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Logarithmic Bose–Einstein condensates with harmonic potential

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Abstract. In this paper, by using a compactness method, we study the Cauchy problem of the logarithmic Schrödinger equation with harmonic potential. We then address the existence of ground states solutions as minimizers of the action on the Nehari manifold. Finally, we explicitly compute ground states (Gausson-type solution) and we show their orbital stability.

Keywords: Logarithmic Schrödinger equation, harmonic potential, stability

1. Introduction

Recently, Zloshchastiev [25] introduced a new Bose–Einstein condensate in a harmonic trap as a candidate structure of physical vacuum, this structure is described by a logarithmic nonlinear Schrödinger equation in presence of a harmonic potential. The main motivation of such condensates lies essentially in their important applications in quantum mechanics, nuclear physics, quantum optics. Extensive details of the physical problem related to logarithmic Bose–Einstein condensate, experimental data and previous numerical studies can be found in [7] and the references therein.

The aim of this work is the study of the existence and stability of the ground states associated with the following nonlinear Schrödinger equation

$$i\partial_t u + \Delta u - V(x)u + u\,\mathrm{Log}\,|u|^2 = 0, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where t is time, $x \in \mathbb{R}^N$ is the spatial coordinate ($N \geqslant 1$) and $u := u(x;t) \in \mathbb{C}$ is the wave function. The local term $u\,\mathrm{Log}\,|u|^2$ describes the short-range interaction forces between particles. The potential

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$V(x)$ describes an electromagnetic field and has the following harmonic confinement

$$V(x) = \gamma(\gamma - 1)|x|^2, \quad \gamma > 1.$$

In the absence of the harmonic potential, Eq. (1.1) now arises from different applications in quantum mechanics, nuclear physics [19], open quantum systems and Bose–Einstein condensation. We refer the readers to [7,14,24,25] for more information on the related physical backgrounds. The classical logarithmic NLS equation was proposed by Bialynicki-Birula and Mycielski [4] as a model of nonlinear wave mechanics.

To the best of our knowledge, existence and stability of the ground states of logarithmic NLS equation (1.1) in presence of a harmonic potential has not been studied in the literature. More precisely, Eq. (1.1) has been previously studied only for $\gamma = 1$ (without the term $V(x)$). Among such works, let us mention [2,3,5,6,10–13]. This type of equations have been of great interest to both the theoretical and applied literature in recent years, see [1,20].

Concerning the Schrödinger equation with power-type nonlinearities and harmonic potential, many authors have been studying the problem of existence and stability of standing waves, see for instance [9,15–17,22,23] and the references therein.

The many-dimensional harmonic oscillator $-\Delta + \gamma(\gamma - 1)|x|^2$ is a self-adjoint operator on $L^2(\mathbb{R}^N)$ with operator domain $\{u \in H^2(\mathbb{R}^N) : |x|^2 u \in L^2(\mathbb{R}^N)\}$ and quadratic form domain

$$\Sigma(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : |x|u \in L^2(\mathbb{R}^N)\}.$$

It is well known that $\Sigma(\mathbb{R}^N)$ is a Hilbert space when is equipped with the norm

$$\|u\|_\Sigma^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |x|^2|u|^2\right) dx,$$

and it is continuously embedded in $H^1(\mathbb{R}^N)$ due to the Hardy inequality. Along the flow of (1.1), we have the conservation of the L^2-norm and of the energy functional associated:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{\gamma(\gamma - 1)}{2}\int_{\mathbb{R}^N} |x|^2|u|^2 \, dx - \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \operatorname{Log}|u|^2 \, dx. \tag{1.2}$$

Note that E is the generating Hamiltonian of (1.1). It is important to note that the logarithmic nonlinearity $z \to z \operatorname{Log}|z|^2$ is not locally Lipschitz continuous due to the singularity of the logarithm at the origin, in particular $E \notin C^1(H^1(\mathbb{R}^N))$. In Section 2, we will show that the energy E is well-defined and of class C^1 on the energy space $\Sigma(\mathbb{R}^N)$, which implies that if $u \in C(\mathbb{R}, \Sigma(\mathbb{R}^N)) \cap C^1(\mathbb{R}, \Sigma'(\mathbb{R}^N))$, then Eq. (1.1) makes sense in the space $\Sigma'(\mathbb{R}^N)$. Here, $\Sigma'(\mathbb{R}^N)$ is the dual space of $\Sigma(\mathbb{R}^N)$.

We have the following result concerning the well-posedness of the Cauchy problem for (1.1) in the energy space $\Sigma(\mathbb{R}^N)$. The proof is done in Section 3.

Proposition 1.1 (Well posedness)**.** *Assume that $\gamma > 1$. Then the Cauchy problem for (1.1) is globally well posed in the energy space $\Sigma(\mathbb{R}^N)$, i.e for every $u_0 \in \Sigma(\mathbb{R}^N)$, there is a unique global solution $u \in C(\mathbb{R}, \Sigma(\mathbb{R}^N)) \cap C^1(\mathbb{R}, \Sigma'(\mathbb{R}^N))$ with $u(0) = u_0$. In addition, the conservation of energy and charge hold, that is*

$$E(u(t)) = E(u_0) \quad \text{and} \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad \text{for all } t \in \mathbb{R}.$$

The most important issue in view of the applications of (1.1) in atomic physics and quantum optics seems to be the study of standing waves solutions of (1.1). In this case they are solutions of (1.1) of the form $u(x,t) = e^{i\omega t}\varphi(x)$, where $\omega \in \mathbb{R}$ and φ is a real valued function which has to solve the following nonlinear scalar field equation

$$-\Delta\varphi + \omega\varphi + \gamma(\gamma-1)|x|^2\varphi - \varphi \operatorname{Log}|\varphi|^2 = 0, \quad x \in \mathbb{R}^N. \tag{1.3}$$

Before stating our results, we introduce some notations to be used throughout the paper. For $\omega \in \mathbb{R}$ and $\gamma > 0$, we define the following functionals of class C^1 on $\Sigma(\mathbb{R}^N)$:

$$S_\omega(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\gamma(\gamma-1)}{2}\|xu\|_{L^2}^2 + \frac{\omega+1}{2}\|u\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}^N}|u|^2 \operatorname{Log}|u|^2\,dx,$$

$$I_\omega(u) := \|\nabla u\|_{L^2}^2 + \gamma(\gamma-1)\|xu\|_{L^2}^2 + \omega\|u\|_{L^2}^2 - \int_{\mathbb{R}^N}|u|^2 \operatorname{Log}|u|^2\,dx.$$

Note the scalar field equation (1.3) is varational in natura, that is, any solution is a critical point of $S_\omega(u)$. It is not difficult to show that $I_\omega(u) = \langle S'_\omega(u), u\rangle$.

For Eq. (1.1), the ground state solution play a crucial role in the dynamics. We recall that a nontrivial solution $\varphi \in \Sigma(\mathbb{R}^N)$ of (1.3) is termed as a ground state if it has some minimal action among all solutions of the nonlinear scalar field equation (1.3). In particular, it is possible to prove existence of ground state solutions solving the constrained variational problem

$$\begin{aligned} d(\omega) &= \inf\{S_\omega(u) : u \in \Sigma(\mathbb{R}^N) \setminus \{0\}, I_\omega(u) = 0\} \\ &= \frac{1}{2}\inf\{\|u\|_{L^2}^2 : u \in \Sigma(\mathbb{R}^N) \setminus \{0\}, I_\omega(u) = 0\}. \end{aligned} \tag{1.4}$$

We define the set of ground states by

$$\mathcal{G}_\omega = \{\varphi \in \Sigma(\mathbb{R}^N) \setminus \{0\} : S_\omega(\varphi) = d(\omega), I_\omega(\varphi) = 0\}.$$

In Section 4, we show that the quantity $d(\omega)$ is positive for every $\omega \in \mathbb{R}$. Indeed, for all $\gamma > 1$, $N \in \mathbb{N}$ and $\omega \in \mathbb{R}$ one has that

$$d(\omega) = \frac{1}{2}\pi^{\frac{N}{2}}\gamma^{-\frac{N}{2}}e^{\omega+\gamma N}.$$

Moreover, we have that any minimizing sequence is compact, the minimum is achieved and we explicitly compute the ground states of (1.3). More precisely,

Theorem 1.2 (Ground states)**.** *Let $N \geqslant 1$, $\gamma > 1$ and $\omega \in \mathbb{R}$. Then we have:*

(i) *Any minimizing sequence of $d(\omega)$ is relativity compact in $\Sigma(\mathbb{R}^N)$. That is, if a sequence $\{u_n\} \subseteq \Sigma(\mathbb{R}^N)$ is such that $I_\omega(u_n) = 0$ and $S_\omega(u_n) \to d(\omega)$ as n goes to $+\infty$, then up to a subsequence there exist $\varphi \in \Sigma(\mathbb{R}^N)$ satisfying $S_\omega(\varphi) = d(\omega)$ and $u_n \to \varphi$ in $\Sigma(\mathbb{R}^N)$.*

(ii) *The set of ground states is given by $\mathcal{G}_\omega = \{e^{i\theta}\phi_\omega : \theta \in \mathbb{R}\}$, where*

$$\phi_\omega(x) := e^{\frac{\omega+\gamma N}{2}}e^{-\frac{\gamma}{2}|x|^2}, \quad x \in \mathbb{R}^N. \tag{1.5}$$

Remark 1.3. Assume that the infimum of $d(\omega)$ is achieved by u. Then, there exist a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $S'_\omega(u) = \Lambda I'_\omega(u)$, which implies that $\langle S'_\omega(u), u\rangle = \Lambda\langle I'_\omega(u), u\rangle$. Hence

$$\langle S'_\omega(u), u\rangle = I_\omega(u) = 0 \quad \text{and} \quad \langle I'_\omega(u), u\rangle = -2\|u\|^2_{L^2} < 0,$$

implies $\Lambda = 0$. Therefore, u is a weak solution to equation (1.3). On the other hand, for any $v \in \Sigma(\mathbb{R}^N) \setminus \{0\}$ satisfying $S'_\omega(v) = 0$, it follows that $I_\omega(v) = 0$. Thus, by definition of $\mathcal{G}_\omega$, we get that u has minimal action among all solutions of (1.3).

We now discuss the notion of stability of standing waves. The basic symmetry associated to equation (1.1) is the *phase-invariance* (while the translation invariance does *not* hold due to the harmonic potential); taking this fact into account, it is reasonable to define orbital stability as follows:

Definition 1.4. We say that a standing wave solution $u(x,t) = e^{i\omega t}\varphi(x)$ of (1.1) is orbitally stable in $\Sigma(\mathbb{R}^N)$ if for any $\varepsilon > 0$ there exist $\eta > 0$ with the following property: if $u_0 \in \Sigma(\mathbb{R}^N)$ satisfies $\|u_0 - \varphi\|_\Sigma < \eta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exist for all $t \in \mathbb{R}$ and satisfies

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \left\| u(t) - e^{i\theta}\varphi \right\|_\Sigma < \varepsilon.$$

Otherwise, the standing wave $e^{i\omega t}\varphi(x)$ is said to be unstable in $\Sigma(\mathbb{R}^N)$.

Our second result shows that, in terms of the Cazenave and Lions' argument, the ground states are orbitally stable.

Theorem 1.5 (Orbital stability)**.** *For any $\omega \in \mathbb{R}$ and $N \geqslant 1$, the standing wave $e^{i\omega t}\phi_\omega(x)$ is orbitally stable in $\Sigma(\mathbb{R}^N)$.*

The paper is organized in the following way: in Section 2, we show that the energy functional E is of class C^1 on $\Sigma(\mathbb{R}^N)$. Moreover, we recall several known results and introduce several notations. In Section 3, we give an idea of the proof of Proposition 1.1. In Section 4 we prove, by variational techniques, the existence of a minimizer for $d(\omega)$ (Theorem 1.2), while in Section 5 the proof of Theorem 1.5 is completed.

Notation. The space $L^2(\mathbb{R}^N, \mathbb{C})$, denoted by $L^2(\mathbb{R}^N)$ for shorthand, is equipped with the norm $\|\cdot\|_{L^2}$. Moreover 2^* is defined by $2^* = 2N/(N-2)$ if $N \geqslant 3$, and $2^* = +\infty$ if $N = 1, 2$. Finally, $\langle\cdot,\cdot\rangle$ is the duality pairing between X' and X, where X is a Banach space and X' is its dual.

2. Preliminary lemmas

In this section we recall several known results, almost all are proved in the paper [10]. Moreover, we show that the energy functional E is of class C^1 on $\Sigma(\mathbb{R}^N)$.

Proposition 2.1. *The energy functional E defined by (1.2) is of class C^1 and for $u \in \Sigma(\mathbb{R}^N)$ the Fréchet derivative of E in u exists and it is given by*

$$E'(u) = -\Delta u + \gamma(\gamma - 1)|x|^2 u - u \operatorname{Log}|u|^2 - u.$$

Before giving the proof of Proposition 2.1, we fix some definitions that will be useful in the sequel. Define

$$F(z) := |z|^2 \operatorname{Log}|z|^2, \quad \text{for every } z \in \mathbb{C},$$

and as in [10], we introduce the functions A, B on $[0, \infty)$ by

$$A(s) := \begin{cases} -s^2 \operatorname{Log}(s^2), & \text{if } 0 \leqslant s \leqslant e^{-3}; \\ 3s^2 + 4e^{-3}s - e^{-6}, & \text{if } s \geqslant e^{-3}; \end{cases} \qquad B(s) := F(s) + A(s). \tag{2.1}$$

We will need the following functions a and b given by

$$a(z) := \frac{z}{|z|^2} A\big(|z|\big) \quad \text{and} \quad b(z) := \frac{z}{|z|^2} B\big(|z|\big), \quad \text{for } z \in \mathbb{C}, z \neq 0. \tag{2.2}$$

Noticing that for any $z \in \mathbb{C}$, $b(z) - a(z) = z \operatorname{Log}|z|^2$. In addition, we note that A is a nonnegative convex and increasing function, and $A \in C^1([0, +\infty)) \cap C^2((0, +\infty))$. We define the following Orlicz space $L^A(\mathbb{R}^N)$ corresponding to A,

$$L^A\big(\mathbb{R}^N\big) := \big\{u \in L^1_{\text{loc}}\big(\mathbb{R}^N\big) : A\big(|u|\big) \in L^1\big(\mathbb{R}^N\big)\big\},$$

equipped with the norm

$$\|u\|_{L^A} := \inf\left\{k > 0 : \int_{\mathbb{R}^N} A\big(k^{-1}\big|u(x)\big|\big)\, dx \leqslant 1\right\}.$$

Here $L^1_{\text{loc}}(\mathbb{R}^N)$ is the space of all locally Lebesgue integrable functions. In [10, Lemma 2.1], the author proved that $(L^A(\mathbb{R}^N), \|\cdot\|_{L^A})$ is a separable reflexive Banach space. Below we describe some properties of $L^A(\mathbb{R}^N)$. See [10, Lemma 2.1] for more details.

Proposition 2.2. *Assume that $\{u_m\}$ is a sequence in $L^A(\mathbb{R}^N)$. Then the following facts hold:*

i) *If $u_m \to u$ in $L^A(\mathbb{R}^N)$, then $A(|u_m|) \to A(|u|)$ in $L^1(\mathbb{R}^N)$ as $m \to \infty$.*
ii) *Let $u \in L^A(\mathbb{R}^N)$. If $u_m \to u$ a.e. in $\mathbb{R}^N$ and if*

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} A\big(\big|u_m(x)\big|\big)\, dx = \int_{\mathbb{R}^N} A\big(\big|u(x)\big|\big)\, dx,$$

then $u_m \to u$ in $L^A(\mathbb{R}^N)$ as $m \to \infty$.

iii) *For any function u in $L^A(\mathbb{R}^N)$, we have the following relationship*

$$\min\big\{\|u\|_{L^A}, \|u\|^2_{L^A}\big\} \leqslant \int_{\mathbb{R}^N} A\big(\big|u(x)\big|\big)\, dx \leqslant \max\big\{\|u\|_{L^A}, \|u\|^2_{L^A}\big\}. \tag{2.3}$$

Remark 2.3. A simple calculation shows that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ with

$$\big|B(z) - B(w)\big| \leqslant C_\varepsilon\big(|z|^{1+\varepsilon} + |w|^{1+\varepsilon}\big)|z - w|, \quad \text{for every } z, w \in \mathbb{C},$$

hence, integrating on $\mathbb{R}^N$ with $\varepsilon = (2^* - 2)/2$ and applying Hölder's inequality and Sobolev's Inequalities we see that

$$\int_{\mathbb{R}^N} \big|B\big(|u|\big) - B\big(|v|\big)\big|\,dx \leqslant C\big(1 + \|u\|^2_{H^1(\mathbb{R}^N)} + \|v\|^2_{H^1(\mathbb{R}^N)}\big)\|u - v\|_{L^2} \tag{2.4}$$

for all $u, v \in H^1(\mathbb{R}^N)$.

Lemma 2.4 (Injections). *Let $N \geqslant 1$. Then the following assertions hold.*

(i) *The embedding $\Sigma(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact, where $2 \leqslant q < 2^*$.*
(ii) *The inclusion map $\Sigma(\mathbb{R}^N) \hookrightarrow L^{2-\delta}(\mathbb{R}^N)$ is continuous, where $\delta = 1/N$.*
(iii) *The inclusion map $\Sigma(\mathbb{R}^N) \hookrightarrow L^A(\mathbb{R}^N)$ is continuous.*

Proof. Item (i) is proved in [22, Lemma 3.1]. Let $u \in \Sigma(\mathbb{R}^N)$. By Hölder's inequality with conjugate exponents $2N/(2N - 1)$, $2N$ we see that

$$\int_{\mathbb{R}^N} \big|u(x)\big|^{2-\frac{1}{N}}\,dx \leqslant \left(\int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^\alpha}\,dx\right)^{\frac{1}{2N}} \left(\int_{\mathbb{R}^N} \big(1 + |x|^2\big)\big|u(x)\big|^2\,dx\right)^{\frac{2N-1}{2N}},$$

where $\alpha = 2N - 1$. Since $\alpha > N/2$, we have that there exists a constant $C > 0$ depending only on N such that $\|u\|_{L^{2-1/N}(\mathbb{R}^N)} \leqslant C\|u\|_\Sigma$, which completes the proof of Item (ii). Concerning (iii), it follows form (2.1) that for every $N \in \mathbb{N}$, there exist $C > 0$ depending only on N such that

$$A\big(|z|\big) \leqslant C\big(|z|^{2+\frac{1}{N}} + |z|^{2-\frac{1}{N}}\big) + B\big(|z|\big) \quad \text{for any } z \in \mathbb{C}.$$

Notice that $2 < 2 + 1/N < 2^*$. Thus from Item (i), Item (ii) and (2.4) we have that if $u_n \to u$ as n goes to $+\infty$ in $\Sigma(\mathbb{R}^N)$, then $A(|u_n - u|) \to 0$ as n goes to $+\infty$ in $L^1(\mathbb{R}^N)$. This implies by (2.3) that $u_n \to u$ as n goes to $+\infty$ in $L^A(\mathbb{R}^N)$. This concludes the proof. $\square$

For a proof of following result, see [10, Lemma 2.5 and Lemma 2.6].

Lemma 2.5. *Let $N \geqslant 1$ and consider the functions a and b defined by (2.2). Then the following is true.*

(i) *The operator $u \to a(u)$ is continuous from $L^A(\mathbb{R}^N)$ into $L^{A'}(\mathbb{R}^N)$. Moreover, the image under a of every bounded subset of $L^A(\mathbb{R}^N)$ is a bounded subset of $L^{A'}(\mathbb{R}^N)$.*
(ii) *The operator $u \to b(u)$ is continuous from $H^1(\mathbb{R}^N)$ into $H^{-1}(\mathbb{R}^N)$. Moreover, the image under b of every bounded subset of $H^1(\mathbb{R}^N)$ is a bounded subset of $H^{-1}(\mathbb{R}^N)$.*

Lemma 2.6. *The operator*

$$L : u \to -\Delta u + \gamma(\gamma - 1)|x|^2 u - u\,\mathrm{Log}\,|u|^2$$

is continuous from $\Sigma(\mathbb{R}^N)$ to $\Sigma'(\mathbb{R}^N)$. The image under L of every bounded subset of $\Sigma(\mathbb{R}^N)$ is a bounded subset of $\Sigma'(\mathbb{R}^N)$.

Proof. It is clear $-\Delta+\gamma(\gamma-1)|x|^2$ is continuous from $\Sigma(\mathbb{R}^N)$ to $\Sigma'(\mathbb{R}^N)$. Hence, we need to prove the continuity of the nonlinearity part of L. Indeed, by Lemma 2.5(i), $u \to a(u)$ is continuous from $L^A(\mathbb{R}^N)$ to $L^{A'}(\mathbb{R}^N)$, which implies by Lemma 2.4(iii), from $\Sigma(\mathbb{R}^N)$ to $\Sigma'(\mathbb{R}^N)$. Finally, applying Lemma 2.5(ii), we see that the operator $u \to a(u) - b(u) = -u \operatorname{Log}|u|^2$ is continuous from $\Sigma(\mathbb{R}^N)$ to $\Sigma'(\mathbb{R}^N)$. This completes the proof of Lemma 2.6. □

Proof of Proposition 2.1. We first show that E is continuous on $\Sigma(\mathbb{R}^N)$. Note that E can be rewritten in the following form

$$E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\gamma(\gamma-1)}{2}\|xu\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^N} A(|u|)\,dx - \frac{1}{2}\int_{\mathbb{R}^N} B(|u|)\,dx. \tag{2.5}$$

The first and second term in the right-hand side of (2.5) are continuous on $\Sigma(\mathbb{R}^N)$. Hence, we need to prove the continuity of the nonlinearity part of E. Combining Proposition 2.2(i) and Lemma 2.4(iii) we obtain that the third term is continuous on $\Sigma(\mathbb{R}^N)$. Moreover, by (2.4) we have that the fourth term in the right-hand side of (2.5) is continuous on $\Sigma(\mathbb{R})$, which implies that $E \in C(\Sigma(\mathbb{R}^N), \mathbb{R})$. Next it is easily seen that, for $u, v \in \Sigma(\mathbb{R}^N)$, $t \in (-1, 1)$ (see [10, Proposition 2.7]),

$$\lim_{t\to 0} \frac{E(u+tv) - E(u)}{t} = \langle (-\Delta u + \gamma(\gamma-1)|x|^2 u - u \operatorname{Log}|u|^2 - u, v\rangle.$$

Thus, E is Gâteaux differentiable. By virtue of Lemma 2.6 we conclude that E is Fréchet differentiable. □

3. The Cauchy problem

In this section we sketch the proof of the global well-posedness of (1.1) for any $\gamma > 1$ as stated in Proposition 1.1. A similar technique was applied by Cazenave [11, Theorem 9.3.4] in the case of the NLS equation (1.1) without the term $V(x)$. We first construct a sequence of global weak solutions of a regularized Cauchy problem in $C(\mathbb{R}, \Sigma(\mathbb{R}^N))$ which converges to a weak solution of the equation (1.1). This produces a weak solution. Then, applying some properties of the logarithmic nonlinearity we show the uniqueness of the weak solution of equation (1.1).

Before outlining the main ideas of the proof of Proposition 1.1, we fix some definitions that will be useful in the sequels. For $z \in \mathbb{C}$ and $m \in \mathbb{N}$, we introduce the functions a_m and b_m by

$$a_m(z) = z\tilde{a}_m(|z|), \qquad \tilde{a}_m(s) := \begin{cases} \frac{A(s)}{s^2}, & \text{if } s \geqslant \frac{1}{m}, \\ m^2 A(\frac{1}{m}), & \text{if } 0 \leqslant s \leqslant \frac{1}{m}, \end{cases}$$

$$b_m(z) = z\tilde{b}_m(|z|), \qquad \tilde{b}_m(s) := \begin{cases} \frac{B(s)}{s^2}, & \text{if } 0 \leqslant s \leqslant m, \\ \frac{B(m)}{m^2}, & \text{if } s \geqslant m, \end{cases}$$

where A and B were defined in (2.2). We will need the following family of nonlinearities given by $g_m(z) = b_m(z) - a_m(z)$ for any fixed $m \in \mathbb{N}$ and for every $z \in \mathbb{C}$.

Now we need to construct an appropriate sequence of weak solutions of the following regularized Cauchy problem

$$i\partial_t u^m + \Delta u^m - \gamma(\gamma-1)|x|^2 u^m + g_m(u^m) = 0, \quad m \in \mathbb{N}. \tag{3.1}$$

Proposition 3.1. *For every $u_0 \in \Sigma(\mathbb{R}^N)$, there is a unique global solution $u^m \in C(\mathbb{R}, \Sigma(\mathbb{R}^N)) \cap C^1(\mathbb{R}, \Sigma'(\mathbb{R}^N))$ of problem (3.1) with $u(0) = u_0$. Furthermore, the mass and total energy associated with (3.1) are conserved in time, namely*

$$E_m(u^m(t)) = E_m(u_0) \quad \text{and} \quad \|u^m(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R}, \tag{3.2}$$

where

$$E_m(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\gamma(\gamma-1)}{2}\|xu\|_{L^2}^2 + \int_{\mathbb{R}^N} \Phi_m(|u|)\,dx - \int_{\mathbb{R}^N} \Psi_m(|u|)\,dx,$$

and the functions $t \mapsto \Phi_m(t)$ and $t \mapsto \Psi_m(t)$ are defined by

$$\Phi_m(t) := \int_0^t s\tilde{a}_m(s)\,ds \quad \text{and} \quad \Psi_m(t) := \int_0^t s\tilde{b}_m(s)\,ds.$$

Proof. Since g_m is globally Lipschitz continuous $\mathbb{C} \to \mathbb{C}$, the global well-posedness follows from Strichartz inequalities and a fixed point argument; see e.g. [11, Theorem 9.2.6 and Remark 9.2.8]. □

In the following we will make use of the following lemma.

Lemma 3.2. *Assume that $\{u^m\}_{m\in\mathbb{N}}$ is a sequence bounded in $L^\infty(\mathbb{R}, \Sigma(\mathbb{R}^N))$ and in $W^{1,\infty}(\mathbb{R}, \Sigma'(\mathbb{R}^N))$. Then there exist a subsequence, which we still denote by $\{u^m\}_{m\in\mathbb{N}}$, and there exist a function $u \in L^\infty(\mathbb{R}, \Sigma(\mathbb{R}^N)) \cap W^{1,\infty}(\mathbb{R}, \Sigma'(\mathbb{R}^N))$ such that the following conclusions are valid.*

(i) *$u^m(t) \rightharpoonup u(t)$ in $\Sigma(\mathbb{R}^N)$ as $m \to \infty$ for all $t \in \mathbb{R}$.*
(ii) *For any $t \in \mathbb{R}$ there is a subsequence m_j with $u^{m_j}(x,t) \to u(x,t)$ as $j \to \infty$, for a.e. $x \in \mathbb{R}^N$.*
(iii) *$u^m(x,t) \to u(x,t)$ as $m \to \infty$, for a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.*

Proof. The proof follows a similar argument used in [11, Lemma 9.3.6], and we omit the details. □

Proof of Proposition 1.1. Here, for simplicity, we assume that $\gamma(\gamma-1) = 1$. We proceed by approximating the equation as follows (see [11, Theorem 9.3.4]): taking into account Lemma 3.1, we get that exists a unique solution $u^m \in C(\mathbb{R}, \Sigma(\mathbb{R}^N)) \cap C^1(\mathbb{R}, \Sigma'(\mathbb{R}^N))$ of the regularized NLS equation (3.1) with $u(0) = u_0$. In turn, by combining the conservation of energy and charge (3.2) we obtain that the sequence of approximating solutions u^m is bounded in $L^\infty(\mathbb{R}, \Sigma(\mathbb{R}^N))$ (see Step 2 of [11, Theorem 9.3.4] for example). On the other hand, notice that the following inequality can be easily shown

$$|g_m(z)|^2 \leqslant C\left(|z|^{2+\frac{1}{N}} + |z|^{2-\frac{1}{N}}\right) \quad \text{for all } z \in \mathbb{C} \text{ and all } m \in \mathbb{N}.$$

Hence, by Lemma 2.4, we get that $g_m(u^m)$ is bounded in $L^\infty(\mathbb{R}, \Sigma'(\mathbb{R}^N))$, which implies by (3.1) that the sequence $\partial_t u^m$ is bounded in $L^\infty(\mathbb{R}, \Sigma'(\mathbb{R}^N))$. We can now conclude that $\{u^m\}_{m\in\mathbb{N}}$ satisfies the conditions of Lemma 3.2. Let u be the limit of u^m.

We claim that the limiting function $u \in L^{\infty}(\mathbb{R}, \Sigma(\mathbb{R}^N))$ is a weak solution of the NLS (1.1). Indeed, it follows from property (i) of Lemma 3.2 that $u(0) = u_0$. Furthermore, by (3.1), for any test function $\psi \in C_0^{\infty}(\mathbb{R}^N)$ and $\phi \in C_0^{\infty}(\mathbb{R})$ we get

$$\int_{\mathbb{R}} \left[-\langle iu^m, \psi \rangle \phi'(t) + \langle u^m, \psi \rangle_{\Sigma} \phi(t) \right] dt + \int_{\mathbb{R}} \int_{\mathbb{R}^N} g_m(u^m) \psi \phi \, dx \, dt = 0. \tag{3.3}$$

Taking into account that $g_m(z) \to z \operatorname{Log} |z|^2$ pointwise in $z \in \mathbb{C}$ as $m \to +\infty$, via properties (i)–(iii) of Lemma 3.2 we get the following integral equation (see Step 3 of [11, Theorem 9.3.4])

$$\int_{\mathbb{R}} \left[-\langle iu, \psi \rangle \phi'(t) + \langle u, \psi \rangle_{\Sigma} \phi(t) \right] dt + \int_{\mathbb{R}} \int_{\mathbb{R}^N} u \operatorname{Log} |u|^2 \psi \phi \, dx \, dt = 0. \tag{3.4}$$

We are now ready to conclude the proof of proposition. Since $u \in L^{\infty}(\mathbb{R}, \Sigma(\mathbb{R}^N))$, in view of Lemma 2.6 and (3.4) we see that $u_t \in L^{\infty}(\mathbb{R}, \Sigma'(\mathbb{R}^N))$ and u is a weak solution of problem (1.1). Next, to prove the uniqueness of the weak solution and the conservation of charge and energy one can follow the argument of [11, Theorem 9.3.4]. The proof is now concluded. □

4. Variational analysis

This section is devoted to the proof of Theorem 1.2. We begin with the logarithmic Sobolev inequality. See [21, Theorem 8.14].

Lemma 4.1. *Let $\alpha > 0$ and assume that $f \in H^1(\mathbb{R}^N) \setminus \{0\}$. Then*

$$\int_{\mathbb{R}^N} \left| f(x) \right|^2 \operatorname{Log} \left| f(x) \right|^2 dx \leqslant \frac{\alpha^2}{\pi} \| \nabla f \|_{L^2}^2 + \left(\operatorname{Log} \| f \|_{L^2}^2 - N(1 + \operatorname{Log} \alpha) \right) \| f \|_{L^2}^2. \tag{4.1}$$

Furthermore, there is equality if and only if the function f is, up to translation, a multiple of $e^{\{-\pi |x|^2/2\alpha^2\}}$.

Lemma 4.2 (Ground energy)**.** *Let $\omega \in \mathbb{R}$ and $\gamma > 1$. Then, the quantity $d(\omega)$ is given by*

$$d(\omega) = \frac{1}{2} \| \phi_{\omega} \|_{L^2}^2 = \frac{1}{2} \pi^{\frac{N}{2}} \gamma^{-\frac{N}{2}} e^{\omega + \gamma N}, \tag{4.2}$$

where ϕ_{ω} is defined by (1.5).

Proof. We observe for further usage that $\| \phi_{\omega} \|_{L^2}^2 = \pi^{\frac{N}{2}} \gamma^{-\frac{N}{2}} e^{\omega + \gamma N}$ for every $\omega \in \mathbb{R}$. We first prove $2d(\omega) \leqslant \| \phi_{\omega} \|_{L^2}^2$. By direct computations, we obtain that $I_{\omega}(\phi_{\omega}) = 0$, which implies that, by the definition of $d(\omega)$, $2d(\omega) \leqslant \| \phi_{\omega} \|_{L^2}^2$. On the other hand, it is easily seen that

$$\inf \left\{ \| \nabla u \|_{L^2}^2 + \gamma^2 \| xu \|_{L^2}^2 : u \in \Sigma\left(\mathbb{R}^N\right), \| u \|_{L^2}^2 = 1 \right\} = \gamma N. \tag{4.3}$$

In particular, multiplying (4.3) by $\gamma^{-1}(\gamma - 1)$ we get

$$(\gamma - 1) N \| u \|_{L^2}^2 \leqslant \gamma^{-1}(\gamma - 1) \| \nabla u \|_{L^2}^2 + \gamma(\gamma - 1) \| xu \|_{L^2}^2. \tag{4.4}$$

Now, let $u \in \Sigma(\mathbb{R}^N) \setminus \{0\}$ be such that $I_\omega(u) = 0$. By virtue of the logarithmic Sobolev inequality with $\alpha^2 = \pi/\gamma$ and inequality (4.4) we get

$$\big(\omega + \gamma N + N \operatorname{Log}(\sqrt{\pi/\gamma})\big)\|u\|_{L^2}^2 \leqslant \big(\operatorname{Log}\|u\|_{L^2}^2\big)\|u\|_{L^2}^2,$$

which implies that $\|u\|_{L^2}^2 \geqslant \|\phi_\omega\|_{L^2}^2$. Then, in view of the definition of $d(\omega)$, it follows that $2d(\omega) \geqslant \|\phi_\omega\|_{L^2}^2$. This conclude the proof. □

Next we give a useful lemma.

Lemma 4.3. *Assume that $\{u_n\}$ is a bounded sequence in $\Sigma(\mathbb{R}^N)$ satisfying as $n \to \infty$, $u_n \to u$ a.e. in $\mathbb{R}^N$. Then $u \in \Sigma(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \big\{|u_n|^2 \operatorname{Log}|u_n|^2 - |u_n - u|^2 \operatorname{Log}|u_n - u|^2\big\}\,dx = \int_{\mathbb{R}^N} |u|^2 \operatorname{Log}|u|^2\,dx.$$

Proof. Taking into account that $\Sigma(\mathbb{R}^N) \hookrightarrow L^A(\mathbb{R}^N)$, the assertion follows by [2, Lemma 2.3] (see also [8]). □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\{u_n\} \subseteq \Sigma(\mathbb{R}^N)$ be a minimizing sequence for $d(\omega)$, namely, $I_\omega(u_n) = 0$ for all n, and $S_\omega(u_n) \to d(\omega)$ as $n \to \infty$. Notice that sequence $\{u_n\}$ is bounded in $\Sigma(\mathbb{R}^N)$. In fact, it is clear that the sequence $\|u_n\|_{L^2}^2$ is bounded. Furthermore, by virtue of the logarithmic Sobolev inequality (4.1) and recalling that $I_\omega(u_n) = 0$, we obtain

$$\left(1 - \frac{\alpha^2}{\pi}\right)\|\nabla u_n\|_{L^2}^2 + \gamma(\gamma - 1)\|x u_n\|_{L^2}^2 \leqslant \operatorname{Log}\left[\left(\frac{e^{-(\omega+N)}}{\alpha^N}\right)\|u_n\|_{L^2}^2\right]\|u_n\|_{L^2}^2.$$

Now by taking sufficiently small positive $\alpha > 0$ enables us to conclude that all minimizing sequences are bounded in $\Sigma(\mathbb{R}^N)$. This implies that there exists some function $\varphi \in \Sigma(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup \varphi$ weakly in $\Sigma(\mathbb{R}^N)$ and this implies, by virtue of Lemma 2.4(i) that as n goes to $+\infty$, $u_n \to \varphi$ in $L^q(\mathbb{R}^N)$ for $2 \leqslant q < 2^*$. In particular, we get $\|\varphi\|_{L^2}^2 = 2d(\omega)$.

Now, let us prove that $I_\omega(\varphi) = 0$. Assume by contradiction that $I_\omega(\varphi) < 0$. Notice that by simple computations, we can see that there is $0 < \lambda < 1$ such that $I_\omega(\lambda\varphi) = 0$. In view of definition of $d(\omega)$, we get

$$d(\omega) \leqslant S_\omega(\lambda\varphi) = \frac{1}{2}\|\lambda\varphi\|_{L^2}^2 < \frac{1}{2}\|\varphi\|_{L^2}^2 = d(\omega),$$

a contradiction. On the other hand, assume that $I_\omega(\varphi) > 0$. Since $u_n \rightharpoonup \varphi$ in $\Sigma(\mathbb{R}^N)$, it follows that

$$\|\nabla u_n\|_{L^2}^2 - \|\nabla u_n - \nabla\varphi\|_{L^2}^2 - \|\nabla\varphi\|_{L^2}^2 \to 0, \tag{4.5}$$

$$\|x u_n\|_{L^2}^2 - \|x u_n - x\varphi\|_{L^2}^2 - \|x\varphi\|_{L^2}^2 \to 0, \tag{4.6}$$

as $n \to \infty$. By combining (4.5) with (4.6) and Lemma 4.3 leads to

$$\lim_{n\to\infty} I_\omega(u_n - \varphi) = \lim_{n\to\infty} I_\omega(u_n) - I_\omega(\varphi) = -I_\omega(\varphi),$$

which combined with $I_\omega(\varphi) > 0$ implies that $I_\omega(u_n - \varphi) < 0$ for sufficiently large n. Then, by arguing as above, we can prove that

$$d(\omega) \leqslant \frac{1}{2} \lim_{n\to\infty} \|u_n - \varphi\|_{L^2}^2 = d(\omega) - \frac{1}{2}\|\varphi\|_{L^2}^2,$$

which is a contradiction. We get $I_\omega(\varphi) = 0$, and this implies, by virtue of the definition of $d(\omega)$, that $\varphi \in \mathcal{G}_\omega$.

Next we prove that $u_n \to \varphi$ in $\Sigma(\mathbb{R}^N)$. Notice that, on one hand, we have $u_n \to \varphi$ in $L^2(\mathbb{R}^N)$. On the other hand, since the sequence $\{u_n\}$ is bounded in $\Sigma(\mathbb{R}^N)$, it follows by (2.4) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} B\big(|u_n(x)|\big)\, dx = \int_{\mathbb{R}^N} B\big(|\varphi(x)|\big)\, dx,$$

which combined with $I_\omega(u_n) = I_\omega(\varphi) = 0$ for any $n \in \mathbb{N}$, gives

$$\begin{aligned} &\lim_{n\to\infty}\left[\|\nabla u_n\|_{L^2}^2 + \gamma(\gamma-1)\|x u_n\|_{L^2}^2 + \int_{\mathbb{R}^N} A\big(|u_n|\big)\, dx\right] \\ &\quad = \|\nabla\varphi\|_{L^2}^2 + \gamma(\gamma-1)\|x\varphi\|_{L^2}^2 + \int_{\mathbb{R}^N} A\big(|\varphi|\big)\, dx, \end{aligned} \tag{4.7}$$

and this implies, by virtue of (4.7), the weak lower semicontinuity and Fatou's Lemma, that (see e.g. [18, Lemma 12 in chapter V])

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2}^2 = \|\nabla\varphi\|_{L^2}^2, \qquad \lim_{n\to\infty} \|x u_n\|_{L^2}^2 = \|x\varphi\|_{L^2}^2. \tag{4.8}$$

Therefore, it follows from (4.8) that $u_n \to \varphi$ in $\Sigma(\mathbb{R}^N)$. This proves the first part of the statement of Theorem 1.2.

Now we claim that $|\varphi| \in \mathcal{G}_\omega$ and $|\varphi|$ is necessarily radially symmetric. Indeed, denoting by φ^* the Schwarz symmetrization of $|\varphi|$, since $A, B \in C^1([0,+\infty))$ are increasing functions with $A(0) = B(0) = 0$, it is follows from Layer cake representation [21, Theorem 1.13] and (2.1) that

$$\int_{\mathbb{R}^N} \big|\varphi^*(x)\big|^2 \operatorname{Log}\big|\varphi^*(x)\big|^2\, dx = \int_{\mathbb{R}^N} \big|\varphi(x)\big|^2 \operatorname{Log}\big|\varphi(x)\big|^2\, dx.$$

Moreover, as it is readily checked,

$$\int_{\mathbb{R}^N} |x|^2\big|\varphi^*(x)\big|^2\, dx < \int_{\mathbb{R}^N} |x|^2\big|\varphi(x)\big|^2\, dx \quad \text{unless} \quad |\varphi| = \varphi^* \quad \text{a.e.}$$

Thus, since we have that $\|\nabla\varphi^*\|_{L^2}^2 \leqslant \|\nabla|\varphi|\|_{L^2}^2 \leqslant \|\nabla\varphi\|_{L^2}^2$ and $\|\varphi^*\|_{L^2}^2 = \|\varphi\|_{L^2}^2$, it follows that if $|\varphi| \neq \varphi^*$, then $I_\omega(\varphi^*) < I_\omega(\varphi) = 0$ with $\|\varphi^*\|_{L^2}^2 = \|\varphi\|_{L^2}^2$, which is a contradiction because $\varphi \in \mathcal{G}_\omega$. This contradiction finishes the proof of claim.

By virtue of Lemma 4.2 it follows that $\{e^{i\theta}\phi_\omega : \theta \in \mathbb{R}\} \subseteq \mathcal{G}_\omega$. Next let us consider $\varphi \in \mathcal{G}_\omega$. Taking into account the definition of $d(\omega)$, $\|\varphi\|_{L^2}^2 = 2d(\omega)$ and $I_\omega(\varphi) = 0$. We claim that the function φ satisfies the equality in (4.1) with $\alpha^2 = \pi/\gamma$. Let us assume the contrary, i.e. suppose that we have the strict inequality in (4.1) with $\alpha^2 = \pi/\gamma$. Since φ satisfies $I_\omega(\varphi) = 0$, a direct computation yields $\|\varphi\|_{L^2}^2 > 2d(\omega)$ (see proof of Lemma 4.2), a contradiction. Thus, in light of Lemma 4.1 we have that there exist $r > 0$, $y \in \mathbb{R}^N$ and $\theta_0 \in \mathbb{R}$ such that

$$\varphi(x) = re^{i\theta_0}e^{-\frac{\gamma}{2}|x-y|^2}, \quad x \in \mathbb{R}^N.$$

Since $|\varphi|$ is radial and $\|\varphi\|_{L^2}^2 = 2d(\omega)$, we conclude that $y = 0$ and $r^2 = e^{\omega+\gamma N}$. Hence, $\varphi(x) = e^{i\theta_0}\phi_\omega(x)$ and the Theorem 1.2 is proved. □

5. Stability of standing waves

Proof of Theorem 1.5. We argue by contradiction. Suppose that ϕ_ω is not stable in $\Sigma(\mathbb{R}^N)$ under flow associated with problem (1.1). Then there exist $\varepsilon > 0$, a sequence of initial data $(u_{n,0})_{n\in\mathbb{N}}$ in $\Sigma(\mathbb{R}^N)$ such that for all $n \geqslant 1$,

$$\|u_{n,0} - \phi_\omega\|_\Sigma < \frac{1}{n}, \tag{5.1}$$

and a sequence $(t_n)_{n\in\mathbb{N}}$ such that

$$\inf_{\theta\in\mathbb{R}}\left\|u_n(t_n) - e^{i\theta}\phi_\omega\right\|_\Sigma \geqslant \varepsilon, \quad \text{for any } n \in \mathbb{N}, \tag{5.2}$$

where u_n denotes the unique solution of problem (1.1) with initial data $u_{n,0}$. Now, setting $v_n(x) = u_n(x, t_n)$ it follows by (5.1) and conservation laws

$$\|v_n\|_{L^2}^2 = \left\|u_n(t_n)\right\|_{L^2}^2 = \|u_{n,0}\|_{L^2}^2 \to \|\phi_\omega\|_{L^2}^2, \tag{5.3}$$

$$E(v_n) = E\big(u_n(t_n)\big) = E(u_{n,0}) \to E(\phi_\omega). \tag{5.4}$$

Consequently, by virtue of (5.3) and (5.4),

$$S_\omega(v_n) \to S_\omega(\phi_\omega) = d(\omega). \tag{5.5}$$

Thus, (5.3) together with (5.5) implies that $I_\omega(v_n) \to 0$ as n goes to $+\infty$. Next, let us set $f_n(x) = \rho_n v_n(x)$ with

$$\rho_n = \exp\left(\frac{I_\omega(v_n)}{2\|v_n\|_{L^2}^2}\right),$$

where $\exp(x)$ represent the exponential function. We know that $\rho_n \to 1$ as n goes to $+\infty$, and $I_\omega(f_n) = 0$ for any $n \in \mathbb{N}$. Since $\{v_n\}$ is bounded in $\Sigma(\mathbb{R}^N)$, it follows immediately that $\|v_n - f_n\|_\Sigma \to 0$. By virtue of (5.5), we see that $\{f_n\}$ is a minimizing sequence for $d(\omega)$. Thanks to Theorem 1.2 we know that, up to a subsequence, there exists $\theta_0 \in \mathbb{R}$ such that

$$\left\| f_n - e^{i\theta_0}\phi_\omega \right\|_\Sigma \to 0, \quad \text{as } n \to +\infty. \tag{5.6}$$

Thus, in view of the triangular inequality, (5.6) and remembering that $v_n = u_n(t_n)$, one can easily proves that

$$\left\| u_n(t_n) - e^{i\theta_0}\phi_\omega \right\|_\Sigma \to 0 \quad \text{as } n \to +\infty,$$

which is a contradiction with (5.2). This completes the proof of the orbital stability of the ground states of (1.1). □

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