tag{4.33}$$

for suitable $a_1, a_2 > 0$. The assertion follows by combining inequalities (4.28), (4.29), (4.30) and (4.33). $\square$

Now, we want to link the critical points of $\widetilde{f}_\chi$ to those ones of f_χ. To this aim we need more information about $\widetilde{f}'_\chi$. Taken $u \in H_0^1(\Omega, \mathbb{R}^N)$, by direct computations we get

$$\begin{aligned}\widetilde{f}'_\chi(u)[u] =&(1+T_1(u))\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx \\ &- (1-\psi(u))\int_\Omega |u|^p \, dx \; - (\psi(u)+T_1(u))\int_\Omega \varphi\cdot u \, dx \\ &- (\psi(u)+T_2(u))\int_\Omega |u+\phi|^{p-2}(u+\phi)\cdot u \, dx\end{aligned} \tag{4.34}$$

where $T_1, T_2 : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ are defined by setting

$$T_1(u) = 4p^2A^2\eta'(\delta(u))\delta(u)\zeta(u)^{-2} f_\chi(u)\int_\Omega \Theta(x,u)\, dx,$$

$$T_2(u) = p\eta'(\delta(u))\zeta(u)^{-1}\int_\Omega \Theta(x,u)\, dx + T_1(u)\,,$$

with $\delta(u) = \zeta(u)^{-1}\int_\Omega |u+\phi|^p \, dx$.

Remark 4.21. To point out some properties of the maps T_1 and T_2 defined above, let us remark that by (4.28) and (4.30) there exist $b_1, b_2 > 0$ such that for all $u \in \operatorname{supp}(\psi)$ it is

$$|T_i(u)| \le b_1|f_\chi(u)|^{-\frac{1}{p}} + b_2|f_\chi(u)|^{-1} \quad \text{for both } i = 1, 2.$$

Therefore, arguing as in [119] (see also [29, Lemma 2.9]), there exist $\alpha_0, M_0 > 0$ such that if $M \ge M_0$ then

$$\widetilde{f}_\chi(u) \ge M, \quad u \in \operatorname{supp}(\psi) \;\Rightarrow\; f_\chi(u) \ge \alpha_0 M;$$

whence, it results $|T_i(u)| \to 0$ as $M \to +\infty$ for $i = 1, 2$ (trivially, it is $T_1(u) = T_2(u) = 0$ if $u \notin \operatorname{supp}(\psi)$).

Theorem 4.22. *There exists $M_1 > 0$ such that if u is a critical point of $\widetilde{f}_\chi$ and $\widetilde{f}_\chi(u) \ge M_1$ then u is a critical point of f_χ and $f_\chi(u) = \widetilde{f}_\chi(u)$.*

Proof. Let $u \in H_0^1(\Omega, \mathbb{R}^N)$ be a critical point of $\widetilde{f}_\chi$. By the definition of ψ it suffices to show that, if $\widetilde{f}_\chi(u) \ge M_1$ for a large enough M_1, then $\delta(u) < 1$, i.e.,

$$\zeta(u)^{-1}\int_\Omega |u+\phi|^p \, dx < 1\,.$$

By (4.34) we have

$$\begin{aligned}f_\chi(u) &= f_\chi(u) - \frac{1}{2(1+T_1(u))}\,\widetilde{f}'_\chi(u)[u] \\ &= -\frac{1}{p}\int_\Omega |u+\phi|^p\, dx - \int_\Omega \varphi\cdot u\, dx + \frac{1-\psi(u)}{2(1+T_1(u))}\int_\Omega |u|^p\, dx \\ &\quad + \frac{\psi(u)+T_1(u)}{2(1+T_1(u))}\int_\Omega \varphi\cdot u\, dx + \frac{\psi(u)+T_2(u)}{2(1+T_1(u))}\int_\Omega |u+\phi|^{p-2}(u+\phi)\cdot u\, dx \\ &= \Big(\frac{1}{2}-\frac{1}{p}\Big)\int_\Omega |u+\phi|^p\, dx - \frac{T_1(u)-T_2(u)}{2(1+T_1(u))}\int_\Omega |u|^p\, dx \\ &\quad + \frac{1}{2}\left(\frac{\psi(u)+T_2(u)}{1+T_1(u)} - 1\right)\int_\Omega (|u+\phi|^p - |u|^p)\, dx\end{aligned}$$

$$- \frac{\psi(u)+T_2(u)}{2(1+T_1(u))}\int_\Omega |u+\phi|^{p-2}(u+\phi)\cdot\phi\,dx$$
$$-\Big(1-\frac{\psi(u)+T_1(u)}{2(1+T_1(u))}\Big)\int_\Omega \varphi\cdot u\,dx.$$

Then, by Remark 4.21 it is possible to choose $M_1 > 0$ so large that

$$\Big|\frac{1-\psi(u)}{1+T_1(u)}\Big| \le 2, \qquad \Big|\frac{\psi(u)+T_1(u)}{1+T_1(u)}\Big| \le 2,$$
$$\Big|\frac{\psi(u)+T_2(u)}{1+T_1(u)}-1\Big| \le 2, \qquad \Big|\frac{\psi(u)+T_2(u)}{1+T_1(u)}\Big| \le 2;$$

so we deduce that for each $\varepsilon > 0$ there exist $h_\varepsilon, \tilde\gamma_\varepsilon(p,\phi,\varphi) > 0$ such that

$$f_\chi(u) \ge \Big(\frac{p-2}{2p} - 2^{p-2}\Big|\frac{T_2(u)-T_1(u)}{1+T_1(u)}\Big| - h_\varepsilon\Big)\int_\Omega |u+\phi|^p\,dx - \tilde\gamma_\varepsilon(p,\phi,\varphi)$$

where $h_\varepsilon \to 0$ as $\varepsilon \to 0$. At this point, choosing a priori ε and M_1 in such a way that

$$2^{p-2}\Big|\frac{T_2(u)-T_1(u)}{1+T_1(u)}\Big| + h_\varepsilon \le \frac{p-2}{4p},$$

we obtain

$$f_\chi(u) \ge \frac{p-2}{4p}\int_\Omega |u+\phi|^p\,dx - \tilde\gamma_\varepsilon(p,\phi,\varphi),$$

which completes the proof if, as in Lemma 4.19, the constant A taken in the definition (4.27) is large enough. □

4.8. **The Palais-Smale condition.** Let us point out that, in the check of the Palais-Smale condition for semi-linear elliptic systems under the assumption (4.24), an important role is played by the so called Gårding's inequality.

Lemma 4.23. *Let $(u^m)_m$ be a bounded sequence in $H^1_0(\Omega,\mathbb{R}^N)$ and let $(w^m)_m$ be a strongly convergent sequence in $H^{-1}(\Omega,\mathbb{R}^N)$ such that*

$$\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h^m D_j v_k\,dx = \langle w^m, v\rangle \quad \text{for all } v \in H^1_0(\Omega,\mathbb{R}^N).$$

Then $(u^m)_m$ has a subsequence $(u^{m_k})_k$ strongly convergent in $H^1_0(\Omega,\mathbb{R}^N)$.

Proof. First of all, in our setting the following Gårding type inequality holds: taken ν as in (4.24) for each $\varepsilon \in]0,\nu[$ there exists $c_\varepsilon \ge 0$ such that

$$\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k\,dx \ge (\nu-\varepsilon)\|Du\|_2^2 - c_\varepsilon\|u\|_2^2$$

for all $u \in H^1_0(\Omega,\mathbb{R}^N)$ (see [106, Theorem 6.5.1]). Therefore, fixed $\varepsilon > 0$, we have

$$\begin{aligned}\langle w^l - w^m, u^l - u^m\rangle &= \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i(u_h^l - u_h^m) D_j(u_k^l - u_k^m)\,dx \\ &\ge (\nu-\varepsilon)\|Du^l - Du^m\|_2^2 - c_\varepsilon\|u^l - u^m\|_2^2\end{aligned}$$

for all $m, l \in \mathbb{N}$. Since $u^m \to u$ in $L^2(\Omega,\mathbb{R}^N)$, up to subsequences, we can conclude that $Du^m \to Du$ in $L^2(\Omega,\mathbb{R}^N)$. □

Now, let $d \geq 0$ be such that

$$\int_\Omega \Big(\sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k + d|u|^2 \Big)\, dx \geq \frac{\nu}{2}\, \|Du\|_2^2 \tag{4.35}$$

for all $u \in H_0^1(\Omega, \mathbb{R}^N)$.

Lemma 4.24. *There exists $M_2 > 0$ such that if $(u^m)_m$ is a $(PS)_c$-sequence of $\widetilde{f}_\chi$ with $c \geq M_2$, then $(u^m)_m$ is bounded in $H_0^1(\Omega, \mathbb{R}^N)$.*

Proof. Let $M_2 > 0$ be fixed and consider $(u^m)_m$, a $(PS)_c$-sequence of $\widetilde{f}_\chi$, with $c \geq M_2$, such that

$$M_2 \leq \widetilde{f}_\chi(u^m) \leq K\,,$$

for a certain $K > M_2$.

First of all, let us remark that if there exists a subsequence $(u^{m_k})_k$ such that $u^{m_k} \notin \operatorname{supp}(\psi)$ for all $k \in \mathbb{N}$, then it is a Palais-Smale sequence for the symmetric functional

$$f_0(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p} \int_\Omega |u|^p \, dx$$

in $H_0^1(\Omega, \mathbb{R}^N)$. Whence, it is easier to prove that such a subsequence is bounded. So, we can assume $u^m \in \operatorname{supp}(\psi)$ for all $m \in \mathbb{N}$. For $m \in \mathbb{N}$ large enough and any $\varrho > 0$, taken d as in (4.35) by (4.34) it results

$$\begin{aligned}
&K + \varrho \|Du^m\|_2 \\
&\geq \widetilde{f}_\chi(u^m) - \varrho \widetilde{f}_\chi'(u^m)[u^m] \\
&= \frac{1}{2}\, (1 - 2\varrho\, (1 + T_1(u^m))) \int_\Omega \Big(\sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h^m D_j u_k^m + d|u^m|^2 \Big) dx \\
&\quad - \frac{d}{2}\, (1 - 2\varrho\, (1 + T_1(u^m)))\, \|u^m\|_2^2 + \Big(\varrho(1 - \psi(u^m)) - \frac{1}{p} \Big) \int_\Omega |u^m|^p \, dx \\
&\quad + \varrho\, (\psi(u^m) + T_2(u^m)) \int_\Omega |u^m + \phi|^{p-2} (u^m + \phi) \cdot u^m \, dx \\
&\quad + \varrho\, (\psi(u^m) + T_1(u^m)) \int_\Omega \varphi \cdot u^m \, dx - \psi(u^m) \int_\Omega \Theta(x, u^m) \, dx.
\end{aligned}$$

Since it is $p > 2$, we can fix, a priori, a constant $\hbar \in]1, \frac{p}{2}[$ such that, taken $\mu \in]0, 1 - 2\frac{\hbar}{p}[$, $\varrho \in]\frac{\hbar}{p}, \frac{1-\mu}{2}[$ and $\bar{\mu} \in]0, \varrho(1 - \frac{1}{\hbar})[$, by Remark 4.21 if M_2 is large enough for all $m \in \mathbb{N}$ we have

$$|T_1(u^m)| < \min\Big\{1, \frac{1-\mu}{2\varrho} - 1\Big\}, \quad |T_2(u^m)| < 1 - \frac{1}{\hbar} - \frac{\bar{\mu}}{\varrho}$$

and then

$$\mu < 1 - 2\varrho(1 + T_1(u^m)) \leq 1, \tag{4.36}$$

$$\bar{\mu} \leq \varrho(1 + T_2(u^m)) - \frac{1}{p}. \tag{4.37}$$

So, by (4.35) and (4.36) we obtain

$$K + \varrho \|Du^m\|_2$$

$$\geq \frac{\nu\mu}{4}\|Du^m\|_2^2 - \frac{d}{2}\|u^m\|_2^2 + \left(\varrho(1-T_2(u^m)) - \frac{1}{p}\right)\int_\Omega |u^m|^p\,dx$$
$$- (\varrho\,(1+|T_1(u^m)|)+1)\int_\Omega |\varphi||u^m|\,dx - \varrho\,(1+|T_2(u^m)|)\int_\Omega |u^m+\phi|^{p-1}|\phi|\,dx$$
$$+\left(\varrho\,(\psi(u^m)+T_2(u^m)) - \frac{\psi(u^m)}{p}\right)\int_\Omega (|u^m+\phi|^p - |u^m|^p)\,dx\,.$$

Hence, fixed any $\varepsilon > 0$, by (4.26), (4.37) and a suitable choice of the positive constants a_1 and a_2^ε there results

$$\begin{aligned}
&K + \varrho\,\|Du^m\|_2 + \frac{d}{2}\|u^m\|_2^2 \\
&\geq \frac{\nu\mu}{4}\|Du^m\|_2^2 + (\bar{\mu} - \varepsilon a_1)\,\|u^m\|_p^p \\
&\quad + \left(\varrho\,(\psi(u^m)+T_2(u^m)) - \frac{\psi(u^m)}{p}\right)\int_\Omega (|u^m+\phi|^p - |u^m|^p)\,dx\ - a_2^\varepsilon.
\end{aligned}$$

Let us point out that, as $u^m \in \mathrm{supp}(\psi)$, (4.28) and (4.33) imply

$$\left(\int_\Omega (|u^m+\phi|^p - |u^m|^p)\,dx\right)_{m\in\mathbb{N}}$$

is bounded. Whence, $p > 2$ and a suitable choice of ε small enough allow to complete the proof. □

Lemma 4.25. *Let M_2 be as in Lemma 4.24 and $c \geq M_2$. Then, taken any $(PS)_c$-sequence $(u^m)_m$ for $\widetilde{f}_\chi$, the sequence*

$$\widehat{g}(x,u^m) = |u^m|^{p-2}u^m + \psi(u^m)\Theta'(x,u^m) + \psi'(u^m)\int_\Omega \Theta(x,u^m)\,dx$$

admits a convergent subsequence in $H^{-1}(\Omega,\mathbb{R}^N)$.

The proof of the above lemma follows the steps in [111, Lemma 3.3].

Theorem 4.26. *The functional $\widetilde{f}_\chi$ satisfies the Palais-Smale condition at each level $c \in \mathbb{R}$ with $c \geq M_2$, where M_2 is as in Lemma 4.24.*

Proof. Let $(u^m)_m$ be a Palais-Smale sequence for $\widetilde{f}_\chi$ at level $c \geq M_2$. Therefore, $(u^m)_m$ is bounded in $H_0^1(\Omega,\mathbb{R}^N)$ and by Lemma 4.25, up to a subsequence, $(\widehat{g}(x,u^m))_m$ is strongly convergent in $H^{-1}(\Omega,\mathbb{R}^N)$. Hence, the assertion follows by Lemma 4.23 applied to $w^m = \widehat{g}(x,u^m) + \widetilde{f}_\chi'(u^m)$ where, by assumption, $\widetilde{f}_\chi'(u^m) \to 0$ in $H^{-1}(\Omega,\mathbb{R}^N)$. □

4.9. **Comparison of growths for min-max values.** In this section we shall build two min-max classes for $\widetilde{f}_\chi$ and then we compare the growth of the associated min-max values.

Let $(\lambda^l, u^l)_l$ be a sequence in $\mathbb{R}\times H_0^1(\Omega,\mathbb{R}^N)$ such that

$$\begin{gathered}
-\Delta u_k^l = \lambda^l u_k^l \quad \text{in } \Omega \\
u^l = 0 \quad \text{on } \partial\Omega, \\
k = 1,\dots N,
\end{gathered}$$

with $(u^l)_l$ orthonormalized. Let us consider the finite dimensional subspaces

$$V_0 := \left\langle u^0 \right\rangle; \quad V_{l+1} := V_l \oplus \mathbb{R}u^{l+1} \quad \text{for any } l \in \mathbb{N}.$$

Fixed $l \in \mathbb{N}$ it is easy to check that some constants $\beta_1, \beta_2, \beta_3, \beta_4 > 0$ exist such that

$$\widetilde{f}_\chi(u) \leq \beta_1 \|u\|_{1,2}^2 - \beta_2 \|u\|_{1,2}^p - \beta_3 \|u\|_{1,2} - \beta_4 , \quad \text{for all } u \in V_l.$$

Then, there exists $R_l > 0$ such that

$$u \in V_l, \quad \|u\|_{1,2} \geq R_l \quad \Rightarrow \quad \widetilde{f}_\chi(u) \leq \widetilde{f}_\chi(0) \leq 0.$$

Definition 4.27. For any $l \geq 1$ we set $D_l = V_l \cap B(0, R_l)$,

$$\Gamma_l = \big\{\gamma \in C(D_l, H_0^1(\Omega, \mathbb{R}^N)) : \gamma \text{ odd and } \gamma_{|\partial B(0,R_l)} = Id\big\},$$

and

$$b_l = \inf_{\gamma \in \Gamma_l} \max_{u \in D_l} \widetilde{f}_\chi(\gamma(u)).$$

To prove some estimates on the growth of the levels b_l, a result due to Tanaka (cf. [140]) implies the following lemma.

Lemma 4.28. *There exist $\beta > 0$ and $l_0 \in \mathbb{N}$ such that*

$$b_l \geq \beta\, l^{\frac{2p}{n(p-2)}} \quad \textit{for all } l \geq l_0.$$

Proof. By (4.35) and simple calculations $a_1, a_2 > 0$ exist such that

$$\widetilde{f}_\chi(u) \geq \frac{\nu}{4} \|Du\|_2^2 - a_1 \|u\|_p^p - a_2 \quad \text{for all } u \in \partial B(0, R_l) \cap V_{l-1}^\perp.$$

Then, it is enough to follow the proof of [140, Theorem 1]. □

Now, let us introduce a second class of min-max values to be compared with b_l.

Definition 4.29. Taken $l \in \mathbb{N}$, define

$$U_l = \{\xi = tu^{l+1} + w : 0 \leq t \leq R_{l+1}, \;\; w \in B(0, R_{l+1}) \cap V_l, \;\; \|\xi\|_{1,2} \leq R_{l+1}\}$$

and

$$\Lambda_l = \big\{\lambda \in C(U_l, H_0^1(\Omega, \mathbb{R}^N)) : \lambda_{|D_l} \in \Gamma_l \text{ and } \lambda|_{\partial B(0,R_{l+1}) \cup ((B(0,R_{l+1}) \setminus B(0,R_l)) \cap V_l)} = Id\big\}.$$

Assume

$$c_l = \inf_{\lambda \in \Lambda_l} \max_{u \in U_l} \widetilde{f}_\chi(\lambda(u)).$$

The following result is the concrete version of Theorem 2.11.

Lemma 4.30. *Assume $c_l > b_l \geq \max\{M_1, M_2\}$. Taken $\delta \in]0, c_l - b_l[$, let us set*

$$\Lambda_l(\delta) = \big\{\lambda \in \Lambda_l : \widetilde{f}_\chi(\lambda(u)) \leq b_l + \delta \textit{ for all } u \in D_l\big\},$$

$$c_l(\delta) = \inf_{\lambda \in \Lambda_l(\delta)} \max_{u \in U_l} \widetilde{f}_\chi(\lambda(u)).$$

Then, $c_l(\delta)$ is a critical value for $\widetilde{f}_\chi$.

The proof of the above lemma can be obtained by arguing as in [119, Lemma 1.57]. Now, we prove that the situation $c_l = b_l$ can not occur for all large l.

Lemma 4.31. *Assume that $c_l = b_l$ for all $l \geq l_1$. Then there exists $\gamma > 0$ with $b_l \leq \gamma\, l^p$.*

Proof. Working as in [119, Lemma 1.64] it is possible to prove that

$$b_{l+1} \leq b_l + \beta\big(|b_l|^{\frac{p-1}{p}} + 1\big) \quad \text{for all } l \geq l_1.$$

The assertion follows by [15, Lemma 5.3]. □

Proof of Theorem 4.16. Observe that the inequality $2 < p < 2\frac{n+1}{n}$ implies

$$p < \frac{2p}{n(p-2)}.$$

Therefore, by Lemmas 4.28 and 4.31 it follows that there exists a diverging sequence $(l_n)_n \subset \mathbb{N}$ such that $c_{l_n} > b_{l_n}$ for all $n \in \mathbb{N}$, then Lemma 4.30 implies that $(c_{l_n}(\delta))_n$ is a sequence of critical values for $\widetilde{f}_\chi$. Whence, by Theorem 4.22 the functional f_χ has a diverging sequence of critical values. □

Remark 4.32. When p goes all the way up to 2^*, in a similar fashion, one can prove that for each $\nu \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $(\mathcal{P}_{\varepsilon\chi,\varepsilon\varphi,N})$ has at least ν distinct solutions in $\mathcal{M}_{\varepsilon\chi}$. This is possible since there exists $\beta > 0$ such that

$$\left|\widetilde{f}^\varepsilon_\chi(u) - \widetilde{f}^\varepsilon_\chi(-u)\right| \leq \varepsilon\beta\left(|\widetilde{f}^\varepsilon_\chi(u)|^{\frac{p-1}{p}} + 1\right),$$

for each $\varepsilon > 0$ and $u \in \operatorname{supp}(\psi)$, where $\widetilde{f}^\varepsilon_\chi : H^1_0(\Omega, \mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\begin{aligned}\widetilde{f}^\varepsilon_\chi(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a^{hk}_{ij}(x) D_i u_h D_j u_k \, dx + \\ -\frac{1}{p}\int_\Omega |u|^p \, dx - \psi_\varepsilon(u)\int_\Omega \Theta_\varepsilon(x,u)\, dx\end{aligned}$$

with

$$\Theta_\varepsilon(x,u) = \frac{|u+\varepsilon\phi|^p}{p} - \frac{|u|^p}{p} + \varepsilon\varphi\cdot u, \quad \psi_\varepsilon(u) = \eta\Big(\zeta(u)^{-1}\int_\Omega |u+\varepsilon\phi|^p\, dx\Big);$$

for more details in the scalar case, see [2, 15].

4.10. **Bolle's method for non-symmetric problems.** In this section we briefly recall from [24] the theory devised by Bolle for dealing with problems with broken symmetry.

The idea is to consider a continuous path of functionals starting from the symmetric functional f_0 and to prove a preservation result for min-max critical levels in order to get critical points also for the end-point functional f_1.

Let $\mathcal{X}$ be a Hilbert space equipped with the norm $\|\cdot\|$ and $f : [0,1] \times \mathcal{X} \to \mathbb{R}$ a C^2-functional. Set $f_\theta = f(\theta, \cdot)$ if $\theta \in [0,1]$.

Assume that $\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_+$ and let $(e_l)_{l\geq 1}$ be an orthonormal base of $\mathcal{X}_+$ such that we can define an increasing sequence of subspaces as follows:

$$\mathcal{X}_0 := \mathcal{X}_-, \quad \mathcal{X}_{l+1} := \mathcal{X}_l \oplus \mathbb{R}e_{l+1} \text{ if } l \in \mathbb{N}.$$

Provided that $\dim(\mathcal{X}_-) < +\infty$, let us set

$$\mathcal{K} = \{\zeta \in C(\mathcal{X}, \mathcal{X}) : \zeta \text{ is odd and } \zeta(u) = u \text{ if } \|u\| \geq R\}$$

for a fixed $R > 0$ and

$$c_l = \inf_{\zeta\in\mathcal{K}} \sup_{u\in\mathcal{X}_l} f_0(\zeta(u)).$$

Assume that

- f satisfies a kind of Palais-Smale condition in $[0,1]\times\mathcal{X}$: any $((\theta^m, u^m))_m$ such that
$$(f(\theta^m, u^m))_m \text{ is bounded and } \quad f'_{\theta^m}(u^m) \to 0 \text{ as } m \to +\infty \tag{4.38}$$
converges up to subsequences;

- for any $b > 0$ there exists $C_b > 0$ such that
$$|f_\theta(u)| \le b \;\Rightarrow\; \left|\frac{\partial}{\partial\theta} f(\theta, u)\right| \le C_b(\|f'_\theta(u)\| + 1)(\|u\| + 1)$$
for all $(\theta, u) \in [0, 1] \times \mathcal{X}$;
- there exist two continuous maps $\eta_1, \eta_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ which are Lipschitz continuous with respect to the second variable and such that $\eta_1 \le \eta_2$. Suppose
$$\eta_1(\theta, f_\theta(u)) \le \frac{\partial}{\partial\theta} f(\theta, u) \le \eta_2(\theta, f_\theta(u)) \tag{4.39}$$
at each critical point u of f_θ;
- f_0 is even and for each finite dimensional subspace $\mathcal{W}$ of $\mathcal{X}$ it results
$$\lim_{\|u\|\to+\infty,\, u\in\mathcal{W}} \sup_{\theta\in[0,1]} f(\theta, u) = -\infty\,.$$

Taken for $i = 1, 2$, let us denote by $\psi_i : [0, 1] \times \mathbb{R} \to \mathbb{R}$ the solutions of
$$\frac{\partial}{\partial\theta}\psi_i(\theta, s) = \eta_i(\theta, \psi_i(\theta, s))$$
$$\psi_i(0, s) = s\,.$$
Note that $\psi_i(\theta, \cdot)$ are continuous, non-decreasing on $\mathbb{R}$ and $\psi_1 \le \psi_2$. Set
$$\overline{\eta}_1(s) = \sup_{\theta\in[0,1]} \eta_1(\theta, s), \quad \overline{\eta}_2(s) = \sup_{\theta\in[0,1]} \eta_2(\theta, s)\,.$$
In this framework, the following abstract result can be proved.

Theorem 4.33. *There exists $C \in \mathbb{R}$ such that if $l \in \mathbb{N}$ then*

(a) *either f_1 has a critical point $\tilde{c}_l$ with $\psi_2(1, c_l) < \psi_1(1, c_{l+1}) \le \tilde{c}_l$,*
(b) *or we have $c_{l+1} - c_l \le C\,(\overline{\eta}_1(c_{l+1}) + \overline{\eta}_2(c_l) + 1)$.*

For the proof of the above theorem, see [24, Theorem 3] and [25, Theorem 2.2].

4.11. **Application to semi-linear elliptic systems.** In this section we want to prove Theorem 4.16 in a simpler fashion by means of the arguments introduced in Section 6.

For $\theta \in [0, 1]$, let us consider the functional $f_\theta : H^1_0(\Omega, \mathbb{R}^N) \to \mathbb{R}$ defined as
$$f_\theta(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a^{hk}_{ij}(x) D_i u_h D_j u_k \,dx - \frac{1}{p}\int_\Omega |u + \theta\phi|^p \,dx - \theta\int_\Omega \varphi \cdot u \,dx\,.$$
It can be proved that all the previous assumptions are satisfied.

Lemma 4.34. *Let $((\theta^m, u^m))_m \subset [0, 1]\times H^1_0(\Omega, \mathbb{R}^N)$ be such that condition (4.38) holds. Then $((\theta^m, u^m))_m$ converges up to subsequences.*

Proof. Let $((\theta^m, u^m))_m$ be such that (4.38) holds. For a suitable $K > 0$ and any $\varrho > 0$ it is
$$\begin{aligned}
&K + \varrho\,\|Du^m\|_2 \ge f_{\theta^m}(u^m) - \varrho f'_{\theta^m}(u^m)[u^m]\\
&= \Big(\frac{1}{2} - \varrho\Big)\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a^{hk}_{ij}(x) D_i u^m_h D_j u^m_k dx + \Big(\varrho - \frac{1}{p}\Big)\int_\Omega |u^m + \theta^m\phi|^p dx\\
&\quad - \theta^m\varrho\int_\Omega |u^m + \theta^m\phi|^{p-2}(u^m + \theta^m\phi)\cdot\phi \,dx
\end{aligned}$$

for all m large enough. Then, fixed any $\varepsilon > 0$ and taken d as in (4.35), (4.26) and simple computations imply

$$\varrho \, \|Du^m\|_2 + \big(\frac{1}{2} - \varrho\big) d \, \|u^m\|_2^2 \geq \big(\frac{1}{2} - \varrho\big) \frac{\nu}{2} \|Du^m\|_2^2 + \frac{1}{2^{p-1}} \big(\varrho(1-\varepsilon) - \frac{1}{p}\big) \|u^m\|_p^p - a_\varepsilon$$

for a certain $a_\varepsilon > 0$. Hence, if we fix $\varrho \in]\frac{1}{p}, \frac{1}{2}[$ and $\varepsilon \in]0, 1 - \frac{1}{\varrho p}[$, by this last inequality it follows that $(u^m)_m$ has to be bounded in $H_0^1(\Omega, \mathbb{R}^N)$.
So, if we assume $w^m = f'_{\theta^m}(u^m) + |u^m + \theta^m \phi|^{p-2}(u^m + \theta^m \phi) + \theta^m \varphi$ it is easy to prove that $(w^m)_m$ strongly converges in $H^{-1}(\Omega, \mathbb{R}^N)$, up to subsequences. Whence, Lemma 4.23 implies that $(u^m)_m$ has a converging subsequence in $H_0^1(\Omega, \mathbb{R}^N)$. □

Lemma 4.35. *For each $b > 0$ there exists $C_b > 0$ such that*

$$|f_\theta(u)| \leq b \;\Rightarrow\; \Big|\frac{\partial}{\partial \theta} f(\theta, u)\Big| \leq C_b(\|f'_\theta(u)\| + 1)(\|u\|_{1,2} + 1)$$

for all $(\theta, u) \in [0,1] \times H_0^1(\Omega, \mathbb{R}^N)$.

Proof. Fix $b > 0$. The condition $|f_\theta(u)| \leq b$ is equivalent to

$$\Big| \int_\Omega \Big(\frac{1}{2} \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k - \frac{1}{p} |u + \theta\phi|^p - \theta\varphi \cdot u \Big) dx \Big| \leq b \tag{4.40}$$

which implies that

$$\begin{aligned} \theta \int_\Omega \varphi \cdot u \, dx \geq \frac{p}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{h,k}(x) D_i u_h D_j u_k \, dx \\ - \int_\Omega |u + \theta\phi|^p \, dx - (p-1)\, \theta \int_\Omega \varphi \cdot u \, dx - p\, b. \end{aligned} \tag{4.41}$$

So, taken d as in (4.35), we have

$$\begin{aligned} -f'_\theta(u)[u] &= -\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx \\ &\quad + \int_\Omega |u + \theta\phi|^{p-2}(u + \theta\phi) \cdot u \, dx + \theta \int_\Omega \varphi \cdot u \, dx \\ &\geq \Big(\frac{p}{2} - 1\Big) \int_\Omega \Big(\sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k + d\, |u|^2 \Big) dx \\ &\quad - \Big(\frac{p}{2} - 1\Big)\, d\, \|u\|_2^2 - \int_\Omega |u + \theta\phi|^{p-2}(u + \theta\phi) \cdot \theta\phi \, dx \\ &\quad - (p-1)\, \theta \int_\Omega \varphi \cdot u \, dx - p\, b \\ &\geq (p-2)\, \frac{\nu}{4} \|Du\|_2^2 - \Big(\frac{p}{2} - 1\Big)\, d\, \|u\|_2^2 \\ &\quad - \int_\Omega |u + \theta\phi|^{p-2}(u + \theta\phi) \cdot \theta\phi \, dx - (p-1)\, \theta \int_\Omega \varphi \cdot u \, dx - p\, b. \end{aligned}$$

By Hölder inequality there exist $c_1, c_2, c_3 > 0$ such that

$$\Big| \int_\Omega |u + \theta\phi|^{p-2}(u + \theta\phi) \cdot \theta\phi \, dx \Big| \leq c_1 \|u + \theta\phi\|_p^{p-1}, \tag{4.42}$$

$$\Big|\int_\Omega \varphi \cdot u\, dx\Big| \le c_2 \|u+\theta\phi\|_p + c_3\,; \tag{4.43}$$

while (4.40) implies

$$\|u+\theta\phi\|_p^p \le c_4 \|Du\|_2^2 + c_5(b) \tag{4.44}$$

for suitable $c_4, c_5(b) > 0$. Then, since Young's inequality yields

$$\begin{aligned} c_1\|u+\theta\phi\|_p^{p-1} &\le \varepsilon\|u+\theta\phi\|_p^p + \widetilde{c}_1(\varepsilon),\\ c_2\|u+\theta\phi\|_p &\le \varepsilon\|u+\theta\phi\|_p^p + \widetilde{c}_2(\varepsilon), \end{aligned} \tag{4.45}$$

for all $\varepsilon > 0$ and certain $\widetilde{c}_1(\varepsilon), \widetilde{c}_2(\varepsilon) > 0$, it can be proved that $c_6, c_7(\varepsilon, b) > 0$ exist such that

$$-f_\theta'(u)[u] \ge \Big((p-2)\frac{\nu}{4} - \varepsilon c_6\Big)\, \|Du\|_2^2 - c_7(\varepsilon, b).$$

So, if ε is small enough, some $\widetilde{c}_6, \widetilde{c}_7(b) > 0$ can be find such that

$$\widetilde{c}_6\, \|Du\|_2^2 - \widetilde{c}_7(b) \le -f_\theta'(u)[u]. \tag{4.46}$$

On the other hand, since

$$\frac{\partial}{\partial\theta} f(\theta, u) = -\int_\Omega |u+\theta\phi|^{p-2}(u+\theta\phi)\cdot\phi\, dx - \int_\Omega \varphi\cdot u\, dx$$

by (4.42) and (4.43) it follows

$$\Big|\frac{\partial}{\partial\theta} f(\theta,u)\Big| \le c_8\|u+\theta\phi\|_p^{p-1} + c_9 \tag{4.47}$$

and then by (4.45)

$$\Big|\frac{\partial}{\partial\theta} f(\theta,u)\Big| \le \varepsilon\|u+\theta\phi\|_p^{p} + c_{10}(\varepsilon)$$

for any $\varepsilon > 0$ and $c_8, c_9, c_{10}(\varepsilon) > 0$ suitable constants. So, for all $\varepsilon > 0$ and a certain $c_{11}(\varepsilon, b) > 0$, (4.44) implies

$$\Big|\frac{\partial}{\partial\theta} f(\theta,u)\Big| \le \varepsilon c_4\, \|Du\|_2^2 + c_{11}(\varepsilon, b). \tag{4.48}$$

Hence, the proof follows by (4.46), (4.48) and a suitable choice of ε. □

Lemma 4.36. *If $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a critical point of f_θ, there exists $\sigma > 0$ such that*

$$\int_\Omega |u+\theta\phi|^p\, dx \le \sigma\left(f_\theta^2(u) + 1\right)^{1/2}.$$

For the proof of the above lemma it suffices to argue as in Lemma 4.19.

Lemma 4.37. *At each critical point u of f_θ the inequality (4.39) holds if η_1, η_2 are defined in $(\theta, s) \in [0,1]\times\mathbb{R}$ as*

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = C\left(s^2+1\right)^{\frac{p-1}{2p}} \tag{4.49}$$

for a suitable constant $C > 0$.

For the proof of this lemma, it is sufficient to combine (4.47) and Lemma 4.36.

New proof of Theorem 4.16. Clearly, f_0 is an even functional. Moreover, by Lemmas 4.34, 4.35 and 4.37 all the hypotheses of the existence theorem are fulfilled. Now, consider $(V_l)_l$, the sequence of subspaces of $H_0^1(\Omega,\mathbb{R}^N)$ introduced in the previous sections. Defined the set of maps $\mathcal{K}$ with $\mathcal{X} = H_0^1(\Omega,\mathbb{R}^N)$, assume

$$c_l = \inf_{\zeta\in\mathcal{K}} \sup_{u\in V_l} f_0(\zeta(u)).$$

Simple computations allow to prove that, taken any finite dimensional subspace $\mathcal{W}$ of H_0^1, some constants $\beta_1, \beta_2, \beta_3 > 0$ exist such that

$$f_\theta(u) \le \beta_1\|u\|_{1,2}^2 - \beta_2\|u\|_{1,2}^p - \beta_3 \quad \text{for all } u \in \mathcal{W}.$$

Then

$$\lim_{\|u\|_{1,2}\to+\infty,\, u\in\mathcal{W}} \ \sup_{\theta\in[0,1]} f_\theta(u) = -\infty.$$

Hence, Theorem 4.33 applies and, by the choice made in (4.49), the condition (b) implies that there exists $\widetilde{C} > 0$ such that

$$|c_{l+1} - c_l| \le \widetilde{C}\left((c_l)^{\frac{p-1}{p}} + (c_{l+1})^{\frac{p-1}{p}} + 1\right), \tag{4.50}$$

which implies $c_l \le \widetilde{\gamma}\, l^p$ for some $\widetilde{\gamma} > 0$ in view of [15, Lemma 5.3]. Taking into account Lemma 4.28 we conclude that (4.50) can not hold provided that

$$\frac{2p}{n(p-2)} > p,$$

namely $p \in]2, 2\,\frac{n+1}{n}[$. Whence, the assertion follows by (a) of Theorem 4.33. $\square$

4.12. **The diagonal case.** Now, we want to prove Theorem 4.17. To this aim let us point out that we deal with the problem

$$\begin{gathered} -\Delta u_k = |u|^{p-2}u_k + \varphi_k(x) \quad \text{in } \Omega \\ u = \chi \quad \text{on } \partial\Omega \\ k = 1,\dots,N \end{gathered} \tag{4.51}$$

and want to prove that (4.51) has an infinite number of solutions if $p \in \left]2, \frac{2n}{n-1}\right[$.

In this case, the functional f_θ defined in the previous section becomes

$$f_\theta(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega |u+\theta\phi|^p\,dx - \theta\int_\Omega \varphi\cdot u\,dx$$

where ϕ solves the system (4.25) with $a_{ij}^{hk} = \delta_{ij}^{hk}$.

By the regularity assumptions we made on $\partial\Omega$, χ and φ the following lemma can be proved.

Lemma 4.38. *There exists $c > 0$ such that if u is a critical point of f_θ, then*

$$\Big|\int_{\partial\Omega}\Big(\frac{1}{2}|\nabla w|^2 - \Big|\frac{\partial w}{\partial n}\Big|^2\Big)d\sigma\Big| \le c\int_\Omega\Big(|\nabla w|^2 + |w|^p + 1\Big)\,dx$$

where $w = u + \theta\phi$.

Proof. If $u \in H_0^1(\Omega,\mathbb{R}^N)$ is such that $f_\theta'(u) = 0$, then some regularity theorems imply that u is a classical solution of the problem

$$\begin{gathered} -\Delta u_k = |u+\theta\phi|^{p-2}(u_k+\theta\phi) + \theta\varphi_k \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

$$k = 1, \dots, N,$$

then $w = u + \theta\phi \in C^2(\Omega, \mathbb{R}^N)$ solves the elliptic system

$$\begin{gathered} -\Delta w_k = |w|^{p-2} w_k + \theta \varphi_k \quad \text{in } \Omega \\ w_k = \theta \phi_k \quad \text{on } \partial\Omega \\ k = 1, \dots, N. \end{gathered} \tag{4.52}$$

Taken $\delta > 0$, let us consider a cut function $\widetilde{\eta} \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\widetilde{\eta}(s) = 1$ for $s \le 0$ and $\widetilde{\eta}(s) = 0$ for $s \ge \delta$. Moreover, taken any $x \in \mathbb{R}^N$, let $d(x, \partial\Omega)$ be the distance of x from the boundary of Ω. Let us point out that, since Ω is smooth enough, δ can be chosen in such a way that $d(\cdot, \partial\Omega)$ is of class C^2 on

$$\overline{\Omega} \cap \{x \in \mathbb{R}^n : d(x, \partial\Omega) < \delta\},$$

and $\widehat{n}(x) = \nabla d(x, \partial\Omega)$ coincides on $\partial\Omega$ with the inner normal. So, defined $g : \mathbb{R}^N \to \mathbb{R}$ as $g(x) = \widetilde{\eta}(d(x, \partial\Omega))$, for each $k = 1, \dots, N$ let us multiply the k-th equation in (4.52) by $g(x)\nabla w_k \cdot \widehat{n}(x)$. Hence, working as in [25, Lemma 4.2] and summing up with respect to k, we get

$$\begin{gathered} \sum_{k=1}^N \int_\Omega -\Delta w_k \, g(x) \nabla w_k \cdot \widehat{n} \, dx = \int_{\partial\Omega} \Big(\frac{1}{2}|\nabla w|^2 - \Big|\frac{\partial w}{\partial n}\Big|^2\Big) d\sigma + O\left(\|\nabla w\|_2^2\right), \\ \sum_{k=1}^N \int_\Omega |w|^{p-2} w_k \, g(x) \nabla w_k \cdot \widehat{n} \, dx = \frac{\theta^p}{p} \int_{\partial\Omega} |\phi|^p \, d\sigma + O\left(\|w\|_p^p\right), \\ \sum_{k=1}^N \int_\Omega \theta \varphi_k(x) \, g(x) \nabla w_k \cdot \widehat{n} \, dx = \theta^2 \int_{\partial\Omega} \varphi \cdot \phi \, d\sigma + O\left(\|w\|_p\right). \end{gathered}$$

Whence, the proof follows by putting together these identities. □

With the stronger assumptions we made in this section, the estimates in Lemma 4.37 can be improved.

Lemma 4.39. *At each critical point u of f_θ the inequality (4.39) holds if η_1, η_2 are defined in $(\theta, s) \in [0, 1] \times \mathbb{R}$ as*

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = C\left(s^2 + 1\right)^{1/4}$$

for a suitable constant $C > 0$.

Proof. Let u be a critical point of f_θ. Then,

$$\frac{\partial}{\partial\theta} f(\theta, u) = \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \phi \, d\sigma + \int_\Omega \varphi \cdot (\theta\phi - u) \, dx;$$

so, taking into account Lemma 4.38, it is enough to argue as in [25, Lemma 4.3]. □

Proof of Theorem 4.17. Arguing as in the proof of Theorem 4.16, we have that the proof of Theorem 4.17 follows by Theorem 4.33 since also in this case the condition (b) can not occur. Let us point out that, by Lemma 4.39, the incompatibility condition is $\frac{2p}{n(p-2)} > 2$, i.e. $p \in \left]2, \frac{2n}{n-1}\right[$. □

5. Problems of Jumping Type

We refer the reader to [80, 81]. Some parts of these publications have been slightly modified to give this collection a more uniform appearance.

5.1. **Fully nonlinear elliptic equation.**

Let us consider the semi-linear elliptic problem

$$\begin{aligned} -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) &= g(x,u)+\omega \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\omega \in H^{-1}(\Omega)$ and $g : \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\lim_{s\to-\infty}\frac{g(x,s)}{s}=\alpha\,,\qquad \lim_{s\to+\infty}\frac{g(x,s)}{s}=\beta. \tag{5.2}$$

Let us denote by (μ_h) the eigenvalues of the linear operator on $H^1_0(\Omega)$

$$u \mapsto -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u).$$

Since 1972, this jumping problem has been widely investigated in the case when some eigenvalue μ_h belongs to the interval $]\beta,\alpha[$ (see e.g. [99, 101, 125] and references therein), starting from the pioneering paper [4] of Ambrosetti and Prodi.

On the other hand, since 1994, several efforts have been devoted to study existence of weak solutions of the quasi-linear problem

$$\begin{aligned} -\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u)+\frac12\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_i u D_j u &= g(x,u)+\omega \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{5.3}$$

via techniques of non-smooth critical point theory (see e.g. [8, 32, 36, 49, 138]).

In particular, a jumping problem for the previous equation has been treated in [31]. More recently, existence for the Euler's equations of multiple integrals of calculus of variations

$$\begin{aligned} -\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big)+D_s\mathcal{L}(x,u,\nabla u) &= g(x,u)+\omega \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{5.4}$$

have also been considered in [6] and in [113, 133] via techniques developed in [36]. In this section we see how the results of [31] may be extended to the more general elliptic problem (5.4). We shall approach the problem from a variational point of view, that is looking for critical points for continuous functionals $f : W^{1,p}_0(\Omega)\to\mathbb{R}$ of type

$$f(u)=\int_\Omega \mathcal{L}(x,u,\nabla u)\,dx-\int_\Omega G(x,u)\,dx-\langle\omega,u\rangle.$$

We point out that, in general, these functionals are not even locally Lipschitzian, so that classical critical point theory fails. Then we shall refer to non-smooth critical point theory, In our main result (Theorem 5.1) we shall prove existence of at least two solutions of the problem by means of a classical min-max theorem in its non-smooth version.

5.2. **The main result.** We assume that Ω is a bounded domain of $\mathbb{R}^n$, $1 < p < n$, $\omega \in W^{-1,p'}(\Omega)$ and $\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and of class C^1 in (s,ξ) a.e. in Ω. Moreover, the function $\mathcal{L}(x,s,\cdot)$ is strictly convex and for each $t \in \mathbb{R}$ $\mathcal{L}(x,s,t\xi) = |t|^p \mathcal{L}(x,s,\xi)$ for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Furthermore, we assume that:

- There exist $\nu > 0$ and $b_1 \in \mathbb{R}$ such that:
$$\nu|\xi|^p \le \mathcal{L}(x,s,\xi) \le b_1|\xi|^p\,, \tag{5.5}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;
- there exist $b_2, b_3 \in \mathbb{R}$ such that:
$$|D_s\mathcal{L}(x,s,\xi)| \le b_2|\xi|^p,$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and
$$\left|\nabla_\xi\mathcal{L}(x,s,\xi)\right| \le b_3|\xi|^{p-1}, \tag{5.6}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;
- there exist $R > 0$ and a bounded Lipschitzian function $\vartheta : \mathbb{R} \to [0,+\infty[$ such that:
$$|s| \ge R \Rightarrow sD_s\mathcal{L}(x,s,\xi) \ge 0\,, \tag{5.7}$$
$$sD_s\mathcal{L}(x,s,\xi) \le s\vartheta'(s)\,\nabla_\xi\mathcal{L}(x,s,\xi)\cdot\xi\,, \tag{5.8}$$
for a.e. $x \in \Omega$ and $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^n$. Without loss of generality, we may take assume that $\vartheta(s) \to \overline{\vartheta}$ as $s \to \pm\infty$;
- $g(x,s)$ is a Carathéodory function and $G(x,s) = \int_0^s g(x,\tau)\,d\tau$. We assume that there exist $a \in L^{np/(n(p-1)+p)}(\Omega)$ and $b \in L^{n/p}(\Omega)$ such that:
$$|g(x,s)| \le a(x) + b(x)|s|^{p-1}\,, \tag{5.9}$$
for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Moreover, there exist $\alpha,\ \beta \in \mathbb{R}$ such that
$$\lim_{s\to-\infty}\frac{g(x,s)}{|s|^{p-2}s} = \alpha\,, \qquad \lim_{s\to+\infty}\frac{g(x,s)}{|s|^{p-2}s} = \beta\,, \tag{5.10}$$
for a.e. $x \in \Omega$.

Let us now suppose that
$$\lim_{s\to+\infty}\mathcal{L}(x,s,\xi) = \lim_{s\to-\infty}\mathcal{L}(x,s,\xi)$$
(both limits exist by (5.7)) and denote by $\mathcal{L}_\infty(x,\xi)$ the common value, that we shall assume to be of the form $a(x)|\xi|^p$ with $a \in L^\infty(\Omega)$. Moreover, assume that
$$s_h \to +\infty, \quad \xi_h \to \xi \Rightarrow \nabla_\xi\mathcal{L}(x,s_h,\xi_h) \to \nabla_\xi\mathcal{L}_\infty(x,\xi). \tag{5.11}$$
Let
$$\lambda_1 = \min\Big\{p\int_\Omega \mathcal{L}_\infty(x,\nabla u)\,dx : u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p\,dx = 1\Big\}, \tag{5.12}$$
be the first eigenvalue of $\{u \longmapsto -\operatorname{div}\left(\nabla_\xi L_\infty(x,\nabla u)\right)\}$.

Observe that by [6, Lemma 1.4] the first eigenfunction ϕ_1 belongs to $L^\infty(\Omega)$ and by [143, Theorem 1.1] is strictly positive.

Under the previous assumptions, we consider problem (5.4) in the case $\omega = t\phi_1^{p-1} + \omega_0$, with $\omega_0 \in W^{-1,p'}(\Omega)$ and $t \in \mathbb{R}$. The following is our main result.

Theorem 5.1. *If $\beta < \lambda_1 < \alpha$ then there exist $\bar{t} \in \mathbb{R}$ and $\underline{t} \in \mathbb{R}$ such that the problem*

$$\begin{gathered} -\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) = g(x,u) + t\phi_1^{p-1} + \omega_0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{5.13}$$

has at least two weak solutions in $W_0^{1,p}(\Omega)$ for $t > \bar{t}$ and no solutions for $t < \underline{t}$.

This result extends [31, Corollary 2.3] dealing with the case $p = 2$ and

$$\mathcal{L}(x,s,\xi) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j - G(x,s)$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

In this particular case, existence of at least three solutions has been recently proved in [34] assuming $\beta < \mu_1$ and $\alpha > \mu_2$ where μ_1 and μ_2 are the first and second eigenvalue of the operator

$$u \mapsto -\sum_{i,j=1}^{n} D_j(A_{ij}D_i u).$$

In our general setting we only have existence of the first eigenvalue λ_1 and it is not clear how to define higher order eigenvalues $\lambda_2, \lambda_3, \dots$. Therefore in our case the comparison of α and β with such eigenvalues is still not possible.

5.3. **The concrete Palais-Smale condition.** The following result is one of the main tool of the section.

Lemma 5.2. *Let (u_h) be a sequence in $W_0^{1,p}(\Omega)$ and $(\varrho_h) \subseteq]0,+\infty[$ with $\varrho_h \to +\infty$ be such that*

$$v_h = \frac{u_h}{\varrho_h} \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega).$$

Let $\gamma_h \rightharpoonup \gamma$ in $L^{n/p}(\Omega)$ with $|\gamma_h(x)| \le c(x)$ for some $c \in L^{n/p}(\Omega)$. Moreover, let

$$\mu_h \to \mu \quad \text{in } L^{np'/(n+p')}(\Omega), \qquad \delta_h \to \delta \quad \text{in } W^{-1,p'}(\Omega)$$

be such that for each $\varphi \in C_c^\infty(\Omega)$:

$$\begin{aligned} &\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\varphi\,dx + \int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)\varphi\,dx \\ &= \int_\Omega \gamma_h|u_h|^{p-2}u_h\varphi\,dx + \varrho_h^{p-1}\int_\Omega \mu_h\varphi\,dx + \langle\delta_h\,,\varphi\rangle. \end{aligned} \tag{5.14}$$

Then, the following facts hold:

(*a*) *(v_h) is strongly convergent to v in $W_0^{1,p}(\Omega)$;*
(*b*) *$(\gamma_h|v_h|^{p-2}v_h)$ is strongly convergent to $\gamma|v|^{p-2}v$ in $W^{-1,p'}(\Omega)$;*
(*c*) *there exist $\eta^+,\ \eta^- \in L^\infty(\Omega)$ such that:*

$$\eta^+(x) = \begin{cases} \exp\left\{-\overline{\vartheta}\right\} & \text{if } v(x) > 0 \\ \exp\{MR\} & \text{if } v(x) < 0\,, \end{cases}$$

$$\exp\{-\overline{\vartheta}\} \le \eta^+(x) \le \exp\{MR\} \quad \text{if } v(x) = 0\,,$$

and

$$\eta^-(x) = \begin{cases} \exp\left\{-\overline{\vartheta}\right\} & \text{if } v(x) < 0 \\ \exp\{MR\} & \text{if } v(x) > 0\,, \end{cases}$$

$$\exp\{-\overline{\vartheta}\} \le \eta^-(x) \le \exp\{MR\} \quad \text{if } v(x) = 0\,,$$

and such that for every $\varphi \in W_0^{1,p}(\Omega)$ *with* $\varphi \ge 0$*:*

$$\int_\Omega \eta^+ \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla\varphi\, dx \ge \int_\Omega \gamma\eta^+ |v|^{p-2} v\varphi\, dx + \int_\Omega \mu\eta^+\varphi\, dx\,,$$
$$\int_\Omega \eta^- \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla\varphi\, dx \le \int_\Omega \gamma\eta^- |v|^{p-2} v\varphi\, dx + \int_\Omega \mu\eta^-\varphi\, dx.$$

Proof. Arguing as in [31, Lemma 3.1], (b) immediately follows. Let us now prove (a). Up to a subsequence, $v_h(x) \to v(x)$ for a.e. $x \in \Omega$. Consider now the function $\zeta : \mathbb{R} \to \mathbb{R}$ defined by

$$\zeta(s) = \begin{cases} Ms & \text{if } 0 < s < R \\ MR & \text{if } s \ge R \\ -Ms & \text{if } -R < s < 0 \\ MR & \text{if } s \le -R\,, \end{cases} \tag{5.15}$$

where $M \in \mathbb{R}$ is such that for a.e. $x \in \Omega$, each $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$|D_s\mathcal{L}(x, s, \xi)| \le M\nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi. \tag{5.16}$$

By [133, Proposition 3.1], we may choose in (5.14) the functions $\varphi = v_h \exp\{\zeta(u_h)\}$ yielding

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v_h \exp\{\zeta(u_h)\}\, dx \\
&\quad + \int_\Omega \big[D_s\mathcal{L}(x, u_h, \nabla u_h) + \zeta'(u_h)\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h\big] v_h \exp\{\zeta(u_h)\}\, dx \\
&= \int_\Omega \gamma_h |u_h|^{p-2} u_h v_h \exp\{\zeta(u_h)\}\, dx + \varrho_h^{p-1} \int_\Omega \mu_h v_h \exp\{\zeta(u_h)\}\, dx \\
&\quad + \langle \delta_h, v_h \exp\{\zeta(u_h)\}\rangle.
\end{aligned}$$

Therefore, taking into account conditions (5.7) and (5.16), we have

$$\begin{aligned}
&\varrho_h^{p-1} \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\}\, dx \\
&\le \varrho_h^{p-1} \int_\Omega \gamma_h |v_h|^p \exp\{\zeta(u_h)\}\, dx + \varrho_h^{p-1} \int_\Omega \mu_h v_h \exp\{\zeta(u_h)\}\, dx \\
&\quad + \langle \delta_h, v_h \exp\{\zeta(u_h)\}\rangle\,.
\end{aligned}$$

After division by ϱ_h^{p-1}, using the hypotheses on γ_h, μ_h and δ_h, we obtain

$$\begin{aligned}
&\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\}\, dx \\
&\le \exp\{MR\}\Big(\int_\Omega \gamma |v|^p\, dx + \int_\Omega \mu v\, dx\Big).
\end{aligned} \tag{5.17}$$

Now, let us consider the function $\vartheta_1 : \mathbb{R} \to \mathbb{R}$ given by

$$\vartheta_1(s) = \begin{cases} \vartheta(s) & \text{if } s \ge 0 \\ Ms & \text{if } -R \le s \le 0 \\ -MR & \text{if } s \le -R\,, \end{cases} \tag{5.18}$$

with ϑ satisfying (5.8). Considering in (5.14) the functions $(v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}$ with $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla(v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\, dx \\
&+ \frac{1}{\varrho_h^{p-1}} \int_\Omega \big[D_s\mathcal{L}(x, u_h, \nabla u_h) - \vartheta_1'(u_h)\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)\cdot \nabla u_h\big] \\
&\times (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\, dx \\
&= \int_\Omega \gamma_h |v_h|^{p-2} v_h (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\, dx + \int_\Omega \mu_h (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\, dx \\
&\quad + \frac{1}{{\varrho_h}^{p-1}} \big\langle \delta_h, (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\big\rangle .
\end{aligned} \tag{5.19}$$

By (5.7), (5.8) and (5.16) it results that for each $h \in \mathbb{N}$

$$\big[D_s\mathcal{L}(x, u_h, \nabla u_h) - \vartheta_1'(u_h)\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)\cdot \nabla u_h\big](v^+ \wedge k)\exp\{-\vartheta_1(u_h)\} \le 0.$$

Taking into account assumptions (5.11) and (5.6), we may apply [54, Theorem 5] and deduce that

$$\text{a.e. in } \Omega \setminus \{v = 0\} : \quad \nabla v_h(x) \to \nabla v(x).$$

Being $u_h(x) \to +\infty$ a.e. in $\Omega \setminus \{v = 0\}$, again recalling (5.11), we have

$$\text{a.e. in } \Omega \setminus \{v = 0\} : \quad \nabla_\xi \mathcal{L}(x, u_h(x), \nabla v_h(x)) \to \nabla_\xi \mathcal{L}_\infty(x, \nabla v(x)).$$

By combining this pointwise convergence with (5.6), we obtain

$$\nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \rightharpoonup \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \quad \text{in } L^{p'}(\Omega). \tag{5.20}$$

Therefore, for each $k \in \mathbb{N}$ we have

$$\begin{aligned}
&\lim_h \nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla(v^+ \wedge k)\exp\{-\vartheta_1(u_h)\} \\
&= \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla(v^+ \wedge k)\exp\{-\overline{\vartheta}\},
\end{aligned}$$

strongly in $L^1(\Omega)$,

$$\lim_h (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\} = (v^+ \wedge k)\exp\{-\overline{\vartheta}\},$$

weakly in $W_0^{1,p}(\Omega)$, using (b)

$$\lim_h \gamma_h |v_h|^{p-2} v_h (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\} = \gamma |v|^{p-2} v (v^+ \wedge k)\exp\{-\overline{\vartheta}\},$$

strongly in $L^1(\Omega)$ and

$$\lim_h \frac{1}{\varrho_h^{p-1}} (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\} = 0,$$

weakly in $W_0^{1,p}(\Omega)$. Therefore, letting $h \to +\infty$ in (5.19), for each $k \in \mathbb{N}$ we get

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla(v^+ \wedge k)\exp\{-\overline{\vartheta}\}\, dx \\
&\ge \int_\Omega \gamma |v|^{p-2} v (v^+ \wedge k)\exp\{-\overline{\vartheta}\}\, dx + \int_\Omega \mu (v^+ \wedge k)\exp\{-\overline{\vartheta}\}\, dx.
\end{aligned}$$

Finally, if we let $k \to +\infty$, after division by $\exp\{-\overline{\vartheta}\}$, we have

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla v^+) \cdot \nabla v^+ \, dx \geq \int_\Omega \gamma |v|^{p-2}(v^+)^2 \, dx + \int_\Omega \mu v^+ \, dx. \tag{5.21}$$

Analogously, if we define a function $\vartheta_2 : \mathbb{R} \to \mathbb{R}$ by

$$\vartheta_2(s) = \begin{cases} \vartheta(s) & \text{if } s \leq 0 \\ -Ms & \text{if } 0 \leq s \leq R \\ -MR & \text{if } s \geq R, \end{cases}$$

and consider in (5.14) the test functions $(v^- \wedge k)\exp\{-\vartheta_2(u_h)\}$ with $k \in \mathbb{N}$, we obtain

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v^- dx \leq -\int_\Omega \gamma |v|^{p-2}(v^-)^2 \, dx + \int_\Omega \mu v^- dx. \tag{5.22}$$

Thus, combining (5.21) and (5.22) yields

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v \, dx \geq \int_\Omega \gamma |v|^p \, dx + \int_\Omega \mu v \, dx. \tag{5.23}$$

Finally, putting together (5.17) and (5.23), we conclude

$$\begin{aligned} &\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \, dx \\ &\leq \exp\{MR\} \int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v \, dx. \end{aligned}$$

In particular, by Fatou's Lemma, it results

$$\begin{aligned} &\exp\{MR\} \int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v \, dx \\ &\leq \liminf_h \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \, dx \\ &\leq \exp\{MR\} \int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v \, dx; \end{aligned}$$

namely,

$$\nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \to \exp\{MR\} \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v.$$

$L^1(\Omega)$. Therefore, since $\nu |\nabla v_h|^p \leq \nabla_\xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\}$, again thanks to Fatou's Lemma, we conclude that

$$\limsup_h \int_\Omega |\nabla v_h|^p \, dx \leq \int_\Omega |\nabla v|^p \, dx,$$

and the proof of (a) is concluded.

Let us now prove assertion (c). Up to a subsequence, $\exp\{-\vartheta_1(u_h)\}$ weakly* converges in $L^\infty(\Omega)$ to some η^+. Of course, we have

$$\eta^+(x) = \begin{cases} \exp\left\{-\overline{\vartheta}\right\} & \text{if } v(x) > 0 \\ \exp\{MR\} & \text{if } v(x) < 0, \end{cases}$$

$$\exp\{-\overline{\vartheta}\} \leq \eta^+(x) \leq \exp\{MR\} \ \text{ if } v(x) = 0.$$

Then, let us consider in (5.14) as test functions:

$$\varphi \exp\{-\vartheta_1(u_h)\}, \quad \varphi \in C_c^\infty(\Omega), \quad \varphi \geq 0.$$

Whence, like in the previous argument, we obtain

$$\int_\Omega \eta^+ \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla\varphi \, dx \geq \int_\Omega \gamma\eta^+ |v|^{p-2} v\varphi \, dx + \int_\Omega \mu\eta^+ \varphi \, dx,$$

for any positive $\varphi \in W_0^{1,p}(\Omega)$. Similarly, by means of the test functions

$$\varphi \exp\{-\vartheta_2(u_h)\}, \quad \varphi \in C_c^\infty(\Omega), \quad \varphi \geq 0,$$

we get for any positive $\varphi \in W_0^{1,p}(\Omega)$

$$\int_\Omega \eta^- \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla\varphi \, dx \leq \int_\Omega \gamma\eta^- |v|^{p-2} v\varphi \, dx + \int_\Omega \mu\eta^- \varphi \, dx,$$

where η^- is the weak* limit of some subsequence of $\exp\{-\vartheta_2(u_h)\}$. □

Consider now

$$g_0(x, s) = g(x, s) - \beta|s|^{p-2}s^+ + \alpha|s|^{p-2}s^-, \quad G_0(x, s) = \int_0^s g_0(x, \tau) \, d\tau.$$

Of course, g_0 is a Carathéodory function satisfying for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$

$$\lim_{|s|\to\infty} \frac{g_0(x, s)}{|s|^{p-2}s} = 0, \quad |g_0(x, s)| \leq a(x) + \tilde{b}(x)|s|^{p-1},$$

with $\tilde{b} \in L^{n/p}(\Omega)$. Since we are interested in weak solutions $u \in W_0^{1,p}(\Omega)$ of the equations

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x, u, \nabla u)\big) + D_s \mathcal{L}(x, u, \nabla u) = g(x, u) + t\phi_1^{p-1} + \omega_0,$$

let us define the associated functional $f_t : W_0^{1,p}(\Omega) \to \mathbb{R}$, by setting

$$\begin{aligned} f_t(u) = &\int_\Omega \mathcal{L}(x, u, \nabla u) \, dx - \frac{\beta}{p} \int_\Omega (u^+)^p \, dx - \frac{\alpha}{p} \int_\Omega (u^-)^p \, dx \\ &- \int_\Omega G_0(x, u) \, dx - |t|^{p-2} t \int_\Omega \phi_1^{p-1} u \, dx - \langle \omega_0, u \rangle. \end{aligned}$$

Lemma 5.3. *Let (u_h) a sequence in $W_0^{1,p}(\Omega)$ and $\varrho_h \subseteq]0, +\infty[$ with $\varrho_h \to +\infty$. Assume that the sequence $(\frac{u_h}{\varrho_h})$ is bounded in $W_0^{1,p}(\Omega)$. Then*

$$\frac{g_0(x, u_h)}{\varrho_h^{p-1}} \to 0 \quad \text{in } L^{\frac{np'}{n+p'}}(\Omega), \qquad \frac{G_0(x, u_h)}{\varrho_h^p} \to 0 \quad \text{in } L^1(\Omega).$$

For the proof of the above lemma, argue as in [31, Lemma 3.3]. We now recall from [133] a compactness property of $(CPS)_c$-sequences.

Theorem 5.4. *Let (u_h) be a bounded sequence in $W_0^{1,p}(\Omega)$ and set*

$$\langle w_h, v \rangle = \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_\Omega D_s \mathcal{L}(x, u_h, \nabla u_h) v \, dx, \tag{5.24}$$

for all $v \in C_c^\infty(\Omega)$. If (w_h) is strongly convergent to some w in $W^{-1,p'}(\Omega)$, then (u_h) admits a strongly convergent subsequence in $W_0^{1,p}(\Omega)$.

For the proof of the above theorem, see [133, Theorem 3.4].

Lemma 5.5. *For each $c, t \in \mathbb{R}$ the following assertions are equivalent:*

(*a*) *f_t satisfies the $(CPS)_c$ condition ;*

(*b*) *every $(CPS)_c$-sequence for f_t is bounded in $W_0^{1,p}(\Omega)$.*

Proof. $(a) \Rightarrow (b)$. It is trivial. $(b) \Rightarrow (a)$. Let (u_h) be a $(CPS)_c$-sequence for f_t. Since (u_h) is bounded in $W_0^{1,p}(\Omega)$, and the map

$$u \longmapsto g(x,u) + t\phi_1^{p-1} + \omega_0\,,$$

is completely continuous by (5.9), up to a subsequence $(g(x,u_h)+t\phi_1^{p-1}+\omega_0)$ is strongly convergent in $L^{\frac{np'}{n+p'}}(\Omega)$, hence in $W^{-1,p'}(\Omega)$. $\square$

We now come to one of the main tool of this section.

Theorem 5.6. *Let $c, t \in \mathbb{R}$. Then f_t satisfies the $(CPS)_c$ condition.*

Proof. If (u_h) is a $(CPS)_c$-sequence for f_t, we have $f_t(u_h) \to c$ and for all $v \in C_0^\infty(\Omega)$:

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla v\,dx + \int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)v\,dx \\
&-\beta\int_\Omega (u_h^+)^{p-1}v\,dx + \alpha\int_\Omega (u_h^-)^{p-1}v\,dx - \int_\Omega g_0(x,u_h)v\,dx - |t|^{p-2}t\int_\Omega \phi_1 v\,dx \\
&= \langle \omega_0+\sigma_h, v\rangle,
\end{aligned}$$

where $\sigma_h \to 0$ in $W^{-1,p'}(\Omega)$. Taking into account Theorem 5.4, by Lemma 5.5 it suffices to show that (u_h) is bounded in $W_0^{1,p}(\Omega)$. Assume by contradiction that, up to a subsequence, $\|u_h\|_{1,p} \to +\infty$ as $h \to +\infty$ and set $v_h = u_h\|u_h\|_{1,p}^{-1}$. By Lemma 5.3, we can apply Lemma 5.2 choosing

$$\gamma_h(x) = \begin{cases} \beta & \text{if } u_h(x) \ge 0 \\ \alpha & \text{if } u_h(x) < 0, \end{cases} \qquad \varrho_h = \|u_h\|_{1,p},$$

$$\mu_h = \frac{g_0(x,u_h)}{\|u_h\|_{1,p}^{p-1}}, \qquad \delta_h = |t|^{p-2}t\phi_1 + \omega_0 + \sigma_h.$$

Then, up to a subsequence, (v_h) strongly converges to some v in $W_0^{1,p}(\Omega)$. Moreover, putting $\varphi = v^+$ in (c) of Lemma 5.2, we get

$$\int_\Omega \eta^-\nabla_\xi\mathcal{L}_\infty(x,\nabla v^+)\cdot\nabla v^+\,dx \le \int_\Omega \beta\eta^-(v^+)^p\,dx\,,$$

hence, taking into account (5.12), we have

$$\lambda_1\int_\Omega (v^+)^p\,dx \le \int_\Omega \nabla_\xi\mathcal{L}_\infty(x,\nabla v^+)\cdot\nabla v^+\,dx \le \beta\int_\Omega (v^+)^p\,dx.$$

Since $\beta < \lambda_1$, then $v^+ = 0$. By using again the first inequality in (c) of Lemma 5.2, for each $\varphi \ge 0$ we get

$$\int_\Omega \eta^+\nabla_\xi\mathcal{L}_\infty(x,\nabla v)\cdot\nabla\varphi\,dx \ge \alpha\int_\Omega \eta^+|v|^{p-2}v\varphi\,dx.$$

namely, since $v \le 0$, we have

$$\int_\Omega \nabla_\xi\mathcal{L}_\infty(x,\nabla v)\cdot\nabla\varphi\,dx \ge \alpha\int_\Omega |v|^{p-2}v\varphi\,dx.$$

In a similar way, by the second inequality in (c) of Lemma 5.2 we get

$$\int_\Omega \nabla_\xi\mathcal{L}_\infty(x,\nabla v)\cdot\nabla\varphi\,dx \le \alpha\int_\Omega |v|^{p-2}v\varphi\,dx.$$

Therefore,

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla \varphi \, dx = \alpha \int_\Omega |v|^{p-2} v \varphi \, dx ,$$

which, in view of [95, Remark 1, pp. 161] is impossible if α differs from λ_1. □

5.4. **Min-Max estimates.** Let us introduce the "asymptotic functional" $f_\infty : W_0^{1,p}(\Omega) \to \mathbb{R}$ by setting

$$f_\infty(u) = \int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx - \frac{\beta}{p} \int_\Omega (u^+)^p \, dx - \frac{\alpha}{p} \int_\Omega (u^-)^p \, dx - \int_\Omega \phi_1^{p-1} u \, dx.$$

Then consider the functional $\widetilde{f}_t : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} \widetilde{f}_t(u) = & \int_\Omega \mathcal{L}(x, tu, \nabla u) \, dx - \frac{\beta}{p} \int_\Omega (u^+)^p \, dx - \frac{\alpha}{p} \int_\Omega (u^-)^p \, dx \\ & - \int_\Omega \frac{G_0(x, tu)}{t^p} - \int_\Omega \phi_1^{p-1} u \, dx - \frac{\langle \omega_0, u \rangle}{t^{p-1}}. \end{aligned}$$

Theorem 5.7. *The following facts hold:*

(*a*) *Assume that* $(t_h) \subset]0, +\infty[$ *with* $t_h \to +\infty$ *and* $u_h \to u$ *in* $W_0^{1,p}(\Omega)$. *Then*

$$\lim_h \widetilde{f}_{t_h}(u_h) = f_\infty(u).$$

(*b*) *Assume that* $(t_h) \subset]0, +\infty[$ *with* $t_h \to +\infty$ *and* $u_h \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$. *Then*

$$f_\infty(u) \le \liminf_h \widetilde{f}_{t_h}(u_h).$$

(*c*) *Assume that* $(t_h) \subset]0, +\infty[$ *with* $t_h \to +\infty$, $u_h \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *and*

$$\limsup_h \widetilde{f}_{t_h}(u_h) \le f_\infty(u).$$

Then (u_h) *strongly converges to* u *in* $W_0^{1,p}(\Omega)$.

Proof. (*a*) It is easy to prove. (*b*) Since $u_h \to u$ in $L^p(\Omega)$, it is sufficient to prove that

$$\int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx \le \liminf_h \int_\Omega \mathcal{L}(x, t_h u_h, \nabla u_h) \, dx.$$

Let us define the Carathéodory function $\widetilde{\mathcal{L}} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by setting

$$\widetilde{\mathcal{L}}(x, s, \xi) := \begin{cases} \mathcal{L}(x, \tan(s), \xi) & \text{if} \quad |s| < \frac{\pi}{2} \\ \mathcal{L}_\infty(x, \xi) & \text{if} \quad |s| \ge \frac{\pi}{2}. \end{cases}$$

Note that $\widetilde{\mathcal{L}} \ge 0$ and $\widetilde{\mathcal{L}}(x, s, \cdot)$ is convex. Up to a subsequence we have

$$t_h u_h \to z \ \text{ a.e. in } \Omega \setminus \{u = 0\}, \quad \nabla u_h \rightharpoonup \nabla u \quad \text{ in } L^p(\Omega \setminus \{u = 0\}),$$

and

$$\arctan(t_h u_h) \to \arctan(z) \quad \text{ in } L^p(\Omega \setminus \{u = 0\}).$$

Therefore, by [86, Theorem 1] we deduce that

$$\int_{\Omega \setminus \{u=0\}} \widetilde{\mathcal{L}}(x, \arctan(z), \nabla u) \, dx \le \liminf_h \int_{\Omega \setminus \{u=0\}} \widetilde{\mathcal{L}}(x, \arctan(t_h u_h), \nabla u_h) \, dx ,$$

that implies

$$\int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx = \int_{\Omega \setminus \{u=0\}} \mathcal{L}_\infty(x, \nabla u) \, dx$$

$$\leq \liminf_h \int_{\Omega\setminus\{u=0\}} \mathcal{L}(x, t_h u_h, \nabla u_h)\, dx$$
$$= \liminf_h \int_{\Omega} \mathcal{L}(x, t_h u_h, \nabla u_h)\, dx.$$

Let us now prove (c). As above, we obtain

$$\liminf_h \int_\Omega \mathcal{L}\Big(x, t_h u_h, \frac{1}{2}\nabla u_h + \frac{1}{2}\nabla u\Big)\, dx \geq \int_\Omega \mathcal{L}_\infty(x, \nabla u)\, dx.$$

Since we have

$$\lim_h \int_\Omega \mathcal{L}(x, t_h u_h, \nabla u)\, dx = \int_\Omega \mathcal{L}_\infty(x, \nabla u)\, dx$$

and

$$\limsup_h \int_\Omega \mathcal{L}(x, t_h u_h, \nabla u_h)\, dx \leq \int_\Omega \mathcal{L}_\infty(x, \nabla u)\, dx\,, \tag{5.25}$$

we get

$$\limsup_h \int_\Omega (\mathcal{L}(x, t_h u_h, \nabla u_h) - \mathcal{L}(x, t_h u_h, \nabla u))\, dx \leq 0.$$

On the other hand, the strict convexity implies that for each $h \in \mathbb{N}$

$$\frac{1}{2}\mathcal{L}(x, t_h u_h, \nabla u_h) + \frac{1}{2}\mathcal{L}(x, t_h u_h, \nabla u) - \frac{1}{2}\mathcal{L}\Big(x, t_h u_h, \frac{1}{2}\nabla u_h + \frac{1}{2}\nabla u\Big) > 0.$$

Therefore, the previous limits yield

$$\int_\Omega \Big\{\frac{1}{2}\mathcal{L}(x, t_h u_h, \nabla u_h) + \frac{1}{2}\mathcal{L}(x, t_h u_h, \nabla u) - \frac{1}{2}\mathcal{L}\Big(x, t_h u_h, \frac{1}{2}\nabla u_h + \frac{1}{2}\nabla u\Big)\Big\}\, dx$$
$$\to 0.$$

In particular, up to a subsequence, we have

$$\frac{1}{2}\mathcal{L}(x, t_h u_h, \nabla u_h) + \frac{1}{2}\mathcal{L}(x, t_h u_h, \nabla u) - \frac{1}{2}\mathcal{L}\Big(x, t_h u_h, \frac{1}{2}\nabla u_h + \frac{1}{2}\nabla u\Big) \to 0,$$

a.e. in Ω. It easily verified that this can be true only if

$$\nabla u_h(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega.$$

Then we have

$$\frac{1}{\nu}\mathcal{L}(x, t_h u_h, \nabla u_h(x)) \to \frac{1}{\nu}\mathcal{L}_\infty(x, \nabla u(x)) \quad \text{for a.e. } x \in \Omega.$$

Taking into account (5.25), we deduce

$$\frac{1}{\nu}\int_\Omega \mathcal{L}(x, t_h u_h, \nabla u_h)\, dx \to \frac{1}{\nu}\int_\Omega \mathcal{L}_\infty(x, \nabla u)\, dx,$$

that by $\nu|\nabla u_h|^p \leq \mathcal{L}(x, t_h u_h, \nabla u_h)$ yields

$$\lim_h \int_\Omega |\nabla u_h|^p\, dx = \int_\Omega |\nabla u|^p\, dx,$$

namely the convergence of u_h to u in $W_0^{1,p}(\Omega)$. □

Remark 5.8. Assume that $\beta < \lambda_1 < \alpha$. Then the following facts hold:

(a) $f'_\infty(\overline{\phi_1})(\phi_1) = 0$;

(b) $\lim_{s\to-\infty} f_\infty(s\phi_1) = -\infty$, where we have set $\overline{\phi_1} = \frac{\phi_1}{(\lambda_1-\beta)^{\frac{1}{p-1}}}$.

Proof. (a) It is easy to prove. (b) A direct computation yields that for $s < 0$

$$f_\infty(s\phi_1) = \frac{\lambda_1 - \alpha}{p}|s|^p - s.$$

Since $\alpha > \lambda_1$, assertion (b) follows. □

Lemma 5.9. *For every $M > 0$ there exists $\varrho > 0$ such that for each $w \in W_0^{1,p}(\Omega)$ with $\|w - \phi_1\|_{1,p} \le \varrho$ we have*

$$\int_\Omega \mathcal{L}_\infty(x, -\nabla w^-)\,dx \ge M \int_\Omega (w^-)^p\,dx.$$

For the proof of the above lemma, we argue as in [31, Lemma 4.1].

Lemma 5.10. *There exists $r > 0$ such that*

(a) *for each $w \in W_0^{1,p}(\Omega)$, $\|w - \overline{\phi_1}\|_{1,p} \le r \Rightarrow f_\infty(w) \ge f_\infty(\overline{\phi_1})$;*
(b) *for each $w \in W_0^{1,p}(\Omega)$, $\|w - \overline{\phi_1}\|_{1,p} = r \Rightarrow f_\infty(w) > f_\infty(\overline{\phi_1})$.*

Proof. Let us fix a $u \in W_0^{1,p}(\Omega)$ and define $\eta_u :]0, +\infty[\to \mathbb{R}$ by setting $\eta_u(t) = f_\infty(tu)$. It is easy to verify that η_u assumes the minimum value:

$$\mathcal{M}(u) = -\big(1 - \frac{1}{p}\big)\big(\frac{1}{p}\big)^{\frac{1}{p-1}} \times \frac{\Big[\int_\Omega \phi_1^{p-1} u\,dx\Big]^{\frac{p}{p-1}}}{\Big[\int_\Omega \mathcal{L}_\infty(x, \nabla u)\,dx - \frac{\beta}{p}\int_\Omega (u^+)^p\,dx - \frac{\alpha}{p}\int_\Omega (u^-)^p\,dx\Big]^{\frac{1}{p-1}}}.$$

Moreover, a direct computation yields for each $u \ne \overline{\phi_1}$

$$f_\infty(\overline{\phi_1}) < \mathcal{M}(u) \tag{5.26}$$

if and only if

$$p\int_\Omega \mathcal{L}_\infty(x, \nabla u)\,dx > \beta \int_\Omega (u^+)^p\,dx + \alpha \int_\Omega (u^-)^p\,dx + (\lambda_1 - \beta)\Big[\int_\Omega \phi_1^{p-1} u\,dx\Big]^p. \tag{5.27}$$

If we now set $W = \Big\{u \in W_0^{1,p}(\Omega) : \int_\Omega \phi_1^{p-1} u\,dx = 0\Big\}$, we obtain

$$W_0^{1,p}(\Omega) = \operatorname{span}(\phi_1) \oplus W. \tag{5.28}$$

Let us now prove that (5.27) is fulfilled in a neighborhood of $\overline{\phi_1}$. Since (5.27) is homogeneous of degree p, we may substitute $\overline{\phi_1}$ with ϕ_1. Let us first consider the case $p \ge 2$ and $\beta > 0$. In view of (5.28), by strict convexity, there exists $\varepsilon_p > 0$ such that for any $w \in W$

$$\begin{aligned}
&\beta \int_\Omega ((\phi_1 + w)^+)^p\,dx + (\lambda_1 - \beta)\int_\Omega \phi_1^p\,dx \\
&\le \beta \int_\Omega ((\phi_1 + w)^+)^p\,dx + (\lambda_1 - \beta)\int_\Omega |\phi_1 + w|^p\,dx - (\lambda_1 - \beta)\varepsilon_p \int_\Omega |w|^p\,dx \\
&\le \frac{\beta}{\lambda_1} p \int_\Omega \mathcal{L}_\infty(x, \nabla(\phi_1 + w)^+)\,dx + \frac{\lambda_1 - \beta}{\lambda_1} p \int_\Omega \mathcal{L}_\infty(x, \nabla(\phi_1 + w))\,dx \\
&\quad - (\lambda_1 - \beta)\varepsilon_p \int_\Omega |w|^p\,dx.
\end{aligned} \tag{5.29}$$

On the other hand, by Lemma (5.9), for a sufficiently large M we get

$$\begin{aligned}\alpha\int_\Omega((\phi_1+w)^-)^p\,dx &\le \frac{1}{M}\int_\Omega \mathcal{L}_\infty(x,-\nabla(\phi_1+w)^-)\,dx\\ &\le \frac{\beta}{\lambda_1}p\int_\Omega \mathcal{L}_\infty(x,-\nabla(\phi_1+w)^-)\,dx\,,\end{aligned}\tag{5.30}$$

for $\|w\|_{1,p}$ small enough. By combining (5.29) and (5.30) we obtain

$$\begin{aligned}&\beta\int_\Omega((\phi_1+w)^+)^p\,dx+\alpha\int_\Omega((\phi_1+w)^-)^p\,dx+(\lambda_1-\beta)\int_\Omega\phi_1^p\,dx\\ &\le p\int_\Omega\mathcal{L}_\infty(x,\nabla(\phi_1+w))\,dx-(\lambda_1-\beta)\varepsilon_p\int_\Omega|w|^p\,dx.\end{aligned}\tag{5.31}$$

Therefore, (5.27) holds in a neighborhood of $\overline{\phi_1}$. In view of (4.4) of [95, Lemma 4.2], the case $1<p<2$ may be treated in a similar fashion. Let us now note that

$$\int_\Omega|\phi_1+w|^p\,dx\ge\int_\Omega\phi_1^p\,dx\quad\forall w\in W.$$

In the case $\beta\le 0$, we have

$$\begin{aligned}&\beta\int_\Omega\big((\phi_1+w)^+\big)^p\,dx+\alpha\int_\Omega((\phi_1+w)^-)^p\,dx+(\lambda_1-\beta)\int_\Omega\phi_1^p\,dx\\ &\le\frac{\lambda_1}{2}\int_\Omega|\phi_1+w|^p\,dx+(\alpha-\beta)\int_\Omega((\phi_1+w)^-)^p\,dx+\big(\lambda_1-\frac{\lambda_1}{2}\big)\int_\Omega\phi_1^p\,dx\end{aligned}$$

so that we reduce to (5.31). $\square$

Proposition 5.11. *Let $r>0$ as in Lemma 5.10. Then there exist $\bar t\in\mathbb{R}^+$ and $\sigma>0$ such that for each $t\ge\bar t$ and $w\in W_0^{1,p}(\Omega)$*

$$\|w-\overline{\phi_1}\|_{1,p}=r\Rightarrow\tilde f_t(w)\ge f_\infty(\overline{\phi_1})+\sigma.$$

Proof. By contradiction, let $(t_h)\subseteq\mathbb{R}$ and $(w_h)\subseteq W_0^{1,p}(\Omega)$ such that $t_h\ge h$ and

$$\|w_h-\overline{\phi_1}\|_{1,p}=r,\quad \tilde f_{t_h}(w_h)<f_\infty(\overline{\phi_1})+\frac{1}{h}.\tag{5.32}$$

Up to a subsequence we have $w_h\rightharpoonup w$ with $\|w-\overline{\phi_1}\|_{1,p}\le r$. Then, by (5.32) and (a) of the previous Lemma we get

$$\limsup_h\tilde f_{t_h}(w_h)\le f_\infty(\overline{\phi_1})\le f_\infty(w).\tag{5.33}$$

In view of (c) of Theorem 5.7, w_h strongly converges to w and then $\|w-\overline{\phi_1}\|_{1,p}=r$. By combining (5.33) with (b) of Lemma 5.10, we get a contradiction. $\square$

Proposition 5.12. *Let σ and $\bar t$ be as in the previous proposition. Then there exists $\tilde t\ge\bar t$ such that for each $t\ge\tilde t$ there exist $v_t,w_t\in W_0^{1,p}(\Omega)$ with*

$$\|v_t-\overline{\phi_1}\|_{1,p}<r,\quad f_t(v_t)\le\frac{\sigma}{2}+f_\infty(\overline{\phi_1}),\tag{5.34}$$

$$\|w_t-\overline{\phi_1}\|_{1,p}>r,\quad f_t(w_t)\le\frac{\sigma}{2}+f_\infty(\overline{\phi_1}).\tag{5.35}$$

Moreover, $\sup_{s\in[0,1]}f_t(sv_t+(1-s)w_t)<+\infty$.

Proof. We argue by contradiction. Set $\tilde{t} = \bar{t} + h$ and suppose that there exists $(t_h) \subseteq \mathbb{R}$ with $t_h \geq \tilde{t}$ such that for every v_{t_h} and w_{t_h} in $W_0^{1,p}(\Omega)$,

$$\|v_{t_h} - \overline{\phi_1}\|_{1,p} < r, \quad f_{t_h}(v_{t_h}) > \frac{\sigma}{2} + f_\infty(\overline{\phi_1}),$$
$$\|w_{t_h} - \overline{\phi_1}\|_{1,p} > r, \quad f_{t_h}(w_{t_h}) > \frac{\sigma}{2} + f_\infty(\overline{\phi_1}).$$

Take now (z_h) going strongly to $\overline{\phi_1}$ in $W_0^{1,p}(\Omega)$. By (a) of Theorem 5.7 we have $\widetilde{f}_{t_h}(z_h) \to f_\infty(\overline{\phi_1})$. On the other hand eventually $\|z_h - \overline{\phi_1}\|_{1,p} < r$ and $f_{t_h}(z_h) \leq \frac{\sigma}{2} + f_\infty(\overline{\phi_1})$, that contradicts our assumptions. Recalling (b) of Remark 5.8, by arguing as in the previous step, it is easy to prove (5.35). The last statement is straightforward. □

5.5. **Proof of the main result.** We now come to the proof of the main result of the section.

Proof of Theorem 5.1. From Theorem 5.6 we know that f_t satisfies the $(CPS)_c$ condition for any $c \in \mathbb{R}$. By Proposition 5.11 and Proposition 5.12 we may apply Theorem 2.9 with $u_0 = \overline{\phi_1}$ and obtain existence of at least two weak solutions $u \in W_0^{1,p}(\Omega)$ of problem (5.13) for $t > \bar{t}$ for a suitable $\bar{t}$.

Let us now prove that there exists $\underline{t}$ such that (5.13) has no solutions for $t < \underline{t}$. If the assertion was false, then we could find a sequence $(t_h) \subseteq \mathbb{R}$ with $t_h \to -\infty$ and a sequence (u_h) in $W_0^{1,p}(\Omega)$ such that for every $v \in C_c^\infty(\Omega)$,

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v\, dx + \int_\Omega D_s \mathcal{L}(x, u_h, \nabla u_h) v\, dx \\
&= \beta \int_\Omega (u_h^+)^{p-1} v\, dx - \alpha \int_\Omega (u_h^-)^{p-1} v\, dx + \int_\Omega g_0(x, u_h) v\, dx \\
&\quad + |t_h|^{p-2} t_h \int_\Omega \phi_1^{p-1} v\, dx + \langle \omega_0, v \rangle
\end{aligned}$$

Let us first consider the case when, up to a subsequence, $\frac{t_h}{\|u_h\|_{1,p}} \to 0$ and set $v_h = \frac{u_h}{\|u_h\|_{1,p}}$. By applying Lemma 5.2 with $\varrho_h = \|u_h\|_{1,p}$, $\delta_h = \omega_0$ and

$$\gamma_h(x) = \begin{cases} \beta & \text{if } u_h(x) \geq 0 \\ \alpha & \text{if } u_h(x) < 0, \end{cases} \qquad \mu_h = \frac{g_0(x, u_h)}{\|u_h\|_{1,p}^{p-1}} + \frac{|t_h|^{p-2} t_h}{\|u_h\|_{1,p}^{p-1}} \phi_1^{p-1},$$

up to a subsequence, (v_h) converges strongly to some v in $W_0^{1,p}(\Omega)$. Then using the same argument as in the proof of Theorem 5.6 we get a contradiction.

Assume now that there exists $M > 0$ such that $\|u_h\|_{1,p} \leq -M t_h$. Then setting $w_h = -u_h t_h^{-1}$, w_h weakly converges to some $w \in W_0^{1,p}(\Omega)$. By applying Lemma 5.2 with $\varrho_h = -t_h$, $\delta_h = \omega_0$ and

$$\gamma_h(x) = \begin{cases} \beta & \text{if } u_h(x) \geq 0 \\ \alpha & \text{if } u_h(x) < 0, \end{cases} \qquad \mu_h = -\frac{g_0(x, u_h)}{|t_h|^{p-2} t_h} - \phi_1^{p-1},$$

we have that w_h strongly converges to w in $W_0^{1,p}(\Omega)$. The choice of the test function $\varphi = w^+$ gives, as in the first case, $w^+ \equiv 0$. Arguing as in the end of the proof of Theorem 5.6 we obtain a contradiction. □

Remark 5.13. Even though we have only considered existence of weak solutions of (5.13), by [6, Lemma 1.4] the weak solutions $u \in W_0^{1,p}(\Omega)$ of (5.4) belong to $L^\infty(\Omega)$. Then some nice regularity results can be found in [91].

5.6. **Fully nonlinear variational inequalities.** Starting from the pioneering paper of Ambrosetti and Prodi [4], jumping problems for semi-linear elliptic equations of the type

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) = g(x,u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega\,,$$

have been extensively studied; see e.g. [85, 99, 101, 125]. Also the case of semi-linear variational inequalities with a situation of jumping type has been discussed in [79, 100]. Very recently, quasi-linear inequalities of the form

$$\begin{aligned}
&\int_\Omega \Big\{ \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j(v-u) \\
&+ \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u (v-u)\Big\}\, dx - \int_\Omega g(x,u)(v-u)\, dx \\
&\ge \langle \omega, v-u\rangle \quad \forall v \in \widetilde{K}_\vartheta\,, \\
&u \in K_\vartheta\,,
\end{aligned}$$

where $K_\vartheta = \{u \in H_0^1(\Omega) : u \ge \vartheta \text{ a.e. in } \Omega\}$, $\widetilde{K}_\vartheta = \{v \in K_\vartheta : (v-u) \in L^\infty(\Omega)\}$ and $\vartheta \in H_0^1(\Omega)$, have been considered in [78].

When $\vartheta \equiv -\infty$, namely we have no obstacle and the variational inequality becomes an equation, the problem has been also studied in [31, 34] by A. Canino and has been extended in [80] to the case of fully nonlinear operators.

The purpose of this section is to study the more general class of nonlinear variational inequalities of the type

$$\begin{aligned}
&\int_\Omega \Big\{\nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla(v-u) + D_s\mathcal{L}(x,u,\nabla u)(v-u)\Big\}\, dx \\
&- \int_\Omega g(x,u)(v-u)\, dx \\
&\ge \langle \omega, v-u\rangle \quad \forall v \in \widetilde{K}_\vartheta\,, \\
&u \in K_\vartheta\,.
\end{aligned} \tag{5.36}$$

In the main result we shall prove the existence of at least two solutions of (5.36). The framework is the same of [80], but technical difficulties arise, mainly in the verification of the Palais-Smale condition. This is due to the fact that such condition is proved in [80] using in a crucial way test functions of exponential type. Such test functions are not admissible for the variational inequality, so that a certain number of modifications is required in particular in the proofs of Theorem 5.18 and Theorem 5.21.

5.7. **The main result.** In the following, Ω will denote a bounded domain of $\mathbb{R}^n$, $1 < p < n$, $\vartheta \in W_0^{1,p}(\Omega)$ with $\vartheta^- \in L^\infty(\Omega)$, $\omega \in W^{-1,p'}(\Omega)$ and

$$\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

is measurable in x for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$ and of class C^1 in (s,ξ) a.e. in Ω. We shall assume that $\mathcal{L}(x,s,\cdot)$ is strictly convex and for each $t \in \mathbb{R}$

$$\mathcal{L}(x,s,t\xi) = |t|^p \mathcal{L}(x,s,\xi) \tag{5.37}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$. Furthermore, we assume that:

- there exist $\nu > 0$ and $b_1 \in \mathbb{R}$ such that
$$\nu|\xi|^p \le \mathcal{L}(x,s,\xi) \le b_1|\xi|^p\,, \tag{5.38}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$;
- there exist $b_2, b_3 \in \mathbb{R}$ such that
$$|D_s\mathcal{L}(x,s,\xi)| \le b_2|\xi|^p, \tag{5.39}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$ and
$$\left|\nabla_\xi\mathcal{L}(x,s,\xi)\right| \le b_3|\xi|^{p-1}, \tag{5.40}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$;
- there exist $R > 0$ and a bounded Lipschitzian function $\psi : [R,+\infty[\to [0,+\infty[$ such that
$$s \ge R \Rightarrow D_s\mathcal{L}(x,s,\xi) \ge 0\,, \tag{5.41}$$
$$s \ge R \Rightarrow D_s\mathcal{L}(x,s,\xi) \le \psi'(s)\,\nabla_\xi\mathcal{L}(x,s,\xi)\cdot\xi\,, \tag{5.42}$$
for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$. We denote by $\overline{\psi}$ the limit of $\psi(s)$ as $s \to +\infty$.
- $g(x,s)$ is a Carathéodory function and $G(x,s) = \int_0^s g(x,\tau)\,d\tau$. We assume that there exist $a \in L^{\frac{np}{n(p-1)+p}}(\Omega)$ and $b \in L^{\frac{n}{p}}(\Omega)$ such that
$$|g(x,s)| \le a(x) + b(x)|s|^{p-1}\,, \tag{5.43}$$
for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Moreover, there exists $\alpha \in \mathbb{R}$ such that
$$\lim_{s\to+\infty}\frac{g(x,s)}{s^{p-1}} = \alpha\,, \tag{5.44}$$
for a.e. $x \in \Omega$.

Set now
$$\lim_{s\to+\infty}\mathcal{L}(x,s,\xi) = \mathcal{L}_\infty(x,\xi)$$
(this limit exists by (5.41)). We also assume that $\mathcal{L}_\infty(x,\cdot)$ is strictly convex for a.e. $x \in \Omega$. Let us remark that we are not assuming the strict convexity uniformly in x so that such $\mathcal{L}_\infty$ is pretty general. Moreover, assume that
$$s_h \to +\infty, \quad \xi_h \to \xi \Rightarrow \nabla_\xi\mathcal{L}(x,s_h,\xi_h) \to \nabla_\xi\mathcal{L}_\infty(x,\xi)\,, \tag{5.45}$$
for a.e. $x \in \Omega$. Let now
$$\lambda_1 = \min\Big\{p\int_\Omega \mathcal{L}_\infty(x,\nabla u)\,dx : u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p\,dx = 1\Big\}, \tag{5.46}$$
be the first (nonlinear) eigenvalue of
$$u \mapsto -\operatorname{div}\big(\nabla_\xi\mathcal{L}_\infty(x,\nabla u)\big)\,.$$
Observe that by [6, Lemma 1.4] the first eigenfunction ϕ_1 belongs to $L^\infty(\Omega)$ and by [143, Theorem 1.1] is strictly positive.

Our purpose is to study (5.36) when $\omega = -t^{p-1}\phi_1^{p-1}$, namely the family of problems
$$\begin{gathered}\int_\Omega\Big\{\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla(v-u) + D_s\mathcal{L}(x,u,\nabla u)(v-u)\Big\}\,dx\\ -\int_\Omega g(x,u)(v-u)\,dx + t^{p-1}\int_\Omega \phi_1^{p-1}(v-u)\,dx \ge 0 \quad \forall v \in \widetilde{K}_\vartheta,\\ u \in K_\vartheta,\end{gathered} \tag{5.47}$$

where $K_\vartheta = \{u \in W_0^{1,p}(\Omega) : u \geq \vartheta \text{ a.e. in } \Omega\}$ and

$$\widetilde{K}_\vartheta = \{v \in K_\vartheta : (v-u) \in L^\infty(\Omega)\}.$$

Under the above assumptions, the following is our main result.

Theorem 5.14. *Assume that $\alpha > \lambda_1$. Then there exists $\bar{t} \in \mathbb{R}$ such that for all $t \geq \bar{t}$ the problem (5.47) has at least two solutions.*

This result extends [78, Theorem 2.1] dealing with Lagrangians of the type

$$\mathcal{L}(x,s,\xi) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j - G(x,s)$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

In this particular case, existence of at least three solutions has been proved in [78] assuming $\alpha > \mu_2$ where μ_2 is the second eigenvalue of the operator

$$u \mapsto -\sum_{i,j=1}^{n} D_j(A_{ij}D_i u).$$

In our general setting, since $\mathcal{L}_\infty$ is not quadratic with respect to ξ, we only have the existence of the first eigenvalue λ_1 and it is not clear how to define higher order eigenvalues $\lambda_2, \lambda_3, \dots$. Therefore in our case the comparison of α with higher eigenvalues has no obvious formulation.

5.8. **The bounded Palais-Smale condition.** In this section we shall consider the more general variational inequalities (5.36). To this aim let us now introduce the functional $f : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$

$$f(u) = \begin{cases} \int_\Omega \mathcal{L}(x,u,\nabla u)\,dx - \int_\Omega G(x,u)\,dx - \langle \omega, u\rangle & u \in K_\vartheta \\ +\infty & u \notin K_\vartheta. \end{cases}$$

Definition 5.15. Let $c \in \mathbb{R}$. A sequence (u_h) in K_ϑ is said to be a concrete Palais-Smale sequence at level c, ($(CPS)_c$-sequence, for short) for f, if $f(u_h) \to c$ and there exists a sequence (φ_h) in $W^{-1,p'}(\Omega)$ such that $\varphi_h \to 0$ and

$$\begin{aligned} &\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(v-u_h)\,dx + \int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)(v-u_h)\,dx \\ &- \int_\Omega g(x,u_h)(v-u_h)\,dx - \langle \omega, v-u_h\rangle \\ &\geq \langle \varphi_h, v-u_h\rangle \quad \forall v \in \widetilde{K}_\vartheta. \end{aligned}$$

We say that f satisfies the concrete Palais-Smale condition at level c, ($(CPS)_c$, for short), if every $(CPS)_c$-sequence for f admits a strongly convergent subsequence in $W_0^{1,p}(\Omega)$.

Theorem 5.16. *Let u in K_ϑ be such that $|df|(u) < +\infty$. Then there exists φ in $W^{-1,p'}(\Omega)$ such that $\|\varphi\|_{-1,p'} \leq |df|(u)$ and*

$$\begin{aligned} &\int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla(v-u)\,dx + \int_\Omega D_s\mathcal{L}(x,u,\nabla u)(v-u)\,dx \\ &- \int_\Omega g(x,u)(v-u)\,dx - \langle \omega, v-u\rangle \\ &\geq \langle \varphi, v-u\rangle \quad \forall v \in \widetilde{K}_\vartheta. \end{aligned}$$

For the proof of the above theorem, we argue as in [78, Theorem 4.6].

Proposition 5.17. *Let $c \in \mathbb{R}$ and assume that f satisfies the $(CPS)_c$ condition. Then f satisfies the $(PS)_c$ condition.*

The above result is an easy consequence of Theorem 5.16.

Let us note that by combining (5.38) with the convexity of $\mathcal{L}(x, s, \cdot)$, we get

$$\nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi \geq \nu |\xi|^p \tag{5.48}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Moreover, there exists $M > 0$ such that

$$|D_s \mathcal{L}(x, s, \xi)| \leq M \nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi \tag{5.49}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

We point out that assumption (5.41) may be strengthened without loss of generality. Suppose that $\vartheta(x) > -R$ for a.e. $x \in \Omega$ and define

$$\widetilde{\mathcal{L}}(x, s, \xi) = \begin{cases} \mathcal{L}(x, s, \xi) & s > -R \\ \mathcal{L}(x, -R, \xi) & s \leq -R. \end{cases}$$

Such $\widetilde{\mathcal{L}}$ satisfy our assumptions. On the other hand, if u satisfies

$$\begin{gathered} \int_\Omega \Big\{ \nabla_\xi \widetilde{\mathcal{L}}(x, u, \nabla u) \cdot \nabla(v - u) + D_s \widetilde{\mathcal{L}}(x, u, \nabla u)(v - u) \Big\}\, dx \\ - \int_\Omega g(x, u)(v - u)\, dx + t^{p-1} \int_\Omega \phi_1^{p-1}(v - u)\, dx \geq 0 \quad \forall v \in \widetilde{K}_\vartheta, \\ u \in K_\vartheta, \end{gathered} \tag{5.50}$$

then u satisfies (5.47). Therefore, up to substituting $\mathcal{L}$ with $\widetilde{\mathcal{L}}$, we can assume that $\mathcal{L}$ satisfies (5.41) for any $s \in \mathbb{R}$ with $|s| > R$. (Actually $\widetilde{\mathcal{L}}$ is only locally Lipschitz in s but one might always define $\widetilde{\mathcal{L}}(x, s, \xi) = \mathcal{L}(x, \sigma(s), \xi)$ for a suitable smooth function σ).

Now, we want to provide in Theorem 5.19 a very useful criterion for the verification of $(CPS)_c$ condition. Let us first prove a local compactness property for $(CPS)_c$-sequences.

Theorem 5.18. *Let (u_h) be a sequence in K_ϑ and (φ_h) a sequence in $W^{-1,p'}(\Omega)$ such that (u_h) is bounded in $W_0^{1,p}(\Omega)$, $\varphi_h \to \varphi$ and*

$$\begin{gathered} \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(v - u_h)\, dx + \int_\Omega D_s \mathcal{L}(x, u_h, \nabla u_h)(v - u_h)\, dx \\ \geq \langle \varphi_h, v - u_h \rangle \quad \forall v \in \widetilde{K}_\vartheta. \end{gathered} \tag{5.51}$$

Then it is possible to extract a subsequence (u_{h_k}) strongly convergent in $W_0^{1,p}(\Omega)$.

Proof. Up to a subsequence, (u_h) converges to some u weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω. Moreover, arguing as in step I of [78, Theorem 4.18] it follows that

$$\nabla u_h(x) \to \nabla u(x) \quad \text{for} \quad \text{a.e.} \quad x \in \Omega.$$

We divide the proof into several steps.

I) Let us prove that

$$\begin{gathered} \limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(-u_h^-) e^{-M(u_h - R)^-}\, dx \\ \leq \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(-u^-) e^{-M(u-R)^-}\, dx \end{gathered} \tag{5.52}$$

where $M > 0$ is defined in (5.49) and $R > 0$ has been introduced in hypothesis (5.41). Consider the test functions

$$v = u_h + \zeta e^{-M(u_h+R)^+}$$

in (5.51) where $\zeta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\zeta \geq 0$. Then

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla\zeta e^{-M(u_h+R)^+}\, dx \\
&+ \int_\Omega [D_s\mathcal{L}(x, u_h, \nabla u_h) - M\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+]\zeta e^{-M(u_h+R)^+}\, dx \\
&\geq \langle \varphi_h, \zeta e^{-M(u_h+R)^+} \rangle.
\end{aligned}$$

From (5.41) and (5.49) we deduce that

$$\left[D_s\mathcal{L}(x, u_h, \nabla u_h) - M\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\right]\zeta e^{-M(u_h+R)^+} \leq 0,$$

so that by the Fatou's Lemma we get

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\zeta e^{-M(u+R)^+}\, dx \\
&+ \int_\Omega [D_s\mathcal{L}(x, u, \nabla u) - M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+]\zeta e^{-M(u+R)^+}\, dx \\
&\geq \langle \varphi, \zeta e^{-M(u+R)^+} \rangle \quad \forall \zeta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ \zeta \geq 0.
\end{aligned} \tag{5.53}$$

Now, let us consider the functions

$$\eta_k = \eta e^{M(u+R)^+}\vartheta_k(u),$$

where $\eta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\eta \geq 0$ and $\vartheta_k \in C^\infty(\mathbb{R})$ is such that $0 \leq \vartheta_k(s) \leq 1$, $\vartheta_k = 1$ on $[-k, k]$, $\vartheta_k = 0$ outside $[-2k, 2k]$ and $|\vartheta_k'| \leq c/k$ for some $c > 0$.

Putting them in (5.53), for each $k \geq 1$ we obtain

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(\eta\vartheta_k(u))\, dx + \int_\Omega D_s\mathcal{L}(x, u, \nabla u)\eta\vartheta_k(u)\, dx \\
&\geq \langle \varphi, \eta\vartheta_k(u) \rangle \quad \forall \eta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ \eta \geq 0.
\end{aligned}$$

Passing to the limit as $k \to +\infty$ we obtain

$$\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\eta\, dx + \int_\Omega D_s\mathcal{L}(x, u, \nabla u)\eta\, dx \geq \langle \varphi, \eta \rangle \tag{5.54}$$

for all $\eta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\eta \geq 0$. Taking $\eta = (\vartheta^- - u^-)e^{-M(u-R)^-} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in (5.54) we get

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(\vartheta^- - u^-)e^{-M(u-R)^-}\, dx \\
&\geq -\int_\Omega \left[D_s\mathcal{L}(x, u, \nabla u) - M\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla(u - R)^-\right] \\
&\quad \times (\vartheta^- - u^-)e^{-M(u-R)^-}\, dx + \langle \varphi, (\vartheta^- - u^-)e^{-M(u-R)^-} \rangle.
\end{aligned} \tag{5.55}$$

On the other hand, taking

$$v = u_h - (\vartheta^- - u_h^-)e^{-M(u_h-R)^-} \geq u_h - (\vartheta^- - u_h^-) = u_h^+ - \vartheta^-$$

in (5.51) we obtain

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(\vartheta^- - u_h^-)e^{-M(u_h-R)^-}\,dx \\
&+\int_\Omega \big[D_s\mathcal{L}(x,u_h,\nabla u_h) - M\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h-R)^-\big] \\
&\times(\vartheta^- - u_h^-)e^{-M(u_h-R)^-}\,dx \\
&\le \langle \varphi_h, (\vartheta^- - u_h^-)e^{-M(u_h-R)^-}\rangle.
\end{aligned} \tag{5.56}$$

From (5.41) and (5.49) we deduce that

$$D_s\mathcal{L}(x,u_h,\nabla u_h) - M\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h-R)^- \ge 0.$$

From (5.56), using Fatou's Lemma and (5.55) we obtain

$$\begin{aligned}
&\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(\vartheta^- - u_h^-)e^{-M(u_h-R)^-}\,dx \\
&\le \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla(\vartheta^- - u^-)e^{-M(u-R)^-}\,dx.
\end{aligned} \tag{5.57}$$

Since

$$\begin{aligned}
&\lim_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\vartheta^- e^{-M(u_h-R)^-}\,dx \\
&= \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla\vartheta^- e^{-M(u-R)^-}\,dx,
\end{aligned}$$

then from (5.57) we deduce (5.52).
II) Let us now prove that

$$\begin{aligned}
&\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h^+ e^{-M(u_h-R)^-}\,dx \\
&\le \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u^+ e^{-M(u-R)^-}\,dx.
\end{aligned} \tag{5.58}$$

We consider the test functions

$$v = u_h - \big[(u_h^+ - \vartheta^+)\wedge k\big]e^{-M(u_h-R)^-} \ge \vartheta + (\vartheta^- - u_h^-)$$

in (5.51). By Fatou's Lemma, we get

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h^+ - \vartheta^+)e^{-M(u_h-R)^-}\,dx \\
&+\int_\Omega \big[D_s\mathcal{L}(x,u_h,\nabla u_h) - M\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h-R)^-\big] \\
&\times(u_h^+ - \vartheta^+)e^{-M(u_h-R)^-}\,dx \\
&\le \langle \varphi_h, (u_h^+ - \vartheta^+)e^{-M(u_h-R)^-}\rangle
\end{aligned} \tag{5.59}$$

from which we deduce that

$$\big[D_s\mathcal{L}(x,u_h,\nabla u_h) - M\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h-R)^-\big](u_h^+ - \vartheta^+)e^{-M(u_h-R)^-}$$

belongs to $L^1(\Omega)$. Using Fatou's Lemma in (5.59) we obtain

$$\begin{aligned}
&\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h^+-\vartheta^+)e^{-M(u_h-R)^-}\,dx\\
&\le -\int_\Omega \big[D_s\mathcal{L}(x,u,\nabla u)-M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla(u-R)^-\big]\\
&\quad\times(u^+-\vartheta^+)e^{-M(u-R)^-}\,dx+\langle\varphi,(u^+-\vartheta^+)e^{-M(u-R)^-}\rangle,
\end{aligned}\tag{5.60}$$

from which we also deduce that

$$\big[D_s\mathcal{L}(x,u,\nabla u)-M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla(u-R)^-\big](u^+-\vartheta^+)e^{-M(u-R)^-}\tag{5.61}$$

belongs to $L^1(\Omega)$. Now, taking $\eta_k=\big[(u^+-\vartheta^+)\wedge k\big]e^{-M(u-R)^-}$ in (5.54), we have

$$\begin{aligned}
&\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\big[(u^+-\vartheta^+)\wedge k\big]e^{-M(u-R)^-}\,dx\\
&+\int_\Omega\big[D_s\mathcal{L}(x,u,\nabla u)-M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla(u-R)^-\big]\\
&\times\big[(u^+-\vartheta^+)\wedge k\big]e^{-M(u-R)^-}\,dx\\
&\ge\Big\langle\varphi,\big[(u^+-\vartheta^+)\wedge k\big]e^{-M(u-R)^-}\Big\rangle.
\end{aligned}\tag{5.62}$$

Using (5.61) and passing to the limit as $k\to+\infty$ in (5.62), it results

$$\begin{aligned}
&\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla(u^+-\vartheta^+)e^{-M(u-R)^-}\,dx\\
&+\int_\Omega\big[D_s\mathcal{L}(x,u,\nabla u)-M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla(u-R)^-\big](u^+-\vartheta^+)e^{-M(u-R)^-}\,dx\\
&\ge\langle\varphi,(u^+-\vartheta^+)e^{-M(u-R)^-}\rangle.
\end{aligned}\tag{5.63}$$

Combining (5.63) with (5.60) we obtain

$$\begin{aligned}
&\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h^+-\vartheta^+)e^{-M(u_h-R)^-}\,dx\\
&\le\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla(u^+-\vartheta^+)e^{-M(u-R)^-}\,dx
\end{aligned}\tag{5.64}$$

Since

$$\begin{aligned}
&\lim_h\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\vartheta^+e^{-M(u_h-R)^-}\,dx\\
&=\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\vartheta^+e^{-M(u-R)^-}\,dx
\end{aligned}$$

from (5.64) we deduce (5.58).

III) Let us prove that $u_h\to u$ strongly in $W_0^{1,p}(\Omega)$. We claim that

$$\begin{aligned}
&\limsup_h\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h e^{-M(u_h-R)^-}\,dx\\
&\le\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u e^{-M(u-R)^-}\,dx.
\end{aligned}$$

In fact using (5.52) and (5.58) we get

$$\begin{aligned}
&\limsup_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h e^{-M(u_h-R)^-}\,dx \\
&\leq \limsup_h \int_{\Omega\cap\{u_h\geq 0\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h^+ e^{-M(u_h-R)^-}\,dx \\
&\quad + \limsup_h \int_{\Omega\cap\{u_h\leq 0\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla(-u_h^-) e^{-M(u_h-R)^-}\,dx \\
&\leq \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u e^{-M(u-R)^-}\,dx
\end{aligned} \tag{5.65}$$

From (5.65) using Fatou's Lemma we get

$$\begin{aligned}
&\lim_h \int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h e^{-M(u_h-R)^-}\,dx \\
&= \int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u e^{-M(u-R)^-}\,dx.
\end{aligned}$$

Therefore, since by (5.48) we have

$$\nu \exp\{-M(R+\|\vartheta^-\|_\infty)\}|\nabla u_h|^p \leq \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h e^{-M(u_h-R)^-}.$$

It follows that

$$\lim_h \int_\Omega |\nabla u_h|^p\,dx = \int_\Omega |\nabla u|^p\,dx,$$

namely the strong convergence of (u_h) to u in $W_0^{1,p}(\Omega)$. □

Theorem 5.19. *For every $c\in\mathbb{R}$ the following assertions are equivalent:*

(a) *f satisfies the $(CPS)_c$ condition ;*

(b) *every $(CPS)_c$-sequence for f is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Since the map $\{u\mapsto g(x,u)\}$ is completely continuous from $W_0^{1,p}(\Omega)$ to $L^{\frac{np'}{n+p'}}(\Omega)$, the proof goes like [78, Theorem 4.37]. □

5.9. **The Palais-Smale condition.** Let us now set

$$g_0(x,s)=g(x,s)-\alpha(s^+)^{p-1},\quad G_0(x,s)=\int_0^s g_0(x,t)dx.$$

Of course, g_0 is a Carathéodory function satisfying

$$\lim_{s\to+\infty}\frac{g_0(x,s)}{s^{p-1}}=0,\quad |g_0(x,s)|\leq a(x)+b(x)|s|^{p-1},$$

for a.e. $x\in\Omega$ and all $s\in\mathbb{R}$ where $a\in L^{\frac{np}{n(p-1)+p}}(\Omega)$ and $b\in L^{\frac{n}{p}}(\Omega)$. Then (5.47) is equivalent to finding $u\in K_\vartheta$ such that

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla(v-u)\,dx+\int_\Omega D_s\mathcal{L}(x,u,\nabla u)(v-u)\,dx \\
&-\alpha\int_\Omega (u^+)^{p-1}(v-u)\,dx-\int_\Omega g_0(x,u)(v-u)\,dx+t^{p-1}\int_\Omega \phi_1^{p-1}(v-u)\,dx\geq 0
\end{aligned}$$

for all $v\in\widetilde{K}_\vartheta$. Let us define the functional $f:W_0^{1,p}(\Omega)\to\mathbb{R}\cup\{+\infty\}$ by setting

$$f(u)=\begin{cases}\int_\Omega \mathcal{L}(x,u,\nabla u)-\frac{\alpha}{p}\int_\Omega u^{+p}-\int_\Omega G_0(x,u)+t^{p-1}\int_\Omega \phi_1^{p-1}u & \text{if } u\in K_\vartheta\\ +\infty & \text{if } u\notin K_\vartheta.\end{cases}$$

In view of Theorem 5.16, any critical point of f is a weak solutions of (P_t). Let us introduce a new functional $f_t : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by setting for each $t > 0$

$$f_t(u) = \begin{cases} \int_\Omega \mathcal{L}(x, tu, \nabla u) - \frac{\alpha}{p} \int_\Omega u^{+p} - \frac{1}{t^p} \int_\Omega G_0(x, tu) + \int_\Omega \phi_1^{p-1} u & \text{if } u \in K_t \\ +\infty & \text{if } u \notin K_t. \end{cases}$$

where we have set

$$K_t = \Big\{ u \in W_0^{1,p}(\Omega) : tu \geq \vartheta \quad \text{a.e. in } \Omega \Big\}.$$

From Theorem 5.16 it follows that if u is a critical point of f_t then tu satisfies (5.47).

Lemma 5.20. *Let (u_h) a sequence in $W_0^{1,p}(\Omega)$ and $\varrho_h \subseteq]0, +\infty[$ with $\varrho_h \to +\infty$. Assume that the sequence $\left(\frac{u_h}{\varrho_h}\right)$ is bounded in $W_0^{1,p}(\Omega)$. Then*

$$\frac{g_0(x, u_h)}{\varrho_h^{p-1}} \to 0 \quad \text{in } L^{\frac{np'}{n+p'}}(\Omega), \qquad \frac{G_0(x, u_h)}{\varrho_h^p} \to 0 \quad \text{in } L^1(\Omega).$$

To prove the above lemma, we argue as in [31, Lemma 3.3]. In view of (5.48) and (5.39), we can extend ψ to $[-N, +\infty[$ where N is such that $\|\vartheta^-\|_\infty \leq N$, so that assumption (5.42) becomes

$$s \geq -N \Rightarrow D_s\mathcal{L}(x, s, \xi) \leq \psi'(s)\, \nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi. \tag{5.66}$$

Theorem 5.21. *Let $\alpha > \lambda_1$, $c \in \mathbb{R}$ and let (u_h) in K_ϑ be a $(CPS)_c$-sequence for f. Then (u_h) is bounded in $W_0^{1,p}(\Omega)$.*

Proof. By Definition 5.15, there exists a sequence (φ_h) in $W^{-1,p'}(\Omega)$ with $\varphi_h \to 0$ and

$$\begin{aligned} &\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(v - u_h)\, dx + \int_\Omega D_s\mathcal{L}(x, u_h, \nabla u_h)(v - u_h)\, dx \\ &- \alpha \int_\Omega (u_h^+)^{p-1}(v - u_h)\, dx - \int_\Omega g_0(x, u_h)(v - u_h)\, dx + t^{p-1} \int_\Omega \phi_1^{p-1}(v - u_h)\, dx \\ &\geq \langle \varphi_h, v - u_h \rangle \quad \forall v \in K_\vartheta : (v - u_h) \in L^\infty(\Omega). \end{aligned} \tag{5.67}$$

We set now $\varrho_h = \|u_h\|_{1,p}$, and suppose by contradiction that $\varrho_h \to +\infty$. If we set $z_h = \varrho_h^{-1} u_h$, up to a subsequence, z_h converges to some z weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω. Note that $z \geq 0$ a.e. in Ω.

We shall divide the proof into several steps.

I) We firstly prove that

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla z) \cdot \nabla z\, dx \geq \alpha \int_\Omega z^p\, dx. \tag{5.68}$$

Consider the test functions $v = u_h + (z \wedge k) \exp\{-\psi(u_h)\}$, where ψ is the function defined in (5.42). Putting such v in (5.67) and dividing by ϱ_h^{p-1}, we obtain

$$\begin{aligned} &\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla z_h) \cdot \nabla(z \wedge k) \exp\{-\psi(u_h)\}\, dx \\ &+ \frac{1}{\varrho_h^{p-1}} \int_\Omega \big[D_s\mathcal{L}(x, u_h, \nabla u_h) - \psi'(u_h) \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \big] \\ &\times (z \wedge k) \exp\{-\psi(u_h)\} dx \end{aligned}$$

$$\geq \alpha \int_\Omega (z_h^+)^{p-1}(z\wedge k)\exp\{-\psi(u_h)\}\,dx + \int_\Omega \frac{g_0(x,u_h)}{\varrho_h^{p-1}}(z\wedge k)\exp\{-\psi(u_h)\}\,dx$$
$$- t^{p-1}\int_\Omega \frac{\phi_1^{p-1}}{\varrho_h^{p-1}}(z\wedge k)\exp\{-\psi(u_h)\}\,dx + \frac{1}{\varrho_h^{p-1}}\langle \varphi_h, (z\wedge k)\exp\{-\psi(u_h)\}\rangle.$$

Observe now that the first term

$$\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla z_h)\cdot\nabla(z\wedge k)\exp\{-\psi(u_h)\}\,dx$$

passes to the limit, yielding

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x,\nabla z)\cdot\nabla(z\wedge k)\exp\{-\overline{\psi}\}\,dx.$$

Indeed, by taking into account assumptions (5.45) and (5.40), we may apply [54, Theorem 5] and deduce that, up to a subsequence,

$$\text{a.e. in } \Omega\setminus\{z=0\}:\quad \nabla z_h(x)\to\nabla z(x).$$

Since of course $u_h(x)\to+\infty$ a.e. in $\Omega\setminus\{z=0\}$, again recalling (5.45), we have

$$\text{a.e. in } \Omega\setminus\{z=0\}:\quad \nabla_\xi\mathcal{L}(x,u_h(x),\nabla z_h(x))\to\nabla_\xi\mathcal{L}_\infty(x,\nabla z(x)).$$

Since by (5.40) the sequence $(\nabla_\xi\mathcal{L}(x,u_h(x),\nabla z_h(x)))$ is bounded in $L^{p'}(\Omega)$, the assertion follows. Note also that the term

$$\frac{1}{\varrho_h^{p-1}}\langle\varphi_h,(z\wedge k)\exp\{-\psi(u_h)\}\rangle\,,$$

goes to 0 even if $1<p<2$. Indeed, in this case, one could use the Cerami-Palais-Smale condition, which yields $\varrho_h\varphi_h\to 0$ in $W_0^{-1,p'}(\Omega)$.

Now, by (5.66) we have

$$D_s\mathcal{L}(x,u_h,\nabla u_h)-\psi'(u_h)\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\leq 0,$$

then, passing to the limit as $h\to+\infty$, we get

$$\int_\Omega \nabla_\xi\mathcal{L}_\infty(x,\nabla z)\cdot\nabla(z\wedge k)\exp\{-\bar\psi\}\,dx\geq\alpha\int_\Omega z^{p-1}(z\wedge k)\exp\{-\bar\psi\}\,dx.$$

Passing to the limit as $k\to+\infty$, we obtain (5.68).

II) Let us prove that $z_h\to z$ strongly in $W_0^{1,p}(\Omega)$, so that of course $\|z\|_{1,p}=1$. Consider the function $\zeta:[-R,+\infty[\to\mathbb{R}$ defined by

$$\zeta(s)=\begin{cases} MR & \text{if } s\geq R\\ Ms & \text{if } |s|<R\end{cases}\tag{5.69}$$

where $M\in\mathbb{R}$ is such that for a.e. $x\in\Omega$, each $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^n$

$$|D_s\mathcal{L}(x,s,\xi)|\leq M\nabla_\xi\mathcal{L}(x,s,\xi)\cdot\xi.$$

If we choose the test functions

$$v=u_h-\frac{u_h-\vartheta}{\exp(MR)}\exp(\zeta(u_h))$$

in (5.67), we have

$$\int_\Omega\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla(u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx$$

$$
\begin{aligned}
&+\int_\Omega \big[D_s\mathcal{L}(x,u_h,\nabla u_h)+\zeta'(u_h)\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\big](u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx\\
&\le \alpha\int_\Omega (u_h^+)^{p-1}(u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx+\int_\Omega g_0(x,u_h)(u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx\\
&\quad -t^{p-1}\int_\Omega \phi_1^{p-1}(u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx+\langle\varphi_h,(u_h-\vartheta)\exp\{\zeta(u_h)\}\rangle\,.
\end{aligned}
$$

Note that

$$
\big[D_s\mathcal{L}(x,u_h,\nabla u_h)+\zeta'(u_h)\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\big](u_h-\vartheta)\ge 0.
$$

Therefore, after division by ϱ_h^p we get

$$
\begin{aligned}
&\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla z_h)\cdot\nabla\big(z_h-\frac{\vartheta}{\varrho_h}\big)\exp\{\zeta(u_h)\}\,dx\\
&\le \alpha\int_\Omega (z_h^+)^{p-1}\big(z_h-\frac{\vartheta}{\varrho_h}\big)\exp\{\zeta(u_h)\}\,dx\\
&\quad+\frac{1}{\varrho_h^{p-1}}\int_\Omega g_0(x,u_h)\big(z_h-\frac{\vartheta}{\varrho_h}\big)\exp\{\zeta(u_h)\}\,dx\\
&\quad-\frac{t^{p-1}}{\varrho_h^{p-1}}\int_\Omega \phi_1^{p-1}\big(z_h-\frac{\vartheta}{\varrho_h}\big)\exp\{\zeta(u_h)\}\,dx+\frac{1}{\varrho_h^{p-1}}\Big\langle\varphi_h,\big(z_h-\frac{\vartheta}{\varrho_h}\big)\exp\{\zeta(u_h)\}\Big\rangle\,,
\end{aligned}
$$

which yields

$$
\limsup_h\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx\le\alpha\exp\{MR\}\int_\Omega z^p\,dx. \tag{5.70}
$$

By combining (5.70) with (5.68) we get

$$
\begin{aligned}
&\limsup_h\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx\\
&\le\exp\{MR\}\int_\Omega \nabla_\xi\mathcal{L}_\infty(x,\nabla z)\cdot\nabla z\,dx
\end{aligned}
$$

In particular, by Fatou's Lemma, it results

$$
\begin{aligned}
&\exp\{MR\}\int_\Omega \nabla_\xi\mathcal{L}_\infty(x,\nabla z)\cdot\nabla z\,dx\\
&\le\liminf_h\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx\\
&\le\limsup_h\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx\\
&\le\exp\{MR\}\int_\Omega \nabla_\xi\mathcal{L}_\infty(x,\nabla z)\cdot\nabla z\,dx,
\end{aligned}
$$

namely, we get

$$
\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx\to\int_\Omega \exp\{MR\}\nabla_\xi\mathcal{L}_\infty(x,\nabla z)\cdot\nabla z\,dx.
$$

Therefore, since

$$
\nu\exp\{-MR\}|\nabla z_h|^p\le\nabla_\xi\mathcal{L}(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,,
$$

thanks to the generalized Lebesgue's theorem, we conclude that

$$\lim_h \int_\Omega |\nabla z_h|^p \, dx = \int_\Omega |\nabla z|^p \, dx,$$

and z_h converges to z in $W_0^{1,p}(\Omega)$.

III) Let us consider the test functions $v = u_h + \varphi \exp\{-\psi(u_h)\}$ such that φ in $W_0^{1,p} \cap L^\infty(\Omega)$ and $\varphi \geq 0$. Taking such v in (5.67) and dividing by ϱ_h^{p-1} we obtain

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla z_h) \cdot \nabla \varphi \exp\{-\psi(u_h)\} \, dx \\
&+ \frac{1}{\varrho_h^{p-1}} \int_\Omega \big[D_s \mathcal{L}(x, u_h, \nabla u_h) - \psi'(u_h) \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h\big] \varphi \exp\{-\psi(u_h)\} \, dx \\
&\geq \alpha \int_\Omega (z_h^+)^{p-1} \varphi \exp\{-\psi(u_h)\} \, dx + \int_\Omega \frac{g_0(x, u_h)}{\varrho_h^{p-1}} \varphi \exp\{-\psi(u_h)\} \, dx \\
&\quad - t^{p-1} \int_\Omega \frac{\varphi^{p-1}}{\varrho_h^{p-1}} \exp\{-\psi(u_h)\} \, dx + \frac{1}{\varrho_h^{p-1}} \langle \varphi_h, \varphi \exp\{-\psi(u_h)\} \rangle .
\end{aligned}$$

Note that, since by step II we have $z_h \to z$ in $W_0^{1,p}(\Omega)$, the term

$$\int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla z_h) \cdot \nabla \varphi \exp\{-\psi(u_h)\} \, dx$$

passes to the limit, yielding

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \varphi \exp\{-\overline{\psi}\} \, dx.$$

By means of (5.66), we have

$$D_s \mathcal{L}(x, u_h, \nabla u_h) - \psi'(u_h) \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \leq 0,$$

then passing to the limit as $h \to +\infty$, we obtain

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \varphi \exp\{-\bar{\psi}\} \, dx - \alpha \int_\Omega z^{p-1} \varphi \exp\{-\bar{\psi}\} \, dx \geq 0 \, ,$$

for each $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ with $\varphi \geq 0$ which yields

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \varphi \, dx \geq \alpha \int_\Omega z^{p-1} \varphi \, dx \tag{5.71}$$

for each $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$.

In a similar fashion, considering in (5.67) the admissible test functions

$$v = u_h - \Big(\varphi \wedge \frac{z_h - \vartheta/\varrho_h}{\exp(\overline{\psi})}\Big) \exp(\psi(u_h))$$

with $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ and $\varphi \geq 0$ and dividing by ϱ_h^{p-1}, recalling that $z_h \to z$ strongly, we get

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla z) \cdot \nabla\Big[\varphi \wedge \frac{z}{\exp \overline{\psi}}\Big] \, dx \leq \alpha \int_\Omega z^{p-1} \Big[\varphi \wedge \frac{z}{\exp \overline{\psi}}\Big] \, dx \, ,$$

for each $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ with $\varphi \geq 0$. Actually this holds for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$. By substituting φ with $t\varphi$ with $t > 0$ we obtain

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla z) \cdot \nabla\Big[\varphi \wedge \frac{z}{t \exp \overline{\psi}}\Big]\, dx \leq \alpha \int_\Omega z^{p-1}\Big[\varphi \wedge \frac{z}{t \exp \overline{\psi}}\Big]\, dx.$$

Letting $t \to +\infty$, and taking into account (5.71), it results

$$\int_\Omega \nabla_\xi \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \varphi \, dx = \alpha \int_\Omega z^{p-1} \varphi \, dx \tag{5.72}$$

for each $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$. Clearly (5.72) holds for any $\varphi \in W_0^{1,p}(\Omega)$, so that z is a positive eigenfunction related to α. This is a contradiction by [95, Remark 1, pp. 161]. □

Theorem 5.22. *Let $c \in \mathbb{R}$, $\alpha > \lambda_1$ and $t > 0$. Then f_t satisfies the $(PS)_c$-condition.*

Proof. Since $f_t(u) = \frac{f(tu)}{t^p}$, it is sufficient to combine Theorem 5.21, Theorem 5.19 and Proposition 5.17. □

5.10. **Min-Max estimates.** Let us first introduce the "asymptotic functional" $f_\infty : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f_\infty(u) = \begin{cases} \int_\Omega \mathcal{L}_\infty(x, \nabla u)\, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx + \int_\Omega \phi_1^{p-1} u \, dx & \text{if } u \in K_\infty \\ +\infty & \text{if } u \notin K_\infty \end{cases}$$

where

$$K_\infty = \Big\{ u \in W_0^{1,p}(\Omega) : u \geq 0 \quad \text{a.e. in } \Omega \Big\}.$$

Proposition 5.23. *There exist $r > 0$, $\sigma > 0$ such that*

(*a*) *for every $u \in W_0^{1,p}(\Omega)$ with $0 < \|u\|_{1,p} \leq r$ then $f_\infty(u) > 0$;*

(*b*) *for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{1,p} = r$ then $f_\infty(u) \geq \sigma > 0$.*

Proof. Let us consider the weakly closed set

$$K^* = \Big\{ u \in K_\infty : \int_\Omega \mathcal{L}_\infty(x, \nabla u)\, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx \leq \frac{1}{2} \int_\Omega \mathcal{L}_\infty(x, \nabla u)\, dx \Big\}.$$

In $K_\infty \setminus K^*$ the statements are evident. On the other hand, it is easy to see that

$$\inf \Big\{ \int_\Omega v \phi_1^{p-1} \, dx : v \in K^*, \quad \|v\|_{1,p} = 1 \Big\} = \varepsilon > 0$$

arguing by contradiction. Therefore for each $u \in K^*$ we have

$$f_\infty(u) = \int_\Omega \mathcal{L}_\infty(x, \nabla u)\, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx + \int_\Omega \phi_1^{p-1} u \, dx \geq c\|u\|_{1,p}^p + \varepsilon \|u\|_{1,p}$$

where $c \in \mathbb{R}$ is a suitable constant. Thus the statements follow. □

Proposition 5.24. *Let $r > 0$ be as in the Proposition 5.23. Then there exist $\bar{t} > 0$, $\sigma' > 0$ such that for every $t \geq \bar{t}$ and for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{1,p} = r$, then $f_t(u) \geq \sigma'$.*

Proof. By contradiction, we can find two sequences $(t_h) \subset \mathbb{R}$ and $(u_h) \subset W_0^{1,p}(\Omega)$ such that $t_h \geq h$ for each $h \in \mathbb{N}$, $\|u_h\|_{1,p} = r$ and $f_{t_h}(u_h) < \frac{1}{h}$. Up to a subsequence, (u_h) weakly converges in $W_0^{1,p}(\Omega)$ to some $u \in K_\infty$. Using (*b*) of [80, Theorem 5], it follows that

$$f_\infty(u) \leq \liminf_h f_{t_h}(u_h) \leq 0.$$

By (a) of Proposition 5.23, we have $u = 0$. On the other hand, since

$$\limsup_h f_{t_h}(u_h) \le 0 = f_\infty(u)\,,$$

using (c) of [80, Theorem 5] we deduce that (u_h) strongly converges to u in $W_0^{1,p}(\Omega)$, namely $\|u\|_{1,p} = r$. This is impossible. □

Proposition 5.25. *Let $\sigma', \bar t$ as in Proposition 5.24. Then there exists $\tilde t \ge \bar t$ such that for every $t \ge \tilde t$ there exist $v_t, w_t \in W_0^{1,p}(\Omega)$ such that $\|v_t\|_{1,p} < r$, $\|w_t\|_{1,p} > r$, $f_t(v_t) \le \frac{\sigma'}{2}$ and $f_t(w_t) \le \frac{\sigma'}{2}$. Moreover we have*

$$\sup\{f_t((1-s)v_t + sw_t) : 0 \le s \le 1\} < +\infty.$$

Proof. We argue by contradiction. We set $\tilde t = \bar t + h$ and suppose that there exists (t_h) such that $t_h \ge h + \bar t$ and such that for every v_{t_h}, w_{t_h} in $W_0^{1,p}(\Omega)$ with $\|v_{t_h}\|_{1,p} < r$, $\|w_{t_h}\|_{1,p} > r$ it results $f_{t_h}(v_{t_h}) > \frac{\sigma'}{2}$ and $f_{t_h}(w_{t_h}) > \frac{\sigma'}{2}$. It is easy to prove that there exists a sequence (u_h) in K_{t_h} which strongly converges to 0 in $W_0^{1,p}(\Omega)$ and therefore $\|u_h\|_{1,p} < r$ and $f_{t_h}(u_{t_h}) \le \frac{\sigma'}{2}$ eventually as $h \to +\infty$. This contradicts our assumptions. In a similar way one can prove the statement for w_t, while the last statement is straightforward. □

5.11. **Proof of the main result.**

Proof of Theorem 5.14. By combining Theorem 5.22, propositions 5.24 and 5.25 we can apply Theorem 2.9 and deduce the assertion. □

6. Problems with Loss of Compactness

The material in this section comes from [128, 131, 73, 108], to which refer the reader. Some parts of these publications have been slightly modified to give this collection a more uniform appearance.

6.1. **Positive entire solutions for fully nonlinear problems.** In the last few years there has been a growing interest in the study of positive solutions to variational quasi-linear equations in unbounded domains of $\mathbb{R}^n$, since these problems are involved in various branches of mathematical physics (see [20]). Since 1988, quasi-linear elliptic equations of the form

$$-\operatorname{div}(\varphi(\nabla u)) = g(x,u) \quad \text{in } \mathbb{R}^n, \tag{6.1}$$

have been extensively treated, among the others, in [14, 45, 69, 94, 145] by means of a combination of topological and variational techniques. Moreover, existence of a positive solution $u \in H^1(\mathbb{R}^n)$ for the more general equation

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_i u D_j u + b(x)u = g(x,u) \quad \text{in } \mathbb{R}^n,$$

behaving asymptotically ($|x| \to +\infty$) like the problem

$$-\Delta u + \lambda u = u^{q-1} \quad \text{in } \mathbb{R}^n\,,$$

for some suitable $\lambda > 0$ and $q > 2$, has been firstly studied in 1996 in [48] via techniques of non-smooth critical point theory. On the other hand, more recently, in a bounded domain Ω of $\mathbb{R}^n$ some existence results for fully nonlinear problems of the type

$$\begin{gathered} -\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) = g(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{6.2}$$

have been established in [6, 113, 133]. The goal of this section is to prove existence of a nontrivial positive solution in $W^{1,p}(\mathbb{R}^n)$ for the nonlinear elliptic equation

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) + b(x)|u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^n, \tag{6.3}$$

behaving asymptotically like the p-Laplacian problem

$$-\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) + \lambda|u|^{p-2}u = u^{q-1} \quad \text{in } \mathbb{R}^n,$$

for some suitable $\lambda > 0$ and $q > p$. In other words, equation (6.3) tends to regularize as $|x| \to +\infty$ together with its associated functional $f : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$

$$f(u) = \int_{\mathbb{R}^n} \mathcal{L}(x,u,\nabla u)\,dx + \frac{1}{p}\int_{\mathbb{R}^n} b(x)|u|^p\,dx - \int_{\mathbb{R}^n} G(x,u)\,dx. \tag{6.4}$$

Since in general f is continuous but not even locally Lipschitzian, unless $\mathcal{L}$ does not depend on u or the growth conditions on $\mathcal{L}$ are very restrictive, we shall refer to the non-smooth critical point theory developed in [36, 50, 58, 87, 88] and we shall follow the approach of [48].

We assume that $1 < p < n$, the function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class C^1 in (s,ξ) for a.e. $x \in \mathbb{R}^n$ and $\mathcal{L}(x,s,\cdot)$ is strictly convex and homogeneous of degree p. Take $b \in L^\infty(\mathbb{R}^n)$ with $\mathbf{b} \le b(x) \le \overline{b}$ for a.e. $x \in \mathbb{R}^n$ for some $\mathbf{b}, \overline{b} > 0$. We shall assume the following:

- There exists $\nu > 0$ such that
$$\nu|\xi|^p \le \mathcal{L}(x,s,\xi) \le \frac{1}{p}|\xi|^p\,, \tag{6.5}$$
for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$;
- there exists $c_1 > 0$ such that:
$$|D_s\mathcal{L}(x,s,\xi)| \le c_1|\xi|^p\,, \tag{6.6}$$
for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$.
Moreover, there exist $c_2 > 0$ and $a \in L^{p'}(\mathbb{R}^n)$ such that
$$\big|\nabla_\xi\mathcal{L}(x,s,\xi)\big| \le a(x) + c_2|s|^{\frac{p^*}{p'}} + c_2|\xi|^{p-1}, \tag{6.7}$$
for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$;
- there exists $R > 0$ such that
$$s \ge R \Rightarrow D_s\mathcal{L}(x,s,\xi)s \ge 0\,, \tag{6.8}$$
for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$.
- uniformly in $s \in \mathbb{R}$ and $\xi,\eta \in \mathbb{R}^n$ with $|\xi| \le 1$ and $|\eta| \le 1$
$$\lim_{|x|\to+\infty} \nabla_\xi\mathcal{L}(x,s,\xi)\cdot\eta = |\xi|^{p-2}\xi\cdot\eta\,, \tag{6.9}$$
$$\lim_{|x|\to+\infty} D_s\mathcal{L}(x,s,\xi)s = 0\,, \tag{6.10}$$
$$\lim_{|x|\to+\infty} b(x) = \lambda\,, \tag{6.11}$$
for some $\lambda > 0$ and with $b(x) \le \lambda$ for a.e. $x \in \mathbb{R}^n$.
- $G : \mathbb{R}^n\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $G(x,s) = \int_0^s g(x,t)\,dt$ and there exist $\beta > 0$ and $q > p$ such that
$$s > 0 \Rightarrow 0 < qG(x,s) \le g(x,s)s\,, \tag{6.12}$$
$$(q-p)\mathcal{L}(x,s,\xi) - D_s\mathcal{L}(x,s,\xi)s \ge \beta|\xi|^p, \tag{6.13}$$
for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$. Moreover there exist $\sigma \in]p,p^*[$ and $c > 0$ such that:
$$|g(x,s)| \le d(x) + c|s|^{\sigma-1}\,, \tag{6.14}$$
for a.e. $x \in \mathbb{R}^n$ and all $s > 0$, where $d \in L^r(\mathbb{R}^n)$ with $r \in \Big[\frac{np'}{n+p'}, p'\Big[$.
- Also
$$\lim_{|x|\to+\infty} \frac{g(x,s)}{s^{q-1}} = 1, \tag{6.15}$$
uniformly in $s > 0$ and
$$\lim_{|s|\to 0} \frac{G(x,s)}{|s|^p} = 0, \tag{6.16}$$
uniformly in $x \in \mathbb{R}^n$ and $g(x,s) \ge s^{q-1}$ for each $s > 0$.

Under the above assumptions, the following is our main result.

Theorem 6.1. *The Euler's equation of f*
$$-\operatorname{div}\big(\nabla_\xi\mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) + b|u|^{p-2}u = g(x,u) \quad \textit{in } \mathbb{R}^n \tag{6.17}$$
admits at least one nontrivial positive solution $u \in W^{1,p}(\mathbb{R}^n)$.

This result extends to a more general setting [48, Theorem 2] dealing with the case

$$\mathcal{L}(x,s,\xi)=\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}(x,s)\xi_i\xi_j\,,$$

and Theorem 2.1 of [45] involving integrands of the type:

$$\mathcal{L}(x,\xi)=\frac{1}{p}a(x)|\xi|^p\,,$$

where $a\in L^\infty(\mathbb{R})$ and $1<p<n$. Let us remark that we assume (6.8) for large values of s, while in [48] it was supposed that for a.e. $x\in\mathbb{R}^n$ and all $\xi\in\mathbb{R}^n$

$$\forall s\in\mathbb{R}:\sum_{i,j=1}^{n}sD_sa_{ij}(x,s)\xi_i\xi_j\geq 0.$$

This assumption has been widely considered in literature, not only in studying existence but also to ensure local boundedness of weak solutions (see e.g. [6]).

Condition (6.13) has been already used in [6, 113, 133] and seems to be a natural extension of what happens in the quasi-linear case [36].

We point out that in a bounded domain, conditions (6.12) and (6.13) may be assumed for large values of s (see e.g. [133]). Finally (6.9), (6.10), (6.11) and (6.15) fix the asymptotic behavior of (6.3). By (6.9) and (6.10) there exist two maps $\varepsilon_1:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $\varepsilon_2:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ such that

$$\nabla_\xi\mathcal{L}(x,s,\xi)\cdot\eta=|\xi|^{p-2}\xi\cdot\eta+\varepsilon_1(x,s,\xi,\eta)|\xi|^{p-1}|\eta| \tag{6.18}$$

$$D_s\mathcal{L}(x,s,\xi)s=\varepsilon_2(x,s,\xi)|\xi|^p \tag{6.19}$$

where $\varepsilon_1(x,s,\xi,\eta)\to 0$ and $\varepsilon_2(x,s,\xi)\to 0$ as $|x|\to+\infty$ uniformly in $s\in\mathbb{R}$ and $\xi,\eta\in\mathbb{R}^n$.

6.2. **The concrete Palais-Smale condition.** Let us now set for a.e. $x\in\mathbb{R}^n$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$:

$$\widetilde{\mathcal{L}}(x,s,\xi)=\begin{cases}\mathcal{L}(x,s,\xi) & \text{if } s\geq 0\\ \mathcal{L}(x,0,\xi) & \text{if } s<0\end{cases}\qquad \tilde{g}(x,s)=\begin{cases}g(x,s) & \text{if } s\geq 0\\ 0 & \text{if } s<0.\end{cases} \tag{6.20}$$

We define a modified functional $\tilde{f}:W^{1,p}(\mathbb{R}^n)\to\mathbb{R}$ by setting

$$\tilde{f}(u)=\int_{\mathbb{R}^n}\widetilde{\mathcal{L}}(x,u,\nabla u)\,dx+\frac{1}{p}\int_{\mathbb{R}^n}b(x)|u|^p\,dx-\int_{\mathbb{R}^n}\widetilde{G}(x,u)\,dx. \tag{6.21}$$

Then the Euler's equation of $\tilde{f}$ is given by :

$$-\operatorname{div}\Big(\nabla_\xi\widetilde{\mathcal{L}}(x,u,\nabla u)\Big)+D_s\widetilde{\mathcal{L}}(x,u,\nabla u)+b(x)|u|^{p-2}u=\tilde{g}(x,u)\quad\text{in }\mathbb{R}^n. \tag{6.22}$$

Lemma 6.2. *If $u\in W^{1,p}(\mathbb{R}^n)$ is a solution of (6.22), then u is a positive solution of (6.17).*

Proof. Let $Q:\mathbb{R}\to\mathbb{R}$ the Lipschitz map defined by:

$$Q(s)=\begin{cases}0 & \text{if } s\geq 0\\ s & \text{if } -1\leq s\leq 0\\ -1 & \text{if } s\leq -1.\end{cases}$$

Testing $\widetilde{f}'(u)$ with $Q(u) \in W^{1,p} \cap L^\infty(\mathbb{R}^n)$ and taking into account (6.20) we have:

$$\begin{aligned}
0 &= \widetilde{f}'(u)(Q(u)) \\
&= \int_{\mathbb{R}^n} \nabla_\xi \widetilde{\mathcal{L}}(x,u,\nabla u)\cdot \nabla Q(u)\,dx \\
&\quad + \int_{\mathbb{R}^n} D_s\widetilde{\mathcal{L}}(x,u,\nabla u)Q(u)\,dx + \int_{\mathbb{R}^n} b(x)|u|^{p-2}uQ(u)\,dx - \int_{\mathbb{R}^n} \widetilde{g}(x,u)Q(u)\,dx \\
&= \int_{\{-1<u<0\}} \nabla_\xi \mathcal{L}(x,0,\nabla u)\cdot \nabla u\,dx + \int_{\{u<0\}} D_s\widetilde{\mathcal{L}}(x,u,\nabla u)Q(u)\,dx \\
&\quad + \int_{\mathbb{R}^n} b(x)|u|^{p-2}uQ(u)\,dx - \int_{\{u<0\}} \widetilde{g}(x,u)Q(u)\,dx \\
&= \int_{\{-1<u<0\}} p\mathcal{L}(x,0,\nabla u)\,dx + \int_{\{u<0\}} b(x)|u|^{p-2}uQ(u)\,dx \\
&\geq \mathbf{b}\int_{\mathbb{R}^n} |u|^{p-2}uQ(u)\,dx \geq 0.
\end{aligned}$$

In particular, it results $Q(u) = 0$, namely $u \geq 0$. □

Therefore, without loss of generality we shall suppose that

$$\forall s \leq 0: \quad g(x,s) = 0, \quad \mathcal{L}(x,s,\xi) = \mathcal{L}(x,0,\xi)$$

for a.e. $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$.

Lemma 6.3. *Let $c \in \mathbb{R}$. Then each $(CPS)_c$-sequence for f is bounded in $W^{1,p}(\mathbb{R}^n)$.*

Proof. If (u_h) is a $(CPS)_c$-sequence for f, arguing as in [48, Lemma 2], since

$$f(u_h) - \frac{1}{q} f'(u_h)(u_h) = c + o(1)$$

as $h \to +\infty$, by (6.12) and (6.13) we get:

$$\beta \int_{\mathbb{R}^n} |\nabla u_h|^p\,dx + \frac{q-p}{p}\mathbf{b}\int_{\mathbb{R}^n} |u_h|^p\,dx \leq C$$

for some $C > 0$, hence the assertion. □

Let us note that there exists $M > 0$ such that:

$$|D_s\mathcal{L}(x,s,\xi)| \leq M\nabla_\xi \mathcal{L}(x,s,\xi)\cdot\xi \tag{6.23}$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

We now prove a local compactness property for $(CPS)_c$-sequences. In the following, $\Omega \Subset \mathbb{R}^n$ will always denote an open and bounded subset of $\mathbb{R}^n$.

Theorem 6.4. *Let (u_h) be a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ and for each $v \in C_c^\infty(\mathbb{R}^n)$ set*

$$\langle w_h, v\rangle = \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla v\,dx + \int_{\mathbb{R}^n} D_s\mathcal{L}(x,u_h,\nabla u_h)v\,dx. \tag{6.24}$$

If (w_h) is strongly convergent to some w in $W^{-1,p'}(\Omega)$ for each $\Omega \Subset \mathbb{R}^n$, then (u_h) admits a strongly convergent subsequence in $W^{1,p}(\Omega)$ for each $\Omega \Subset \mathbb{R}^n$.

Proof. Since (u_h) is bounded in $W^{1,p}(\mathbb{R}^n)$, we find a u in $W^{1,p}(\mathbb{R}^n)$ such that, up to a subsequence, $u_h \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$. Moreover, for each $\Omega \Subset \mathbb{R}^n$ we have:

$$u_h \to u \quad \text{in } L^p(\Omega), \quad u_h(x) \to u(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

By a natural extension of [22, Theorem 2.1] to unbounded domains, we have $\nabla u_h(x) \to \nabla u(x)$ for a.e. $x \in \mathbb{R}^n$. Then, following the blueprint of [133, Theorem 3.2] we obtain for each $v \in C_c^\infty(\mathbb{R}^n)$

$$\langle w, v\rangle = \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla v\, dx + \int_{\mathbb{R}^n} D_s\mathcal{L}(x,u,\nabla u) v\, dx. \tag{6.25}$$

Choose now $\Omega \Subset \mathbb{R}^n$ and fix a positive smooth cut-off function η on $\mathbb{R}^n$ with $\eta = 1$ on Ω. Moreover, let $\vartheta : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\vartheta(s) = \begin{cases} Ms & \text{if } 0 < s < R \\ MR & \text{if } s \geq R \\ -Ms & \text{if } -R < s < 0 \\ MR & \text{if } s \leq -R, \end{cases} \tag{6.26}$$

where M is as in (6.23). Since by [133, Proposition 3.1] $v_h = \eta u_h \exp\{\vartheta(u_h)\}$ are admissible test functions for (6.24), we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h \eta \exp\{\vartheta(u_h)\}\, dx - \langle w_h, \eta u_h \exp\{\vartheta(u_h)\}\rangle \\ &+ \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla \eta u_h \exp\{\vartheta(u_h)\}\, dx \\ &+ \int_{\mathbb{R}^n} \left[D_s\mathcal{L}(x,u_h,\nabla u_h) + \vartheta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h\right] \eta u_h \exp\{\vartheta(u_h)\}\, dx \\ &= 0. \end{aligned}$$

Let us observe that

$$\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h \to \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

Since for each $h \in \mathbb{N}$ we have

$$\left[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \vartheta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h\right] \eta u_h \exp\{\vartheta(u_h)\} \leq 0,$$

Fatou's Lemma yields

$$\begin{aligned} &\limsup_h \int_{\mathbb{R}^n} \left[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \vartheta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h\right] \\ &\times \eta u_h \exp\{\vartheta(u_h)\}\, dx \\ &\leq \int_{\mathbb{R}^n} \left[-D_s\mathcal{L}(x,u,\nabla u) - \vartheta'(u)\nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u\right] \eta u \exp\{\vartheta(u)\}\, dx. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} &\limsup_h \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h \eta \exp\{\vartheta(u_h)\}\, dx \\ &\leq \limsup_h \Big\{ \int_{\mathbb{R}^n} \left[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \vartheta'(u_h)\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h\right] \\ &\quad \times \eta u_h \exp\{\vartheta(u_h)\}\, dx + \langle w_h, \eta u_h \exp\{\vartheta(u_h)\}\rangle \\ &\quad - \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla \eta u_h \exp\{\vartheta(u_h)\}\, dx \Big\} \end{aligned}$$

$$\begin{aligned}
&\le \Big\{ \int_{\mathbb{R}^n} \big[-D_s\mathcal{L}(x,u,\nabla u) - \vartheta'(u)\nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u \big] \eta u \exp\{\vartheta(u)\}\, dx \\
&\quad + \langle w, \eta u \exp\{\vartheta(u)\}\rangle - \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla \eta u \exp\{\vartheta(u)\}\, dx \Big\} \\
&= \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u \eta \exp\{\vartheta(u)\}\, dx,
\end{aligned}$$

where we used (6.25) with $v = \eta u \exp\{\vartheta(u)\}$. In particular, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u\, \eta \exp\{\vartheta(u)\}\, dx \\
&\le \liminf_h \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h \eta \exp\{\vartheta(u_h)\}\, dx \\
&\le \limsup_h \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h \eta \exp\{\vartheta(u_h)\}\, dx \\
&\le \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u\, \eta \exp\{\vartheta(u)\}\, dx\,,
\end{aligned}$$

namely

$$\begin{aligned}
&\lim_h \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h \eta \exp\{\vartheta(u_h)\}\, dx \\
&= \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u \eta \exp\{\vartheta(u)\}\, dx.
\end{aligned}$$

Since $\mathcal{L}(x,s,\cdot)$ is p-homogeneous, by (6.5) for each $h\in\mathbb{N}$ we have

$$\nu\eta p|\nabla u_h|^p \le \eta \exp\{\vartheta(u_h)\}\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot \nabla u_h\,,$$

by the generalized Lebesgue's theorem we deduce that:

$$\lim_h \int_{\mathbb{R}^n} \eta|\nabla u_h|^p\, dx = \int_{\mathbb{R}^n} \eta|\nabla u|^p\, dx.$$

Up to substituting η with η^p, we get:

$$\lim_h \int_{\mathbb{R}^n} |\eta\nabla u_h|^p\, dx = \int_{\mathbb{R}^n} |\eta\nabla u|^p\, dx\,,$$

which implies that

$$\eta\nabla u_h \to \eta\nabla u \quad \text{in } L^p(\mathbb{R}^n)\,,$$

namely $\nabla u_h \to \nabla u$ in $L^p(\Omega)$. □

Let us remark that, in general, since the imbedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is not compact, we cannot have strong convergence of $(CPS)_c$ sequences on unbounded domains of $\mathbb{R}^n$. Nevertheless, we have the following result.

Lemma 6.5. *Let (u_h) be a $(CPS)_c$-sequence for f. Then there exists u in $W^{1,p}(\mathbb{R}^n)$ such that, up to a subsequence, the following facts hold:*

- (a) *(u_h) converges to u weakly in $W^{1,p}(\mathbb{R}^n)$;*
- (b) *(u_h) converges to u strongly in $W^{1,p}(\Omega)$ for each $\Omega \Subset \mathbb{R}^n$;*
- (c) *u is a positive weak solution to (6.3).*

Proof. Since by Lemma 6.3 the sequence (u_h) is bounded in $W^{1,p}(\mathbb{R}^n)$, of course (a) holds. Now, fixed $\Omega \Subset \mathbb{R}^n$, if we set

$$w_h = \gamma_h + g(x, u_h) - b|u_h|^{p-2}u_h \in W^{-1,p'}(\Omega), \quad \gamma_h \to 0 \quad \text{in } W^{-1,p'}(\Omega),$$

(b) follows by Theorem 6.4 with $w = g(x,u) - b|u|^{p-2}u$. Finally by Lemma 6.2, (c) is a consequence of equation (6.25). □

Let us now prove a technical Lemma that we shall use later.

Lemma 6.6. *Let $c \in \mathbb{R}$ and (u_h) be a bounded $(CPS)_c$-sequence for f. Then for each $\varepsilon > 0$ there exists $\varrho > 0$ such that*

$$\int_{\{|u_h| \le \varrho\}} |\nabla u_h|^p \, dx \le \varepsilon$$

for each $h \in \mathbb{N}$.

Proof. Let $\varepsilon, \varrho > 0$ and define for $\delta \in]0,1[$ the function $\vartheta_\delta : \mathbb{R} \to \mathbb{R}$ by setting

$$\vartheta_\delta(s) = \begin{cases} s & \text{if } |s| \le \varrho \\ \varrho + \delta\varrho - \delta s & \text{if } \varrho < s < \varrho + \frac{\varrho}{\delta} \\ -\varrho - \delta\varrho - \delta s & \text{if } -\varrho - \frac{\varrho}{\delta} < s < -\varrho \\ 0 & \text{if } |s| \ge \varrho + \frac{\varrho}{\delta}. \end{cases} \tag{6.27}$$

Since $\vartheta_\delta(u_h) \in W^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we get

$$\begin{aligned} \langle w_h, \vartheta_\delta(u_h)\rangle = &\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h) \, dx \\ &+ \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h)\vartheta_\delta(u_h) \, dx + \int_{\mathbb{R}^n} b|u_h|^{p-2}u_h\vartheta_\delta(u_h) \\ &- \int_{\mathbb{R}^n} g(x, u_h)\vartheta_\delta(u_h) \, dx. \end{aligned}$$

Then condition (6.8), $b(x) > 0$ and $|\vartheta_\delta(u_h)| \le \varrho$ yield

$$\begin{aligned} &\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h) \, dx \\ &\le \int_{\mathbb{R}^n} g(x, u_h)\vartheta_\delta(u_h) \, dx + \varrho\|u_h\|_{1,p}^p + \frac{1}{p' p^{\frac{p'}{p}} \delta^{\frac{p'}{p}}} \|w_h\|_{-1,p'}^{p'} + \delta\|u_h\|_{1,p}^p. \end{aligned}$$

Since (u_h) is bounded in $W^{1,p}(\mathbb{R}^n)$, there exists $\delta > 0$ such that $\delta\|u_h\|_{1,p}^p \le \varepsilon\nu/8$ and

$$\delta \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \le \frac{\varepsilon\nu}{2}, \tag{6.28}$$

uniformly with $h \in \mathbb{N}$ so large that $\frac{1}{p' p^{\frac{p'}{p}} \delta^{\frac{p'}{p}}} \|w_h\|_{-1,p'}^{p'} \le \frac{\varepsilon\nu}{8}$. Now, since

$$\begin{aligned} \int_{\mathbb{R}^n} g(x, u_h)\vartheta_\delta(u_h) \, dx &\le \int_{\{|u_h| \le \varrho + \frac{\varrho}{\delta}\}} g(x, u_h)u_h \, dx \\ &\le \|d\|_r \Big(\int_{\{|u_h| \le \varrho + \frac{\varrho}{\delta}\}} |u_h|^{r'} \, dx\Big)^{1/r'} + c\int_{\{|u_h| \le \varrho + \frac{\varrho}{\delta}\}} |u_h|^\sigma \, dx, \end{aligned}$$

we can find $\varrho > 0$ such that

$$\int_{\mathbb{R}^n} g(x, u_h)\vartheta_\delta(u_h) \, dx \le \frac{\varepsilon\nu}{8}$$

and $\varrho\|u_h\|_{1,p}^p \leq \frac{\varepsilon\nu}{8}$. Therefore we obtain

$$\int_{\{|u_h|\leq\varrho+\frac{\varrho}{\delta}\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\vartheta_\delta(u_h)\,dx \leq \frac{\varepsilon\nu}{2},$$

namely, taking into account (6.28)

$$\int_{\{|u_h|\leq\varrho\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx \leq \varepsilon\nu.$$

By (6.5) the proof is complete. □

Let us now introduce the "asymptotic functional" $f_\infty : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ by setting

$$f_\infty(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p\,dx + \frac{\lambda}{p}\int_{\mathbb{R}^n} |u|^p\,dx - \frac{1}{q}\int_{\mathbb{R}^n} |u^+|^q\,dx$$

and consider the associated p-Laplacian problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda|u|^{p-2}u = u^{q-1} \quad \text{in } \mathbb{R}^n.$$

(See [45] for the case $p>2$ and [19] for the case $p=2$).

We now investigate the behavior of the functional f over its $(CPS)_c$-sequences.

Lemma 6.7. *Let (u_h) be a $(CPS)_c$-sequence for f and u its weak limit. Then*

$$f(u_h) \approx f(u) + f_\infty(u_h - u), \tag{6.29}$$

$$f'(u_h)(u_h) \approx f'(u)(u) + f'_\infty(u_h-u)(u_h-u) \tag{6.30}$$

as $h\to+\infty$, where the notation $A_h \approx B_h$ means $A_h - B_h \to 0$.

Proof. By [37, Lemma 2.2] we have the splitting:

$$\int_{\mathbb{R}^n} G(x,u_h)\,dx - \int_{\mathbb{R}^n} G(x,u)\,dx - \frac{1}{q}\int_{\mathbb{R}^n} |(u_h-u)^+|^q\,dx = o(1),$$

as $h\to+\infty$. Moreover, we easily get:

$$\int_{\mathbb{R}^n} b|u_h|^p\,dx - \int_{\mathbb{R}^n} b|u|^p\,dx - \lambda\int_{\mathbb{R}^n} |u_h-u|^p\,dx = o(1),$$

as $h\to+\infty$. Observe now that thanks to (6.18) we have

$$\int_{\{|x|>\varrho\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx - \int_{\{|x|>\varrho\}} |\nabla u_h|^p\,dx \to 0, \ \text{ as } \varrho\to+\infty,$$

uniformly in $h\in\mathbb{N}$ and

$$\int_{\{|x|>\varrho\}} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\,dx - \int_{\{|x|>\varrho\}} |\nabla u|^p\,dx \to 0, \ \text{ as } \varrho\to+\infty.$$

Therefore, taking into account that for each $\sigma>0$ there exists $c_\sigma>0$ with

$$|\nabla u_h|^p \leq c_\sigma|\nabla u|^p + (1+\sigma)|\nabla u_h - \nabla u|^p\,,$$

we deduce that for each $\varepsilon>0$ there exists $\varrho>0$ such that for each $h\in\mathbb{N}$

$$\int_{\{|x|>\varrho\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx - \int_{\{|x|>\varrho\}} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\,dx$$
$$-\int_{\{|x|>\varrho\}} |\nabla(u_h-u)|^p\,dx < \tilde{c}\varepsilon\,,$$

for some $\widetilde{c} > 0$. On the other hand, since by Lemma 6.5 we have

$$\nabla u_h \to \nabla u \quad \text{in } L^p(B(0,\varrho),\mathbb{R}^n),$$

we deduce

$$\int_{\{|x|\le\varrho\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx = \int_{\{|x|\le\varrho\}} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\,dx + o(1),$$

as $h \to +\infty$. Then, for each $\varepsilon > 0$ there exists $\overline{h} \in \mathbb{N}$ such that

$$\begin{aligned}
&\int_{\{|x|\le\varrho\}} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx - \int_{\{|x|\le\varrho\}} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\,dx\\
&-\int_{\{|x|\le\varrho\}} |\nabla(u_h-u)|^p\,dx < \widehat{c}\varepsilon\,,
\end{aligned}$$

for each $h \ge \overline{h}$, for some $\widehat{c} > 0$. Putting the previous inequalities together, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\\
&= \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u\,dx + \int_{\mathbb{R}^n} |\nabla(u_h-u)|^p\,dx + o(1)
\end{aligned}$$

as $h \to +\infty$. Taking into account that $\mathcal{L}(x,s,\cdot)$ is homogeneous of degree p, (6.29) is proved. To prove (6.30), by the previous step and condition (6.15), it suffices to show that

$$\int_{\mathbb{R}^n} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx = \int_{\mathbb{R}^n} D_s\mathcal{L}(x,u,\nabla u)u\,dx + o(1), \tag{6.31}$$

as $h \to +\infty$. By (6.19), we find $b_1, b_2 > 0$ such that for each $\varepsilon > 0$ there exists $\varrho > 0$ with

$$\int_{\{|x|>\varrho\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx \le b_1\varepsilon, \quad \int_{\{|x|>\varrho\}} D_s\mathcal{L}(x,u,\nabla u)u\,dx \le b_2\varepsilon,$$

uniformly in $h \in \mathbb{N}$. On the other hand, combining (b) of Lemma 6.5 with (6.13), the generalized Lebesgue's Theorem yields

$$\int_{\{|x|\le\varrho\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx = \int_{\{|x|\le\varrho\}} D_s\mathcal{L}(x,u,\nabla u)u\,dx + o(1),$$

as $h \to +\infty$. Then, (6.30) follows by the arbitrariness of ε. □

Let us recall from [98, Lemma I.1] the following result.

Lemma 6.8. *Let $1 < p \le \infty$ and $1 \le q < \infty$ with $q \ne p^*$. Assume that (u_h) is a bounded sequence in $L^q(\mathbb{R}^n)$ with (∇u_h) bounded in $L^p(\mathbb{R}^n)$ and there exists $R > 0$ such that:*

$$\sup_{y\in\mathbb{R}^n}\int_{y+B_R} |u_h|^q\,dx = o(1)\,,$$

as $h \to +\infty$. Then $u_h \to 0$ in $L^\alpha(\mathbb{R}^n)$ for each $\alpha \in]q,p^*[$.

Let (u_h) denote a concrete Palais-Smale sequence for f and let us assume that its weak limit u is 0. If $\frac{np'}{n+p'} < r < p'$, recalling that by (6.31) it results

$$\int_{\mathbb{R}^n} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx = o(1),$$

as $h \to +\infty$, we get

$$\begin{aligned} pc = pf(u_h) - f'(u_h)(u_h) + o(1) \leq \int_{\mathbb{R}^n} g(x, u_h) u_h \, dx + \\ + o(1) \leq \|d\|_r \|u_h\|_{r'} + c\|u_h\|_\sigma^\sigma + o(1). \end{aligned}$$

Hence, either $\|u_h\|_{r'}$ or $\|u_h\|_\sigma$ does not converge strongly to 0. If we now apply Lemma 6.8 with $p = q$ (note also that $p < r', \sigma < p^*$), taking into account that (u_h) is bounded in $W^{1,p}(\mathbb{R}^n)$ we find $C > 0$ and a sequence $(y_h) \subset \mathbb{R}^n$ with $|y_h| \to +\infty$ such that

$$\int_{y_h + B_R} |u_h|^p \, dx \geq C \, ,$$

for some $R > 0$. In particular, if $\tau_h u_h(x) = u_h(x - y_h)$, we have

$$\int_{B_R} |\tau_h u_h|^p \, dx \geq C$$

and there exists $\overline{u} \not\equiv 0$ such that:

$$\tau_h u_h \rightharpoonup \overline{u} \ \text{ in } \ W^{1,p}(\mathbb{R}^n). \tag{6.32}$$

If $r = \frac{np'}{n+p'}$, the same can be obtained in a similar fashion since for each $\varepsilon > 0$ there exist

$$d_{1,\varepsilon} \in L^\ell(\mathbb{R}^n) \quad \ell \in \Big]\frac{np'}{n+p'}, p'\Big[, \quad d_{2,\varepsilon} \in L^{\frac{np'}{n+p'}}(\mathbb{R}^n)$$

such that

$$d = d_{1,\varepsilon} + d_{2,\varepsilon}, \quad \|d_{2,\varepsilon}\|_{\frac{np'}{n+p'}} \leq \varepsilon.$$

We now show that $\overline{u}$ is a weak solution of

$$-\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) + \lambda |u|^{p-2}u = u^{q-1} \quad \text{in } \mathbb{R}^n. \tag{6.33}$$

Lemma 6.9. *Let (u_h) a $(CPS)_c$-sequence for f with $u_h \rightharpoonup 0$. Then $\overline{u}$ is a weak solution of (6.33). Moreover $\overline{u} > 0$.*

Proof. For all $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $h \in \mathbb{N}$ we set

$$\forall x \in \mathbb{R}^n : \quad (\tau^h \varphi)(x) := \varphi(x + y_h).$$

Since (u_h) is a $(CPS)_c$-sequence for f, we have that

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n) : \quad f'(u_h)(\tau^h \varphi) = o(1),$$

namely, as $h \to +\infty$

$$\begin{aligned} &\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \tau^h \varphi \, dx + \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) \tau^h \varphi \, dx \\ &+ \int_{\mathbb{R}^n} b(x)|u_h|^{p-2} u_h \tau^h \varphi \, dx - \int_{\mathbb{R}^n} g(x, u_h) \tau^h \varphi \, dx = o(1). \end{aligned}$$

Of course, as $h \to +\infty$ we have

$$\begin{aligned} \int_{\mathbb{R}^n} b(x)|u_h|^{p-2} u_h \tau^h \varphi \, dx &= \int_{\operatorname{supp} \varphi} b(x - y_h)|\tau_h u_h|^{p-2} \tau_h u_h \varphi \, dx \\ &\to \lambda \int_{\mathbb{R}^n} |\overline{u}|^{p-2} \overline{u} \, \varphi \, dx, \end{aligned}$$

$$\int_{\mathbb{R}^n} g(x, u_h) \tau^h \varphi \, dx = \int_{\operatorname{supp} \varphi} g(x - y_h, \tau_h u_h) \varphi \, dx \to \int_{\mathbb{R}^n} |\overline{u}^+|^{q-1} \varphi \, dx.$$

Next, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \tau^h \varphi \, dx \\ &= \int_{\operatorname{supp}\varphi} \nabla_\xi \mathcal{L}(x - y_h, \tau_h u_h, \nabla \tau_h u_h) \cdot \nabla \varphi \, dx \\ &\to \int_{\mathbb{R}^n} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \varphi \, dx.\end{aligned}$$

Now, for each $\varepsilon > 0$, Lemma 6.6 gives a $\varrho > 0$ such that

$$\int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) \tau^h \varphi \, dx \le \tilde{c}\varepsilon + \int_{\{|u_h| > \varrho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) \tau^h \varphi \, dx.$$

On the other hand, by (6.10) we have

$$\begin{aligned}&\int_{\{|u_h| > \varrho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) \tau^h \varphi \, dx \\ &= \int_{\operatorname{supp}\varphi \cap \{|\tau_h u_h| > \varrho\}} D_s \mathcal{L}(x - y_h, \tau_h u_h, \nabla \tau_h u_h) \varphi \, dx = o(1),\end{aligned}$$

as $h \to +\infty$. By arbitrariness of ε we conclude the proof. Finally $\overline{u} \ge 0$ follows by Lemma 6.2 and $\overline{u} > 0$ follows by [143, Theorem 1.1]. □

Lemma 6.10. *Let (u_h) be a $(CPS)_c$-sequence for f with $u_h \rightharpoonup 0$. Then*

$$f_\infty(\overline{u}) \le \liminf_h f_\infty(\tau_h u_h).$$

Proof. Since (u_h) weakly goes to 0, Lemma 6.7 gives $f'_\infty(u_h)(u_h) \to 0$ as $h \to +\infty$, so that

$$f'_\infty(\tau_h u_h)(\tau_h u_h) \to 0 \quad \text{as} \quad h \to +\infty,$$

namely

$$\int_{\mathbb{R}^n} |\nabla \tau_h u_h|^p \, dx + \lambda \int_{\mathbb{R}^n} |\tau_h u_h|^p \, dx - \int_{\mathbb{R}^n} (\tau_h u_h^+)^q \, dx \to 0$$

as $h \to +\infty$. Therefore

$$f_\infty(\tau_h u_h) - \Big(\frac{1}{p} - \frac{1}{q}\Big) \int_{\mathbb{R}^n} (\tau_h u_h^+)^q \, dx \to 0.$$

Similarly, Lemma 6.9 yields

$$f_\infty(\overline{u}) = \Big(\frac{1}{p} - \frac{1}{q}\Big) \int_{\mathbb{R}^n} |\overline{u}|^q \, dx,$$

and the assertion follows by Fatou's Lemma. □

Lemma 6.11. *If (u_h) is a $(CPS)_c$-sequence for f with $u_h \rightharpoonup 0$, then $f_\infty(\overline{u}) \le c$.*

Proof. Since Lemma 6.7 yields

$$f(u_h) \approx f_\infty(\tau_h u_h), \quad \text{as} \quad h \to +\infty,$$

by the previous Lemma we conclude the proof. □

We finally come to the proof of the main result of this section.

Proof of Theorem 6.1. Since G is super-linear at $+\infty$, (6.12), for all u in the space $W^{1,p}(\mathbb{R}^n)\setminus\{0\}$,

$$u \geq 0 \;\Rightarrow\; \lim_{t\to+\infty} f(tu) = -\infty.$$

Let $v \in C_c^\infty(\mathbb{R}^n)$ positive be such that for all $t > 1 : f(tv) < 0$, and define the min-max class

$$\Gamma = \Big\{\gamma \in C([0,1], W^{1,p}(\mathbb{R}^n)) : \gamma(0) = 0, \quad \gamma(1) = v\Big\},$$

and the min-max value

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)).$$

Let us remark that for each $u \in W^{1,p}(\mathbb{R}^n)$

$$f(u) \geq \nu\|\nabla u\|_p^p + \frac{\mathbf{b}}{p}\|u\|_p^p - \int_{\mathbb{R}^n} G(x,u)\,dx.$$

Then, since by (6.16) it results

$$\lim_h \frac{\int_{\mathbb{R}^n} G(x,w_h)}{\|w_h\|_{1,p}^p} = 0$$

for each (w_h) that goes to 0 in $W^{1,p}(\mathbb{R}^n)$, f has a mountain pass geometry, and by the deformation Lemma of [36] there exists a $(CPS)_c$-sequence $(u_h) \subset W^{1,p}(\mathbb{R}^n)$ for f. By Lemma 6.5 it results that (u_h) converges weakly to a positive weak solution u of (6.3). Therefore, if $u \neq 0$, we are done. On the other hand, if $u = 0$ let us consider $\overline{u}$. We now prove that $\overline{u}$ is a weak solution to our problem. Since we have for each $u \in W^{1,p}(\mathbb{R}^n)\setminus\{0\}$

$$u \geq 0 \Rightarrow \lim_{t\to+\infty} f_\infty(tu) = -\infty,$$

we find $R > 0$ so large that

$$\forall a, b \geq 0 : \quad a + b = R \Rightarrow f_\infty(a\overline{u} + bv) < 0.$$

Define the path $\gamma : [0,1] \to W^{1,p}(\mathbb{R}^n)$ by

$$\gamma(t) = \begin{cases} 3Rt\overline{u} & \text{if } t \in \left[0, \frac{1}{3}\right] \\ (3t-1)Rv + (2-3t)R\overline{u} & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ (3R + 3t - 3Rt - 2)v & \text{if } t \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Of course we have $\gamma \in \Gamma$, $f_\infty(\gamma(t)) < 0$ for each $t \in]\frac{1}{3}, 1]$ and by [45, Lemma 2.4]

$$\max_{t\in[0,\frac{1}{3}]} f_\infty(\gamma(t)) = f_\infty(\overline{u}).$$

Hence, by Lemma (6.11) and the assumptions on $\mathscr{L}$ and g, we have

$$c \leq \max_{t\in[0,1]} f(\gamma(t)) \leq \max_{t\in[0,1]} f_\infty(\gamma(t)) = f_\infty(\overline{u}) \leq c.$$

Therefore, since γ is an optimal path in Γ, by the non-smooth deformation Lemma of [36], there exists $\overline{t} \in]0,1[$ such that $\gamma(\overline{t})$ is a critical point of f at level c. Moreover $\gamma(\overline{t}) = \overline{u}$, otherwise

$$f(\gamma(\overline{t})) \leq f_\infty(\gamma(\overline{t})) < f_\infty(\overline{u}) = c,$$

in contradiction with $f(\gamma(\overline{t})) = c$. Then $\overline{u}$ is a positive solution to (6.3). $\square$

Remark 6.12. Let $1 < p < n$, $q > p$ and $\lambda > 0$. As a by-product of Theorem 6.1, taking

$$\mathcal{L}(x,s,\xi) = \frac{1}{p}|\xi|^p + \frac{\lambda}{p}|s|^p - \frac{1}{q}|s|^q,$$

we deduce that the problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda|u|^{p-2}u = |u|^{q-2}u \quad \text{in } \mathbb{R}^n\,, \tag{6.34}$$

has at least one nontrivial positive solution $u \in W^{1,p}(\mathbb{R}^n)$. (see also [45, 145]).

In some sense, Theorem 6.1 implies that the ε-perturbed problem

$$-\operatorname{div}\left((1+\varepsilon(x,u,\nabla u))|\nabla u|^{p-2}\nabla u\right) + \lambda|u|^{p-2}u = |u|^{q-2}u \quad \text{in } \mathbb{R}^n\,, \tag{6.35}$$

has at least one nontrivial positive solution $u \in W^{1,p}(\mathbb{R}^n)$.

Remark 6.13. By [6, Lemma 1.4] we have a local boundedness property for solutions of problem (6.3), namely for each $\Omega \Subset \mathbb{R}^n$ each weak solution $u \in W^{1,p}(\Omega)$ of (6.3) belongs to $L^\infty(\Omega)$ provided that in (6.14) is $d \in L^s(\Omega)$ for a sufficiently large s. (see [6, 36]).

6.3. **Fully nonlinear problems at critical growth.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 < p < n$ and $p < q < p^* = \frac{np}{n-p}$. In this section we are concerned with the existence of two nontrivial solutions in $W_0^{1,p}(\Omega)$ of the problem (6.36),

$$\begin{gathered} -\operatorname{div}\left(\nabla_\xi\mathcal{L}(x,u,\nabla u)\right) + D_s\mathcal{L}(x,u,\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u + \varepsilon h \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{6.36}$$

with $h \in L^{p'}(\Omega)$, $h \not\equiv 0$, provided that $\varepsilon > 0$ is small and $\lambda > 0$ is large.

Motivations for investigating problems as (6.36) come from various situations in geometry and physics which present lack of compactness (see e.g. [28]). A typical example is Yamabe's problem, i.e. find $u > 0$ such that

$$-4\frac{n-1}{n-2}\Delta_M u = R' u^{(n+2)/(n-2)} - R(x)u \quad \text{on } M,$$

for some constant R', where M is an n-dimensional Riemannian manifold, $R(x)$ its scalar curvature and $-\Delta_M$ is the Laplace-Beltrami operator on M. Since p^* is the critical Sobolev exponent for which the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ fails to be compact, as known, one encounters serious difficulties in applying variational methods to (6.36). As known, in general, if $h \equiv 0$ and $\lambda = 0$, to obtain a solution of

$$\begin{gathered} -\Delta_p u = |u|^{p^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

one has to consider in detail the geometry of Ω (see e.g. [16]) or has to replace the critical term u^{p^*-1} with $u^{p^*-1-\varepsilon}$ and then investigate the limits of u_ε as $\varepsilon \to 0$ (nearly critical growth, see [72] and references therein). Let us now assume that $h \equiv 0$ and $\lambda \neq 0$. As we showed in Corollary 9.21 by the general Pohožaev identity of Pucci and Serrin [117], if

$$p^*\nabla_x\mathcal{L}(x,s,\xi)\cdot x - nD_s\mathcal{L}(x,s,\xi)s \geq 0,$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, then (6.36) admits no nontrivial smooth solution for each $\lambda \leq 0$ when the domain Ω is star-shaped and $\mathcal{L}$ is sufficiently smooth. Therefore, in this case we are reduced to consider positive λ. Let us briefly recall the historical background of existence results for problems at critical growth with lower-order perturbations. In 1983, in a pioneering paper [28], Brézis and Nirenberg proved that the problem

$$\begin{gathered} -\Delta u = u^{(n+2)/(n-2)} + \lambda u \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

has at least one solution $u \in H_0^1(\Omega)$ provided that

$$\lambda \in \begin{cases} (0, \lambda_1) & \text{if } n \geq 4, \\ (\lambda_1/4,\ \lambda_1) & \text{if } n = 3 \text{ and } \Omega = B(0, R), \end{cases}$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω. The extension to the p-Laplacian was achieved by Garcia Azorero and Peral Alonso in [70, 71] (see also [11]). Namely, they proved the existence of a nontrivial solution of:

$$\begin{gathered} -\Delta_p u = |u|^{p^*-2} u + \lambda |u|^{q-2} u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

provided that

$$\lambda \in \begin{cases} (0, \lambda_1) & \text{if } 1 < p = q < p^* \text{ and } p^2 \leq n\,; \\ (\lambda_0, \infty) & \text{if } 1 < p < q < p^* \text{ and } p^2 > n\,; \\ (0, \infty) & \text{if } 1 < p < q < p^* \text{ and } p^2 \leq n\,; \\ (0, \infty) & \text{if } \max\{p, p^* - \frac{p}{p-1}\} < q < p^*\,, \end{cases}$$

where λ_1 is the first eigenvalue of $-\Delta_p$ and λ_0 is a suitable positive real number. Finally, for bifurcation and multiplicity results in the semi-linear case ($p = 2$), we refer to the paper of Cerami, Fortunato and Struwe [38].

Let us now assume $h \not\equiv 0$. Then, a natural question is whether inhomogeneous problems like (6.36) have more than one solution. For bounded domains one of the first answers was given in 1992 by Tarantello in [141], where it is shown that the problem

$$\begin{gathered} -\Delta u = |u|^{2^*-2} u + h(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

admits two distinct solutions $u_1, u_2 \in H_0^1(\Omega)$ if $\|h\|_2$ is small. The existence of two nontrivial solutions for the p-Laplacian problem

$$\begin{gathered} -\Delta_p u = |u|^{p^*-2} u + \lambda |u|^{q-2} u + h(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

for $1 < p < q < p^*$, λ large and $\|h\|_{p'}$ small enough, has been proven in 1995 by Chabrowski in [41]. This achievement has been recently extended by Zhou in [150] to the equation:

$$-\Delta_p u + c|u|^{p-2} u = |u|^{p^*-2} u + f(x, u) + h(x)$$

on the entire $\mathbb{R}^n$, where $f(x, u)$ is a lower-order perturbation of $|u|^{p^*-2}u$. This case involves a double loss of compactness, one due to the unboundedness of the domain and the other due to the critical Sobolev exponent. Now, more recently, some results for the more general problem

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x, u, \nabla u)\big) + D_s \mathcal{L}(x, u, \nabla u) = g(x, u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

with g subcritical and super-linear have been considered in [6, 113] and [133]. It is therefore natural to see what happens when g has a critical growth.

A first answer was given in 1998 by Arioli and Gazzola in [10], where they proved the existence of a nontrivial solution $u \in H_0^1(\Omega)$ for a class of quasi-linear equations of the type

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju = |u|^{2^*-2}u + \lambda u, \tag{6.37}$$

where the coefficients $(a_{ij}(x,s))$ satisfy some suitable assumptions, including a semi-linear asymptotic behavior as $s \to +\infty$ (see remark 6.15).

Now, in view of the above mentioned results for $-\Delta$, $-\Delta_p$, we expect that problems $(\mathcal{P}_{\varepsilon,\lambda})$ admits at least two nontrivial solutions for ε small and λ large. To prove this, we shall argue on the functional $f_{\varepsilon,\lambda} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$f_{\varepsilon,\lambda}(u) = \int_\Omega \mathcal{L}(x,u,\nabla u)\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \varepsilon\int_\Omega hu\,dx, \tag{6.38}$$

where $W_0^{1,p}(\Omega)$ will be endowed with the norm $\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$.

The first solution is obtained via a local minimization argument while the second solution will follow by the mountain pass theorem without Palais-Smale condition in its non-smooth version (see [36]).

In general, under reasonable assumptions on $\mathcal{L}$, $f_{\varepsilon,\lambda}$ is continuous but not even locally Lipschitzian unless $\mathcal{L}$ does not depend on u or is subjected to some very restrictive growth conditions. Then, we shall refer to the non-smooth critical point theory developed in [36, 50, 58].

We assume that $\mathcal{L}(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class C^1 in s and of class C^2 in ξ and that $\mathcal{L}(x,s,\cdot)$ is strictly convex and p-homogeneous with $\mathcal{L}(x,s,0) = 0$. Moreover, we shall assume that:

- There exists $\nu > 0$ such that
$$\mathcal{L}(x,s,\xi) \geq \frac{\nu}{p}|\xi|^p$$
a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;
- there exists $c_1, c_2 \in \mathbb{R}$ such that
$$|D_s\mathcal{L}(x,s,\xi)| \leq c_1|\xi|^p$$
a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and
$$\left|\nabla^2_{\xi\xi}\mathcal{L}(x,s,\xi)\right| \leq c_2|\xi|^{p-2} \tag{6.39}$$
a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;
- there exist $R > 0$ and $\gamma \in]0, q-p[$ such that
$$|s| \geq R \Rightarrow D_s\mathcal{L}(x,s,\xi)s \geq 0 \tag{6.40}$$
a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and
$$D_s\mathcal{L}(x,s,\xi)s \leq \gamma\mathcal{L}(x,s,\xi) \tag{6.41}$$
a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

Assumptions (6.40) and (6.41) have already been considered in literature (see [6, 113, 133]). Under the previous assumptions, the following is our main result:

Theorem 6.14. *There exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exists $\varepsilon_0 > 0$ such that (6.36) has at least two nontrivial solutions in $W_0^{1,p}(\Omega)$ for each $0 < \varepsilon < \varepsilon_0$.*

This result extends the achievements of [41, 141] to a more general class of elliptic boundary value problems. We stress that, unlike in [41], we proved our result without any use of concentration-compactness techniques. Indeed, to prove the existence of the first solution as a local minimum of $f_{\varepsilon,\lambda}$, we showed that our functional is weakly lower semi-continuous on small balls of $W_0^{1,p}(\Omega)$. From this point of view, our approach seems to be simpler and more direct. Furthermore, we gave in Theorem 6.25 a precise range of compactness for $f_{\varepsilon,\lambda}$. This, to our knowledge, has not been previously stated for fully nonlinear elliptic problems and not even for the quasi-linear elliptic equation (6.37). In fact, in [10] it was only found a "nontrivial energy range" for the functional, inside which weak limits of Palais-Smale sequences are nontrivial and are solutions of (6.37).

Remark 6.15. Note that no asymptotic behavior has been assumed on $\mathcal{L}(x,s,\xi)$ and $D_s\mathcal{L}(x,s,\xi)s$ when s goes to $+\infty$, while in [10], to prove that problem (6.37) has a solution, it was assumed that

$$\lim_{s\to+\infty} a_{ij}(x,s) = \delta_{ij}, \quad \lim_{s\to+\infty} sD_s a_{ij}(x,s) = 0, \quad (i,j = 1,\dots,n)$$

uniformly with respect to $x \in \Omega$, namely problem (6.37) converges "in some sense" to the semi-linear equation $-\Delta u = |u|^{2^*-2}u + \lambda u$.

Remark 6.16. We point out that we assumed (6.40) just for $|s| \geq R$, while in [10], for problem (6.37), it was assumed that:

$$\forall s \in \mathbb{R} : \sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i\xi_j \geq 0$$

for a.e. $x \in \Omega$ and each $\xi \in \mathbb{R}^n$.

6.4. **The first solution.** Let us note that by combining $\mathcal{L}(x,s,0) = 0$ and (6.39), one finds $b_1, b_2 > 0$ such that:

$$\mathcal{L}(x,s,\xi) \leq b_1|\xi|^p, \tag{6.42}$$

for a.e. $x \in \Omega$ and each $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$ and

$$|\nabla_\xi \mathcal{L}(x,s,\xi)| \leq b_2|\xi|^{p-1} \tag{6.43}$$

for a.e. $x \in \Omega$ and each $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$. We now prove a weakly lower semi-continuity property for $f_{\varepsilon,\lambda}$.

Theorem 6.17. *There exists $\varrho > 0$ such that the functional $f_{\varepsilon,\lambda}$ is weakly lower semi-continuous on $\{u \in W_0^{1,p}(\Omega) : \|u\|_{1,p} \leq \varrho\}$, for each $\lambda \in \mathbb{R}$ and $\varepsilon > 0$.*

Proof. Let $(u_h) \subset W_0^{1,p}(\Omega)$ and u with $u_h \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\|u_h\|_{1,p} \leq \varrho$. Taking into account that up to a subsequence we have

$$u_h \to u \quad \text{in } L^p(\Omega), \quad \nabla u_h \rightharpoonup \nabla u \quad \text{in } L^p(\Omega), \tag{6.44}$$

and $u_h(x) \to u(x)$ for a.e. $x \in \Omega$, by the growth condition (6.42), it results:

$$\int_\Omega \mathcal{L}(x,u_h,\nabla u)\,dx = \int_\Omega \mathcal{L}(x,u,\nabla u)\,dx + o(1),$$

as $h \to +\infty$. Also note that

$$\int_\Omega |u_h|^q\,dx = \int_\Omega |u|^q\,dx + o(1),$$

$$\int_\Omega hu_h\,dx = \int_\Omega hu\,dx + o(1),$$

as $h \to +\infty$. In particular, it suffices to show that for ϱ small:

$$\begin{aligned}\liminf_h \Big\{ \int_\Omega \mathcal{L}(x, u_h, \nabla u_h)\,dx - \int_\Omega \mathcal{L}(x, u_h, \nabla u)\,dx \\ - \frac{1}{p^*}\int_\Omega |u_h|^{p^*}\,dx + \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx \Big\} \geq 0\end{aligned} \tag{6.45}$$

Let us now consider for each $k \geq 1$ the function $T_k : \mathbb{R} \to \mathbb{R}$ given by

$$T_k(s) = \begin{cases} -k & \text{if } s \leq -k \\ s & \text{if } -k \leq s \leq k \\ k & \text{if } s \geq k \end{cases}$$

and let $R_k : \mathbb{R} \to \mathbb{R}$ be the map defined by $R_k = Id - T_k$, namely

$$R_k(s) = \begin{cases} s+k & \text{if } s \leq -k \\ 0 & \text{if } -k \leq s \leq k \\ s-k & \text{if } s \geq k. \end{cases}$$

It is easily seen that

$$\int_\Omega \mathcal{L}(x, u_h, \nabla u_h)\,dx = \int_\Omega \mathcal{L}(x, u_h, \nabla T_k(u_h))\,dx + \int_\Omega \mathcal{L}(x, u_h, \nabla R_k(u_h))\,dx, \tag{6.46}$$

for each $k \in \mathbb{N}$. Of course, we also have

$$\int_\Omega \mathcal{L}(x, u_h, \nabla u)\,dx = \int_\Omega \mathcal{L}(x, u_h, \nabla T_k(u))\,dx + \int_\Omega \mathcal{L}(x, u_h, \nabla R_k(u))\,dx, \tag{6.47}$$

for each $k \in \mathbb{N}$. Now, taking into account that

$$\int_\Omega |u|^{p^*-1}|u_h - u|\,dx = o(1)$$

as $h \to +\infty$, and that for any $k \in \mathbb{N}$

$$\int_\Omega |T_k(u_h) - T_k(u)|^{p^*}\,dx = o(1)$$

as $h \to +\infty$, there exist $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned} &\frac{1}{p^*}\int_\Omega |u_h|^{p^*}\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx \\ &\leq c_1 \int_\Omega \Big(|u_h|^{p^*-1} + |u|^{p^*-1}\Big)|u_h - u|\,dx \\ &\leq c_2 \int_\Omega |u_h - u|^{p^*}\,dx + o(1) \\ &\leq c_3 \int_\Omega |T_k(u_h) - T_k(u)|^{p^*}\,dx + c_3 \int_\Omega |R_k(u_h) - R_k(u)|^{p^*}\,dx + o(1) \\ &= c_3 \int_\Omega |R_k(u_h) - R_k(u)|^{p^*}\,dx + o(1) \end{aligned} \tag{6.48}$$

for any k fixed, as $h \to +\infty$. For each $h, k \in \mathbb{N}$ we have

$$\int_\Omega \mathcal{L}(x, u_h, \nabla R_k(u_h))\,dx \geq \frac{\nu}{p}\int_\Omega |\nabla R_k(u_h)|^p\,dx.$$

On the other hand, by the definition of R_k we have

$$\int_\Omega \mathcal{L}(x,u_h,\nabla R_k(u))\,dx \le c_1\int_\Omega |\nabla R_k(u)|^p\,dx \le \ \le \frac{\nu}{p}\int_\Omega |\nabla R_k(u)|^p\,dx + o(1)\,,$$

as $k\to+\infty$, uniformly in $h\in\mathbb{N}$. In particular, since for each $k\in\mathbb{N}$ it holds

$$\liminf_h \Big\{\int_\Omega \mathcal{L}(x,u_h,T_k(\nabla u_h))\,dx - \int_\Omega \mathcal{L}(x,u_h,T_k(\nabla u))\,dx\Big\}\ge 0,$$

by (6.46), (6.47) and (6.48) there exists $c_p>0$ such that:

$$\begin{aligned}
&\liminf_h \Big\{\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx - \int_\Omega \mathcal{L}(x,u_h,\nabla u)\,dx \\
&-\frac{1}{p^*}\int_\Omega |u_h|^{p^*}\,dx + \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx\Big\} \\
&\ge \liminf_h \Big\{\int_\Omega \mathcal{L}(x,u_h,\nabla R_k(u_h))\,dx - \int_\Omega \mathcal{L}(x,u_h,\nabla R_k(u))\,dx \\
&\quad - c_3\int_\Omega |R_k(u_h)-R_k(u)|^{p^*}\,dx\Big\} \\
&\ge \liminf_h \Big\{\frac{\nu}{p}\int_\Omega |\nabla R_k(u_h)|^p\,dx - \frac{\nu}{p}\int_\Omega |\nabla R_k(u)|^p\,dx \\
&\quad - c_3\int_\Omega |R_k(u_h)-R_k(u)|^{p^*}\,dx\Big\} - o(1) \ge \\
&\ge \liminf_h \Big\{c_p\int_\Omega |\nabla R_k(u_h)-\nabla R_k(u)|^p\,dx \\
&\quad - c_3\int_\Omega |R_k(u_h)-R_k(u)|^{p^*}\,dx\Big\} - o(1)
\end{aligned} \tag{6.49}$$

as $k\to+\infty$. Now, by Sobolev inequality, we find $b_1,b_2>0$ with

$$\begin{aligned}
&\liminf_h \Big\{c_p\int_\Omega |\nabla R_k(u_h)-\nabla R_k(u)|^p\,dx - c_3\int_\Omega |R_k(u_h)-R_k(u)|^{p^*}\,dx\Big\} \\
&\ge \liminf_h \|R_k(u_h)-R_k(u)\|_{p^*}^p\Big\{b_1 - b_2\|R_k(u_h)-R_k(u)\|_{p^*}^{p^*-p}\Big\}\ge 0
\end{aligned}$$

provided that $\|u_h\|_{1,p}\le\varrho$ with ϱ sufficiently small and independent of ε and λ. In particular, (6.45) follows by (6.49) by the arbitrariness of k. □

Lemma 6.18. *For each $\lambda\in\mathbb{R}$ there exist $\varepsilon>0$ and $\varrho,\eta>0$ such that*

$$\forall u\in W_0^{1,p}(\Omega):\ \|u\|_{1,p}=\varrho \Rightarrow f_{\varepsilon,\lambda}(u)>\eta.$$

Proof. Since

$$f_{\varepsilon,\lambda}(u)\ge \frac{\nu}{p}\int_\Omega |\nabla u|^p\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \varepsilon\int_\Omega hu\,dx,$$

arguing as in [41, Lemma 2], one gets

$$f_{\varepsilon,\lambda}(u)\ge \|u\|_{1,p}\Big\{\|u\|_{1,p}^{p-1}\varphi_\lambda(\|u\|_{1,p}) - \varepsilon\|h\|_{p'}c\mathcal{L}^n(\Omega)^{\frac{p^*-p}{pp^*}}\Big\} \tag{6.50}$$

where $\varphi_\lambda:[0,+\infty[\to\mathbb{R}$ is given by

$$\varphi_\lambda(\tau)=\frac{\nu}{p}-\frac{S^{-p^*}}{p^*}\tau^{p^*-p}-\frac{\lambda}{q}c^q\mathcal{L}^n(\Omega)^{\frac{p^*-q}{p^*}}\tau^{q-p}$$

for some $c > 0$. The assertion now follows. □

Proposition 6.19. *For each $\lambda \in \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that (6.36) admits at least one nontrivial solution $u_1 \in W_0^{1,p}(\Omega)$ for each $\varepsilon < \varepsilon_0$. Moreover $f_{\varepsilon,\lambda}(u_1) < 0$.*

Proof. Let us choose $\phi \in W_0^{1,p}(\Omega)$ in such a way that: $\int_\Omega h\phi\, dx > 0$. Therefore, since for each $t > 0$ it results

$$f_{\varepsilon,\lambda}(t\phi) = t^p \int_\Omega \mathcal{L}(x, t\phi, \nabla\phi)\, dx - \frac{t^{p^*}}{p^*} \int_\Omega |\phi|^{p^*}\, dx - \frac{\lambda t^q}{q} \int_\Omega |\phi|^q\, dx - \varepsilon t \int_\Omega h\phi\, dx,$$

there exists $t_{\varepsilon,\lambda} > 0$ such that $f_{\varepsilon,\lambda}(t\phi) < 0$ for each $t \in]0, t_{\varepsilon,\lambda}[$. In particular,

$$\inf_{\|u\|_{1,p} \le \varrho} f_{\varepsilon,\lambda}(u) < 0\,,$$

for each $\varrho > 0$ sufficiently small. Now, by Theorem 6.17 there exist $\varrho > 0$ and $u_1 \in W_0^{1,p}(\Omega)$ with $\|u_1\|_{1,p} \le \varrho$ such that:

$$f_{\varepsilon,\lambda}(u_1) = \min_{\|u\|_{1,p} \le \varrho} f_{\varepsilon,\lambda}(u) < 0.$$

Moreover, up to reducing ϱ, it has to be $\|u_1\|_{1,p} < \varrho$ if $\varepsilon > 0$ is small enough, otherwise by Lemma 6.18 we would get $f_{\varepsilon,\lambda}(u_1) \ge 0$. In particular, u_1 is a solution of (6.36). □

Remark 6.20. Note that by (6.50), one can get a weak solution of (6.36) for each $\varepsilon > 0$ on domains Ω with $\mathcal{L}^n(\Omega)$ sufficiently small.

Remark 6.21. Following Lemmas 3 and 4 in [41], one obtains existence of a weak solution also in the case $p \ge q$. On the other hand we remark that if $p \ge q$ and $\lambda > 0$ one has to require that $\mathcal{L}^n(\Omega)$ is sufficiently small.

6.5. **The concrete Palais-Smale condition.** In this section we prove that $f_{\varepsilon,\lambda}$ satisfies the concrete Palais-Smale condition at levels c within a suitable range of values.

Lemma 6.22. *Let $c \in \mathbb{R}$. Then each $(CPS)_c$-sequence for $f_{\varepsilon,\lambda}$ is bounded.*

Proof. Let $c \in \mathbb{R}$ and let (u_h) be a $(CPS)_c$-sequence for $f_{\varepsilon,\lambda}$. Set:

$$\begin{aligned}\langle w_h, \varphi\rangle &= \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)\cdot\nabla\varphi\, dx + \int_\Omega D_s\mathcal{L}(x, u_h, \nabla u_h)\varphi\, dx \\ &\quad - \int_\Omega g_{\varepsilon,\lambda}(x, u_h)\varphi\, dx - \int_\Omega |u_h|^{p^*-2}u_h\varphi\, dx\end{aligned}$$

for all $\varphi \in C_c^\infty(\Omega)$ where $\|w_h\|_{-1,p'} \to 0$ as $h \to +\infty$ and

$$g_{\varepsilon,\lambda}(x, s) = \lambda|s|^{q-2}s + \varepsilon h(x).$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. It is easily verified that for each $\alpha \in [p, p^*[$ there exists $b_\alpha \in L^1(\Omega)$ such that:

$$g_{\varepsilon,\lambda}(x, s)s + |s|^{p^*} \ge \alpha\Big\{\frac{\lambda}{q}|s|^q + \frac{1}{p^*}|s|^{p^*} + \varepsilon h(x)s\Big\} - b_\alpha(x)$$

a.e. in Ω and for each $s \in \mathbb{R}$. Now, from $\frac{f'_{\varepsilon,\lambda}(u_h)(u_h)}{\|u_h\|_{1,p}} = o(1)$ as $h \to +\infty$, one deduces that

$$\begin{aligned}&\int_\Omega p\mathcal{L}(x, u_h, \nabla u_h)\, dx + \int_\Omega D_s\mathcal{L}(x, u_h, \nabla u_h)u_h\, dx \\ &= \int_\Omega g_{\varepsilon,\lambda}(x, u_h)u_h\, dx + \int_\Omega |u_h|^{p^*}\, dx + \langle w_h, u_h\rangle\end{aligned}$$

$$
\begin{aligned}
&\geq \alpha\Big\{\frac{\lambda}{q}\int_\Omega |u_h|^q\,dx+\frac{1}{p^*}\int_\Omega |u_h|^{p^*}\,dx+\varepsilon\int_\Omega hu_h\,dx\Big\}\\
&\quad-\int_\Omega b_\alpha(x)\,dx+\langle w_h,u_h\rangle\\
&\geq \alpha\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx-\alpha f_{\varepsilon,\lambda}(u_h)-\int_\Omega b_\alpha(x)\,dx+\langle w_h,u_h\rangle.
\end{aligned}
$$

On the other hand, by (6.41) one obtains

$$
\begin{aligned}
\frac{\nu}{p}(\alpha-\gamma-p)\int_\Omega |\nabla u_h|^p\,dx &\leq (\alpha-\gamma-p)\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx\\
&\leq \alpha f_{\varepsilon,\lambda}(u_h)+\int_\Omega b_\alpha(x)\,dx+\|w_h\|_{-1,p'}\|u_h\|_{1,p}.
\end{aligned}
$$

Choosing now $\alpha > p$ in such a way that $\alpha-\gamma-p>0$, one obtains the assertion. □

Remark 6.23. By exploiting the proof of Lemma 6.22 one notes that

$$
\sup\Big\{\Big|\int_\Omega hu\,dx\Big| : u \text{ is a critical point of } f_{\varepsilon,\lambda} \text{ at level } c\in\mathbb{R}\Big\}\leq\sigma
$$

for some $\sigma>0$ independent on $\varepsilon>0$ and $\lambda>0$.

Remark 6.24. Let $1\leq p<\infty$. It is readily seen that the following proposition holds: assume that $u_h\to u$ strongly in $L^p(\Omega)$ and $v_h\to v$ weakly in $L^{p'}(\Omega)$ and a.e. in Ω. Then $u_hv_h\to uv$ strongly in $L^1(\Omega)$.

Let now S denote the best Sobolev constant (cf. [139])

$$
S=\inf\big\{\|\nabla u\|_p^p : u\in W_0^{1,p}(\Omega),\quad \|u\|_{p^*}=1\big\}.
$$

The next result is the main technical tool of this section.

Theorem 6.25. *There exist $K>0$ and $\varepsilon_0>0$ such that $f_{\varepsilon,\lambda}$ satisfies $(CPS)_c$ with*

$$
0<c<\frac{p^*-\gamma-p}{p^*(\gamma+p)}(\nu S)^{n/p}-K\varepsilon \tag{6.51}
$$

for each $\varepsilon<\varepsilon_0$ and $\lambda>0$.

Proof. Let (u_h) be a concrete Palais-Smale sequence for $f_{\varepsilon,\lambda}$ at level c. Since (u_h) is bounded in $W_0^{1,p}(\Omega)$ by Lemma 6.22, up to a subsequence we have

$$
u_h\to u \quad \text{in } L^p(\Omega), \qquad \nabla u_h\rightharpoonup\nabla u \quad \text{in } L^p(\Omega).
$$

Moreover, as shown in [22], we also have:

$$
\text{for}\quad \text{a.e.}\quad x\in\Omega : \nabla u_h(x)\to\nabla u(x).
$$

Arguing as in [133, Theorem 3.2] we get

$$
\langle w_{\varepsilon,\lambda},u\rangle+\|u\|_{p^*}^{p^*}=\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u\,dx+\int_\Omega D_s\mathcal{L}(x,u,\nabla u)u\,dx\,,
$$

where $w_{\varepsilon,\lambda}\in W^{-1,p'}(\Omega)$ is defined by

$$
\langle w_{\varepsilon,\lambda},v\rangle=\lambda\int_\Omega |u|^{q-2}uv\,dx+\varepsilon\int_\Omega hv\,dx.
$$

This, following again [133, Theorem 3.2], yields the existence of $d \in \mathbb{R}$ with

$$\begin{aligned}&\limsup_h \Big\{\int_\Omega \nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h - \int_\Omega |u_h|^{p^*}\,dx\Big\}\\ &\leq d \leq \Big\{\int_\Omega \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u - \int_\Omega |u|^{p^*}\,dx\Big\}.\end{aligned} \tag{6.52}$$

Of course, we have

$$\Big\{\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h) - \nabla_\xi \mathcal{L}(x,u_h,\nabla(u_h-u))\Big\} \rightharpoonup \nabla_\xi \mathcal{L}(x,u,\nabla u)$$

in $L^{p'}(\Omega)$. Let us note that it actually holds the strong limit

$$\Big\{\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h) - \nabla_\xi \mathcal{L}(x,u_h,\nabla(u_h-u))\Big\} \to \nabla_\xi \mathcal{L}(x,u,\nabla u)$$

in $L^{p'}(\Omega)$, since by (6.39) there exist $\tau \in]0,1[$ and $c > 0$ with

$$\begin{aligned}&\big|\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h) - \nabla_\xi \mathcal{L}(x,u_h,\nabla(u_h-u))\big|\\ &\leq \Big|\nabla^2_{\xi\xi} \mathcal{L}(x,u_h,\nabla u_h + (\tau-1)\nabla u)\Big|\,|\nabla u|\\ &\leq c|\nabla u_h|^{p-2}|\nabla u| + c|\nabla u|^{p-1}.\end{aligned}$$

Therefore, by Remark 6.24, we have

$$\begin{aligned}&\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\\ &= \nabla_\xi \mathcal{L}(x,u_h,\nabla(u_h-u))\cdot\nabla u_h + \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u_h + o(1)\\ &= \nabla_\xi \mathcal{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) + \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u + o(1) \quad \text{in } L^1(\Omega)\,,\end{aligned}$$

as $h \to +\infty$, namely

$$\nabla_\xi \mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h - \nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot\nabla u \tag{6.53}$$

$$= \nabla_\xi \mathcal{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) + o(1) \quad \text{in } L^1(\Omega)\,, \tag{6.54}$$

as $h \to +\infty$. In a similar way, since there exists $\tilde{c} > 0$ with

$$\Big||u_h|^{p^*} - |u_h|^{p^*-p}|u_h-u|^p\Big| \leq \tilde{c}\Big[|u_h|^{p^*-p}(|u_h|^{p-1} + |u|^{p-1})\Big]|u|,$$

one obtains

$$\Big\{|u_h|^{p^*} - |u_h|^{p^*-p}|u_h-u|^p\Big\} \to |u|^{p^*} \quad \text{in } L^1(\Omega). \tag{6.55}$$

In particular, by combining (6.52), (6.53) and (6.55), it results:

$$\limsup_h \int_\Omega \Big[\nabla_\xi \mathcal{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) - |u_h|^{p^*-p}|u_h-u|^p\Big]\,dx \leq 0. \tag{6.56}$$

On the other hand, by Hölder and Sobolev inequalities, we get

$$\int_\Omega \Big[\nabla_\xi \mathcal{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) - |u_h|^{p^*-p}|u_h-u|^p\Big]\,dx \tag{6.57}$$

$$\geq \nu\|\nabla(u_h-u)\|_p^p - \frac{1}{S}\|u_h\|_{p^*}^{p^*-p}\|\nabla(u_h-u)\|_p^p \tag{6.58}$$

$$= \{\nu - \frac{1}{S}\|u_h\|_{p^*}^{p^*-p}\}\|\nabla(u_h-u)\|_p^p\,, \tag{6.59}$$

which turns out to be coercive if

$$\limsup_h \|u_h\|_{p^*}^{p^*} < (\nu S)^{n/p}. \tag{6.60}$$

Now, from $f_{\varepsilon,\lambda}(u_h) \to c$ we deduce

$$\int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx - \frac{1}{p^*}\|u_h\|_{p^*}^{p^*} = \frac{\lambda}{q}\|u\|_q^q + \varepsilon \int_\Omega hu\, dx + c + o(1)\,, \tag{6.61}$$

as $h \to +\infty$. On the other hand, by using (6.41), from $f'_{\varepsilon,\lambda}(u_h)(u_h) \to 0$ we obtain

$$\frac{\gamma + p}{p} \int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx - \frac{1}{p}\|u_h\|_{p^*}^{p^*} \geq \frac{\lambda}{p}\|u\|_q^q + \frac{\varepsilon}{p} \int_\Omega hu\, dx + o(1)\,, \tag{6.62}$$

as $h \to +\infty$. Multiplying (6.61) by $\frac{\gamma+p}{p}$, we obtain

$$\frac{\gamma + p}{p} \int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx - \frac{\gamma + p}{pp^*}\|u_h\|_{p^*}^{p^*} \tag{6.63}$$

$$= \frac{\gamma + p}{pq}\lambda\|u\|_q^q + \frac{\gamma + p}{p}\varepsilon \int_\Omega hu + \frac{\gamma + p}{p}c + o(1)\,, \tag{6.64}$$

as $h \to +\infty$. Therefore, by combining (6.63) with (6.62), one gets

$$\frac{p^* - \gamma - p}{pp^*}\|u_h\|_{p^*}^{p^*} \leq -\frac{q - \gamma - p}{pq}\lambda\|u\|_q^q + c'\varepsilon \int_\Omega hu\, dx + \frac{\gamma + p}{p}c + o(1) \tag{6.65}$$

$$\leq c'\varepsilon \int_\Omega hu\, dx + \frac{\gamma + p}{p}c + o(1)\,, \tag{6.66}$$

as $h \to +\infty$. Now, taking into account Remark 6.23, we deduce

$$\|u_h\|_{p^*}^{p^*} \leq \frac{p^*(\gamma + p)}{p^* - \gamma - p}c + \widetilde{K}\varepsilon + o(1),$$

as $h \to +\infty$ for some $\widetilde{K} > 0$. In particular, condition (6.60) is fulfilled if

$$\frac{p^*(\gamma + p)}{p^* - \gamma - p}c + \widetilde{K}\varepsilon < (\nu S)^{n/p}$$

which yields range (6.51) for ε small and a suitable $K > 0$. By combining (6.56) and (6.57) we conclude that u_h goes to u strongly in $W_0^{1,p}(\Omega)$. $\square$

Remark 6.26. We observe that for the equation

$$-\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u + \varepsilon h \quad \text{in } \Omega\,,$$

being $\gamma = 0$ and $\nu = 1$, our range of compactness (6.51) reduces to:

$$0 < c < \frac{S^{n/p}}{n} - K\varepsilon.$$

See also the results of [41].

6.6. **The second solution.** Let us finally come to the proof of Theorem 6.14.

Proof. Let us choose $\phi \in W_0^{1,p} \cap L^\infty(\Omega)$ such that

$$\|\phi\|_{p^*} = 1 \quad \text{and} \quad \int_\Omega h\phi\, dx < 0.$$

It is easily seen that

$$\lim_{t\to+\infty} f_{\varepsilon,\lambda}(t\phi) = -\infty\,,$$

so that there exists $t_{\lambda,\varepsilon} > 0$ with

$$f_{\varepsilon,\lambda}(t_{\lambda,\varepsilon}\phi) = \sup_{t\geq 0} f_{\varepsilon,\lambda}(t\phi) > 0. \tag{6.67}$$

Taking into account (6.41), the value $t_{\lambda,\varepsilon}$ must satisfy

$$\begin{aligned}\varepsilon\int_\Omega h\phi &= t_{\lambda,\varepsilon}^{q-1}\Big\{t_{\lambda,\varepsilon}^{p-q}\Big[\int_\Omega p\mathcal{L}(x,t_{\lambda,\varepsilon}\phi,\nabla\phi)\,dx \\ &\quad+\int_\Omega D_s\mathcal{L}(x,t_{\lambda,\varepsilon}\phi,\nabla\phi)t_{\lambda,\varepsilon}\phi\,dx\Big]-t_{\lambda,\varepsilon}^{p^*-q}-\lambda\int_\Omega|\phi|^q\,dx\Big\} \\ &\le t_{\lambda,\varepsilon}^{q-1}\Big\{t_{\lambda,\varepsilon}^{p-q}M\int_\Omega|\nabla\phi|^p\,dx-t_{\lambda,\varepsilon}^{p^*-q}-\lambda\int_\Omega|\phi|^q\,dx\Big\},\end{aligned}$$

for some $M>0$. Now, being

$$\lim_{\lambda\to+\infty}\Big\{t_{\lambda,\varepsilon}^{p^*-q}+\lambda\int_\Omega|\phi|^q\,dx\Big\}=+\infty$$

it has to be $t_{\lambda,\varepsilon}\to 0$ as $\lambda\to+\infty$. In particular, by (6.67) we obtain

$$\lim_{\lambda\to+\infty}\sup_{t\ge0}f_{\varepsilon,\lambda}(t\phi)=0\,,$$

so that there exists $\lambda_0>0$ such that:

$$0<\sup_{t\ge0}f_{\varepsilon,\lambda}(t\phi)<\frac{p^*-\gamma-p}{p^*(\gamma+p)}(\nu S)^{n/p}-K\varepsilon \tag{6.68}$$

for each $\lambda\ge\lambda_0$ and $\varepsilon<\varepsilon_0$. Let $w=t\phi$ with t so large that $f_{\varepsilon,\lambda}(w)<0$ and set

$$\Phi=\big\{\gamma\in C([0,1],W_0^{1,p}(\Omega)):\gamma(0)=0,\quad\gamma(1)=w\big\}$$

and

$$\beta_{\varepsilon,\lambda}=\inf_{\gamma\in\Phi}\max_{t\in[0,1]}f_{\varepsilon,\lambda}(\gamma(t))$$

Taking into account Lemma 6.18, by Theorem 2.10 one finds $(u_h)\subset W_0^{1,p}(\Omega)$ with:

$$f_{\varepsilon,\lambda}(u_h)\to\beta_{\varepsilon,\lambda},\quad|df_{\varepsilon,\lambda}|(u_h)\to0\,,$$

$$0<\eta\le\beta_{\varepsilon,\lambda}=\inf_{\gamma\in\Phi}\max_{t\in[0,1]}f_{\varepsilon,\lambda}(\gamma(t))\le\sup_{t\ge0}f_{\varepsilon,\lambda}(t\phi). \tag{6.69}$$

By Theorem 6.25 $f_{\varepsilon,\lambda}$ satisfies $(CPS)_{\beta_{\varepsilon,\lambda}}$, since by (6.68) and (6.69)

$$\lambda\ge\lambda_0\Rightarrow 0<\beta_{\varepsilon,\lambda}<\frac{p^*-\gamma-p}{p^*(\gamma+p)}(\nu S)^{n/p}-K\varepsilon$$

for each $\varepsilon<\varepsilon_0$. Therefore there exist a subsequence of $(u_h)\subset W^{1,p}(\Omega)$ strongly convergent to some u_2 which solves (6.36). Since $f_{\varepsilon,\lambda}(u_1)<0$ and $f_{\varepsilon,\lambda}(u_2)>0$, of course $u_1\not\equiv u_2$. □

Remark 6.27. In the case $1<q\le p<p^*$, in general, our method is inconclusive since it may happen that

$$\lim_{\lambda\to+\infty}\sup_{t\ge0}f_{\varepsilon,\lambda}(t\phi)\ne0.$$

See section 4 of [41] where this is discussed for the p-Laplacian.

6.7. **One solution for a more general nonlinearity.** Assume that $\mathcal{L}(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is measurable in x for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$, of class C^1 in (s,ξ) and that $\mathcal{L}(x,s,\cdot)$ is strictly convex and $p-$homogeneous with $\mathcal{L}(x,s,0)=0$. Moreover:

- there exist $\nu>0$ and $c_1,c_2>0$ such that:
$$\mathcal{L}(x,s,\xi)\geq\frac{\nu}{p}|\xi|^p,\quad |D_s\mathcal{L}(x,s,\xi)|\leq c_1|\xi|^p, \tag{6.70}$$
a.e. in Ω and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$ and
$$|\nabla_\xi\mathcal{L}(x,s,\xi)|\leq c_2|\xi|^{p-1}, \tag{6.71}$$
a.e. in Ω and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$;
- there exist $R,R'>0$ and $\gamma\in(0,p^*-p)$ such that:
$$|s|\geq R\Rightarrow D_s\mathcal{L}(x,s,\xi)s\geq 0, \tag{6.72}$$
$$|s|\geq R'\Rightarrow D_s\mathcal{L}(x,s,\xi)s\leq\gamma\mathcal{L}(x,s,\xi), \tag{6.73}$$
a.e. in Ω and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$;
- Let λ_1 be the first eigenvalue of $-\Delta_p$ with homogeneous boundary conditions.

Let $g:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that
$$\forall\varepsilon>0\quad\exists a_\varepsilon\in L^{\frac{np}{n(p-1)+p}}(\Omega):|g(x,s)|\leq a_\varepsilon(x)+\varepsilon|s|^{p^*-1}, \tag{6.74}$$
$$\limsup_{s\to 0}\frac{G(x,s)}{|s|^p}<\frac{\nu\lambda_1}{p},\qquad G(x,s)\geq 0, \tag{6.75}$$
uniformly for a.e. $x\in\Omega$ and each $s\in\mathbb{R}$. Moreover, we assume that there exists a nonempty open set $\Omega_0\subset\Omega$ such that

- if $n<p^2$ (critical dimensions),
$$\lim_{s\to+\infty}\frac{G(x,s)}{s^{p(np+p-2n)/(p-1)(n-p)}}=+\infty \tag{6.76}$$
uniformly for a.e. $x\in\Omega_0$.
- if $n=p^2$: $\exists\mu>0$, there exist $\mu,a>0$ such that
$$\forall s\in[0,a]:G(x,s)\geq\mu|s|^p\quad\text{or}\quad\forall s\geq a:G(x,s)\geq\mu(|s|^p-a^p), \tag{6.77}$$
for a.e. $x\in\Omega_0$.
- if $n>p^2$, there exists $\mu>0$ and $b>a$ such that
$$\forall s\in[a,b]:G(x,s)\geq\mu \tag{6.78}$$
for a.e. $x\in\Omega_0$.

Conditions (6.70), (6.71), (6.72) and (6.73) have already been considered in [6, 133], while assumptions (6.74), (6.75), (6.76), (6.77) and (6.78) can be found in [11]. Note that $g(x,u)$ is neither assumed to be positive nor homogeneous in u.

Under additional assumptions (6.81) and (6.82), that will be stated in the next sections, we have the following result.

Theorem 6.28. *$\mathcal{E}_g$ admits at least one nontrivial solution.*

This result extends the achievements of [10, 11] to a more general class of elliptic boundary value problems. We remark that we assume (6.72) and (6.73) for $|s|\geq R$, while in [10] these assumptions are requested for each $s\in\mathbb{R}$.

6.8. **Existence of one nontrivial solution.** Let us first prove that the concrete Palais-Smale sequences of

$$f(u) = \int_\Omega \mathcal{L}(x, u, \nabla u)\, dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\, dx - \int_\Omega G(x, u)\, dx \tag{6.79}$$

are bounded. We will make a new choice of test function, which also removes some of the technicalities involved in [133].

Lemma 6.29. *Let $c \in \mathbb{R}$. Then each $(CPS)_c$-sequence for f is bounded.*

Proof. Let $c \in \mathbb{R}$ and let (u_h) be a $(CPS)_c$-sequence for f. In the usual notations, one has $\|w_h\|_{-1,p'} \to 0$ as $h \to +\infty$. It is easily verified that for each $\alpha \in [p, p^*[$ there exists $b_\alpha \in L^1(\Omega)$ with

$$g(x, s)s + |s|^{p^*} \geq \alpha\{G(x, s) + \frac{1}{p^*}|s|^{p^*}\} - b_\alpha(x)$$

a.e. in Ω and for each $s \in \mathbb{R}$. Let now $M > 0$, $k \geq 1$ and $\vartheta_k : \mathbb{R} \to \mathbb{R}$,

$$\vartheta_k(s) = \begin{cases} s & \text{if } s \geq kM \\ \frac{M}{M-1}s - \frac{M}{M-1}k & \text{if } k \leq s \leq kM \\ 0 & \text{if } -k \leq s \leq k \\ \frac{M}{M-1}s + \frac{M}{M-1}k & \text{if } -kM \leq s \leq -k \\ s & \text{if } s \leq -kM \end{cases}$$

Since for each $k \in \mathbb{N}$ we have $f'(u_h)(\vartheta_k(u_h)) = o(1)$ as $h \to +\infty$, there exists $C_{k,M} > 0$ such that

$$\begin{aligned}
&\int_{\{|u_h| \geq kM\}} p\mathcal{L}(x, u_h, \nabla u_h) + \frac{M}{M-1}\int_{\{k \leq |u_h| \leq kM\}} p\mathcal{L}(x, u_h, \nabla u_h) \\
&+ \int_{\{|u_h| \geq kM\}} D_s\mathcal{L}(x, u_h, \nabla u_h)u_h \\
&+ \frac{M}{M-1}\int_{\{k \leq |u_h| \leq kM\}} D_s\mathcal{L}(x, u_h, \nabla u_h)(u_h \pm k) \\
&= \int_{\{|u_h| \geq kM\}} g(x, u_h)u_h\, dx + \frac{M}{M-1}\int_{\{k \leq |u_h| \leq kM\}} g(x, u_h)(u_h \pm k) \\
&\quad + \int_{\{|u_h| \geq kM\}} |u_h|^{p^*} + \frac{M}{M-1}\int_{\{k \leq |u_h| \leq kM\}} |u_h|^{p^*-2}u_h(u_h \pm k) + \langle w_h, \vartheta_k(u_h)\rangle \\
&\geq \int_\Omega g(x, u_h)u_h - kM\int_{\{|u_h| \leq kM\}} |g(x, u_h)| \\
&\quad + \frac{M}{M-1}\int_{\{k \leq |u_h| \leq kM\}} g(x, u_h)(u_h \pm k) + \int_\Omega |u_h|^{p^*}\, dx \\
&\quad - kM\int_{\{|u_h| \leq kM\}} |u_h|^{p^*-1} + \frac{M}{M-1}\int_{\{k \leq |u_h| \leq kM\}} |u_h|^{p^*-2}u_h(u_h \pm k)\, dx \\
&\quad + \langle w_h, \vartheta_k(u_h)\rangle \\
&\geq \alpha\Big[\int_\Omega G(x, u_h) + \frac{1}{p^*}\int_\Omega |u_h|^{p^*}\, dx\Big]\int_\Omega b_\alpha(x) \\
&\quad - kM\int_{\{|u_h| \leq kM\}} |g(x, u_h)|\, dx + \frac{M}{M-1}\int_{\{k \leq |u_h| \leq kM\}} g(x, u_h)(u_h \pm k)
\end{aligned}$$

$$
\begin{aligned}
&- kM \int_{\{|u_h|\leq kM\}} |u_h|^{p^*-1} + \frac{M}{M-1}\int_{\{k\leq |u_h|\leq kM\}} |u_h|^{p^*-2}u_h(u_h \pm k) \\
&+ \langle w_h, \vartheta_k(u_h)\rangle \\
\geq\ & \alpha \int_\Omega \mathcal{L}(x,u_h,\nabla u_h) - \alpha f(u_h) - \int_\Omega b_\alpha(x) - C_{k,M} + \langle w_h, \vartheta_k(u_h)\rangle.
\end{aligned}
$$

On the other hand, by (6.73) and (6.72) one obtains

$$
\int_{\{|u_h|\geq \overline{k}\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx \leq \gamma \int_{\{|u_h|\geq \overline{k}\}} \mathcal{L}(x,u_h,\nabla u_h)\,dx, \tag{6.80}
$$

and

$$
\begin{gathered}
-\overline{k}\int_{\{\overline{k}\leq u_h\leq \overline{k}M\}} D_s\mathcal{L}(x,u_h,\nabla u_h)\,dx \leq 0, \\
\overline{k}\int_{\{-\overline{k}M\leq u_h\leq -\overline{k}\}} D_s\mathcal{L}(x,u_h,\nabla u_h)\,dx \leq 0,
\end{gathered}
$$

for some $\overline{k} \geq 1$ so that $\overline{k} \geq \max\{R, R'\}$. Therefore, we find $\widetilde{C}_{\overline{k},M} > 0$ with

$$
\begin{aligned}
&\frac{\nu}{p}\Big(\alpha - \frac{M}{M-1}\gamma - \frac{M}{M-1}p\Big)\int_\Omega |\nabla u_h|^p\,dx \\
&\leq \Big(\alpha - \frac{M}{M-1}\gamma - \frac{M}{M-1}p\Big)\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx \\
&\leq \alpha f(u_h) + \int_\Omega b_\alpha(x)\,dx + \widetilde{C}_{\overline{k},M} + \|w_h\|_{-1,p'}\|\vartheta_{\overline{k}}(u_h)\|_{1,p}.
\end{aligned}
$$

To conclude, choose $\alpha \in]p, p^*[$ and $M > 0$ so that $\alpha - \frac{M}{M-1}\gamma - \frac{M}{M-1}p > 0$. $\square$

Remark 6.30. It has to be pointed out that with the choice of test function ϑ_k there is no need of using [133, Lemma 3.3], which involves lots of very technical computations.

Lemma 6.31. *Let $c \in \mathbb{R}$ and let (u_h) be a $(CPS)_c$-sequence for f such that $u_h \rightharpoonup 0$. Then for each $\varepsilon > 0$ and $\varrho > 0$ we have*

$$
\int_{\{|u_h|\leq \varrho\}} \mathcal{L}(x,u_h,\nabla u_h)\,dx \leq \varepsilon \int_{\{|u_h|>\varrho\}} \mathcal{L}(x,u_h,\nabla u_h)\,dx + o(1),
$$

uniformly as $h \to +\infty$.

Proof. It is a consequence of [133, Lemma 3.3], taking into account that

$$
\int_\Omega (g(x,u_h) + |u_h|^{p^*-1})\vartheta_\delta(u_h)\,dx \to 0
$$

as $h \to +\infty$ (where ϑ_δ is the bounded test function defined in the proof). $\square$

Assume now furthermore that (asymptotic behavior):

$$
\lim_{s\to+\infty} \mathcal{L}(x,s,\xi) = \frac{1}{p}|\xi|^p, \tag{6.81}
$$

$$
\lim_{s\to+\infty} D_s\mathcal{L}(x,s,\xi)s = 0, \tag{6.82}
$$

uniformly with respect to $x \in \Omega$ and to $\xi \in \mathbb{R}^n$ with $|\xi| \leq 1$. This means that there exist $\varepsilon_1 : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ and $\varepsilon_2 : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ such that

$$
\mathcal{L}(x,s,\xi) = \frac{1}{p}|\xi|^p + \varepsilon_1(x,s,\xi)|\xi|^p
$$

$$D_s\mathcal{L}(x,s,\xi)s = \varepsilon_2(x,s,\xi)|\xi|^p$$

where $\varepsilon_{1,2}(x,s,\xi) \to 0$ as $s \to +\infty$ uniformly in $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Let S denote the best Sobolev constant

$$S = \inf\Big\{\|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega), \quad \|u\|_{p^*} = 1\Big\}.$$

Lemma 6.32. *Let $(u_h) \subset W_0^{1,p}(\Omega)$ be a concrete Palais-Smale sequence for f at level c with*

$$0 < c < \frac{1}{n} S^{n/p}.$$

Assume that $u_h \rightharpoonup u$. Then $u \not\equiv 0$.

Proof. Assume by contradiction that $u \equiv 0$. In particular, $u \to 0$ in $L^s(\Omega)$ for each $1 \le s < p^*$. Therefore, taking into account (6.74) and the p-homogeneity of $\mathcal{L}$ with respect to ξ, from $f'(u_h)(u_h) \to 0$ we obtain

$$\int_\Omega p\mathcal{L}(x,u_h,\nabla u_h)\,dx + \int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx - \int_\Omega |u_h|^{p^*}\,dx = o(1), \quad (6.83)$$

as $h \to +\infty$. Let us now prove that for each $\varrho > 0$

$$\lim_h \Big|\int_{\{|u_h|\le\varrho\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx\Big| \le \frac{C''}{\varrho}, \quad (6.84)$$

for some $C'' > 0$. Indeed, since $u_h \rightharpoonup 0$, by Lemma 6.31 and (6.70), one has

$$\begin{aligned}\Big|\int_{\{|u_h|\le\varrho\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx\Big| &\le C\varrho\int_{\{|u_h|\le\varrho\}}\mathcal{L}(x,u_h,\nabla u_h)\,dx\\ &\le C\varrho\varepsilon\int_{\{|u_h|>\varrho\}}\mathcal{L}(x,u_h,\nabla u_h)\,dx + o(1)\\ &\le C'\varrho\varepsilon\int_\Omega|\nabla u_h|^p\,dx + o(1) \le C''\varrho\varepsilon + o(1),\end{aligned}$$

for each $\varrho > 0$ and $\varepsilon > 0$ uniformly as $h \to +\infty$. Then (6.84) follows by choosing $\varepsilon = 1/\varrho^2$. In particular, since condition (6.82) yields

$$\lim_{\varrho\to+\infty}\int_{\{|u_h|>\varrho\}} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx = 0, \quad (6.85)$$

uniformly in $h \in \mathbb{N}$, by combining (6.84) with (6.85), one gets

$$\lim_h \int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx = 0. \quad (6.86)$$

In a similar way, by (6.81), one shows that, as $h \to +\infty$,

$$\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx = \frac{1}{p}\int_\Omega|\nabla u_h|^p\,dx + o(1). \quad (6.87)$$

Therefore, by (6.83) one gets

$$\|u_h\|_{1,p}^p - \|u_h\|_{p^*}^{p^*} = o(1),$$

as $h \to +\infty$. In particular, from the definition of S, it holds

$$\|u_h\|_{1,p}^p\Big(1 - S^{-p^*/p}\|u_h\|_{1,p}^{p^*-p}\Big) \le o(1),$$

as $h \to +\infty$. Since $c > 0$ it has to be

$$\|u_h\|_{1,p}^p \ge S^{n/p} + o(1), \quad \|u_h\|_{p^*}^{p^*} \ge S^{n/p} + o(1),$$

as $h \to +\infty$. Hence, by (6.86) and (6.87) one deduces that

$$f(u_h) = \frac{1}{n}\|u_h\|_{1,p}^p + \frac{1}{p^*}(\|u_h\|_{1,p}^p + \|u_h\|_{p^*}^{p^*}) + o(1) \geq \frac{1}{n} S^{n/p},$$

contradicting the assumption. □

Proof of Theorem 6.28. Let us consider the min-max class

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \quad \gamma(1) = w\}$$

with $f(tw) < 0$ for t large and

$$\beta = \inf_{\gamma \in \Phi} \max_{t \in [0,1]} f(\gamma(t)).$$

Then, by the mountain pass theorem in its non-smooth version (see [36]), one finds a Palais-Smale sequence for f at level β. We have to prove that

$$0 < \beta < \frac{1}{n} S^{n/p}.$$

Consider the family of maps on $\mathbb{R}^n$

$$T_{\delta,x_0}(x) = \frac{c_n \delta^{\frac{n-p}{p(p-1)}}}{\left(\delta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}}$$

with $\delta > 0$ and $x_0 \in \mathbb{R}^n$. T_{δ,x_0} is a solution of $-\Delta_p u = u^{p^*-1}$ on $\mathbb{R}^n$. Taking a function $\phi \in C_c^\infty(\Omega)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ in a neighborhood of x_0 and setting $v_\delta = \phi T_{\delta,x_0}$, it results

$$\|v_\delta\|_{1,p}^p = S^{n/p} + o\left(\varepsilon^{(n-p)/(p-1)}\right), \quad \|v_\delta\|_{p^*}^{p^*} = S^{n/p} + o\left(\varepsilon^{n/(p-1)}\right) \tag{6.88}$$

as $\delta \to 0$, so that, as $\delta \to 0$,

$$\frac{t_\delta^p}{p}\|v_\delta\|_{1,p}^p - \frac{t_\delta^{p^*}}{p^*}\|v_\delta\|_{p^*}^{p^*} \leq \frac{1}{n} S^{n/p} + o\left(\varepsilon^{(n-p)/(p-1)}\right). \tag{6.89}$$

Assume by contradiction that for each $\delta > 0$ there exists $t_\delta > 0$ with

$$f(t_\delta v_\delta) = \frac{t_\delta^p}{p}\|v_\delta\|_{1,p}^p + t_\delta^p \int_\Omega \left\{\mathcal{L}(x, t_\delta v_\delta, \nabla v_\delta) - \frac{1}{p}|\nabla v_\delta|^p\right\} dx \tag{6.90}$$

$$- \int_\Omega G(x, t_\delta v_\delta)\, dx - \frac{t_\delta^{p^*}}{p^*}\|v_\delta\|_{p^*}^{p^*} \geq \frac{1}{n} S^{n/p} \tag{6.91}$$

In particular, there exist $M_1, M_2 > 0$ with $M_1 \leq t_\delta \leq M_2$. Moreover, as proved in [11, Lemma 5], there exists $\tau : [0,1] \to \mathbb{R}$ with $\tau(\varepsilon) \to +\infty$ and

$$\int_\Omega G(x, t_\delta v_\delta)\, dx \geq \tau(\varepsilon)\varepsilon^{(n-p)/(p-1)}. \tag{6.92}$$

as $\varepsilon \to 0$. By (6.72) and (6.81) one also has

$$\int_\Omega \left\{\mathcal{L}(x, t_\delta v_\delta, \nabla v_\delta) - \frac{1}{p}|\nabla v_\delta|^p\right\} dx \leq 0 \tag{6.93}$$

for each $\delta > 0$. By putting together (6.89), (6.90), (6.92), (6.93), one concludes

$$f(t_\delta v_\delta) \leq \frac{1}{n} S^{n/p} + (C - \tau(\varepsilon))\varepsilon^{(n-p)/(p-1)}$$

which contradict (6.90) for ε sufficiently small. □

6.9. **Problems with nearly critical growth.** Let Ω be a bounded domain of $\mathbb{R}^n$, $1 < p < n$ and $p^* = \frac{np}{n-p}$. In 1989 Guedda and Veron [82] proved that the p-Laplacian problem at critical growth

$$\begin{gathered} -\Delta_p u = u^{p^*-1} \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{6.94}$$

has no non-trivial solution $u \in W_0^{1,p}(\Omega)$ if the domain Ω is star-shaped. As known, this non-existence result is due to the failure of compactness for the critical Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, which causes a loss of global Palais-Smale condition for the functional associated with (6.94). On the other hand, if for instance one considers annular domains

$$\Omega_{r_1,r_2} = \{x \in \mathbb{R}^n : 0 < r_1 < |x| < r_2\},$$

then the radial embedding

$$W^{1,p}_{0,rad}(\Omega_{r_1,r_2}) \hookrightarrow L^q(\Omega_{r_1,r_2})$$

is compact for each $q < +\infty$ and one can find a non-trivial radial solution of (6.94) (see [89]). Therefore, we see how the existence of non-trivial solutions of (6.94) is related to the shape of the domain and not just to the topology. In the case $p = 2$, the problem

$$\begin{gathered} -\Delta u = u^{(n+2)/(n-2)} \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{6.95}$$

has been deeply studied and existence results have been obtained provided that Ω satisfies suitable assumptions. In a striking paper [16], Bahri and Coron have proved that if Ω has a non-trivial topology, i.e. if Ω has a non-trivial homology in some positive dimension, then (6.95) always admits a non-trivial solution. Moreover, Dancer [56] constructed for each $n \geq 3$ a contractible domain Ω_n, homeomorphic to a ball, for which (6.95) has a non-trivial solution. See also [112] and references therein for more recent existence and multiplicity results.

We remark that, to our knowledge, this type of achievements are not known when $p \neq 2$. In our opinion, one of the main difficulties is the fact, that differently from the case $p = 2$, it is not proven that all positive solutions of $-\Delta_p u = u^{p^*-1}$ in $\mathbb{R}^n$ are Talenti's radial functions, which attain the best Sobolev constant (see Proposition 6.37).

Now, there is a second approach in the study of problem (6.94), which in general does not require any geometrical or topological assumption on Ω, namely to investigate the asymptotic behavior of solutions u_ε of problems with nearly critical growth

$$\begin{gathered} -\Delta_p u = |u|^{p^*-2-\varepsilon} u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{6.96}$$

as ε goes to 0. If Ω is a ball and $p = 2$, Atkinson and Peletier [12] showed in 1987 the blow-up of a sequence of radial solutions. The extension to the case $p \neq 2$ was achieved by Knaap and Peletier [90] in 1989. On a general bounded domain, instead, the study of limits of solutions of (6.96) was performed by Garcia Azorero and Peral Alonso [72] around 1992.

Let now $\varepsilon > 0$ and consider the following general class of Euler-Lagrange equations with nearly critical growth

$$\begin{gathered} -\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) = |u|^{p^*-2-\varepsilon}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{6.97}$$

associated with the functional $f_\varepsilon : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$f_\varepsilon(u) = \int_\Omega \mathcal{L}(x,u,\nabla u)\,dx - \frac{1}{p^*-\varepsilon}\int_\Omega |u|^{p^*-\varepsilon}\,dx. \tag{6.98}$$

As noted in [133], in general these functionals are not even locally Lipschitzian under natural growth assumptions. Nevertheless, via techniques of non-smooth critical point theory (see [133] and references therein) it can be shown that (6.97) admits a non-trivial solution $u_\varepsilon \in W_0^{1,p}(\Omega)$.

Let $(u_\varepsilon)_{\varepsilon>0}$ denote a sequence of solutions of (6.97). The main goal of this section is to prove that if the weak limit of $(|\nabla u_\varepsilon|^p)_{\varepsilon>0}$ has no blow-up points in Ω, then the limit problem

$$\begin{gathered} -\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) = |u|^{p^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{6.99}$$

has a non-trivial solution (the weak limit of $(u_\varepsilon)_{\varepsilon>0}$), provided that $f_\varepsilon(u_\varepsilon) \to c$ with

$$\frac{p^*-p-\gamma}{pp^*}(\nu S)^{n/p} < c < 2\frac{p^*-p-\gamma}{pp^*}(\nu S)^{n/p}, \tag{6.100}$$

where $\nu > 0$ and $\gamma \in (0, p^*-p)$ will be defined later. In our framework (6.100) plays the role of a generalized second critical energy range (if $\gamma = 0$ and $\nu = 1$, one finds the usual range $\frac{S^{n/p}}{n} < c < 2\frac{S^{n/p}}{n}$ for problem (6.96)).

The plan is as follows: in Section 6.10 we shall state our main results; in Section 6.11 we shall collect the main tools, namely the lower bounds on the non-vanishing Dirac masses and on the non-trivial weak limits; in Section 6.12 we shall prove our main results; finally, in Section 6.13 we shall see that at the mountain pass levels the sequence $(u_\varepsilon)_{\varepsilon>0}$ blows up. Moreover, we shall state a non-existence results obtained via the Pucci-Serrin variational identity.

In the following, we shall always consider the space $W_0^{1,p}(\Omega)$ endowed with the standard norm $\|u\|_{1,p}^p = \int_\Omega |\nabla u|^p\,dx$ and we shall denote by $\|\cdot\|_p$ the usual norm of $L^p(\Omega)$.

6.10. **The main results.** Let Ω be any bounded domain of $\mathbb{R}^n$ and assume that $\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class C^1 in (s,ξ) a.e. in Ω, that $\mathcal{L}(x,s,\cdot)$ is strictly convex and p-homogeneous with $\mathcal{L}(x,s,0) = 0$. Moreover:

- There exist $b_0 > 0$ and $\nu > 0$ such that
$$\frac{\nu}{p}|\xi|^p \le \mathcal{L}(x,s,\xi) \le b_0|s|^p + b_0|\xi|^p \tag{6.101}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;
- there exists $b_1 > 0$ such that for each $\delta > 0$ there exists $a_\delta \in L^1(\Omega)$ with
$$|D_s\mathcal{L}(x,s,\xi)| \le a_\delta(x) + \delta|s|^{p^*} + b_1|\xi|^p \tag{6.102}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, and
$$\big|\nabla_\xi \mathcal{L}(x,s,\xi)\big| \le a_1(x) + b_1|s|^{\frac{p^*}{p'}} + b_1|\xi|^{p-1} \tag{6.103}$$
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, where $a_1 \in L^{p'}(\Omega)$;

- for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$,
$$D_s \mathcal{L}(x, s, \xi)s \geq 0 \tag{6.104}$$
and there exists $\gamma \in (0, p^* - p)$ such that:
$$D_s \mathcal{L}(x, s, \xi)s \leq \gamma \mathcal{L}(x, s, \xi) \tag{6.105}$$
for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

The previous assumptions are natural in the quasi-linear setting and were considered in [133] and in a stronger form in [6].

We stress that although as noted in the introduction f_ε fails to be differentiable on $W_0^{1,p}(\Omega)$, one may compute the derivatives along the L^∞-directions; namely $\forall u \in W_0^{1,p}(\Omega)$, $\forall \varphi \in W_0^{1,p} \cap L^\infty(\Omega)$:
$$\begin{aligned} &f_\varepsilon'(u)(\varphi) \\ &= \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u)\varphi \, dx - \int_\Omega |u|^{p^*-2-\varepsilon} u\varphi \, dx. \end{aligned}$$
By combining the following proposition with (3.25), one can also compute $f_\varepsilon'(u)(u)$.

Proposition 6.33. *Let $u, v \in W_0^{1,p}(\Omega)$ be such that $D_s \mathcal{L}(x, u, \nabla u)v \geq 0$ and*
$$\langle w, \varphi \rangle = \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u)\varphi \, dx. \tag{6.106}$$
for all $\varphi \in C_c^\infty(\Omega)$ and with $w \in W^{-1,p'}(\Omega)$. Then $D_s \mathcal{L}(x, u, \nabla u)v \in L^1(\Omega)$ and one can take $\varphi = v$ in (6.106).

For the proof of the above proposition, see [133, Proposition 3.1].

Under the preceding assumptions, by [133, Theorem 1.1], for each $\varepsilon > 0$ one deduces that (6.97) admits at least one non-trivial solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ (by solution we shall always mean weak solution, namely $f_\varepsilon'(u_\varepsilon) = 0$ in the sense of distributions). We point out that the technical aspects in the verification of the Palais-Smale condition are, in our opinion, interesting and not trivial. As a starting point, let us show that (u_ε) is bounded in $W_0^{1,p}(\Omega)$.

Lemma 6.34. *Let $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ be a sequence of solutions of (6.97) such that*
$$\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) < +\infty.$$
Then $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$.

Proof. If u_ε is a solution of (6.97), we have $f_\varepsilon'(u_\varepsilon)(\varphi) = 0$ for each $\varphi \in C_c^\infty(\Omega)$. On the other hand, taking into account (6.104), by Proposition 6.33 one can choose $\varphi = u_\varepsilon$. Therefore, in view of (6.105) and the p-homogeneity of $\mathcal{L}(x, s, \cdot)$, one obtains
$$\begin{aligned} \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon \to 0} \Big(f_\varepsilon(u_\varepsilon) - \frac{1}{p^* - \varepsilon} f_\varepsilon'(u_\varepsilon)(u_\varepsilon) \Big) \\ &= \lim_{\varepsilon \to 0} \Big(\int_\Omega \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \, dx - \frac{p}{p^* - \varepsilon} \int_\Omega \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \, dx \\ &\quad - \frac{1}{p^* - \varepsilon} \int_\Omega D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx \Big) \\ &\geq \lim_{\varepsilon \to 0} \frac{p^* - p - \varepsilon - \gamma}{p^* - \varepsilon} \int_\Omega \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \, dx \end{aligned}$$

$$\geq \frac{p^* - p - \gamma}{pp^*} \nu \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^p \, dx.$$

In particular, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$. □

As a consequence, one may apply P.L. Lions' concentration-compactness principle (see [96, 97]) and obtain a subsequence of $(u_\varepsilon)_{\varepsilon>0}$, $u \in W_0^{1,p}(\Omega)$ and two bounded positive measures μ and σ such that:

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \quad u_\varepsilon \to u \quad \text{in } L^q(\Omega), \quad 1 < q < p^*, \tag{6.107}$$

$$|\nabla u_\varepsilon|^p \rightharpoonup \mu, \quad |u_\varepsilon|^{p^*} \rightharpoonup \sigma \quad \text{(in the sense of measures)}, \tag{6.108}$$

$$\mu \geq |\nabla u|^p + \sum_{j=1}^\infty \mu_j \delta_{x_j}, \quad \mu_j \geq 0, \tag{6.109}$$

$$\sigma = |u|^{p^*} + \sum_{j=1}^\infty \sigma_j \delta_{x_j}, \quad \sigma_j \geq 0, \tag{6.110}$$

$$\mu_j \geq S \sigma_j^{\frac{p}{p^*}}, \tag{6.111}$$

where δ_{x_j} denotes the Dirac measure at $x_j \in \overline{\Omega}$ and S denotes the best Sobolev constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ (see e.g. [139]).

The following is our main result.

Theorem 6.35. *Let $(u_\varepsilon)_{\varepsilon>0}$ be any sequence of solutions of (6.97) with $f_\varepsilon(u_\varepsilon) \to c$ and*

$$\frac{p^* - p - \gamma}{pp^*} (\nu S)^{n/p} < c < 2 \frac{p^* - p - \gamma}{pp^*} (\nu S)^{n/p}.$$

Then $\mu_j = 0$ for $j \geq 2$ and the following alternative holds:

(a) *$\mu_1 = 0$ and u is a non-trivial solution of (6.99);*
(b) *$\mu_1 \neq 0$ and $u \equiv 0$.*

This result extends [72, Theorem 9] to fully nonlinear elliptic problems.

Theorem 6.36. *Let $(u_\varepsilon)_{\varepsilon>0}$ be any sequence of solutions of (6.97) with*

$$\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \frac{p^* - p - \gamma}{pp^*} (\nu S)^{n/p}.$$

Then $u \equiv 0$.

As we shall see in section (6.13), this is also the behavior when one considers critical levels of mountain-pass type.

6.11. **The weak limit.** Let us briefly summarize the main properties of the best Sobolev constant.

Proposition 6.37. *Let $1 < p < n$ and S be the best Sobolev constant, i.e.*

$$S = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^{p^*} \, dx = 1 \right\}. \tag{6.112}$$

Then, the following facts hold:

(a) *S is independent on $\Omega \subset \mathbb{R}^n$; it depends only on the dimension n;*
(b) *the infimum (6.112) is never achieved on bounded domains $\Omega \subset \mathbb{R}^n$;*

(c) *the infimum (6.112) is achieved if $\Omega = \mathbb{R}^n$ by the family of functions on $\mathbb{R}^n$*

$$T_{\delta,x_0}(x) = \Big(n\delta\big(\frac{n-p}{p-1}\big)^{p-1}\Big)^{\frac{n-p}{p^2}}(\delta + |x-x_0|^{\frac{p}{p-1}})^{-\frac{n-p}{p}} \tag{6.113}$$

with $\delta > 0$ and $x_0 \in \mathbb{R}^n$. Moreover for each $\delta > 0$ and $x_0 \in \mathbb{R}^n$, T_{δ,x_0} is a solution of the equation $-\Delta_p u = u^{p^*-1}$ on $\mathbb{R}^n$.

For the proof of the above proposition, see [139]. The next result establishes uniform lower bounds for the Dirac masses.

Lemma 6.38. *If $\sigma_j \neq 0$, then $\sigma_j \geq \nu^{\frac{n}{p}} S^{\frac{n}{p}}$ and $\mu_j \geq \nu^{\frac{n}{p^*}} S^{\frac{n}{p}}$.*

Proof. Let $x_j \in \overline{\Omega}$ the point which supports the Dirac measure of coefficient $\sigma_j \neq 0$. Denoting with $B(x_j,\delta)$ the open ball of center x_j and radius $\delta > 0$, we can consider a function $\psi_\delta \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \psi_\delta \leq 1$, $|\nabla\psi_\delta| \leq \frac{2}{\delta}$, $\psi_\delta(x) = 1$ if $x \in B(x_j,\delta)$ and $\psi_\delta(x) = 0$ if $x \notin B(x_j, 2\delta)$.

By Proposition 6.33 and the p-homogeneity of $\mathcal{L}(x,s,\cdot)$, we have

$$\begin{aligned}
0 &= f_\varepsilon'(u_\varepsilon)(\psi_\delta u_\varepsilon) \\
&= \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi_\delta\,dx + p\int_\Omega \psi_\delta\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\,dx \\
&\quad + \int_\Omega \psi_\delta D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon\,dx - \int_\Omega |u_\varepsilon|^{p^*-\varepsilon}\psi_\delta\,dx
\end{aligned} \tag{6.114}$$

Applying Hölder inequality and (6.103) to the first term of the decomposition and keeping into account that $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$ and $u_\varepsilon \to u$ in $L^q(\Omega)$ for every $q < p^*$, one find $c_1 > 0$ and $c_2 > 0$ with

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big|\int_\Omega u_\varepsilon\nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi_\delta\,dx\Big| \\
&\leq \Big(\int_{B(x_j,2\delta)}|a_1|^{\frac{p}{p-1}}\,dx\Big)^{\frac{p-1}{p}}\Big(\int_{B(x_j,2\delta)}|u|^{p^*}\,dx\Big)^{\frac{1}{p^*}}\Big(\int_{B(x_j,2\delta)}|\nabla\psi_\delta|^n\,dx\Big)^{\frac{1}{n}} \\
&+ b_1\Big(\int_{B(x_j,2\delta)}|u|^{p^*}\,dx\Big)^{\frac{n-1}{n}}\Big(\int_{B(x_j,2\delta)}|\nabla\psi_\delta|^n\,dx\Big)^{\frac{1}{n}} \\
&+ \tilde{b}_1\Big(\int_{B(x_j,2\delta)}|u|^{p^*}\,dx\Big)^{\frac{1}{p^*}}\Big(\int_{B(x_j,2\delta)}|\nabla\psi_\delta|^n\,dx\Big)^{\frac{1}{n}} \\
&\leq c_1\Big(\int_{B(x_j,2\delta)}|u|^{p^*}\,dx\Big)^{\frac{1}{p^*}} + c_2\Big(\int_{B(x_j,2\delta)}|u|^{p^*}\,dx\Big)^{\frac{n-1}{n}} = \beta_\delta
\end{aligned} \tag{6.115}$$

with $\beta_\delta \to 0$ as $\delta \to 0$. Then, taking into account (6.104) and (6.101) one has

$$\begin{aligned}
0 &\geq -\beta_\delta + \lim_{\varepsilon\to 0}\nu\int_\Omega|\nabla u_\varepsilon|^p\psi_\delta\,dx - \lim_{\varepsilon\to 0}\mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}}\Big(\Big(\int_\Omega|u_\varepsilon|^{p^*}\psi_\delta\,dx\Big)\Big)^{\frac{p^*-\varepsilon}{p^*}} \\
&\geq -\beta_\delta + \nu\int_\Omega\psi_\delta\,d\mu - \int_\Omega\psi_\delta\,d\sigma.
\end{aligned}$$

Letting $\delta \to 0$, it results $\nu\mu_j \leq \sigma_j$. By means of (6.111) one concludes the proof. $\square$

Next result establishes uniform lower bounds for the non-zero weak limits.

Lemma 6.39. *If $u \neq 0$, then $\int_\Omega |\nabla u|^p\,dx > \nu^{\frac{n}{p^*}} S^{n/p}$ and $\int_\Omega |u|^{p^*}\,dx > \nu^{n/p}S^{n/p}$.*

Proof. By Lemma 6.38, we may assume that μ has at most r Dirac masses $\mu_1, \dots, \mu_r$ at $x_1, \dots, x_r$. Let now $0 < \delta < \frac{1}{4} \min_{i \neq j} |x_i - x_j|$ and $\psi_\delta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \psi_\delta \leq 1$, $|\nabla \psi_\delta| \leq \frac{2}{\delta}$, $\psi_\delta(x) = 1$ if $x \in B(x_j, \delta)$ and $\psi_\delta(x) = 0$ if $x \notin B(x_j, 2\delta)$. Taking into account (6.104), for each $\varepsilon, \delta > 0$ we have

$$\int_\Omega D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon (1 - \psi_\delta)\, dx \geq 0.$$

Then, since one can choose $(1 - \psi_\delta) u_\varepsilon$ as test, one obtains

$$\begin{aligned}
0 &= f_\varepsilon'(u_\varepsilon)((1 - \psi_\delta) u_\varepsilon) \\
&= \int_\Omega p \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon)(1 - \psi_\delta)\, dx - \int_\Omega \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta u_\varepsilon\, dx \\
&\quad + \int_\Omega D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon (1 - \psi_\delta)\, dx - \int_\Omega |u_\varepsilon|^{p^* - \varepsilon} (1 - \psi_\delta)\, dx \\
&\geq \nu \int_\Omega |\nabla u_\varepsilon|^p (1 - \psi_\delta)\, dx - \int_\Omega \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta u_\varepsilon\, dx \\
&\quad - \mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}} \Big(\int_\Omega |u_\varepsilon|^{p^*} (1 - \psi_\delta)\, dx \Big)^{\frac{p^* - \varepsilon}{p^*}} .
\end{aligned} \tag{6.116}$$

On the other hand, arguing as for (6.115), one gets

$$\lim_{\varepsilon \to 0} \Big| \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta\, dx \Big| \leq \beta_\delta \tag{6.117}$$

for each $\delta > 0$. Now, it results

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^p (1 - \psi_\delta)\, dx &= \int_\Omega (1 - \psi_\delta)\, d\mu \\
&\geq \int_\Omega |\nabla u|^p (1 - \psi_\delta)\, dx + \sum_{j=1}^r \mu_j (1 - \psi_\delta(x_j)) \\
&= \int_\Omega |\nabla u|^p\, dx + o(1)
\end{aligned} \tag{6.118}$$

as $\delta \to 0$ and

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_\Omega |u_\varepsilon|^{p^*} (1 - \psi_\delta)\, dx &= \int_\Omega (1 - \psi_\delta)\, d\sigma \\
&= \int_\Omega |u|^{p^*} (1 - \psi_\delta)\, dx + \sum_{j=1}^r \sigma_j (1 - \psi_\delta(x_j)) \\
&= \int_\Omega |u|^{p^*}\, dx + o(1)
\end{aligned} \tag{6.119}$$

as $\delta \to 0$. Therefore, in view of (6.117), (6.118) and (6.119), by letting $\delta \to 0$ and $\varepsilon \to 0$ in (6.116), one concludes that

$$\nu \int_\Omega |\nabla u|^p\, dx \leq \int_\Omega |u|^{p^*}\, dx. \tag{6.120}$$

As Ω is bounded, by (b) of Proposition 6.37 one has

$$\int_\Omega |\nabla u|^p\, dx > S \Big(\int_\Omega |u|^{p^*}\, dx \Big)^{p/p^*},$$

which combined with (6.120) yields the assertion. □

In the next result we show that weak limits of $(u_\varepsilon)_{\varepsilon>0}$ are indeed solutions of (6.99).

Lemma 6.40. *Let $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ be a sequence of solutions of (6.97) and let u be its weak limit. Then u is a solution of (6.99).*

Proof. For each $\varepsilon > 0$ one has for all $\varphi \in C_c^\infty(\Omega)$:

$$\begin{aligned}&\int_\Omega \nabla_\xi \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi\,dx + \int_\Omega D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\varphi\,dx\\ &= \int_\Omega |u_\varepsilon|^{p^*-2-\varepsilon}u_\varepsilon\varphi\,dx.\end{aligned}\tag{6.121}$$

Since $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$, up to a subsequence, as $\varepsilon\to 0$, u satisfies

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^p(\Omega), \quad u_\varepsilon \to u \quad \text{in } L^p(\Omega), \quad u_\varepsilon(x)\to u(x) \ \text{ for a.e. } x\in\Omega.$$

Moreover, by [22, Theorem 1], up to a subsequence, we have $\nabla u_\varepsilon(x)\to\nabla u(x)$ for a.e. $x\in\Omega$. Therefore, in view of (6.103) one deduces that

$$\nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon) \rightharpoonup \nabla_\xi\mathcal{L}(x,u,\nabla u) \quad \text{in } L^{p'}(\Omega,\mathbb{R}^n).\tag{6.122}$$

By (6.101) and (6.102) one finds $M>0$ such that for each $\delta>0$

$$|D_s\mathcal{L}(x,s,\xi)| \le M\nabla_\xi\mathcal{L}(x,s,\xi)\cdot\xi + a_\delta(x) + \delta|s|^{p^*}\tag{6.123}$$

for a.e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$. If we test equation (6.121) with the functions

$$\varphi_\varepsilon = \varphi\exp\{-Mu_\varepsilon^+\}, \quad \varphi\in W_0^{1,p}\cap L^\infty(\Omega), \quad \varphi\ge 0$$

for each $\varepsilon>0$ we obtain

$$\begin{aligned}&\int_\Omega \nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi\exp\{-Mu_\varepsilon^+\}\,dx - \int_\Omega |u_\varepsilon|^{p^*-2-\varepsilon}u_\varepsilon\varphi\exp\{-Mu_\varepsilon^+\}\,dx\\ &+\int_\Omega \big[D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon) - M\nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^+\big]\varphi\exp\{-Mu_\varepsilon^+\}\,dx = 0.\end{aligned}$$

Since by inequalities (6.104) and (6.123) for each $\varepsilon>0$ and $\delta>0$ we have

$$\big[D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon) - M\nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^+\big]\varphi\exp\{-Mu_\varepsilon^+\} \le a_\delta(x)+\delta|u_\varepsilon|^{p^*},$$

arguing as in [133, Theorem 3.4], one obtains

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_\Omega \big[D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon) - M\nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^+\big]\varphi\exp\{-Mu_\varepsilon^+\}\,dx\\ &\le \int_\Omega \big[D_s\mathcal{L}(x,u,\nabla u) - M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u^+\big]\varphi\exp\{-Mu^+\}\,dx.\end{aligned}$$

Therefore, taking into account (6.122) and since as $\varepsilon\to 0$,

$$\int_\Omega |u_\varepsilon|^{p^*-2-\varepsilon}u_\varepsilon\varphi\,dx \to \int_\Omega |u|^{p^*-2}u\varphi\,dx$$

for each $\varphi\in W_0^{1,p}\cap L^\infty(\Omega)$ positive, one may conclude that

$$\begin{aligned}&\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\varphi\exp\{-Mu^+\}\,dx - \int_\Omega |u|^{p^*-2}u\varphi\exp\{-Mu^+\}\,dx\\ &+\int_\Omega \big[D_s\mathcal{L}(x,u,\nabla u) - M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u^+\big]\varphi\exp\{-Mu^+\}\,dx \ge 0.\end{aligned}$$

for each $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ positive. Testing now (6.121) with

$$\varphi_k = \varphi\vartheta(\frac{u}{k}) \exp\{Mu^+\}, \quad \varphi \in C_c^\infty(\Omega), \quad \varphi \geq 0,$$

where ϑ is smooth, $\vartheta = 1$ in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\vartheta = 0$ in $]-\infty, -1] \cup [1, +\infty[$, it follows that

$$\begin{aligned}
&\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\varphi_k \exp\{-Mu^+\}\, dx - \int_\Omega |u|^{p^*-2} u\varphi\, \vartheta(\frac{u}{k})\, dx \\
&+ \int_\Omega \left[D_s \mathcal{L}(x, u, \nabla u) - M \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u^+\right] \varphi\, \vartheta(\frac{u}{k})\, dx \geq 0.
\end{aligned}$$

which, arguing again as [133, Theorem 3.4], yields as $k \to +\infty$

$$\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\varphi\, dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u)\varphi\, dx \geq \int_\Omega |u|^{p^*-2} u\varphi\, dx.$$

for each $\varphi \in C_c^\infty(\Omega)$ positive. Working analogously with $\varphi_\varepsilon = \varphi \exp\{-Mu_\varepsilon^-\}$, one obtains the opposite inequality, i.e. u is a solution of (6.99). $\square$

6.12. **Proof of the main results.** Let us now consider a sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions of (6.97) with $f_\varepsilon(u_\varepsilon) \to c$ and

$$\frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p} < c < 2\frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p}. \tag{6.124}$$

Then, there exist a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ and two bounded positive measures μ and σ verifying (6.107), (6.108), (6.109), (6.110) and (6.111).

Proof of Theorem 6.35. Let us first show that there exists at most one j such that $\mu_j \neq 0$. Suppose that $\mu_j \neq 0$ for every $j = 1, \dots r$; in view of Lemma 6.38 one has that $\mu_j \geq \nu^{\frac{n}{p^*}} S^{\frac{n}{p}}$. Following the proof of Lemma 6.34, we obtain

$$\begin{aligned}
c = \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) &\geq \frac{p^* - p - \gamma}{pp^*}\nu \lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^p\, dx \\
&\geq \frac{p^* - p - \gamma}{pp^*}\nu \int_\Omega d\mu \\
&\geq \frac{p^* - p - \gamma}{pp^*}\nu \sum_{j=1}^r \mu_j \\
&\geq r\frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}}.
\end{aligned}$$

Taking into account (6.124) one has

$$2\frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p} > c \geq r\frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}},$$

hence $r \leq 1$. Now, arguing as in Lemma 6.34 one obtains

$$\begin{aligned}
2\,\frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p} > c &= \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \\
&= \lim_{\varepsilon\to 0} \left[f_\varepsilon(u_\varepsilon) - \frac{1}{p^* - \varepsilon} f'_\varepsilon(u_\varepsilon)(u_\varepsilon)\right] \\
&\geq \frac{p^* - p - \gamma}{pp^*}\nu \lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^p\, dx
\end{aligned}$$

$$\geq \frac{p^*-p-\gamma}{pp^*}\Big(\nu\int_\Omega |\nabla u|^p\,dx+\nu\mu_1\Big).$$

If both summands were non-zero, by Lemma 6.38 and Lemma 6.39 we would obtain

$$\nu\int_\Omega |\nabla u|^p\,dx > (\nu S)^{n/p}, \quad \nu\mu_1\geq (\nu S)^{\frac{n}{p}}$$

and thus a contradiction. Viceversa, let us assume that $u\equiv 0$ and $\mu_1=0$. Let $\psi\in C_c^1(\Omega)$ with $\psi\geq 0$. By testing our equation with ψu_ε and using Hölder inequality, one gets

$$\begin{aligned}
&\int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi\,dx + p\int_\Omega \psi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\,dx\\
&+\int_\Omega D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\psi u_\varepsilon\,dx\\
&=\int_\Omega |u_\varepsilon|^{p^*-\varepsilon}\psi\,dx\\
&\leq \Big(\int_\Omega |u_\varepsilon|^{p^*}\psi\,dx\Big)^{\frac{p^*-\varepsilon}{p^*}}\mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}}.
\end{aligned}\tag{6.125}$$

Since $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$, by (6.103) there exists $C>0$ such that

$$\Big|\int_\Omega u_\varepsilon\nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi\,dx\Big|\leq C\,\|u_\varepsilon\|_p\,,$$

which, by $u_\varepsilon\to 0$ in $L^p(\Omega)$, yields

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon\nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi\,dx=0.$$

Moreover, since by (6.104) we get

$$\int_\Omega D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\psi u_\varepsilon\,dx\geq 0,$$

taking into account (6.101) and passing to the limit in (6.125), we get

$$\forall\psi\in C_c(\Omega):\psi\geq 0\Rightarrow \nu\int_\Omega\psi\,d\mu\leq\int_\Omega\psi\,d\sigma.\tag{6.126}$$

On the other hand $\mu_1=0$ and $u=0$ imply $\sigma=0$. Then, since $\mu\geq 0$, by (6.126), we get $\mu=0$. In particular, one gets

$$\begin{aligned}
c&=\lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon)\\
&=\lim_{\varepsilon\to 0}\Big[\frac{p^*-p-\varepsilon}{p^*-\varepsilon}\int_\Omega\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\,dx \qquad -\frac{1}{p^*-\varepsilon}\int_\Omega D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon\,dx\Big]\\
&\leq\frac{pb_0}{n}\lim_{\varepsilon\to 0}\Big(\int_\Omega|u_\varepsilon|^p\,dx+\int_\Omega|\nabla u_\varepsilon|^p\,dx\Big)\\
&=\frac{pb_0}{n}\int_\Omega d\mu=0
\end{aligned}$$

which is not possible. Therefore, either $\mu_1=0$ and $u\not\equiv 0$, or $\mu_1\neq 0$ and $u\equiv 0$. □

Remark 6.41. If (6.124) is replaced by the $(k+1)$-th critical energy range

$$k\frac{p^*-p-\gamma}{pp^*}(\nu S)^{n/p}<c<(k+1)\frac{p^*-p-\gamma}{pp^*}(\nu S)^{n/p}$$

for $k \in \mathbb{N}$, $k \geq 1$, in a similar way one can prove that $\mu_j = 0$ for any $j \geq k+1$ and

(*a*) if $\mu_j = 0$ for every $j \geq 1$, then u is a non-trivial solution of (6.99);
(*b*) if $\mu_j \neq 0$ for every $1 \leq j \leq k$, then $u \equiv 0$.

Remark 6.42. Let $f_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the functional associated with (6.99) and $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, a solution of (6.99) (obtained as weak limit of $(u_\varepsilon)_{\varepsilon>0}$). Then

$$f_0(u) > \frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p}. \tag{6.127}$$

Indeed,

$$\begin{aligned} f_0(u) &= f_0(u) - \frac{1}{p^*} f_0'(u)(u) \\ &\geq \frac{p^* - p - \gamma}{p^*} \int_\Omega \mathcal{L}(x, u, \nabla u)\, dx \\ &\geq \frac{p^* - p - \gamma}{pp^*} \nu \int_\Omega |\nabla u|^p\, dx\,, \end{aligned}$$

which yields (6.127) in view of Lemma 6.39. This, in some sense, explains why one chooses c greater than $\frac{p^*-p-\gamma}{pp^*}(\nu S)^{n/p}$ in Theorem 6.35.

Let now $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of solutions of (6.97) with $f_\varepsilon(u_\varepsilon) \to c$ and

$$\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p}.$$

Proof of Theorem 6.36. Let us first note that

$$f_0(u) \leq \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) + \frac{1}{p^*}\sum_{j=1}^{\infty} \sigma_j. \tag{6.128}$$

Indeed, taking into account that by [53, Theorem 3.4]

$$\int_\Omega \mathcal{L}(x, u, \nabla u)\, dx \leq \lim_{\varepsilon \to 0} \int_\Omega \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon)\, dx,$$

(6.128) follows by combining Hölder inequality with (6.110).

Now assume by contradiction that $u \not\equiv 0$. Then, there exists $j_0 \in \mathbb{N}$ such that $\mu_{j_0} \neq 0$ and $\sigma_{j_0} \neq 0$ otherwise, by Remark 6.42 and (6.128) we would get

$$\frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p} < f_0(u) \leq \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p}.$$

Then, arguing as in Lemma 6.34 and applying Lemma 6.38, we obtain

$$\begin{aligned} \frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p} &= \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) \\ &\geq \frac{p^* - p - \gamma}{pp^*}\Big(\nu \int_\Omega |\nabla u|^p\, dx + \nu \mu_{j_0}\Big) \\ &\geq \frac{p^* - p - \gamma}{pp^*} \nu \int_\Omega |\nabla u|^p\, dx + \frac{p^* - p - \gamma}{pp^*}(\nu S)^{n/p}\,, \end{aligned}$$

which implies $u \equiv 0$, a contradiction. □

6.13. **Mountain-pass critical values.** In this section, we shall investigate the asymptotics of (u_ε) in the case of critical levels of min-max type. We assume that $\mathcal{L}$ satisfies a stronger assumption, i.e.

$$\mathcal{L}(x,s,\xi) \le \frac{1}{p}|\xi|^p \tag{6.129}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. In particular, it results that $\nu \le 1$. Let u_ε be a critical point of f_ε associated with the mountain pass level

$$c_\varepsilon = \inf_{\eta \in \mathcal{C}_\varepsilon} \max_{t\in[0,1]} f_\varepsilon(\eta(t)), \tag{6.130}$$

where

$$\mathcal{C}_\varepsilon = \{\eta \in C([0,1], W_0^{1,p}(\Omega)) : \eta(0) = 0, \quad \eta(1) = w_\varepsilon\}$$

and $w_\varepsilon \in W_0^{1,p}(\Omega)$ is chosen in such a way that $f_\varepsilon(w_\varepsilon) < 0$. If u is the weak limit of $(u_\varepsilon)_{\varepsilon>0}$, as before one can apply P.L. Lions' concentration-compactness principle.

Lemma 6.43. $\lim\limits_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \le \frac{1}{n} S^{n/p}$.

Proof. Let $x_0 \in \Omega$ and $\delta > 0$ and consider the functions T_{δ,x_0} as in (6.113). By (c) of Proposition 6.37, one has:

$$\|\nabla T_{\delta,x_0}\|^p_{p,\mathbb{R}^n} = \|T_{\delta,x_0}\|^{p^*}_{p^*,\mathbb{R}^n} = S^{\frac{n}{p}}.$$

Moreover, taking a function $\phi \in C_c^\infty(\Omega)$ with $0 \le \phi \le 1$ and $\phi = 1$ in a neighborhood of x_0 and setting $v_\delta = \phi T_{\delta,x_0}$, it results

$$\|\nabla v_\delta\|_p^p = S^{\frac{n}{p}} + o(1), \qquad \|v_\delta\|^{p^*}_{p^*} = S^{\frac{n}{p}} + o(1)\,, \tag{6.131}$$

as $\delta \to 0$ (see [82, Lemma 3.2]). We want to prove that, for any $t \ge 0$,

$$\lim_{\varepsilon\to 0} f_\varepsilon(tv_\delta) \le \frac{1}{n} S^{\frac{n}{p}} + o(1)$$

as $\delta \to 0$. By (6.129) one has

$$\begin{aligned}\lim_{\varepsilon\to 0} f_\varepsilon(tv_\delta) &= t^p \int_\Omega \mathcal{L}(x, tv_\delta, \nabla v_\delta)\,dx - \lim_{\varepsilon\to 0} \frac{t^{p^*-\varepsilon}}{p^*-\varepsilon} \int_\Omega |v_\delta|^{p^*-\varepsilon}\,dx \\ &\le \frac{t^p}{p}\int_\Omega |\nabla v_\delta|^p\,dx - \frac{t^{p^*}}{p^*}\int_\Omega |v_\delta|^{p^*}\,dx.\end{aligned}$$

Keeping into account (6.131) and the fact that $\frac{t^p}{p} - \frac{t^{p^*}}{p^*} \le \frac{1}{n}$ for every $t \ge 0$, one gets

$$\lim_{\varepsilon\to 0} f_\varepsilon(tv_\delta) \le \frac{t^p}{p} S^{\frac{n}{p}} - \frac{t^{p^*}}{p^*} S^{\frac{n}{p}} + o(1) \le \frac{1}{n} S^{\frac{n}{p}} + o(1)$$

as $\delta \to 0$. Now choose $t_0 > 0$ such that $f_\varepsilon(t_0 v_\delta) < 0$; by (6.130) we have that

$$\lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \le \lim_{\varepsilon\to 0} \max_{s\in[0,1]} f_\varepsilon(st_0v_\delta) \le \frac{1}{n} S^{\frac{n}{p}} + o(1)$$

and conclude the proof letting $\delta \to 0$. □

Theorem 6.44. *Suppose that the number of non-zero Dirac masses is*

$$\Big[\frac{pp^*}{(p^*-p-\gamma)n\nu^{\frac{n}{p}}}\Big]$$

where $[x]$ *denotes the integer part of* x. *Then* $u \equiv 0$.

Proof. Taking into account the previous lemma and arguing as in Lemma 6.34, one obtains

$$\begin{aligned}\frac{1}{n}S^{\frac{n}{p}} &\geq \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon)\\ &\geq \frac{p^*-p-\gamma}{pp^*}\nu\Big(\int_\Omega |\nabla u|^p\,dx+\sum_{j=1}^r \mu_j\Big)\\ &\geq \frac{p^*-p-\gamma}{pp^*}\nu\int_\Omega |\nabla u|^p\,dx + r\frac{p^*-p-\gamma}{pp^*}\nu^{n/p}S^{\frac{n}{p}},\end{aligned}$$

where r denotes the number of non-vanishing masses. Hence it must be

$$0\leq r\leq \Big[\frac{pp^*}{(p^*-p-\gamma)n\nu^{\frac{n}{p}}}\Big].$$

In particular, if r is maximum and $u\not\equiv 0$, by virtue of Lemma 6.39 one obtains

$$\frac{p^*-p-\gamma}{pp^*}\nu^{n/p}S^{\frac{n}{p}} > \frac{p^*-p-\gamma}{pp^*}\nu\int_\Omega |\nabla u|^p\,dx > \frac{p^*-p-\gamma}{pp^*}\nu^{n/p}S^{\frac{n}{p}},$$

which is a contradiction. □

7. The Singularly Perturbed Case, I

Let Ω be a possibly unbounded smooth domain of $\mathbb{R}^N$ with $N \geq 3$. Since the pioneering work of Floer and Einstein [68] in the one space dimension, much interest has been directed in the last decade to singularly perturbed elliptic problems of the form

$$\begin{gathered} -\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{7.1}$$

for a super-linear and subcritical nonlinearity f with $f(s)/s$ nondecreasing.

Typically, there exists a family of solutions $(u_\varepsilon)_{\varepsilon>0}$ which exhibits a spike shape around the local minima (possibly degenerate) of the function $V(x)$ and decade elsewhere as ε goes to zero (see e.g. [3, 62, 63, 64, 84, 109, 110, 120, 126, 127, 147] and references therein). A natural question is now whether these concentration phenomena are a special feature of the semi-linear case or we can expect a similar behavior to hold for more general elliptic equations which possess a variational structure.

In this section we will give a positive answer to this question for the following class of singularly perturbed quasi-linear elliptic problems

$$\begin{gathered} -\varepsilon^2 \sum_{i,j=1}^N D_j(a_{ij}(x,u)D_i u) + \frac{\varepsilon^2}{2} \sum_{i,j=1}^N D_s a_{ij}(x,u) D_i u D_j u + V(x)u = f(u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{7.2}$$

under suitable assumptions on the functions a_{ij}, V and f. Notice that if $a_{ij}(x,s) = \delta_{ij}$ then equation (7.2) reduces to (7.1), in which case the problem originates from different physical and biological models and, in particular, in the study of the so called *standing waves* for the nonlinear Schrödinger equation.

Existence and multiplicity results for equations like (7.2) have been object of a very careful analysis since 1994 (see e.g. [6, 7, 33, 36, 133] for the case where Ω is bounded and [48, 131] for Ω unbounded). On the other hand, to the author's knowledge, no result on the *asymptotic behavior* of the solutions (as ε vanishes) of (7.2) can be found in literature. In particular no achievement is known so far concerning the concentration phenomena for the solutions u_ε of (7.2) around the local minima, not necessarily non-degenerate, of V.

We stress that various difficulties arise in comparison with the study of the semi-linear equation (7.1) (see Section 7.4 for a list of properties which are not known to hold in our framework).

A crucial step in proving our main result is to show that the *Mountain-Pass* energy level of the functional J associated with the autonomous limiting equation

$$-\sum_{i,j=1}^N D_j(a_{ij}(\widehat{x},u)D_i u) + \frac{1}{2} \sum_{i,j=1}^N D_s a_{ij}(\widehat{x},u) D_i u D_j u + V(\widehat{x})u = f(u) \quad \text{in } \mathbb{R}^N \tag{7.3}$$

with $\widehat{x} \in \mathbb{R}^N$, is the *least* among other nontrivial critical values (Lemma 7.10). Notice that, no uniqueness result is available, to our knowledge, for this general equation (on the contrary in the semi-linear case some uniqueness theorems for ground state solutions have been obtained by performing an ODE analysis in radial coordinates, see e.g. [44]). The least energy problem for (7.3) is also related to the fact:

$$u \in H^1(\mathbb{R}^N),\ u \geq 0 \text{ and } u \text{ solution of (7.3) implies that } J(u) = \max_{t\geq 0} J(tu) \tag{7.4}$$

Unfortunately, as remarked in [48, section 3], if one assumes that condition (7.10) holds, then property (7.4) *cannot* hold true even if the map $s \mapsto f(s)/s$ is nondecreasing.

To show the minimality property for the Mountain-Pass level and to study the uniform limit of u_ε on $\partial\Lambda$, inspired by the recent work of Jeanjean and Tanaka [83], we make a repeated use of the Pucci-Serrin identity [117], which has turned out to be a very powerful tool (Lemmas 7.10 and 7.11).

Notice that the functional associated with (7.2) (see (7.16)) is not even locally Lipschitz and tools of non-smooth critical point theory will be employed (see [50, 58] and references therein). Also the proof of a suitable Palais-Smale type condition for a modification of the functional I_ε becomes more involved.

We assume that $f \in C^1(\mathbb{R}^+)$ and there exist $1 < p < \frac{N+2}{N-2}$ and $2 < \vartheta \leq p+1$ with

$$\lim_{s\to+\infty} \frac{f(s)}{s^p} = 0, \qquad \lim_{s\to 0^+} \frac{f(s)}{s} = 0, \tag{7.5}$$

$$0 < \vartheta F(s) \leq f(s)s \quad \text{for every } s \in \mathbb{R}^+, \tag{7.6}$$

where $F(s) = \int_0^s f(t)\,dt$ for every $s \in \mathbb{R}^+$.

Furthermore, let $V : \mathbb{R}^N \to \mathbb{R}$ be a locally Hölder continuous function bounded below away from zero, that is, there exists $\alpha > 0$ with

$$V(x) \geq \alpha \quad \text{for every } x \in \mathbb{R}^N. \tag{7.7}$$

The functions $a_{ij}(x,s) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are continuous in x and of class C^1 with respect to s, $a_{ij}(x,s) = a_{ji}(x,s)$ for every $i, j = 1, \dots, N$ and there exists a positive constant C with

$$|a_{ij}(x,s)| \leq C, \quad |D_s a_{ij}(x,s)| \leq C$$

for every $x \in \Omega$ and $s \in \mathbb{R}^+$. Finally, let $R, \nu > 0$ and $0 < \gamma < \vartheta - 2$ be such that

$$\sum_{i,j=1}^N a_{ij}(x,s)\xi_i\xi_j \geq \nu|\xi|^2, \tag{7.8}$$

$$\sum_{i,j=1}^N sD_s a_{ij}(x,s)\xi_i\xi_j \leq \gamma \sum_{i,j=1}^N a_{ij}(x,s)\xi_i\xi_j, \tag{7.9}$$

$$s \geq R \ \Rightarrow \ \sum_{i,j=1}^N D_s a_{ij}(x,s)\xi_i\xi_j \geq 0 \tag{7.10}$$

for every $x \in \Omega$, $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$.

Hypothesis (7.5), (7.6) and (7.7) on f and V are standard. Observe that neither *monotonicity* assumptions on the function $f(s)/s$ nor *uniqueness* conditions on the limiting equation (7.3) are considered. Finally, (7.9) and (7.10) have already been used, for instance in [6, 7, 33, 36, 48], in order to tackle these general equations.

Let $H_V(\Omega)$ be the weighted Hilbert space defined by

$$H_V(\Omega) = \left\{ u \in H_0^1(\Omega) : \int_\Omega V(x)u^2 < +\infty \right\},$$

endowed with the scalar product $(u,v)_V = \int_\Omega DuDv + V(x)uv$ and denote by $\|\cdot\|_{H_V(\Omega)}$ the corresponding norm.

Let Λ be a compact subset of Ω such that there exists $x_0 \in \Lambda$ with

$$V(x_0) = \min_{\Lambda} V < \min_{\partial\Lambda} V, \tag{7.11}$$

$$\sum_{i,j=1}^{N} a_{ij}(x_0,s)\xi_i\xi_j = \min_{x\in\Lambda} \sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j \tag{7.12}$$

for every $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$. Let us set

$$\sigma := \sup\big\{s > 0 : f(t) \le tV(x_0) \text{ for every } t \in [0,s]\big\}, \tag{7.13}$$

$$\mathcal{M} := \big\{x \in \Lambda : V(x) = V(x_0)\big\}. \tag{7.14}$$

The following is the main result of the section.

Theorem 7.1. *Assume that conditions (7.5), (7.6), (7.7), (7.8), (7.9), (7.10), (7.11), (7.12) hold. Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0,\varepsilon_0)$, there exist $u_\varepsilon \in H_V(\Omega) \cap C(\overline{\Omega})$ and $x_\varepsilon \in \Lambda$ satisfying the following properties:*

(a) *u_ε is a weak solution of the problem*

$$\begin{gathered} -\varepsilon^2 \sum_{i,j=1}^{N} D_j(a_{ij}(x,u)D_iu) + \frac{\varepsilon^2}{2}\sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_iuD_ju + V(x)u = f(u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,; \end{gathered} \tag{7.15}$$

(b) *there exists $\sigma' > 0$ such that*

$$u_\varepsilon(x_\varepsilon) = \sup_\Omega u_\varepsilon, \quad \sigma < u_\varepsilon(x_\varepsilon) < \sigma', \quad \lim_{\varepsilon\to 0} d(x_\varepsilon,\mathcal{M}) = 0$$

where σ is as in (7.13) and $\mathcal{M}$ is as in (7.14);

(c) *for every $\varrho > 0$ we have*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Omega\setminus B_\varrho(x_\varepsilon))} = 0\,;$$

(d) *we have*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{H_V(\Omega)} = 0$$

and, as a consequence, $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^q(\Omega)} = 0$ for every $2 \le q < +\infty$.

The proof of the theorem is variational and in the spirit of a well-known paper by del Pino and Felmer [62], where it was successfully developed into a *local* setting the global approach initiated by Rabinowitz [120].

We will consider the functional $I_\varepsilon : H_V(\Omega) \to \mathbb{R}$ associated with the problem (7.15),

$$I_\varepsilon(u) := \frac{\varepsilon^2}{2}\sum_{i,j=1}^{N}\int_\Omega a_{ij}(x,u)D_iuD_ju + \frac{1}{2}\int_\Omega V(x)u^2 - \int_\Omega F(u) \tag{7.16}$$

and construct a new functional J_ε which satisfies the Palais-Smale condition (in a suitable sense) at every level (I_ε does not, in general) and to which the (non-smooth) Mountain-Pass Theorem can be directly applied to get a critical point u_ε with precise energy estimates.

Then we will prove that u_ε goes to zero uniformly on $\partial\Lambda$ as ε goes to zero (this is the hardest step, here we repeatedly use the Pucci-Serrin identity in a suitable form) and show that u_ε is actually a solution of the original problem with all of the stated properties.

Remark 7.2. We do not know whether the solutions of problem (7.15) obey to the following *exponential decay*

$$u_\varepsilon(x) \le \alpha \exp\Big\{-\frac{\beta}{\varepsilon}|x-x_\varepsilon|\Big\} \quad \text{for every } x \in \Omega, \text{ for some } \alpha, \beta \in \mathbb{R}^+, \tag{7.17}$$

which is a typical feature in the semi-linear case. This fact would follow if we had a suitable Gidas-Ni-Nirenberg [77] type result for the equation (7.3) to be combined with some results by Rabier and Stuart [118] on the exponential decay of second order elliptic equations.

As pointed out in [64], the concentration around the minima of the potential is, in some sense, a *model situation* for other phenomena such as concentration around the maxima of $d(x, \partial\Omega)$. Furthermore it seems to be the *technically simplest* case, thus suitable for a first investigation in the quasi-linear case.

7.1. **The del Pino-Felmer penalization scheme.** We now define a suitable modification of the functional I_ε in order to regain the (concrete) Palais-Smale condition at any level and apply the Mountain Pass Theorem. Let us consider the positive constant

$$\ell := \sup\Big\{s > 0 : \frac{f(t)}{t} \le \frac{\alpha}{k} \quad \text{for every } 0 \le t \le s\Big\}$$

for some $k > \vartheta/(\vartheta - 2)$. We define the function $\tilde f : \mathbb{R}^+ \to \mathbb{R}$ by setting

$$\tilde f(s) := \begin{cases} \frac{\alpha}{k} s & \text{if } s > \ell \\ f(s) & \text{if } 0 \le s \le \ell \end{cases}$$

and the maps $g, G : \Omega \times \mathbb{R}^+ \to \mathbb{R}$,

$$g(x,s) := \chi_\Lambda(x) f(s) + (1 - \chi_\Lambda(x))\tilde f(s), \quad G(x,s) = \int_0^s g(x,\tau)\,d\tau$$

for every $x \in \Omega$. Then the function $g(x,s)$ is measurable in x, of class C^1 in s and it satisfies the following assumptions:

$$\lim_{s\to+\infty} \frac{g(x,s)}{s^p} = 0, \quad \lim_{s\to 0^+} \frac{g(x,s)}{s} = 0 \quad \text{uniformly in } x \in \Omega, \tag{7.18}$$

$$0 < \vartheta G(x,s) \le g(x,s)s \quad \text{for every } x \in \Lambda \text{ and } s \in \mathbb{R}^+, \tag{7.19}$$

$$0 \le 2G(x,s) \le g(x,s)s \le \frac{1}{k} V(x)s^2 \quad \text{for every } x \in \Omega \setminus \Lambda \text{ and } s \in \mathbb{R}^+. \tag{7.20}$$

Without loss of generality, we may assume that

$$g(x,s) = 0 \quad \text{for every } x \in \Omega \text{ and } s < 0,$$
$$a_{ij}(x,s) = a_{ij}(x,0) \quad \text{for every } x \in \Omega, s < 0 \text{ and } i,j = 1,\dots,N.$$

Let $J_\varepsilon : H_V(\Omega) \to \mathbb{R}$ be the functional

$$J_\varepsilon(u) := \frac{\varepsilon^2}{2} \sum_{i,j=1}^N \int_\Omega a_{ij}(x,u) D_i u D_j u + \frac{1}{2}\int_\Omega V(x)u^2 - \int_\Omega G(x,u).$$

The next result provides the link between the critical points of the modified functional J_ε and the solutions of the original problem.

Proposition 7.3. *Assume that $u_\varepsilon \in H_V(\Omega)$ is a critical point of J_ε and that there exists a positive number ε_0 such that*

$$u_\varepsilon(x) \le \ell \quad \text{for every } \varepsilon \in (0, \varepsilon_0) \text{ and } x \in \Omega \setminus \Lambda.$$

Then u_ε is a solution of (7.15).

Proof. By assertion (a) of Corollary 2.25, it results that u_ε is a solution of the penalized problem. Since $u_\varepsilon \le \ell$ on $\Omega \setminus \Lambda$, we have

$$G(x, u_\varepsilon(x)) = F(u_\varepsilon(x)) \quad \text{for every } x \in \Omega.$$

Moreover, by arguing as in the proof of [131, Lemma 1], one gets $u_\varepsilon > 0$ in Ω. Then u_ε is a solution of (7.15). □

The next Lemma - which is nontrivial - provides a local compactness property for bounded concrete Palais-Smale sequences of J_ε. For the proof, we refer the reader to [131, Theorem 2 and Lemma 3].

Lemma 7.4. *Assume that conditions (7.5), (7.6), (7.7), (7.8), (7.9), (7.10) hold. Let $\varepsilon > 0$. Assume that $(u_h) \subset H^1(\mathbb{R}^N)$ is a bounded sequence and*

$$\langle w_h, \varphi\rangle = \varepsilon^2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij}(x, u_h) D_i u_h D_j \varphi + \frac{\varepsilon^2}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_s a_{ij}(x, u_h) D_i u_h D_j u_h \varphi$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, where (w_h) is strongly convergent in $H^{-1}(\widetilde{\Omega})$ for a given bounded domain $\widetilde{\Omega}$ of $\mathbb{R}^N$.

Then (u_h) admits a strongly convergent subsequence in $H^1(\widetilde{\Omega})$. In particular, if (u_h) is a bounded concrete Palais-Smale condition for J_ε at level c and u is its weak limit, then, up to a subsequence, $Du_h \to Du$ in $L^2(\widetilde{\Omega}, \mathbb{R}^N)$ for every bounded subset $\widetilde{\Omega}$ of Ω.

Since Ω may be unbounded, in general, the original functional I_ε does not satisfy the concrete Palais-Smale condition. In the following Lemma we prove that, instead, the functional J_ε satisfies it for every $\varepsilon > 0$ at every level $c \in \mathbb{R}$.

Lemma 7.5. *Assume that conditions (7.5), (7.6), (7.7), (7.8), (7.9), (7.10) hold. Let $\varepsilon > 0$. Then J_ε satisfies the concrete Palais-Smale condition at every level $c \in \mathbb{R}$.*

Proof. Let $(u_h) \subset H_V(\Omega)$ be a concrete Palais-Smale sequence for J_ε at level c. We divide the proof into two steps:
Step I. Let us prove that (u_h) is bounded in $H_V(\Omega)$. Since $J_\varepsilon(u_h) \to c$, from inequalities (7.19) and (7.20), we get

$$\begin{aligned} &\frac{\vartheta\varepsilon^2}{2} \sum_{i,j=1}^N \int_\Omega a_{ij}(x, u_h) D_i u_h D_j u_h + \frac{\vartheta}{2} \int_\Omega V(x) u_h^2 \\ &\le \int_\Lambda g(x, u_h) u_h + \frac{\vartheta}{2k} \int_{\Omega\setminus\Lambda} V(x) u_h^2 + \vartheta c + o(1) \end{aligned} \tag{7.21}$$

as $h \to +\infty$. Moreover, we have $J_\varepsilon'(u_h)(u_h) = o(\|u_h\|_{H_V(\Omega)})$ as $h \to +\infty$. Then, again by virtue of (7.20), we deduce

$$\varepsilon^2 \sum_{i,j=1}^N \int_\Omega a_{ij}(x, u_h) D_i u_h D_j u_h + \frac{\varepsilon^2}{2} \sum_{i,j=1}^N \int_\Omega D_s a_{ij}(x, u_h) u_h D_i u_h D_j u_h + \int_\Omega V(x) u_h^2$$

$$\geq \int_\Lambda g(x,u_h)u_h + o(\|u_h\|_{H_V(\Omega)}),$$

as $h \to +\infty$, which, by (7.9), yields

$$\begin{aligned}(\frac{\gamma}{2}+1)\varepsilon^2 \sum_{i,j=1}^N \int_\Omega a_{ij}(x,u_h)D_iu_hD_ju_h + \int_\Omega V(x)u_h^2 \\ \geq \int_\Lambda g(x,u_h)u_h + o(\|u_h\|_{H_V(\Omega)})\end{aligned} \tag{7.22}$$

as $h \to +\infty$. Then, in view of (7.8), by combining inequalities (7.21) and (7.22) one gets

$$\begin{aligned}&\min\Big\{(\frac{\vartheta}{2}-\frac{\gamma}{2}-1)\nu\varepsilon^2, \frac{\vartheta}{2}-\frac{\vartheta}{2k}-1\Big\}\int_\Omega \Big(|Du_h|^2+V(x)u_h^2\Big) \\ &\leq \vartheta c + o(\|u_h\|_{H_V(\Omega)}) + o(1)\end{aligned} \tag{7.23}$$

as $h \to +\infty$, which implies the boundedness of (u_h) in $H_V(\Omega)$.

Step II. By virtue of Step I, there exists $u \in H_V(\Omega)$ such that, up to a subsequence, (u_h) weakly converges to u in $H_V(\Omega)$.

Let us now prove that actually (u_h) converges strongly to u in $H_V(\Omega)$. By taking into account Lemma 7.4 (applied with $\widetilde{\Omega} = B_\varrho(0)$ for every $\varrho > 0$), it suffices to prove that for every $\delta > 0$ there exists $\varrho > 0$ such that

$$\limsup_h \int_{\Omega\setminus B_\varrho(0)} \Big(|Du_h|^2 + V(x)u_h^2\Big) < \delta. \tag{7.24}$$

We may assume that $\Lambda \subset B_{\varrho/2}(0)$. Consider a cut-off function $\psi_\varrho \in C^\infty(\Omega)$ with $\psi_\varrho = 0$ on $B_{\varrho/2}(0)$, $\psi_\varrho = 1$ on $\Omega \setminus B_\varrho(0)$, $|D\psi_\varrho| \leq c/\varrho$ on Ω for some positive constant c. Let M be a positive number such that

$$\Big|\frac{1}{2}\sum_{i,j=1}^N D_sa_{ij}(x,s)\xi_i\xi_j\Big| \leq M\sum_{i,j=1}^N a_{ij}(x,s)\xi_i\xi_j \tag{7.25}$$

for every $x \in \Omega$, $s \in \mathbb{R}^+$, $\xi \in \mathbb{R}^N$ and let $\zeta : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$\zeta(s) := \begin{cases} 0 & \text{if } s < 0 \\ Ms & \text{if } 0 \leq s < R \\ MR & \text{if } s \geq R, \end{cases} \tag{7.26}$$

being $R > 0$ the constant defined in (7.10). Notice that

$$\sum_{i,j=1}^N \Big[\frac{1}{2}D_sa_{ij}(x,s) + \zeta'(s)a_{ij}(x,s)\Big]\xi_i\xi_j \geq 0, \tag{7.27}$$

for every $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$. Of course $J_\varepsilon'(u_h)(\psi_\varrho u_h \exp\{\zeta(u_h)\})$ can be computed. Since (u_h) is bounded in $H_V(\Omega)$ and (7.27) holds, we get

$$\begin{aligned}o(1) &= J_\varepsilon'(u_h)(\psi_\varrho u_h \exp\{\zeta(u_h)\}) \\ &= \varepsilon^2\sum_{i,j=1}^N \int_\Omega a_{ij}(x,u_h)D_iu_hD_ju_h\psi_\varrho \exp\{\zeta(u_h)\} \\ &\quad + \varepsilon^2\sum_{i,j=1}^N \int_\Omega \Big[\frac{1}{2}D_sa_{ij}(x,u_h) + \zeta'(u_h)a_{ij}(x,u_h)\Big]D_iu_hD_ju_hu_h\psi_\varrho \exp\{\zeta(u_h)\}\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^2\sum_{i,j=1}^N\int_\Omega a_{ij}(x,u_h)u_hD_iu_hD_j\psi_\varrho\exp\{\zeta(u_h)\}+\int_\Omega V(x)u_h^2\psi_\varrho\exp\{\zeta(u_h)\}\\
&-\int_\Omega g(x,u_h)u_h\psi_\varrho\exp\{\zeta(u_h)\}\\
\ge&\int_\Omega\left(\varepsilon^2\nu|Du_h|^2+V(x)u_h^2\right)\psi_\varrho\exp\{\zeta(u_h)\}\\
&+\varepsilon^2\sum_{i,j=1}^N\int_\Omega a_{ij}(x,u_h)u_hD_iu_hD_j\psi_\varrho\exp\{\zeta(u_h)\}\\
&-\int_\Omega g(x,u_h)u_h\psi_\varrho\exp\{\zeta(u_h)\}.
\end{aligned}
$$

Therefore, in view of (7.20), it results

$$
\begin{aligned}
o(1)\ge&\int_\Omega\left(\varepsilon^2\nu|Du_h|^2+V(x)u_h^2\right)\psi_\varrho\exp\{\zeta(u_h)\}\\
&+\varepsilon^2\sum_{i,j=1}^N\int_\Omega a_{ij}(x,u_h)u_hD_iu_hD_j\psi_\varrho\exp\{\zeta(u_h)\}\\
&-\frac{1}{k}\int_\Omega V(x)u_h^2\psi_\varrho\exp\{\zeta(u_h)\}
\end{aligned}
$$

as $\varrho\to+\infty$. Taking into account that

$$
\Big|\sum_{i,j=1}^N\int_\Omega a_{ij}(x,u_h)u_hD_iu_hD_j\psi_\varrho\exp\{\zeta(u_h)\}\Big|\le\frac{\exp\{MR\}\widetilde{C}}{\varrho}\|Du_h\|_2\|u_h\|_2,
$$

there exists $C'>0$ (which depends only on ε, ν and k) such that, as $\varrho\to+\infty$,

$$
\limsup_h\int_{\Omega\setminus B_\varrho(0)}\left(|Du_h|^2+V(x)u_h^2\right)\le\frac{C'}{\varrho},
$$

which yields (7.24). Therefore $u_h\to u$ strongly in $H_V(\Omega)$ and the proof is complete. □

7.2. **Energy estimates and concentration.** Let us now introduce the functional $J_0:H^1(\mathbb{R}^N)\to\mathbb{R}$ defined by

$$
J_0(u):=\frac12\sum_{i,j=1}^N\int_{\mathbb{R}^N}a_{ij}(x_0,u)D_iuD_ju+\frac12\int_{\mathbb{R}^N}V(x_0)\,u^2-\int_{\mathbb{R}^N}F(u)
$$

where x_0 is as in (7.11). Let us set

$$
\bar c:=\inf_{\gamma\in\mathcal{P}_0}\sup_{t\in[0,1]}J_0(\gamma(t)),
$$

where $\mathcal{P}_0$ is the family defined by

$$
\mathcal{P}_0:=\Big\{\gamma\in C([0,1],H_V(\mathbb{R}^N)):\gamma(0)=0,\quad J_0(\gamma(1))<0\Big\}. \tag{7.28}
$$

Let us also set

$$
\mathcal{P}_\varepsilon:=\{\gamma\in C([0,1],H_V(\Omega)):\gamma(0)=0,\quad J_\varepsilon(\gamma(1))<0\}. \tag{7.29}
$$

In the following, if necessary, we will assume that, for every $\gamma\in\mathcal{P}_\varepsilon$, for every $t\in[0,1]$ the map $\gamma(t)$ is extended to zero outside Ω.

In the next Lemma we get a critical point u_ε of J_ε with a precise energy upper bound.

Lemma 7.6. *For $\varepsilon > 0$ sufficiently small J_ε admits a critical point $u_\varepsilon \in H_V(\Omega)$ such that*

$$J_\varepsilon(u_\varepsilon) \leq \varepsilon^N \bar{c} + o(\varepsilon^N). \tag{7.30}$$

Proof. Let $\varepsilon > 0$. By Lemma 7.5 the functional J_ε satisfies the concrete Palais-Smale condition at every level $c \in \mathbb{R}$. Moreover, since $g(x,s) = o(s)$ as $s \to 0$ uniformly in x, it is readily seen that J_ε verifies the Mountain-Pass geometry. Finally, if z is a positive function in $H_V(\Omega) \setminus \{0\}$ such that $\mathrm{supt}(z) \subset \Lambda$, by (7.6) it results $J_\varepsilon(tz) \to -\infty$ as $t \to +\infty$. Therefore, by minimaxing over the family (7.29), the functional J_ε admits a nontrivial critical point $u_\varepsilon \in H_V(\Omega)$ such that

$$J_\varepsilon(u_\varepsilon) = \inf_{\gamma \in \mathcal{P}_\varepsilon} \sup_{t \in [0,1]} J_\varepsilon(\gamma(t)).$$

Since $\bar{c}$ is the Mountain-Pass value of the limiting functional J_0, for every $\delta > 0$ there exists a continuous path $\gamma : [0,1] \to H_V(\mathbb{R}^N)$ such that

$$\bar{c} \leq \sup_{t\in[0,1]} J_0(\gamma(t)) \leq \bar{c} + \delta, \quad \gamma(0) = 0, \quad J_0(\gamma(1)) < 0. \tag{7.31}$$

Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function with $\zeta = 1$ in a neighborhood U of x_0 in Λ. We define the continuous path $\Gamma_\varepsilon : [0,1] \to H_V(\Omega)$ by setting $\Gamma_\varepsilon(\tau)(x) := \zeta(x)\gamma(\tau)\big(\frac{x-x_0}{\varepsilon}\big)$ for every $\tau \in [0,1]$ and $x \in \Omega$. Then, for every $\tau \in [0,1]$, after extension to zero outside Ω, we have

$$\begin{aligned}
&J_\varepsilon(\Gamma_\varepsilon(\tau))\\
&= \frac{\varepsilon^2}{2}\sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij}\Big(x, \zeta(x)\gamma(\tau)\big(\frac{x-x_0}{\varepsilon}\big)\Big)\, D_i\zeta D_j\zeta\, \gamma^2(\tau)\big(\frac{x-x_0}{\varepsilon}\big)\\
&+ \frac{1}{2}\sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij}\Big(x, \zeta(x)\gamma(\tau)\big(\frac{x-x_0}{\varepsilon}\big)\Big)\, (D_i\gamma(\tau))\big(\frac{x-x_0}{\varepsilon}\big)(D_j\gamma(\tau))\big(\frac{x-x_0}{\varepsilon}\big)\,\zeta^2\\
&+ \varepsilon\sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij}\Big(x, \zeta(x)\gamma(\tau)\big(\frac{x-x_0}{\varepsilon}\big)\Big)\, D_i\zeta(D_j\gamma(\tau))\big(\frac{x-x_0}{\varepsilon}\big)\zeta\gamma(\tau)\big(\frac{x-x_0}{\varepsilon}\big)\\
&+ \frac{1}{2}\int_{\mathbb{R}^N} V(x)\zeta^2(x)\gamma^2(\tau)\big(\frac{x-x_0}{\varepsilon}\big) - \int_{\mathbb{R}^N} G\Big(x, \zeta(x)\gamma(\tau)\big(\frac{x-x_0}{\varepsilon}\big)\Big).
\end{aligned}$$

Then, after the change of coordinates, for every $\tau \in [0,1]$, we get

$$\begin{aligned}
J_\varepsilon(\Gamma_\varepsilon(\tau)) = \frac{\varepsilon^{N+2}}{2} &\sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij}\,(\varepsilon y + x_0, \zeta(\varepsilon y + x_0)\gamma(\tau)(y))\, D_i\zeta(\varepsilon y + x_0)\\
&\times D_j\zeta(\varepsilon y + x_0)\,\gamma^2(\tau)(y)\\
&+ \varepsilon^{N+1}\sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij}\,(\varepsilon y + x_0, \zeta(\varepsilon y + x_0)\gamma(\tau)(y))\, D_i\zeta(\varepsilon y + x_0)\\
&\times D_j\gamma(\tau)(y)\zeta(\varepsilon y + x_0)\gamma(\tau)(y)\\
&+ \frac{\varepsilon^N}{2}\sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij}\,(\varepsilon y + x_0, \zeta(\varepsilon y + x_0)\gamma(\tau)(y))\, D_i\gamma(\tau)(y)
\end{aligned}$$

$$\times D_j\gamma(\tau)(y)\zeta^2(\varepsilon y+x_0)+\frac{\varepsilon^N}{2}\int_{\mathbb{R}^N}V(\varepsilon y+x_0)\zeta^2(\varepsilon y+x_0)\gamma^2(\tau)(y)$$
$$-\varepsilon^N\int_{\mathbb{R}^N}G(\varepsilon y+x_0,\zeta(\varepsilon y+x_0)\gamma(\tau)(y)).$$

Taking into account that for every $\tau\in[0,1]$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}V(\varepsilon y+x_0)\zeta^2(\varepsilon y+x_0)\gamma^2(\tau)(y)=\int_{\mathbb{R}^N}V(x_0)\gamma^2(\tau)(y),$$
$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}G(\varepsilon y+x_0,\zeta(\varepsilon y+x_0)\gamma(\tau)(y))=\int_{\mathbb{R}^N}F(\gamma(\tau)(y)),$$

and

$$\lim_{\varepsilon\to 0}\sum_{i,j=1}^N\int_{\mathbb{R}^N}a_{ij}(\varepsilon y+x_0,\zeta(\varepsilon y+x_0)\gamma(\tau)(y))D_i\gamma(\tau)(y)D_j\gamma(\tau)(y)\zeta^2(\varepsilon y+x_0)$$
$$=\sum_{i,j=1}^N\int_{\mathbb{R}^N}a_{ij}(x_0,\gamma(\tau)(y))D_i\gamma(\tau)(y)D_j\gamma(\tau)(y),$$

we obtain

$$J_\varepsilon(\Gamma_\varepsilon(\tau))=\varepsilon^N\Big\{\frac12\sum_{i,j=1}^N\int_{\mathbb{R}^N}a_{ij}(x_0,\gamma(\tau)(y))D_i\gamma(\tau)(y)D_j\gamma(\tau)(y)$$
$$+\frac12\int_{\mathbb{R}^N}V(x_0)\gamma^2(\tau)(y)-\int_{\mathbb{R}^N}F(\gamma(\tau)(y))\Big\}+o(\varepsilon^N)$$

as $\varepsilon\to 0$, namely

$$J_\varepsilon(\Gamma_\varepsilon(\tau))=\varepsilon^N J_0(\gamma(\tau))+o(\varepsilon^N) \tag{7.32}$$

as $\varepsilon\to 0$, where $o(\varepsilon^N)$ is independent of τ (by a compactness argument). Then, by (7.31) and (7.32), it follows that $\Gamma_\varepsilon\in\mathcal{P}_\varepsilon$ for every $\varepsilon>0$ sufficiently small and,

$$\begin{aligned}J_\varepsilon(u_\varepsilon)&=\inf_{\gamma\in\mathcal{P}_\varepsilon}\sup_{t\in[0,1]}J_\varepsilon(\gamma(t))\le\sup_{t\in[0,1]}J_\varepsilon(\Gamma_\varepsilon(t))\\&=\varepsilon^N\sup_{t\in[0,1]}J_0(\gamma(t))+o(\varepsilon^N)\\&\le\varepsilon^N\bar c+o(\varepsilon^N)+\delta\varepsilon^N\quad\text{for every }\delta>0.\end{aligned}$$

By the arbitrariness of δ one concludes the proof. □

In the following result we get some priori estimates for the rescalings of u_ε.

Corollary 7.7. *Let $(\varepsilon_h)\subset\mathbb{R}^+$, $(x_h)\subset\Lambda$ and assume that $(u_{\varepsilon_h})\subset H_V(\Omega)$ is as in Lemma 7.6. Let us set*

$$v_h\in H_V(\Omega_h),\quad \Omega_h:=\varepsilon_h^{-1}(\Omega-x_h),\quad v_h(x):=u_{\varepsilon_h}(x_h+\varepsilon_h x)$$

and put $v_h=0$ outside Ω_h. Then there exists a positive constant C such that for every $h\in\mathbb{N}$,

$$\|v_h\|_{H^1(\mathbb{R}^N)}\le C\,. \tag{7.33}$$

Proof. We consider the functional $J_h : H_V(\Omega_h) \to \mathbb{R}$ given by

$$J_h(v) := \frac{1}{2} \sum_{i,j=1}^{N} \int_{\Omega_h} a_{ij}(x_h + \varepsilon_h x, v) D_i v D_j v \tag{7.34}$$

$$+ \frac{1}{2} \int_{\Omega_h} V(x_h + \varepsilon_h x) v^2 - \int_{\Omega_h} G(x_h + \varepsilon_h x, v). \tag{7.35}$$

Since $J_h(v_h) = \varepsilon_h{}^{-N} J_{\varepsilon_h}(u_{\varepsilon_h})$, by virtue of Lemma 7.6 we have $J_h(v_h) \leq \bar{c} + o(1)$ as $h \to +\infty$. Therefore, if we set $\Lambda_h = \varepsilon_h^{-1}(\Lambda - x_h)$, from inequalities (7.19) and (7.20), we get

$$\begin{aligned} &\frac{\vartheta}{2} \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} a_{ij}(x_h + \varepsilon_h x, v_h) D_i v_h D_j v_h + \frac{\vartheta}{2} \int_{\mathbb{R}^N} V(x_h + \varepsilon_h x) v_h^2 \\ &\leq \int_{\Lambda_h} g(x_h + \varepsilon_h x, v_h) v_h + \frac{\vartheta}{2k} \int_{\mathbb{R}^N \setminus \Lambda_h} V(x_h + \varepsilon_h x) v_h^2 + \vartheta \bar{c} + o(1) \end{aligned} \tag{7.36}$$

as $h \to +\infty$. Moreover, since it results $J'_h(v_h)(v_h) = 0$ for every $h \in \mathbb{N}$, again by (7.20), we get

$$\begin{aligned} &\sum_{i,j=1}^{N} \int_{\mathbb{R}^N} a_{ij}(x_h + \varepsilon_h x, v_h) D_i v_h D_j v_h \\ &+ \frac{1}{2} \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} D_s a_{ij}(x_h + \varepsilon_h x, v_h) v_h D_i v_h D_j v_h + \int_{\mathbb{R}^N} V(x_h + \varepsilon_h x) v_h^2 \\ &\geq \int_{\Lambda_h} g(x_h + \varepsilon_h x, v_h) v_h, \end{aligned}$$

which, in view of (7.9), yields

$$\begin{aligned} &\Big(\frac{\gamma}{2} + 1\Big) \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} a_{ij}(x_h + \varepsilon_h x, v_h) D_i v_h D_j v_h + \int_{\mathbb{R}^N} V(x_h + \varepsilon_h x) v_h^2 \\ &\geq \int_{\Lambda_h} g(x_h + \varepsilon_h x, v_h) v_h. \end{aligned} \tag{7.37}$$

Then, recalling (7.7) and (7.8), by combining inequality (7.36) and (7.37) one gets

$$\min\Big\{\big(\frac{\vartheta}{2} - \frac{\gamma}{2} - 1\big)\nu, \big(\frac{\vartheta}{2} - \frac{\vartheta}{2k} - 1\big)\alpha\Big\} \int_{\mathbb{R}^N} \Big(|Dv_h|^2 + v_h^2\Big) \leq \vartheta \bar{c} + o(1) \tag{7.38}$$

as $h \to +\infty$, which yields the assertion. □

Corollary 7.8. *Assume that $(u_\varepsilon)_{\varepsilon>0} \subset H_V(\Omega)$ is as in Lemma 7.6. Then*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{H_V(\Omega)} = 0.$$

Proof. We may argue as in Step I of Lemma 7.5 with u_h replaced by u_ε and c replaced by $J_\varepsilon(u_\varepsilon)$. Thus, from inequality (7.23), for every $\varepsilon > 0$ we get

$$\int_{\Omega} \Big(|Du_\varepsilon|^2 + V(x) u_\varepsilon^2\Big) \leq \frac{\vartheta}{\min\Big\{\Big(\frac{\vartheta}{2} - \frac{\gamma}{2} - 1\Big)\nu\varepsilon^2, \frac{\vartheta}{2} - \frac{\vartheta}{2k} - 1\Big\}} J_\varepsilon(u_\varepsilon).$$

By virtue of Lemma 7.6, this yields

$$\int_\Omega \left(|Du_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \le \frac{2\vartheta\bar{c}}{(\vartheta-\gamma-2)\nu}\,\varepsilon^{N-2} + o(\varepsilon^{N-2})$$

for every ε sufficiently small, which implies the assertion. □

Let $\mathcal{L} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a function of class C^1 such that the function $\nabla_\xi \mathcal{L}$ is of class C^1 and let $\varphi \in L^\infty_{\rm loc}(\mathbb{R}^N)$. We now recall the Pucci-Serrin variational identity [117].

Lemma 7.9. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a C^2 solution of*

$$-\operatorname{div}\left(D_\xi \mathcal{L}(x,u,Du)\right) + D_s\mathcal{L}(x,u,Du) = \varphi \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Then for every $h \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$,

$$\begin{aligned} &\sum_{i,j=1}^N \int_{\mathbb{R}^N} D_i h^j\, D_{\xi_i}\mathcal{L}(x,u,Du) D_j u \\ &- \int_{\mathbb{R}^N} \left[(\operatorname{div} h)\,\mathcal{L}(x,u,Du) + h\cdot D_x\mathcal{L}(x,u,Du)\right] = \int_{\mathbb{R}^N} (h\cdot Du)\varphi\,. \end{aligned} \tag{7.39}$$

We refer the reader to [59], where the above variational relation is proved for C^1 solutions. We now derive an important consequence of the previous identity which will play an important role in the proof of Lemma 7.11.

Lemma 7.10. *Let $\mu > 0$ and $h, H : \mathbb{R}^+ \to \mathbb{R}$ be the continuous functions defined by*

$$h(s) = -\mu s + f(s), \quad H(s) = \int_0^s h(t)\,dt,$$

where f satisfies (7.5) and (7.6). Moreover, let $b_{ij} \in C^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ with $b'_{ij} \in L^\infty(\mathbb{R}^+)$ and assume that there exist $\nu' > 0$ and $R' > 0$ with

$$\sum_{i,j=1}^N b_{ij}(s)\xi_i\xi_j \ge \nu'|\xi|^2, \quad s \ge R' \;\Rightarrow\; \sum_{i,j=1}^N b'_{ij}(s)\xi_i\xi_j \ge 0 \tag{7.40}$$

for every $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$. Let $u \in H^1(\mathbb{R}^N)$ be any nontrivial positive solution of the equation

$$-\sum_{i,j=1}^N D_j(b_{ij}(u)D_iu) + \frac{1}{2}\sum_{i,j=1}^N b'_{ij}(u)D_iuD_ju = h(u) \quad \text{in } \mathbb{R}^N. \tag{7.41}$$

We denote by $\widehat{J}$ the associated functional

$$\widehat{J}(u) := \frac{1}{2}\sum_{i,j=1}^N \int_{\mathbb{R}^N} b_{ij}(u)D_iuD_ju - \int_{\mathbb{R}^N} H(u). \tag{7.42}$$

Then it results $\widehat{J}(u) \ge b$, where

$$b := \inf_{\gamma\in\widehat{\mathcal{P}}}\ \sup_{t\in[0,1]} \widehat{J}(\gamma(t)),$$

$$\widehat{\mathcal{P}} := \left\{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \quad \widehat{J}(\gamma(1)) < 0\right\}.$$

Proof. By condition (7.40),

$$\widehat{J}(v) \geq \frac{1}{2} \min\{\nu', \mu\} \|v\|^2_{H^1(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(v) \quad \text{for every } v \in H^1(\mathbb{R}^N).$$

Then, since for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with

$$0 \leq F(s) \leq \varepsilon s^2 + C_\varepsilon |s|^{\frac{2N}{N-2}} \quad \text{for every } s \in \mathbb{R}^+,$$

it is readily seen that there exist $\varrho_0 > 0$ and $\delta_0 > 0$ such that $\widehat{J}(v) \geq \delta_0$ for every v with $\|v\|_{1,2} = \varrho_0$. In particular $\widehat{J}$ has a Mountain-Pass geometry. As we will see, $\widehat{\mathcal{P}} \neq \emptyset$, so that b is well defined. Let u be a nontrivial positive solution of (7.41) and consider the dilation path

$$\gamma(t)(x) := \begin{cases} u(\frac{x}{t}) & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Notice that $\|\gamma(t)\|^2_{H^1} = t^{N-2}\|Du\|_2^2 + t^N\|u\|_2^2$ for every $t \in \mathbb{R}^+$, which implies that the curve γ belongs to $C([0, +\infty[, H^1(\mathbb{R}^N))$. For every $t \in \mathbb{R}^+$ it results that

$$\begin{aligned} \widehat{J}(\gamma(t)) &= \frac{1}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} b_{ij}(\gamma(t)) D_i\gamma(t) D_j\gamma(t) - \int_{\mathbb{R}^N} H(\gamma(t)) \\ &= \frac{t^{N-2}}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} b_{ij}(u) D_i u D_j u - t^N \int_{\mathbb{R}^N} H(u) \end{aligned}$$

which yields, for every $t \in \mathbb{R}^+$

$$\frac{d}{dt}\widehat{J}(\gamma(t)) = \frac{N-2}{2} t^{N-3} \sum_{i,j=1}^N \int_{\mathbb{R}^N} b_{ij}(u) D_i u D_j u - N t^{N-1} \int_{\mathbb{R}^N} H(u). \tag{7.43}$$

By (7.40), arguing like at the end of Step I of Lemma 7.11 (namely using the local Serrin estimates) it results that $u \in L^\infty_{\rm loc}(\mathbb{R}^N)$. Hence by the regularity results of [91], it follows that u is of class C^2. Then we can use Lemma 7.9 by choosing $\varphi = 0$,

$$\mathcal{L}(s, \xi) := \frac{1}{2} \sum_{i,j=1}^N b_{ij}(s)\xi_i\xi_j - H(s) \quad \text{for every } s \in \mathbb{R}^+ \text{ and } \xi \in \mathbb{R}^N, \tag{7.44}$$

$$h(x) := h_k(x) = T(\frac{x}{k})x \quad \text{for every } x \in \mathbb{R}^N \text{ and } k \geq 1, \tag{7.45}$$

being $T \in C^1_c(\mathbb{R}^N)$ such that $T(x) = 1$ if $|x| \leq 1$ and $T(x) = 0$ if $|x| \geq 2$. In particular, it results that $h_k \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$ for every $k \geq 1$ and

$$\begin{gathered} D_i h_k^j(x) = D_i T(\frac{x}{k})\frac{x_j}{k} + T(\frac{x}{k})\delta_{ij} \quad \text{for every } x \in \mathbb{R}^N \text{ and } i, j = 1, \dots, N \\ (\operatorname{div} h_k)(x) = DT(\frac{x}{k}) \cdot \frac{x}{k} + NT(\frac{x}{k}) \quad \text{for every } x \in \mathbb{R}^N. \end{gathered}$$

Then, since $D_x\mathcal{L}(u, Du) = 0$, it follows by (7.39) that for every $k \geq 1$

$$\begin{aligned} &\sum_{i,j=1}^n \int_{\mathbb{R}^N} D_i T(\frac{x}{k})\frac{x_j}{k} D_j u D_{\xi_i}\mathcal{L}(u, Du) + \int_{\mathbb{R}^N} T(\frac{x}{k}) D_\xi \mathcal{L}(u, Du) \cdot Du \\ &- \int_{\mathbb{R}^N} DT(\frac{x}{k}) \cdot \frac{x}{k}\, \mathcal{L}(u, Du) - \int_{\mathbb{R}^N} NT(\frac{x}{k})\mathcal{L}(u, Du) = 0. \end{aligned}$$

Since there exists $C > 0$ with

$$D_i T\Big(\frac{x}{k}\Big)\frac{x_j}{k} \leq C \quad \text{for every } x \in \mathbb{R}^N,\ k \geq 1 \text{ and } i, j = 1, \dots, N,$$

by the Dominated Convergence Theorem, letting $k \to +\infty$, we obtain

$$\int_{\mathbb{R}^N} \Big[N\mathcal{L}(u, Du) - D_\xi \mathcal{L}(u, Du) \cdot Du\Big] = 0,$$

namely, by (7.44),

$$\frac{N-2}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} b_{ij}(u) D_i u D_j u = N \int_{\mathbb{R}^N} H(u). \tag{7.46}$$

By plugging this formula into (7.43), we obtain

$$\frac{d}{dt}\widehat{J}(\gamma(t)) = N(1-t^2)t^{N-3} \int_{\mathbb{R}^N} H(u)$$

which yields $\frac{d}{dt}\widehat{J}(\gamma(t)) > 0$ for $t < 1$ and $\frac{d}{dt}\widehat{J}(\gamma(t)) < 0$ for $t > 1$, i.e.

$$\sup_{t\in[0,L]} \widehat{J}(\gamma(t)) = \widehat{J}(\gamma(1)) = \widehat{J}(u).$$

Moreover, observe that

$$\gamma(0) = 0 \text{ and } \widehat{J}(\gamma(T)) < 0 \text{ for } T > 0 \text{ sufficiently large.}$$

Then, after a suitable scale change in t, $\gamma \in \widehat{\mathcal{P}}$ and the assertion follows. □

The following is one of the main tools of the section.

Lemma 7.11. *Assume that $(u_\varepsilon)_{\varepsilon>0} \subset H_V(\Omega)$ is as in Lemma 7.6. Then*

$$\lim_{\varepsilon\to 0} \max_{\partial\Lambda} u_\varepsilon = 0. \tag{7.47}$$

Proof. The following auxiliary fact is sufficient to prove assertion (7.47): if $\varepsilon_h \to 0$ and $(x_h) \subset \Lambda$ are such that $u_{\varepsilon_h}(x_h) \geq c$ for some $c > 0$, then

$$\lim_h V(x_h) = \min_\Lambda V. \tag{7.48}$$

Indeed, assume by contradiction that there exist $(\varepsilon_h) \subset \mathbb{R}^+$ with $\varepsilon_h \to 0$ and $(x_h) \subset \partial\Lambda$ such that $u_{\varepsilon_h}(x_h) \geq c$ for some $c > 0$. Up to a subsequence, we have $x_h \to \bar{x} \in \partial\Lambda$. Then by (7.48) it results

$$\min_{\partial\Lambda} V \leq V(\bar{x}) = \lim_h V(x_h) = \min_\Lambda V$$

which contradicts assumption (7.11).

We divide the proof of (7.48) into four steps:

Step I. Up to a subsequence, $x_h \to \widehat{x}$ for some $\widehat{x} \in \Lambda$. By contradiction, we assume that

$$V(\widehat{x}) > \min_\Lambda V = V(x_0).$$

Since for every $h \in \mathbb{N}$ the function u_{ε_h} solves (P_{ε_h}), the sequence

$$v_h \in H_V(\Omega_h), \quad \Omega_h = \varepsilon_h^{-1}(\Omega - x_h), \quad v_h(x) = u_{\varepsilon_h}(x_h + \varepsilon_h x)$$

satisfies

$$-\sum_{i,j=1}^N D_j(a_{ij}(x_h + \varepsilon_h x, v_h) D_i v_h) + \frac{1}{2}\sum_{i,j=1}^N D_s a_{ij}(x_h + \varepsilon_h x, v_h) D_i v_h D_j v_h = w_h$$

in Ω_h, $v_h > 0$ in Ω_h and $v_h = 0$ on $\partial\Omega_h$, where we have set

$$w_h := g(x_h + \varepsilon_h x, v_h) - V(x_h + \varepsilon_h x)v_h \quad \text{for every } h \in \mathbb{N}.$$

Setting $v_h = 0$ outside $\mathbb{R}^N$, by Corollary 7.7, up to a subsequence, $v_h \to v$ weakly in $H^1(\mathbb{R}^N)$. Notice that the sequence $(\chi_\Lambda(x_h + \varepsilon_h x))$ converges weak* in L^∞ to a measurable function $0 \le \chi \le 1$. In particular, taking into account that $|w_h| \le c_1|v_h| + c_2|v_h|^p$, (w_h) is strongly convergent in $H^{-1}(\widetilde{\Omega})$ for every bounded subset $\widetilde{\Omega}$ of $\mathbb{R}^N$. Therefore, by a simple variant of Lemma 7.4, we conclude that (v_h) is strongly convergent to v in $H^1(\widetilde{\Omega})$ for every bounded subset $\widetilde{\Omega} \subset \mathbb{R}^N$ (actually, as we will see, $v_h \to v$ uniformly over compacts). Then it follows that the limit v is a solution of the equation

$$-\sum_{i,j=1}^N D_j(a_{ij}(\widehat{x}, v)D_i v) + \frac{1}{2}\sum_{i,j=1}^N D_s a_{ij}(\widehat{x}, v)D_i v D_j v + V(\widehat{x})v = g_0(x, v) \quad \text{in } \mathbb{R}^N \tag{7.49}$$

where $g_0(x,s) := \chi(x)f(s) + (1-\chi(x))\widetilde{f}(s)$ for every $x \in \mathbb{R}^N$ and $s \in \mathbb{R}^+$.

We now prove that $v \neq 0$. Let us set

$$d_h(x) := \begin{cases} V(x_h + \varepsilon_h x) - \frac{g(x, v_h(x))}{v_h(x)} & \text{if } v_h(x) \neq 0 \\ 0 & \text{if } v_h(x) = 0, \end{cases}$$

$$A_j(x,s,\xi) := \sum_{i=1}^N a_{ij}(x_h + \varepsilon_h x, s)\xi_i \quad \text{for } j = 1,\dots,N,$$

$$B(x,s,\xi) := d_h(x)s,$$

$$C(x,s) := \frac{1}{2}\sum_{i,j=1}^N D_s a_{ij}(x_h + \varepsilon_h x, s)D_i v_h(x)D_j v_h(x)$$

for every $x \in \mathbb{R}^N$, $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$. Taking into account the assumptions on the coefficients $a_{ij}(x,s)$, it results that

$$A(x,s,\xi)\cdot\xi \ge \nu|\xi|^2, \quad |A(x,s,\xi)| \le c|\xi|, \quad |B(x,s,\xi)| \le d_h(x)|s|.$$

Moreover, by (7.10) we have

$$s \ge R \ \Rightarrow\ C(x,s)s \ge 0$$

for every $x \in \mathbb{R}^N$ and $s \in \mathbb{R}^+$. By the growth condition on g, $d_h \in L^{\frac{N}{2-\delta}}(B_{2\varrho}(0))$ for every $\varrho > 0$ and

$$S = \sup_h \|d_h\|_{L^{\frac{N}{2-\delta}}(B_{2\varrho}(0))} \le D_\varrho \sup_{h\in\mathbb{N}} \|v_h\|_{L^{2^*}(B_{2\varrho}(0))} < +\infty$$

for some $\delta > 0$ sufficiently small. Since $\operatorname{div}(A(x, v_h, Dv_h)) = B(x, v_h, Dv_h) + C(x, v_h)$ for every $h \in \mathbb{N}$, by virtue of [123, Theorem 1 and Remark at p.261] there exists a positive constant $M(\delta, N, c, \varrho^\delta S)$ and a radius $\varrho > 0$, sufficiently small, such that

$$\sup_{h\in\mathbb{N}} \max_{x\in B_\varrho(0)} |v_h(x)| \le M(\delta, N, c, \varrho^\delta S)(2\varrho)^{-N/2} \sup_{h\in\mathbb{N}} \|v_h\|_{L^2(B_{2\varrho}(0))} < +\infty$$

so that (v_h) is uniformly bounded in $B_\varrho(0)$. Then, by [123, Theorem 8], (v_h) is bounded in some $C^{1,\alpha}(\overline{B_{\varrho/2}(0)})$. Up to a subsequence this implies that (v_h) converges uniformly to v in $\overline{B_{\varrho/2}(0)}$. This yields $v(0) = \lim_h v_h(0) = \lim_h u_{\varepsilon_h}(x_h) \ge c > 0$.

In a similar fashion one shows that $v_h \to v$ uniformly over compacts.

Step II. We prove that v actually solves the following equation

$$-\sum_{i,j=1}^{N} D_j(a_{ij}(\widehat{x},v)D_iv)+\frac{1}{2}\sum_{i,j=1}^{N} D_sa_{ij}(\widehat{x},v)D_ivD_jv+V(\widehat{x})v=f(v) \quad \text{in } \mathbb{R}^N. \tag{7.50}$$

In general the function χ of Step I is given by $\chi=\chi_{T_\Lambda(\widehat{x})}$, where $T_\Lambda(\widehat{x})$ is the tangent cone of Λ at $\widehat{x}$. On the other hand, since we may assume without loss of generality that Λ is smooth, it results (up to a rotation) that $\chi(x)=\chi_{\{x_1<0\}}(x)$ for every $x\in\mathbb{R}^N$. In particular, v is a solution of the problem

$$\begin{aligned} &-\sum_{i,j=1}^{N} D_j(a_{ij}(\widehat{x},v)D_iv)+\frac{1}{2}\sum_{i,j=1}^{N} D_sa_{ij}(\widehat{x},v)D_ivD_jv+V(\widehat{x})v \\ &=\chi_{\{x_1<0\}}(x)f(v)+\chi_{\{x_1>0\}}(x)\widetilde{f}(v) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{7.51}$$

Let us first show that $v(x)\le\ell$ on $\{x_1=0\}$. To this aim, let us use again Lemma 7.9, by choosing this time

$$\begin{gathered} \varphi(x):=\chi_{\{x_1<0\}}(x)f(v(x))+\chi_{\{x_1>0\}}(x)\widetilde{f}(v(x)) \quad \text{for every } x\in\mathbb{R}^N \\ \mathcal{L}(s,\xi):=\frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(\widehat{x},s)\xi_i\xi_j+\frac{V(\widehat{x})}{2}s^2 \quad \text{for every } s\in\mathbb{R} \text{ and } \xi\in\mathbb{R}^N, \\ h(x):=h_k(x)=\Big(T(\frac{x}{k}),0,\dots,0\Big) \quad \text{for every } x\in\mathbb{R}^N \text{ and } k\ge 1. \end{gathered}$$

Then $h_k\in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$ and, since $D_x\mathcal{L}(v,Dv)=0$, for every $k\ge1$, it results

$$\begin{aligned} &\int_{\mathbb{R}^N}\Big[\frac{1}{k}\sum_{i=1}^{N} D_iT(\frac{x}{k})D_1vD_{\xi_i}\mathcal{L}(v,Dv)-D_1T(\frac{x}{k})\frac{1}{k}\mathcal{L}(v,Dv)\Big] \\ &=\int_{\mathbb{R}^N} T(\frac{x}{k})\varphi(x,v)D_1v. \end{aligned}$$

Again by the Dominated Convergence Theorem, letting $k\to+\infty$, it results

$$\int_{\mathbb{R}^N}\varphi(x,v)D_1v=0,$$

that is, after integration by parts,

$$\int_{\mathbb{R}^{N-1}}\Big[F(v(0,x'))-\widetilde{F}(v(0,x'))\Big]\,dx'=0.$$

Taking into account that $F(s)\ge\widetilde{F}(s)$ with equality only if $s\le\ell$, we get $v(0,x')\le\ell$ for every $x'\in\mathbb{R}^{N-1}$. To prove that actually $v(x_1,x')\le\ell$ for every $x_1>0$ and $x'\in\mathbb{R}^{N-1}$, we test (7.51) with

$$\eta(x):=\begin{cases}0 & \text{if } x_1<0\\ (v(x_1,x')-\ell)^+\exp\{\zeta(v(x_1,x'))\} & \text{if } x_1>0\end{cases}$$

where $\zeta(s)$ is as in (7.26) and then we argue as in Section 7.3 (see the computations in formula (7.58)). In particular,

$$\varphi(x,v(x))=f(v(x)) \quad \text{for every } x\in\mathbb{R}^N, \tag{7.52}$$

so that v is a nontrivial solution of (7.50).

Step III If $J_h : H_V(\Omega_h) \to \mathbb{R}$ is as in (7.34), the function v_h is a critical point of J_h and $J_h(v_h) = \varepsilon_h{}^{-N} J_{\varepsilon_h}(u_{\varepsilon_h})$ for every $h \in \mathbb{N}$. Let us consider the functional $J_{\widehat{x}} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$J_{\widehat{x}}(u) := \frac{1}{2} \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} a_{ij}(\widehat{x}, u) D_i u D_j u + \frac{1}{2} \int_{\mathbb{R}^N} V(\widehat{x}) u^2 - \int_{\mathbb{R}^N} F(u).$$

We now want to prove that

$$J_{\widehat{x}}(v) \le \liminf_h J_h(v_h). \tag{7.53}$$

Let us set for every $h \in \mathbb{N}$ and $x \in \Omega_h$

$$\xi_h(x) := \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}(x_h + \varepsilon_h x, v_h) D_i v_h D_j v_h + \frac{1}{2} V(x_h + \varepsilon_h x) v_h^2 - G(x_h + \varepsilon_h x, v_h). \tag{7.54}$$

Since $v_h \to v$ in H^1 over compact sets, in view of (7.52), for every $\varrho > 0$ one gets

$$\lim_h \int_{B_\varrho(0)} \xi_h(x) = \frac{1}{2} \int_{B_\varrho(0)} \Big(\sum_{i,j=1}^{N} a_{ij}(\widehat{x}, v) D_i v D_j v + V(\widehat{x}) v^2 \Big) - \int_{B_\varrho(0)} F(v).$$

Moreover, as v belongs to $H^1(\mathbb{R}^N)$,

$$\frac{1}{2} \int_{B_\varrho(0)} \Big(\sum_{i,j=1}^{N} a_{ij}(\widehat{x}, v) D_i v D_j v + V(\widehat{x}) v^2 \Big) - \int_{B_\varrho(0)} F(v) = J_{\widehat{x}}(v) - o(1)$$

as $\varrho \to +\infty$. Therefore, it suffices to show that for every $\delta > 0$ there exists $\varrho > 0$ with

$$\liminf_h \int_{\Omega_h \setminus B_\varrho(0)} \xi_h(x) \ge -\delta. \tag{7.55}$$

Consider a function $\eta_\varrho \in C^\infty(\mathbb{R}^N)$ such that $0 \le \eta_\varrho \le 1$, $\eta_\varrho = 0$ on $B_{\varrho-1}(0)$, $\eta_\varrho = 1$ on $\mathbb{R}^N \setminus B_\varrho(0)$ and $|D\eta_\varrho| \le c$. Let us set for every $h \in \mathbb{N}$,

$$\begin{aligned}
\beta_h(\varrho) := & \sum_{i,j=1}^{N} \int_{B_\varrho(0) \setminus B_{\varrho-1}(0)} a_{ij}(x_h + \varepsilon_h x, v_h) D_i v_h D_j(\eta_\varrho v_h) \\
& + \frac{1}{2} \sum_{i,j=1}^{N} \int_{B_\varrho(0) \setminus B_{\varrho-1}(0)} D_s a_{ij}(x_h + \varepsilon_h x, v_h) \eta_\varrho v_h D_i v_h D_j v_h \\
& + \int_{B_\varrho(0) \setminus B_{\varrho-1}(0)} V(x_h + \varepsilon_h x) v_h^2 \eta_\varrho - \int_{B_\varrho(0) \setminus B_{\varrho-1}(0)} g(x_h + \varepsilon_h x, v_h) \eta_\varrho v_h.
\end{aligned}$$

After some computations, in view of (7.9) and (7.54), one gets

$$\begin{aligned}
& -\beta_h(\varrho) + J_h'(v_h)(\eta_\varrho v_h) \\
& \le (\gamma + 2) \int_{\Omega_h \setminus B_\varrho(0)} \xi_h(x) - \frac{\gamma}{2} \int_{\Omega_h \setminus B_\varrho(0)} V(x_h + \varepsilon_h x) v_h^2 \\
& \quad + (\gamma + 2) \int_{\Omega_h \setminus B_\varrho(0)} G(x_h + \varepsilon_h x, v_h) - \int_{\Omega_h \setminus B_\varrho(0)} g(x_h + \varepsilon_h x, v_h) v_h
\end{aligned}$$

Notice that, by virtue of (7.19), for ϱ large enough, setting $\Lambda_h = \varepsilon_h^{-1}(\Lambda - x_h)$, we get

$$\begin{aligned}
&-\frac{\gamma}{2}\int_{\Lambda_h\setminus B_\varrho(0)} V(x_h+\varepsilon_h x)v_h^2 + (\gamma+2)\int_{\Lambda_h\setminus B_\varrho(0)} G(x_h+\varepsilon_h x, v_h)\\
&-\int_{\Lambda_h\setminus B_\varrho(0)} g(x_h+\varepsilon_h x, v_h)v_h\\
&\le -(\vartheta-2-\gamma)\int_{\Lambda_h\setminus B_\varrho(0)} G(x_h+\varepsilon_h x, v_h) \le 0.
\end{aligned}$$

Analogously, in view of (7.20), we obtain

$$\begin{aligned}
&-\frac{\gamma}{2}\int_{\Omega_h\setminus(B_\varrho(0)\cup\Lambda_h)} V(x_h+\varepsilon_h x)v_h^2 + (\gamma+2)\int_{\Omega_h\setminus(B_\varrho(0)\cup\Lambda_h)} G(x_h+\varepsilon_h x, v_h)\\
&-\int_{\Omega_h\setminus(B_\varrho(0)\cup\Lambda_h)} g(x_h+\varepsilon_h x, v_h)v_h\\
&\le -\frac{\gamma}{2}\int_{\Omega_h\setminus(B_\varrho(0)\cup\Lambda_h)} V(x_h+\varepsilon_h x)v_h^2 + \frac{\gamma}{2k}\int_{\Omega_h\setminus(B_\varrho(0)\cup\Lambda_h)} V(x_h+\varepsilon_h x)v_h^2 \le 0.
\end{aligned}$$

Therefore, since $J_h'(v_h)(\eta_\varrho v_h) = 0$ for every $h \in \mathbb{N}$ and

$$\limsup_h \beta_h(\varrho) = o(1) \quad \text{as } \varrho \to +\infty,$$

inequality (7.55) follows and thus (7.53) holds true.

Step IV. In this step we get the desired contradiction. By combining Lemma 7.6 with the inequality (7.53), one immediately gets

$$J_{\widehat{x}}(v) \le \bar{c} = \inf_{\gamma\in\mathcal{P}_0}\sup_{t\in[0,1]} J_0(\gamma(t)). \tag{7.56}$$

Since v is a nontrivial solution of (7.50), by applying Lemma 7.10 with

$$\mu = V(\widehat{x}), \quad \nu' = \nu, \quad R' = R, \quad b_{ij}(s) = a_{ij}(\widehat{x}, s),$$

being $\widehat{\mathcal{P}} \subset \mathcal{P}_0$, $V(\widehat{x}) > V(x_0)$ and, by (7.12),

$$\sum_{i,j=1}^N a_{ij}(\widehat{x}, s)\xi_i\xi_j \ge \sum_{i,j=1}^N a_{ij}(x_0, s)\xi_i\xi_j \quad \text{for every } s \in \mathbb{R}^+ \text{ and } \xi \in \mathbb{R}^N,$$

it follows that

$$J_{\widehat{x}}(v) \ge \inf_{\gamma\in\widehat{\mathcal{P}}}\sup_{t\in[0,1]} J_{\widehat{x}}(\gamma(t)) > \inf_{\gamma\in\mathcal{P}_0}\sup_{t\in[0,1]} J_0(\gamma(t)) = \bar{c}, \tag{7.57}$$

which contradicts (7.56). □

7.3. **Proof of the main result.** We are now ready to prove Theorem 7.1.

Step I. We prove that (a) holds. By Lemma 7.11 there exists $\varepsilon_0 > 0$ such that

$$u_\varepsilon(x) < \ell \quad \text{for every } \varepsilon \in (0, \varepsilon_0) \text{ and } x \in \partial\Lambda.$$

Then, since $u_\varepsilon \in H_V(\Omega)$, for every $\varepsilon \in (0, \varepsilon_0)$, if ζ is defined as in (7.26), the function

$$v_\varepsilon(x) := \begin{cases} 0 & \text{if } x \in \Lambda\\ (u_\varepsilon(x) - \ell)^+ \exp\{\zeta(u_\varepsilon(x))\} & \text{if } x \in \Omega\setminus\Lambda \end{cases}$$

belongs to $H_0^1(\Omega)$ and it is an admissible test for the equation

$$-\varepsilon^2\sum_{i,j=1}^N D_j(a_{ij}(x,u_\varepsilon)D_iu_\varepsilon)+\frac{\varepsilon^2}{2}\sum_{i,j=1}^N D_sa_{ij}(x,u_\varepsilon)D_iu_\varepsilon D_ju_\varepsilon+V(x)u_\varepsilon=g(x,u_\varepsilon).$$

After some computations, one obtains

$$\begin{aligned}&\varepsilon^2\sum_{i,j=1}^N\int_{\Omega\setminus\Lambda}a_{ij}(x,u_\varepsilon)D_i\big[(u_\varepsilon-\ell)^+\big]D_j\big[(u_\varepsilon-\ell)^+\big]\exp\{\zeta(u_\varepsilon)\}\\&+\varepsilon^2\sum_{i,j=1}^N\int_{\Omega\setminus\Lambda}\Big[\frac{1}{2}D_sa_{ij}(x,u_\varepsilon)+\zeta'(u_\varepsilon)a_{ij}(x,u_\varepsilon)\Big]D_iu_\varepsilon D_ju_\varepsilon(u_\varepsilon-\ell)^+\exp\{\zeta(u_\varepsilon)\}\\&+\int_{\Omega\setminus\Lambda}\Phi_\varepsilon(x)\big[(u_\varepsilon-\ell)^+\big]^2\exp\{\zeta(u_\varepsilon)\}+\int_{\Omega\setminus\Lambda}\Phi_\varepsilon(x)\ell(u_\varepsilon-\ell)^+\exp\{\zeta(u_\varepsilon)\}=0,\end{aligned}\tag{7.58}$$

where $\Phi_\varepsilon:\Omega\to\mathbb{R}$ is the function given by

$$\Phi_\varepsilon(x):=V(x)-\frac{g(x,u_\varepsilon(x))}{u_\varepsilon(x)}.$$

Notice that, by virtue of condition (7.20), one has

$$\Phi_\varepsilon(x)>0\quad\text{for every } x\in\Omega\setminus\Lambda.$$

Therefore, taking into account (7.27), all the terms in (7.58) must be equal to zero. We conclude that $(u_\varepsilon-\ell)^+=0$ on $\Omega\setminus\Lambda$, namely,

$$u_\varepsilon(x)\le\ell\quad\text{for every } \varepsilon\in(0,\varepsilon_0)\text{ and } x\in\Omega\setminus\Lambda.\tag{7.59}$$

Hence, by Proposition 7.3, u_ε is a positive solution of the original problem (7.15). Moreover, by virtue of (7.10), using again the argument at the end of Step I of Lemma 7.11 it results that $u_\varepsilon\in L^\infty_{\rm loc}(\Omega)$, which, by the regularity results of [91], yields $u_\varepsilon\in C(\overline{\Omega})$. Notice that by arguing in a similar fashion testing with

$$v_\varepsilon(x):=\begin{cases}0 & \text{if } x\in\Lambda\\ (u_\varepsilon(x)-\sup_{\partial\Lambda}u_\varepsilon)^+\exp\{\zeta(u_\varepsilon(x))\} & \text{if } x\in\Omega\setminus\Lambda\end{cases}$$

it results $u_\varepsilon\to 0$ uniformly outside Λ.

Step II. We prove that (b) holds. If x_ε denotes the maximum of u_ε in Λ, since $u_\varepsilon\to 0$ uniformly outside Λ, it results that $u_\varepsilon(x_\varepsilon)=\sup_\Omega u_\varepsilon$. By arguing as at the end of Step I of Lemma 7.11, setting $v_\varepsilon(x)=u_\varepsilon(x_\varepsilon+\varepsilon x)$ it results that the sequence $(v_\varepsilon(0))$ is bounded in $\mathbb{R}$. Then there exists $\sigma'>0$ such that $u_\varepsilon(x_\varepsilon)=v_\varepsilon(0)\le\sigma'$. Assume now by contradiction that $u_\varepsilon(x_\varepsilon)\le\sigma$ for some $\varepsilon\in(0,\varepsilon_0)$. Then, taking into account the definition of σ and that $u_\varepsilon\to 0$ uniformly outside Λ, it holds (with strict inequality in some subset of Ω)

$$V(x)-\frac{f(u_\varepsilon(x))}{u_\varepsilon(x)}\ge 0\quad\text{for every } x\in\Omega.\tag{7.60}$$

Let $\zeta:\mathbb{R}^+\to\mathbb{R}$ be the map defined in (7.26). Then the function $u_\varepsilon\exp\{\zeta(u_\varepsilon)\}$ can be chosen as an admissible test in the equation

$$-\varepsilon^2\sum_{i,j=1}^N D_j(a_{ij}(x,u_\varepsilon)D_iu_\varepsilon)+\frac{\varepsilon^2}{2}\sum_{i,j=1}^N D_sa_{ij}(x,u_\varepsilon)D_iu_\varepsilon D_ju_\varepsilon+V(x)u_\varepsilon=f(u_\varepsilon).$$

After some computations, one obtains

$$\begin{aligned} &\varepsilon^2 \sum_{i,j=1}^N \int_\Omega a_{ij}(x,u_\varepsilon) D_i u_\varepsilon D_j u_\varepsilon \exp\{\zeta(u_\varepsilon)\} \\ &+ \varepsilon^2 \sum_{i,j=1}^N \int_\Omega \Big[\frac{1}{2} D_s a_{ij}(x,u_\varepsilon) + \zeta'(u_\varepsilon) a_{ij}(x,u_\varepsilon)\Big] D_i u_\varepsilon D_j u_\varepsilon u_\varepsilon \exp\{\zeta(u_\varepsilon)\} \\ &+ \int_\Omega \Big(V(x) - \frac{f(u_\varepsilon)}{u_\varepsilon}\Big) u_\varepsilon^2 \exp\{\zeta(u_\varepsilon)\} = 0. \end{aligned} \tag{7.61}$$

Then, by (7.8), (7.27) and (7.60) all the terms in equation (7.61) must be equal to zero, namely $u_\varepsilon \equiv 0$, which is not possible. Then $u_\varepsilon(x_\varepsilon) \geq \sigma$ for every $\varepsilon \in (0, \varepsilon_0)$ and by (7.48) we also get $d(x_\varepsilon, \mathcal{M}) \to 0$ as $\varepsilon \to 0$.

Step III. We prove that (c) holds. Assume by contradiction that there exists $\varrho > 0$, $\delta > 0$, $\varepsilon_h \to 0$ and $y_h \in \Lambda \setminus B_\varrho(x_{\varepsilon_h})$ such that

$$\limsup_h u_{\varepsilon_h}(y_h) \geq \delta. \tag{7.62}$$

Then, arguing as in Lemma 7.11, we can assume that $y_h \to y$, $x_{\varepsilon_h} \to \tilde{y}$ and $v_h(y) := u_{\varepsilon_h}(y_h + \varepsilon_h y) \to v$, $\tilde{v}_h(y) := u_{\varepsilon_h}(x_{\varepsilon_h} + \varepsilon_h y) \to \tilde{v}$ strongly in $H^1_{\rm loc}(\mathbb{R}^N)$, where v is a solution of

$$-\sum_{i,j=1}^N D_j(a_{ij}(y,v)D_i v) + \frac{1}{2}\sum_{i,j=1}^N D_s a_{ij}(y,v) D_i v D_j v + V(y)v = f(v) \quad \text{in } \mathbb{R}^N$$

and $\tilde{v}$ is a solution of

$$-\sum_{i,j=1}^N D_j(a_{ij}(\tilde{y},v)D_i v) + \frac{1}{2}\sum_{i,j=1}^N D_s a_{ij}(\tilde{y},v) D_i v D_j v + V(\tilde{y})v = f(v) \quad \text{in } \mathbb{R}^N.$$

Observe that $v \neq 0$ and $\tilde{v} \neq 0$. Indeed, arguing as in Step I of Lemma 7.11 it results that (v_h) and $(\tilde{v}_h)$ converge uniformly in a neighborhood of zero, so that from (7.62) and $u_{\varepsilon_h}(x_{\varepsilon_h}) \geq \sigma$ we get $v(0) \geq \delta$ and $\tilde{v}(0) \geq \sigma$. Now, setting $z_h := \frac{x_{\varepsilon_h} - y_h}{\varepsilon_h}$ and

$$\xi_h(y) := \frac{1}{2}\sum_{i,j=1}^N a_{ij}(y_h + \varepsilon_h y, v_h) D_i v_h D_j v_h + \frac{1}{2} V(y_h + \varepsilon_h y) v_h^2 - G(y_h + \varepsilon_h y, v_h),$$

if $\psi \in C^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$, $\psi(s) = 0$ for $s \leq 1$ and $\psi(s) = 1$ for $s \geq 2$, arguing as in Lemma 7.11 by testing the equation satisfied by v_h with

$$\varphi_{h,R}(y) = v_h(y)\big[\psi\big(\frac{|y|}{R}\big) + \psi\big(\frac{|y - z_h|}{R}\big) - 1\big],$$

taking into account that

$$\lim_h \Big| \int_{B_{2R}(0) \cup B_{2R}(z_h) \setminus (B_R(0) \cup B_R(z_h))} \xi_h(y) \Big| = o(1)$$

as $R \to +\infty$, it turns out that for every $\delta > 0$ there exists $R > 0$ with

$$\liminf_h \int_{\Omega_h \setminus (B_R(0) \cup B_R(z_h))} \xi_h(y) \geq -\delta.$$

Moreover, for every $R > 0$, we have

$$\begin{aligned}\liminf_h \int_{B_R(0)\cup B_R(z_h)} \xi_h(y) &= \liminf_h \int_{B_R(0)} \frac{1}{2}\sum_{i,j=1}^N a_{ij}(y_h+\varepsilon_h y, v_h) D_i v_h D_j v_h \\ &\quad + \frac{1}{2} V(y_h+\varepsilon_h y) v_h^2 - G(y_h+\varepsilon_h y, v_h) \\ &\quad + \liminf_h \int_{B_R(0)} \frac{1}{2}\sum_{i,j=1}^N a_{ij}(x_{\varepsilon_h}+\varepsilon_h y, \widetilde{v}_h) D_i \widetilde{v}_h D_j \widetilde{v}_h \\ &\quad + \frac{1}{2} V(x_{\varepsilon_h}+\varepsilon_h y) \widetilde{v}_h^2 - G(x_{\varepsilon_h}+\varepsilon_h y, \widetilde{v}_h) \\ &= \int_{B_R(0)} \frac{1}{2}\sum_{i,j=1}^N a_{ij}(y, v) D_i v D_j v + \frac{1}{2} V(y) v^2 - F(v) \\ &\quad + \int_{B_R(0)} \frac{1}{2}\sum_{i,j=1}^N a_{ij}(\widetilde{y}, \widetilde{v}) D_i \widetilde{v} D_j \widetilde{v} + \frac{1}{2} V(\widetilde{y}) \widetilde{v}^2 - F(\widetilde{v}).\end{aligned}$$

Therefore, we deduce that

$$\liminf_h \varepsilon_h{}^{-N} J_{\varepsilon_h}(u_{\varepsilon_h}) = \liminf_h \int_{\Omega_h} \xi_h(y) \geq J_y(v) + J_{\widetilde{y}}(\widetilde{v}).$$

Let b_y and $b_{\widetilde{y}}$ be the Mountain-Pass values of J_y and $J_{\widetilde{y}}$. By Lemma 7.10, (7.11) and (7.12) we have $J_y(v) \geq b_y \geq \bar{c}$ and $J_{\widetilde{y}}(\widetilde{v}) \geq b_{\widetilde{y}} \geq \bar{c}$. Therefore we conclude that

$$\liminf_h \varepsilon_h{}^{-N} J_{\varepsilon_h}(u_{\varepsilon_h}) \geq 2\bar{c},$$

which contradicts Lemma 7.6.

Step IV. We prove that (d) holds. By Corollary 7.8, we have $\|u_\varepsilon\|_{H_V(\Omega)} \to 0$. In particular, $u_\varepsilon \to 0$ in $L^q(\Omega)$ for every $2 \leq q \leq 2^*$. As a consequence $u_\varepsilon \to 0$ in $L^q(\Omega)$ also for every $q > 2^*$. Indeed, if $q > 2^*$, we have

$$\int_\Omega |u_\varepsilon|^q = \int_\Omega |u_\varepsilon|^{q-2^*} |u_\varepsilon|^{2^*} \leq \sigma'^{q-2^*} \int_\Omega |u_\varepsilon|^{2^*} \to 0$$

as $\varepsilon \to 0$. The proof is now complete. □

7.4. **A few related open problems.** We quote here a few (open) problems related to the main result.

Problem 7.12. Under suitable assumptions, does a Gidas-Ni-Nirenberg [77] type result (radial symmetry) hold for the solutions of autonomous equations of the type

$$-\sum_{i,j=1}^N D_j(b_{ij}(u) D_i u) + \frac{1}{2}\sum_{i,j=1}^N b'_{ij}(u) D_i u D_j u = h(u) \quad \text{in } \mathbb{R}^N? \tag{7.63}$$

Problem 7.13. Under suitable assumptions on b_{ij} and h, is it possible to prove, as in the semi-linear case, a *uniqueness* result for the solutions of equation (7.63)?

Problem 7.14. Is it true that for each $\varepsilon > 0$ the solution u_ε of problem (7.15) admits a *unique* maximum point inside Λ?

Problem 7.15. Is it true that the solutions u_ε of problem (7.15) decay exponentially as for the semi-linear case (see formula (7.17))?

8. The Singularly Perturbed Case, II

In this section we turn to a more delicate situation, namely the study of the multi-peak case, also for possibly degenerate operators. We refer the reader to [75]. Some parts of this publication has been slightly modified to give this collection a more uniform appearance.

Assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a C^1 function and there exists a positive constant α such that

$$V(x) \geq \alpha \quad \text{for every } x \in \mathbb{R}^N. \tag{8.1}$$

Moreover let $\Lambda_1, \dots, \Lambda_k$ be k disjoint compact subsets of Ω and $x_i \in \Lambda_i$ with

$$V(x_i) = \min_{\Lambda_i} V < \min_{\partial\Lambda_i} V, \quad i = 1, \dots, k. \tag{8.2}$$

Let us set for all $i = 1, \dots, k$

$$\mathcal{M}_i := \{x \in \Lambda_i : V(x) = V(x_i)\}. \tag{8.3}$$

Let $1 < p < N$, $p^* = \frac{Np}{N-p}$ and let $W_V(\Omega)$ be the weighted Banach space

$$W_V(\Omega) := \Big\{u \in W_0^{1,p}(\Omega) : \int_\Omega V(x)|u|^p < +\infty\Big\}$$

endowed with the natural norm $\|u\|^p_{W_V} := \int_\Omega |Du|^p + \int_\Omega V(x)|u|^p$. For all $A, B \subset \mathbb{R}^N$, let us denote their distance by $\operatorname{dist}(A, B)$.

The following is the first of our main results.

Theorem 8.1. *Assume that (8.1) and (8.2) hold and let $1 < p \leq 2$, $p < q < p^*$.*

Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exist u_ε in $W_V(\Omega) \cap C^{1,\beta}_{\text{loc}}(\Omega)$ and k points $x_{\varepsilon,i} \in \Lambda_i$ satisfying the following properties:

(a) *u_ε is a weak solution of the problem*

$$\begin{gathered} -\varepsilon^p \Delta_p u + V(x)u^{p-1} = u^{q-1} \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega; \end{gathered} \tag{8.4}$$

(b) *there exist $\sigma, \sigma' \in]0, +\infty[$ such that for every $i = 1, \dots, k$ we have*

$$u_\varepsilon(x_{\varepsilon,i}) = \sup_{\Lambda_i} u_\varepsilon, \quad \sigma < u_\varepsilon(x_{\varepsilon,i}) < \sigma', \quad \lim_{\varepsilon\to 0} \operatorname{dist}(x_{\varepsilon,i}, \mathcal{M}_i) = 0$$

where $\mathcal{M}_i$ is as in (8.3);

(c) *for every $r < \min\{\operatorname{dist}(\mathcal{M}_i, \mathcal{M}_j) : i \neq j\}$ we have*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Omega \setminus \bigcup_{i=1}^k B_r(x_{\varepsilon,i}))} = 0;$$

(d) *it results*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{W_V} = 0.$$

Moreover, if $k = 1$ the assertions hold for every $1 < p < N$.

Actually, this result will follow by a more general achievement involving a larger class of quasi-linear operators. Before stating it, we make a few assumptions. Assume that $1 < p < N$, $f \in C^1(\mathbb{R}^+)$ and there exist $p < q < p^*$ and $p < \vartheta \leq q$ with

$$\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0, \quad \lim_{s\to+\infty} \frac{f(s)}{s^{q-1}} = 0, \tag{8.5}$$

$$0 < \vartheta F(s) \leq f(s)s \quad \text{for every } s \in \mathbb{R}^+, \tag{8.6}$$

where $F(s) = \int_0^s f(t)\,dt$ for every $s \in \mathbb{R}^+$.

The function $j(x,s,\xi) : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ is continuous in x and of class C^1 with respect to s and ξ, the function $\{\xi \mapsto j(x,s,\xi)\}$ is strictly convex and p-homogeneous and there exist two positive constants c_1, c_2 with

$$|j_s(x,s,\xi)| \le c_1|\xi|^p, \quad |j_\xi(x,s,\xi)| \le c_2|\xi|^{p-1} \tag{8.7}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^+$, $\xi \in \mathbb{R}^N$ (j_s and j_ξ denote the derivatives of j with respect of s and ξ respectively). Let $R, \nu > 0$ and $0 < \gamma < \vartheta - p$ with

$$j(x,s,\xi) \ge \nu|\xi|^p, \tag{8.8}$$

$$j_s(x,s,\xi)s \le \gamma j(x,s,\xi) \tag{8.9}$$

a.e. in Ω, for every $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$, and

$$j_s(x,s,\xi) \ge 0 \quad \text{for every } s \ge R \tag{8.10}$$

a.e. in Ω and for every $\xi \in \mathbb{R}^N$. For every fixed $\bar{x} \in \Omega$, the limiting equation

$$-\operatorname{div}(j_\xi(\bar{x},u,Du)) + j_s(\bar{x},u,Du) + V(\bar{x})u^{p-1} = f(u) \quad \text{in } \mathbb{R}^N \tag{8.11}$$

admits a unique positive solution (up to translations). Finally, we assume that

$$j(x_i,s,\xi) = \min_{x \in \Lambda_i} j(x,s,\xi), \quad i = 1,\dots,k \tag{8.12}$$

for every $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$, where the x_is are as in (8.2).

We point out that assumptions (8.1), (8.2), (8.5) and (8.6) are the same as in [62, 63]. Conditions (8.7)-(8.10) are natural assumption, already used, throughout this monograph.

The following result is an extension of Theorem 8.1.

Theorem 8.2. *Assume that* (8.1), (8.2), (8.5), (8.6), (8.7), (8.8), (8.9), (8.10), (8.11), (8.12) *hold. Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0,\varepsilon_0)$, there exist u_ε in $W_V(\Omega) \cap C^{1,\beta}_{\rm loc}(\Omega)$ and k points $x_{\varepsilon,i} \in \Lambda_i$ satisfying the following properties:*

(a) *u_ε is a weak solution of the problem*

$$\begin{gathered} -\varepsilon^p \operatorname{div}(j_\xi(x,u,Du)) + \varepsilon^p j_s(x,u,Du) + V(x)u^{p-1} = f(u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,; \end{gathered} \tag{8.13}$$

(b) *there exist $\sigma, \sigma' \in]0,+\infty[$ such that for every $i = 1,\dots,k$ we have*

$$u_\varepsilon(x_{\varepsilon,i}) = \sup_{\Lambda_i} u_\varepsilon, \quad \sigma < u_\varepsilon(x_{\varepsilon,i}) < \sigma', \quad \lim_{\varepsilon \to 0} \operatorname{dist}(x_{\varepsilon,i}, \mathcal{M}_i) = 0$$

where $\mathcal{M}_i$ is as in (8.3);

(c) *for every $r < \min\{\operatorname{dist}(\mathcal{M}_i,\mathcal{M}_j) : i \neq j\}$ we have*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\Omega \setminus \bigcup_{i=1}^k B_r(x_{\varepsilon,i}))} = 0\,;$$

(d) *it results*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{W_V} = 0.$$

Notice that if $k = 1$ assumption (8.11) can be dropped: in fact following the arguments of [132] it is possible to prove that the previous result holds without any uniqueness assumption, which instead, as in the semi-linear case, seems to be necessary for the case $k > 1$. This holds true for the p-Laplacian problem (8.4) and for more general situation we refer the reader to [124].

Various difficulties arise in comparison with the semi-linear framework (see also Section 5 of [132]). To study the concentration properties of u_ε inside the Λ_is (see Section 8.3), inspired by the recent work of Jeanjean and Tanaka [83], we make a repeated use of a Pucci-Serrin type identity [59] which has turned out to be a very powerful tool (see Section 8.2). It has to be pointed out that, in our possibly degenerate setting, we cannot hope to have C^2 solutions, but at most $C^{1,\beta}$ solutions (see [65, 142]). Therefore, the classical Pucci-Serrin identity [117] is not applicable in our framework. On the other hand, it has been recently shown in [59] that, under minimal regularity assumptions, the identity holds for locally Lipschitz solutions, provided that the operator is strictly convex in the gradient, which, from our viewpoint, is a very natural requirement (see Theorem 8.6). Under uniqueness assumptions this identity has also turned out to be useful in characterizing the exact energy level of the solution of (8.11). More precisely, we prove that (8.11) admits a least energy solution having the Mountain-Pass energy level (see Theorem 8.7).

8.1. **Penalization and compactness.** In this section, following the approach of del Pino and Felmer [63], we define a suitable penalization of the functional $I_\varepsilon : W_V(\Omega) \to \mathbb{R}$ associated with the problem (8.13),

$$I_\varepsilon(u) := \varepsilon^p \int_\Omega j(x,u,Du) + \frac{1}{p}\int_\Omega V(x)|u|^p - \int_\Omega F(u).$$

By the growth condition on j, it is easily seen that I_ε is a continuous functional.

Let $\alpha > 0$ be as in (8.1) and consider the positive constant

$$\ell := \sup\Big\{s > 0 : \frac{f(t)}{t^{p-1}} \le \frac{\alpha}{\kappa} \quad \text{for every } 0 \le t \le s\Big\} \tag{8.14}$$

for some fixed $\kappa > \vartheta/(\vartheta - p)$. We define the function $\widetilde{f} : \mathbb{R}^+ \to \mathbb{R}$ by setting

$$\widetilde{f}(s) := \begin{cases} \frac{\alpha}{\kappa} s^{p-1} & \text{if } s > \ell \\ f(s) & \text{if } 0 \le s \le \ell \end{cases}$$

and the map $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ as

$$g(x,s) := 1_\Lambda(x) f(s) + (1 - 1_\Lambda(x))\widetilde{f}(s), \quad \Lambda = \bigcup_{i=1}^k \Lambda_i$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^+$. The function $g(x,s)$ is measurable in x, of class C^1 in s and it satisfies the following properties:

$$\lim_{s\to+\infty} \frac{g(x,s)}{s^{q-1}} = 0, \quad \lim_{s\to 0^+} \frac{g(x,s)}{s^{p-1}} = 0 \quad \text{uniformly in } x, \tag{8.15}$$

$$0 < \vartheta G(x,s) \le g(x,s)s \quad \text{for } x \in \Lambda \text{ and } s \in \mathbb{R}^+, \tag{8.16}$$

$$0 \le pG(x,s) \le g(x,s)s \le \frac{1}{\kappa} V(x)s^p \quad \text{for } x \in \Omega \setminus \Lambda \text{ and } s \in \mathbb{R}^+, \tag{8.17}$$

where we have set $G(x,s) := \int_0^s g(x,\tau)\,d\tau$.

Without loss of generality, we may assume that

$$g(x,s) = 0 \quad \text{for a.e. } x \in \Omega \text{ and every } s < 0, \tag{8.18}$$

$$j(x,s,\xi) = j(x,0,\xi) \quad \text{for every } x \in \Omega,\ s < 0 \text{ and } \xi \in \mathbb{R}^N. \tag{8.19}$$

Let now $J_\varepsilon : W_V(\Omega) \to \mathbb{R}$ be the functional defined as

$$J_\varepsilon(u) := \varepsilon^p \int_\Omega j(x,u,Du) + \frac{1}{p}\int_\Omega V(x)|u|^p - \int_\Omega G(x,u).$$

If $\bar{x}$ is in one of the Λ_is, we also consider the "limit" functionals on $W^{1,p}(\mathbb{R}^N)$,

$$I_{\bar{x}}(u) := \int_{\mathbb{R}^N} j(\bar{x},u,Du) + \frac{1}{p}\int_{\mathbb{R}^N} V(\bar{x})|u|^p - \int_{\mathbb{R}^N} F(u) \tag{8.20}$$

whose positive critical points solve equation (8.11). We denote by $c_{\bar{x}}$ the Mountain-Pass value of $I_{\bar{x}}$, namely

$$c_{\bar{x}} := \inf_{\gamma\in\mathcal{P}_{\bar{x}}} \sup_{t\in[0,1]} I_{\bar{x}}(\gamma(t)), \tag{8.21}$$

$$\mathcal{P}_{\bar{x}} := \Big\{\gamma \in C([0,1], W^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \quad I_{\bar{x}}(\gamma(1)) < 0\Big\}. \tag{8.22}$$

We set $c_i := c_{x_i}$ for every $i = 1,\dots,k$. Considering $\sigma_i > 0$ such that

$$\sum_{i=1}^k \sigma_i < \frac{1}{2}\min\big\{c_i : i = 1,\dots,k\big\},$$

we claim that, up to making Λ_is smaller, we may assume that

$$c_i \le c_{\bar{x}} \le c_i + \sigma_i \quad \text{for all } \bar{x} \in \Lambda_i. \tag{8.23}$$

In fact $c_i \le c_{\bar{x}}$ follows because x_i is a minimum of V in Λ_i and (8.12) holds. On the other hand, let us consider $\bar{x}_h \to x_i$ such that $\lim_h c_{\bar{x}_h} = \limsup_{\bar{x}\to x_i} c_{\bar{x}}$. Let $\gamma \in \mathcal{P}_{\bar{x}}$ be such that $\max_{\tau\in[0,1]} I_{x_i}(\gamma(\tau)) \le c_i + \sigma_i$. Since $I_{\bar{x}_h} \to I_{x_i}$ uniformly on γ, we have that for h large enough, $\gamma \in \mathcal{P}_{\bar{x}_h}$ and there exists $\tau_h \in [0,1]$ such that

$$c_{\bar{x}_h} \le I_{\bar{x}_h}(\gamma(\tau_h)) \le I_{x_i}(\gamma(\tau_h)) + o(1) \le c_i + \sigma_i + o(1).$$

We deduce that $\limsup_{\bar{x}\to x_i} c_{\bar{x}} \le c_i + \sigma_i$ so that the claim is proved.

If $\widehat{\Lambda}_i$ denote mutually disjoint open sets compactly containing Λ_i, we introduce the functionals $J_{\varepsilon,i} : W^{1,p}(\widehat{\Lambda}_i) \to \mathbb{R}$ as

$$J_{\varepsilon,i}(u) := \varepsilon^p \int_{\widehat{\Lambda}_i} j(x,u,Du) + \frac{1}{p}\int_{\widehat{\Lambda}_i} V(x)|u|^p - \int_{\widehat{\Lambda}_i} G(x,u) \tag{8.24}$$

for every $i = 1,\dots,k$.

Finally, let us define the penalized functional $E_\varepsilon : W_V(\Omega) \to \mathbb{R}$ by setting

$$E_\varepsilon(u) := J_\varepsilon(u) + P_\varepsilon(u), \tag{8.25}$$

$$P_\varepsilon(u) := M \sum_{i=1}^k \Big((J_{\varepsilon,i}(u)_+)^{1/2} - \varepsilon^{N/2}(c_i+\sigma_i)^{1/2}\Big)_+^2, \tag{8.26}$$

where $M > 0$ is chosen so that

$$M > \frac{c_1 + \dots + c_k}{\min_{i=1,\dots,k}\big\{(2c_i)^{1/2} - (c_i+\sigma_i)^{1/2}\big\}}.$$

The functionals J_ε, $J_{\varepsilon,i}$ and E_ε are merely continuous.

The next result provides the link between the critical points of E_ε and the weak solutions of the original problem.

Proposition 8.3. *Let $u_\varepsilon \in W_V(\Omega)$ be any critical point of E_ε and assume that there exists a positive number ε_0 such that the following conditions hold*

$$u_\varepsilon(x) < \ell \quad \text{for every } \varepsilon \in (0, \varepsilon_0) \text{ and } x \in \Omega \setminus \Lambda, \tag{8.27}$$

$$\varepsilon^{-N} J_{\varepsilon,i}(u_\varepsilon) < c_i + \sigma_i \quad \text{for every } \varepsilon \in (0, \varepsilon_0) \text{ and } i = 1, \dots, k. \tag{8.28}$$

Then, for every $\varepsilon \in (0, \varepsilon_0)$, u_ε is a solution of (8.13).

Proof. Let $\varepsilon \in (0, \varepsilon_0)$. By condition (8.28) and the definition of $P(u_\varepsilon)$, u_ε is actually a critical point of J_ε. In view of (a) of Proposition 2.25, u_ε is a weak solution of

$$-\varepsilon^p \operatorname{div}(j_\xi(x, u, Du)) + \varepsilon^p j_s(x, u, Du) + V(x)|u|^{p-2}u = G(x, u).$$

Moreover, by (8.27) and the definition of $\widetilde{f}$, it results $G(x, u_\varepsilon(x)) = F(u_\varepsilon(x))$ for a.e. $x \in \Omega$. By (8.18) and (8.19) and arguing as in the proof of [131, Lemma 1], one gets $u_\varepsilon > 0$ in Ω. Thus u_ε is a solution of (8.13). $\square$

The next Lemma is a variant of a local compactness property for bounded concrete Palais-Smale sequences (cf. [131, Theorem 2 and Lemma 3]; see also [48]).

Lemma 8.4. *Assume that* (8.7), (8.8), (8.10) *hold and let $(\psi_h) \subset L^\infty(\mathbb{R}^N)$ bounded with $\psi_h(x) \geq \lambda > 0$. Let $\varepsilon > 0$ and assume that $(u_h) \subset W^{1,p}(\mathbb{R}^N)$ is a bounded sequence such that*

$$\langle w_h, \varphi \rangle = \varepsilon^p \int_{\mathbb{R}^N} \psi_h(x) j_\xi(x, u_h, Du_h) \cdot D\varphi + \varepsilon^p \int_{\mathbb{R}^N} \psi_h(x) j_s(x, u_h, Du_h)\varphi$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, where (w_h) is strongly convergent in $W^{-1,p'}(\widetilde{\Omega})$ for a given bounded domain $\widetilde{\Omega}$ of $\mathbb{R}^N$. Then (u_h) admits a strongly convergent subsequence in $W^{1,p}(\widetilde{\Omega})$.

Since Ω may be unbounded, in general the original functional I_ε does not satisfy the concrete Palais-Smale condition. In the following Lemma we prove that, instead, for every $\varepsilon > 0$ the functional E_ε satisfies it at every level $c \in \mathbb{R}$.

Lemma 8.5. *Assume that conditions* (8.1), (8.5), (8.6), (8.7), (8.8), (8.9), (8.10) *hold. Let $\varepsilon > 0$.*

Then E_ε satisfies the concrete Palais-Smale condition at every level $c \in \mathbb{R}$.

Proof. Let $(u_h) \subset W_V(\Omega)$ be a concrete Palais-Smale sequence for E_ε at level c. We divide the proof into two steps:

Step I. We prove that (u_h) is bounded in $W_V(\Omega)$. From (8.16) and (8.17), we get

$$\begin{aligned} &\vartheta \varepsilon^p \int_\Omega j(x, u_h, Du_h) + \frac{\vartheta}{p} \int_\Omega V(x)|u_h|^p \\ &\leq \int_\Lambda g(x, u_h)u_h + \frac{\vartheta}{p\kappa} \int_{\Omega\setminus\Lambda} V(x)|u_h|^p + \vartheta J_\varepsilon(u_h) \end{aligned} \tag{8.29}$$

for every $h \in \mathbb{N}$. Moreover, for every $h \in \mathbb{N}$ we can compute $J'_\varepsilon(u_h)(u_h)$; in view of (8.17) we obtain

$$\begin{aligned} &\int_\Lambda g(x, u_h)u_h + J'_\varepsilon(u_h)[u_h] \\ &\leq \varepsilon^p \int_\Omega j_\xi(x, u_h, Du_h) \cdot Du_h + \varepsilon^p \int_\Omega j_s(x, u_h, Du_h)u_h + \int_\Omega V(x)|u_h|^p \end{aligned}$$

for every $h \in \mathbb{N}$. Notice that by (8.9) and the p-homogeneity of the map $\{\xi \mapsto j(x,s,\xi)\}$, it results

$$j_s(x,u_h,Du_h)u_h \le \gamma j(x,u_h,Du_h),$$
$$j_\xi(x,u_h,Du_h)\cdot Du_h = pj(x,u_h,Du_h)$$

for every $h \in \mathbb{N}$. Therefore,

$$\int_\Lambda g(x,u_h)u_h + J'_\varepsilon(u_h)[u_h] \le (\gamma+p)\varepsilon^p \int_\Omega j(x,u_h,Du_h) + \int_\Omega V(x)|u_h|^p \tag{8.30}$$

for every $h \in \mathbb{N}$. In view of (8.8), by combining inequalities (8.29) and (8.30) one gets

$$\begin{aligned} &\min\Big\{ (\vartheta-\gamma-p)\,\nu\varepsilon^p, \frac{\vartheta}{p} - \frac{\vartheta}{p\kappa} - 1\Big\} \int_\Omega \Big(|Du_h|^p + V(x)|u_h|^p\Big) \\ &\le \vartheta J_\varepsilon(u_h) - J'_\varepsilon(u_h)[u_h] \end{aligned} \tag{8.31}$$

for every $h \in \mathbb{N}$. In a similar fashion, arguing on the functionals $J_{\varepsilon,i}$, it results

$$\begin{aligned} &\min\Big\{ (\vartheta-\gamma-p)\,\nu\varepsilon^p, \frac{\vartheta}{p} - \frac{\vartheta}{p\kappa} - 1\Big\} \int_{\widehat{\Lambda}_i} \Big(|Du_h|^p + V(x)|u_h|^p\Big) \\ &\le \vartheta J_{\varepsilon,i}(u_h) - J'_{\varepsilon,i}(u_h)[u_h] \quad \text{for every } h \in \mathbb{N} \text{ and } i = 1,\dots,k. \end{aligned} \tag{8.32}$$

In particular, notice that one obtains

$$\bar\vartheta J_{\varepsilon,i}(u_h) - J'_{\varepsilon,i}(u_h)[u_h] \ge 0 \quad \text{for every } h \in \mathbb{N} \text{ and } i = 1,\dots,k$$

and every $\gamma + p < \bar\vartheta < \vartheta$. Then, after some computations, one gets

$$\begin{aligned} &\bar\vartheta P_\varepsilon(u_h) - P'_\varepsilon(u_h)[u_h] \\ &\ge -\bar\vartheta M\varepsilon^{N/2}\sum_{i=1}^k (c_i+\sigma_i)^{1/2}\Big((J_{\varepsilon,i}(u_h)_+)^{1/2} - \varepsilon^{N/2}(c_i+\sigma_i)^{1/2}\Big)_+ \\ &\ge -C\varepsilon^{N/2}P_\varepsilon(u_h)^{1/2} \end{aligned}$$

which implies, by Young's inequality, the existence of a constant $d > 0$ such that

$$\vartheta P_\varepsilon(u_h) - P'_\varepsilon(u_h)[u_h] \ge -d\varepsilon^N \tag{8.33}$$

for every $h \in \mathbb{N}$. By combining (8.31) with (8.33), since

$$E_\varepsilon(u_h) = c + o(1), \quad E'_\varepsilon(u_h)[u_h] = o(\|u_h\|_{W_V})$$

as $h \to +\infty$, one obtains

$$\begin{aligned} &\int_\Omega \Big(|Du_h|^p + V(x)|u_h|^p\Big) \\ &\le \frac{\vartheta c + d\varepsilon^N}{\min\Big\{(\vartheta-\gamma-p)\,\nu\varepsilon^p, \frac{\vartheta}{p} - \frac{\vartheta}{p\kappa} - 1\Big\}} + o(\|u_h\|_{W_V}) + o(1) \end{aligned} \tag{8.34}$$

as $h \to +\infty$, which yields the boundedness of (u_h) in $W_V(\Omega)$.

Step II. By virtue of Step I, there exists $u \in W_V(\Omega)$ such that, up to a subsequence, (u_h) weakly converges to u in $W_V(\Omega)$. Let us now prove that actually (u_h) converges strongly to u in $W_V(\Omega)$. If we define for every $h \in \mathbb{N}$ the weights

$$\theta_{h,i} = M\big[(J_{\varepsilon,i}(u_h)_+)^{1/2} - \varepsilon^{N/2}(c_i+\sigma_i)^{1/2}\big]_+ (J_{\varepsilon,i}(u_h)_+)^{-1/2}, \quad i = 1,\dots,k$$

and put $\theta_h(x) = \sum_{i=1}^{k} \theta_{h,i} 1_{\widehat{\Lambda}_i}(x)$ with $0 \le \theta_{h,i} \le M$. After a few computations, one gets

$$\langle w_h, \varphi\rangle = \varepsilon^p \int_\Omega (1+\theta_h) j_\xi(x,u_h,Du_h)\cdot D\varphi + \varepsilon^p \int_\Omega (1+\theta_h) j_s(x,u_h,Du_h)\varphi$$

for every $\varphi \in C_c^\infty(\Omega)$, where

$$w_h = (1+\theta_h) g(x,u_h) - (1+\theta_h)V(x)|u_h|^{p-2}u_h + \xi_h,$$

with $\xi_h \to 0$ strongly in $W^{-1,p'}(\Omega)$. Since, up to a subsequence, (w_h) strongly converges to $w := (1+\bar\theta)g(x,u) - (1+\bar\theta)V(x)|u|^{p-2}u$ in $W^{-1,p'}(B_\varrho)$ for every $\varrho > 0$, by applying Lemma 8.4 with $\widetilde\Omega = B_\varrho \cap \Omega$ and $\psi_h(x) = 1+\theta_h(x)$, it suffices to show that, for every $\delta > 0$, there exists $\varrho > 0$ such that

$$\limsup_h \int_{\Omega\setminus B_\varrho} \Big(|Du_h|^p + V(x)|u_h|^p\Big) < \delta. \tag{8.35}$$

Consider a cut-off function $\chi_\varrho \in C^\infty(\mathbb{R}^N)$ with $0 \le \chi_\varrho \le 1$, $\chi_\varrho = 0$ on $B_{\varrho/2}$, $\chi_\varrho = 1$ on $\mathbb{R}^N \setminus B_\varrho$ and $|D\chi_\varrho| \le a/\varrho$ for some $a > 0$. By taking ϱ large enough, we have

$$\bigcup_{i=1}^{k} \widehat{\Lambda}_i \cap \operatorname{supt}(\chi_\varrho) = \emptyset. \tag{8.36}$$

Let now $\zeta : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$\zeta(s) := \begin{cases} 0 & \text{if } s < 0 \\ \bar M s & \text{if } 0 \le s < R \\ \bar M R & \text{if } s \ge R, \end{cases} \tag{8.37}$$

being $R > 0$ the constant defined in (8.10) and $\bar M$ a positive number (which exists by the growths (8.7) and (8.8)) such that

$$|j_s(x,s,\xi)| \le p\bar M j(x,s,\xi) \tag{8.38}$$

for every $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. Notice that, by combining (8.10) and (8.38), we obtain

$$j_s(x,s,\xi) + p\zeta'(s) j(x,s,\xi) \ge 0 \quad \text{for every } x \in \Omega,\ s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N. \tag{8.39}$$

By (8.36) it is easily proved that $P_\varepsilon'(u_h)(\chi_\varrho u_h e^{\zeta(u_h)}) = 0$ for every h. Therefore, since the sequence $(\chi_\varrho u_h e^{\zeta(u_h)})$ is bounded in $W_V(\Omega)$, taking into account (8.39) and (8.19) we obtain

$$\begin{aligned}
o(1) &= J_\varepsilon'(u_h)(\chi_\varrho u_h e^{\zeta(u_h)}) \\
&= \varepsilon^p \int_\Omega j_\xi(x,u_h,Du_h)\cdot Du_h \chi_\varrho e^{\zeta(u_h)} \\
&\quad + \varepsilon^p \int_\Omega j_\xi(x,u_h,Du_h)\cdot D\chi_\varrho u_h e^{\zeta(u_h)} \\
&\quad + \varepsilon^p \int_\Omega \big[j_s(x,u_h,Du_h) + p\zeta'(u_h) j(x,u_h,Du_h)\big] u_h \chi_\varrho e^{\zeta(u_h)} \\
&\quad + \int_\Omega V(x)|u_h|^p \chi_\varrho e^{\zeta(u_h)} - \int_\Omega g(x,u_h)u_h \chi_\varrho e^{\zeta(u_h)} \\
&\ge \int_\Omega \Big(p\varepsilon^p j(x,u_h,Du_h) + V(x)|u_h|^p\Big)\chi_\varrho e^{\zeta(u_h)}
\end{aligned}$$

$$+\varepsilon^p \int_\Omega j_\xi(x,u_h,Du_h)\cdot D\chi_\varrho u_h e^{\zeta(u_h)} - \int_\Omega g(x,u_h)u_h\chi_\varrho e^{\zeta(u_h)}$$

as $h \to +\infty$. Therefore, in view of (8.17) and (8.36), it results

$$\begin{aligned} o(1) \geq & \int_\Omega \Big(p\varepsilon^p\nu|Du_h|^p + V(x)|u_h|^p\Big)\chi_\varrho e^{\zeta(u_h)} \\ & + \varepsilon^p \int_\Omega j_\xi(x,u_h,Du_h)\cdot D\chi_\varrho u_h e^{\zeta(u_h)} - \frac{1}{\kappa}\int_\Omega V(x)|u_h|^p\chi_\varrho e^{\zeta(u_h)} \end{aligned}$$

as $h \to +\infty$ for ϱ large enough. Since by (8.7) we have

$$\Big|\int_\Omega j_\xi(x,u_h,Du_h)\cdot D\chi_\varrho u_h e^{\zeta(u_h)}\Big| \leq \frac{C}{\varrho}\|Du_h\|_p^{p-1}\|u_h\|_p \leq \frac{\widetilde{C}}{\varrho},$$

there exists a positive constant C' such that

$$\limsup_h \int_{\Omega\setminus B_\varrho}\Big(|Du_h|^p + V(x)|u_h|^p\Big) \leq \frac{C'}{\varrho}$$

which yields (8.35). The proof is now complete. □

8.2. **Two consequences of the Pucci-Serrin identity.** Let $\mathscr{L} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a function of class C^1 such that the function $\{\xi \mapsto \mathscr{L}(x,s,\xi)\}$ is strictly convex for every $(x,s) \in \mathbb{R}^N \times \mathbb{R}$, and let $\varphi \in L^\infty_{\rm loc}(\mathbb{R}^N)$.

We now recall a Pucci-Serrin variational identity for locally Lipschitz continuous solutions of a general class of Euler equations, recently obtained in [59]. Notice that the classical identity [117] is not applicable here, since it requires the C^2 regularity of the solutions while in our degenerate setting the maximal regularity is $C^{1,\beta}_{\rm loc}$ (see [65, 142]).

Theorem 8.6. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz solution of*

$$-\operatorname{div}\big(D_\xi\mathscr{L}(x,u,Du)\big) + D_s\mathscr{L}(x,u,Du) = \varphi \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

Then for every $h \in C^1_c(\mathbb{R}^N,\mathbb{R}^N)$,

$$\begin{aligned} &\sum_{i,j=1}^N \int_{\mathbb{R}^N} D_i h^j D_{\xi_i}\mathscr{L}(x,u,Du)D_j u \\ &- \int_{\mathbb{R}^N}\big[(\operatorname{div} h)\,\mathscr{L}(x,u,Du) + h\cdot D_x\mathscr{L}(x,u,Du)\big] = \int_{\mathbb{R}^N}(h\cdot Du)\varphi\,. \end{aligned} \tag{8.40}$$

We want to derive two important consequences of the previous variational identity. In the first we show that the Mountain-Pass value associated with a large class of elliptic autonomous equations is the minimal among other nontrivial critical values.

Theorem 8.7. *Let $\bar{x} \in \mathbb{R}^N$ and assume that conditions (8.1), (8.5), (8.6), (8.7), (8.8), (8.9), (8.10) hold. Then the equation*

$$-\operatorname{div}(j_\xi(\bar{x},u,Du)) + j_s(\bar{x},u,Du) + V(\bar{x})u^{p-1} = f(u) \quad \text{in } \mathbb{R}^N \tag{8.41}$$

admits a least energy solution $u \in W^{1,p}(\mathbb{R}^N)$, that is

$$I_{\bar{x}}(u) = \inf\big\{I_{\bar{x}}(w) : w \in W^{1,p}(\mathbb{R}^N)\setminus\{0\} \text{ is a solution of } (8.41)\big\},$$

where $I_{\bar{x}}$ is as in (8.20). Moreover, $I_{\bar{x}}(u) = c_{\bar{x}}$, that is u is at the Mountain-Pass level.

Proof. We divide the proof into two steps. **Step I**. Let u be any nontrivial solution of (8.41), and let us prove that $I_{\bar{x}}(u) \geq c_{\bar{x}}$. By the assumptions on V and f, it is readily seen that there exist $\varrho_0 > 0$ and $\delta_0 > 0$ such that $I_{\bar{x}}(v) \geq \delta_0$ for every $v \in W^{1,p}(\mathbb{R}^N)$ with $\|v\|_{1,p} = \varrho_0$. In particular $I_{\bar{x}}$ has a Mountain-Pass geometry. As we will see, $\mathcal{P}_{\bar{x}} \neq \emptyset$, so that $c_{\bar{x}}$ is well defined. Let now u be a positive solution of (8.41) and consider the dilation path

$$\gamma(t)(x) := \begin{cases} u(x/t) & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Notice that $\|\gamma(t)\|_{1,p}^p = t^{N-p}\|Du\|_p^p + t^N\|u\|_p^p$ for every $t \in \mathbb{R}^+$, which implies that the curve γ belongs to $C(\mathbb{R}^+, W^{1,p}(\mathbb{R}^N))$. For the sake of simplicity, we consider the continuous function $H : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$H(s) = \int_0^s h(t)\,dt, \quad \text{where } h(s) = -V(\bar{x})s^{p-1} + f(s).$$

For every $t \in \mathbb{R}^+$ it results that

$$\begin{aligned} I_{\bar{x}}(\gamma(t)) &= \int_{\mathbb{R}^N} j(\bar{x}, \gamma(t), D\gamma(t)) - \int_{\mathbb{R}^N} H(\gamma(t)) \\ &= t^{N-p}\int_{\mathbb{R}^N} j(\bar{x}, u, Du) - t^N\int_{\mathbb{R}^N} H(u) \end{aligned}$$

which yields, for every $t \in \mathbb{R}^+$

$$\frac{d}{dt} I_{\bar{x}}(\gamma(t)) = (N-p)t^{N-p-1}\int_{\mathbb{R}^N} j(\bar{x}, u, Du) - Nt^{N-1}\int_{\mathbb{R}^N} H(u). \tag{8.42}$$

By virtue of (8.8) and (8.10), a standard argument yields $u \in L^\infty_{\rm loc}(\mathbb{R}^N)$ (see [123, Theorem 1]); by the regularity results of [65, 142], it follows that $u \in C^{1,\beta}_{\rm loc}(\mathbb{R}^N)$ for some $0 < \beta < 1$. Then, since $\{\xi \mapsto j(x, s, \xi)\}$ is strictly convex, we can use Theorem 8.6 by choosing in (8.40) $\varphi = 0$ and

$$\begin{aligned} &\mathcal{L}(s, \xi) := j(\bar{x}, s, \xi) - H(s) \quad \text{for every } s \in \mathbb{R}^+ \text{ and } \xi \in \mathbb{R}^N, \\ &h(x) = h_k(x) := T(\frac{x}{k})x \quad \text{for every } x \in \mathbb{R}^N \text{ and } k \geq 1, \end{aligned} \tag{8.43}$$

being $T \in C_c^1(\mathbb{R}^N)$ such that $T(x) = 1$ if $|x| \leq 1$ and $T(x) = 0$ if $|x| \geq 2$. In particular, for every k we have that $h_k \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\begin{aligned} D_i h_k^j(x) &= D_i T(\frac{x}{k})\frac{x_j}{k} + T(\frac{x}{k})\delta_{ij} \quad \text{for every } x \in \mathbb{R}^N, i, j = 1, \dots, N, \\ (\operatorname{div} h_k)(x) &= DT(\frac{x}{k}) \cdot \frac{x}{k} + NT(\frac{x}{k}) \quad \text{for every } x \in \mathbb{R}^N. \end{aligned}$$

Then, since $D_x\mathcal{L}(u, Du) = 0$, it follows by (8.40) that

$$\begin{aligned} &\sum_{i,j=1}^n \int_{\mathbb{R}^N} D_i T(\frac{x}{k})\frac{x_j}{k} D_j u D_{\xi_i}\mathcal{L}(u, Du) + \int_{\mathbb{R}^N} T(\frac{x}{k}) D_\xi \mathcal{L}(u, Du) \cdot Du \\ &- \int_{\mathbb{R}^N} DT(\frac{x}{k}) \cdot \frac{x}{k}\,\mathcal{L}(u, Du) - \int_{\mathbb{R}^N} NT(\frac{x}{k})\mathcal{L}(u, Du) = 0 \end{aligned}$$

for every $k \geq 1$. Since there exists $C > 0$ with

$$D_i T(\frac{x}{k})\frac{x_j}{k} \leq C \quad \text{for every } x \in \mathbb{R}^N, k \geq 1 \text{ and } i, j = 1, \dots, N,$$

by the Dominated Convergence Theorem, letting $k \to +\infty$, we obtain

$$\int_{\mathbb{R}^N} \Big[N\mathcal{L}(u, Du) - D_\xi \mathcal{L}(u, Du) \cdot Du \Big] = 0,$$

namely, by (8.43) and the p-homogeneity of $\{\xi \mapsto j(x, s, \xi)\}$,

$$(N - p) \int_{\mathbb{R}^N} j(\bar{x}, u, Du) = N \int_{\mathbb{R}^N} H(u). \tag{8.44}$$

In particular notice that $\int_{\mathbb{R}^N} H(u) > 0$. By plugging this formula into (8.42), we obtain

$$\frac{d}{dt} I_{\bar{x}}(\gamma(t)) = N(1 - t^p)t^{N-p-1} \int_{\mathbb{R}^N} H(u)$$

which yields $\frac{d}{dt} I_{\bar{x}}(\gamma(t)) > 0$ for $0 < t < 1$ and $\frac{d}{dt} I_{\bar{x}}(\gamma(t)) < 0$ for $t > 1$, namely

$$\sup_{t \in [0, +\infty[} I_{\bar{x}}(\gamma(t)) = I_{\bar{x}}(\gamma(1)) = I_{\bar{x}}(u).$$

Moreover, observe that $\gamma(0) = 0$ and $I_{\bar{x}}(\gamma(T)) < 0$ for $T > 0$ sufficiently large. Then, after a suitable scale change in t, $\gamma \in \mathcal{P}_{\bar{x}}$ and the assertion follows.

Step II Let us now prove that (8.41) has a nontrivial solution $u \in W^{1,p}(\mathbb{R}^N)$ such that $c_{\bar{x}} \geq I_{\bar{x}}(u)$. Let (u_h) be a Palais-Smale sequence for $I_{\bar{x}}$ at the level $c_{\bar{x}}$. Since (u_h) is bounded in $W^{1,p}(\mathbb{R}^N)$, considering the test $u_h e^{\zeta(u_h)}$ with ζ as in (8.37), and recalling (8.39), we have

$$\begin{aligned}
pc_{\bar{x}} + o(1) &= pI_{\bar{x}}(u_h) - I'_{\bar{x}}(u_h)[u_h e^{\zeta(u_h)}] \\
&= \int_{\mathbb{R}^N} p(1 - e^{\zeta(u_h)}) j(\bar{x}, u_h, Du_h) + \int_{\mathbb{R}^N} (1 - e^{\zeta(u_h)}) V(\bar{x})|u_h|^p \\
&\quad - \int_{\mathbb{R}^N} \big[p\zeta'(u_h) j(\bar{x}, u_h, Du_h) + j_s(\bar{x}, u_h, Du_h) \big] u_h e^{\zeta(u_h)} \\
&\quad - \int_{\mathbb{R}^N} pF(u_h) + \int_{\mathbb{R}^N} f(u_h) u_h e^{\zeta(u_h)} \\
&\leq - \int_{\mathbb{R}^N} pF(u_h) + \int_{\mathbb{R}^N} f(u_h) u_h e^{\zeta(u_h)} \\
&\leq C \int_{\mathbb{R}^N} |u_h|^p + |u_h|^q
\end{aligned}$$

for some $C > 0$. By [98, Lemma I.1], we conclude that (u_h) may not vanish in L^p, that is there exists $x_h \in \mathbb{R}^N$, $R > 0$ and $\lambda > 0$ such that for h large

$$\int_{x_h + B_R} |u_h|^p \geq \lambda. \tag{8.45}$$

Let $v_h(x) := u_h(x_h + x)$ and let $u \in W^{1,p}(\mathbb{R}^N)$ be such that $v_h \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^N)$. Since v_h is a Palais-Smale sequence for $I_{\bar{x}}$ at level $c_{\bar{x}}$, by Lemma 8.4, we have that $v_h \to u$ strongly in $W^{1,p}_{\rm loc}(\mathbb{R}^N)$. By (8.45), we deduce that u is a nontrivial solution of (8.41). Let $\delta > 0$; we claim that there exists $\varrho > 0$ such that

$$\liminf_h \int_{\mathbb{R}^N \setminus B_\varrho} \Big[j(\bar{x}, v_h, Dv_h) + \frac{1}{p} V(\bar{x})|v_h|^p - F(v_h) \Big] \geq -\delta. \tag{8.46}$$

In fact, let $\varrho > 0$, and let η_ϱ be a smooth function such that $0 \le \eta_\varrho \le 1$, $\eta_\varrho = 0$ on $B_{\varrho-1}$, $\eta_\varrho = 1$ on $\mathbb{R}^N \setminus B_\varrho$ and $\|D\eta_\varrho\|_\infty \le 2$. Testing with $\eta_\varrho v_h$, we get

$$\begin{aligned}
&\langle w_h, \eta_\varrho v_h\rangle - \int_{B_\varrho \setminus B_{\varrho-1}} \big[j_\xi(\bar{x}, v_h, Dv_h) \cdot D(\eta_\varrho v_h) \\
&+ j_s(\bar{x}, v_h, Dv_h)\eta_\varrho v_h + V(\bar{x})|v_h|^p \eta_\varrho - f(v_h)v_h\eta_\varrho\big] \\
&= \int_{\mathbb{R}^N \setminus B_\varrho} \big[j_\xi(\bar{x}, v_h, Dv_h) \cdot D(\eta_\varrho v_h) + j_s(\bar{x}, v_h, Dv_h)\eta_\varrho v_h \\
&\quad + V(\bar{x})|v_h|^p \eta_\varrho - f(v_h)v_h\eta_\varrho\big]
\end{aligned}$$

where $w_h \to 0$ strongly in $W^{-1,p'}(\mathbb{R}^N)$. For the right hand side we have

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_\varrho} \big[j_\xi(\bar{x}, v_h, Dv_h) \cdot D(\eta_\varrho v_h) + j_s(\bar{x}, v_h, Dv_h)\eta_\varrho v_h \\
&\quad + V(\bar{x})|v_h|^p \eta_\varrho - f(v_h)v_h\eta_\varrho\big] \\
&= \int_{\mathbb{R}^N \setminus B_\varrho} \big[pj(\bar{x}, v_h, Dv_h) + j_s(\bar{x}, v_h, Dv_h)v_h + V(\bar{x})|v_h|^p - f(v_h)v_h\big],
\end{aligned}$$

and by (8.9) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_\varrho} \big[pj(\bar{x}, v_h, Dv_h) + j_s(\bar{x}, v_h, Dv_h)v_h + V(\bar{x})|v_h|^p - f(v_h)v_h\big] \\
&\le (p+\gamma)\int_{\mathbb{R}^N \setminus B_\varrho} j(\bar{x}, v_h, Dv_h) + \int_{\mathbb{R}^N \setminus B_\varrho} V(\bar{x})|v_h|^p - f(v_h)v_h \\
&= (p+\gamma)\int_{\mathbb{R}^N \setminus B_\varrho} \big[j(\bar{x}, v_h, Dv_h) + \frac{1}{p}V(\bar{x})|v_h|^p - F(v_h)\big] \\
&\quad - \frac{p+\gamma}{p}\int_{\mathbb{R}^N \setminus B_\varrho} V(\bar{x})|v_h|^p + \int_{\mathbb{R}^N \setminus B_\varrho} V(\bar{x})|v_h|^p \\
&\quad + \int_{\mathbb{R}^N \setminus B_\varrho} \big[(p+\gamma)F(v_h) - f(v_h)v_h\big] \\
&\le (p+\gamma)\int_{\mathbb{R}^N \setminus B_\varrho} \big[j(\bar{x}, v_h, Dv_h) + \frac{1}{p}V(\bar{x})|v_h|^p - F(v_h)\big] \\
&\quad + \int_{\mathbb{R}^N \setminus B_\varrho} \big[(p+\gamma)F(v_h) - \vartheta F(v_h)\big] \\
&\le (p+\gamma)\int_{\mathbb{R}^N \setminus B_\varrho} \big[j(\bar{x}, v_h, Dv_h) + \frac{1}{p}V(\bar{x})|v_h|^p - F(v_h)\big].
\end{aligned}$$

We conclude that

$$\begin{aligned}
&(p+\gamma)\int_{\mathbb{R}^N \setminus B_\varrho} \big[j(\bar{x}, v_h, Dv_h) + \frac{1}{p}V(\bar{x})|v_h|^p - F(v_h)\big] \\
&\ge \langle w_h, \eta_\varrho v_h\rangle - \int_{B_\varrho \setminus B_{\varrho-1}} \big[j_\xi(\bar{x}, v_h, Dv_h) \cdot D(\eta_\varrho v_h) + j_s(\bar{x}, v_h, Dv_h)\eta_\varrho v_h \\
&\quad + V(\bar{x})|v_h|^p \eta_\varrho - f(v_h)v_h\eta_\varrho\big].
\end{aligned}$$

Since by Lemma 8.4 we have $v_h \to u$ strongly in $W^{1,p}(B_\varrho)$, we get

$$\lim_h \int_{B_\varrho \setminus B_{\varrho-1}} \big[j_\xi(\bar{x}, v_h, Dv_h) \cdot D(\eta_\varrho v_h) + j_s(\bar{x}, v_h, Dv_h)\eta_\varrho v_h$$

$$
\begin{aligned}
&+ V(\bar{x})|v_h|^p\eta_\varrho - f(v_h)v_h\eta_\varrho\big] \\
&= \int_{B_\varrho\setminus B_{\varrho-1}} \big[j_\xi(\bar{x},u,Du)\cdot D(\eta_\varrho u) + j_s(\bar{x},u,Du)\eta_\varrho u + V(\bar{x})|u|^p\eta_\varrho - f(u)u\eta_\varrho\big],
\end{aligned}
$$

and so we deduce that for every $\delta > 0$ there exists $\bar{\varrho} > 0$ such that for all $\varrho > \bar{\varrho}$ we have

$$
\liminf_h \int_{\mathbb{R}^N\setminus B_\varrho} \big[j(\bar{x},v_h,Dv_h) + \frac{1}{p}V(\bar{x})|v_h|^p - F(v_h)\big] \geq -\delta.
$$

Furthermore we have

$$
\lim_h \int_{B_\varrho} \big[j(\bar{x},v_h,Dv_h) + \frac{1}{p}V(\bar{x})|v_h|^p - F(v_h)\big] = I_{\bar{x}}(u,B_\varrho),
$$

where

$$
I_{\bar{x}}(u,B_\varrho) := \int_{B_\varrho} \big[j(\bar{x},u,Du) + \frac{1}{p}V(\bar{x})|u|^p - F(u)\big],
$$

and so we conclude that for all $\varrho > \bar{\varrho}$

$$
c_{\bar{x}} \geq I_{\bar{x}}(u,B_\varrho) - \delta.
$$

Letting $\varrho \to +\infty$ and since δ is arbitrary, we get $c_{\bar{x}} \geq I_{\bar{x}}(u)$, and the proof is complete. $\square$

The second result can be considered as an extension (also with a different proof) of [63, Lemma 2.3] to a general class of elliptic equations. Again we stress that, in this degenerate setting, Theorem 8.6 plays an important role.

Lemma 8.8. *Let $u \in W^{1,p}(\mathbb{R}^N)$ be a positive solution of the equation*

$$
\begin{aligned}
&-\operatorname{div}(j_\xi(\bar{x},u,Du)) + j_s(\bar{x},u,Du) + V(\bar{x})u^{p-1} \\
&= 1_{\{x_1<0\}}(x)f(u) + 1_{\{x_1>0\}}(x)\tilde{f}(u) \quad \text{in } \mathbb{R}^N.
\end{aligned}
\tag{8.47}
$$

Then u is actually a solution of the equation

$$
-\operatorname{div}(j_\xi(\bar{x},u,Du)) + j_s(\bar{x},u,Du) + V(\bar{x})u^{p-1} = f(u) \quad \text{in } \mathbb{R}^N. \tag{8.48}
$$

Proof. Let us first show that $u(x) \leq \ell$ on the set $\{x_1 = 0\}$. As in the proof of Theorem 8.7 it follows that $u \in C^{1,\beta}_{\rm loc}(\mathbb{R}^N)$ for some $0 < \beta < 1$. Then we can apply again Theorem 8.6 by choosing this time in (8.40):

$$
\begin{gathered}
\mathcal{L}(s,\xi) := j(\bar{x},s,\xi) + \frac{V(\bar{x})}{p}s^p \quad \text{for every } s \in \mathbb{R}^+ \text{ and } \xi \in \mathbb{R}^N, \\
\varphi(x) := 1_{\{x_1<0\}}(x)f(u(x)) + 1_{\{x_1>0\}}(x)\tilde{f}(u(x)) \quad \text{for every } x \in \mathbb{R}^N, \\
h(x) = h_k(x) := \Big(T(\frac{x}{k}), 0, \dots, 0\Big) \quad \text{for every } x \in \mathbb{R}^N \text{ and } k \geq 1
\end{gathered}
$$

being $T \in C^1_c(\mathbb{R}^N)$ such that $T(x) = 1$ if $|x| \leq 1$ and $T(x) = 0$ if $|x| \geq 2$. Then $h_k \in C^1_c(\mathbb{R}^N,\mathbb{R}^N)$ and, taking into account that $D_x\mathcal{L}(u,Du) = 0$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Big[\frac{1}{k}\sum_{i=1}^N D_iT(\frac{x}{k})D_1uD_{\xi_i}\mathcal{L}(u,Du) - D_1T(\frac{x}{k})\frac{1}{k}\mathcal{L}(u,Du)\Big] \\
&= \int_{\mathbb{R}^N} T(\frac{x}{k})\varphi(x)D_1u
\end{aligned}
$$

for every $k \geq 1$. Again by the Dominated Convergence Theorem, letting $k \to +\infty$, it follows $\int_{\mathbb{R}^N} \varphi(x) D_{x_1} u = 0$, that is, after integration by parts,

$$\int_{\mathbb{R}^{N-1}} \Big[F(u(0,x')) - \widetilde{F}(u(0,x')) \Big] \, dx' = 0.$$

Taking into account that $F(s) \geq \widetilde{F}(s)$ with equality only if $s \leq \ell$, we get

$$u(0,x') \leq \ell \quad \text{for every } x' \in \mathbb{R}^{N-1}. \tag{8.49}$$

To prove that actually

$$u(x_1,x') \leq \ell \quad \text{for every } x_1 > 0 \text{ and } x' \in \mathbb{R}^{N-1}, \tag{8.50}$$

let us test equation (8.47) with the function

$$\eta(x) = \begin{cases} 0 & \text{if } x_1 < 0 \\ (u(x_1,x') - \ell)^+ e^{\zeta(u(x_1,x'))} & \text{if } x_1 > 0, \end{cases}$$

where $\zeta : \mathbb{R}^+ \to \mathbb{R}$ is the map defined in (8.37). Notice that, in view of (8.49), the function η belongs to $W^{1,p}(\mathbb{R}^N)$. After some computations, one obtains

$$\begin{aligned} &\int_{\{x_1>0\}} p j(\bar{x}, u, D(u-\ell)^+) e^{\zeta(u)} \\ &+ \int_{\{x_1>0\}} \big[j_s(\bar{x}, u, Du) + p\zeta'(u) j(\bar{x}, u, Du) \big] (u-\ell)^+ e^{\zeta(u)} \\ &+ \int_{\{x_1>0\}} \Big[V(\bar{x}) - \frac{\alpha}{\kappa} \Big] u^{p-1} (u-\ell)^+ e^{\zeta(u)} = 0. \end{aligned} \tag{8.51}$$

By (8.1) and (8.39) all the terms in (8.51) must be equal to zero. We conclude that $(u - \ell)^+ = 0$ on $\{x_1 > 0\}$, namely (8.50) holds. In particular $\varphi(x) = f(u(x))$ for every $x \in \mathbb{R}^N$, so that u is a solution of (8.48). $\square$

8.3. **Energy estimates.** Let $d_{\varepsilon,i}$ be the Mountain-Pass critical value which corresponds to the functional $J_{\varepsilon,i}$ defined in (8.24). More precisely,

$$d_{\varepsilon,i} := \inf_{\gamma_i \in \Gamma_i} \sup_{t\in[0,1]} J_{\varepsilon,i}(\gamma_i(t)) \tag{8.52}$$

where

$$\Gamma_i := \big\{ \gamma_i \in C([0,1], W^{1,p}(\widehat{\Lambda}_i)) : \gamma_i(0) = 0,\ J_{\varepsilon,i}(\gamma_i(1)) < 0 \big\}.$$

Then the following result holds.

Lemma 8.9. *For $i = 1, \dots, k$, we have*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-N} d_{\varepsilon,i} = c_i\,.$$

Proof. The inequality

$$d_{\varepsilon,i} \leq \varepsilon^N c_i + o(\varepsilon^N) \tag{8.53}$$

can be easily derived (see the first part of the proof of Lemma 8.10). Let us prove the opposite inequality, which is harder. To this aim, we divide the proof into two steps.
Step I. Let w_ε be a Mountain-Pass critical point for $J_{\varepsilon,i}$. We have $w_\varepsilon \geq 0$, and by regularity results $w_\varepsilon \in L^\infty(\widehat{\Lambda}_i) \cap C^{1,\alpha}_{\rm loc}(\widehat{\Lambda}_i)$. Let us define

$$M_\varepsilon := \sup_{x\in\widehat{\Lambda}_i} w_\varepsilon(x) < +\infty,$$

and for all $\delta > 0$ define the set

$$U_\delta := \{x \in \widehat{\Lambda}_i : w_\varepsilon(x) > M_\varepsilon - \delta\}.$$

We may use the following nontrivial test for the equation satisfied by w_ε

$$\varphi_\delta := [w_\varepsilon - (M_\varepsilon - \delta)]^+ e^{\zeta(w_\varepsilon)},$$

where the map $\zeta : \mathbb{R}^+ \to \mathbb{R}$ is defined as in (8.37). We have

$$D\varphi_\delta = e^{\zeta(w_\varepsilon)} Dw_\varepsilon 1_{U_\delta} + \varphi_\delta \zeta'(w_\varepsilon) Dw_\varepsilon,$$

and so we obtain

$$\varepsilon^p \int_{U_\delta} pj(x, w_\varepsilon, Dw_\varepsilon) e^{\zeta(w_\varepsilon)} + \varepsilon^p \int_{U_\delta} [p\zeta'(w_\varepsilon) j(x, w_\varepsilon, Dw_\varepsilon) + j_s(x, w_\varepsilon, Dw_\varepsilon)]\varphi_\delta$$
$$= \int_{U_\delta} \Big[-V(x) w_\varepsilon^{p-1} + g(x, w_\varepsilon)\Big] \varphi_\delta.$$

Then, by (8.39), it results

$$\int_{U_\delta} \Big[-V(x) w_\varepsilon^{p-1} + g(x, w_\varepsilon)\Big] \varphi_\delta \geq \varepsilon^p \int_{U_\delta} pj(x, w_\varepsilon, Dw_\varepsilon) e^{\zeta(w_\varepsilon)} > 0. \tag{8.54}$$

Suppose that $U_\delta \cap \Lambda_i = \emptyset$ for some $\delta > 0$; we have that $g(x, w_\varepsilon) = \widetilde{f}(w_\varepsilon)$ on U_δ, so that

$$\int_{U_\delta} \Big[-V(x) w_\varepsilon^{p-1} + \widetilde{f}(w_\varepsilon)\Big] \varphi_\delta > 0. \tag{8.55}$$

On the other hand, we note that by construction $\widetilde{f}(w_\varepsilon) \leq \frac{1}{k} V(x) w_\varepsilon^{p-1}$ with strict inequality on an open subset of U_δ. We deduce that (8.55) cannot hold, and so $U_\delta \cap \Lambda_i \neq \emptyset$ for all δ. Since Λ_i is compact, we conclude that w_ε admits a maximum point x_ε in Λ_i. Moreover, we have $w_\varepsilon(x_\varepsilon) \geq \ell$, where ℓ is as in (8.14), since otherwise (8.54) cannot hold.

Let us now consider the functions $v_\varepsilon(y) := w_\varepsilon(x_\varepsilon + \varepsilon y)$ and let $\varepsilon_j \to 0$. We have that, up to a subsequence, $x_{\varepsilon_j} \to \bar{x} \in \Lambda_i$. Since w_ε is a Mountain-Pass critical point of $J_{\varepsilon,i}$, arguing as in Step I of Lemma 8.5 there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} \Big(\varepsilon^p |Dw_\varepsilon|^p + V(x)|w_\varepsilon|^p\Big) \leq C d_{\varepsilon,i},$$

which, by (8.53) implies, up to subsequences, $v_{\varepsilon_j} \rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}^N)$. We now prove that $v \neq 0$. Let us set

$$d_j(y) := \begin{cases} V(x_{\varepsilon_j} + \varepsilon_j y) - \frac{g(x_{\varepsilon_j} + \varepsilon_j y, v_{\varepsilon_j}(y))}{v_{\varepsilon_j}^{p-1}(y)} & \text{if } v_{\varepsilon_j}(y) \neq 0 \\ 0 & \text{if } v_{\varepsilon_j}(y) = 0, \end{cases}$$
$$A(y, s, \xi) := j_\xi(x_{\varepsilon_j} + \varepsilon_j y, s, \xi),$$
$$B(y, s, \xi) := d_j(y) s^{p-1},$$
$$C(y, s) := j_s(x_{\varepsilon_j} + \varepsilon_j y, s, Dv_{\varepsilon_j}(y))$$

for every $y \in \mathbb{R}^N$, $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$. Taking into account the growth of condition on j_ξ, the strict convexity of j in ξ and condition (8.8), we get

$$A(y, s, \xi) \cdot \xi \geq \nu |\xi|^p, \quad |A(y, s, \xi)| \leq c_2 |\xi|^{p-1}, \quad |B(y, s, \xi)| \leq |d_j(y)||s|^{p-1}.$$

Moreover, by condition (8.10) we have

$$s \geq R \ \Rightarrow \ C(y, s) \geq 0$$

for every $y \in \mathbb{R}^N$ and $s \in \mathbb{R}^+$. By the growth of conditions on g, we have that for δ sufficiently small $d_j \in L^{\frac{N}{p-\delta}}(B_{2\varrho})$ for every $\varrho > 0$ and

$$S = \sup_j \|d_j\|_{L^{\frac{N}{p-\delta}}(B_{2\varrho})} \le D\big(1 + \sup_{j\in\mathbb{N}} \|v_{\varepsilon_j}\|_{L^{p^*}(B_{2\varrho})}\big) < +\infty$$

for some $D = D_\varrho > 0$. Since we have $\operatorname{div}(A(y, v_{\varepsilon_j}, Dv_{\varepsilon_j})) = B(y, v_{\varepsilon_j}, Dv_{\varepsilon_j}) + C(y, v_{\varepsilon_j})$ for every $j \in \mathbb{N}$, by virtue of [123, Theorem 1 and Remark at p.261] there exists a radius $\varrho > 0$ and a positive constant $M = M(\nu, c_2, S\varrho^\delta)$ such that

$$\sup_{j\in\mathbb{N}} \max_{y\in B_\varrho} |v_{\varepsilon_j}(y)| \le M(2\varrho)^{-N/p} \sup_{j\in\mathbb{N}} \|v_{\varepsilon_j}\|_{L^p(B_{2\varrho})} < +\infty$$

so that (v_{ε_j}) is uniformly bounded in B_ϱ. Then, by [123, Theorem 8], up to a subsequence (v_{ε_j}) converges uniformly to v in a small neighborhood of zero. This yields $v(0) = \lim_j v_{\varepsilon_j}(0) = \lim_j w_{\varepsilon_j}(x_{\varepsilon_j}) \ge \ell$.

Up to a rotations and translation, it is easily seen that v is a positive solution of

$$-\operatorname{div}(j_\xi(\bar{x}, v, Dv)) + j_s(\bar{x}, v, Dv) + V(\bar{x})v^{p-1} = 1_{\{x_1<0\}} f(v) + 1_{\{x_1>0\}} \widetilde{f}(v).$$

By Lemma 8.8 it follows that v is actually a nontrivial solution of

$$-\operatorname{div}(j_\xi(\bar{x}, v, Dv)) + j_s(\bar{x}, v, Dv) + V(\bar{x})v^{p-1} = f(v).$$

Then, by Theorem 8.7 and (8.23), we have $I_{\bar{x}}(v) = c_{\bar{x}} \ge c_i$. To conclude the proof, it is sufficient to prove that

$$\liminf_j \varepsilon_j^{-N} d_{\varepsilon_j,i} = \liminf_j \varepsilon_j^{-N} J_{\varepsilon_j,i}(w_{\varepsilon_j}) \ge I_{\bar{x}}(v). \tag{8.56}$$

Step II. We prove (8.56). It results

$$\begin{aligned} \varepsilon_j^{-N} J_{\varepsilon_j,i}(w_{\varepsilon_j}) = &\int_{\widehat{\Lambda}_{\varepsilon_j,i}} j(x_{\varepsilon_j} + \varepsilon_j y, v_{\varepsilon_j}, Dv_{\varepsilon_j}) \\ &+ \frac{1}{p}\int_{\widehat{\Lambda}_{\varepsilon_j,i}} V(x_{\varepsilon_j} + \varepsilon_j y)v_{\varepsilon_j}^p - \int_{\widehat{\Lambda}_{\varepsilon_j,i}} G(x_{\varepsilon_j} + \varepsilon_j y, v_{\varepsilon_j}) \end{aligned}$$

where $\widehat{\Lambda}_{\varepsilon_j,i} = \{y \in \mathbb{R}^N : x_{\varepsilon_j} + \varepsilon_j y \in \widehat{\Lambda}_i\}$. By Lemma 8.4, we have $v_{\varepsilon_j} \to v$ strongly in $W^{1,p}_{\rm loc}(\mathbb{R}^N)$. Following the same computations of Theorem 8.7, Step II, we deduce that for all $\delta > 0$ there exists $\bar{\varrho} > 0$ such that for all $\varrho > \bar{\varrho}$ we have

$$\liminf_j \int_{\widehat{\Lambda}_{\varepsilon_j,i}\setminus B_\varrho} \Big[j(x_{\varepsilon_j}+\varepsilon_j y, v_{\varepsilon_j}, Dv_{\varepsilon_j}) + \frac{1}{p}V(x_{\varepsilon_j}+\varepsilon_j y)v_{\varepsilon_j}^p - G(x_{\varepsilon_j}+\varepsilon_j y, v_{\varepsilon_j})\Big] \ge -\delta.$$

Furthermore,

$$\begin{aligned} &\lim_j \int_{B_\varrho} \big[j(x_{\varepsilon_j} + \varepsilon_j y, v_{\varepsilon_j}, Dv_{\varepsilon_j}) + \frac{1}{p}V(x_{\varepsilon_j} + \varepsilon_j y)v_{\varepsilon_j}^p - G(x_{\varepsilon_j} + \varepsilon_j y, v_{\varepsilon_j})\big] \\ &= I_{\bar{x}}(v, B_\varrho), \end{aligned}$$

where

$$I_{\bar{x}}(v, B_\varrho) := \int_{B_\varrho} \big[j(\bar{x}, v, Dv) + \frac{1}{p}V(\bar{x})v^p - F(v)\big].$$

We conclude that for all $\varrho > \bar{\varrho}$

$$\liminf_j \varepsilon_j^{-N} J_{\varepsilon_j,i}(w_{\varepsilon_j}) \ge I_{\bar{x}}(v, B_\varrho) - \delta,$$

and (8.56) follows letting $\varrho \to +\infty$ and $\delta \to 0$. □

Let us now consider the class

$$\Gamma_\varepsilon := \big\{\gamma \in C([0,1]^k, W_V(\Omega)) : \gamma \text{ satisfies conditions } (a), (b), (c), (d)\big\},$$

where:

(a) $\gamma(t) = \sum_{i=1}^k \gamma_i(t_i)$ for every $t \in \partial[0,1]^k$, with $\gamma_i \in C([0,1], W_V(\Omega))$;
(b) $\mathrm{supt}(\gamma_i(t_i)) \subset \Lambda_i$ for every $t_i \in [0,1]$ and $i = 1,\dots,k$;
(c) $\gamma_i(0) = 0$ and $J_\varepsilon(\gamma_i(1)) < 0$ for every $i = 1,\dots,k$;
(d) $\varepsilon^{-N} E_\varepsilon(\gamma(t)) \le \sum_{i=1}^k c_i + \sigma$ for every $t \in \partial[0,1]^k$,

where $0 < \sigma < \frac{1}{2}\min\{c_i : i = 1,\dots,k\}$. We set

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \sup_{t\in[0,1]^k} E_\varepsilon(\gamma(t)). \tag{8.57}$$

Lemma 8.10. *For ε small enough $\Gamma_\varepsilon \ne \emptyset$ and*

$$\lim_{\varepsilon\to0^+} \varepsilon^{-N} c_\varepsilon = \sum_{i=1}^k c_i. \tag{8.58}$$

Proof. Firstly, let us prove that for ε small $\Gamma_\varepsilon \ne \emptyset$ and

$$c_\varepsilon \le \varepsilon^N \sum_{i=1}^k c_i + o(\varepsilon^N). \tag{8.59}$$

By definition of c_i, for all $\delta > 0$ there exists $\gamma_i \in \mathcal{P}_i$ with

$$c_i \le \max_{\tau\in[0,1]} I_{x_i}(\gamma_i(\tau)) \le c_i + \frac{\delta}{2k} \tag{8.60}$$

where the x_is are as in (8.2) and

$$\mathcal{P}_i := \big\{\gamma_i \in C([0,1], W^{1,p}(\mathbb{R}^N)) : \gamma_i(0) = 0,\ I_{x_i}(\gamma_i(1)) < 0\big\}.$$

We choose δ so that $\delta < \min\{\sigma, k\sigma_i\}$. Let us set

$$\widehat{\gamma}_i(\tau)(x) = \eta_i(x)\gamma_i(\tau)\big(\frac{x - x_i}{\varepsilon}\big) \quad \text{for every } \tau \in [0,1] \text{ and } x \in \Omega,$$

where $\eta_i \in C_c^\infty(\mathbb{R}^N)$, $0 \le \eta_i \le 1$, $\mathrm{supp}\,\eta_i \subset \Lambda_i$, and $x_i \in \mathrm{int}(\{\eta_i = 1\})$. We have

$$J_\varepsilon(\widehat{\gamma}_i(\tau)) = \int_\Omega \varepsilon^p j(x, \widehat{\gamma}_i(\tau), D\widehat{\gamma}_i(\tau)) + \frac{1}{p}\int_\Omega V(x)|\widehat{\gamma}_i(\tau)|^p - \int_\Omega G(x, \widehat{\gamma}_i(\tau)). \tag{8.61}$$

Since it results

$$D\widehat{\gamma}_i(\tau) = D\eta_i(x)\gamma_i(\tau)\big(\frac{x - x_i}{\varepsilon}\big) + \frac{1}{\varepsilon}\eta_i(x)D\gamma_i(\tau)\big(\frac{x - x_i}{\varepsilon}\big),$$

and for all $\xi_1, \xi_2 \in \mathbb{R}^N$ there exists $t \in [0,1]$ with

$$j(x, s, \xi_1 + \xi_2) = j(x, s, \xi_2) + j_\xi(x, s, t\xi_1 + \xi_2)\cdot\xi_1,$$

taking into account the p-homogeneity of j, the term

$$\varepsilon^p \int_\Omega j(x, \widehat{\gamma}_i(\tau), D\widehat{\gamma}_i(\tau))$$

has the same behavior of

$$\int_\Omega j\Big(x, \eta_i(x)\gamma_i(\tau)\big(\frac{x - x_i}{\varepsilon}\big), \eta_i(x)D\gamma_i\big(\frac{x - x_i}{\varepsilon}\big)\Big) \tag{8.62}$$

up to an error given by

$$\varepsilon^p \int_\Omega j_\xi(x, s(x), t(x)\xi_1(x) + \xi_2(x)) \cdot \xi_1(x), \tag{8.63}$$

where we have set

$$\begin{aligned} s(x) &:= \widehat{\gamma}_i(\tau)(x), \\ \xi_1(x) &:= D\eta_i(x)\gamma_i(\tau)\big(\frac{x - x_i}{\varepsilon}\big), \\ \xi_2(x) &:= \frac{1}{\varepsilon}\eta_i(x)D\gamma_i(\tau)\big(\frac{x - x_i}{\varepsilon}\big), \end{aligned}$$

and $t(x)$ is a function with $0 \le t(x) \le 1$ for every $x \in \Omega$. We proceed in the estimation of (8.63). We obtain

$$\begin{aligned} &\varepsilon^p \Big| \int_\Omega j_\xi(x, s(x), t(x)\xi_1(x) + \xi_2(x)) \cdot \xi_1(x) \Big| \\ &\le \widetilde{c}_2 \varepsilon^p \int_\Omega |\xi_1(x)|^p + \widetilde{c}_2 \varepsilon^p \int_\Omega |\xi_2(x)|^{p-1} |\xi_1(x)|. \end{aligned}$$

Making the change of variable $y = \frac{x - x_i}{\varepsilon}$, we obtain

$$\begin{aligned} &\varepsilon^p \Big| \int_\Omega j_\xi(x, s(x), t(x)\xi_1(x) + \xi_2(x)) \cdot \xi_1(x) \Big| \\ &\le \widetilde{c}_2 \varepsilon^{p+N} \int_{\mathbb{R}^N} |D\eta_i(x_i + \varepsilon y)|^p |\gamma_i(\tau)(y)|^p \\ &\quad + \widetilde{c}_2 \varepsilon^{N+1} \int_{\mathbb{R}^N} |\eta_i(x_i + \varepsilon y)|^{p-1} |D\gamma_i(\tau)(y)|^{p-1} |D\eta_i(x_i + \varepsilon y)| |\gamma_i(\tau)(y)| \\ &= o(\varepsilon^N) \end{aligned}$$

where $o(\varepsilon^N)$ is independent of τ, since γ_i has compact values in $W^{1,p}(\mathbb{R}^N)$. Changing the variable also in (8.62) yields

$$\begin{aligned} &\int_\Omega j\, \Big(x, \eta_i(x)\gamma_i(\tau)\big(\frac{x - x_i}{\varepsilon}\big), \eta_i(x) D\gamma_i(\tau)\big(\frac{x - x_i}{\varepsilon}\big)\Big) \\ &= \varepsilon^N \int_{\mathbb{R}^N} j(x_i + \varepsilon y, \eta_i(x_i + \varepsilon y)\gamma_i(\tau)(y), \eta_i(x_i + \varepsilon y) D\gamma_i(\tau)(y)). \end{aligned}$$

By the Dominated Convergence Theorem we get

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} j(x_i + \varepsilon y, \eta_i(x_i + \varepsilon y)\gamma_i(\tau)(y), \eta_i(x_i + \varepsilon y) D\gamma_i(\tau)(y)) \\ &= \int_{\mathbb{R}^N} j(x_i, \gamma_i(\tau)(y), D\gamma_i(\tau)(y)) \end{aligned}$$

uniformly with respect to τ. Reasoning in a similar fashion for the other terms in (8.61), we conclude that for ε small enough

$$J_\varepsilon(\widehat{\gamma}_i(\tau)) = \varepsilon^N I_{x_i}(\gamma_i(\tau)) + o(\varepsilon^N) \tag{8.64}$$

for every $\tau \in [0, 1]$ with $o(\varepsilon^N)$ independent of τ. Let us now set

$$\gamma_0(\tau_1, \dots, \tau_k) := \sum_{i=1}^k \widehat{\gamma}_i(\tau_i).$$

Since $\operatorname{supp}\widehat{\gamma}_i(\tau)\subset\Lambda_i$ for every τ, we have that $J_{\varepsilon,i}(\widehat{\gamma}_i(\tau))=J_\varepsilon(\widehat{\gamma}_i(\tau))$; then, by the choice of δ, we get for ε small

$$\begin{aligned}[J_{\varepsilon,i}(\widehat{\gamma}_i(\tau))_+]^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}&=[J_\varepsilon(\widehat{\gamma}_i(\tau))_+]^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\\&=\varepsilon^{\frac N2}[I_{x_i}(\gamma_i(\tau))+o(1)]^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\\&\le\varepsilon^{\frac N2}\Big[c_i+\frac{\delta}{2k}+o(1)\Big]^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\le 0,\end{aligned}$$

and

$$E_\varepsilon(\gamma_0(\tau_1,\dots,\tau_k))=J_\varepsilon(\gamma_0(\tau_1,\dots,\tau_k))=\sum_{i=1}^k J_\varepsilon(\widehat{\gamma}_i(\tau_i)).$$

By (8.60) and (8.64) we obtain that for ε small enough

$$E_\varepsilon(\gamma_0(\tau))\le\varepsilon^N\sum_{i=1}^k\big(c_i+\frac{\delta}{2k}\big)\le\varepsilon^N\Big(\sum_{i=1}^k c_i+\sigma\Big)$$

so that the class Γ_ε is not empty. Moreover, we have

$$\limsup_{\varepsilon\to0^+}\frac{c_\varepsilon}{\varepsilon^N}\le\sum_{i=1}^k c_i+\delta$$

and, by the arbitrariness of δ, we have conclude that (8.59) holds. Let us now prove that

$$c_\varepsilon\ge\varepsilon^N\sum_{i=1}^k c_i+o(\varepsilon^N). \tag{8.65}$$

Given $\gamma\in\Gamma_\varepsilon$, by a variant of [51, Proposition 3.4] there exists $\bar t\in[0,1]^k$ such that

$$J_{\varepsilon,i}(\gamma(\bar t))\ge d_{\varepsilon,i}$$

for all $i=1,\dots,k$, where the $d_{\varepsilon,i}$s are as in (8.52). Then we have by Lemma 8.9

$$\sup_{t\in[0,1]^k}J_\varepsilon(\gamma(t))\ge\sup_{t\in[0,1]^k}\sum_{i=1}^k J_{\varepsilon,i}(\gamma(t))\ge\sum_{i=1}^k d_{\varepsilon,i}=\varepsilon^N\sum_{i=1}^k c_i+o(\varepsilon^N),$$

which implies the assertion. □

Corollary 8.11. *For every $\varepsilon>0$ there exists a critical point $u_\varepsilon\in W_V(\Omega)$ of the functional E_ε such that $c_\varepsilon=E_\varepsilon(u_\varepsilon)$. Moreover $\|u_\varepsilon\|_{W_V}\to0$ as $\varepsilon\to0$.*

Proof. By Lemma 8.5 it results that E_ε satisfies the Palais-Smale condition for every $c\in\mathbb{R}$ (see Definition 2.15). Then, by Lemma 8.10, for every $\varepsilon>0$ the (non-smooth) Mountain-Pass Theorem (see [50]) for the class Γ_ε provides the desired critical point u_ε of E_ε. To prove the second assertion we may argue as in Step I of Lemma 8.5 with u_h replaced by u_ε and c replaced by $E_\varepsilon(u_\varepsilon)$. Thus, from inequality (8.34), for every $\varepsilon>0$ we get

$$\int_\Omega\Big(|Du_\varepsilon|^p+V(x)|u_\varepsilon|^p\Big)\le\frac{\vartheta E_\varepsilon(u_\varepsilon)+d\varepsilon^N}{\min\Big\{(\vartheta-\gamma-p)\,\nu\varepsilon^p,\ \frac{\vartheta}{p}-\frac{\vartheta}{2\kappa}-1\Big\}}. \tag{8.66}$$

By virtue of Lemma 8.10, this yields

$$\int_\Omega\Big(|Du_\varepsilon|^p+V(x)|u_\varepsilon|^p\Big)\le\Big\{\frac{\vartheta(c_1+\dots+c_k)+d}{(\vartheta-\gamma-p)\nu}\Big\}\varepsilon^{N-p}+o(\varepsilon^{N-p}),$$

as $\varepsilon\to0$, which implies the assertion. □

Let us now set

$$\Omega_\varepsilon := \{y \in \mathbb{R}^N : \varepsilon y \in \Omega\}, \quad v_\varepsilon(y) := u_\varepsilon(\varepsilon y) \in W^{1,p}(\Omega_\varepsilon),$$
$$\widehat{\Lambda}_{\varepsilon,i} := \{y \in \mathbb{R}^N : \varepsilon y \in \widehat{\Lambda}_i\}, \quad \Lambda_\varepsilon := \{y \in \mathbb{R}^N : \varepsilon y \in \Lambda\}.$$

Lemma 8.12. *The function v_ε is a solution of the equation*

$$\begin{aligned} &-\operatorname{div}\big((1+\theta_\varepsilon(\varepsilon y))j_\xi(\varepsilon y, v, Dv)\big) + (1+\theta_\varepsilon(\varepsilon y))j_s(\varepsilon y, v, Dv)\\ &+ (1+\theta_\varepsilon(\varepsilon y))V(\varepsilon y)v^{p-1}\\ &= (1+\theta_\varepsilon(\varepsilon y))g(\varepsilon y, v) \quad \text{in } \Omega_\varepsilon, \end{aligned} \tag{8.67}$$

where for every $\varepsilon > 0$

$$\begin{gathered} \theta_\varepsilon(x) := \sum_{i=1}^k \theta_{\varepsilon,i} 1_{\widehat{\Lambda}_i}(x), \quad \theta_{\varepsilon,i} \in [0, M],\\ \theta_{\varepsilon,i} := M\big[(J_{\varepsilon,i}(u_\varepsilon)_+)^{1/2} - \varepsilon^{N/2}(c_i + \sigma_i)^{1/2}\big]_+ (J_{\varepsilon,i}(u_\varepsilon)_+)^{-1/2}. \end{gathered} \tag{8.68}$$

Proof. It suffices to expand $E'_\varepsilon(u_\varepsilon)(\varphi) = 0$ for every $\varphi \in C_c^\infty(\Omega)$. □

Corollary 8.13. *The sequence (v_ε) is bounded in $W^{1,p}(\mathbb{R}^N)$.*

Proof. It suffices to combine Lemma 8.10 with the inequality

$$\int_{\mathbb{R}^N} \Big(|Dv_\varepsilon|^p + V(x)|v_\varepsilon|^p\Big) \le \frac{\vartheta\varepsilon^{-N}c_\varepsilon + d}{\min\big\{(\vartheta - \gamma - p)\,\nu, \frac{\vartheta}{p} - \frac{\vartheta}{2\kappa} - 1\big\}}$$

which follows by (8.66). □

The following lemma "kills" the second penalization term of E_ε.

Lemma 8.14. *For $i = 1, \dots, k$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} J_{\varepsilon,i}(u_\varepsilon) = c_i\,. \tag{8.69}$$

Proof. Let us first prove that, as $\varrho \to +\infty$,

$$\limsup_{\varepsilon\to 0^+} \int_{\Omega_\varepsilon\setminus\mathcal{N}_\varrho(\Lambda_\varepsilon)} \Big(|Dv_\varepsilon|^p + |v_\varepsilon|^p\Big) = o(1), \tag{8.70}$$

where $\mathcal{N}_\varrho(\Lambda_\varepsilon) := \{y \in \mathbb{R}^N : \operatorname{dist}(y, \Lambda_\varepsilon) < \varrho\}$. We can test equation (8.67) with $\psi_{\varepsilon,\varrho}v_\varepsilon e^{\zeta(v_\varepsilon)}$, where $\psi_{\varepsilon,\varrho} = 1 - \sum_{i=1}^k \psi^i_{\varepsilon,\varrho}$, $\psi^i_{\varepsilon,\varrho} \in C^\infty(\mathbb{R}^N)$,

$$\psi^i_{\varepsilon,\varrho} = \begin{cases} 1 & \text{if } \operatorname{dist}(y, \Lambda_{\varepsilon,i}) < \varrho/2,\\ 0 & \text{if } \operatorname{dist}(y, \Lambda_{\varepsilon,i}) > \varrho \end{cases}$$

and the function ζ is defined as in (8.37). By virtue of (8.1), (8.7), the boundedness of (v_ε) in $W^{1,p}(\mathbb{R}^N)$ and (8.39) there exist $C, C' > 0$ such that

$$\begin{aligned} &C\int_{\Omega_\varepsilon\setminus\mathcal{N}_\varrho(\Lambda_\varepsilon)} \Big(|Dv_\varepsilon|^p + |v_\varepsilon|^p\Big)\\ &\le \int_{\Omega_\varepsilon\setminus\Lambda_\varepsilon} (1+\theta_\varepsilon(\varepsilon y))\Big[pj(\varepsilon y, v_\varepsilon, Dv_\varepsilon) + \Big\{V(\varepsilon y) - \frac{\widetilde{f}(v_\varepsilon)}{v_\varepsilon^{p-1}}\Big\}v_\varepsilon^p\Big]\psi_{\varepsilon,\varrho}e^{\zeta(v_\varepsilon)}\\ &= -\int_{\Omega_\varepsilon\setminus\Lambda_\varepsilon} (1+\theta_\varepsilon(\varepsilon y))\big[j_s(\varepsilon y, v_\varepsilon, Dv_\varepsilon) + p\zeta'(v_\varepsilon)j(\varepsilon y, v_\varepsilon, Dv_\varepsilon)\big]v_\varepsilon\psi_{\varepsilon,\varrho}e^{\zeta(v_\varepsilon)} \end{aligned}$$

$$
\begin{aligned}
&-\int_{\Omega_\varepsilon\setminus\Lambda_\varepsilon}(1+\theta_\varepsilon(\varepsilon y))j_\xi(\varepsilon y,v_\varepsilon,Dv_\varepsilon)\cdot D\psi_{\varepsilon,\varrho}v_\varepsilon e^{\zeta(v_\varepsilon)}\\
&\leq 2e^{\bar M R}\int_{\Omega_\varepsilon\setminus\Lambda_\varepsilon}|D\psi_{\varepsilon,\varrho}||j_\xi(\varepsilon y,v_\varepsilon,Dv_\varepsilon)|v_\varepsilon\leq\frac{\widetilde C}{\varrho}\|Dv_\varepsilon\|_p^{p-1}\|v_\varepsilon\|_p\leq\frac{C'}{\varrho},
\end{aligned}
$$

which implies (8.70). Now, to prove (8.69), we adapt the argument of [63, Lemma 2.1] to our context. It is sufficient to prove that

$$\lim_{\varepsilon\to 0}\varepsilon^{-N}J_{\varepsilon,i}(u_\varepsilon)\leq c_i+\sigma_i \tag{8.71}$$

for every $i=1,\dots,k$. Then (8.69) follows by arguing exactly as in [63, Lemma 2.4]. By contradiction, let us suppose that for some $\varepsilon_j\to 0$ we have

$$\limsup_j \varepsilon_j^{-N}J_{\varepsilon_j,i}(u_{\varepsilon_j})>c_i+\sigma_i. \tag{8.72}$$

Then there exists $\lambda>0$ with

$$\int_{\widehat\Lambda_{\varepsilon_j,i}}\Big(|Dv_{\varepsilon_j}|^p+|v_{\varepsilon_j}|^p\Big)\geq\lambda,$$

and so by (8.70) there exists $\varrho>0$ such that for j large enough

$$\int_{\mathcal N_\varrho(\Lambda_{\varepsilon_j,i})}\Big(|Dv_{\varepsilon_j}|^p+|v_{\varepsilon_j}|^p\Big)\geq\frac{\lambda}{2}.$$

Following [63, Lemma 2.1], P.L. Lions' concentration compactness argument [98] yields the existence of $S>0$, $\rho>0$ and a sequence $y_j\in\Lambda_{\varepsilon_j,i}$ such that for j large enough

$$\int_{B_S(y_j)}v_{\varepsilon_j}^p\geq\rho. \tag{8.73}$$

Let us set $v_j(y):=v_{\varepsilon_j}(y_j+y)$, and let $\varepsilon_j y_j\to\bar x\in\Lambda_i$. By Corollary 8.13, we may assume that v_j weakly converges to some v in $W^{1,p}(\mathbb R^N)$. By Lemma 8.4, we have that $v_j\to v$ strongly in $W^{1,p}_{\rm loc}(\mathbb R^N)$; note that $v\neq 0$ by (8.73). In the case $\operatorname{dist}(y_j,\partial\Lambda_{\varepsilon_j,i})\to+\infty$, since v_j satisfies in $-y_j+\Lambda_{\varepsilon_j,i}$ the equation

$$-\operatorname{div}(j_\xi(\varepsilon_j y_j+\varepsilon_j y,v_j,Dv_j))+j_s(\varepsilon_j y_j+\varepsilon_j y,v_j,Dv_j)+V(\varepsilon_j y_j+\varepsilon_j y)v^{p-1}=f(v_j),$$

v satisfies on $\mathbb R^N$ the equation

$$-\operatorname{div}(j_\xi(\bar x,v,Dv))+j_s(\bar x,v,Dv)+V(\bar x)v^{p-1}=f(v). \tag{8.74}$$

If $\operatorname{dist}(y_j,\partial\Lambda_{\varepsilon_j,i})\leq C<+\infty$, we deduce that v satisfies an equation of the form (8.47), and by Lemma 8.8, we conclude that v satisfies equation (8.74). Since v is a nontrivial critical point for $I_{\bar x}$, by (8.11) and Theorem 8.7, recalling that $c_i\leq c_{\bar x}\leq c_i+\sigma_i$, we get $c_i\leq I_{\bar x}(v)\leq c_i+\sigma_i$. Then we can find a sequence $R_j\to+\infty$ such that

$$\lim_j\int_{B_{R_j}(y_j)}j(\varepsilon_j y,v_{\varepsilon_j},Dv_{\varepsilon_j})+\frac1p V(\varepsilon_j y)|v_{\varepsilon_j}|^p-G(\varepsilon_j y,v_{\varepsilon_j})=I_{\bar x}(v)\leq c_i+\sigma_i.$$

Then by (8.72) we deduce that for j large enough

$$\int_{\widehat\Lambda_{\varepsilon_j,i}\setminus B_{R_j}(y_j)}\Big(|Dv_{\varepsilon_j}|^p+|v_{\varepsilon_j}|^p\Big)\geq\lambda>0.$$

Reasoning as above, there exist $\widetilde{S}, \widetilde{\rho} > 0$ and a sequence $\widetilde{y}_j \in \Lambda_{\varepsilon_j,i} \setminus B_{R_j}(y_j)$ such that

$$\int_{B_{\widetilde{S}}(\widetilde{y}_j)} v_{\varepsilon_j}^p \geq \widetilde{\rho} > 0. \tag{8.75}$$

Let $\varepsilon_j \widetilde{y}_j \to \widetilde{x} \in \Lambda_i$; then we have $\widetilde{v}_j(y) := v_{\varepsilon_j}(\widetilde{y}_j + y) \to \widetilde{v}$ weakly in $W^{1,p}(\mathbb{R}^N)$, where $\widetilde{v}$ is a nontrivial solution of the equation

$$-\operatorname{div}(j_\xi(\widetilde{x}, v, Dv)) + j_s(\widetilde{x}, v, Dv) + V(\widetilde{x})v^{p-1} = f(v).$$

As before we get $I_{\widetilde{x}}(\widetilde{v}) \geq c_i$. We are now in a position to deduce that

$$\liminf_j \varepsilon_j^{-N} J_{\varepsilon_j,i}(u_\varepsilon) > I_{\bar{x}}(v) + I_{\widetilde{x}}(\widetilde{v}) \geq 2c_i.$$

In fact, v_{ε_j} satisfies in $\widehat{\Lambda}_{\varepsilon_j,i}$ the equation

$$-\operatorname{div}(j_\xi(\varepsilon_j y, v_{\varepsilon_j}, Dv_{\varepsilon_j})) + j_s(\varepsilon_j y, v_{\varepsilon_j}, Dv_{\varepsilon_j}) + V(\varepsilon_j y)v_{\varepsilon_j}^{p-1} = g(\varepsilon_j y, v_{\varepsilon_j}). \tag{8.76}$$

Since $y_j, \widetilde{y}_j \in \Lambda_{\varepsilon_j,i}$, for j large enough $B_{j,R} := B(y_j, R) \cup B(\widetilde{y}_j, R) \subset \widehat{\Lambda}_{\varepsilon_j,i}$, and so we can test (8.76) with

$$\varphi(y) = \Big[\psi\Big(\frac{|y - y_j|}{R}\Big) + \psi\Big(\frac{|y - \widetilde{y}_j|}{R}\Big) - 1\Big] v_{\varepsilon_j}(y)$$

where $\psi \in C^\infty(\mathbb{R})$ with $0 \leq \psi \leq 1$, $\psi(s) = 0$ for $s \leq 1$ and $\psi(s) = 1$ for $s \geq 2$. Reasoning as in Lemma 8.9, we have that for all $\delta > 0$ there exists $\bar{R}$ such that for all $R > \bar{R}$ we have

$$\int_{\widehat{\Lambda}_{\varepsilon_j,i} \setminus B_{j,R}} \Big[j(\varepsilon_j y, v_{\varepsilon_j}, Dv_{\varepsilon_j}) + \frac{1}{p} V(\varepsilon_j y)|v_{\varepsilon_j}|^p - G(\varepsilon_j y, v_{\varepsilon_j})\Big] \geq -\delta$$

so that

$$\liminf_j \varepsilon_j^{-N} J_{\varepsilon_j,i}(u_{\varepsilon_j}) \geq I_{\bar{x}}(v, B_R) + I_{\widetilde{x}}(\widetilde{v}, B_R) - \delta.$$

Letting $R \to +\infty$ and $\delta \to 0$, we get

$$\liminf_j \varepsilon_j^{-N} J_{\varepsilon_j,i}(u_{\varepsilon_j}) > 2c_i. \tag{8.77}$$

The same arguments apply to the functional J_ε: we have that

$$\liminf_j \varepsilon_j^{-N} J_{\varepsilon_j}(u_{\varepsilon_j}) \geq 2c_i. \tag{8.78}$$

Then by combining (8.77) and (8.78) we obtain

$$\liminf_j \varepsilon_j^{-N} E_{\varepsilon_j}(u_{\varepsilon_j}) \geq 2c_i + M\Big[(2c_i)^{1/2} - (c_i + \sigma_i)^{1/2}\Big]_+^2.$$

By Lemma 8.10, we have

$$M\Big[(2c_i)^{1/2} - (c_i + \sigma_i)^{1/2}\Big]_+^2 \leq \sum_{i=1}^k c_i,$$

against the choice of M. ☐

8.4. **Proofs of the main results.** We are now ready to prove the main results of the section.

Proof of Theorem 8.2. Let us consider the sequence (u_ε) of critical points of E_ε given by Corollary 8.11. We have that $\|u_\varepsilon\|_{W_V} \to 0$. Since u_ε satisfies

$$\begin{aligned}
&-\operatorname{div}\big((1+\theta_\varepsilon(x))j_\xi(x,v,Dv)\big) + (1+\theta_\varepsilon(x))j_s(x,v,Dv) + (1+\theta_\varepsilon(x))V(x)v^{p-1}\\
&= (1+\theta_\varepsilon(x))g(x,v) \quad \text{in } \Omega,
\end{aligned}$$

with θ_ε defined as in (8.68), by the regularity results of [91] u_ε is locally Hölder continuous in Ω. We claim that there exists $\sigma > 0$ such that

$$u_\varepsilon(x_{\varepsilon,i}) = \sup_{\Lambda_i} u_\varepsilon > \sigma > 0 \tag{8.79}$$

for every ε sufficiently small and $i = 1,\dots,k$: moreover

$$\lim_{\varepsilon\to 0} \operatorname{dist}(x_{\varepsilon,i}, \mathcal{M}_i) = 0 \tag{8.80}$$

for $i = 1,\dots,k$, where the $\mathcal{M}_i$s are the sets of minima of V in Λ_i. In fact, let us assume that there exists $i_0 \in \{1,\dots,k\}$ such that $u_\varepsilon(x_{\varepsilon,i_0}) \to 0$ as $\varepsilon \to 0$. Therefore, $u_\varepsilon \to 0$ uniformly on Λ_{i_0} as $\varepsilon \to 0$, which implies that

$$\sup_{y\in\Lambda_{\varepsilon,i_0}} v_\varepsilon(y) \to 0 \quad \text{as } \varepsilon \to 0, \tag{8.81}$$

where $v_\varepsilon(y) := u_\varepsilon(\varepsilon y)$. On the other hand, since by (8.69) we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} J_{\varepsilon,i_0}(u_\varepsilon) = c_{i_0} > 0,$$

considering $\widetilde{\Lambda}_{i_0}$ relatively compact in Λ_{i_0}, following the proof of Lemma 8.14, we find $S > 0$ and $\varrho > 0$ such that

$$\sup_{y\in\widetilde{\Lambda}_{\varepsilon,i_0}} \int_{B_S(y)} v_\varepsilon^p \ge \varrho$$

for every $\varepsilon \in (0,\varepsilon_0)$, which contradicts (8.81). We conclude that (8.79) holds. To prove (8.80), it is sufficient to prove that

$$\lim_{\varepsilon\to 0} V(x_{\varepsilon,i}) = \min_{\Lambda_i} V$$

for every $i = 1,\dots,k$. Assume by contradiction that for some i_0

$$\lim_{\varepsilon\to 0} V(x_{\varepsilon,i_0}) > \min_{\Lambda_{i_0}} V = b_{i_0}.$$

Then, up to a subsequence, $x_{\varepsilon_j,i_0} \to x_{i_0} \in \Lambda_{i_0}$ and $V(x_{i_0}) > b_{i_0}$. Then, arguing as in the proof of Lemma 8.14 and using Theorem 8.7, we would get

$$\liminf_j \varepsilon_j^{-N} J_{\varepsilon_j,i_0}(u_{\varepsilon_j}) \ge I_{x_{i_0}}(v) = c_{x_{i_0}} > c_{i_0}$$

which is impossible, in view of (8.69).

We now prove that

$$\lim_{\varepsilon\to 0} u_\varepsilon = 0 \quad \text{uniformly on } \Omega \setminus \bigcup_{i=1}^k \operatorname{int}(\Lambda_i). \tag{8.82}$$

Let us first prove that

$$\lim_{\varepsilon\to 0} \sup_{\partial\Lambda_i} u_\varepsilon = 0 \quad \text{for } i = 1,\dots,k.$$

By contradiction, let $i_0 \in \{1,\dots,k\}$ and $\sigma > 0$ with $u_{\varepsilon_j}(x_j) \geq \sigma$ for $(x_j) \subset \partial\Lambda_{i_0}$. Up to a subsequence, $x_j \to x_0 \in \partial\Lambda_{i_0}$. Therefore, taking into account Lemma 8.8 and the local regularity estimates of [123] (see also the end of Step I of Lemma 8.9), the sequence $v_j(y) := u_{\varepsilon_j}(x_j + \varepsilon_j y)$ converges weakly to a nontrivial solution $v \in W^{1,p}(\mathbb{R}^N)$ of

$$-\operatorname{div}(j_\xi(x_0, v, Dv)) + j_s(x_0, v, Dv) + V(x_0)v^{p-1} = f(v) \quad \text{in } \mathbb{R}^N.$$

As $V(x_0) > V(x_{i_0})$, we have

$$\liminf_j \varepsilon_j^{-N} J_{\varepsilon_j, i_0}(u_{\varepsilon_j}) \geq I_{x_0}(v) > c_{i_0},$$

which violates (8.69). Testing the equation with

$$(u_\varepsilon - \max_i \sup_{\partial\Lambda_i} u_\varepsilon)^+ 1_{\Omega\setminus\Lambda} e^{\zeta(u_\varepsilon)},$$

as in Lemma 8.8, this yields that $u_\varepsilon(x) \leq \max_i \sup_{\partial\Lambda_i} u_\varepsilon$ for every $x \in \Omega \setminus \Lambda$, so that (8.82) holds.

By Proposition 8.3, u_ε is actually a solution of the original problem (8.13) because the penalization terms are neutralized by the facts $J_{\varepsilon,i}(u_\varepsilon) < c_i + \sigma_i$ and $u_\varepsilon < \ell$ on $\Omega \setminus \Lambda$ for ε small. By regularity results, it follows $u_\varepsilon \in C^{1,\beta}_{\rm loc}(\Omega)$, and so point (a) is proved. Taking into account (8.79) and (8.82), we get that u_ε has a maximum $\bar{x}_\varepsilon \in \Omega$ which coincides with one of the $x_{\varepsilon,i}$s. Considering $\bar{v}_\varepsilon(y) := u_\varepsilon(x_{\varepsilon,i} + \varepsilon y)$, since $\bar{v}_\varepsilon$ is uniformly bounded in $W^{1,p}_{\rm loc}(\mathbb{R}^N)$, by the local regularity estimates [123], there exists σ' with

$$u_\varepsilon(x_{\varepsilon,i}) \leq \sigma'$$

for all $i = 1,\dots,k$. In view of (8.79), (8.80) and Corollary 8.11, we conclude that points (b) and (d) are proved. Let us now come to point (c). Let us assume by contradiction that there exists $\bar{r}$, δ, i_0 and $\varepsilon_j \to 0$ such that there exists $y_j \in \Lambda_{i_0} \setminus B_{\bar{r}}(x_{\varepsilon_j, i_0})$ with

$$\limsup_j u_{\varepsilon_j}(y_j) \geq \delta.$$

We may assume that $y_j \to \bar{y}$, $x_{\varepsilon_j,i_0} \to \bar{x}$, and $\bar{v}_j(y) := u_{\varepsilon_j}(y_j + \varepsilon_j y) \to \bar{v}$, $v_j(y) := u_{\varepsilon_j}(x_{\varepsilon_j,i_0} + \varepsilon_j y) \to v$ strongly in $W^{1,p}_{\rm loc}(\mathbb{R}^N)$: then, arguing as in Lemma 8.14, it turns out that

$$\liminf_j \varepsilon_j^{-N} J_{\varepsilon_j,i_0}(u_{\varepsilon_j}) \geq I_{\bar{x}}(v) + I_{\bar{y}}(\bar{v}) \geq 2c_{i_0}$$

which is against (8.69). We conclude that point (c) holds, and the proof is concluded. □

Proof of Theorem 8.1. If $1 < p \leq 2$ and $p < q < p^*$, the equation

$$-\Delta_p u + V(\bar{x})u^{p-1} = u^{q-1} \quad \text{in } \mathbb{R}^N \tag{8.83}$$

admits a unique positive C^1 solution (up to translations). Indeed, a solution $u \in C^1(\mathbb{R}^N)$ of (8.83) exists by Theorem 8.7. By [94, Theorem 1] we have $u(x) \to 0$ as $|x| \to \infty$. Moreover, by [55, Theorem 1.1], the solution u is radially symmetric about some point $x_0 \in \mathbb{R}^N$ and radially decreasing. Then u is a radial ground state solution of (8.83). By [124, Theorem 1], u is unique (up to translations). Then (8.11) is satisfied and the assertions follow by Theorem 8.2 applied to the functions $j(x,s,\xi) = \frac{1}{p}|\xi|^p$ and $f(s) = s^{q-1}$. □

9. Nonexistence Results

Some parts in this section have been slightly modified to give this collection a more uniform appearance. We refer the reader to source in [59].

9.1. **A general Pucci-Serrin type identity.** In 1965 Pohozǎev discovered a very important identity for solutions of the problem

$$\begin{gathered}\Delta u + g(u) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

This variational identity enabled him to show that the above problem has no nontrivial solution provided that Ω is a bounded star-shaped domain of $\mathbb{R}^n$ and g satisfies

$$\forall s \in \mathbb{R}: \; s \neq 0 \Rightarrow (n-2)sg(s) - 2nG(s) > 0$$

where G is the primitive of g with $G(0) = 0$.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and outer normal ν. Assume that $\mathcal{L}$ is a function of class C^1 on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ with $\mathcal{L}(x, 0, 0) = 0$ and that the vector valued function

$$\nabla_\xi \mathcal{L}(x, s, \xi) = \Big(\frac{\partial \mathcal{L}}{\partial \xi_1}(x, s, \xi), \cdots, \frac{\partial \mathcal{L}}{\partial \xi_n}(x, s, \xi) \Big)$$

is of class C^1 in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Moreover, let $\mathcal{G}$ be a continuous function in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Consider the problem

$$\begin{gathered}-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x, u, \nabla u)\big) + D_s \mathcal{L}(x, u, \nabla u) = \mathcal{G}(x, u, \nabla u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{9.1}$$

Let us recall the celebrated identity proved by Pucci and Serrin [117].

Theorem 9.1. *Assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of (9.1). Then*

$$\begin{aligned}&\int_{\partial\Omega} \big[\mathcal{L}(x, 0, \nabla u) - \nabla_\xi \mathcal{L}(x, 0, \nabla u) \cdot \nabla u\big] h \cdot \nu \, d\mathcal{H}^{n-1} \\ &= \int_\Omega \big[\mathcal{L}(x, u, \nabla u) \operatorname{div} h + h \cdot \nabla_x \mathcal{L}(x, u, \nabla u)\big]\, dx \\ &\quad - \sum_{i,j=1}^n \int_\Omega \big[D_j u D_i h_j + u D_i a\big] D_{\xi_i} \mathcal{L}(x, u, \nabla u)\, dx \\ &\quad - \int_\Omega a\big[\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u + u D_s \mathcal{L}(x, u, \nabla u)\big]\, dx \\ &+ \int_\Omega [h \cdot \nabla u + au] \mathcal{G}(x, u, \nabla u)\, dx\end{aligned} \tag{9.2}$$

for each $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

Remark 9.2. Identity (9.2) follows by testing the equation with $h \cdot \nabla u + au$. More generally, it is satisfied by solutions $u \in C^1(\overline{\Omega}) \cap W^{2,2}_{loc}(\Omega)$.

Theorem 9.1 generalizes a well-known identity of Pohožaev [116] which has turned out to be a powerful tool in proving non-existence of solutions of problem (9.1). On the other hand, in some cases the requirement that u is of class $C^2(\Omega)$ seems too restrictive, while $C^1(\overline{\Omega})$ is not (cf. [142]). See e.g. problems in which the p-Laplacian operator is involved [82].

The aim of this section is to remove the $C^2(\Omega)$ assumption on u, by imposing the strict convexity of $\mathcal{L}(x,s,\cdot)$. The main result is the following:

Theorem 9.3. *Assume that $u \in C^1(\overline{\Omega})$ is a solution of (9.1) and that the map*

$$\xi \mapsto \mathcal{L}(x,s,\xi)$$

is strictly convex for each $(x,s) \in \overline{\Omega} \times \mathbb{R}$. Then (9.2) holds for all $a \in C^1(\overline{\Omega})$, $h \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

Let us observe that the strict convexity of $\mathcal{L}(x,s,\cdot)$ is indeed usually assumed in the applications and it is also natural if one expects the solution u to be of class $C^1(\overline{\Omega})$. In some particular situations (see Section 9.3), we are also able to assume only the convexity of $\mathcal{L}(x,s,\cdot)$. This is the case, for instance, if one takes

$$\mathcal{L}(x,s,\xi) = \alpha(x,s)\beta(\xi) + \gamma(x,s).$$

Note that if the test functions a and h have compact support in Ω, we obtain the variational identity also when u is only locally Lipschitz in Ω. This seems to be useful in particular when $\mathcal{L}(x,s,\cdot)$ is merely convex, as a C^1 regularity of u cannot be expected.

Finally, we refer the reader to [117] for various applications of the previous result to non-existence theorems.

9.2. **The approximation argument.** Let Ω be an open subset of $\mathbb{R}^n$, not necessarily bounded. Assume that $\xi \mapsto \mathcal{L}(x,s,\xi)$ is strictly convex for each $(x,s) \in \overline{\Omega} \times \mathbb{R}$.

Lemma 9.4. *Let $u : \Omega \to \mathbb{R}$ be a locally Lipschitz solution of*

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) = \mathcal{G}(x,u,\nabla u) \quad \text{in } \Omega. \tag{9.3}$$

Then

$$\begin{aligned} &\int_\Omega \big(\mathcal{L}(x,u,\nabla u)\operatorname{div} h + h\cdot\nabla_x\mathcal{L}(x,u,\nabla u)\big)\,dx \\ &= \sum_{i,j=1}^n \int_\Omega D_i h_j D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u\,dx - \int_\Omega \mathcal{G}(x,u,\nabla u) h\cdot\nabla u\,dx \end{aligned} \tag{9.4}$$

for every $h \in C^1_c(\Omega,\mathbb{R}^n)$.

Proof. Since h has compact support, without loss of generality we may assume that Ω is bounded. Let $R > 0$ with $|\nabla u(x)| \le R$ for every $x \in \operatorname{supt} h$ and let $\vartheta \in C^\infty(\mathbb{R})$ be such that $\vartheta = 1$ on $[-R,R]$ and $\vartheta = 0$ outside $[-R-1, R+1]$. Define now $\overline{\mathcal{L}}(x,s,\xi)$ by

$$\overline{\mathcal{L}}(x,s,\xi) = \mathcal{L}(x,s,\vartheta(|\xi|)\xi)$$

for each $(x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Then there exists $\omega > 0$ such that

$$\sum_{i,j=1}^n \nabla^2_{\xi_i\xi_j}\overline{\mathcal{L}}(x,s,\xi)\eta_i\eta_j > -\omega|\eta|^2$$

for each $(x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$. Let us now introduce $\Lambda \in C^1([0,+\infty[)$ by

$$\Lambda(\tau) = \begin{cases} 0 & \text{if } 0 \le \tau \le R \\ \omega'(\tau - R)^2 & \text{if } \tau \ge R, \end{cases}$$

where $\omega' > \omega$. Moreover, let $\widetilde{\mathcal{L}} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be given by

$$\widetilde{\mathcal{L}}(x,\xi) = \overline{\mathcal{L}}(x,u(x),\xi) + \Lambda(|\xi|) \tag{9.5}$$

for each $(x,\xi)\in\Omega\times\mathbb{R}^n$. Then $\widetilde{\mathcal{L}}(x,\cdot)$ is strictly convex and there are $\nu,c>0$ with

$$\widetilde{\mathcal{L}}(x,\xi)\geq\nu|\xi|^2-c$$

for each $(x,\xi)\in\Omega\times\mathbb{R}^n$. In particular, since u solves (9.3), then it is the unique minimum of the functional $f:H_0^1(\Omega)\to\mathbb{R}$ given by

$$f(w)=\int_\Omega\widetilde{\mathcal{L}}(x,\nabla w)\,dx+\int_\Omega\big[D_s\mathcal{L}(x,u,\nabla u)-\mathcal{G}(x,u,\nabla u)\big]w\,dx.$$

On the other hand, if $u_k\in H_0^1(\Omega)$ denotes the minimum of the modified functional

$$f_k(w)=f(w)+\frac{1}{k}\int_\Omega|\nabla w|^2\,dx,$$

by standard regularity arguments, $u_k\in C^1(\overline{\Omega})\cap W^{2,2}_{loc}(\Omega)$. Since $f(u_k)\to f(u)$ as $k\to+\infty$, we get $u_k\rightharpoonup u$ in $H_0^1(\Omega)$ and

$$\int_\Omega\widetilde{\mathcal{L}}(x,\nabla u_k)\,dx\to\int_\Omega\widetilde{\mathcal{L}}(x,\nabla u)\,dx,$$

which by [146, Theorem 3] implies $u_k\to u$ in $H_0^1(\Omega)$. In particular, $\nabla u_k(x)\to\nabla u(x)$ a.e. in Ω, up to a subsequence. Put now

$$\widehat{\mathcal{L}}(x,\xi)=\widetilde{\mathcal{L}}(x,\xi)+\frac{1}{k}|\xi|^2.$$

Since u_k satisfies the Euler's equation of f_k

$$\operatorname{div}(\nabla_\xi\widehat{\mathcal{L}}(x,\nabla u_k))=D_s\mathcal{L}(x,u,\nabla u)-\mathcal{G}(x,u,\nabla u),$$

by (9.2) it results

$$\begin{aligned}
&\int_\Omega\Big(\widehat{\mathcal{L}}(x,\nabla u_k)\operatorname{div}h+h\cdot\nabla_x\widehat{\mathcal{L}}(x,\nabla u_k)\Big)\,dx\\
&=\sum_{i,j=1}^n\int_\Omega D_ih_jD_{\xi_i}\widehat{\mathcal{L}}(x,\nabla u_k)D_ju_k\,dx\\
&\quad+\int_\Omega\big[D_s\mathcal{L}(x,u,\nabla u)-\mathcal{G}(x,u,\nabla u)\big]h\cdot\nabla u_k\,dx,
\end{aligned}$$

namely

$$\begin{aligned}
&\int_\Omega\mathcal{L}(x,u(x),\vartheta(|\nabla u_k|)\nabla u_k)\operatorname{div}h\,dx+\int_\Omega\Lambda(|\nabla u_k|)\operatorname{div}h\,dx\\
&+\frac{1}{k}\int_\Omega|\nabla u_k|^2\operatorname{div}h\,dx+\int_\Omega h\cdot\nabla_x\widetilde{\mathcal{L}}(x,\nabla u_k)\,dx\\
&-\sum_{i,j=1}^n\int_\Omega D_ih_jD_{\xi_i}\overline{\mathcal{L}}(x,u(x),\nabla u_k)D_ju_k\,dx-\sum_{i,j=1}^n\int_\Omega D_ih_jD_{\xi_i}\Lambda(|\nabla u_k|)D_ju_k\,dx\\
&-\frac{2}{k}\sum_{i,j=1}^n\int_\Omega D_ih_jD_iu_kD_ju_k\,dx-\int_\Omega\big[D_s\mathcal{L}(x,u,\nabla u)-\mathcal{G}(x,u,\nabla u)\big]h\cdot\nabla u_k\,dx\\
&=0\,.
\end{aligned}$$

Notice that

$$\frac{1}{k}\int_\Omega |\nabla u_k|^2 \operatorname{div} h\,dx \to 0, \quad \frac{2}{k}\sum_{i,j=1}^n \int_\Omega D_i h_j\, D_i u_k\, D_j u_k\,dx \to 0$$

as $k \to +\infty$. Moreover, since $D_j u_k \to D_j u$ and $D_{\xi_i}\Lambda(|\nabla u_k|) \rightharpoonup D_{\xi_i}\Lambda(|\nabla u|)$ in $L^2(\Omega)$,

$$\sum_{i,j=1}^n \int_\Omega D_i h_j\, D_{\xi_i}\Lambda(|\nabla u_k|) D_j u_k\,dx \to \sum_{i,j=1}^n \int_\Omega D_i h_j\, D_{\xi_i}\Lambda(|\nabla u|) D_j u\,dx = 0$$

as $k \to +\infty$ and

$$\int_\Omega \Lambda(|\nabla u_k|)\operatorname{div} h\,dx \to \int_\Omega \Lambda(|\nabla u|)\operatorname{div} h\,dx = 0$$

as $k \to +\infty$. Since

$$\begin{aligned}\int_\Omega h\cdot\nabla_x\widetilde{\mathcal{L}}(x,\nabla u_k)\,dx &= \int_\Omega h\cdot\nabla_x\mathcal{L}(x,u(x),\vartheta(|\nabla u_k|)\nabla u_k)\,dx\\ &\quad+\int_\Omega D_s\mathcal{L}(x,u(x),\vartheta(|\nabla u_k|)\nabla u_k)h\cdot\nabla u\,dx\end{aligned}$$

and being

$$|\nabla_x\mathcal{L}(x,u(x),\vartheta(|\nabla u_k|)\nabla u_k)| \le c_1, \quad |D_s\mathcal{L}(x,u(x),\vartheta(|\nabla u_k|)\nabla u_k)| \le c_2$$

for some $c_1, c_2 > 0$, one obtains

$$\int_\Omega h\cdot\nabla_x\widetilde{\mathcal{L}}(x,\nabla u_k)\,dx \to \int_\Omega h\cdot\nabla_x\mathcal{L}(x,u,\nabla u)\,dx + \int_\Omega D_s\mathcal{L}(x,u,\nabla u)h\cdot\nabla u\,dx.$$

Furthermore, since there exists $c_3 > 0$ with $|\mathcal{L}(x,u(x),\vartheta(|\nabla u_k|)\nabla u_k)| \le c_3$, one gets

$$\int_\Omega \mathcal{L}(x,u,\vartheta(|\nabla u_k|)\nabla u_k)\operatorname{div} h\,dx \to \int_\Omega \mathcal{L}(x,u,\nabla u)\operatorname{div} h\,dx.$$

Taking into account that there exists $c_4 > 0$ with

$$\Big|D_{\xi_i}\mathcal{L}(x,u(x),\vartheta(|\nabla u_k|)\nabla u_k)\Big[\vartheta'(|\nabla u_k|)\frac{D_i u_k\nabla u_k}{|\nabla u_k|} + \vartheta(|\nabla u_k|)e_i\Big]\Big| \le c_4$$

and that $D_j u_k \to D_j u$ in $L^2(\Omega)$, one deduces

$$\sum_{i,j=1}^n \int_\Omega D_i h_j\, D_{\xi_i}\overline{\mathcal{L}}(x,u,\nabla u_k) D_j u_k\,dx \to \sum_{i,j=1}^n \int_\Omega D_i h_j\, D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u\,dx$$

as $k \to +\infty$. Noting that, of course

$$\int_\Omega \big[D_s\mathcal{L}(x,u,\nabla u) - \mathcal{G}(x,u,\nabla u)\big]h\cdot\nabla u_k dx$$

converges to

$$\int_\Omega \big[D_s\mathcal{L}(x,u,\nabla u) - \mathcal{G}(x,u,\nabla u)\big]h\cdot\nabla u dx$$

as $k \to \infty$, the proof is complete. □

Remark 9.5. Let us observe that a (different) approximation technique was also used by Guedda and Véron [82] to deal with the particular case $\mathcal{L}(x,s,\xi) = \frac{1}{p}|\xi|^p$.

Let us now assume that Ω is bounded with Lipschitz boundary and let $\nu(x)$ denote the outer normal to $\partial\Omega$ at x (which exists for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$).

Lemma 9.6. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (9.1). Then*

$$\begin{aligned}
&\int_\Omega \big(\mathcal{L}(x,u,\nabla u)\, \operatorname{div} h + h\cdot \nabla_x \mathcal{L}(x,u,\nabla u)\big)\, dx \\
&= \sum_{i,j=1}^n \int_\Omega D_i h_j\, D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u\, dx - \int_\Omega \mathcal{G}(x,u,\nabla u) h\cdot \nabla u\, dx \\
&\quad + \int_{\partial\Omega} \big[\mathcal{L}(x,0,\nabla u) - \nabla_\xi \mathcal{L}(x,0,\nabla u)\cdot \nabla u\big]\, (h\cdot \nu)\, d\mathcal{H}^{n-1}
\end{aligned}$$

for every $h \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

Proof. Let $k \geq 1$ and $\varphi_k : \mathbb{R} \to [0,1]$ be given by

$$\varphi_k(s) = \begin{cases} 0 & \text{if } s \leq \frac{1}{k} \\ ks - 1 & \text{if } \frac{1}{k} \leq s \leq \frac{2}{k} \\ 1 & \text{if } s \geq \frac{2}{k}, \end{cases} \tag{9.6}$$

and define the Lipschitz map $\psi_k : \mathbb{R}^n \to [0,1]$ by setting

$$\psi_k(x) = \varphi_k(d(x, \mathbb{R}^n \setminus \Omega)).$$

Applying Lemma 9.4 on $\mathbb{R}^n$ with $\psi_k h$ in place of h, one deduces

$$\begin{aligned}
&\int_{\mathbb{R}^n} \psi_k \mathcal{L}(x,u,\nabla u)\, \operatorname{div} h\, dx + \int_{\mathbb{R}^n} \mathcal{L}(x,u,\nabla u)\, \nabla\psi_k \cdot h\, dx \\
&+ \int_{\mathbb{R}^n} \psi_k h\cdot \nabla_x \mathcal{L}(x,u,\nabla u)\, dx \\
&= \sum_{i,j=1}^n \int_{\mathbb{R}^n} h_j D_i\psi_k D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u\, dx \\
&\quad + \sum_{i,j=1}^n \int_{\mathbb{R}^n} \psi_k D_i h_j D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u\, dx - \int_{\mathbb{R}^n} \mathcal{G}(x,u,\nabla u)\psi_k h\cdot \nabla u\, dx.
\end{aligned}$$

Taking into account that (ψ_k) is bounded in $BV(\mathbb{R}^n)$ and

$$\forall \eta \in C(\mathbb{R}^n, \mathbb{R}^n) : \int_{\mathbb{R}^n} \nabla\psi_k \cdot \eta\, dx \to -\int_{\partial\Omega} \eta\cdot \nu\, d\mathcal{H}^{n-1},$$

one has

$$\int_{\mathbb{R}^n} \mathcal{L}(x,u,\nabla u)\, \nabla\psi_k \cdot h\, dx \to -\int_{\partial\Omega} \mathcal{L}(x,0,\nabla u)(h\cdot\nu)\, d\mathcal{H}^{n-1}$$

as $k \to +\infty$ and

$$\sum_{i,j=1}^n \int_{\mathbb{R}^n} h_j D_i\psi_k D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u\, dx \to -\sum_{i,j=1}^n \int_{\partial\Omega} \nu_i h_j D_{\xi_i}\mathcal{L}(x,0,\nabla u) D_j u\, dx.$$

As observed in [117], clearly one has

$$\sum_{i,j=1}^n \nu_i h_j D_{\xi_i}\mathcal{L}(x,0,\nabla u) D_j u = \nabla_\xi \mathcal{L}(x,0,\nabla u)\cdot \nabla u\, (h\cdot\nu) \quad \text{on } \partial\Omega.$$

Since of course $\psi_k(x) \to \chi_\Omega(x)$ for each $x \in \mathbb{R}^n$, the proof is complete. □

Proof of Theorem 9.3. Clearly, if $u \in C^1(\overline{\Omega})$ is a solution of (9.1) one has

$$\begin{aligned}&\int_\Omega a\big[\nabla_\xi \mathcal{L}(x,u,\nabla u)\cdot \nabla u + uD_s\mathcal{L}(x,u,\nabla u) - u\,\mathcal{G}(x,u,\nabla u)\big]\,dx\\&+\int_\Omega u\nabla a\cdot\nabla_\xi\mathcal{L}(x,u,\nabla u)\,dx = 0\end{aligned}\tag{9.7}$$

for each $a \in C^1(\overline{\Omega})$. The assertion follows by combining (9.7) with Lemma 9.6. □

Remark 9.7. Let $N \geq 2$. It is easily seen that Theorem 9.3 has a vectorial counterpart for solutions $u \in C^1(\overline{\Omega}, \mathbb{R}^N)$ of the system

$$\begin{gathered}-\operatorname{div}\big(\nabla_{\xi_k}\mathcal{L}(x,u,\nabla u)\big) + D_{s_k}\mathcal{L}(x,u,\nabla u) = \mathcal{G}_k(x,u,\nabla u)\quad\text{in }\Omega\\ u=0\quad\text{on }\partial\Omega\\ k=1,\dots,N.\end{gathered}$$

See also [117, Proposition 3].

9.3. **Non-strict convexity in some particular cases.** In this section we will see that, in some particular cases, the assumption of strict convexity of $\mathcal{L}(x,s,\cdot)$ can be relaxed to the weaker assumption of convexity. Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary.

Lemma 9.8. *Let $\mathcal{F} : \Omega\times\mathbb{R}^n \to \mathbb{R}$ be a function with $\mathcal{F}(x,\cdot)$ convex and C^1 and $\mathcal{F}(\cdot,\xi)$ measurable. Assume that there exist $a_0 \in L^1(\Omega)$, $a_1 \in L^{p'}(\Omega)$, $1 < p < +\infty$, and $b, d > 0$ with*

$$|\nabla_\xi\mathcal{F}(x,\xi)| \leq a_1(x) + b|\xi|^{p-1}, \tag{9.8}$$

$$\mathcal{F}(x,\xi) \geq d|\xi|^p - a_0(x) \tag{9.9}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$. Let $(w_k) \subset L^p(\Omega,\mathbb{R}^n)$ and w be such that

$$w_k \rightharpoonup w \quad \text{in } L^p(\Omega,\mathbb{R}^n),\qquad \int_\Omega \mathcal{F}(x,w_k)\,dx \to \int_\Omega \mathcal{F}(x,w)\,dx$$

as $k\to+\infty$. Then

$$\mathcal{F}(x,w_k) \rightharpoonup \mathcal{F}(x,w)\quad \text{in } L^1(\Omega), \tag{9.10}$$

$$\nabla_\xi\mathcal{F}(x,w_k) \to \nabla_\xi\mathcal{F}(x,w)\quad \text{in } L^{p'}(\Omega) \tag{9.11}$$

as $k\to+\infty$. Moreover, up to a subsequence, $|w_k|^p \leq \psi$ for some $\psi \in L^1(\Omega)$.

Proof. Let us define $\widetilde{\mathcal{F}} : \Omega\times\mathbb{R}^n \to \mathbb{R}$ by setting

$$\widetilde{\mathcal{F}}(x,\xi) = \mathcal{F}(x,w(x)+\xi) - \mathcal{F}(x,w(x)) - \nabla_\xi\mathcal{F}(x,w(x))\cdot\xi.$$

Note that $\widetilde{\mathcal{F}} \geq 0$, $\widetilde{\mathcal{F}}(x,0)=0$, $\nabla_\xi\widetilde{\mathcal{F}}(x,0)=0$ and

$$\int_\Omega \widetilde{\mathcal{F}}(x,w_k-w)\,dx \to 0 \quad \text{as } k\to+\infty. \tag{9.12}$$

Therefore, since for each $\varphi \in L^\infty(\Omega)$

$$\int_\Omega \varphi\nabla_\xi\mathcal{F}(x,w)\cdot(w_k-w)\,dx \to 0 \quad \text{as } k\to+\infty,$$

one has

$$\int_\Omega \varphi\big[\mathcal{F}(x,w_k)-\mathcal{F}(x,w)\big]\,dx \to 0 \quad \text{as } k\to+\infty,$$

which proves (9.10).

Note that, in view of (9.12), up to a subsequence one has $\widetilde{\mathcal{F}}(x, w_k(x) - w(x)) \to 0$ for a.e. $x \in \Omega$. Fix now such an x; then by (9.9) up to a subsequence $w_k(x) \to y$ for some $y \in \mathbb{R}^n$, which yields $\widetilde{\mathcal{F}}(x, y - w(x)) = 0$. In particular, $y - w(x)$ is a local minimum for $\widetilde{\mathcal{F}}(x, \cdot)$, so that $\nabla_\xi \widetilde{\mathcal{F}}(x, y - w(x)) = 0$. Hence we conclude

$$\nabla_\xi \mathcal{F}(x, w_k(x)) \to \nabla_\xi \mathcal{F}(x, w(x)). \tag{9.13}$$

Now, since by (9.12) there exists $\widetilde{\psi} \in L^1(\Omega)$ such that

$$\mathcal{F}(x, w_k) - \mathcal{F}(x, w) - \nabla_\xi \mathcal{F}(x, w) \cdot (w_k - w) \leq \widetilde{\psi},$$

by (9.9) and Young's inequality one finds $c_1, c_2 > 0$ such that

$$c_1 |w_k|^p \leq a_0 + \mathcal{F}(x, w) - \nabla_\xi \mathcal{F}(x, w) \cdot w + \widetilde{\psi} + c_2 |\nabla_\xi \mathcal{F}(x, w)|^{p'}$$

In particular, in view of (9.8) one deduces $|\nabla_\xi \mathcal{F}(x, w_k)| \leq g$ for some $g \in L^{p'}(\Omega)$, which combined with (9.13) yields the second assertion. □

9.4. **The splitting case.** In this subsection we shall deal with the case when $\mathcal{L}(x, s, \xi)$ is of the form $\alpha(x, s)\beta(\xi) + \gamma(x, s)$.

Lemma 9.9. *Let $\alpha, \gamma \in W^{1,\infty}(\Omega)$ with $\alpha \geq 0$ and $\beta \in C^1(\mathbb{R}^n)$ convex such that*

$$\mathcal{F}(x, \xi) = \alpha(x)\beta(\xi) + \gamma(x)$$

with $d|\xi|^p - b \leq \beta(\xi) \leq b(1 + |\xi|^p)$, $1 < p < +\infty$, for some $b, d > 0$. Let (w_k) and w with

$$w_k \rightharpoonup w \quad \text{in } L^p(\Omega, \mathbb{R}^n), \qquad \int_\Omega \mathcal{F}(x, w_k)\, dx \to \int_\Omega \mathcal{F}(x, w)\, dx$$

as $k \to +\infty$. Then

$$\beta(w_k)\nabla\alpha(x) \rightharpoonup \beta(w)\nabla\alpha(x) \quad \text{in } L^1(\Omega), \tag{9.14}$$

as $k \to +\infty$.

Proof. If Ω_0 denotes the set where $\alpha = 0$, one may argue on

$$\Omega \setminus \Omega_0 = \bigcup_{h=1}^{+\infty} \Omega_h, \quad \Omega_h = \Big\{x \in \Omega : \alpha(x) > \frac{1}{h}\Big\}.$$

By Lemma 9.8 there exists $\psi \in L^1(\Omega)$ such that

$$\chi_{\Omega \setminus \Omega_h}(x)\beta(w_k(x))\nabla\alpha(x) \leq \psi(x)$$

up to a subsequence; hence for each $\varepsilon > 0$ one finds $h_0 \geq 1$ such that

$$\int_{\Omega \setminus \Omega_{h_0}} \beta(w_k(x))\nabla\alpha(x)\, dx < \varepsilon$$

uniformly with respect to k. On the other hand, again by Lemma 9.8 one knows that

$$\mathcal{F}(x, w_k) \rightharpoonup \mathcal{F}(x, w) \quad \text{in } L^1(\Omega_{h_0})$$

as $k \to +\infty$, which implies

$$\alpha(x)\beta(w_k) \rightharpoonup \alpha(x)\beta(w) \quad \text{in } L^1(\Omega_{h_0}).$$

Then since $1/\alpha \in L^\infty(\Omega_{h_0})$ one gets $\beta(w_k) \rightharpoonup \beta(w)$ in $L^1(\Omega_{h_0})$, which yields (9.14). □

Theorem 9.10. *Let $u : \Omega \to \mathbb{R}$ be a locally Lipschitz solution of (9.3). Assume that there exist $\alpha, \gamma \in C^1(\overline{\Omega} \times \mathbb{R})$ and $\beta \in C^2(\mathbb{R}^n)$ convex such that $\alpha \geq 0$, $\beta(0) = 0$ and*

$$\mathcal{L}(x, s, \xi) = \alpha(x, s)\beta(\xi) + \gamma(x, s).$$

Then

$$\begin{aligned}
&\sum_{i,j=1}^{n} \int_\Omega \big[D_j u D_i h_j + u D_i a\big] D_{\xi_i} \mathcal{L}(x, u, \nabla u)\, dx \\
&+ \int_\Omega a\big[\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u + u D_s \mathcal{L}(x, u, \nabla u)\big]\, dx \\
&= \int_\Omega \big[\mathcal{L}(x, u, \nabla u) \operatorname{div} h + h \cdot \nabla_x \mathcal{L}(x, u, \nabla u)\big]\, dx \\
&= \int_\Omega \big[h \cdot \nabla u + au\big] \mathcal{G}(x, u, \nabla u)\, dx
\end{aligned} \tag{9.15}$$

holds for each $a \in C_c^1(\Omega)$ and $h \in C_c^1(\Omega, \mathbb{R}^n)$.

Proof. Let θ, Λ, $\widetilde{\mathcal{L}}$ and $(u_k) \subset H_0^1(\Omega)$ be as in Lemma 9.4. We apply Lemma 9.8 choosing

$$w_k = \nabla u_k, \quad \mathcal{F}(x, \xi) = \Lambda(|\xi|) \quad \text{or} \quad \mathcal{F}(x, \xi) = \beta(\theta(|\xi|)\xi).$$

By (9.11) one has

$$\sum_{i,j=1}^{n} \int_\Omega D_i h_j\, D_{\xi_i} \Lambda(|\nabla u_k|) D_j u_k\, dx \to \sum_{i,j=1}^{n} \int_\Omega D_i h_j\, D_{\xi_i} \Lambda(|\nabla u|) D_j u\, dx = 0,$$

and the term

$$\sum_{i,j=1}^{n} \int_\Omega D_i h_j \alpha(x, u) D_{\xi_i} \beta(\vartheta(|\nabla u_k|)\nabla u_k) D_j u_k\, dx$$

goes to

$$\sum_{i,j=1}^{n} \int_\Omega D_i h_j \alpha(x, u) D_{\xi_i} \beta(\nabla u) D_j u\, dx$$

as $k \to +\infty$. Moreover, by (9.10) one obtains

$$\int_\Omega \Lambda(|\nabla u_k|) \operatorname{div} h\, dx \to \int_\Omega \Lambda(|\nabla u|) \operatorname{div} h\, dx = 0,$$

$$\int_\Omega \alpha(x, u)\beta(\vartheta(|\nabla u_k|)\nabla u_k) \operatorname{div} h\, dx \to \int_\Omega \alpha(x, u)\beta(\nabla u) \operatorname{div} h\, dx$$

as $k \to +\infty$. Finally, by (9.14) of Lemma 9.9 one gets

$$\begin{aligned}
\int_\Omega h \cdot \nabla_x \widetilde{\mathcal{L}}(x, \nabla u_k)\, dx \to &\int_\Omega \big[h \cdot \nabla_x \alpha(x, u) + D_s \alpha(x, u) h \cdot \nabla u\big] \beta(\nabla u)\, dx \\
&+ \int_\Omega [\nabla_x \gamma(x, u) + D_s \gamma(x, u)]\, h \cdot \nabla u\, dx
\end{aligned}$$

as $k \to +\infty$. Then (9.15) follows by exploiting the proof of Lemma 9.4. □

At this point, arguing as in Lemma 9.6 and taking into account (9.7), we obtain the following result.

Theorem 9.11. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (9.1)*. Let $\alpha, \gamma \in C^1(\overline{\Omega} \times \mathbb{R})$ and $\beta \in C^2(\mathbb{R}^n)$ convex such that $\alpha \geq 0$, $\beta(0) = 0$ and*

$$\mathcal{L}(x, s, \xi) = \alpha(x, s)\beta(\xi) + \gamma(x, s).$$

Then (9.2) *holds for each $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega}, \mathbb{R}^n)$.*

9.5. **The one-dimensional case.** In this subsection we assume that Ω is an interval in $\mathbb{R}$ and $\mathcal{L} : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1 with $\mathcal{L}(x, s, \cdot)$ convex and $D_\xi \mathcal{L}$ of class C^1.

Theorem 9.12. *Let $u : \Omega \to \mathbb{R}$ be a locally Lipschitz solution of* (9.3)*. Then* (9.15) *holds for each $a \in C^1_c(\Omega)$ and $h \in C^1_c(\Omega)$.*

Theorem 9.13. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (9.1)*. Then* (9.2) *holds for each $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega})$.*

Taking into account next result, the above theorems follow arguing as in the proof of Lemmas 9.4 and 9.6.

Lemma 9.14. *Let $\mathcal{F} : \Omega \times \mathbb{R} \to \mathbb{R}$ be a C^1 function with $\mathcal{F}(x, \cdot)$ convex. Let $(w_k) \subset L^p(\Omega)$ and w be such that*

$$w_k \rightharpoonup w \quad \text{in } L^p(\Omega), \quad \int_\Omega \mathcal{F}(x, w_k)\, dx \to \int_\Omega \mathcal{F}(x, w)\, dx$$

as $k \to +\infty$ and assume that $(D_x \mathcal{F}(x, w_k))$ is bounded in L^q for some $q > 1$. Then

$$D_x \mathcal{F}(x, w_k) \rightharpoonup D_x \mathcal{F}(x, w) \quad \text{in } L^1(\Omega) \tag{9.16}$$

as $k \to +\infty$.

Proof. Let us set, for each $x \in \Omega$,

$$y_\flat(x) = \liminf_k w_k(x), \quad y_\sharp(x) = \limsup_k w_k(x).$$

Notice that one has

$$y_\flat(x) \leq w(x) \leq y_\sharp(x) \tag{9.17}$$

for a.e. $x \in \Omega$. Without loss of generality, one can replace $w_k(x)$ by its projection onto $[y_\flat(x), y_\sharp(x)]$; in particular

$$y_\flat(x) \leq w_k(x) \leq y_\sharp(x) \tag{9.18}$$

for a.e. $x \in \Omega$. Arguing as in the proof of Lemma 9.8 one obtains

$$\tilde{\mathcal{F}}(x, y_\flat(x) - w(x)) = 0, \quad \tilde{\mathcal{F}}(x, y_\sharp(x) - w(x)) = 0$$

for a.e. $x \in \Omega$. Then, by $\tilde{\mathcal{F}} \geq 0$ and the convexity of $\tilde{\mathcal{F}}(x, \cdot)$ one has

$$\tilde{\mathcal{F}}(x, (1-\theta)y_\flat(x) + \theta y_\sharp(x) - w(x)) = 0$$

for every $\theta \in [0, 1]$ and a.e. $x \in \Omega$. This yields

$$\mathcal{F}\big(x, (1-\vartheta)y_\flat(x) + \vartheta y_\sharp(x)\big) = (1-\vartheta)\mathcal{F}(x, y_\flat(x)) + \vartheta \mathcal{F}(x, y_\sharp(x)) \tag{9.19}$$

for a.e. $x \in \Omega$. For each $m \geq 1$ let us set

$$\Omega_m = \big\{x \in \Omega : y_\sharp(x) - y_\flat(x) \geq \frac{1}{m}\big\}.$$

By Lusin's theorem, for each $\varepsilon > 0$ there exists a closed subset $C_{m,\varepsilon} \subset \Omega_m$ such that

$$y_\flat\big|_{C_{m,\varepsilon}}, \; y_\sharp\big|_{C_{m,\varepsilon}} \quad \text{are continuous}, \quad \mathcal{L}^1(\Omega_m \setminus C_{m,\varepsilon}) < \varepsilon,$$

where $\mathscr{L}^1$ denotes the one-dimensional Lebesgue measure. We also cut off from $C_{m,\varepsilon}$ the negligible set of isolated points. Let us now take $x \in C_{m,\varepsilon}$ and $(x_k) \subset C_{m,\varepsilon}$ with $x_k \to x$. If $\delta > 0$ is sufficiently small, by continuity one has

$$y_\flat(x_k) \leq y_\flat(x) + \delta < y_\sharp(x) - \delta \leq y_\sharp(x_k) \tag{9.20}$$

for each $k \in \mathbb{N}$ large enough. By (9.19), for each $\vartheta \in [0, 1]$ one obtains

$$\begin{aligned}&\mathscr{F}(x, (1-\vartheta)(y_\flat(x) + \delta) + \vartheta(y_\sharp(x) - \delta))\\&= (1-\vartheta)\mathscr{F}(x, y_\flat(x) + \delta) + \vartheta\mathscr{F}(x, y_\sharp(x) - \delta).\end{aligned}$$

Moreover, (9.20) implies

$$\begin{aligned}&\mathscr{F}(x_k, (1-\vartheta)(y_\flat(x) + \delta) + \vartheta(y_\sharp(x) - \delta))\\&= (1-\vartheta)\mathscr{F}(x_k, y_\flat(x) + \delta) + \vartheta\mathscr{F}(x_k, y_\sharp(x) - \delta).\end{aligned}$$

Therefore, combining the previous identities yields

$$\begin{aligned}&D_x\mathscr{F}(x, (1-\vartheta)(y_\flat(x) + \delta) + \vartheta(y_\sharp(x) - \delta))\\&= (1-\vartheta)D_x\mathscr{F}(x, y_\flat(x) + \delta) + \vartheta D_x\mathscr{F}(x, y_\sharp(x) - \delta)\end{aligned}$$

for each $\vartheta \in [0, 1]$. Letting $\delta \to 0$ one obtains

$$D_x\mathscr{F}(x, (1-\vartheta)y_\flat(x) + \vartheta y_\sharp(x)) = (1-\vartheta)D_x\mathscr{F}(x, y_\flat(x)) + \vartheta D_x\mathscr{F}(x, y_\sharp(x))$$

for each $\vartheta \in [0, 1]$. By (9.17) and (9.18) we can choose

$$\overline{\vartheta} = \frac{w(x) - y_\flat(x)}{y_\sharp(x) - y_\flat(x)}, \quad \overline{\vartheta}_k = \frac{w_k(x) - y_\flat(x)}{y_\sharp(x) - y_\flat(x)}.$$

Then one gets

$$D_x\mathscr{F}(x, w(x)) = \frac{y_\sharp(x) - w(x)}{y_\sharp(x) - y_\flat(x)}D_x\mathscr{F}(x, y_\flat(x)) + \frac{w(x) - y_\flat(x)}{y_\sharp(x) - y_\flat(x)}D_x\mathscr{F}(x, y_\sharp(x))$$

and

$$D_x\mathscr{F}(x, w_k(x)) = \frac{y_\sharp(x) - w_k(x)}{y_\sharp(x) - y_\flat(x)}D_x\mathscr{F}(x, y_\flat(x)) + \frac{w_k(x) - y_\flat(x)}{y_\sharp(x) - y_\flat(x)}D_x\mathscr{F}(x, y_\sharp(x)).$$

In particular, one concludes

$$\begin{aligned}&D_x\mathscr{F}(x, w_k(x))\\&= D_x\mathscr{F}(x, w(x)) + (w_k(x) - w(x))\frac{D_x\mathscr{F}(x, y_\sharp(x)) - D_x\mathscr{F}(x, y_\flat(x))}{y_\sharp(x) - y_\flat(x)}\end{aligned}$$

for all $x \in C_{m,\varepsilon}$, which implies that

$$\forall\varphi \in L^\infty(C_{m,\varepsilon}) : \int_{C_{m,\varepsilon}} D_x\mathscr{F}(x, w_k)\varphi\,dx \to \int_{C_{m,\varepsilon}} D_x\mathscr{F}(x, w)\varphi\,dx$$

as $k \to +\infty$. On the other hand, since $(D_x\mathscr{F}(x, w_k))$ is bounded in $L^q(\Omega)$, for any $\varphi \in L^\infty(C_{m,\varepsilon})$ there exists $c > 0$ such that

$$\Big|\int_{\Omega_m\setminus C_{m,\varepsilon}} D_x\mathscr{F}(x, w_k)\varphi\,dx\Big| \leq c\,\mathscr{L}^1(\Omega_m \setminus C_{m,\varepsilon}) < c\,\varepsilon.$$

Letting $\varepsilon \to 0$, one gets

$$\forall\varphi \in L^\infty(\Omega_m) : \int_{\Omega_m} D_x\mathscr{F}(x, w_k)\varphi\,dx \to \int_{\Omega_m} D_x\mathscr{F}(x, w)\varphi\,dx$$

for each $m \geq 1$. Moreover, since on the set

$$\Omega_\infty = \{x \in \Omega : y_\sharp(x) = y_\flat(x)\}$$

one has $w_k \to w$ pointwise, then

$$\forall \varphi \in L^\infty(\Omega_\infty) : \int_{\Omega_\infty} D_x \mathcal{F}(x, w_k)\varphi \, dx \to \int_{\Omega_\infty} D_x \mathcal{F}(x, w)\varphi \, dx$$

which concludes the proof. □

9.6. **Non-existence results.** In the following we want to recall from [117] a general variational identity that holds both for scalar-valued and vector-valued extremals of multiple integrals of calculus of variations that will allow us to get non-existence results for various classes of problems.

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$ and $k \geq 1$. For each $\alpha \in \mathbb{N}^n$ we set

$$\xi^\alpha := \xi_i \cdots \xi_k, \quad C_\alpha := \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!}, \quad \mathcal{L}_{\xi^\alpha} := C_\alpha D_{\xi_{i\dots k}} \mathcal{L}.$$

Let now $f : W_0^{k,p}(\Omega) \to \mathbb{R}$ be the k-th order functional of calculus of variations

$$f(u) = \int_\Omega \mathcal{L}(x, u, \nabla u, \dots, \nabla^k u) \, dx.$$

By direct calculation, the Euler-Lagrange's equation of f is given by

$$\sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha \mathcal{L}_{\xi^\alpha}(x, u, \dots, \nabla^k u) = 0 \quad \text{in } \Omega. \tag{9.21}$$

If $u \in W_0^{k,p}(\Omega)$ is a weak solution to (9.21) and $\lambda \in C^k(\Omega)$, $v \in C^k(\Omega, \mathbb{R}^n)$, we set

$$\vartheta := v \cdot \nabla u + \lambda u, \quad \xi^\gamma := \sum_{i=1}^{n} \xi^{\alpha+\beta} \xi_i, \quad \partial^\alpha_{\lambda,v} := [D^\alpha(v \cdot D + \lambda) - (v \cdot D) D^\alpha].$$

We now recall the following Pohozǎev-type identity for general lagrangians.

Proposition 9.15. *Assume that $u \in C^k(\Omega)$ is a weak solution to (9.21). Then*

$$\begin{aligned} &\operatorname{div} \Big\{ v\mathcal{L}(x, u, \dots, \nabla^k u) - \sum_{|\alpha+\beta|=0}^{k-1} (-1)^{|\beta|} \frac{C_\alpha C_\beta}{C_\gamma} D^\alpha \vartheta D^\beta \mathcal{L}_{\xi^\gamma}(x, u, \dots, \nabla^k u) \Big\} \\ &= \operatorname{div}(v) \mathcal{L}(x, u, \dots, \nabla^k u) + v \cdot \nabla_x \mathcal{L}(x, u, \dots, \nabla^k u) \\ &\quad - \sum_{|\alpha|=0}^{k} \partial^\alpha_{\lambda,v} u \cdot \mathcal{L}_{\xi^\alpha}(x, u, \dots, \nabla^k u) \end{aligned}$$

for a.e. $x \in \Omega$ and for each $v \in C^k(\Omega, \mathbb{R}^n)$.

The proof of the above identity follows by direct computation. See [117, section 5].

We now come to the main non-existence result for first order scalar-valued extremals.

Theorem 9.16. *Assume that Ω is star-shaped with respect to 0. Suppose also that*

$$\xi \cdot \nabla_\xi \mathcal{L}(x, 0, \xi) - \mathcal{L}(x, 0, \xi) \geq 0 \tag{9.22}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$ and that there exists $\lambda \in \mathbb{R}$ such that

$$n\mathcal{L}(x, s, \xi) + x \cdot \nabla_x \mathcal{L}(x, s, \xi) - \lambda s D_s \mathcal{L}(x, s, \xi) - (\lambda + 1)\xi \cdot \nabla_\xi \mathcal{L}(x, s, \xi) \geq 0 \tag{9.23}$$

for a.e. $x \in \Omega$ and each $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Let $s = 0$ or $\xi = 0$ whenever equality holds in (8.65). *Then the elliptic boundary value problem*

$$-\operatorname{div}\big(\nabla_\xi \mathcal{L}(x,u,\nabla u)\big) + D_s\mathcal{L}(x,u,\nabla u) = 0 \quad \text{in } \Omega \tag{9.24}$$

has no weak solution $u \in C^1(\overline{\Omega})$.

Proof. Let $u \in C^1(\overline{\Omega})$ be a weak solution of (9.21). By applying the divergence Theorem to identity of Proposition 9.15 choosing $v(x) = x$ and $k = 1$, since $u = 0$ on $\partial\Omega$, we get

$$\begin{aligned}
&\int_{\partial\Omega} \big[\mathcal{L}(x,0,\nabla u) - \nabla_\xi \mathcal{L}(x,0,\nabla u)\cdot\nabla u\big](x\cdot\nu)\, d\mathcal{H}^{n-1} \\
&\int_\Omega \Big\{n\mathcal{L}(x,u,\nabla u) + x\cdot\nabla_x\mathcal{L}(x,u,\nabla u) \\
&\quad - \lambda u D_s\mathcal{L}(x,u,\nabla u) - (\lambda+1)\nabla u\cdot\nabla_\xi\mathcal{L}(x,u,\nabla u)\Big\}\, dx.
\end{aligned}$$

Taking into account that on $\partial\Omega$ it is $(x\cdot\nu) > 0$, conditions (9.22) and (9.23) yield a contradiction. ☐

Corollary 9.17. *Assume that there exists $\lambda \in \mathbb{R}$ such that*

$$\sum_{i,j=1}^n \big((n-2\lambda-2)a_{ij}(x,s) + x\cdot\nabla_x a_{ij}(x,s) - \lambda s D_s a_{ij}(x,s)\big)\xi_i\xi_j \ge 0$$

for a.e. $x \in \Omega$ and each $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$ and

$$\lambda s g(x,s) - nG(x,s) - x\cdot\nabla_x G(x,s) > 0$$

and for a.e. $x \in \Omega$ and each $s \in \mathbb{R}\setminus\{0\}$. Then the quasi-linear problem

$$-\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u)D_i u D_j u = g(x,u) \quad \text{in } \Omega \tag{9.25}$$

has no weak solution $u \in C^1(\overline{\Omega})$.

Proof. It comes straightforward from the previous result taking

$$\mathcal{L}(x,s,\xi) = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j$$

for a.e. $x \in \Omega$ and each $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$. ☐

We now come to the main non-existence result for first order vector-valued extremals.

Theorem 9.18. *Assume that Ω is star-shaped with respect to 0. Suppose also that*

$$\xi\cdot\nabla_\xi\mathcal{L}(x,0,\xi) - \mathcal{L}(x,0,\xi) \ge 0, \tag{9.26}$$

for a.e. $x \in \Omega$ and each $\xi \in \mathbb{R}^{nN}$ and that there exists $\lambda \in \mathbb{R}$ such that

$$n\mathcal{L}(x,s,\xi) + x\cdot\nabla_x\mathcal{L}(x,s,\xi) - \lambda u\cdot\nabla_s\mathcal{L}(x,s,\xi) - (\lambda+1)\xi\cdot\nabla_\xi\mathcal{L}(x,s,\xi) \ge 0 \tag{9.27}$$

for a.e. $x \in \Omega$ and each $(s,\xi) \in \mathbb{R}^N\times\mathbb{R}^{nN}$. Assume further that equality holds only when either $s = 0$ or $\xi = 0$. Then the nonlinear elliptic system

$$\operatorname{div}(\nabla_\xi\mathcal{L}(x,u,\nabla u)) + \nabla_s\mathcal{L}(x,u,\nabla u) = 0 \tag{9.28}$$

has no weak solution $u \in C^2(\Omega,\mathbb{R}^N)\cap C^1(\overline{\Omega},\mathbb{R}^N)$.

Proof. Arguing as in the scalar case we obtain the following variational identity

$$\begin{aligned}&D_i\left\{v_i\mathcal{L}(x,u,\nabla u)-\left(v_jD_ju^k+\lambda u^k\right)D_{\xi_i^k}\mathcal{L}(x,u,\nabla u)\right\}\\&=D_iv_i\mathcal{L}(x,u,\nabla u)+v_iD_{x_i}\mathcal{L}(x,u,\nabla u)-\left(D_ju^kD_iv_j+u^kD_i\lambda\right)D_{\xi_i^k}\mathcal{L}(x,u,\nabla u)\\&\quad-\lambda\left(D_iu^kD_{\xi_i^k}\mathcal{L}(x,u,\nabla u)+u^kD_{s_k}\mathcal{L}(x,u,\nabla u)\right)\end{aligned}$$

where i,j are understood to be summed from 1 to n and k from 1 to N. Therefore, it suffices to argue as in Theorem 9.16. □

Corollary 9.19. *Assume that there exists $\lambda\in\mathbb{R}$ such that*

$$\sum_{i,j=1}^{n}\sum_{h,k=1}^{N}\left((n-2\lambda-2)a_{ij}^{hk}(x,s)+x\cdot\nabla_x a_{ij}^{hk}(x,s)-\lambda s\cdot D_sa_{ij}^{hk}(x,s)\right)\xi_i^h\xi_j^k\geq 0$$

for a.e. $x\in\Omega$ and each $(s,\xi)\in\mathbb{R}^N\times\mathbb{R}^{nN}$ and

$$\lambda s\cdot g(x,s)-nG(x,s)-x\cdot\nabla_xG(x,s)>0.$$

and for a.e. $x\in\Omega$ and each $s\in\mathbb{R}^N\setminus\{0\}$. Then the quasi-linear system ($\ell=1,\dots,N$)

$$-\sum_{i,j=1}^{n}\sum_{h=1}^{N}D_j(a_{ij}^{h\ell}(x,u)D_iu_h)+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{h,k=1}^{N}D_{s_\ell}a_{ij}^{hk}(x,u)D_iu_hD_ju_k=g_\ell(x,u)\tag{9.29}$$

has no weak solution $u\in C^2(\Omega,\mathbb{R}^N)\cap C^1(\overline{\Omega},\mathbb{R}^N)$.

Proof. It follows by Theorem (9.18) choosing

$$\mathcal{L}(x,s,\xi)=\frac{1}{2}\sum_{i,j=1}^{n}\sum_{h,k=1}^{N}a_{ij}^{hk}(x,s)\xi_i^h\xi_j^k-G(x,s)$$

for a.e. $x\in\Omega$ and each $(s,\xi)\in\mathbb{R}^n\times\mathbb{R}^{nN}$. □

Theorem 9.20. *Let Ω be star-shaped with respect to the origin and*

$$\nabla_x\mathcal{L}(x,s,\xi)\cdot x-\frac{n}{p^*}D_s\mathcal{L}(x,s,\xi)s+\left\{\frac{n}{p^*}-\frac{n}{q}\right\}\lambda\,|s|^q\geq 0\,,\tag{9.30}$$

for a.e. $x\in\Omega$ and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$. Then $(\mathcal{P}_{0,\lambda})$ has no nontrivial solution $u\in C^1(\overline{\Omega})$.

Proof. If we define $\mathcal{F}:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ by setting

$$\forall(x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n:\mathcal{F}(x,s,\xi)=\mathcal{L}(x,s,\xi)-\frac{\lambda}{q}|s|^q-\frac{1}{p^*}|s|^{p^*},$$

the first assertion follows, after some computations, by the inequality

$$n\mathcal{F}+\nabla_x\mathcal{F}\cdot x-aD_s\mathcal{F}s-(a+1)\nabla_\xi\mathcal{F}\cdot\xi\geq 0$$

where we have chosen $a=(n-p)/p$ (see [117, Theorem 1]). □

Corollary 9.21. *Let Ω be star-shaped with respect to the origin, $\lambda\leq 0$ and*

$$p^*\nabla_x\mathcal{L}(x,s,\xi)\cdot x-nD_s\mathcal{L}(x,s,\xi)s\geq 0,\tag{9.31}$$

for a.e. $x\in\Omega$ and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$. Then $(\mathcal{P}_{0,\lambda})$ admits no nontrivial solution $u\in C^1(\overline{\Omega})$.

Proof. Since $q<p^*$ and $\lambda\leq 0$, condition (9.31) implies condition (9.30). □

Assume that $\lambda \leq 0$ and $\mathcal{L}$ does not depend on x. Then, by the previous result, the non-existence condition becomes $D_s\mathcal{L}(s, \xi)s \leq 0$. Note that this is precisely the contrary of our assumption (6.40). Then, from this point of view (6.40) seems to be natural.

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MARCO SQUASSINA
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA
VIA R. COZZI 53, I-20125 MILANO, ITALY
E-mail address: marco.squassina@unimib.it