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Research Article

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The Brezis–Nirenberg problem for nonlocal systems

Abstract: By means of variational methods we investigate existence, nonexistence as well as regularity of weak solutions for a system of nonlocal equations involving the fractional laplacian operator and with nonlinearity reaching the critical growth and interacting, in a suitable sense, with the spectrum of the operator.

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1 Introduction and results

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. In 1983, Brezis and Nirenberg, in the seminal paper [3], showed that the critical growth semi-linear problem

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

admits a solution provided that $\lambda \in (0, \lambda_1)$ and $N \geq 4$, λ_1 being the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions and $2^* = 2N/(N-2)$ the critical Sobolev exponent. Furthermore, in dimension $N = 3$, the same existence result holds provided that $\mu < \lambda < \lambda_1$, for a suitable $\mu > 0$ (if Ω is a ball, then $\mu = \lambda_1/4$ is sharp). By Pohožaev identity, if $\lambda \notin (0, \lambda_1)$ and Ω is a star-shaped domain, then problem (1.1) admits no solution. Later on, in 1984, Cerami, Fortunato and Struwe obtained in [6] multiplicity results for the nontrivial solutions of

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

when λ belongs to a left neighborhood of an eigenvalue of $-\Delta$. In 1985, Capozzi, Fortunato and Palmieri proved in [5] the existence of a nontrivial solution of (1.2) for all $\lambda > 0$ and $N \geq 5$ or for $N \geq 4$ and λ different from an eigenvalue of $-\Delta$. Let $s \in (0, 1)$ and $N > 2s$. The aim of this paper is to obtain a Brezis–Nirenberg-type

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result for the fractional system

$$\begin{cases} (-\Delta)^s u = au + bv + \dfrac{2p}{p+q}|u|^{p-2}u|v|^q & \text{in } \Omega, \\ (-\Delta)^s v = bu + cv + \dfrac{2q}{p+q}|u|^p|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where $(-\Delta)^s$ is defined, on smooth functions, by

$$(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N,$$

$C(N,s)$ being a suitable positive constant and $p, q > 1$ are such that $p+q$ is compared to $2_s^* := 2N/(N-2s)$, the fractional critical Sobolev exponent [8]. The corresponding system in the local case was studied in [1]. For positive solutions, system (1.3) turns into

$$\begin{cases} (-\Delta)^s u = au + bv + \dfrac{2p}{p+q}u^{p-1}v^q & \text{in } \Omega, \\ (-\Delta)^s v = bu + cv + \dfrac{2q}{p+q}u^p v^{q-1} & \text{in } \Omega, \\ u > 0, \quad v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

In the following we shall assume that Ω is a smooth bounded domain of $\mathbb{R}^N$ with $N > 2s$ and we shall denote by $(\lambda_{i,s})$ the sequence of eigenvalues of $(-\Delta)^s$ with homogeneous Dirichlet-type boundary condition and by μ_1 and μ_2 the real eigenvalues of the matrix

$$A := \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, b, c \in \mathbb{R}.$$

Without loss of generality, we will assume $\mu_1 \le \mu_2$. By solution we shall always mean *weak* solution in the sense specified in Section 2, where the functional space $X(\Omega)$ is fully described. It is known that the first eigenvalue $\lambda_{1,s}$ is positive, simple and characterized by

$$\lambda_{1,s} = \inf_{u \in X(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\, dx}{\int_{\mathbb{R}^N} |u|^2\, dx}. \tag{1.5}$$

The following are the main results of the paper.

Theorem 1.1 (Existence I). *Assume that $b \ge 0$, $\mu_2 < \lambda_{1,s}$ and*

$$p + q < 2_s^*.$$

Then (1.4) *admits a solution.*

Theorem 1.2 (Existence II). *Assume that $b \ge 0$ and*

$$p + q = 2_s^*.$$

Then the following facts hold:

(1) *If $N \ge 4s$ and $0 < \mu_1 \le \mu_2 < \lambda_{1,s}$, then* (1.4) *admits a solution.*
(2) *If $2s < N < 4s$, there is $\bar{\mu} > 0$ such that* (1.4) *admits a solution if $\bar{\mu} < \mu_1 \le \mu_2 < \lambda_{1,s}$.*

Theorem 1.3 (Nonexistence). *Assume that $p + q = 2_s^*$ and one of the following facts hold:*
(1) *Ω is star-shaped with respect to the origin and $\mu_2 < 0$.*
(2) *Ω is star-shaped with respect to the origin and A is the zero matrix.*
(3) *$b \geq 0$ and $\mu_1 \geq \lambda_{1,s} - |a - c|$.*
Then (1.4) *does not admit any solution. Furthermore, if $\mu_2 \leq 0$ and*

$$p + q > 2_s^*,$$

then (1.4) *does not admit any bounded solution if Ω is star-shaped with respect to the origin.*

Theorem 1.4 (Regularity). *Assume that $p + q \leq 2_s^*$. If (u, v) is a solution to problem* (1.3)*, then $u, v \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for $s \in (0, 1/2)$ and $u, v \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ for $s \in (1/2, 1)$. In particular, (u, v) solves* (1.3) *in the classical sense.*

The nonexistence result stated in (3) of Theorem 1.3 holds in any bounded domain. For $b = 0$ it reads as $\mu_2 \geq \lambda_{1,s}$, properly complementing the assertions of Theorem 1.2. The above results provide a full extension of the classical results of Brezis and Nirenberg [3] for the local case $s = 1$. We point out that we adopt in the paper the integral definition of the fractional laplacian in a bounded domain and we do not exploit any *localization procedure* based upon the Caffarelli–Silvestre extension [4], as done e.g. in [2]. See [13] for a nice comparison between these two different notions of fractional laplacian in bounded domains. By choosing $p = q = 2_s^*/2$, system (1.4) reduces to

$$\begin{cases} (-\Delta)^s u = au + bv + u^{2s/(N-2s)} v^{N/(N-2s)} & \text{in } \Omega, \\ (-\Delta)^s v = bu + cv + u^{N/(N-2s)} v^{2s/(N-2s)} & \text{in } \Omega, \\ u > 0, \quad v > 0, & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.6}$$

which, in the particular case of $a = c$, setting $u = v$, boils down to the scalar equation

$$\begin{cases} (-\Delta)^s u = \lambda u + u^{2_s^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.7}$$

which is the natural fractional counterpart for the classical Brezis–Nirenberg problem [3]. For existence results for this problem, we refer to [12, 14] and to the references therein.

2 Preliminary stuff

2.1 Notations and setting

We refer the reader to [8] for further details about the functional framework that follows. For any measurable function $u : \mathbb{R}^N \to \mathbb{R}$ we define the Gagliardo seminorm by setting

$$[u]_s := \left(\frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \right)^{\frac{1}{2}}.$$

The second equality follows by [8, Proposition 3.6] when the above integrals are finite. Then, we introduce the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\}, \quad \|u\|_{H^s} = (\|u\|_{L^2}^2 + [u]_s^2)^{\frac{1}{2}},$$

which is a Hilbert space, and we consider the closed subspace

$$X(\Omega) := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}. \tag{2.1}$$

Due to the fractional Sobolev inequality, $X(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle_X := \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy, \tag{2.2}$$

which induces the norm $\|\cdot\|_X = [\,\cdot\,]_s$. Now, we consider the Hilbert space given by the product

$$Y(\Omega) := X(\Omega) \times X(\Omega), \tag{2.3}$$

equipped with the inner product

$$\langle (u,v), (\varphi,\psi)\rangle_Y := \langle u,\varphi\rangle_X + \langle v,\psi\rangle_X \tag{2.4}$$

and the norm

$$\|(u,v)\|_Y := (\|u\|_X^2 + \|v\|_X^2)^{\frac{1}{2}}. \tag{2.5}$$

We shall consider $L^m(\Omega) \times L^m(\Omega)$ $(m > 1)$ equipped with the standard product norm

$$\|(u,v)\|_{L^m \times L^m} := (\|u\|_{L^m}^2 + \|v\|_{L^m}^2)^{\frac{1}{2}}. \tag{2.6}$$

We recall that we have

$$\mu_1 |U|^2 \le (AU, U)_{\mathbb{R}^2} \le \mu_2 |U|^2 \quad \text{for all } U := (u,v) \in \mathbb{R}^2. \tag{2.7}$$

In this paper, we consider the following notation for product space $\mathfrak{F} \times \mathfrak{F} := \mathfrak{F}^2$ and set

$$w^+(x) := \max\{w(x), 0\}, \quad w^-(x) := \min\{w(x), 0\},$$

for positive and negative part of a function w. Consequently, we get $w = w^+ + w^-$. During chains of inequalities, universal constants will be denoted by the same letter C even if their numerical value may change from line to line.

2.2 Weak solutions

Consider the system

$$\begin{cases} (-\Delta)^s u = f(u,v) & \text{in } \Omega, \\ (-\Delta)^s v = g(u,v) & \text{in } \Omega, \\ \quad u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.8}$$

where $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory mappings which satisfies, respectively, the growths conditions

$$|f(z,w)| \le C(1 + |z|^{2_s^*-1} + |w|^{2_s^*-1}) \quad \text{for all } (z,w) \in \mathbb{R}^2, \tag{2.9}$$

$$|g(z,w)| \le C(1 + |z|^{2_s^*-1} + |w|^{2_s^*-1}) \quad \text{for all } (z,w) \in \mathbb{R}^2. \tag{2.10}$$

Definition 2.1. We say that $(u,v) \in Y(\Omega)$ is a weak solutions of (2.8) if

$$\langle (u,v), (\varphi,\psi)\rangle_Y = \int_\Omega f(u,v)\varphi\,dx + \int_\Omega g(u,v)\psi\,dx \tag{2.11}$$

for all $(\varphi,\psi) \in Y(\Omega)$.

2.3 A priori bounds

We introduce some notation: for all $t \in \mathbb{R}$ and $k > 0$, we set

$$t_k := \operatorname{sgn}(t)\min\{|t|, k\}. \tag{2.12}$$

From [9, Lemma 3.1] we recall the following lemma.

Lemma 2.2. *For all $a, b \in \mathbb{R}$, $r \geq 2$, and $k > 0$ we have*

$$(a-b)(a|a|_k^{r-2} - b|b|_k^{r-2}) \geq \frac{4(r-1)}{r^2}\Big(a|a|_k^{\frac{r}{2}-1} - b|b|_k^{\frac{r}{2}-1}\Big)^2.$$

In the following, we prove an L^∞-bound on the weak solutions of (2.8) which will be needed in order to get nonexistence and regularity results.

Lemma 2.3. *Assume that f and g satisfy* (2.9)–(2.10) *and let $(u, v) \in Y(\Omega)$ be a weak solution to* (2.8). *Then we have $u, v \in L^\infty(\Omega)$.*

Proof. For all $r \geq 2$ and $k > 0$, the map $t \mapsto t|t|_k^{r-2}$ is Lipschitz in $\mathbb{R}$. Then

$$(u|u|_k^{r-2}, 0) \in Y(\Omega), \quad (0, v|v|_k^{r-2}) \in Y(\Omega).$$

We test equation (2.11) with $(u|u|_k^{r-2}, 0)$, we apply the fractional Sobolev inequality, Young's inequality, Lemma 2.2, and use (2.9) to end up with

$$\begin{aligned}
\|u|u|_k^{\frac{r}{2}-1}\|_{L^{2_s^*}}^2 &\leq C\|u|u|_k^{\frac{r}{2}-1}\|_X^2 \\
&\leq \frac{Cr^2}{r-1}\langle u, u|u|_k^{r-2}\rangle_X \\
&\leq Cr \int_\Omega |f(u,v)||u||u|_k^{r-2}\,dx \\
&\leq Cr \int_\Omega (|u||u|_k^{r-2} + |u|^{2_s^*}|u|_k^{r-2} + |v|^{2_s^*-1}|u||u|_k^{r-2})\,dx \\
&\leq Cr \int_\Omega (|u|^{r-1} + |u|^{2_s^*+r-2} + |v|^{2_s^*+r-2})\,dx
\end{aligned} \tag{2.13}$$

for some $C > 0$ independent of $r \geq 2$ and $k > 0$. Then, Fatou's lemma, as $k \to \infty$, yields

$$\|u\|_{L^{\gamma^2 r}}^r \leq Cr\bigg(\int_\Omega (|u|^{r-1} + |u|^{2_s^*+r-2} + |v|^{2_s^*+r-2})\,dx\bigg), \tag{2.14}$$

where $\gamma = (2_s^*/2)^{\frac{1}{2}}$ (the right-hand side may at this stage be ∞). Now, in a similar way, test (2.11) with $(0, v|v|_k^{r-2})$ to obtain for some $C > 0$ independent of $r \geq 2$ that

$$\|v\|_{L^{\gamma^2 r}}^r \leq Cr\bigg(\int_\Omega (|v|^{r-1} + |u|^{2_s^*+r-2} + |v|^{2_s^*+r-2})\,dx\bigg) \tag{2.15}$$

(the right-hand side may be ∞). By (2.14) and (2.15) we get

$$\|u\|_{L^{\gamma^2 r}}^r + \|v\|_{L^{\gamma^2 r}}^r \leq Cr\bigg(\int_\Omega (|u|^{r-1} + |v|^{r-1} + |u|^{2_s^*+r-2} + |v|^{2_s^*+r-2})\,dx\bigg). \tag{2.16}$$

Our aim is to develop a suitable bootstrap argument to prove that $u, v \in L^p(\Omega)$ for all $p \geq 1$. We start from (2.16), with $r = 2_s^* + 1 > 2$, and fix $\sigma > 0$ such that $Cr\sigma < \frac{1}{2}$. Then there exists a constant $K_0 > 0$ (depending on u and v) such that

$$\bigg(\int_{\{|u|>K_0\}} |u|^{2_s^*}\,dx\bigg)^{1-\frac{2}{2_s^*}} + \bigg(\int_{\{|v|>K_0\}} |v|^{2_s^*}\,dx\bigg)^{1-\frac{2}{2_s^*}} \leq \sigma. \tag{2.17}$$

By Hölder's inequality and (2.17) we have

$$\begin{aligned}
\int_\Omega |u|^{2_s^*+r-2}\,dx &\leq K_0^{2_s^*+r-2}|\{|u| \leq K_0\}| + \int_{\{|u|>K_0\}} |u|^{2_s^*+r-2}\,dx \\
&\leq K_0^{2_s^*+r-2}|\Omega| + \bigg(\int_\Omega (u^r)^{\frac{2_s^*}{2}}\,dx\bigg)^{\frac{2}{2_s^*}}\bigg(\int_{\{|u|>K_0\}} |u|^{2_s^*}\,dx\bigg)^{1-\frac{2}{2_s^*}} \\
&\leq K_0^{2_s^*+r-2}|\Omega| + \sigma\|u\|_{L^{\gamma^2 r}}^r
\end{aligned} \tag{2.18}$$

and

$$\int_\Omega |v|^{2_s^*+r-2}\,dx \le K_0^{2_s^*+r-2}|\Omega| + \sigma\|v\|^r_{L^{\gamma^2 r}}. \tag{2.19}$$

By (2.16), (2.18) and (2.19), we have

$$\frac{1}{2}(\|u\|^r_{L^{\gamma^2 r}} + \|v\|^r_{L^{\gamma^2 r}}) \le Cr\bigg(\int_\Omega (|u|^{r-1} + |v|^{r-1})\,dx + K_0^{2_s^*+r-2}\bigg). \tag{2.20}$$

Since $r = 2_s^* + 1$, we get $u, v \in L^{\frac{2_s^*(2_s^*+1)}{2}}(\Omega)$. We define a sequence $\{r_n\}$ with

$$r_0 = 2_s^* + 1, \quad r_{n+1} = \gamma^2 r_n - 2_s^* + 2.$$

Since

$$2_s^* + r_0 - 2 < \frac{2_s^*(2_s^*+1)}{2},$$

we get

$$\|u\|_{L^{2_s^*+r_0-2}} + \|v\|_{L^{2_s^*+r_0-2}} < +\infty.$$

Hence, we aim to begin an iteration in order to get the L^∞-bounds of u and v. Using formula (2.16) and the Hölder inequality, we obtain

$$\begin{aligned}\|u\|_{L^{\gamma^2 r}} + \|v\|_{L^{\gamma^2 r}} &\le (Cr)^{\frac{1}{r}}\Big(|\Omega|^{\frac{2_s^*-1}{2_s^*+r-2}}(\|u\|^{r-1}_{L^{2_s^*+r-2}} + \|v\|^{r-1}_{L^{2_s^*+r-2}}) + (\|u\|^{2_s^*+r-2}_{L^{2_s^*+r-2}} + \|v\|^{2_s^*+r-2}_{L^{2_s^*+r-2}})\Big)^{\frac{1}{r}}\\ &\le (Cr)^{\frac{1}{r}}\Big((1+|\Omega|^{\frac{2_s^*-1}{2_s^*}})(\|u\|_{L^{2_s^*+r-2}} + \|v\|_{L^{2_s^*+r-2}})^{r-1} + (\|u\|_{L^{2_s^*+r-2}} + \|v\|_{L^{2_s^*+r-2}})^{2_s^*+r-2}\Big)^{\frac{1}{r}}.\end{aligned}$$

Substituting r_{n+1} for r, since $\gamma^2 r_n = 2_s^* + r_{n+1} - 2$, we get

$$\|u\|_{L^{\gamma^2 r_{n+1}}} + \|v\|_{L^{\gamma^2 r_{n+1}}} \le (Cr_{n+1})^{\frac{1}{r_{n+1}}}\Big(C(\|u\|_{L^{\gamma^2 r_n}} + \|v\|_{L^{\gamma^2 r_n}})^{r_{n+1}-1} + (\|u\|_{L^{\gamma^2 r_n}} + \|v\|_{L^{\gamma^2 r_n}})^{\gamma^2 r_n}\Big)^{\frac{1}{r_{n+1}}}. \tag{2.21}$$

Denote

$$T_n := \max\{1, \|u\|_{L^{\gamma^2 r_n}} + \|v\|_{L^{\gamma^2 r_n}}\}.$$

Then (2.21) can be written as

$$T_{n+1} \le (1+C)^{\frac{1}{r_{n+1}}} r_{n+1}^{\frac{1}{r_{n+1}}} T_n^{\frac{\gamma^2 r_n}{r_{n+1}}}. \tag{2.22}$$

Since $r_{n+1} = \gamma^2 r_n - 2_s^* + 2$, by induction it is possible to prove that

$$\frac{r_{n+1}}{\gamma^{2n+2}} = 2_s^* - 1 + 2\gamma^{-2n-2}, \quad n \in \mathbb{N}.$$

If $n = 0$, the assertion follows by a direct calculation. Assume now that the assertion holds for a given $n \ge 1$ and let us prove it for $n + 1$. We get

$$\begin{aligned}\frac{r_{n+2}}{\gamma^{2n+4}} &= \frac{r_{n+1}}{\gamma^{2n+2}} - \frac{2_s^*-2}{\gamma^{2n+4}}\\ &= 2_s^* - 1 + 2\gamma^{-2n-2} - \frac{2_s^*-2}{\gamma^{2n+4}}\\ &= 2_s^* - 1 + 2\gamma^{-2n-4},\end{aligned}$$

which proves the claim. In particular, $\frac{r_{n+1}}{\gamma^{2n+2}} \approx 2_s^* - 1$. From (2.22), we also have

$$\begin{aligned}T_{n+1} &\le (1+C)^{\frac{1}{r_{n+1}}} r_{n+1}^{\frac{1}{r_{n+1}}} T_n^{\frac{\gamma^2 r_n}{r_{n+1}}}\\ &\le (1+C)^{\frac{1}{r_{n+1}}} r_{n+1}^{\frac{1}{r_{n+1}}}\Big((1+C)^{\frac{1}{r_n}} r_n^{\frac{1}{r_n}} T_{n-1}^{\frac{\gamma^2 r_{n-1}}{r_n}}\Big)^{\frac{\gamma^2 r_n}{r_{n+1}}}\\ &= (1+C)^{\frac{1+\gamma^2}{r_{n+1}}} r_{n+1}^{\frac{1}{r_{n+1}}} r_n^{\frac{\gamma^2}{r_{n+1}}} T_{n-1}^{\frac{\gamma^4 r_{n-1}}{r_{n+1}}}\\ &\le \cdots \le (1+C)^{\frac{1+\gamma^2+\gamma^4+\cdots+\gamma^{2n}}{r_{n+1}}}\Big(r_{n+1}^{\frac{1}{r_{n+1}}} r_n^{\frac{\gamma^2}{r_{n+1}}} r_{n-1}^{\frac{\gamma^4}{r_{n+1}}} \cdots r_1^{\frac{\gamma^{2n}}{r_{n+1}}}\Big) T_0^{\frac{\gamma^{2n+2} r_0}{r_{n+1}}}\\ &= (1+C)^{\frac{\gamma^{2n+2}-1}{(\gamma^2-1)r_{n+1}}}\Big(\prod_{i=0}^n r_{i+1}^{\gamma^{2(n-i)}}\Big)^{\frac{1}{r_{n+1}}} T_0^{\frac{\gamma^{2n+2} r_0}{r_{n+1}}}.\end{aligned}$$

We can easily compute that

$$\frac{\gamma^{2n+2}-1}{(\gamma^2-1)r_{n+1}} \approx \frac{2}{(2_s^*-1)(2_s^*-2)}, \qquad \frac{\gamma^{2n+2}r_0}{r_{n+1}} \approx \frac{2_s^*+1}{2_s^*-1}.$$

Moreover, $r_{i+1} < r_0\gamma^{2i+2}$ for every $i \in \mathbb{N}$, since

$$\frac{r_{i+1}}{\gamma^{2i+2}} = \frac{r_i}{\gamma^{2i}} - \frac{2_s^*-2}{\gamma^{2i+2}} < \frac{r_i}{\gamma^{2i}} < \cdots < r_0,$$

and $r_{n+1} > \gamma^{2n+2}$ eventually for n large since $\frac{r_{n+1}}{\gamma^{2n+2}} \approx 2_s^* - 1 > 1$, so that

$$\Big(\prod_{i=0}^{n} r_{i+1}^{\gamma^{2(n-i)}}\Big)^{\frac{1}{r_{n+1}}} < \Big(\prod_{i=0}^{n}(r_0\gamma^{2i+2})^{\gamma^{2(n-i)}}\Big)^{\frac{1}{\gamma^{2n+2}}} \le r_0^{\sum_{i=0}^{\infty}\gamma^{-2i-2}}\gamma^{\sum_{i=0}^{\infty}\frac{2i+2}{\gamma^{2i+2}}} < +\infty.$$

Hence (T_n) remains uniformly bounded and the assertion follows. Notice that the L^∞-bound depends on T_0 which depends on u (and not only on $\|u\|_{2_s^*}$) through the presence of $K_0 > 0$ in estimate (2.20). □

3 Pohǒzaev identity and nonexistence

The purpose of this section is to prove Theorem 1.3, for this we need the following auxiliary result known as Pohǒzaev identity for systems involving the Laplacian fractional operator.

Lemma 3.1. *Let Ω be a bounded $C^{1,1}$ domain and let $F \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ be such that F_u and F_v satisfy the growth conditions* (2.9) *and* (2.10). *Let $(u, v) \in Y(\Omega)$ be a solution to the system*

$$\begin{cases} (-\Delta)^s u = F_u(u,v) & \text{in } \Omega,\\ (-\Delta)^s v = F_v(u,v) & \text{in } \Omega,\\ \quad u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.1}$$

Then $u, v \in C^s(\mathbb{R}^N)$, $u, v \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for $s \in (0, \frac{1}{2})$, $u, v \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ for $s \in (\frac{1}{2}, 1)$ and

$$\frac{u}{\delta^s}\Big|_\Omega, \frac{v}{\delta^s}\Big|_\Omega \in C^\alpha(\overline{\Omega}) \quad \text{for some } \alpha \in (0,1), \tag{3.2}$$

where $\delta(x) := \operatorname{dist}(x, \partial\Omega)$, meaning that $\frac{u}{\delta^s}|_\Omega$ and $\frac{v}{\delta^s}|_\Omega$ admit a continuous extension to $\overline{\Omega}$ which is $C^\alpha(\overline{\Omega})$. Moreover, the following identity holds:

$$\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2)\,dx - 2_s^*\int_\Omega F(u,v)\,dx + \frac{\Gamma(1+s)^2}{N-2s}\int_{\partial\Omega}\Big[\Big(\frac{u}{\delta^s}\Big)^2 + \Big(\frac{v}{\delta^s}\Big)^2\Big](x,\nu)_{\mathbb{R}^N}\,d\sigma = 0, \tag{3.3}$$

where Γ is the Gamma function.

Proof. In light of Lemma 2.3, we learn that $u, v \in L^\infty(\Omega)$. Then, $F_u(u,v)$ and $F_v(u,v)$ belong to $L^\infty(\Omega)$ too. In turn, by [10, Theorem 1.2 and Corollary 1.6], we have that u and v satisfy the regularity conclusions stated in (3.2). In particular, the system is satisfied in the classical sense. Whence, we are allowed to apply [11, Proposition 1.6] to both components u and v, obtaining

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\,dx = \frac{2s-N}{2}\int_\Omega u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\Big(\frac{u}{\delta^s}\Big)^2(x,\nu)_{\mathbb{R}^N}\,d\sigma,$$

$$\int_\Omega (x\cdot\nabla v)(-\Delta)^s v\,dx = \frac{2s-N}{2}\int_\Omega v(-\Delta)^s v\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\Big(\frac{v}{\delta^s}\Big)^2(x,\nu)_{\mathbb{R}^N}\,d\sigma.$$

Then, since $(-\Delta)^s u = F_u(u,v)$ and $(-\Delta)^s v = F_v(u,v)$ weakly in Ω and recalling that

$$\int_\Omega u(-\Delta)^s u\,dx = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\,dx, \qquad \int_\Omega v(-\Delta)^s v\,dx = \int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}v|^2\,dx,$$

we get

$$\int_\Omega (x\cdot\nabla u)F_u(u,v)\,dx = \frac{2s-N}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\Big(\frac{u}{\delta^s}\Big)^2(x,\nu)_{\mathbb{R}^N}\,d\sigma,$$
$$\int_\Omega (x\cdot\nabla v)F_v(u,v)\,dx = \frac{2s-N}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}v|^2\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\Big(\frac{v}{\delta^s}\Big)^2(x,\nu)_{\mathbb{R}^N}\,d\sigma.$$

Observing that $\nabla F(u,v)\cdot x = F_u(u,v)\nabla u\cdot x + F_v(u,v)\nabla v\cdot x$, integrating by parts we get

$$(2s-N)\int_{\mathbb{R}^N}\big(|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2\big)\,dx + 2N\int_\Omega F(u,v)\,dx = \Gamma(1+s)^2\int_{\partial\Omega}\Big[\Big(\frac{u}{\delta^s}\Big)^2 + \Big(\frac{v}{\delta^s}\Big)^2\Big](x,\nu)_{\mathbb{R}^N}\,d\sigma,$$

which concludes the proof. □

3.1 Proof of nonexistence

Consider first the case $p+q=2_s^*$ with assumption (1) and assume by contradiction that (1.4) admits a positive solution $(u,v)\in Y(\Omega)$. Consider the functions $f,g:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ defined by

$$f(z,w) = az + bw + \frac{2p}{p+q}z^{p-1}w^q,\quad g(z,w) = bz + cw + \frac{2q}{p+q}z^p w^{q-1}.$$

Then, setting

$$F(z,w) = \frac{a}{2}z^2 + bzw + \frac{c}{2}w^2 + \frac{2}{p+q}z^p w^q = \frac{1}{2}(A(z,w),(z,w))_{\mathbb{R}^2} + \frac{2}{p+q}z^p w^q,$$

we obtain that $F\in C^1(\mathbb{R}^+\times\mathbb{R}^+)$, $F_z = f$ and $F_w = g$ satisfy the growth conditions (2.9) and (2.10) and (u,v) is a weak solution to (2.8). Then, the components u,v enjoy the regularity (3.2) stated in Lemma 3.1 and identity (3.3) holds. Testing (2.11) with $(\varphi,\psi)=(u,v)$, yields

$$\begin{aligned}\int_{\mathbb{R}^N}\big(|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2\big)\,dx &= \int_\Omega f(u,v)u\,dx + \int_\Omega g(u,v)u\,dx\\ &= \int_\Omega (AU,U)_{\mathbb{R}^2}\,dx + 2\int_\Omega u^p v^q\,dx,\end{aligned}$$

which substituted in (3.3), yields, recalling that $p+q=2_s^*$,

$$\Big(1-\frac{2_s^*}{2}\Big)\int_\Omega (AU,U)_{\mathbb{R}^2}\,dx + \frac{\Gamma(1+s)^2}{N-2s}\int_{\partial\Omega}\Big[\Big(\frac{u}{\delta^s}\Big)^2 + \Big(\frac{v}{\delta^s}\Big)^2\Big](x,\nu)_{\mathbb{R}^N}\,d\sigma = 0.$$

Since Ω is star-shaped with respect to the origin, the equation above yields

$$\int_\Omega (AU,U)_{\mathbb{R}^2}\,dx \ge 0.$$

This is a contradiction with (2.7), because $\mu_2<0$ and $u,v>0$.

Now we cover case (2). If A is the zero matrix, we get

$$\int_{\partial\Omega}\Big[\Big(\frac{u}{\delta^s}\Big)^2 + \Big(\frac{v}{\delta^s}\Big)^2\Big](x,\nu)_{\mathbb{R}^N}\,d\sigma = 0,$$

which contradicts the fractional version of Hopf's lemma, see [9, Lemma 2.7], since $(-\Delta)^s u\ge 0$ and $(-\Delta)^s v\ge 0$ weakly yield $\frac{u}{\delta^s}\ge\omega$ and $\frac{v}{\delta^s}\ge\omega'$, for some positive constants ω,ω'.

Let us turn to case (3). If $\varphi_1 > 0$ is the first eigenfunction corresponding to $\lambda_{1,s}$ and we assume that a solution of (1.4) exists, by choosing $(\varphi_1, 0)$ and $(0, \varphi_1)$ respectively in (2.11), we get

$$\lambda_{1,s} \int_\Omega u\varphi_1\,dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi_1\,dx = \int_\Omega \Big(au\varphi_1 + bv\varphi_1 + \frac{2p}{p+q} u^{p-1} v^q \varphi_1 \Big)\,dx,$$
$$\lambda_{1,s} \int_\Omega v\varphi_1\,dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} \varphi_1\,dx = \int_\Omega \Big(bu\varphi_1 + cv\varphi_1 + \frac{2q}{p+q} u^{p} v^{q-1} \varphi_1 \Big)\,dx.$$

Then, since $b \geq 0$ and $u, v > 0$, we get

$$\lambda_{1,s} \int_\Omega u\varphi_1\,dx > a \int_\Omega u\varphi_1\,dx, \quad \lambda_{1,s} \int_\Omega v\varphi_1\,dx > c \int_\Omega v\varphi_1\,dx,$$

that is $\max\{a, c\} < \lambda_{1,s}$. On the other hand, by assumption and a direct calculation

$$\lambda_{1,s} - |a-c| \leq \mu_1 = \frac{(a+c) - \sqrt{(a-c)^2 + 4b^2}}{2} \leq \frac{(a+c) - |a-c|}{2} = \min\{a, c\},$$

which yields $\max\{a, c\} \geq \lambda_{1,s}$, namely a contradiction.

Finally, we prove the last assertion. In the case $p + q > 2_s^*$, any bounded solution of system (1.4) is smooth according to Lemma 3.1 and arguing as above yields the identity

$$\Big(1 - \frac{2_s^*}{2}\Big) \int_\Omega (AU, U)_{\mathbb{R}^2}\,dx + 2\Big(1 - \frac{2_s^*}{p+q}\Big) \int_\Omega u^p v^q\,dx + \frac{\Gamma(1+s)^2}{N-2s} \int_{\partial\Omega} \Big[\Big(\frac{u}{\delta^s}\Big)^2 + \Big(\frac{v}{\delta^s}\Big)^2 \Big] (x, \nu)_{\mathbb{R}^N}\,d\sigma = 0.$$

This yields $\int_\Omega (AU, U)_{\mathbb{R}^2}\,dx > 0$, contradicting $\mu_2 \leq 0$ via (2.7). This concludes the proof. □

Proof of Theorem 1.4. The assertion follows as a particular case of Lemma 3.1. □

4 Existence I, subcritical case

In this section, we will prove Theorem 1.1 which guarantees the existence of solutions for problem (1.4) involving subcritical nonlinearity.

4.1 Proof of existence I

Let Ω be a bounded domain and suppose that

$$b \geq 0, \tag{4.1}$$
$$\mu_2 < \lambda_{1,s}, \tag{4.2}$$
$$p + q < 2_s^*. \tag{4.3}$$

Consider the functional $I : Y(\Omega) \to \mathbb{R}$ defined by

$$I(U) := \frac{1}{2}\|U\|_Y^2 - \frac{1}{2} \int_\Omega (AU, U)_{\mathbb{R}^2}\,dx.$$

We shall minimize the functional I restricted to the set

$$\mathcal{M} := \Big\{ U = (u, v) \in Y(\Omega) : \int_\Omega (u^+)^p (v^+)^q\,dx = 1 \Big\}.$$

By virtue of (4.2) the embedding $X(\Omega) \hookrightarrow L^2(\Omega)$ (with the sharp constant $\lambda_{1,s}$), we have

$$I(U) \geq \frac{1}{2} \min\Big\{1, \Big(1 - \frac{\mu_2}{\lambda_{1,s}}\Big)\Big\} \|U\|_Y^2 \geq 0. \tag{4.4}$$

So define

$$I_0 := \inf_{\mathcal{M}} I,$$

and let $(U_n) = (u_n, v_n) \subset \mathcal{M}$ be a minimizing sequence for I_0. Then $I(U_n) = I_0 + o_n(1) \le C$ for some $C > 0$ (where $o_n(1) \to 0$, as $n \to \infty$) and consequently by (4.4), we get

$$[u_n]_s^2 + [v_n]_s^2 = \|u_n\|_X^2 + \|v_n\|_X^2 = \|U_n\|_Y^2 \le C'. \tag{4.5}$$

Hence, there are two subsequences of $(u_n) \subset X(\Omega)$ and $(v_n) \subset X(\Omega)$ (that we will still label as u_n and v_n) such that $U_n = (u_n, v_n)$ converges to some $U = (u, v)$ in $Y(\Omega)$ weakly and

$$[u]_s^2 \le \liminf_n \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy, \tag{4.6}$$

$$[v]_s^2 \le \liminf_n \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy. \tag{4.7}$$

Furthermore, in view of the compact embedding $X(\Omega) \hookrightarrow L^\sigma(\Omega)$ for all $\sigma < 2_s^*$ (cf. [8, Corollary 7.2]), we get that $U_n = (u_n, v_n)$ converges to (u, v) strongly in $(L^{p+q}(\Omega))^2$, as $n \to \infty$. Of course, up to a further subsequence, we have that $(u_n(x), v_n(x))$ converges to $(u(x), v(x))$ for a.e. $x \in \mathbb{R}^N$. Now we will show that $U := (u, v) \in \mathcal{M}$. Indeed, since $(U_n) \subset \mathcal{M}$, we have

$$\int_\Omega (u_n^+)^p (v_n^+)^q\,dx = 1. \tag{4.8}$$

Since

$$\lim_n \int_\Omega |u_n|^{p+q}\,dx = \int_\Omega |u|^{p+q}\,dx, \quad \lim_n \int_\Omega |v_n|^{p+q}\,dx = \int_\Omega |v|^{p+q}\,dx,$$

we have in particular $|u_n|^{p+q} \le \eta_1$ and $|v_n|^{p+q} \le \eta_2$, for some $\eta_i \in L^1(\Omega)$ and any $n \in \mathbb{N}$. Then

$$(u_n^+)^p(x)(v_n^+)^q(x) \le \frac{p}{p+q}|u_n(x)|^{p+q} + \frac{q}{p+q}|v_n(x)|^{p+q} \le \eta_1(x) + \eta_2(x) \quad \text{for a.e. in } \Omega.$$

In turn, by the Dominated Convergence Theorem, passing to the limit in (4.8), we obtain

$$\int_\Omega (u^+)^p (v^+)^q\,dx = 1,$$

and, consequently $U = (u, v) \in \mathcal{M}$ with $u, v \ne 0$. We now show that $U = (u, v)$ is, indeed, a minimizer for I on $\mathcal{M}$ and both the components u, v are nonnegative. By passing to the limit in $I(U_n) = I_0 + o_n(1)$, where $o_n(1) \to 0$ as $n \to \infty$, using (4.6) and (4.7) and the strong convergence of (u_n, v_n) to (u, v) in $(L^2(\Omega))^2$, as $n \to \infty$, we conclude that $I(U) \le I_0$. Moreover, since $U \in \mathcal{M}$ and $I_0 = \inf_{\mathcal{M}} I \le I(U)$, we achieve that $I(U) = I_0$. This proves the minimality of $U \in \mathcal{M}$. On the other hand, let

$$G(U) = \int_\Omega (u^+)^p (v^+)^q\,dx - 1,$$

where $U = (u, v) \in Y(\Omega)$. Note that $G \in C^1$ and since $U \in \mathcal{M}$,

$$G'(U)U = (p+q) \int_\Omega (u^+)^p (v^+)^q\,dx = p + q \ne 0,$$

hence, by the Lagrange Multiplier Theorem, there exists a multiplier $\mu \in \mathbb{R}$ such that

$$I'(U)(\varphi, \psi) = \mu G'(U)(\varphi, \psi) \quad \text{for all } (\varphi, \psi) \in Y(\Omega). \tag{4.9}$$

Taking $(\varphi, \psi) = (u^-, v^-) := U^-$ in (4.9), we get

$$\begin{aligned}\|U^-\|_Y^2 = {} & \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}}\,dx\,dy \\ & + \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2s}}\,dx\,dy + \int_\Omega (AU, U^-)_{\mathbb{R}^2}\,dx.\end{aligned}$$

Dropping this formula into the expression of $I(U^-)$, we have

$$I(U^-) = \frac{b}{2}\int_\Omega (v^+u^- + u^+v^-)\,dx + \frac{C(N,s)}{4}\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}}\,dx\,dy + \frac{C(N,s)}{4}\int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2s}}\,dx\,dy \le 0, \tag{4.10}$$

since $b \ge 0$, $w^- \le 0$ and $w^+ \ge 0$. Furthermore,

$$I(U^-) \ge \min\Big\{1, \Big(1 - \frac{\mu_2}{\lambda_1}\Big)\Big\}\|U^-\|_Y^2 \ge 0$$

and using (4.10), we get $U^- = (u^-, v^-) = (0,0)$ and therefore $u, v \ge 0$. We now prove the existence of a positive solution to (1.3). Using again (4.9), we see that

$$\|U\|_Y^2 - \int_\Omega (AU, U)_{\mathbb{R}^2}\,dx - \mu(p+q)\int_\Omega u^p v^q\,dx = 0,$$

and since $U \in \mathcal{M}$, we conclude that

$$I_0 = I(U) = \frac{\mu(p+q)}{2} > 0,$$

since I_0 is positive, via (4.2). Then, by (4.9), U satisfies the following system, weakly:

$$\begin{cases} (-\Delta)^s u = au + bv + \dfrac{2pI_0}{p+q}u^{p-1}v^q & \text{in } \Omega,\\ (-\Delta)^s v = bu + cv + \dfrac{2qI_0}{p+q}u^p v^{q-1} & \text{in } \Omega,\\ u = v = 0 & \text{in } \mathbb{R}^N\setminus\Omega. \end{cases}$$

Now using the homogeneity of system, we get $\tau > 0$ such that $W = (I_0)^\tau U$ is a solution of (1.4). Since $b \ge 0$ and $u, v \ge 0$, we get, in the weak sense,

$$\begin{cases} (-\Delta)^s u \ge au & \text{in } \Omega,\\ (-\Delta)^s v \ge cv & \text{in } \Omega,\\ u \ge 0, \quad v \ge 0 & \text{in } \Omega,\\ u = v = 0 & \text{in } \mathbb{R}^N\setminus\Omega. \end{cases}$$

By the strong maximum principle (cf. [9, Theorem 2.5]), we conclude $u, v > 0$ in Ω. □

5 Existence II, critical case

Next we turn to Theorem 1.2, for the critical case $p + q = 2_s^*$. The variational tool used is the Mountain Pass Theorem. The embedding $X(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$ is *not* compact, but we will show that, below a certain level c, the associated functional satisfies the Palais–Smale condition.

5.1 Preliminary results

We will make use of the definition

$$\mathcal{S}_s := \inf_{u \in X(\Omega)\setminus\{0\}} \mathcal{S}_s(u), \tag{5.1}$$

where

$$\mathcal{S}_s(u) := \Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\,dx\Big)\Big(\int_{\mathbb{R}^N} |u(x)|^{2_s^*}\,dx\Big)^{-\frac{2}{2_s^*}} \tag{5.2}$$

is the associated Rayleigh quotient. We also define the following related minimizing problems:

$$\mathcal{S}_{p+q}(\Omega) := \inf_{u\in X(\Omega)\setminus\{0\}}\left(\left(\int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\,dx\right)\left(\int\limits_{\mathbb{R}^N} |u(x)|^{p+q}\,dx\right)^{-\frac{2}{p+q}}\right) \tag{5.3}$$

and

$$\widetilde{\mathcal{S}}_{p,q}(\Omega) := \inf_{u,v\in X(\Omega)\setminus\{0\}}\left(\left(\int\limits_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2)\,dx\right)\left(\int\limits_{\mathbb{R}^N} |u(x)|^p|v(x)|^q\,dx\right)^{-\frac{2}{p+q}}\right). \tag{5.4}$$

We shall also agree that

$$\mathcal{S}_s = \mathcal{S}_{p+q}(\Omega), \quad \widetilde{\mathcal{S}}_s := \widetilde{\mathcal{S}}_{p,q}(\Omega), \quad \text{if } p+q = 2_s^*.$$

The following result, in the local case, was proved in [1]. The proof follows by arguing as it was made in [1], but, for the sake of completeness, we present its proof.

Lemma 5.1. *Let Ω be a domain, not necessarily bounded, and $p+q \le 2_s^*$. Then*

$$\widetilde{\mathcal{S}}_{p,q}(\Omega) = \left[\left(\frac{p}{q}\right)^{\frac{q}{p+q}} + \left(\frac{p}{q}\right)^{\frac{-p}{p+q}}\right]\mathcal{S}_{p+q}(\Omega). \tag{5.5}$$

Moreover, if w_0 realizes $\mathcal{S}_{p+q}(\Omega)$ then (Bw_0, Cw_0) realizes $\widetilde{\mathcal{S}}_{p,q}(\Omega)$, for all positive constants B and C such that $B/C = \sqrt{p/q}$.

Proof. Let $\{w_n\} \subset X(\Omega)\setminus\{0\}$ be a minimizing sequence for $\mathcal{S}_{p+q}(\Omega)$. Define $u_n := sw_n$ and $v_n := tw_n$, where $s, t > 0$ will be chosen later on. By definition (5.4), we get

$$g\left(\frac{s}{t}\right)\left(\int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w_n|^2\,dx\right)\left(\int\limits_{\mathbb{R}^N} |w_n(x)|^{p+q}\,dx\right)^{-\frac{2}{p+q}} \ge \widetilde{\mathcal{S}}_{p,q}(\Omega), \tag{5.6}$$

where $g:\mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$g(x) := x^{\frac{2q}{p+q}} + x^{-\frac{2p}{p+q}}, \quad x > 0.$$

The minimum value is assumed by g at the point $x = \sqrt{p/q}$, and it is given by

$$g\left(\sqrt{\frac{p}{q}}\right) = \left(\left(\frac{p}{q}\right)^{\frac{q}{p+q}} + \left(\frac{p}{q}\right)^{-\frac{p}{p+q}}\right).$$

Whence, by choosing s, t in (5.6) so that $s/t = \sqrt{p/q}$, and passing to the limit, we obtain

$$\widetilde{\mathcal{S}}_{p,q}(\Omega) \le g\left(\sqrt{\frac{p}{q}}\right)\mathcal{S}_{p+q}(\Omega).$$

In order to prove the reverse inequality, let $\{(u_n, v_n)\} \subset (X(\Omega)\setminus\{0\})^2$ be a minimizing sequence for $\widetilde{\mathcal{S}}_{p,q}(\Omega)$ and define $z_n := s_n v_n$ for some $s_n > 0$ such that

$$\int\limits_{\mathbb{R}^N} |u_n|^{p+q}\,dx = \int\limits_{\mathbb{R}^N} |z_n|^{p+q}\,dx.$$

Then, by Young's inequality, we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^N} |u_n|^p|z_n|^q\,dx &\le \frac{p}{p+q}\int\limits_{\mathbb{R}^N} |u_n|^{p+q}\,dx + \frac{q}{p+q}\int\limits_{\mathbb{R}^N} |z_n|^{p+q}\,dx\\ &= \int\limits_{\mathbb{R}^N} |u_n|^{p+q}\,dx\\ &= \int\limits_{\mathbb{R}^N} |z_n|^{p+q}\,dx.\end{aligned} \tag{5.7}$$

Thus, using (5.7), we obtain

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u_n|^2+|(-\Delta)^{\frac{s}{2}}v_n|^2)\,dx\Big)\Big(\int_{\mathbb{R}^N}|u_n(x)|^p|v_n(x)|^q\,dx\Big)^{-\frac{2}{p+q}}\\
&\quad= s_n^{\frac{2q}{p+q}}\Big(\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u_n|^2+|(-\Delta)^{\frac{s}{2}}v_n|^2)\,dx\Big)\Big(\int_{\mathbb{R}^N}|u_n(x)|^p|z_n(x)|^q\,dx\Big)^{-\frac{2}{p+q}}\\
&\quad\ge s_n^{\frac{2q}{p+q}}\Big(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\,dx\Big)\Big(\int_{\mathbb{R}^N}|u_n(x)|^{p+q}\,dx\Big)^{-\frac{2}{p+q}}+s_n^{-\frac{2p}{p+q}}\Big(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}z_n|^2\,dx\Big)\Big(\int_{\mathbb{R}^N}|z_n(x)|^{p+q}\,dx\Big)^{-\frac{2}{p+q}}\\
&\quad\ge g(s_n)\mathcal{S}_{p+q}(\Omega)\\
&\quad\ge g\Big(\sqrt{\frac{p}{q}}\Big)\mathcal{S}_{p+q}(\Omega).
\end{aligned}$$

Therefore, letting $n\to\infty$ in the above inequality, we get the reverse inequality, as desired. From (5.5), the last assertion immediately follows and the proof is concluded. □

From [7, Theorem 1.1], we learn that $\mathcal{S}_s$ is attained. Precisely $\mathcal{S}_s=\mathcal{S}_s(\tilde{u})$, where

$$\tilde{u}(x)=\frac{k}{(\mu^2+|x-x_0|^2)^{\frac{N-2s}{2}}},\qquad x\in\mathbb{R}^N,\ k\in\mathbb{R}\setminus\{0\},\ \mu>0,\ x_0\in\mathbb{R}^N. \tag{5.8}$$

Equivalently,

$$\mathcal{S}_s=\inf_{\substack{u\in X(\Omega)\setminus\{0\}\\ \|u\|_{L^{2_s^*}}=1}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\,dx=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\overline{u}|^2\,dx,$$

where

$$\overline{u}(x)=\frac{\tilde{u}(x)}{\|\tilde{u}\|_{L^{2_s^*}}}.$$

In what follows, we suppose that, up to a translation, $x_0=0$ in (5.8). The function

$$u^*(x):=\overline{u}\Big(\frac{x}{\mathcal{S}_s^{\frac{1}{2s}}}\Big),\qquad x\in\mathbb{R}^N,$$

is a solution to the problem

$$(-\Delta)^s u=|u|^{2_s^*-2}u\quad\text{in }\mathbb{R}^N, \tag{5.9}$$

verifying the property

$$\|u^*\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*}=\mathcal{S}_s^{\frac{N}{2s}}. \tag{5.10}$$

Define the family of functions

$$U_\varepsilon(x)=\varepsilon^{-\frac{N-2s}{2}}u^*\Big(\frac{x}{\varepsilon}\Big),\qquad x\in\mathbb{R}^N.$$

Then U_ε is a solution of (5.9) and verifies, for all $\varepsilon>0$,

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}U_\varepsilon|^2\,dx=\int_{\mathbb{R}^N}|U_\varepsilon(x)|^{2_s^*}\,dx=\mathcal{S}_s^{\frac{N}{2s}}. \tag{5.11}$$

Fix $\delta>0$ such that $B_{4\delta}\subset\Omega$ and $\eta\in C^\infty(\mathbb{R}^N)$ a cut-off function such that $0\le\eta\le1$ in $\mathbb{R}^N$, $\eta=1$ in B_δ and $\eta=0$ in $B_{2\delta}^c=\mathbb{R}^N\setminus B_{2\delta}$, where $B_r=B_r(0)$ is the ball centered at origin with radius $r>0$. Now define the family of nonnegative truncated functions

$$u_\varepsilon(x)=\eta(x)U_\varepsilon(x),\qquad x\in\mathbb{R}^N, \tag{5.12}$$

and note that $u_\varepsilon\in X(\Omega)$. The following result was proved in [14] and it constitutes the natural fractional counterpart of those proved for the local case in [3].

Proposition 5.2. *Let $s \in (0,1)$ and $N > 2s$. Then the following facts hold:*
(a) *As $\varepsilon \to 0$,*

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \, dx \le \mathcal{S}_s^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{N-2s}).$$

(b) *As $\varepsilon \to 0$,*

$$\int_{\mathbb{R}^N} |u_\varepsilon(x)|^2 \, dx \ge \begin{cases} C_s \varepsilon^{2s} + \mathcal{O}(\varepsilon^{N-2s}) & \text{if } N > 4s, \\ C_s \varepsilon^{2s} |\log \varepsilon| + \mathcal{O}(\varepsilon^{2s}) & \text{if } N = 4s, \\ C_s \varepsilon^{N-2s} + \mathcal{O}(\varepsilon^{2s}) & \text{if } 2s < N < 4s. \end{cases}$$

Here C_s is a positive constant depending only on s.
(c) *As $\varepsilon \to 0$,*

$$\int_{\mathbb{R}^N} |u_\varepsilon(x)|^{2_s^*} \, dx = \mathcal{S}_s^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^N).$$

Consider now, for any $\lambda \ge 0$, the minimization problem

$$\mathcal{S}_{s,\lambda} := \inf_{v \in X(\Omega) \setminus \{0\}} \mathcal{S}_{s,\lambda}(v),$$

where

$$\mathcal{S}_{s,\lambda}(v) := \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx - \lambda \int_{\mathbb{R}^N} |v(x)|^2 \, dx \right) \left(\int_{\mathbb{R}^N} |v(x)|^{2_s^*} \, dx \right)^{-\frac{2}{2_s^*}}.$$

The following result was proved in [14, Propositions 21–22] for the first assertion, and in [12, Corollary 8] for the second assertion.

Proposition 5.3. *Let $s \in (0,1)$ and $N > 2s$. Then the following facts hold:*
(a) *For $N \ge 4s$,*

$$\mathcal{S}_{s,\lambda}(u_\varepsilon) < \mathcal{S}_s \quad \text{for all } \lambda > 0 \text{ and any } \varepsilon > 0 \text{ sufficiently small.}$$

(b) *For $2s < N < 4s$, there exists $\lambda_s > 0$ such that*

$$\mathcal{S}_{s,\lambda}(u_\varepsilon) < \mathcal{S}_s \quad \text{for all } \lambda > \lambda_s \text{ and any } \varepsilon > 0 \text{ sufficiently small.}$$

Proof. For the sake of the completeness, we sketch the proof.

Case: $N > 4s$. By Proposition 5.2, we infer

$$\mathcal{S}_{s,\lambda}(u_\varepsilon) \le \frac{\mathcal{S}_s^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{N-2s}) - \lambda C_s \varepsilon^{2s}}{\left(\mathcal{S}_s^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^N)\right)^{\frac{2}{2_s^*}}} \le \mathcal{S}_s + \mathcal{O}(\varepsilon^{N-2s}) - \lambda \widetilde{C}_s \varepsilon^{2s} \le \mathcal{S}_s + \varepsilon^{2s}(\mathcal{O}(\varepsilon^{N-4s}) - \lambda \widetilde{C}_s) < \mathcal{S}_s$$

for all $\lambda > 0$ and $\varepsilon > 0$ small enough and some $\widetilde{C}_s > 0$.

Case: $N = 4s$. We have

$$\begin{aligned} \mathcal{S}_{s,\lambda}(u_\varepsilon) &\le \frac{\mathcal{S}_s^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{N-2s}) - \lambda C_s \varepsilon^{2s} |\log \varepsilon| + \mathcal{O}(\varepsilon^{2s})}{\left(\mathcal{S}_s^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^N)\right)^{\frac{2}{2_s^*}}} \le \mathcal{S}_s + \mathcal{O}(\varepsilon^{2s}) - \lambda \widetilde{C}_s \varepsilon^{2s} |\log \varepsilon| \\ &\le \mathcal{S}_s + \varepsilon^{2s}(\mathcal{O}(1) - \lambda \widetilde{C}_s |\log \varepsilon|) < \mathcal{S}_s \end{aligned}$$

for all $\lambda > 0$ and $\varepsilon > 0$ small enough and some $\widetilde{C}_s > 0$.

Case: $2s < N < 4s$. We have

$$\mathcal{S}_{s,\lambda}(u_\varepsilon) \le \frac{\mathcal{S}_s^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{N-2s}) - \lambda C_s \varepsilon^{N-2s} + \mathcal{O}(\varepsilon^{2s})}{\left(\mathcal{S}_s^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^N)\right)^{\frac{2}{2_s^*}}} \le \mathcal{S}_s + \varepsilon^{N-2s}(\mathcal{O}(1) - \lambda \widetilde{C}_s) + \mathcal{O}(\varepsilon^{2s}) < \mathcal{S}_s$$

for all $\lambda > 0$ large enough ($\lambda \ge \lambda_s$), ε sufficiently small and some $\widetilde{C}_s > 0$.

This concludes the sketch. □

Even if it is not strictly necessary for the proof of our main result, we state the following corollary for possible future usage.

Corollary 5.4. *Suppose that μ_1 given in (2.7) is positive and let*

$$\widetilde{\mathcal{S}}_{s,A} = \inf_{u,v\, \in X(\Omega)\setminus\{0\}} \mathcal{S}_{s,A}(u,v),$$

where

$$\mathcal{S}_{s,A}(u,v) = \bigg(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2)\,dx - \int_{\mathbb{R}^N} (A(u(x),v(x)),(u(x),v(x)))_{\mathbb{R}^2}\,dx\bigg)\bigg(\int_{\mathbb{R}^N} |u(x)|^p|v(x)|^q\,dx\bigg)^{-\frac{2}{2_s^*}},$$

where $p+q = 2_s^*$. Then the following facts holds:
(a) *If $N \geq 4s$, then*

$$\widetilde{\mathcal{S}}_{s,A} < \widetilde{\mathcal{S}}_s.$$

(b) *For $2s < N < 4s$, there exists $\mu_{1,s} > 0$, such that if $\mu_1 > \mu_{1,s}$, we have*

$$\widetilde{\mathcal{S}}_{s,A} < \widetilde{\mathcal{S}}_s.$$

Proof. From Proposition 5.3, we have:
(a) For $N \geq 4s$, we have

$$\mathcal{S}_{s,\mu_1}(u_\varepsilon) < \mathcal{S}_s, \quad \text{if } \mu_1 > 0 \text{ and provided } \varepsilon > 0 \text{ is sufficiently small.}$$

(b) For $2s < N < 4s$, there exists $\mu_{1,s} > 0$, such that if $\mu_1 > \mu_{1,s}$, we have

$$\mathcal{S}_{s,\mu_1}(u_\varepsilon) < \mathcal{S}_s, \quad \text{provided } \varepsilon > 0 \text{ is sufficiently small.}$$

Let $B, C > 0$ be such that $\frac{B}{C} = \sqrt{\frac{p}{q}}$. From (2.7) and the above inequalities, we infer that

$$\begin{aligned}
\widetilde{\mathcal{S}}_{s,A} &\leq \mathcal{S}_{s,A}(Bu_\varepsilon, Cu_\varepsilon)\\
&\leq (B^2+C^2)\bigg(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\,dx - \mu_1\int_{\mathbb{R}^N} |u_\varepsilon(x)|^2\,dx\bigg)(B^pC^q)^{-\frac{2}{2_s^*}}\bigg(\int_{\mathbb{R}^N} |u_\varepsilon(x)|^{2_s^*}\,dx\bigg)^{-\frac{2}{2_s^*}}\\
&= \bigg[\Big(\frac{p}{q}\Big)^{\frac{q}{p+q}} + \Big(\frac{p}{q}\Big)^{-\frac{p}{p+q}}\bigg]\mathcal{S}_{s,\mu_1}(u_\varepsilon)\\
&< \bigg[\Big(\frac{p}{q}\Big)^{\frac{q}{p+q}} + \Big(\frac{p}{q}\Big)^{-\frac{p}{p+q}}\bigg]\mathcal{S}_s = \widetilde{\mathcal{S}}_s.
\end{aligned}$$

This concludes the proof. □

5.2 Proof of existence II

In order to get weak solutions to system (1.4), we now define the functional $J : Y(\Omega) \to \mathbb{R}$ by setting

$$J(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2)\,dx - \frac{1}{2}\int_{\mathbb{R}^N} (A(u,v),(u,v))_{\mathbb{R}^2}\,dx - \frac{2}{2_s^*}\int_{\mathbb{R}^N} (u^+)^p(v^+)^q\,dx,$$

whose Gateaux derivative is given by

$$\begin{aligned}
J'(u,v)(\varphi,\psi) &= \frac{C(N,s)}{2}\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y)) + (v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad - \int_\Omega (A(u,v),(\varphi,\psi))_{\mathbb{R}^2}\,dx - \frac{2p}{2_s^*}\int_\Omega (u^+)^{p-1}(v^+)^q\varphi\,dx - \frac{2q}{2_s^*}\int_\Omega (v^+)^{q-1}(u^+)^p\psi\,dx
\end{aligned} \tag{5.13}$$

for every $(\varphi,\psi) \in Y(\Omega)$. We shall observe that the weak solutions of problem (1.4) correspond to the critical points of the functional J. Under hypothesis $0 < \mu_1 \leq \mu_2 < \lambda_{1,s}$, our goal is to prove Theorem 1.2. We first show that J satisfies the Mountain Pass Geometry.

Proposition 5.5. *Suppose $\mu_2 < \lambda_{1,s}$. The functional J satisfies the following:*
(a) *there exist $\beta, \rho > 0$ such that $J(u,v) \geq \beta$ if $\|(u,v)\|_Y = \rho$,*
(b) *there exists $(e_1, e_2) \in Y(\Omega)\backslash\{(0,0)\}$ with $\|(e_1,e_2)\|_Y > \rho$ such that $J(e_1,e_2) \leq 0$.*

Proof. (a) By means of (2.7), using

$$(u^+)^p(v^+)^q \leq |u|^{p+q} + |v|^{p+q} = |u|^{2_s^*} + |v|^{2_s^*}$$

and the Poincaré inequality, we have

$$J(u,v) \geq \frac{1}{2}\Big(1 - \frac{\mu_2}{\lambda_{1,s}}\Big)\|(u,v)\|_Y^2 - C\|(u,v)\|_Y^{2_s^*},$$

where $C > 0$ is a constant.

(b) Choose $(\tilde{u}_0, \tilde{v}_0) \in Y(\Omega) \setminus \{(0,0)\}$ with $\tilde{u}_0 \geq 0$, $\tilde{v}_0 \geq 0$ a.e. and $\tilde{u}_0\tilde{v}_0 \neq 0$. Then

$$J(t\tilde{u}_0, t\tilde{v}_0) = \frac{t^2}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\tilde{u}_0|^2 + |(-\Delta)^{\frac{s}{2}}\tilde{v}_0|^2)\,dx - \frac{t^2}{2}\int_{\mathbb{R}^N} (A(\tilde{u}_0,\tilde{v}_0), (\tilde{u}_0,\tilde{v}_0))\,dx - \frac{2t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N} \tilde{u}_0^p\tilde{v}_0^q\,dx,$$

by choosing $t > 0$ sufficiently large, the assertion follows. This concludes the proof. □

Therefore, by the previous facts, from the Mountain Pass Theorem it follows that there exists a sequence $\{(u_n, v_n)\} \subset Y(\Omega)$, called $(\mathrm{PS})_c$-Palais–Smale sequence at level c, such that

$$J(u_n, v_n) \to c, \quad \|J'(u_n, v_n)\| \to 0, \tag{5.14}$$

where c is given by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1], Y(\Omega)) : \gamma(0) = (0,0) \text{ and } J(\gamma(1)) \leq 0\}.$$

Next we turn to the boundedness of $\{(u_n, v_n)\}$ in $Y(\Omega)$.

Lemma 5.6 (Boundedness). *The* (PS) *sequence $\{(u_n, v_n)\} \subset Y(\Omega)$ is bounded.*

Proof. We have for every $n \in \mathbb{N}$,

$$\begin{aligned}C + C\|(u_n,v_n)\|_Y &\geq J(u_n,v_n) - \frac{1}{2_s^*}J'(u_n,v_n)(u_n,v_n)\\ &= \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)\|(u_n,v_n)\|_Y^2 - \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)\int_{\mathbb{R}^N} (A(u_n,v_n),(u_n,v_n))_{\mathbb{R}^2}\,dx\\ &\geq \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)\Big(1 - \frac{\mu_2}{\lambda_{1,s}}\Big)\|(u_n,v_n)\|_Y^2.\end{aligned}$$

Since $\mu_2 < \lambda_{1,s}$, the assertion follows. □

The next result is useful to get nonnegative solutions as weak limits of Palais–Smale sequences. The same argument shows that a critical point of J corresponds to a nonnegative solution to (1.3).

Lemma 5.7. *Assume that $b \geq 0$ and $\mu_2 < \lambda_{1,s}$. Let $\{(u_n, v_n)\} \subset Y(\Omega)$ be a Palais–Smale sequence for the functional J. Then*

$$\lim_n \|(u_n^-, v_n^-)\|_Y = 0.$$

In particular, the weak limit (u,v) of the Palais–Smale sequence $\{(u_n, v_n)\}$ has nonnegative components.

Proof. By choosing $\varphi := u^- \in X(\Omega)$ and $\psi := v^- \in X(\Omega)$ as test functions in (5.13) and using the elementary inequality

$$(a-b)(a^- - b^-) \geq (a^- - b^-)^2 \quad \text{for all } a, b \in \mathbb{R},$$

we obtain

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(u^-(x)-u^-(y)) + (v(x)-v(y))(v^-(x)-v^-(y))}{|x-y|^{N+2s}}\,dx\,dy \geq \int_{\mathbb{R}^{2N}} \frac{(u^-(x)-u^-(y))^2 + (v^-(x)-v^-(y))^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Now, note that, since $b \geq 0$ and $w^- \leq 0$ and $w^+ \geq 0$, it holds

$$\int_{\mathbb{R}^N} (A(u,v),(u^-,v^-))_{\mathbb{R}^2}\,dx \leq \int_{\mathbb{R}^N} (A(u^-,v^-),(u^-,v^-))_{\mathbb{R}^2}\,dx.$$

In fact, we have

$$(A(u,v),(u^-,v^-))_{\mathbb{R}^2} = (A(u^-,v^-),(u^-,v^-))_{\mathbb{R}^2} + b((v^+)u^- + (u^+)v^-) \leq (A(u^-,v^-),(u^-,v^-))_{\mathbb{R}^2}.$$

In turn, from the formula for $J'(u,v)(u^-,v^-)$, it follows that

$$\begin{aligned} J'(u,v)(u^-,v^-) &\geq \frac{C(N,s)}{2}\int_{\mathbb{R}^{2N}} \frac{(u^-(x)-u^-(y))^2 + (v^-(x)-v^-(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \int_{\Omega} (A(u^-,v^-),(u^-,v^-))_{\mathbb{R}^2}\,dx \\ &\geq I(u^-) + I(v^-), \end{aligned}$$

where we have set

$$I(w) := \frac{C(N,s)}{2}\int_{\mathbb{R}^{2N}} \frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \mu_2 \int_{\Omega} |w|^2\,dx = [w]_s^2 - \mu_2\|w\|^2_{L^2(\Omega)}.$$

On the other hand, by the definition of $\lambda_{1,s}$, we have

$$I(w) \geq \Big(1 - \frac{\mu_2}{\lambda_{1,s}}\Big)[w]_s^2,$$

which finally yields the inequality

$$J'(u,v)(u^-,v^-) \geq \Big(1 - \frac{\mu_2}{\lambda_{1,s}}\Big)([u^-]_s^2 + [v^-]_s^2).$$

Since $\{(u_n,v_n)\} \subset Y(\Omega)$ is a Palais–Smale sequence, we get $J'(u_n,v_n)(u_n^-,v_n^-) = o_n(1)$, from which that assertion immediately follows. □

From the boundedness of Palais–Smale sequences (see Lemma 5.6) and compact embedding theorems, passing to a subsequence if necessary, there exists $(u_0,v_0) \in Y(\Omega)$ which, by Lemma 5.7, satisfies $u_0, v_0 \geq 0$, such that $(u_n,v_n) \rightharpoonup (u_0,v_0)$ weakly in $Y(\Omega)$ as $n \to \infty$, $(u_n,v_n) \to (u_0,v_0)$ a.e. in Ω and strongly in $L^r(\Omega)$ for $1 \leq r < 2_s^*$. Recalling that the sequences

$$w_n^1 := (u_n^+)^{p-1}(v_n^+)^q, \quad w_n^2 := (v_n^+)^{q-1}(u_n^+)^p, \quad p+q = 2_s^*,$$

are uniformly bounded in $L^{(2_s^*)'}(\Omega)$ and converge pointwisely to $w_0^1 = u_0^{p-1}v_0^q$ and $w_0^2 = v_0^{q-1}u_0^p$ respectively, we obtain

$$(w_n^1, w_n^2) \rightharpoonup (w_0^1, w_0^2), \quad \text{weakly in } L^{(2_s^*)'}(\Omega), \quad \text{as } n \to \infty.$$

Hence, passing to the limit in

$$J'(u_n,v_n)(\varphi,\psi) = o_n(1) \quad \text{for all } (\varphi,\psi) \in Y(\Omega), \quad \text{as } n \to \infty,$$

we infer that (u_0,v_0) is a nonnegative weak solution. Now, to conclude the proof, it is sufficient to prove that the solution is nontrivial.

Claim. *We have* $(u_0,v_0) \neq (0,0)$.

Notice that if (u_0,v_0) is a solution of system with $u_0 = 0$, then $v_0 = 0$. The same holds for the reversed situation.

In fact, suppose $u_0 = 0$. Then, if $b > 0$, it follows that $v_0 = 0$. If, instead, $b = 0$, then $c \in \{\mu_1, \mu_2\} < \lambda_{1,s}$. Since v_0 is a solution of equation

$$\begin{cases} (-\Delta)^s v_0 = c v_0 & \text{in } \Omega, \\ v_0 = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

we have that $v_0 = 0$. Therefore, we may suppose that $(u_0, v_0) = (0, 0)$. Define, as in [3],

$$L := \lim_{n\to\infty} \int\limits_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2)\, dx,$$

from $J'(u_n, v_n)(u_n, v_n) = o_n(1)$, we get

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} (u_n^+)^p (v_n^+)^q\, dx = \frac{L}{2}.$$

Recalling that $J(u_n, v_n) = c + o_n(1)$, thus

$$c = \frac{sL}{N}. \tag{5.15}$$

From the definition of (5.4), we have

$$\int\limits_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2)\, dx \ge \widetilde{\mathcal{S}}_s \bigg(\int\limits_{\mathbb{R}^N} |u_n(x)|^p |v_n(x)|^q\, dx \bigg)^{\frac{2}{2_s^*}}$$

and passing to the limit the inequality above, we get

$$L \ge \widetilde{\mathcal{S}}_s \Big(\frac{L}{2}\Big)^{\frac{2}{2_s^*}}.$$

Now, combining this estimate with (5.15), it follows that

$$c \ge \frac{2s}{N} \bigg(\frac{\widetilde{\mathcal{S}}_s}{2}\bigg)^{\frac{N}{2s}}. \tag{5.16}$$

Take $B, C > 0$ with $B/C = \sqrt{p/q}$ and let $u_\varepsilon \ge 0$ as in Proposition 5.2. Fix $\varepsilon > 0$ sufficiently small so that Proposition 5.3 holds and define $v_\varepsilon := u_\varepsilon / \|u_\varepsilon\|_{L^{2_s^*}}$. Using the definition of $\mathcal{S}_{s,\lambda}(u)$, for every $t \ge 0$, we obtain

$$\begin{aligned} J(tBv_\varepsilon, tCv_\varepsilon) &\le \frac{t^2(B^2 + C^2)}{2} \bigg(\int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2\, dx - \mu_1 \int\limits_{\mathbb{R}^N} |v_\varepsilon|^2\, dx \bigg) - \frac{2t^{2_s^*} B^p C^q}{2_s^*} \\ &= \frac{t^2(B^2 + C^2)}{2} \mathcal{S}_{s,\mu_1}(u_\varepsilon) - \frac{2t^{2_s^*} B^p C^q}{2_s^*} \\ &=: \psi(t), \quad t \ge 0. \end{aligned}$$

Thus, an elementary calculation yields

$$\psi_{\max} = \max_{\mathbb{R}^+} \psi = \frac{2s}{N} \bigg\{ \frac{(B^2 + C^2)}{2(B^p C^q)^{2/2_s^*}} \mathcal{S}_{s,\mu_1}(u_\varepsilon) \bigg\}^{\frac{N}{2s}}.$$

By Lemma 5.1 and Proposition 5.3, we conclude that, for $\varepsilon > 0$ small,

$$\psi_{\max} < \frac{2s}{N} \bigg\{ \frac{(B^2 + C^2)}{2(B^p C^q)^{2/2_s^*}} \mathcal{S}_s \bigg\}^{\frac{N}{2s}} = \frac{2s}{N} \bigg\{ \frac{1}{2} \bigg[\Big(\frac{p}{q}\Big)^{\frac{q}{p+q}} + \Big(\frac{p}{q}\Big)^{-\frac{p}{p+q}} \bigg] \mathcal{S}_s \bigg\}^{\frac{N}{2s}} = \frac{2s}{N} \bigg(\frac{\widetilde{\mathcal{S}}_s}{2} \bigg)^{\frac{N}{2s}}.$$

Let now $\gamma \in C([0, 1], Y(\Omega))$ be defined by

$$\gamma(t) := (\tau t B v_\varepsilon, \tau t C v_\varepsilon), \quad t \in [0, 1],$$

where $\tau > 0$ is sufficiently large so that $J(\tau B v_\varepsilon, \tau C v_\varepsilon) \le 0$. Hence, $\gamma \in \Gamma$ and we conclude that

$$c \le \sup_{t\in[0,1]} J(\gamma(t)) \le \sup_{t\ge 0} J(tBv_\varepsilon, tCv_\varepsilon) \le \psi_{\max} < \frac{2s}{N} \bigg(\frac{\widetilde{\mathcal{S}}_s}{2} \bigg)^{\frac{N}{2s}},$$

which contradicts (5.16). Hence $(u_0, v_0) \ne (0, 0)$ and the proof is complete. Finally, that $u_0 > 0$ and $v_0 > 0$ follows as in the sub-critical case. □

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