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Asymptotic symmetries for fractional operators

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In this paper, we study equations driven by a non-local integrodifferential operator $\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. More precisely, we study the problem

$$\begin{cases} -\mathcal{L}_K u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $2 < p < 2_s^* = \frac{2N}{N-2s}$, Ω is an open bounded domain in $\mathbb{R}^N$ for $N \geqslant 2$ and V is a L^∞ potential such that $-\mathcal{L}_K + V$ is positive definite. As a particular case, we study the problem

$$\begin{cases} (-\Delta)^s u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $(-\Delta)^s$ denotes the fractional Laplacian (with $0 < s < 1$). We give assumptions on V, Ω and K such that ground state solutions (resp. least energy nodal solutions) respect the symmetries of some first (resp. second) eigenfunctions of $-\mathcal{L}_K + V$, at least for p close to 2. We study the uniqueness, up to a multiplicative factor, of those types of solutions. The results extend those obtained for the local case.

1. Introduction

Non-local operators arise naturally in many different topics in physics, engineering and even finance. For examples, they have applications in crystal dislocation, soft thin films, obstacle problems [1,2], continuum

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mechanics [3], chaotic dynamics of classical conservative systems [4] and graph theory [5]. In this paper, we shall consider the non-local counterpart of semi-linear elliptic equations of the type

$$\begin{cases} -\Delta u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain with Lipschitz boundary, $2 < p < 2^*$ is a subcritical exponent (where $2^* := 2N/(N-2)$ if $N \geqslant 3$, $2^* = +\infty$ if $N = 2$) and $V \in L^\infty$ is such that $-\Delta + V$ is positive definite. Precisely, we are predominantly interested in the qualitative behavior of solutions to

$$\begin{cases} (-\Delta)^s u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.2}$$

where $(-\Delta)^s$ denotes the fractional Laplacian (with $0 < s < 1$) and $2 < p < 2_s^* := \frac{2N}{N-2s}$. Let us recall that, up to a normalization factor, $(-\Delta)^s$ may be defined [6] as follows: for $x \in \mathbb{R}^N$,

$$(-\Delta)^s u(x) := -c_{N,s} \lim_{\varepsilon \to 0} \int_{\complement B(x,\varepsilon)} \frac{u(y) - u(x)}{|y - x|^{N+2s}} \, \mathrm{d}y = -\tfrac{1}{2} c_{N,s} \int_{\mathbb{R}^N} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{N+2s}} \, \mathrm{d}y$$

where $c_{N,s} := s2^{2s}\, \Gamma(\frac{N+2s}{2}) / \big(\pi^{N/2}\, \Gamma(1-s)\big)$ is a positive constant chosen [7] to be coherent with the Fourier definition of $(-\Delta)^s$. This problem is variational and a ground state (resp. a least energy nodal solution) can be defined from the associated Euler–Lagrange functional—see [8] (resp. Section 2) for more details. In this paper, we would like to study the symmetries of those two types of variational solutions. In fact, we consider a more general setting: we are dealing with ground state and least energy nodal solutions to the following equation:

$$\begin{cases} -\mathcal{L}_K u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

where $\mathcal{L}_K$ is the non-local operator defined as follows

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} \big(u(x+y) - 2u(x) + u(x-y)\big) K(y) \, \mathrm{d}y.$$

We shall assume that $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ is a function such that $mK \in L^1(\mathbb{R}^N)$ where $m(x) := \min\{|x|^2, 1\}$ and we require the existence of $\theta > 0$ and $s \in (0, 1)$ such that $K(x) \geqslant \theta |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \backslash \{0\}$. We also require that $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \backslash \{0\}$. In particular, we can consider $K(x) = \frac{1}{2} c_{N,s} |x|^{-(N+2s)}$ so that $-\mathcal{L}_K$ is exactly the fractional Laplacian operator $(-\Delta)^s$ as defined in (1.1) and (1.3) boils down to (1.2).

Let us point out that, in the current literature, there are several notions of fractional Laplacian, all of which agree when the problems are set on the whole $\mathbb{R}^N$, but some of them disagree in a bounded domain. The values $(-\Delta)^s u(x)$ are, as we said, consistent with the Fourier definition of $(-\Delta)^s$, namely $\mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u)$ and also agree with the local formulation due to Caffarelli–Silvestre [9],

$$(-\Delta)^s u(x) = -C \lim_{t \to 0} \Big(t^{1-2s} \frac{\partial U}{\partial t}(x, t) \Big),$$

where $U : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}$ is the solution to $\operatorname{div}(t^{1-2s} \nabla U) = 0$ and $U(x, 0) = u(x)$. The fractional Laplacian defined in this way is also called *integral*. In a bounded domain Ω, as in [10], we choose to operate with it on restrictions to Ω of functions defined on $\mathbb{R}^N$ which are equal to zero on $\complement\Omega$. A different operator $(-\Delta)^s_{\text{spec}}$ called *regional, local* or *spectral* fractional Laplacian, largely utilized in literature, can be defined as the power of the Laplace operator $-\Delta$ via the spectral decomposition theorem. Let $(\lambda_k)_{k \geqslant 1}$ and $(e_k)_{k \geqslant 1}$

be the eigenvalues and eigenfunctions of $-\Delta$ in Ω with Dirichlet boundary condition on $\partial\Omega$, normalized in such a way that $|e_k|_2 = 1$. Then, for every $s \in (0,1)$ and all $u \in H_0^1(\Omega)$ with

$$u(x) = \sum_{k=1}^{\infty} \gamma_k e_k(x), \quad x \in \Omega,$$

one considers the operator

$$(-\Delta)^s_{\mathrm{spec}} u(x) = \sum_{j=1}^{\infty} \gamma_j \lambda_j^s e_j(x), \quad x \in \Omega.$$

Of course, in this way, the eigenfunctions of $(-\Delta)^s_{\mathrm{spec}}$ agree with the eigenfunction e_k of $-\Delta$. The operators $(-\Delta)^s$ and $(-\Delta)^s_{\mathrm{spec}}$ are different, in spite of the current literature where they are sometimes erroneously interchanged. In [11], the authors were able to recover also for the spectral fractional Laplacian the aforementioned local realization procedure. We refer the interested reader to [12] for a careful comparison of eigenvalues and eigenvectors of these two operators and to [13] for further discussions about the correlations among physically relevant nonlocal operators and the introduction of a notion of fractional Laplacian for Neumann boundary conditions.

Under the assumptions on K stated above, the Problem (1.3) is variational (see [8, Section 2]). The energy is defined on the space H of Lebesgue measurable functions $g : \mathbb{R}^N \to \mathbb{R}$ such that g is zero almost everywhere outside Ω, its restriction to Ω belongs to $L^2(\Omega)$ and, furthermore, the map $(x, y) \mapsto \big(g(x) - g(y)\big)\sqrt{K(x-y)} \in L^2\big(\mathbb{R}^{2N} \backslash (\complement\Omega \times \complement\Omega)\big)$ (we write $\complement\Omega := \mathbb{R}^N \backslash \Omega$). The inner product of H is defined as

$$\langle u, v\rangle_H := \int_Q \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) K(x-y)\, \mathrm{d}x\, \mathrm{d}y, \tag{1.4}$$

where $Q := \mathbb{R}^{2N} \backslash (\complement\Omega \times \complement\Omega)$ (see e.g. [8, Section 2] for more details on $\langle\cdot,\cdot\rangle_H$). The corresponding norm will be written $\|\cdot\|_H$. The existence of ground state solutions has been proved in [8] while the existence of least energy nodal solutions is established in this paper (see Section 2).

We now state the main results. For $k \geqslant 1$, we let λ_k (resp. φ_k) be the kth eigenvalue s counted without multiplicity (resp. eigenfunction) of the operator $-\mathcal{L}_K + V$ with "Dirichlet boundary conditions" in Ω in the sense that $\varphi_k = 0$ in $\complement\Omega$. We also consider E_k the eigenspace associated to λ_k.

Theorem 1.1. *Assume that $-\mathcal{L}_K + V$ is positive definite. If $(u_p)_{p>2}$ is a family of ground state (resp. least energy nodal) solutions to Problem (1.3), then*

$$\|u_p\|_H + |u_p|_2 \leqslant C\lambda_i^{1/(p-2)}, \quad \lambda_i = \lambda_1 \text{ (resp. } \lambda_2).$$

If $p_n \to 2$ and $\lambda_i^{1/(2-p_n)} u_{p_n} \rightharpoonup u_*$ in H (the weak convergence necessarily holds, up to a subsequence), then $\lambda_i^{1/(2-p_n)} u_{p_n} \to u_*$ in H and $u_* \neq 0$ satisfies

$$\begin{cases} -\mathcal{L}_K u_* + V u_* = \lambda_i u_*, & \text{in } \Omega, \\ u_* = 0, & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

Assume that λ_1 (resp. λ_2) is simple. Then, for p close to 2 and any reflection R such that $R(\Omega) = \Omega$, ground state solutions (resp. least energy nodal solutions) to Problem (1.3) possess the same symmetry or antisymmetry as φ_1 (resp. φ_2) with respect to R. Moreover, this type of solution is unique up to its sign.

The proof of the previous theorem makes use of the implicit function theorem (see Sections 3 and 4). In particular, since it is known that φ_1 is a positive eigenfunction when $V \equiv 0$ (see [8, Proposition 9, assertion (c)]) and thus λ_1 is simple. As we show in Lemma 3.2, λ_1 is also simple when $V \in L^\infty$ and $-\mathcal{L}_K + V$ is positive definite. In these cases, we have the following.

Corollary 1.2. *Assume that $-\mathcal{L}_K + V$ is positive definite. If $(u_p)_{p>2}$ are ground state solutions to*

$$\begin{cases} -\mathcal{L}_K u + Vu = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

then, for p close to 2 and any reflection R such that $R(\Omega) = \Omega$, ground state solutions of (1.5) possess the same symmetry or antisymmetry as φ_1 with respect to R. Moreover, this type of solution is unique up to its sign.

When λ_2 is not simple, then we cannot use the implicit function theorem. In this case, we just are able to conclude the following result which no longer asserts uniqueness (see Section 5).

Theorem 1.3. *Assume that $-\mathcal{L}_K + V$ is positive definite and that the zero set of any function in $E_2 \backslash \{0\}$ has zero Lebesgue measure. For p close to 2, least energy nodal solutions u_p to Problem (1.3) possess the same symmetries and antisymmetries of their orthogonal projection on E_2.*

Notice that the result is obtained for a general nonlocal operator and that the assumption about the zero sets is known to hold when $-\mathcal{L}_K = (-\Delta)^s$ (see [14]). We are also able to localize least energy nodal solutions when $p \approx 2$, see Theorem 5.5.

These theorems are a generalization of the corresponding results for the semi-linear case (1.1). In [15–18], this equation has been extensively studied for several choices of potentials V, boundary conditions and non-linearities. We refer the reader to the references therein.

In the final Section 6, we also present numerical experiments that have not been performed elsewhere in dimension higher than one. The ideas to obtain them are more than simply an adaptation of the ones present in the current literature.

In this paper, $|\cdot|_p$ will denote the traditional norm in L^p. The notation $f'(u)[v]$ stands for the Fréchet derivative of the function f at u in the direction v.

2. Formulation, ground state and least energy nodal solutions

In [8], it is proved that H is a Hilbert space when endowed with the norm

$$\|u\|_H^2 = \int_Q |u(x) - u(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Moreover, the embedding $H \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for $q \in [1, 2_s^*]$ and it is compact when $q < 2_s^*$. In particular, the Sobolev's inequality holds: there exists $C > 0$ such that, for any $u \in H$, $|u|_{2_s^*} \leqslant C\|u\|_H$.

At this point, we may define the functional

$$\mathcal{E}_p : H \to \mathbb{R} : u \mapsto \frac{1}{2}\|u\|_H^2 + \frac{1}{2}\int_\Omega V(x)u^2\,\mathrm{d}x - \frac{1}{p}|u|_p^p \tag{2.1}$$

whose corresponding Euler–Lagrange equation is the weak formulation of Problem (1.3) (see [19]): for $u \in H$,

$$\forall \varphi \in H, \quad \int_{\mathbb{R}^{2N}} \big(u(y) - u(x)\big)\big(\varphi(y) - \varphi(x)\big)K(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega V(x)u\varphi\,\mathrm{d}x = \int_\Omega |u|^{p-2}u\varphi\,\mathrm{d}x.$$

To establish the existence of ground state solutions, R. Servadei and E. Valdinoci [8] assume that $-\mathcal{L}_K + V$ is positive definite and make use of the traditional Mountain Pass Theorem. They minimize $\mathcal{E}_p$ on the following Nehari manifold

$$\mathcal{N}_p := \Big\{u \in H \backslash \{0\} : \int_{\mathbb{R}^{2N}} \big(u(x) - u(y)\big)^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega V(x)u^2\,\mathrm{d}x = \int_\Omega |u|^p\,\mathrm{d}x\Big\}. \tag{2.2}$$

Up to our knowledge, there is no characterization of sign-changing solutions mentioned in the literature for the non-local case. We prove hereafter that a least energy nodal solution exists and may be characterized as a minimum of $\mathcal{E}_p$ on the traditional nodal Nehari set

$$\mathcal{M}_p := \big\{u \in H\backslash\{0\} : u^\pm \neq 0 \text{ and } \mathcal{E}_p(u) = \max_{t^+,\ t^->0} \mathcal{E}_p(t_p^+ u^+ + t_p^- u^-)\big\}, \tag{2.3}$$

where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. Let us remark that an important technical difference with the classical local semilinear occurs: $\langle u^+, u^-\rangle_H = -2\int_\Omega\int_\Omega u^+(x)u^-(y)K(x-y)\,\mathrm{d}x\,\mathrm{d}y > 0$ for any sign-changing solutions $u \in H$.

Let us mention that $u \in \mathcal{M}_p$ is equivalent to $u^\pm \neq 0$ and $\mathcal{E}_p'(u)[u^\pm] = 0$. Indeed, it is clear that if $u \in \mathcal{M}_p$ then $\mathcal{E}_p'(u)[u^\pm] = 0$. For the other direction, let us first remark that, for any sign-changing function $u \in H$, there exists $t^+ > 0$ and $t^- > 0$ such that $t^+u^+ + t^-u^- \in \mathcal{M}_p$. This comes easily from the fact that $\mathcal{E}(u) \to -\infty$ when $\|u\|_H \to +\infty$ while being constrained to any finite dimensional subspace, and the fact that the maximum cannot be on the boundary of the cone because $\mathcal{E}_p'(u^+)[u^-] = \langle u^+, u^-\rangle_H > 0$ and $\mathcal{E}_p'(u^-)[u^+] > 0$. It is thus sufficient to show that the system $\mathcal{E}_p'(t^+u^+ + t^-u^-)[u^\pm] = 0$ has at most one non-trivial solution (t^+, t^-) with $t^\pm > 0$. Developing the equations $\mathcal{E}_p'(t^+u^+ + t^-u^-)[u^\pm] = 0$ leads to a system of the type

$$\begin{cases} t^- = A(t^+)^{p-1} - Bt^+, \\ t^+ = C(t^-)^{p-1} - Dt^-, \end{cases}$$

for some $A, B, C, D > 0$. As a quick drawing will convince you, one can see the solutions of this system as the intersection of two increasing functions of t^+, one that it super-quadratic and one that is sub-quadratic. Hence a single intersection exists.

Note that, contrarily to the local case, it is not true that $u^\pm \in \mathcal{N}_p$ if and only if $u^+ + u^- \in \mathcal{M}_p$. This is again a consequence of the fact that $\langle u^+, u^-\rangle_H \neq 0$ for sign-changing functions.

Now let us show that (1.3) possesses at least one least energy nodal solution. In doing so, we will prove again, as a byproduct, that non-negative and non-positive solutions also exist. We take our inspiration from [20].

Let $\|\cdot\|$ be the norm on H, equivalent to $\|\cdot\|_H$ (see Section 3), induced by the inner product

$$\langle u, v\rangle = \langle u, v\rangle_H + \int_\Omega V(x)uv\,\mathrm{d}x. \tag{2.4}$$

We know that $\nabla\mathcal{E}_p(u) = u - A(u)$, namely

$$\mathcal{E}_p'(u)[\varphi] = \langle u - A(u), \varphi\rangle, \qquad \langle A(u), \varphi\rangle := \int_\Omega |u|^{p-2}u\varphi, \quad u, \varphi \in H.$$

If $H^\pm$ denote the positive and negative cones of H, we set

$$H_\varepsilon^\pm := \big\{u \in H : \mathrm{dist}(u, H^\pm) < \varepsilon\big\}.$$

Then, we first state the following

Lemma 2.1 (*Order Preserving Property*). *Assume that $-\mathcal{L}_k + V$ is positive definite and let $u \in H$ be such that*

$$\langle u, \varphi\rangle \geqslant 0, \quad \textit{for every } \varphi \in H \textit{ with } \varphi \geqslant 0. \tag{2.5}$$

Then $u \geqslant 0$.

Proof. Testing inequality (2.5) with $\varphi = -u^- \in H^+$ yields

$$\langle u, -u^- \rangle = -\langle u^+, u^- \rangle - \|u^-\|^2 \geqslant 0.$$

As $\langle u^+, u^- \rangle = \langle u^+, u^- \rangle_H \geqslant 0$, we get $\|u^-\|^2 = 0$ and thus $u^- = 0$ since $-\mathcal{L}_k + V$ is positive definite. □

Lemma 2.2. *For every $\varepsilon > 0$ sufficiently small, $A(\partial H_\varepsilon^\pm) \subseteq H_\varepsilon^\pm$. In particular, if $u \in H_\varepsilon^\pm$ is a critical point of $\mathcal{E}_p$, namely $A(u) = u$, then $u \in H^\pm$.*

Proof. We have, for $2 \leqslant p \leqslant 2_s^*$,

$$\forall u \in H, \quad |u^+|_p = \min_{w \in H^-} |u - w|_p \leqslant C \min_{w \in H^-} \|u - w\| = C \operatorname{dist}(u, H^-). \tag{2.6}$$

Let P_+ denotes the metric projector on the positive cone H^+ for the norm $\|\cdot\|$. The metric projection on the convex set H^+ is characterized by

$$\forall \varphi \in H^+, \quad \langle u - P_+ u,\, \varphi - P_+ u \rangle \leqslant 0. \tag{2.7}$$

Because H^+ is a cone pointed at 0, this is equivalent to

$$\langle u - P_+ u,\, P_+ u \rangle = 0 \quad \text{and} \quad \forall \varphi \in H^+, \quad \langle u - P_+ u,\, \varphi \rangle \leqslant 0. \tag{2.8}$$

The implication (2.8) ⇒ (2.7) is obvious. For (2.7) ⇒ (2.8), taking $\varphi = tP_+ u$ with $t > 0$ in (2.7) yields $(t-1)\langle u - P_+ u,\, P_+ u \rangle \leqslant 0$, whence $\langle u - P_+ u,\, P_+ u \rangle = 0$.

Consequently, if we set $P_- u := u - P_+ u$, then $P_- u$ is orthogonal to $P_+ u$. Moreover, since $\langle P_- u, \varphi \rangle \leqslant 0$ for every $\varphi \in H^+$, it follows that $P_- u \leqslant 0$ by virtue of Lemma 2.1. If $v := A(u)$, then taking inequality (2.6) into account,

$$\begin{aligned} \operatorname{dist}\big(A(u), H^-\big)\|P_+ v\| &\leqslant \|v - P_- v\| \|P_+ v\| = \|P_+ v\|^2 \\ &= \langle v, P_+ v \rangle = \int_\Omega |u|^{p-2} u \, P_+ v \leqslant \int_\Omega |u^+|^{p-2} u^+ \, P_+ v \\ &\leqslant |u^+|_p^{p-1} \, |P_+ v|_p \leqslant C \operatorname{dist}(u, H^-)^{p-1} \|P_+ v\|. \end{aligned}$$

If $\operatorname{dist}(u, H^-)$ is small enough, we have $\operatorname{dist}(A(u), H^-) \leqslant \frac{1}{2} \operatorname{dist}(u, H^-)$, concluding the proof. □

According to [21, Lemma 3.2], Lemma 2.2 implies the existence of a pseudo-gradient vector field $\mathcal{G}$ such that $H_\varepsilon^\pm$ are (forward) invariant for the descending flow. Let us denote the flow by η i.e., $\eta(\cdot, u)$ is the maximal solution to

$$\begin{cases} \partial_t \eta(t, u) = -\mathcal{G}\big(\eta(t, u)\big), \\ \eta(0, u) = 0, \end{cases}$$

defined on the interval $\big[0, T(u)\big)$. It follows that $\partial H_\varepsilon^\pm \subseteq \mathscr{A}(H_\varepsilon^\pm)$, where $\mathscr{A}(H_\varepsilon^\pm)$ stands for the basin of attraction of $H_\varepsilon^\pm$ for the flow η.

First, we need the following

Lemma 2.3. *For $\varepsilon > 0$ sufficiently small, $\overline{H_\varepsilon^+} \cap \overline{H_\varepsilon^-} \subseteq \mathscr{A}_0$, where $\mathscr{A}_0$ is the basin of attraction of 0 . In particular, $\mathcal{E}_p(u) > 0$ for every $u \in \overline{H_\varepsilon^+} \cap \overline{H_\varepsilon^-} \setminus \{0\}$.*

Proof. We know that, for $\epsilon > 0$ small enough, $H_\varepsilon^+ \cap H_\varepsilon^-$ is (forward) invariant for η and the sole critical point it contains is 0. The map η being a pseudo-gradient flow and the Palais–Smale condition is satisfied, either $\mathcal{E}_p\big(\eta(t, u)\big) \to -\infty$ as $t \to T(u)$ or $\eta(t, u)$ possesses a limit point u^* as $t \to T(u)$ which is a critical point of $\mathcal{E}_p$, in which case $T(u) = +\infty$. Also, if $u \in H_\varepsilon^+ \cap H_\varepsilon^-$, then the second case rephrases as $\eta(t, u) \to 0$

as $t \to T(u)$. To conclude, we need to rule out the first case. Taking into account inequalities (2.6), it follows that

$$\mathcal{E}_p(u) \geqslant -\frac{1}{p}|u|_p^p \geqslant -\frac{1}{p}\big(|u^+|_p + |u^-|_p\big)^p \geqslant -\frac{C}{p}\big(\operatorname{dist}(u, H^-) + \operatorname{dist}(u, H^-)\big)^p \geqslant -\frac{C\,(2\varepsilon)^p}{p},$$

whenever $u \in H_\varepsilon^+ \cap H_\varepsilon^-$. Finally, the flow decreases the functional $\mathcal{E}_p$, so $\mathcal{E}_p(u) > \mathcal{E}_p\big(\eta(t,u)\big) > \mathcal{E}_p(0) = 0$ for all $t > 0$. □

We now state the following

Theorem 2.4. *There exist a solution to non-negative, a non-positive and a sign-changing solution to Problem (1.3).*

Proof. There exists a solution to Problem (1.3) in $H_\varepsilon^+ \backslash \overline{H_\varepsilon^-}$, a solution in $H_\varepsilon^- \backslash \overline{H_\varepsilon^+}$ and one solution in $H \backslash (\overline{H_\varepsilon^+} \cup \overline{H_\varepsilon^-})$. This follows directly from [21, Theorem 3.2], provided that one shows the existence of a path $h : [0,1] \to H$ such that $h(0) \in H_\varepsilon^+ \backslash H_\varepsilon^-$, $h(1) \in H_\varepsilon^- \backslash H_\varepsilon^+$ and

$$0 = \inf_{\overline{H_\varepsilon^+} \cap \overline{H_\varepsilon^-}} \mathcal{E}_p > \sup_{t \in [0,1]} \mathcal{E}_p\big(h(t)\big).$$

It is readily seen that for any finite dimensional subspace E of H, there exists $R > 0$ such that $u \in E$ and $\|u\| \geqslant R$ imply that $\mathcal{E}_p(u) < 0$. Pick $E := \operatorname{span}\{u_0, u_1\} \subseteq H$, where $u_0 \in H^+ \backslash \{0\}$ and $u_1 \in H^- \backslash \{0\}$ are non-collinear given elements. If $R > 0$ is the corresponding radius, set

$$h(t) := R^*\big((1-t)u_0 + tu_1\big) \in H \backslash \{0\}, \quad t \in [0,1],$$

where R^* is large enough so that $\min\big\{\|h(t)\| : t \in [0,1]\big\} \geqslant R$. This ends the proof, thanks to Lemma 2.2. □

Finally, we state the following

Theorem 2.5. *There exists a sign-changing solution u^* to (1.3) with minimal energy among all sign-changing solutions. In addition, $u^* \in \mathcal{M}_p$ achieves the minimum of $\mathcal{E}_p$ on $\mathcal{M}_p$.*

Proof. Lemma 2.2 says that the only critical points of $\mathcal{E}_p$ inside $H_\varepsilon^+ \cup H_\varepsilon^-$ are those belonging to $H^+ \cup H^-$. Thus, all the sign-changing critical points of $\mathcal{E}_p$ belong to the closed set $H \backslash (H_\varepsilon^+ \cup H_\varepsilon^-)$. Let us set

$$c := \inf\{\mathcal{E}_p(u) : u \text{ is a sign-changing critical point of } \mathcal{E}_p\}.$$

Since Theorem 2.4 says that the set is non-empty, there exists a sequence $(u_n) \subseteq H \backslash (H_\varepsilon^+ \cup H_\varepsilon^-)$ such that $\mathcal{E}_p(u_n) \to c$, as $n \to \infty$. Since $\mathcal{E}_p$ satisfies the Palais–Smale condition, it follows that (u_n) admits a subsequence which converges strongly in H to a limit point $u^* \in H \backslash (H_\varepsilon^+ \cup H_\varepsilon^-)$, such that $\mathcal{E}_p'(u^*) = 0$ and $\mathcal{E}_p(u^*) = c$.

As u^* changes sign and $\mathcal{E}_p'(u^*)[u^\pm] = 0$, $u \in \mathcal{M}_p$ (see the discussion following (2.3)). To show that u^* has minimal energy on $\mathcal{M}_p$, pick any $v \in \mathcal{M}_p$ and consider the rectangle $C := \{t^+v^+ + t^-v^- : t^+, t^- \in [0,R]\}$ where R is large enough so that $\mathcal{E}_p(t^+v^+ + t^-v^-) < 0$ whenever $t^+ = R$ or $t^- = R$. We will show that there exists $w \in C \backslash (H_\varepsilon^+ \cup H_\varepsilon^-)$ such that $\big(\mathcal{E}_p(\eta(t,w))\big)_{t \geqslant 0}$ is bounded from below. This will conclude the proof because, η being a gradient flow, $\eta(t,w)$ must then possess a limit point $w^* \in H \backslash (H_\varepsilon^+ \cup H_\varepsilon^-)$ which is a sign-changing critical point of $\mathcal{E}_p$. Thus $\mathcal{E}_p(v) \geqslant \mathcal{E}_p(w) \geqslant \mathcal{E}_p(\eta(t,w)) \geqslant \mathcal{E}_p(w^*) \geqslant c$ for all $t \in [0, +\infty)$.

To prove the existence of w, we follow an argument similar to the last one of the proof of Theorem 3.1 in [21] which we shall briefly explain. Let us start by considering $\mathscr{A}_0$, the basin of attraction of 0. The set $O := C \cap \mathscr{A}_0$ is a non-empty open subset in C on which $\mathcal{E}_p \geqslant 0$. By the choice of R, $\big\{t^+v^+ + t^-v^- : (t^+, t^-) \in (\{R\} \times [0,R]) \cup ([0,R] \times \{R\})\big\} \cap O = \varnothing$. Consequently, there exists a connected component Γ of ∂O intersecting both $[0,R]v^+$ and $[0,R]v^-$. Thus $\Gamma \cap H^+ \neq \varnothing$ and $\Gamma \cap H^- \neq \varnothing$. Let $\mathscr{A}(H^+) = \mathscr{A}(H^+_\epsilon)$ (resp. $\mathscr{A}(H^-) = \mathscr{A}(H^-_\epsilon)$) be the basin of attraction of H^+ (resp. H^-), where ϵ is small enough. The sets $\Gamma \cap \mathscr{A}(H^+)$ and $\Gamma \cap \mathscr{A}(H^-)$ are non-empty open subsets of Γ. Moreover they are disjoint because, if they were not, Lemma 2.3 would imply that $\Gamma \cap \mathscr{A}_0 \neq 0$ but, on the other hand, $\Gamma \subseteq \partial O \subseteq \partial \mathscr{A}_0$ implies $\Gamma \cap \mathscr{A}_0 = \varnothing$. In conclusion there exists $w \in \Gamma \backslash \big(\mathscr{A}(H^+) \cap \mathscr{A}(H^-)\big)$. It remains to show that $\big(\mathcal{E}_p(\eta(t,w))\big)_{t \geqslant 0}$ is bounded from below. But this is clear because $\partial \mathscr{A}_0$ is forward invariant and, thanks again to Lemma 2.3, $\mathcal{E}_p(u) > 0$ for all $u \in \partial \mathscr{A}_0$. □

Remark 2.6. Following the ideas of [22, proposition 3.1], it can be shown that any minimizer of $\mathcal{E}_p$ on $\mathcal{M}_p$ is a sign-changing critical point of $\mathcal{E}_p$. Therefore, least energy nodal solutions can be characterized as for the local problem, namely as minimizers of the functional on the nodal Nehari set. This is important from a numerical point of view as it gives a natural procedure for seeking such solutions.

3. A priori estimates

3.1. Equivalence between norms

In this section, we prove that the norm $\|\cdot\|$ corresponding to the inner product (2.4) and the traditional norm $\|\cdot\|_H$ are equivalent.

Proposition 3.1. *The norms $\|\cdot\|$ and $\|\cdot\|_H$ are equivalent when $V \in L^\infty(\Omega)$ and $-\mathcal{L}_K + V$ is positive definite.*

Proof. As $V \in L^\infty$ and H embeds continuously in L^2, there exists a constant $C > 0$ such that, for any $u \in H$, one has $\|u\|^2 \leqslant (1 + C|V|_\infty)\|u\|^2_H$. Moreover, for any $\varepsilon \in (0,1)$ and $u \in H$, we have

$$\begin{aligned}\|u\|^2 &= \varepsilon \|u\|^2_H + (1-\varepsilon)\|u\|^2 + \varepsilon \int_\Omega V(x) u^2 \,\mathrm{d}x \\ &\geqslant \varepsilon \|u\|^2_H + \big(\lambda_1 - \varepsilon\lambda_1 - \varepsilon |V|_\infty\big) \int_\Omega u^2\end{aligned}$$

where $\lambda_1 > 0$ since the operator $-\mathcal{L}_K + V$ is positive definite. Taking ε small, we conclude the proof. □

Thus, for this new norm $\|\cdot\|$ on H, we can use Poincaré's and Sobolev's inequalities. In the following, we assume that we work with H endowed with the norm $\|\cdot\|$ and the inner product (2.4).

3.2. Upper bound

In this section, let us consider $(u_p)_{2<p<2^*_s}$ a family of ground state solutions (resp. least energy nodal solutions) for the problem

$$\begin{cases} -\mathcal{L}_K u(x) + V(x)u(x) = \lambda |u(x)|^{p-2} u(x), & \text{for } x \in \Omega, \\ u(x) = 0, & \text{for } x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.1}$$

where $\lambda = \lambda_1$ (resp. λ_2), then the $v_p := \lambda^{1/(p-2)} u_p$ are solutions to Problem (1.3). Let us note $\tilde{\mathcal{E}}_p$ the functional associated to (3.1):

$$\tilde{\mathcal{E}}_p(u) = \frac{1}{2}\|u\|^2_H + \frac{1}{2}\int_\Omega V(x)u^2 \,\mathrm{d}x - \frac{\lambda}{p}|u|^p_p = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}|u|^p_p,$$

and let $\tilde{\mathcal{N}}_p$ (resp. $\tilde{\mathcal{M}}_p$) be its corresponding Nehari manifold (resp. nodal Nehari set). As the symmetries of u_p and v_p are the same and $\mathcal{E}_p(v_p) = \lambda^{2/(p-2)} \tilde{\mathcal{E}}_p(u_p)$, it suffices to study the ground state and least energy nodal solutions to (3.1).

Lemma 3.2. *Assume $-\mathcal{L}_K + V$ is positive definite. All eigenfunctions in $E_1\backslash\{0\}$ are nonnegative or nonpositive. Thus* $\dim E_1 = 1$. *All eigenfunctions of $E_2\backslash\{0\}$ change sign.*

Proof. Since $-\mathcal{L}_K+V$ is positive definite, $\|\cdot\|$ is a norm. Suppose on the contrary that there exists $u \in E_1\backslash\{0\}$ with both $u^+ \neq 0$ and $u^- \neq 0$. Then, one has

$$\begin{aligned}\frac{\|u\|^2}{|u|_2^2} = \frac{\|u^+ + u^-\|^2}{|u^+ + u^-|_2^2} &= \frac{\|u^+\|^2 + 2\langle u^+, u^-\rangle_H + \|u^-\|^2}{|u^+|_2^2 + |u^-|_2^2}\\ &> \frac{\|u^+\|^2 + \|u^-\|^2}{|u^+|_2^2 + |u^-|_2^2} \geqslant \min\left\{\frac{\|u^+\|^2}{|u^+|_2^2}, \frac{\|u^-\|^2}{|u^-|_2^2}\right\}\end{aligned}$$

which contradicts the variational characterization of λ_1. Because the cone $K := \{u : u \geqslant 0\}$ is closed and pointed, it is standard to show that the fact that E_1 is made only of elements of K or $-K$ implies $\dim E_1 \leqslant 1$.

Finally, let $\varphi_2 \in E_2\backslash\{0\}$ and suppose on the contrary that $\varphi_2 \geqslant 0$ (the case $\varphi_2 \leqslant 0$ is similar). Because $\varphi_2 \perp E_1$ in $L^2(\Omega)$, one concludes that $\varphi_2 = 0$ a.e. on $\{\varphi_1 > 0\}$. Thus, $0 = \langle\varphi_1, \varphi_2\rangle = \langle\varphi_1, \varphi_2\rangle_H = -2\int_{\mathbb{R}^N\times\mathbb{R}^N} \varphi_1(x)\varphi_2(y)K(x-y)\,\mathrm{d}(x, y) < 0$, a contradiction. □

Proposition 3.3. *The family $(u_p)_{2<p<\bar{p}}$ is bounded in H for the norm $\|\cdot\|$, for any $\bar{p} < 2_s^*$.*

Proof. Let us start with ground state solutions. Consider $\varphi_1 \in E_1$ such that $\|\varphi_1\| = 1$. If

$$t_p := \left(\frac{1}{\lambda_1|\varphi_1|_p^p}\right)^{1/(p-2)} > 0,$$

then $t_p\varphi_1 \in \mathcal{N}_p$ i.e., $t_p^2\|\varphi_1\|^2 = t_p^p\lambda_1|\varphi_1|_p^p$. We shall prove that $p \mapsto t_p : (2, \bar{p}] \to \mathbb{R}$ is bounded. By continuity, it is enough to check that t_p converges to some $t_* < +\infty$ as $p \to 2$. We have

$$\lim_{p\to 2} \ln t_p = -\lim_{p\to 2} \frac{\ln(\lambda_1|\varphi_1|_p^p)}{p-2}.$$

Since $\lambda_1|\varphi_1|_2^2 = 1$, we can use L'Hospital's rule. Remark that $\partial_p \int_\Omega |\varphi_1|^p = \int_\Omega \ln|\varphi_1|\,|\varphi_1|^p$ by Lebesgue's dominated convergence theorem. Then, for $p \to 2$,

$$\lim_{p\to 2} t_p = \exp\left(-\frac{\int_\Omega \ln|\varphi_1|\,|\varphi_1|^2}{\int_\Omega |\varphi_1|^2}\right) < +\infty.$$

Since $u_p \in \tilde{\mathcal{N}_p}$ has the lowest energy, $\left(\frac{1}{2} - \frac{1}{p}\right)\|u_p\|^2 = \tilde{\mathcal{E}}_p(u_p) \leqslant \tilde{\mathcal{E}}_p(t_p\varphi_1) = \left(\frac{1}{2} - \frac{1}{p}\right)t_p^2$ concluding this case.

Let us now treat the case of least energy nodal solutions. Pick $\varphi_2 \in E_2\backslash\{0\}$ and let $t_p^+ > 0$ and $t_p^- > 0$ be such that $t_p^+\varphi_2^+ + t_p^-\varphi_2^- \in \tilde{\mathcal{M}}_p$ (they exist because φ_2 changes sign, see Section 2). Expanding the equations $\tilde{\mathcal{E}}_p'(t_p^+\varphi_2^+ + t_p^-\varphi_2^-)[\varphi_2^\pm] = 0$ yields

$$t_p^+\|\varphi_2^+\|^2 + t_p^-\langle\varphi_2^+, \varphi_2^-\rangle - \lambda_2(t_p^+)^{p-1}|\varphi_2^+|_p^p = 0 \tag{3.2}$$

$$t_p^-\|\varphi_2^-\|^2 + t_p^+\langle\varphi_2^+, \varphi_2^-\rangle - \lambda_2(t_p^-)^{p-1}|\varphi_2^-|_p^p = 0. \tag{3.3}$$

The fact that φ_2 is a second eigenfunction reads $\langle\varphi_2, w\rangle = \lambda_2\int_\Omega \varphi_2 w$ for all $w \in H$. In particular, taking w as φ_2^+ and φ_2^- yields

$$\|\varphi_2^+\|^2 = -\langle\varphi_2^+, \varphi_2^-\rangle + \lambda_2|\varphi_2^+|_2^2 \quad \text{and} \quad \|\varphi_2^-\|^2 = -\langle\varphi_2^+, \varphi_2^-\rangle + \lambda_2|\varphi_2^-|_2^2. \tag{3.4}$$

Substituting back in (3.2)–(3.3), one deduces that

$$t_p^+\left(|\varphi_2^+|_2^2 - (t_p^+)^{p-2}|\varphi_2^+|_p^p\right) = -t_p^-\left(|\varphi_2^-|_2^2 - (t_p^-)^{p-2}|\varphi_2^-|_p^p\right).$$

Thus $|\varphi_2^+|_2^2 - (t_p^+)^{p-2}|\varphi_2^+|_p^p$ and $|\varphi_2^-|_2^2 - (t_p^-)^{p-2}|\varphi_2^-|_p^p$ always have opposite signs. Let us show that t_p^+ and t_p^- are bounded as $p \to 2$. Let us split these families into (possibly) two sub-families according to the sign of

$|\varphi_2^+|_2^2 - (t_p^+)^{p-2}|\varphi_2^+|_p^p$. We deal with the subfamily for which $|\varphi_2^+|_2^2 - (t_p^+)^{p-2}|\varphi_2^+|_p^p \geqslant 0$ (for the other one, this expression is <0, so $|\varphi_2^-|_2^2 - (t_p^-)^{p-2}|\varphi_2^-|_p^p > 0$ and the argument is similar). This inequality can be rewritten as

$$t_p^+ \leqslant \left(\frac{|\varphi_2^+|_2^2}{|\varphi_2^+|_p^p}\right)^{1/(p-2)} \xrightarrow[p\to 2]{} \exp\left(-\frac{\int_\Omega \ln|\varphi_2^+|\,|\varphi_2^+|^2}{|\varphi_2^+|_2^2}\right) \tag{3.5}$$

(with $s\ln|s|$ understood as 0 when $s=0$) where the convergence results from arguments similar to those used in the ground state case. Thus t_p^+ is bounded for p close to 2 and Eq. (3.2) implies that the same holds for t_p^-.

The conclusion follows easily since

$$\left(\frac{1}{2}-\frac{1}{p}\right)\|u_p\|^2 = \tilde{\mathcal{E}}_p(u_p) \leqslant \tilde{\mathcal{E}}_p(\hat{v}_p) = \left(\frac{1}{2}-\frac{1}{p}\right)\|\hat{v}_p\|^2$$

and $\|\hat{v}_p\|$ is bounded for p close to 2 because t_p^+ and t_p^- are and $v_p \to u_*$. □

We may thus assume that u_p weakly converges, up to a subsequence, to some $u_* \in H$ as $p\to 2$.

Proposition 3.4. *If (u_p) converges weakly in H to u_* as $p\to 2$ then $u_* \in E_1$ (resp. E_2).*

Proof. For every $v \in H$, one has

$$0 = \tilde{\mathcal{E}}_p'(u_p)[v] = \langle u_p, v\rangle - \lambda \int_\Omega |u_p|^{p-2} u_p v$$

where $\lambda = \lambda_1$ (resp. $\lambda = \lambda_2$) for ground state solutions (resp. least energy nodal solutions). Since u_p converges weakly to u_*, the first term converges to $\langle u_*, v\rangle$. Moreover, $u_p \to u_*$ in $L^q(\Omega)$ for $1 \leqslant q < 2_s^*$. So, up to a subsequence, $u_p \to u_*$ a.e. and there is $f \in L^2(\Omega)$ such that $|u_p| \leqslant f$ almost everywhere. By Lebesgue's dominated convergence theorem, the second term converges to $\int_\Omega u_* v$ as $\big||u_p|^{p-2}u_p v\big| \leqslant |\max\{f,1\}|^{\bar{p}-1}\,|v| \in L^1(\Omega)$ when $p \leqslant \bar{p} < 2_s^*$. As the limit does not depend on the subsequence, the whole sequence converges. Thus, u_* is a weak solution to $-\mathcal{L}_K u + V(x)u = \lambda u$. □

3.3. Lower bound

Proposition 3.5. *If (u_p) converges weakly to u_* in H as $p\to 2$ then $u_* \neq 0$.*

Proof. We first treat the case when u_p is a ground state solution. By Hölder's inequality, we have $|u_p|_p^2 \leqslant |u_p|_2^{2(1-\omega)}\,|u_p|_{2_s^*}^{2\omega}$ with $\omega = \frac{2_s^*}{2_s^*-2}\frac{p-2}{p}$. Then, by using Poincaré and Sobolev inequalities and since u_p belongs to the Nehari manifold, we have

$$|u_p|_p^2 \leqslant \left(\lambda_1^{-1}\|u_p\|^2\right)^{1-\omega}\left(S^{-1}\|u_p\|^2\right)^{\omega} = \left(|u_p|_p^p\right)^{1-\omega}|u_p|_p^{p\omega}(S^{-1}\lambda_1)^{\omega} = |u_p|_p^p\,(S^{-1}\lambda_1)^{\omega}.$$

Thus, $|u_p|_p^p \geqslant (S\lambda_1^{-1})^{2_s^*/(2_s^*-2)}$. Using the compact embeddings and Lebesgue's dominated convergence theorem, one has $|u_*|_2^2 = \lim_{p\to 2}|u_p|_p^p > 0$.

In the case of least energy nodal solutions, we claim that there exists $v_p = t_p^+ u_p^+ + t_p^- u_p^- \in \mathcal{N}_p \cap E_1^\perp$ such that $\|v_p\| \leqslant \|u_p\|$. Then, by the same argument as for the ground state case (with λ_2 instead of λ_1 because $v_p \perp E_1$), we get that v_p stays away from zero which is enough to conclude. To prove the claim, consider the line segment

$$T : [0,1] \to H\backslash\{0\} : \alpha \mapsto (1-\alpha)u_p^+ + \alpha u_p^-.$$

For all $\alpha \in [0,1]$, there exists a unique $t_\alpha > 0$ such that $t_\alpha T(\alpha) \in \mathcal{N}_p$. This t_α can be written explicitly and is easily seen to be continuous w.r.t. α. For $\alpha = 0$, we have $\int_\Omega t_\alpha u_p^+ \varphi_1 > 0$ and, for $\alpha = 1$, we have

$\int_\Omega t_\alpha u_p^- \varphi_1 < 0$. So, by continuity, there is a $\alpha^* \in (0,1)$ such that $\int_\Omega t_{\alpha^*} T(\alpha^*)\varphi_1 = 0$ and $t_{\alpha^*}T(\alpha^*) \in \mathcal{N}_p$. We just set $t_p^+ := t_{\alpha^*}(1-\alpha^*)$ and $t_p^- := t_{\alpha^*}\alpha^*$ to conclude. By definition of $\mathcal{M}_p \subseteq \mathcal{N}_p$, we get that $\tilde{\mathcal{E}}_p(v_p) \leqslant \tilde{\mathcal{E}}_p(u_p)$ and so, $\|v_p\| \leqslant \|u_p\|$. □

Proposition 3.6. *Ground state solutions (resp. least energy nodal solutions) to (3.1) converge, up to a subsequence, in H to some $\varphi_1^* \in E_1 \backslash \{0\}$ (resp. $\varphi_2^* \in E_2\backslash\{0\}$).*

Proof. Let (u_p) be a family of ground state solutions of (3.1). The argument is identical for least energy nodal solutions. By Proposition 3.3, for any sequence $p_n \to 2$, there exists a subsequence, still denoted p_n, such that u_{p_n} converges weakly in H to some $u^* \in H$. Propositions 3.4 and 3.5 imply that $u_* \in E_1\backslash\{0\}$. Finally, the compact embedding of H into L^q for $1 \leqslant q < 2_s^*$ and

$$\begin{aligned} 0 &= \tilde{\mathcal{E}}_{p_n}'(u_{p_n})[u_{p_n} - u^*] - \tilde{\mathcal{E}}_2'(u^*)[u_{p_n} - u^*] \\ &= \|u_{p_n} - u^*\|^2 - \frac{\lambda_1}{p_n}\int_\Omega |u_{p_n}|^{p_n-2}u_{p_n}(u_{p_n} - u^*) + \frac{\lambda_1}{2}\int_\Omega u^*(u_{p_n} - u^*) \end{aligned}$$

show that $u_{p_n} \to u_*$ in H. □

Remark that, from Propositions 3.3–3.5, we get the first conclusion of Theorem 1.1.

4. Symmetries and uniqueness via implicit function theorem

In this section, we prove the uniqueness (up to its sign) in H of a ground state solution (resp. least energy nodal solution) to Problem (3.1) when $\dim E_1 = 1$ (resp. $\dim E_2 = 1$). To start, we consider the following family of problems parametrized by $2 < p < 2_s^*$ and $\lambda \in \mathbb{R}$:

$$\begin{cases} (-\mathcal{L}_K + V)u = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega, \\ \|u\| = 1. \end{cases} \tag{4.1}$$

Proposition 4.1. *When $\dim E_1 = 1$ (resp. $\dim E_2 = 1$), there exists a unique curve of solutions $p \mapsto (p, u_p^*, \lambda_p)$ solving (4.1) starting from $(2, \varphi_1, \lambda_1)$ (resp. $(2,\varphi_2,\lambda_2)$) where $\varphi_1 \in E_1$ with $\|\varphi_1\| = 1$ (resp. $\varphi_2 \in E_2$ with $\|\varphi_2\| = 1$). There is also a unique curve of solutions starting from $(2, -\varphi_1, \lambda_1)$ (resp. $(2, -\varphi_2, \lambda_2)$) which is given by $p \mapsto (p, -u_p^*, \lambda_p)$.*

Proof. We make the proof for the ground states, the other case being similar. Let ψ be the function

$$\psi : (2, 2_s^*) \times H \times \mathbb{R} \to H \times \mathbb{R} : (p, u, \lambda) \mapsto \big(u - \lambda(-\mathcal{L}_K + V)^{-1}(|u|^{p-2}u),\ \|u\|^2 - 1\big),$$

so that (p, u, λ) is a root of ψ if and only if u is a solution to (4.1). To pursue our goal, we shall use the implicit function theorem as well as the closed graph theorem. First, we have to show that the Fréchet derivative of ψ at $(2, \varphi_1, \lambda_1)$ with respect to (u, λ) is bijective on $H \times \mathbb{R}$. Let us remark that

$$\partial_{(u,\lambda)}\psi(2,\varphi_1,\lambda_1)[(v,t)] = \big(v - \lambda_1(-\mathcal{L}_K + V)^{-1}v - t(-\mathcal{L}_K + V)^{-1}\varphi_1,\ 2\langle\varphi_1, v\rangle\big). \tag{4.2}$$

For injectivity, let us start by showing that $\partial_{(u,\lambda)}\psi(2,\varphi_1,\lambda_1)[(v,t)] = 0$ if and only if

$$\begin{cases} v - \lambda_1(-\mathcal{L}_K + V)^{-1}v = 0, \\ t = 0, \\ v \text{ is orthogonal to } \varphi_1 \text{ in } H. \end{cases} \tag{4.3}$$

Clearly, (4.3) is sufficient. For its necessity, observe that the second component of (4.2) implies that φ_1 is orthogonal to v in H and thus also in $L^2(\Omega)$ because φ_1 is an eigenfunction. Taking the L^2-inner product of the first component of (4.2) with φ_1 yields $t = 0$, hence completing the equivalence. Now, the only solution of (4.3) is $(v, t) = (0, 0)$ because the first equation and the dimension 1 of E_1 imply $v = \alpha\varphi_1$ for some $\alpha \in \mathbb{R}$ and then the third property implies $v = 0$. This concludes the proof of the injectivity. Let us now show that, for any $(w, s) \in H \times \mathbb{R}$, the equation $\partial_{(u,\lambda)}\psi(2, \varphi_1, \lambda_1)[(v, t)] = (w, s)$ always possesses at least one solution $(v, t) \in H \times \mathbb{R}$. One can write $w = \bar{w}\varphi_1 + \widetilde{w}$ for some $\bar{w} \in \mathbb{R}$ and $\widetilde{w} \in H$ orthogonal to φ_1 in H. Similarly, one can decompose $v = \bar{v}\varphi_1 + \widetilde{v}$. Arguing as for the first part, the equation can be written

$$\begin{cases} \widetilde{v} - \lambda_1(-\mathcal{L}_K + V)^{-1}\widetilde{v} = \widetilde{w}, \\ t = -\lambda_1\bar{w}, \\ \bar{v} = s/2. \end{cases} \tag{4.4}$$

The existence of the solution $\widetilde{v}$ results from the Fredholm alternative. This concludes the proof that $\partial_{(u,\lambda)}\psi(2, \varphi_1, \lambda_1)$ is onto and thus of the existence and uniqueness of the branch $p \mapsto (p, u_p^*, \lambda_p)$ emanating from $(2, \varphi_1, \lambda_1)$. It is clear that $p \mapsto (p, -u_p^*, \lambda_p)$ is a branch emanating from $(2, -\varphi_1, \lambda_1)$ and, using as above the implicit function theorem at that point, we know it is the only one. $\square$

Theorem 4.2. *Assume* $\dim E_1 = 1$ *(resp.* $\dim E_2 = 1$*). For p close to 2, ground state solutions (resp. least energy nodal solutions) to (3.1) are unique (up to their sign) and possess the same symmetries as* φ_1 *(resp.* φ_2*).*

Proof. We make the argument for the ground state solutions as it is identical for the other case. Let $(u_p)_{2<p<2_s^*}$ be a family of ground state solutions to Problem (3.1) and $p_n \to 2$. It suffices to show that, up to a subsequence, (u_{p_n}) possess the same symmetries as φ_1. Thanks to Proposition 3.6, we can assume without loss of generality that $u_{p_n} \to u_* \in E_1\setminus\{0\}$. Thus $u_* = \alpha\varphi_1$ for some $\alpha \neq 0$. Notice that u is a solution to (3.1) if and only if $u/\|u\|$ is a solution to (4.1) with $\lambda = \lambda_1\|u\|^{p-2}$. Also, since the family (u_p) remains bounded away from 0, one has $u_{p_n}\|u_{p_n}\|^{-1} \to \operatorname{sign}(\alpha)\varphi_1$ and $\lambda_1\|u_{p_n}\|^{p_n-2} \to \lambda_1$. Then, for n large, Proposition 4.1 implies $u_{p_n}\|u_{p_n}\|^{-1} = \operatorname{sign}(\alpha)\, u_{p_n}^*$. Hence u_{p_n} is unique up to its sign. Also, u_{p_n} respects the (anti-)symmetries of φ_1. Indeed, let us consider a direction d such that φ_1 is symmetric (resp. anti-symmetric) with respect to d. If u_{p_n} is not, let us consider u'_{p_n} the symmetric (resp. anti-symmetric) image of u_{p_n}. Because φ_1 is symmetric (resp. anti-symmetric) in the direction d, $u'_{p_n} \to \alpha\varphi_1$ (resp. $u'_{p_n} \to -\alpha\varphi_1$). Arguing as before, we conclude that $u_p\|u_p\|^{-1} = \pm\operatorname{sign}(\alpha)\, u_p^* = \pm u'_p\|u'_p\|^{-1}$, which concludes the proof. $\square$

This directly gives the second conclusion of Theorem 1.1 and thus completes it proof.

5. Asymptotic symmetries: Lyapunov-type reduction

In this section, we present an abstract symmetry result which is useful when $\dim E_1 \neq 1$ or $\dim E_2 \neq 1$. By Propositions 3.3 and 3.5, it will give the proof of Theorem 1.3. The idea is to show that, for p close to 2, a priori bounded solutions of (1.3) can be distinguished by their projections on the eigenspaces E_i. This will follow from Proposition 5.2 below.

Lemma 5.1. *Let* $i \geqslant 1$*. There exists* $\varepsilon > 0$ *such that if* $a \in L^{N/(2s)}(\Omega)$ *satisfies* $|a - \lambda_i|_{N/(2s)} < \varepsilon$ *and* u *solves*

$$\begin{cases} -\mathcal{L}_K u + V u = a(x)u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $P_{E_i}u = 0 \Rightarrow u = 0$ *where* P_{E_i} *is the orthogonal projector on* E_i*.*

Proof. Assume by contradiction that there exists a nontrivial solution u such that $P_{E_i}u = 0$. Let $w = P_{E_1\oplus\cdots\oplus E_{i-1}}u$ (with $w = 0$ if $i = 1$ so one does not need (5.1)) and $z = P_{(E_1\oplus\cdots\oplus E_i)^\perp}u$. Taking successively w and z as test functions and using Poincaré, Sobolev and Hölder inequalities, we infer that

$$\|w\|^2 = \lambda_i|w|_2^2 + \int_\Omega (a(x) - \lambda_i)uw\,\mathrm{d}x \geqslant \frac{\lambda_i}{\lambda_{i-1}}\|w\|^2 - C|a(x) - \lambda_i|_{\frac{N}{2s}}\|w\|\,\|u\|,$$

$$\|z\|^2 = \lambda_i|z|_2^2 + \int_\Omega (a(x) - \lambda_i)uz\,\mathrm{d}x \leqslant \frac{\lambda_i}{\lambda_{i+1}}\|z\|^2 + C|a(x) - \lambda_i|_{\frac{N}{2s}}\|z\|\,\|u\|.$$

We deduce that

$$\|w\| \leqslant \frac{\lambda_{i-1}C}{\lambda_i - \lambda_{i-1}}|a - \lambda_i|_{\frac{N}{2s}}\|u\|, \tag{5.1}$$

$$\|z\| \leqslant \frac{\lambda_{i+1}C}{\lambda_{i+1} - \lambda_i}|a - \lambda_i|_{\frac{N}{2s}}\|u\|. \tag{5.2}$$

Since $\|u\|^2 = \|w\|^2 + \|z\|^2$, we get a contradiction when $|a - \lambda_i|_{N/(2s)}$ is small enough for the coefficients of $\|u\|$ in (5.1)–(5.2) to be less than 1. □

The next result must be compared with the use of the implicit function theorem in the previous section. Note that, this time, uniqueness is not guaranteed.

Proposition 5.2. *Let $i \geqslant 1$. Let $(u_p)_{2<p<2_s^*}$ and $(v_p)_{2<p<2_s^*}$ be two families of solutions to*

$$\begin{cases} -\mathcal{L}_K u + Vu = \lambda_i|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases}$$

Let $p_n \to 2$ be such that $u_{p_n} \rightharpoonup \varphi_i$ for some $\varphi_i \in E_i\backslash\{0\}$, (v_{p_n}) is bounded in H, and the Lebesgue measure of the zero set of φ_i, namely $\{x \in \Omega : \varphi_i(x) = 0\}$, is zero. If, for n large, $P_{E_i}u_{p_n} = P_{E_i}v_{p_n}$, then, for all n large enough, $u_{p_n} = v_{p_n}$.

Proof. Suppose on the contrary that there is a subsequence, still denoted (p_n), such that, for all n, $P_{E_i}u_{p_n} = P_{E_i}v_{p_n}$ and $u_{p_n} \neq v_{p_n}$. Since (v_{p_n}) is bounded in H, up to a subsequence, $v_{p_n} \rightharpoonup v_*$ in H for some $v_* \in H$. Clearly $\varphi_i = P_{E_i}\varphi_i = P_{E_i}v_*$. Compact embeddings of H imply that $v_{p_n} \to v_*$ in $L^q(\Omega)$ for every $q \in [1, 2_s^*)$ and thus $v_* \in E_i$. Therefore $v_* = \varphi_i$. Observe that

$$\begin{cases} (-\mathcal{L}_K + V)(u_p - v_p) = a_p(x)(u_p - v_p), & \text{in } \Omega, \\ u_p - v_p = 0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{5.3}$$

where

$$a_p(x) := (p-1)\int_0^1 \left|v_p(x) + \theta(u_p(x) - v_p(x))\right|^{p-2}\mathrm{d}\theta.$$

It is readily seen that $a_{p_n}(x) \to \lambda_i$ for a.e. x such that $\varphi_i(x) \neq 0$. Noting that $\left|v_p(x)+\theta(u_p(x)-v_p(x))\right|^{p-2} \leqslant |v_p(x)|^{p-2} + |u_p(x)|^{p-2}$, we can apply Lebesgue's dominated convergence theorem to deduce that $a_{p_n} \to \lambda_i$ in $L^{N/(2s)}(\Omega\backslash\{\varphi_i = 0\})$. Since $\{\varphi_i = 0\}$ has zero measure, this convergence also holds in $L^{N/(2s)}(\Omega)$.

In particular, for n large enough, $|a_p - \lambda_i|_{N/(2s)} < \varepsilon$ where $\varepsilon > 0$ is given by Lemma 5.1. Since $P_{E_i}(u_{p_n} - v_{p_n}) = 0$, Lemma 5.1 implies $u_{p_n} = v_{p_n}$. This contradiction concludes the proof. □

Remark 5.3. In the nonlocal setting, the unique continuation property is a difficult subject and it has only recently been investigated in [14]. In particular, by [14, Theorem 1.4], the Lebesgue measure of $\{x \in \Omega : \varphi_i(x) = 0\}$ is indeed equal to zero for the model operator $-\mathcal{L}_K = (-\Delta)^s$.

Theorem 5.4. *Let $(u_p)_{2<p<2_s^*}$ be a family of ground state (resp. least energy nodal) solutions to Problem (3.1) and let $i = 1$ (resp. $i = 2$). Let G be a group acting on H in such a way that there exists $C > 0$ so that, for every $g \in G$, $u \in H$, and p close to 2,*

$$\text{(i) } g(E_i) = E_i, \qquad \text{(ii) } g(E_i^\perp) = E_i^\perp, \qquad \text{(iii) } \mathcal{E}_p(gu) = \mathcal{E}_p(u), \qquad \text{(iv) } \|gu\| \leqslant C\|u\|.$$

Assume the zero set of any functions in $E_i \backslash \{0\}$ has zero Lebesgue measure. Then, for p close enough to 2, u_p is invariant under the isotropy group $G_{\alpha_p} = \{g \in G : g\alpha_p = \alpha_p\}$ of $\alpha_p := P_{E_i} u_p$.

Proof. Suppose on the contrary that there exists sequences $p_n \to 2$ and $g_n \in G_{\alpha_{p_n}}$, where $\alpha_{p_n} := P_{E_i} u_{p_n}$, such that $g_n u_{p_n} \neq u_{p_n}$ for all n. According to Proposition 3.6, one can assume w.l.o.g. that $u_{p_n} \to \varphi_i^* \in E_i \backslash \{0\}$.

It follows from (iii) that, for all $v \in H$, $\mathcal{E}'_{p_n}(g u_{p_n})[v] = \mathcal{E}'_{p_n}(u_{p_n})[g^{-1} v]$, so $g_n u_{p_n}$ are also solutions to Problem (3.1). Moreover, given that

$$g u_p = g(P_{E_i} u_p) + g(P_{E_i^\perp} u_p) \quad \text{with } g(P_{E_i} u_p) \in E_i \text{ and } g(P_{E_i^\perp} u_p) \in E_i^\perp,$$

one deduces that $P_{E_i}(g u_p) = g(P_{E_i} u_p)$. In particular, $P_{E_i}(g_n u_{p_n}) = g_n \alpha_{p_n} = \alpha_{p_n}$. As a consequence, $P_{E_i}(g_n u_{p_n}) = P_{E_i}(u_{p_n})$. Moreover, property (iv) implies that $(g_n u_{p_n})$ is bounded in H. Proposition 5.2 thus implies that $g_n u_{p_n} = u_{p_n}$ for n large which contradicts our initial negation of the thesis. $\square$

Theorem 5.5 (*Localization of Limit Functions*)**.** *Let $(u_p)_{p>2}$ be a family of least energy nodal solutions to Problem (3.1). Let $p_n \to 2$ be such that $u_{p_n} \to u_*$ in H. Then $u_* \in E_2 \backslash \{0\}$ and it achieves the minimum of the reduced functional*

$$\mathcal{E}_* : E_2 \to \mathbb{R} : u \mapsto \tfrac{1}{2} \int_\Omega u^2 - u^2 \ln u^2$$

subject to the constraint $u \in \mathcal{N}_*$ where $\mathcal{N}_*$ is the reduced Nehari manifold

$$\mathcal{N}_* := \big\{ u \in E_2 \backslash \{0\} : \mathcal{E}'_*(u)[u] = 0 \big\}.$$

In particular, u_* satisfies

$$\begin{cases} (-\mathcal{L}_K + V) u_* = \lambda_2 u_* & \text{in } \Omega, \\ u_* = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \\ \displaystyle\int_\Omega u_* \ln |u_*|\, v = 0 & \text{for all } v \in E_2. \end{cases} \tag{5.4}$$

Remark 5.6. 1. Quantities like $s \ln s$ are understood as being 0 when $s = 0$.

2. For all $v \in E_2 \backslash \{0\}$, there exists a unique $t_v > 0$ such that $t_v v \in \mathcal{N}_*$. This t_v is given by the explicit formula $t_v = \exp\big(-\int_\Omega v^2 \ln |v| \,\mathrm{d}x / |v|_2^2\big)$. Since $v \mapsto t_v$ is continuous and $\mathcal{N}_*$ is the image of the unit sphere of E_2 under the map $v \mapsto t_v\, v$, $\mathcal{N}_*$ is compact. Therefore, there exists a $v_* \in \mathcal{N}_*$ that achieves the minimum of $\mathcal{E}_*$ on $\mathcal{N}_*$. Moreover, for all $v \in \mathcal{N}_*$ and all $t \geqslant 0$, $\mathcal{E}_*(tv) = \frac{1}{2} t^2 (1 - \ln t^2) |v|_2^2 \leqslant \mathcal{E}_*(v)$ so the reduced functional $\mathcal{E}_*$ possesses a Mountain-Pass structure.

Proof. Propositions 3.4 and 3.5 imply that $u_* \in E_2 \backslash \{0\}$. Let $v \in E_2$. We have

$$\begin{aligned} 0 &= \frac{1}{2 - p_n} \mathcal{E}'_{p_n}(u_{p_n})[v] = \lambda_2 \int_\Omega \frac{u_{p_n} - |u_{p_n}|^{p_n - 2} u_{p_n}}{2 - p_n}\, v \\ &= \lambda_2 \int_\Omega \frac{1}{2 - p_n} \int_2^{p_n} \ln |u_{p_n}|\, |u_{p_n}|^{q-2} u_{p_n} v \,\mathrm{d}q \,\mathrm{d}x \xrightarrow[n \to \infty]{} \lambda_2 \int_\Omega u_* \ln |u_*|\, v. \end{aligned}$$

Thus u_* satisfies (5.4). Since $\mathcal{E}_*'(u)[v] = -2\int_\Omega u \ln|u|\, v$, u_* is a critical point of $\mathcal{E}_*$ and, in particular, $u_* \in \mathcal{N}_*$. It remains to show that u_* achieves the minimal value of $\mathcal{E}_*$ on $\mathcal{N}_*$.

Let $v \in \mathcal{N}_*$. Set $v_p := t_p^+ v^+ + t_p^- v^-$ where $t^\pm > 0$ are the unique positive reals such that $v_p \in \mathcal{M}_p$ (they exist because v changes sign, see Section 2). Let $p_n \to 2$. Arguing as in the proof of Proposition 3.3, one can show that $(t_{p_n}^\pm)$ are bounded. So, up to subsequences, $t_{p_n}^\pm \to t^\pm$ for some $t^\pm \in [0,\infty)$. Passing to the limit on Eq. (3.2) and using (3.4), one finds that $(t^- - t^+)\langle v^+, v^-\rangle = 0$ and so that $t^+ = t^-$. In addition, as in the proof of Proposition 3.3, we can also assume w.l.o.g. that

$$\forall n, \quad \delta_n := t_{p_n}^+\big(|v^+|_2^2 - (t_{p_n}^+)^{p_n-2}|v^+|_{p_n}^{p_n}\big) = -t_{p_n}^-\big(|v^-|_2^2 - (t_{p_n}^-)^{p_n-2}|v^-|_{p_n}^{p_n}\big) \geqslant 0. \tag{5.5}$$

Using the fact that the bracket of the right expression is non-positive and passing to the limit (similarly to Eq. (3.5)) yields

$$t^+ = t^- \geqslant \exp\Big(-\frac{\int_\Omega \ln|v^-|\,|v^-|^2}{|v^-|_2^2}\Big) > 0.$$

Thus $(t_{p_n}^\pm)^{p_n-2} \to 1$ and so $\delta_n \to 0$. Dividing (5.5) by $2 - p_n$ and passing to the limit gives

$$t^+\Big(\ln t^+ |v^+|_2^2 + \int_\Omega \ln|v^+|\,|v^+|^2\Big) = -t^-\Big(\ln t^- |v^-|_2^2 + \int_\Omega \ln|v^-|\,|v^-|^2\Big) \tag{5.6}$$

where we used the elementary identity $t^{p-2}|v|_p^p - |v|_2^2 = \int_2^p t^{q-2}\ln t\,|v|_q^q + t^{q-2}\int_\Omega \ln|v|\,|v|^q\,\mathrm{d}x\,\mathrm{d}q$, for all $v \in H$ and $t > 0$, to compute the limit. Since $t^+ = t^-$, (5.6) can be rewritten

$$\ln t^+\, |v|_2^2 + \int_\Omega \ln|v|\,|v|^2 = 0.$$

Recalling that $v \in \mathcal{N}_*$ means $\int \ln|v|\,|v|^2 = 0$, one deduces that $t^+ = 1 = t^-$. Thus $v_p \to v$.

Because u_{p_n} has least energy on $\mathcal{M}_{p_n}$, $\tilde{\mathcal{E}}_{p_n}(u_{p_n}) \leqslant \tilde{\mathcal{E}}_{p_n}(v_{p_n})$. Because u_{p_n} and v_{p_n} belong to $\mathcal{N}_{p_n}$, this is equivalent to $|u_{p_n}|_{p_n}^{p_n} \leqslant |v_{p_n}|_{p_n}^{p_n}$. Passing to the limit and using the fact that $u_*, v \in \mathcal{N}_*$ yield the desired inequality $2\mathcal{E}_*(u_*) = |u_*|_2^2 \leqslant |v|_2^2 = 2\mathcal{E}_*(v)$. $\square$

6. Numerical examples

In this section, we illustrate our results by numerical computations. We consider the particular case of the fractional Laplacian problem (1.2) for some values of $s \in (0,1]$. The functional and its derivatives are computed thanks to the Finite Element Method. Ground states (resp. least energy nodal solutions) are approximated using the Mountain-Pass Algorithm (resp. the Modified Mountain-Pass Algorithm) [23,24].

Let us give some details on the computation of the various quantities. Given a mesh of the domain Ω, the integrals $\int_\Omega V u^2$, $\int_\Omega |u|^p, \ldots$ are approximated using standard quadrature rules on each element of the mesh. The hardest part for evaluating the functional $\mathcal{E}_p$ and its derivatives is clearly the computation of the stiffness matrix. More precisely, if $(\varphi_i)_{i=1}^n$ denotes the usual FEM basis consisting of "hat functions" for each interior node of the mesh, we need to compute

$$\langle \varphi_i, \varphi_j\rangle_H = \int_{\mathbb{R}^N\times\mathbb{R}^N} \big(\varphi_i(x) - \varphi_i(y)\big)\big(\varphi_j(x) - \varphi_j(y)\big) K(x-y)\,\mathrm{d}(x,y). \tag{6.1}$$

The two difficulties are that the kernel K is singular and the domain is unbounded. The convergence of the finite element method for this type of non-local operator was proved by Marta D'Elia and Max Gunzburger [25]. In order to compute (6.1), they restrict their attention to $N = 1$, use an "interaction domain" $\Omega_{\mathcal{I}} \subseteq \mathbb{R}^N \setminus \Omega$ and assume that K vanishes outside a ball of "large" radius. In this paper, we deal

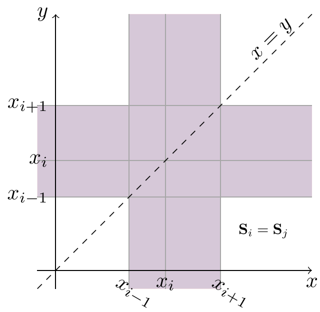

Fig. 1. Case $i = j$.

directly with (6.1) posed on $\mathbb{R}^N \times \mathbb{R}^N$ with $K(x) = \frac{1}{2} c_{N,s} |x|^{-N-2s}$. The reason it is possible results from a couple of remarks. First notice that

$$\mathbf{S}_i := \operatorname{supp}\bigl((x, y) \mapsto \varphi_i(y) - \varphi_i(x)\bigr) = (\operatorname{supp}\varphi_i \times \mathbb{R}^N) \cup (\mathbb{R}^N \times \operatorname{supp}\varphi_i).$$

Thus the integral in (6.1) has only to be considered on $\mathbf{S}_i \cap \mathbf{S}_j$. For brevity, let us write $S_i := \operatorname{supp}\varphi_i$. Expanding $\mathbf{S}_i \cap \mathbf{S}_j$ and remarking that the integral (6.1) is unchanged if one swaps x and y, one deduces that

$$\int_{\mathbf{S}_i \cap \mathbf{S}_j} = 2 \int_{(S_i \cap S_j) \times \complement(S_i \cup S_j)} + \int_{(S_i \cap S_j)^2} + 2 \int_{(S_i \setminus S_j) \times S_j} + 2 \int_{(S_i \cap S_j) \times (S_j \setminus S_i)} .$$

Among these four sets, the sole unbounded one is $(S_i \cap S_j) \times \complement(S_i \cup S_j)$. If $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j$ has zero Lebesgue measure, i.e., if i and j are not indices of neighboring nodes, then the integral boils down to

$$\langle \varphi_i, \varphi_j \rangle_H = -2 \int_{\operatorname{supp}\varphi_i \times \operatorname{supp}\varphi_j} \varphi_i(x) \varphi_j(y) K(x - y) \, \mathrm{d}(x, y),$$

where $K(x - y)$ is non-singular except when $x = y \in \operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j$ (often empty) where both φ_i and φ_j vanish. If $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j$ has non-zero measure, then the integral on the unbounded set must be taken into account. However, it simplifies to

$$\begin{aligned} &\int_{(S_i \cap S_j) \times \complement(S_i \cup S_j)} \bigl(\varphi_i(x) - \varphi_i(y)\bigr)\bigl(\varphi_j(x) - \varphi_j(y)\bigr) K(x - y) \, \mathrm{d}(x, y) \\ &\quad = \int_{S_i \cap S_j} \varphi_i(x) \varphi_j(x) \biggl(\int_{\complement(S_i \cup S_j)} K(x - y) \, \mathrm{d}y \biggr) \mathrm{d}x, \end{aligned} \tag{6.2}$$

and therefore to estimate this integral it is enough to be able to estimate the integral of K in a neighborhood of infinity.

For the one-dimensional case ($N = 1$) where Ω is an interval, the mesh is simply given by points $x_1 < x_2 < \cdots < x_M$ such that $\Omega =]x_1, x_M[$. The various possibilities for the sets $\mathbf{S}_i \cap \mathbf{S}_j$ are depicted in Figs. 1–3. For $K(x) = \frac{1}{2} c_{1,s} |x|^{-1-2s}$, the integrals on the various rectangles or unbounded strips in Figs. 1–3, amount to compute

$$\int_a^b \int_c^d \frac{\sum_{i,j=0}^{2} q_{ij} x^i y^j}{|y - x|^p} \, \mathrm{d}y \, \mathrm{d}x \tag{6.3}$$

where $-\infty \leqslant a < b \leqslant c < d \leqslant +\infty$, and $p \in \mathbb{R}$. Note that, thanks to the symmetry w.r.t. the diagonal, one may only integrate on $\{(x, y) \mid y \geqslant x\}$ and remove the absolute value. It is tedious but elementary to explicitly compute integrals of the type (6.3) and thus to have a precise estimate of the stiffness matrix at a low cost.

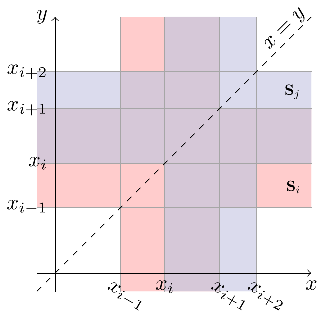

Fig. 2. Case $i + 1 = j$.

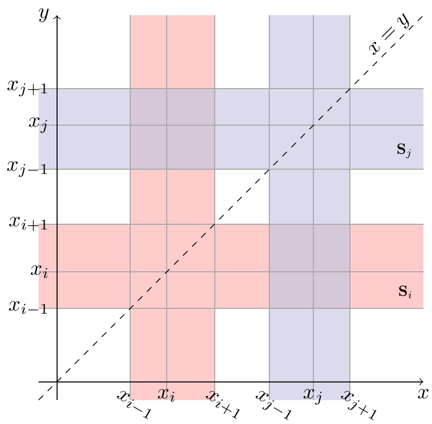

Fig. 3. Case $i + 1 < j$.

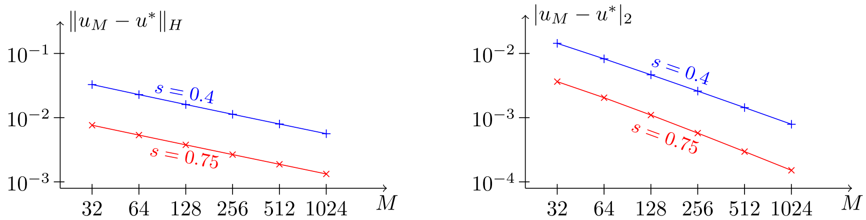

Fig. 4. Errors $\|u_M - u^*\|_H$ and $|u_M - u^*|_2$ w.r.t. the number of nodes M.

As Marta D'Elia and Max Gunzburger [25] did, one can judge the convergence of the method by comparing the FEM solution to the explicit solution to $(-\Delta)^s u = 1$ on $\Omega = B(0, R)$, namely

$$u^*(x) = 2^{-2s} \frac{\Gamma(N/2)}{\Gamma(N/2 + s)\Gamma(1 + s)} \big(R^2 - |x|^2\big)^s, \quad x \in B(0, R). \tag{6.4}$$

For a given s, let us denote u_M the FEM solution to $(-\Delta)^s u = 1$ on a mesh with M nodes. Fig. 4 shows the errors $\|u_M - u^*\|_H$ and $|u_M - u^*|_2$ as functions of M. These graphs suggest that $\|u_M - u^*\|_H = O(M^{-0.5})$ and $|u_M - u^*|_2 = O(M^{-0.8})$.

Let us now turn to the non-linear problem (1.2) with $V = 0$, $p = 4$ and $\Omega =]-1, 1[$. The initial function for the Mountain-Pass Algorithm (resp. the Modified Mountain-Pass Algorithm) is $u_0(x) = \cos(\pi x/2)$ (resp. $u_0(x) = \sin(\pi x)$) and the algorithms stop when $\|\nabla \mathcal{E}_p\|_H \leqslant 10^{-2}$. The ground state and l.e.n.s. are plotted in Fig. 5 for several values of s. Some characteristics of the solutions are given in Table 1. Note that, for

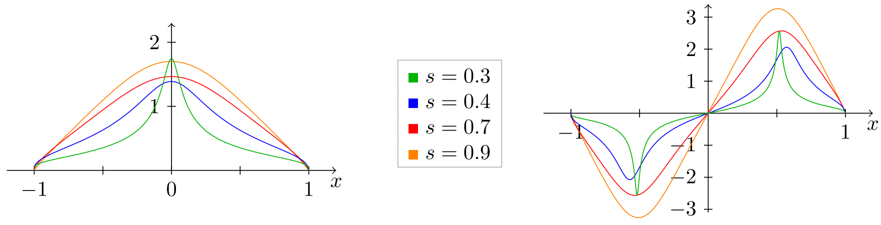

Fig. 5. Ground state and l.e.n.s. for $s \in \{0.3, 0.4, 0.7, 0.9\}$.

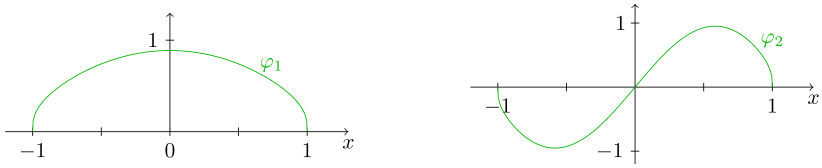

Fig. 6. First and second eigenfunctions for $s = 0.3$.

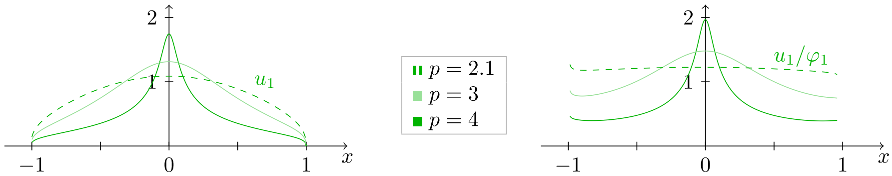

Fig. 7. Comparison of the ground state u_1 with φ_1 for $s = 0.3$ and $p \in \{2.1, 3, 4\}$.

Table 1
Characteristics of the ground state u_1 and the l.e.n.s. u_2 for $\Omega =]-1, 1[$.

s	$\mathcal{E}_4(u_1)$	$\max u_1$	$\mathcal{E}_4(u_2)$	$\max u_2$	$\min u_2$
0.3	0.29	1.7	0.74	2.5	−2.5
0.4	0.38	1.4	1.41	2.1	−2.1
0.7	0.76	1.5	6.45	2.6	−2.6
0.9	1.39	1.7	18.30	3.3	−3.3

p fixed, the smaller s is, the more concentrated around 0 (resp. around $\pm 1/2$) the ground state (resp. the l.e.n.s.) becomes. This contrasts with the linear case $(-\Delta)^s u = 1$ where the solution (6.4) goes to 1 as $s \to 0$.

If one looks at the first and second eigenfunctions φ_1 and φ_2 (see Fig. 6), the concentration phenomena may be surprising as one expects u_1 (resp. u_2) to resemble φ_1 (resp. φ_2). However, the above results say that the latter is true for s fixed and $p \to 2$. If one set s to, say, 0.3, and let $p \to 2$, one clearly sees on Fig. 7 that the ground state goes to a multiple of φ_1.

For the two dimensional case, the computation of the stiffness matrix (6.1) is more challenging [25, p. 1259]. The reason is that there are no longer explicit formulas for the integrals and $\complement(S_i \cup S_j)$ is not a simple shape. Let us give some information on how we estimate the stiffness matrix (6.1). The functions of the space H are approximated by P^1-finite elements on a triangular mesh $\mathcal{T}$ of Ω (i.e., continuous functions that are affine on each triangle of the mesh $\mathcal{T}$). We require that these functions vanish on (the piecewise affine approximation of) $\partial\Omega$.

To deal with the singular kernel, we use a generalized Duffy transformation. Let us explain how it works to compute $\int_T \int_T \big(\varphi_i(x) - \varphi_i(y)\big)\big(\varphi_j(x) - \varphi_j(y)\big) K(x-y)\,\mathrm{d}x\,\mathrm{d}y$ where T is a triangle of the mesh of $\mathcal{T}$. For the outer integral, we use a standard second order integration scheme which evaluates the function at the

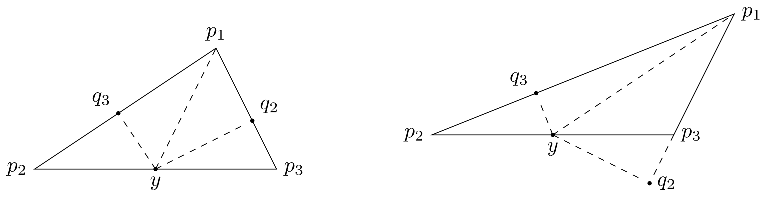

Fig. 8. Splitting the integral on right triangles.

middle of the edges of T. For the inner one, we first make use of the fact that φ_i (as well as φ_j) is affine on T so that $\varphi_i(x) - \varphi_i(y) = \nabla\varphi_i \cdot (x-y)$ where $\nabla\varphi_i$ is constant on T, so the integral boils down to

$$\tfrac{1}{2}c_{N,s}\int_T \nabla\varphi_i \cdot e_x\, \nabla\varphi_j \cdot e_x \frac{1}{|x-y|^{2s}}\,\mathrm{d}x \quad \text{where } e_x := \frac{x-y}{|x-y|}. \tag{6.5}$$

For each $y \in \partial T$ considered for the outer integral approximation, one can project orthogonally y on the two opposite sides of T and compute the integral on T as a sum or difference (depending on whether the projection falls or not inside T) of integrals on right triangles yq_3p_2, yq_3p_1, yq_2p_1 and yq_2p_3 (see Fig. 8). It thus remains to compute (6.5) on a right triangle to which y is a non-right corner. So let T be the triangle yqp with a right angle at q. We perform the following change of variable, dubbed generalized Duffy transformation,

$$x = y + u^\beta(q-y) + u^\beta v(p-q), \quad (u,v) \in (0,1)^2,$$

so that (6.5) becomes

$$c_{N,s}|T|\int_0^1\int_0^1 \nabla\varphi_i \cdot e_v\, \nabla\varphi_j \cdot e_v \frac{\beta u^{2\beta(1-s)-1}}{(|q-y|^2 + v^2|p-q|^2)^s}\,\mathrm{d}u\,\mathrm{d}v \quad \text{where } e_v := \frac{q-y+v(p-q)}{\sqrt{|q-y|^2+v^2|p-q|^2}}$$

and $|T|$ denotes the area of T. Taking $\beta := 1/(2(1-s))$, so that $u^{2\beta(1-s)-1} \equiv 1$, gives a smooth integrand so that the integral can be estimated by standard means.

To deal with the unboundedness of $\complement(S_i \cup S_j)$ in (6.2), we first integrate on $\Omega\backslash(S_i \cup S_j)$. Then, for each x used to compute the outer integral of (6.2), we mesh $B(x,R)\backslash\Omega$, where R is large enough so that $B(x,R) \supset \Omega$. The integral on $B(x,R)\backslash\Omega$ is computed using that mesh. For the remaining set, $\complement B(x,R)$, the integral is computed explicitly:

$$\int_{\complement B(x,R)} K(x-y)\,\mathrm{d}y = \tfrac{1}{2}c_{N,s}\int_{\mathbb{S}^{N-1}}\mathrm{d}\theta\int_R^\infty r^{-N-2s}r^{N-1}\,\mathrm{d}r = c_{N,s}\,|\mathbb{S}^{N-1}|\,\frac{1}{4sR^{2s}}.$$

Note that this approach could be extended to kernels that are well approximated by functions "of separated variables" in a neighborhood of infinity: $K(x) \approx \Theta(\theta)/r^{N+2s}$ when $|x| = r \to +\infty$.

In Figs. 9–10, you can see the computed ground state and least energy nodal solutions for $s \in \{0.6, 0.9\}$ on the unit ball $B(0,1)$ for $p = 4$. The behavior is similar to the one-dimensional case, namely the ground state is rotationally invariant and the least energy nodal solution looks Schwarz foliated symmetric. Moreover, both solutions concentrate as s becomes smaller.

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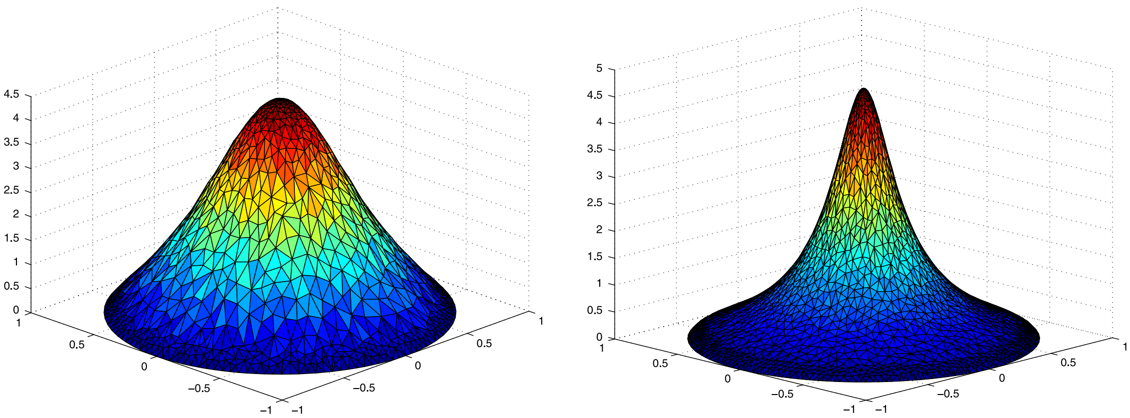

Fig. 9. Ground state solution for $s = 0.9$ (left) and $s = 0.6$ (right).

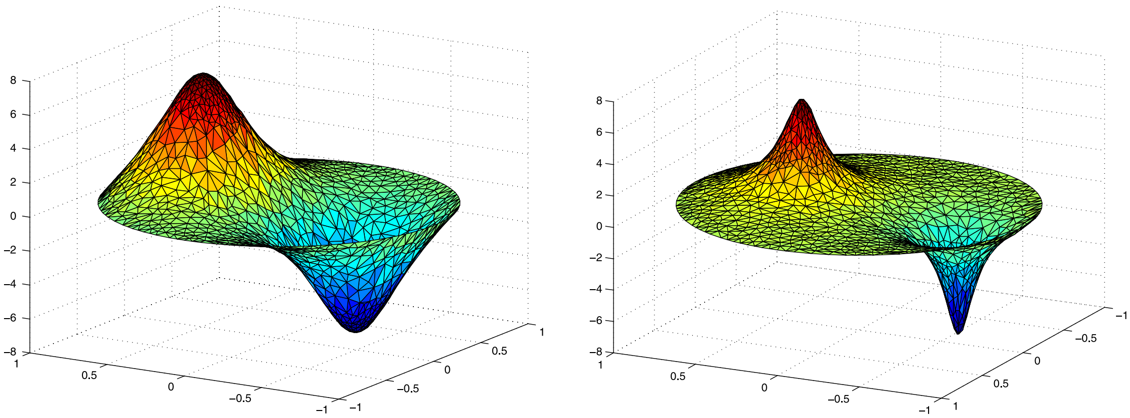

Fig. 10. Least energy nodal solution for $s = 0.9$ (left) and $s = 0.6$ (right).

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