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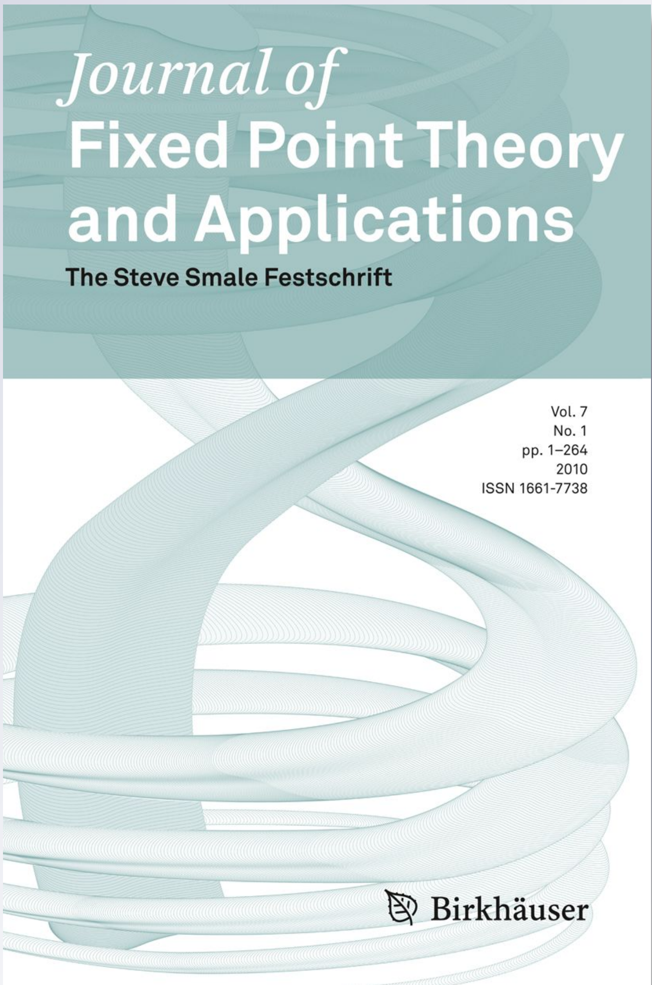

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Diffeomorphism-invariant properties for quasi-linear elliptic operators

Viviana Solferino and Marco Squassina

To Professor Dick Palais

Abstract. For quasi-linear elliptic equations we detect relevant properties which remain invariant under the action of a suitable class of diffeomorphisms. This yields a connection between existence theories for equations with degenerate and nondegenerate coerciveness.

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1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. In the study of the nonlinear equation

$$-\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) \quad \text{in } \Omega, \tag{1.1}$$

an important role is played by the coerciveness feature of j, namely the fact that there exists a positive constant $\sigma > 0$ such that

$$j(x,s,\xi) \geq \sigma|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N. \tag{1.2}$$

Under condition (1.2) and other suitable assumptions, including the boundedness of the map $s \mapsto j(x,s,\xi)$, equation (1.1) has been deeply investigated in the last twenty years by means of variational methods and tools of nonsmooth critical point theory, essentially via two different approaches (see, e.g., [3, 11] and the references therein). More recently, it was also covered the case where the map $s \mapsto j(x,s,\xi)$ is unbounded (see, e.g., [4, 21], again via different strategies). The situation is by far more delicate under the assumption of degenerate coerciveness, namely for some function $\sigma : \mathbb{R} \to \mathbb{R}^+$ with $\sigma(s) \to 0$ as $s \to \infty$,

$$j(x,s,\xi) \geq \sigma(s)|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N. \tag{1.3}$$

To the authors' knowledge, in this setting, for j of the form

$$(b(x)+|s|)^{-2\beta}|\xi|^2/2,$$

the first contribution to minimization problems is [9], while for existence of mountain pass type solutions, we refer the reader to [5], the main point being the fact that cluster points of arbitrary Palais–Smale sequences are bounded. See [1] for more general existence statements and [7, 6] for regularity results.

Relying upon a solid background for the treatment of (1.1) in the coercive case, the main goal of this paper is that of building a bridge between the theory for nondegenerate coerciveness problems and that for problems with degenerate coerciveness. Roughly speaking, we see a solution to a degenerate problem as related to a solution of a corresponding nondegenerate problem, preserving at the same time the main structural assumptions typically assumed for these classes of equations. To this aim, we introduce a suitable class of diffeomorphisms $\varphi \in C^2(\mathbb{R})$ and consider the functions $j^\sharp : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g^\sharp : \Omega \times \mathbb{R} \to \mathbb{R}$, defined as

$$j^\sharp(x,s,\xi) = j(x,\varphi(s),\varphi'(s)\xi), \qquad g^\sharp(x,s) = g(x,\varphi(s))\varphi'(s)$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$. Then, if (1.3) holds, we can find $\sigma^\sharp > 0$ such that

$$j^\sharp(x,s,\xi) \geq \sigma^\sharp|\xi|^2$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, thus recovering the nondegenerate coerciveness from the original degenerate framework. We write the corresponding Euler's equation as

$$-\operatorname{div}(j^\sharp_\xi(x,v,\nabla v)) + j^\sharp_s(x,v,\nabla v) = g^\sharp(x,v) \quad \text{in } \Omega. \tag{1.4}$$

A first natural issue is the correspondence between the solutions of (1.1) and the solutions of (1.4) through the diffeomorphism φ. Roughly speaking, the natural connection is that $u = \varphi(v)$ is a solution of (1.1) when v is a solution to (1.4), in some sense. On the other hand, in general, $\varphi(v) \notin H^1_0(\Omega)$ although $v \in H^1_0(\Omega)$. Hence, the notion of solution for functions in the Sobolev space $H^1_0(\Omega)$ cannot remain invariant under the action of φ, unless $v \in L^\infty(\Omega)$. In fact, we provide a new definition of generalized solution which is partly based upon the notion of renormalized solution introduced in [13] in the study of elliptic equations with general measure data and partly on the variational formulation adopted in [21]. The new notion turns out to be invariant under diffeomorphisms (Proposition 2.6) as well as conveniently related to the machinery developed in [21]. Moreover, we detect two relevant invariant conditions. The first (Proposition 2.11) is a modification of the standard (noninvariant) sign condition

$$j_s(x,s,\xi)s \geq 0 \quad \text{for all } |s| \geq R \text{ and some } R \geq 0, \tag{1.5}$$

namely there exist $\varepsilon \in (0,1)$ and $R \geq 0$ such that

$$(1-\varepsilon)j_\xi(x,s,\xi)\cdot\xi + j_s(x,s,\xi)s \geq 0 \tag{1.6}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $|s| \geq R$. Condition (1.5) is well known [3, 4, 5, 11, 21, 8, 24] and plays an important role in the study of both existence and summability issues for (1.1). In [17], Frehse provides a counterexample to the $L^\infty(\Omega)$-boundedness of the solutions when (1.5) is dropped. The second one (Proposition 2.15) is the generalized Ambrosetti–Rabinowitz condition [2]: there exist $\delta > 0$, $\nu > 2$ and $R \geq 0$ such that

$$\nu j(x,s,\xi) - (1+\delta) j_\xi(x,s,\xi)\cdot\xi - j_s(x,s,\xi)s - \nu G(x,s) + g(x,s)s \geq 0 \quad (1.7)$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $|s| \geq R$. Typically, this condition guarantees that an arbitrary Palais–Smale sequence is bounded [3, 4, 11, 21]. The invariant properties for growth conditions are stated in Propositions 2.3, 2.9 and 2.10. In the situations where

$$j_s^\sharp(x,s,\xi)s \geq 0 \quad \text{for all } |s| \geq R^\sharp \text{ and some } R^\sharp \geq 0,$$

the results of our paper allow to obtain existence and multiplicity of solutions for problems with degenerate coercivity by a *direct* application of the results of [21] (see Theorem 3.1). This is new compared with the results of [5], since the technique adopted therein does not allow to obtain multiplicity results. In addition, contrary to [5], under certain assumptions on the nonlinearity g, the solutions need not be bounded. The further development of the ideas in this paper is related to strengthening some of the results of [21] in order to allow the weaker sign condition (1.6) to replace the standard sign condition (1.5). Then existence and multiplicity theorems for coercive equations with unbounded coefficients automatically recover existence and multiplicity theorems for equations with degenerate coercivity. This will be the subject of a further investigation.

The plan of the paper is as follows. In Section 2.1 we introduce a new notion of generalized solution for (1.1) and prove that it is invariant under the action of φ. In Section 2.2 we show how φ affects some useful growth conditions. In Section 2.3 we study the invariance of the sign condition (1.6) and get some related summability results. In Section 2.4 we consider the invariance of an Ambrosetti–Rabinowitz (AR, in brief) type inequality (1.7). Finally, in Section 3 we get a new existence results for multiple, possibly unbounded, generalized solutions of (1.1).

2. Invariant properties

Now let Ω be a smooth bounded domain in $\mathbb{R}^N$. We consider $j : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ with $j(\cdot,s,\xi)$ measurable in Ω for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ and $j(x,\cdot,\cdot)$ of class C^1 for a.e. $x \in \Omega$. Moreover, we assume that the map $\xi \mapsto j(x,s,\xi)$ is strictly convex and there exist $\alpha,\gamma,\mu : \mathbb{R}^+ \to \mathbb{R}^+$ continuous with $\alpha(s) \geq 1$ for all $s \in \mathbb{R}^+$ and such that

$$\frac{1}{\alpha(|s|)}|\xi|^2 \leq j(x,s,\xi) \leq \alpha(|s|)|\xi|^2, \quad (2.1)$$

$$|j_s(x,s,\xi)| \leq \gamma(|s|)|\xi|^2, \qquad |j_\xi(x,s,\xi)| \leq \mu(|s|)|\xi| \quad (2.2)$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$. Actually, the second inequality of (2.2) can be deduced by the strict convexity of $\xi \mapsto j(x,s,\xi)$ and the right inequality of (2.1). Furthermore, again by the strict convexity of $\xi \mapsto j(x,s,\xi)$ and the left inequality of (2.1) it holds that

$$j_\xi(x,s,\xi)\cdot\xi \geq \frac{1}{\alpha(|s|)}|\xi|^2; \tag{2.3}$$

see [21, Remarks 4.1 and 4.3]. Without loss of generality, one may assume that $\alpha,\gamma,\mu:\mathbb{R}^+\to\mathbb{R}^+$ appearing in the growth conditions of j, j_s, j_ξ are monotonically increasing. Indeed, we can always replace them by the increasing functions $\alpha_0,\gamma_0,\mu_0:\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$\alpha_0(r) = \sup_{s\in[-r,r]} \alpha(|s|),$$
$$\gamma_0(r) = \sup_{s\in[-r,r]} \gamma(|s|),$$
$$\mu_0(r) = \sup_{s\in[-r,r]} \mu(|s|).$$

We also assume that $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that

$$\sup_{|t|\leq s}|g(\cdot,t)| \in L^1(\Omega) \quad \text{for every } s\in\mathbb{R}^+, \tag{2.4}$$

and we set $G(x,s)=\int_0^s g(x,t)dt$ for every $s\in\mathbb{R}$.

Definition 2.1. For an odd diffeomorphism $\varphi:\mathbb{R}\to\mathbb{R}$ of class C^2 such that $\varphi(0)=0$, we consider the following properties:

$$\varphi'(s) \geq \sigma\sqrt{\alpha(|\varphi(s)|)} \quad \text{for all } s\in\mathbb{R} \text{ and some } \sigma>0, \tag{2.5}$$

$$\lim_{s\to+\infty}\frac{s\varphi'(s)}{\varphi(s)} = 1 + \lim_{s\to+\infty}\frac{s\varphi''(s)}{\varphi'(s)} = \frac{1}{1-\beta} \quad \text{for some } \beta\in[0,1). \tag{2.6}$$

A simple model satisfying the requirements of Definition 2.1 is the function

$$\varphi(s) = s(1+s^2)^{\frac{\beta}{2(1-\beta)}} \quad \text{for all } s\in\mathbb{R},\ 0\leq\beta<1, \tag{2.7}$$

in the case when $\alpha(t)=C(1+t)^{2\beta}$ for some $C>0$.

Definition 2.2. Consider the functions

$$j:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}, \quad g:\Omega\times\mathbb{R}\to\mathbb{R}, \quad G:\Omega\times\mathbb{R}\to\mathbb{R},$$

and let $\varphi\in C^2(\mathbb{R})$ be a diffeomorphism according to Definition 2.1. We define

$$j^\sharp:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}, \quad g^\sharp:\Omega\times\mathbb{R}\to\mathbb{R}, \quad G^\sharp:\Omega\times\mathbb{R}\to\mathbb{R}$$

by setting

$$j^\sharp(x,s,\xi) = j(x,\varphi(s),\varphi'(s)\xi)$$

for a.e. $x\in\Omega$ and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and

$$g^\sharp(x,s) = g(x,\varphi(s))\varphi'(s), \qquad G^\sharp(x,s) = \int_0^s g^\sharp(x,t)dt = G(x,\varphi(s))$$

for a.e. $x\in\Omega$ and all $s\in\mathbb{R}$.

Now we see that φ turns a degenerate problem associated with j into a nondegenerate one associated with $j^\sharp$, and that $j^\sharp, j_s^\sharp$ and $j_\xi^\sharp$ satisfy growths analogous to those of j, j_s and j_ξ.

Proposition 2.3. *Let $\varphi \in C^2(\mathbb{R})$ be a diffeomorphism which satisfies the properties of Definition* 2.1*. Assume that $\alpha, \gamma, \mu : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the growth conditions* (2.1)–(2.2)*. Then there exist continuous functions $\alpha^\sharp, \gamma^\sharp, \mu^\sharp : \mathbb{R}^+ \to \mathbb{R}^+$ and $\sigma^\sharp > 0$ such that*

$$\sigma^\sharp|\xi|^2 \le j^\sharp(x, s, \xi) \le \alpha^\sharp(|s|)|\xi|^2,$$
$$|j_s^\sharp(x, s, \xi)| \le \gamma^\sharp(|s|)|\xi|^2, \qquad |j_\xi^\sharp(x, s, \xi)| \le \mu^\sharp(|s|)|\xi|$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

Proof. In light of (2.1) and of (2.5) of Definition 2.1, for $\sigma^\sharp = \sigma^2$, we have

$$\sigma^\sharp|\xi|^2 \le \frac{\varphi'(s)^2}{\alpha(|\varphi(s)|)}|\xi|^2 \le j(x, \varphi(s), \varphi'(s)\xi) \le \alpha(|\varphi(s)|)\varphi'(s)^2|\xi|^2$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Furthermore, by virtue of (2.2), we have

$$|j_\xi^\sharp(x, s, \xi)| \le (\varphi'(s))^2\mu(|\varphi(s)|)|\xi|$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, as well as

$$|j_s^\sharp(x, s, \xi)| \le \Big[|\varphi''(s)|\mu(|\varphi(s)|)\varphi'(s) + (\varphi'(s))^3\gamma(|\varphi(s)|)\Big]|\xi|^2$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. The assertions follow with $\alpha^\sharp, \gamma^\sharp, \mu^\sharp : \mathbb{R} \to \mathbb{R}^+$,

$$\begin{aligned} \alpha^\sharp(s) &= \alpha(|\varphi(s)|)\varphi'(s)^2, \\ \gamma^\sharp(s) &= |\varphi''(s)|\mu(|\varphi(s)|)\varphi'(s) + (\varphi'(s))^3\gamma(|\varphi(s)|), \\ \mu^\sharp(s) &= (\varphi'(s))^2\mu(|\varphi(s)|) \end{aligned}$$

for all $s \in \mathbb{R}$. Of course, without loss of generality, one can then substitute $\alpha^\sharp, \gamma^\sharp, \mu^\sharp$ with even functions satisfying the same growth controls. ☐

2.1. Generalized solutions

For any $k > 0$, consider the truncation $T_k : \mathbb{R} \to \mathbb{R}$,

$$T_k(s) = \begin{cases} s & \text{for } |s| \le k, \\ k\,\mathrm{sign}(s) & \text{for } |s| \ge k. \end{cases}$$

Moreover, as in [21], for a measurable function $u : \Omega \to \mathbb{R}$, let us consider the space

$$V_u = \big\{v \in H_0^1(\Omega) \cap L^\infty(\Omega) : u \in L^\infty(\{v \ne 0\})\big\}. \tag{2.8}$$

This functional space was originally introduced by Degiovanni and Zani [16] for functions u of $H_0^1(\Omega)$, in which case V_u turns out to be a dense subspace of $H_0^1(\Omega)$. Observe that, in view of conditions (2.2) and (2.4), it follows that

$$j_\xi(x, u, \nabla u) \cdot \nabla v \in L^1(\Omega), \quad j_s(x, u, \nabla u)v \in L^1(\Omega), \quad g(x, u)v \in L^1(\Omega)$$

for every $v \in V_u$ and any measurable $u : \Omega \to \mathbb{R}$ with $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$. For such functions, according to [13], the meaning of ∇u will be made clear in the proof of Proposition 2.6.

In the spirit of [13], where the notion of renormalized solution is introduced, and [21], where the notion of generalized solution is given, based upon V_u, we now introduce the following definition.

Definition 2.4. We say that u is a generalized solution to

$$\begin{cases} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

if u is a measurable function finite a.e. such that

$$T_k(u) \in H_0^1(\Omega) \quad \text{for all } k > 0, \tag{2.10}$$

and, furthermore,

$$j_\xi(x,u,\nabla u) \cdot \nabla u \in L^1(\Omega), \qquad j_s(x,u,\nabla u)u \in L^1(\Omega), \tag{2.11}$$

and

$$\int_\Omega j_\xi(x,u,\nabla u) \cdot \nabla w + \int_\Omega j_s(x,u,\nabla u)w = \int_\Omega g(x,u)w \quad \text{for all } w \in V_u. \tag{2.12}$$

Remark 2.5. We point out that, in [21, Definition 1.1], a different notion of generalized solution of problem (2.9) is introduced when u belongs to the Sobolev space $H_0^1(\Omega)$. On the other hand, actually, by [21, Theorem 4.8] the two notions agree, whenever $u \in H_0^1(\Omega)$. Also, the variational formulation (2.12) with test functions in V_u is conveniently related to the weak slope [15, 12] of the functional associated with (2.9); see [21, Proposition 4.5] (see also Proposition 2.13).

In the framework of the previous definition, we provide in the following a suitable meaning for the gradient of a function u which satisfies (2.10). As proved in [13], for a measurable function u on Ω, finite a.e., with $T_k(u) \in H_0^1(\Omega)$ for any $k > 0$, there exists a unique $\omega : \Omega \to \mathbb{R}^N$, measurable and such that

$$\nabla T_k(u) = \omega \chi_{\{|u| \le k\}} \quad \text{a.e. in } \Omega \text{ and for all } k > 0. \tag{2.13}$$

Then, the gradient ∇u of u is naturally defined by setting $\nabla u = \omega$. Assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism with $\varphi(0) = 0$ and that for a measurable function v on Ω, it holds that $T_k(v) \in H_0^1(\Omega)$ for every $k > 0$. Then, setting $u = \varphi(v)$, it follows that $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$. In fact, given $k > 0$, there exists $h > 0$ such that $T_k(u) = (T_k \circ \varphi) \circ T_h(v)$. Since $T_k \circ \varphi : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous function which is zero at zero, it follows that $T_k(u) \in H_0^1(\Omega)$ for all $k > 0$. Moreover, if ∇u and ∇v denote the gradients of u and v, respectively, in the sense pointed out above, we get the following chain rule:

$$\nabla u = \varphi'(v)\nabla v \quad \text{a.e. in } \Omega. \tag{2.14}$$

In fact, for all $k > 0$, since $T_k(u), T_h(v) \in H_0^1(\Omega)$, from $T_k(u) = (T_k \circ \varphi) \circ T_h(v)$ we can write

$$\nabla T_k(u) = (T_k \circ \varphi)'(T_h(v)) \nabla T_h(v)$$

for every $k > 0$, namely, by (2.13),

$$\nabla u \chi_{\{|\varphi(v)| \le k\}} = (T_k \circ \varphi)'(T_h(v)) \nabla v \chi_{\{|v| \le h\}} \quad \text{a.e. in } \Omega. \tag{2.15}$$

Let now $x \in \Omega$ be an arbitrary point with $|v(x)| \le h$. In turn, by construction, $|\varphi(v(x))| \le k$, and formula (2.15) directly yields

$$\nabla u = (T_k \circ \varphi)'(v) \nabla v \quad \text{a.e. in } \{|v| \le h\}. \tag{2.16}$$

Formula (2.14) then follows by taking into account that $(T_k \circ \varphi)'(v(x)) = \varphi'(v(x))$ a.e. in $\{|v| \le h\}$ and by the arbitrariness of $h > 0$.

The following proposition establishes a link between the generalized solutions of the problem under the change-of-variable procedure.

Proposition 2.6. *Let $\varphi \in C^2(\mathbb{R})$ be a diffeomorphism which satisfies the properties of Definition 2.1. Assume that v is a generalized solution to*

$$\begin{cases} -\operatorname{div}(j_\xi^\sharp(x, v, \nabla v)) + j_s^\sharp(x, v, \nabla v) = g^\sharp(x, v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

Then $u = \varphi(v)$ is a generalized solution to

$$\begin{cases} -\operatorname{div}(j_\xi(x, u, \nabla u)) + j_s(x, u, \nabla u) = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

If in addition $v \in H_0^1 \cap L^\infty(\Omega)$, then $u \in H_0^1 \cap L^\infty(\Omega)$ is a distributional solution to (2.18).

Proof. Let v be a generalized solution to (2.17), so that $T_k(v) \in H_0^1(\Omega)$ for all $k > 0$. As pointed out above, it follows that $T_k(u) \in H_0^1(\Omega)$ too, for every $k > 0$ and the chain rule $\nabla u = \varphi'(v) \nabla v$ holds a.e. in Ω. From the definition of generalized solution we learn that

$$j_\xi^\sharp(x, v, \nabla v) \cdot \nabla v \in L^1(\Omega), \qquad j_s^\sharp(x, v, \nabla v) v \in L^1(\Omega), \tag{2.19}$$

as well as

$$\int_\Omega j_\xi^\sharp(x, v, \nabla v) \cdot \nabla w + \int_\Omega j_s^\sharp(x, v, \nabla v) w = \int_\Omega g^\sharp(x, v)\, w \quad \text{for all } w \in V_v. \tag{2.20}$$

Notice that, for any $w \in V_v$, the integrands in (2.20) are in $L^1(\Omega)$, by Proposition 2.3, the definition of V_v and $\nabla v = \nabla T_k(v) \in L^2(\{w \ne 0\})$ for any $k > \|v\|_{L^\infty(\{w \ne 0\})}$. In light of (2.14) and (2.19), it follows that

$$j_\xi(x, u, \nabla u) \cdot \nabla u = j_\xi^\sharp(x, v, \nabla v) \cdot \nabla v \in L^1(\Omega).$$

Moreover, a simple computation yields

$$j_s^\sharp(x, v, \nabla v) v = \left[\frac{v \varphi'(v)}{\varphi(v)} \chi_{\{v \ne 0\}} \right] j_s(x, u, \nabla u) u + \left[\frac{v \varphi''(v)}{\varphi'(v)} \right] j_\xi(x, u, \nabla u) \cdot \nabla u.$$

Hence, in view of (2.6), it follows that $j_s(x,u,\nabla u)u \in L^1(\Omega)$, being

$$j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega) \qquad \text{and} \qquad j_s^\sharp(x,v,\nabla v)v \in L^1(\Omega).$$

This yields the desired summability conditions. For any $w \in V_v$, consider now $\hat{w} = \varphi'(v)w$. We have $\hat{w} \in V_u$. In fact, since $v \in L^\infty(\{w \neq 0\})$, we obtain $\hat{w} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $u = \varphi(v) \in L^\infty(\{w \neq 0\}) = L^\infty(\{\hat{w} \neq 0\})$, since φ' is positive by virtue of (2.5). Of course, we have $\hat{w} = \varphi'(T_k(v))w$, for all $k > \|v\|_{L^\infty(\{w\neq 0\})}$. Hence, recalling (2.13), from

$$\nabla(\varphi'(T_k(v))w) = w\varphi''(T_k(v))\nabla v\chi_{\{|v|\leq k\}} + \varphi'(T_k(v))\nabla w \quad \text{for any } k > 0,$$

by choosing $k > \|v\|_{L^\infty(\{w\neq 0\})}$, we conclude that

$$\nabla\hat{w} = w\varphi''(v)\nabla v + \varphi'(v)\nabla w \quad \text{a.e. in } \Omega.$$

Therefore, by easy computations, we get

$$j_\xi(x,u,\nabla u)\cdot\nabla\hat{w} = j_\xi^\sharp(x,v,\nabla v)\cdot\nabla w + \frac{\varphi''(v)w}{\varphi'(v)} j_\xi(x,u,\nabla u)\cdot\nabla u, \tag{2.21}$$

$$j_s(x,u,\nabla u)\hat{w} = j_s^\sharp(x,v,\nabla v)w - \frac{\varphi''(v)w}{\varphi'(v)} j_\xi(x,u,\nabla u)\cdot\nabla u, \tag{2.22}$$

yielding

$$j_\xi(x,u,\nabla u)\cdot\nabla\hat{w} \in L^1(\Omega), \qquad j_s(x,u,\nabla u)\hat{w} \in L^1(\Omega),$$

since $j_\xi^\sharp(x,v,\nabla v)\cdot\nabla w \in L^1(\Omega)$, $j_s^\sharp(x,v,\nabla v)w \in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega \left|\frac{\varphi''(v)w}{\varphi'(v)}\, j_\xi(x,u,\nabla u)\cdot\nabla u\right| &= \int_{\{w\neq 0\}} \left|\frac{\varphi''(v)w}{\varphi'(v)}\, j_\xi(x,u,\nabla u)\cdot\nabla u\right| \\ &\leq C\int_\Omega \big|\, j_\xi(x,u,\nabla u)\cdot\nabla u\,\big|.\end{aligned}$$

By adding identities (2.21)–(2.22) and recalling the definition of $g^\sharp(x,v)$, we get from (2.20)

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\hat{w} + \int_\Omega j_s(x,u,\nabla u)\hat{w} = \int_\Omega g(x,u)\hat{w}, \quad \hat{w} = \varphi'(v)w \in V_u.$$

Given any $z \in V_u$, we have

$$w = \frac{z}{\varphi'(v)} = \frac{z}{\varphi'(T_k(v))} \in V_v \quad \text{for } k > \|v\|_{L^\infty(\{z\neq 0\})}.$$

In turn,

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla z + \int_\Omega j_s(x,u,\nabla u)z = \int_\Omega g(x,u)z \quad \text{for every } z \in V_u,$$

yielding the assertion. Finally, if v is a bounded generalized solution to (2.17), $u \in H_0^1(\Omega)$ is bounded too and it follows that $u = \varphi(v)$ is a distributional solution to (2.18). $\square$

Remark 2.7. The gradient $\nabla u = \omega$ does not agree, in general, with the one in the sense of distributions, since it could be either $u \notin L^1_{\rm loc}(\Omega)$ or $\omega \notin L^1_{\rm loc}(\Omega, \mathbb{R}^N)$. If $\omega \in L^1_{\rm loc}(\Omega, \mathbb{R}^N)$, then $u \in W^{1,1}_{\rm loc}(\Omega)$ and ω agrees with the distributional gradient [13, Remark 2.10].

Under natural regularity assumptions, a generalized solution is, actually, distributional.

Proposition 2.8. *Assume that u is a generalized solution to problem* (2.9) *and that, in addition,*

$$j_\xi(x,u,\nabla u) \in L^1_{\rm loc}(\Omega;\mathbb{R}^N), \quad j_s(x,u,\nabla u) \in L^1_{\rm loc}(\Omega), \quad g(x,u) \in L^1_{\rm loc}(\Omega). \tag{2.23}$$

Then u solves problem (2.9) *in the sense of distributions.*

Proof. Let $H : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function such that $0 \leq H \leq 1$, $H(s) = 1$ for $|s| \leq 1$ and $H(s) = 0$ for $|s| \geq 2$. Given $k > 0$ and $\varphi \in C^\infty_c(\Omega)$, consider in formula (2.12) the admissible test functions $w = w_k = H(T_{2k+1}(u)/k)\varphi \in V_u$. Whence, for every $k > 0$, it holds that

$$\begin{aligned} &\int_\Omega j_\xi(x,u,\nabla u)\cdot H(T_{2k+1}(u)/k)\nabla\varphi \\ &\quad + \int_\Omega j_\xi(x,u,\nabla u)\cdot H'(T_{2k+1}(u)/k)1/k\nabla T_{2k+1}(u)\varphi \\ &\quad + \int_\Omega j_s(x,u,\nabla u)H(T_{2k+1}(u)/k)\varphi = \int_\Omega g(x,u)H(T_{2k+1}(u)/k)\varphi. \end{aligned} \tag{2.24}$$

Taking into account that $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ and by (2.13), for all $k > 0$, we have

$$\begin{aligned} &\big|j_\xi(x,u,\nabla u)\cdot H'(T_{2k+1}(u)/k)1/k\nabla T_{2k+1}(u)\varphi\big| \\ &\qquad\qquad \leq C\big|j_\xi(x,u,\nabla u)\cdot\nabla u\big| \in L^1(\Omega), \end{aligned}$$

yielding, by the Dominated Convergence Theorem,

$$\lim_k \int_\Omega j_\xi(x,u,\nabla u)\cdot H'(T_{2k+1}(u)/k)1/k\nabla T_{2k+1}(u)\varphi = 0.$$

On account of assumptions (2.23), the assertion follows by letting $k \to \infty$ in (2.24), again in light of the Dominated Convergence Theorem. $\square$

2.2. Further growth conditions

The next proposition is useful for the study of the mountain pass geometry of the functional associated with problem (1.1).

Proposition 2.9. *Let $\varphi \in C^2(\mathbb{R})$ be a diffeomorphism satisfying the properties of Definition* 2.1 *and such that*

$$0 < \lim_{s\to+\infty} \frac{\varphi(s)}{s^{\frac{1}{1-\beta}}} < +\infty, \tag{2.25}$$

and let $\alpha^\sharp : \mathbb{R}^+ \to \mathbb{R}^+$ *be the function introduced in Proposition* 2.3. *Let* $\nu > 2(1-\beta)$, $k_1 \in L^\infty(\Omega)$ *with* $k_1 > 0$, $k_2 \in L^1(\Omega)$, $k_3 \in L^{2N/(N+2)}(\Omega)$. *Assume that*

$$\lim_{s\to\infty} \frac{\alpha(|s|)}{|s|^{\nu-2}} = 0 \quad \text{and} \quad G(x,s) \geq k_1(x)|s|^\nu - k_2(x) - k_3(x)|s|^{1-\beta} \tag{2.26}$$

for a.e. $x \in \Omega$ *and all* $s \in \mathbb{R}$. *Then there exists* $\nu^\sharp > 2$ *such that*

$$\lim_{s\to\infty} \frac{\alpha^\sharp(|s|)}{|s|^{\nu^\sharp-2}} = 0 \quad \text{and} \quad G^\sharp(x,s) \geq k_1^\sharp(x)|s|^{\nu^\sharp} - k_2^\sharp(x) - k_3^\sharp(x)|s|$$

for a.e. $x \in \Omega$ *and all* $s \in \mathbb{R}$, *for some* $k_1^\sharp \in L^\infty(\Omega)$, $k_1^\sharp > 0$, $k_2^\sharp \in L^1(\Omega)$ *and* $k_3^\sharp \in L^{\frac{2N}{N+2}}(\Omega)$.

Proof. By assumptions (2.25) and (2.6), for $\nu^\sharp = \frac{\nu}{1-\beta}$, we have

$$\lim_{s\to+\infty} \frac{\alpha^\sharp(s)}{s^{\nu^\sharp-2}} = \lim_{s\to\infty} \frac{\alpha(\varphi(s))}{\varphi(s)^{\nu-2}} \cdot \lim_{s\to\infty} \frac{\varphi(s)^{\nu-2}\varphi'(s)^2}{s^{\nu^\sharp-2}} = 0.$$

Finally, if $G(x,s) \geq k_1(x)|s|^\nu - k_2(x) - k_3(x)|s|^{1-\beta}$, condition (2.25) yields

$$\begin{aligned} G^\sharp(x,s) &\geq k_1(x)|\varphi(s)|^\nu - k_2(x) - k_3(x)|\varphi(s)|^{1-\beta} \\ &\geq k_1^\sharp(x)|s|^{\nu^\sharp} - k_2^\sharp(x) - k_3^\sharp(x)|s| \end{aligned}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, for suitable $k_j^\sharp : \Omega \to \mathbb{R}$, $j = 1, 2, 3$, with the stated summability. □

Now, we see how the nonlinearity g gets modified under the action of a diffeomorphism.

Proposition 2.10. *Let* $\varphi \in C^2(\mathbb{R})$ *be a diffeomorphism which satisfies the properties of Definition* 2.1 *with* $0 \leq \beta < 2/N$, $N \geq 3$ *and such that* (2.25) *holds. Let* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfy*

$$|g(x,s)| \leq a(x) + b|s|^{p-1} \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}, \tag{2.27}$$

for some $a \in L^{q+\beta q(p-1)^{-1}}(\Omega)$, $q \geq \frac{2N}{N+2}$, $b \geq 0$ *with* $2 < p \leq 2^*(1-\beta)$. *Then, we have*

$$|g^\sharp(x,s)| \leq a^\sharp(x) + b|s|^{p^\sharp-1} \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R},$$

for some $2 < p^\sharp \leq 2^*$ *and* $a^\sharp \in L^q(\Omega)$.

Proof. Taking into account (2.25) and (2.6), for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ we have

$$\begin{aligned} |g^\sharp(x,s)| &\leq a(x)\varphi'(s) + b|\varphi(s)|^{p-1}\varphi'(s) \\ &\leq Ca(x) + C + Ca(x)^{\frac{p+\beta-1}{p-1}} + C|s|^{\frac{p}{1-\beta}-1}, \end{aligned}$$

yielding the assertion with $p^\sharp = \frac{p}{1-\beta}$ and $a^\sharp = Ca + C + Ca^{\frac{p+\beta-1}{p-1}}$. □

2.3. Sign conditions

The classical sign condition (1.5) is *not* invariant under diffeomorphism as Proposition 3.5 shows. The next proposition introduces a different kind of sign condition that remains invariant under the effect of φ.

Proposition 2.11. *Let $\varphi \in C^2(\mathbb{R})$ be a diffeomorphism which satisfies the properties of Definition* 2.1. *Assume that there exist $\varepsilon \in (0, 1-\beta]$ and $R \geq 0$ such that*

$$(1-\varepsilon) j_\xi(x, s, \xi) \cdot \xi + j_s(x, s, \xi) s \geq 0 \tag{2.28}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ with $|s| \geq R$.

Then there exist $\varepsilon^\sharp \in (0, 1]$ and $R^\sharp > 0$ such that

$$(1-\varepsilon^\sharp) j^\sharp_\xi(x, s, \xi) \cdot \xi + j^\sharp_s(x, s, \xi) s \geq 0$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ with $|s| \geq R^\sharp$.

Proof. Let us write $\varepsilon = \varepsilon_0(1-\beta)$ for some $\varepsilon_0 \in (0, 1]$. By taking into account (2.6), there exist $0 < \delta < \varepsilon_0(1 + \varepsilon_0(1-\beta))^{-1}$ and $R^\sharp > 0$ sufficiently large such that

$$1 + \frac{\varphi''(s)s}{\varphi'(s)} \geq \frac{\varphi'(s)s}{\varphi(s)} - \delta, \qquad \frac{\varphi'(s)s}{\varphi(s)} \geq \frac{1}{1-\beta} - \delta,$$

and $|\varphi(s)| \geq R$ for all $s \in \mathbb{R}$ with $|s| \geq R^\sharp$. Then, in turn, we get

$$\begin{aligned}
j^\sharp_\xi(x, s, \xi) &\cdot \xi + j^\sharp_s(x, s, \xi) s \\
&= \left(1 + \frac{\varphi''(s)s}{\varphi'(s)}\right) j_\xi(x, \varphi(s), \varphi'(s)\xi) \cdot \varphi'(s)\xi \\
&\quad + \frac{\varphi'(s)s}{\varphi(s)} j_s(x, \varphi(s), \varphi'(s)\xi)\varphi(s) \\
&\geq \frac{\varphi'(s)s}{\varphi(s)} \big(j_\xi(x, \varphi(s), \varphi'(s)\xi) \cdot \varphi'(s)\xi \\
&\quad + j_s(x, \varphi(s), \varphi'(s)\xi)\varphi(s) \big) \\
&\quad - \delta j_\xi(x, \varphi(s), \varphi'(s)\xi) \cdot \varphi'(s)\xi
\end{aligned}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ with $|s| \geq R^\sharp$. Setting

$$\varepsilon^\sharp = \varepsilon_0 - \delta(1 + \varepsilon_0(1-\beta)) \in (0, 1],$$

it follows by assumption that

$$\begin{aligned}
j^\sharp_\xi(x, s, \xi) \cdot \xi + j^\sharp_s(x, s, \xi) s &\geq \left(\varepsilon \frac{\varphi'(s)s}{\varphi(s)} - \delta\right) j_\xi(x, \varphi(s), \varphi'(s)\xi) \cdot \varphi'(s)\xi \\
&\geq \varepsilon^\sharp j^\sharp_\xi(x, s, \xi) \cdot \xi
\end{aligned}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ with $|s| \geq R^\sharp$. This concludes the proof. □

Remark 2.12. In the literature of quasi-linear problems like (1.1) the (say, positive) sign condition $j_s(x, s, \xi)s \geq 0$ is a classical assumption (cf. [3, 11] and the references therein), helping to achieve both existence and summability of the solutions. On the other hand, in [20], when $j(x, s, \xi) = A(x, s)\xi \cdot \xi$, the existence of solutions is obtained either with the opposite sign condition or even without any sign hypothesis at all. To handle this situation, alternative conditions as [20, Assumption 1.5] are assumed, which imply (2.28) (at least for $s \geq R$) for suitable ε, as it can be easily verified.

Under the generalized sign condition (2.28), we get a summability result which improves [21, Lemma 4.6]. This also shows that condition (2.11) in Definition 2.4 is natural. For a function f, the notation $|df|(u)$ stands for the weak slope of f at u (cf., e.g., [12, 15]).

Proposition 2.13. *Assume that* (2.2) *holds and that there exist* $\varepsilon \in (0, 1)$ *and* $R \geq 0$ *such that* (2.28) *holds. Let us set*

$$I(u) = \int_\Omega j(x, u, \nabla u), \quad u \in H_0^1(\Omega).$$

Then, for every $u \in \mathrm{dom}(I)$ *with* $|dI|(u) < +\infty$*, we have*

$$\int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla u + j_s(x, u, \nabla u)u \leq |dI|(u)\|u\|_{1,2}. \tag{2.29}$$

In particular, there holds

$$j_\xi(x, u, \nabla u) \cdot \nabla u \in L^1(\Omega), \qquad j_s(x, u, \nabla u)u \in L^1(\Omega),$$

and there exists $\Psi \in H^{-1}(\Omega)$ *with* $\|\Psi\|_{H^{-1}} \leq |dI|(u)$ *such that*

$$\int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla w + \int_\Omega j_s(x, u, \nabla u)w = \langle \Psi, w\rangle \quad \textit{for all } w \in V_u.$$

Proof. Let $b \in \mathbb{R}$ be such that $b > I(u)$. Notice first that if u is such that

$$\int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla u + j_s(x, u, \nabla u)u \leq 0,$$

then the conclusion holds. Otherwise, let σ be an arbitrary positive number such that

$$\int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla u + j_s(x, u, \nabla u)u > \sigma\|u\|_{1,2}.$$

Fixed $\eta > 0$, we set $\alpha^{-1} = \|u\|_{1,2}(1 + \eta)$. Let us prove that there exists $\delta > 0$ such that, for all $v \in B(u, \delta)$ and for any $\tau \in L^\infty(\Omega)$ with $\|\tau\|_\infty < \delta$, it follows that

$$\int_\Omega \Big[j_s(x, w, (1 - \alpha\tau)\nabla v)v + j_\xi(x, w, (1 - \alpha\tau)\nabla v) \cdot \nabla v\Big] > \sigma\|u\|_{1,2}, \tag{2.30}$$

where $w = (1 - \alpha\tau)v$. In fact, assume by contradiction that this is not the case. Then, we find a sequence $(v_n) \subset H_0^1(\Omega)$ with $\|v_n - u\|_{1,2} \to 0$ as $n \to \infty$

and a sequence $(\tau_n) \subset L^\infty(\Omega)$ with $\|\tau_n\|_\infty \to 0$ as $n \to \infty$ such that, denoting $w_n = (1-\alpha\tau_n)v_n$ for all $n \geq 1$, it holds that

$$\int_\Omega \Big[j_s(x, w_n, (1-\alpha\tau_n)\nabla v_n)v_n + j_\xi(x, w_n, (1-\alpha\tau_n)\nabla v_n)\cdot\nabla v_n \Big] \leq \sigma\|u\|_{1,2}. \tag{2.31}$$

Since $v_n \to u$ in $H_0^1(\Omega)$ and $\tau_n \to 0$ in $L^\infty(\Omega)$ as $n \to \infty$, a.e. in Ω, we have that

$$\begin{aligned} j_s(x, w_n, (1-\alpha\tau_n)\nabla v_n)v_n &+ j_\xi(x, w_n, (1-\alpha\tau_n)\nabla v_n)\cdot\nabla v_n \\ &\longrightarrow j_s(x, u, \nabla u)u + j_\xi(x, u, \nabla u)\cdot\nabla u. \end{aligned}$$

Moreover, there exists a positive constant $C(R)$ such that, for every $n \geq 1$,

$$j_s(x, w_n, (1-\alpha\tau_n)\nabla v_n)v_n + j_\xi(x, w_n, (1-\alpha\tau_n)\nabla v_n)\cdot\nabla v_n \geq -C(R)|\nabla v_n|^2. \tag{2.32}$$

In fact, if $|w_n(x)| \geq R$, from condition (2.28) the left-hand side is nonnegative. If instead $|w_n(x)| \leq R$, we can assume $|v_n(x)| \leq 2R$, and by (2.2) we get

$$\begin{aligned} &\Big| j_s(x, w_n, (1-\alpha\tau_n)\nabla v_n)v_n + j_\xi(x, w_n, (1-\alpha\tau_n)\nabla v_n)\cdot\nabla v_n \Big| \\ &\quad \leq \gamma(|w_n|)|v_n|\,|\nabla v_n|^2 + \mu(|w_n|)|\nabla v_n|^2 \leq (2\gamma(R)R + \mu(R))|\nabla v_n|^2. \end{aligned}$$

Then, we are allowed to apply Fatou's lemma, yielding

$$\begin{aligned} \liminf_{n\to\infty} &\int_\Omega \Big[j_s(x, w_n, (1-\alpha\tau_n)\nabla v_n)v_n + j_\xi(x, w_n, (1-\alpha\tau_n)\nabla v_n)\cdot\nabla v_n \Big] \\ &\geq \int_\Omega j_s(x, u, \nabla u)u + j_\xi(x, u, \nabla u)\cdot\nabla u > \sigma\|u\|_{1,2}, \end{aligned}$$

which immediately yields a contradiction with (2.31). Hence (2.30) holds, for some $\delta > 0$. Observe that, since $j(x,\cdot,\cdot)$ is of class C^1 for a.e. $x \in \Omega$ then, for any $t \in [0,1]$ and every $v \in \mathrm{dom}(I)$, there exists $0 \leq \tau(x,t) \leq t$ such that

$$\begin{aligned} &j(x, (1-\alpha t)v, (1-\alpha t)\nabla v) - j(x, v, \nabla v) \\ &\quad = -\alpha t\Big[j_s(x, (1-\alpha\tau)v, (1-\alpha\tau)\nabla v)v + j_\xi(x, (1-\alpha\tau)v, (1-\alpha\tau)\nabla v)\cdot\nabla v \Big]. \end{aligned} \tag{2.33}$$

As for inequality (2.32), for some $C(R) > 0$, for t small enough it holds that

$$\begin{aligned} &j_s(x, (1-\alpha\tau)v, (1-\alpha\tau)\nabla v)v \\ &\qquad + j_\xi(x, (1-\alpha\tau)v, (1-\alpha\tau)\nabla v)\cdot\nabla v \geq -C(R)|\nabla v|^2. \end{aligned}$$

Whence, if $v \in \mathrm{dom}(I)$, by (2.33) it follows that $(1-\alpha t)v \in \mathrm{dom}(I)$ for all $t \in [0,\delta]$ and

$$j_s(x, (1-\alpha\tau)v, (1-\alpha\tau)\nabla v)v + j_\xi(x, (1-\alpha\tau)v, (1-\alpha\tau)\nabla v)\cdot\nabla v \in L^1(\Omega). \tag{2.34}$$

Up to reducing δ, we may assume that $\delta < \eta\|u\|_{1,2}$. Then, for all $v \in B(u,\delta)$, we have $\|v\|_{1,2} \leq (1+\eta)\|u\|_{1,2} = \alpha^{-1}$. Consider the continuous map $\mathcal{H} : B(u,\delta) \cap I^b \times [0,\delta] \to H_0^1(\Omega)$ defined as $\mathcal{H}(v,t) = (1-\alpha t)v$, where $I^b = \{v \in H_0^1(\Omega) : I(v) \leq b\}$. From (2.30) (applied, for each $t \in [0,\delta]$, with the function

$\tau(\cdot,t) \in L^\infty(\Omega,[0,\delta])$ for which identity (2.33) holds) and identity (2.33), for every $t \in [0,\delta]$ and $v \in B(u,\delta) \cap I^b$ we have

$$\|\mathcal{H}(v,t) - v\|_{1,2} \leq t, \qquad I(\mathcal{H}(v,t)) \leq I(v) - \frac{\sigma}{1+\eta}t.$$

Then, by means of [15, Proposition 2.5] and exploiting the arbitrariness of η, we get $|dI|(u) \geq \sigma$. In turn, (2.29) follows from the arbitrariness of σ. Concerning the second part of the statement, since $|dI|(u) < +\infty$, from (2.28) and (2.29),

$$j_\xi(x,u,\nabla u)\cdot\nabla u + j_s(x,u,\nabla u)u \in L^1(\Omega). \tag{2.35}$$

In turn, using again (2.28), it follows that $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$, since

$$\begin{aligned}\varepsilon j_\xi(x,u,\nabla u)\cdot\nabla u &\leq \varepsilon\mu(R)|\nabla u|^2 + \varepsilon j_\xi(x,u,\nabla u)\cdot\nabla u\chi_{\{|u|\geq R\}} \\ &\leq \varepsilon\mu(R)|\nabla u|^2 + |j_s(x,u,\nabla u)u + j_\xi(x,u,\nabla u)\cdot\nabla u|.\end{aligned}$$

Then, by exploiting (2.35) again, $j_s(x,u,\nabla u)u \in L^1(\Omega)$. The final assertion does not rely upon any sign condition and follows directly from [21, Proposition 4.5]. This concludes the proof. □

In the next result we show that it is possible to enlarge the class of admissible test functions. In order to do this, suppose we have a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla z + \int_\Omega j_s(x,u,\nabla u)z = \langle w,z\rangle \quad \text{for all } z \in V_u \tag{2.36}$$

for $w \in H^{-1}(\Omega)$. Under suitable assumptions, if (2.28) holds true, we can use $\zeta u \in H_0^1(\Omega)$ with $\zeta \in L^\infty(\Omega)$ as an admissible test functions in (2.36), generalizing [21, Theorem 4.8].

Proposition 2.14. *Assume that* (2.2) *and* (2.28) *hold. Let* $w \in H^{-1}(\Omega)$, *and let* $u \in H_0^1(\Omega)$ *be such that* (2.36) *is satisfied. Moreover, suppose that* $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ *and that there exist* $v \in H_0^1(\Omega)$ *and* $\eta \in L^1(\Omega)$ *such that*

$$j_s(x,u,\nabla u)v \geq \eta \qquad \textit{and} \qquad j_\xi(x,u,\nabla u)\cdot\nabla v \geq \eta. \tag{2.37}$$

Then $j_s(x,u,\nabla u)v \in L^1(\Omega)$, $j_\xi(x,u,\nabla u)\cdot\nabla v \in L^1(\Omega)$ *and*

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle w,v\rangle. \tag{2.38}$$

In particular, if $\zeta \in L^\infty(\Omega)$, $\zeta \geq 0$, $\zeta u \in H_0^1(\Omega)$ *and* $j_\xi(x,u,\nabla u)\cdot\nabla(\zeta u) \in L^1(\Omega)$, *then it follows that* $j_s(x,u,\nabla u)\zeta u \in L^1(\Omega)$ *and*

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla(\zeta u) + \int_\Omega j_s(x,u,\nabla u)\zeta u = \langle w,\zeta u\rangle. \tag{2.39}$$

Proof. The first part of the statement follows by means of [21, Theorem 4.8]. By assumption (2.28) and since ζ is nonnegative and bounded, we have

$$\begin{aligned} j_s(x,u,\nabla u)\zeta u &= \zeta j_s(x,u,\nabla u)u\chi_{\{|u|\le R\}} + \zeta j_s(x,u,\nabla u)u\chi_{\{|u|\ge R\}} \\ &\ge -R\gamma(R)\|\zeta\|_{L^\infty(\Omega)}|\nabla u|^2 \\ &\quad - (1-\varepsilon)\zeta j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega). \end{aligned}$$

The last assertion of the statement then follows from the first one. □

2.4. AR-type conditions

Some AR-type conditions, typically used in order to guarantee the boundedness of Palais–Smale sequences, remain invariant.

Proposition 2.15. *Let $\varphi \in C^2(\mathbb{R})$ be a diffeomorphism which satisfies the properties of Definition 2.1. Assume that there exist $\delta > 0$, $\nu > 2(1-\beta)$ and $R \ge 0$ such that*

$$\nu j(x,s,\xi) - (1+\delta)j_\xi(x,s,\xi)\cdot\xi - j_s(x,s,\xi)s - \nu G(x,s) + g(x,s)s \ge 0$$

and $G(x,s) \ge 0$ for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $|s| \ge R$.

Then there exist $\delta^\sharp > 0$, $\nu^\sharp > 2$ and $R^\sharp > 0$ such that

$$\begin{aligned} \nu^\sharp j^\sharp(x,s,\xi) - (1+\delta^\sharp)j^\sharp_\xi(x,s,\xi)\cdot\xi - j^\sharp_s(x,s,\xi)s& \\ - \nu^\sharp G^\sharp(x,s) + g^\sharp(x,s)s &\ge 0 \end{aligned}$$

and $G^\sharp(x,s) \ge 0$ for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $|s| \ge R^\sharp$.

Proof. A direct calculation yields

$$\begin{aligned} &\frac{\nu}{1-\beta}\, j^\sharp(x,s,\xi) - j^\sharp_\xi(x,s,\xi)\cdot\xi - j^\sharp_s(x,s,\xi)s - \frac{\nu}{1-\beta}\, G^\sharp(x,s) + g^\sharp(x,s)s \\ &\quad= \frac{\nu}{1-\beta}\, j(x,\varphi(s),\varphi'(s)\xi) - \left(1 + \frac{\varphi''(s)s}{\varphi'(s)}\right) j_\xi(x,\varphi(s),\varphi'(s)\xi)\cdot\varphi'(s)\xi \\ &\qquad - \frac{\varphi'(s)s}{\varphi(s)}\, j_s(x,\varphi(s),\varphi'(s)\xi)\varphi(s) - \frac{\nu}{1-\beta}\, G(x,\varphi(s)) \\ &\qquad + \frac{\varphi'(s)s}{\varphi(s)}\, g(x,\varphi(s))\varphi(s) \\ &\quad= \frac{\varphi'(s)s}{\varphi(s)} \left(\frac{\varphi(s)}{\varphi'(s)s}\frac{\nu}{1-\beta} j(x,\varphi(s),\varphi'(s)\xi) \right. \\ &\qquad\qquad - \frac{\varphi(s)}{\varphi'(s)s}\left(1 + \frac{\varphi''(s)s}{\varphi'(s)}\right) j_\xi(x,\varphi(s),\varphi'(s)\xi)\cdot\varphi'(s)\xi \\ &\qquad\qquad - j_s(x,\varphi(s),\varphi'(s)\xi)\varphi(s) \\ &\qquad\qquad \left. - \frac{\nu}{1-\beta}\frac{\varphi(s)}{\varphi'(s)s}\, G(x,\varphi(s)) + g(x,\varphi(s))\varphi(s) \right) \end{aligned}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $s \ne 0$. We recall that $j(x,\tau,\zeta) \ge 0$, $j_\xi(x,\tau,\zeta)\cdot\zeta \ge 0$ and that the map $s \mapsto s\varphi(s)$ is nonnegative. Therefore, on account of condition (2.6), for all $\eta > 0$ small enough, there exists $R^\sharp > 0$ large enough such that $|\varphi(s)| \geq R$ for all $s \in \mathbb{R}$ with $|s| \geq R^\sharp$ and

$$
\begin{aligned}
&\frac{\nu}{1-\beta} j^\sharp(x,s,\xi) - j^\sharp_\xi(x,s,\xi)\cdot\xi - j^\sharp_s(x,s,\xi)s - \frac{\nu}{1-\beta}G^\sharp(x,s) + g^\sharp(x,s)s \\
&\quad\geq \frac{\varphi'(s)s}{\varphi(s)}\bigg(\nu j(x,\varphi(s),\varphi'(s)\xi) - \eta(1-\beta)j(x,\varphi(s),\varphi'(s)\xi) \\
&\qquad\qquad - j_\xi(x,\varphi(s),\varphi'(s)\xi)\cdot\varphi'(s)\xi \\
&\qquad\qquad - \eta(1-\beta)j_\xi(x,\varphi(s),\varphi'(s)\xi)\cdot\varphi'(s)\xi \\
&\qquad\qquad - j_s(x,\varphi(s),\varphi'(s)\xi)\varphi(s) - \nu G(x,\varphi(s)) \\
&\qquad\qquad - \eta(1-\beta)G(x,\varphi(s)) + g(x,\varphi(s))\varphi(s)\bigg) \\
&\quad\geq ((1-\beta)^{-1}-\eta)(\delta-\eta(1-\beta))j^\sharp_\xi(x,s,\xi)\cdot\xi \\
&\qquad - \frac{\varphi'(s)s}{\varphi(s)}(1-\beta)\eta j^\sharp(x,s,\xi) - \frac{\varphi'(s)s}{\varphi(s)}(1-\beta)\eta G^\sharp(x,s) \\
&\quad\geq ((1-\beta)^{-1}-\eta)(\delta-\eta(1-\beta))j^\sharp_\xi(x,s,\xi)\cdot\xi - 2\eta j^\sharp(x,s,\xi) - 2\eta G^\sharp(x,s)
\end{aligned}
$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $|s| \geq R^\sharp$. Finally, since by convexity of $j^\sharp$ and $j^\sharp(x,s,0) = 0$ we have $j^\sharp_\xi(x,s,\xi)\cdot\xi \geq j^\sharp(x,s,\xi)$, we get

$$
\begin{aligned}
&\frac{\nu}{1-\beta} j^\sharp(x,s,\xi) - j^\sharp_\xi(x,s,\xi)\cdot\xi - j^\sharp_s(x,s,\xi)s - \frac{\nu}{1-\beta}G^\sharp(x,s) + g^\sharp(x,s)s \\
&\qquad\qquad \geq \delta^\sharp j^\sharp_\xi(x,s,\xi)\cdot\xi + 2\eta j^\sharp(x,s,\xi) - 2\eta G^\sharp(x,s).
\end{aligned}
$$

In turn, choosing η small enough and setting

$$
\delta^\sharp = (1-\beta)^{-1}\delta - \eta(5+\delta) + \eta^2(1-\beta) > 0, \qquad \nu^\sharp = \nu(1-\beta)^{-1} - 2\eta > 2,
$$

the assertion follows. □

Corollary 2.16. *Let $\varphi \in C^2(\mathbb{R})$ be a diffeomorphism satisfying the properties of Definition 2.1. Assume that $\xi \mapsto j(x,s,\xi)$ is homogeneous of degree two and that there are $\nu > 2$ and $R > 0$ with*

$$
j_s(x,s,\xi)s \leq 0, \qquad 0 \leq \nu G(x,s) \leq g(x,s)s \tag{2.40}
$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $|s| \geq R$. Then

$$
\nu^\sharp j^\sharp(x,s,\xi) - (1+\delta^\sharp)j^\sharp_\xi(x,s,\xi)\cdot\xi - j^\sharp_s(x,s,\xi)s - \nu^\sharp G^\sharp(x,s) + g^\sharp(x,s)s \geq 0
$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $|s| \geq R^\sharp$, for some $\delta^\sharp > 0$, $R^\sharp > 0$ and $\nu^\sharp > 2$.

Proof. Since $\xi \mapsto j(x,s,\xi)$ is 2-homogeneous and $\nu > 2$, there exists $\delta > 0$ with

$$
\nu j(x,s,\xi) - (1+\delta)j_\xi(x,s,\xi)\cdot\xi = (\nu-2-2\delta)j(x,s,\xi) \geq 0
$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Hence, by assumptions (2.40), we get

$$\nu j(x, s, \xi) - (1 + \delta) j_\xi(x, s, \xi) \cdot \xi - j_s(x, s, \xi)s - \nu G(x, s) + g(x, s)s \geq 0$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ with $|s| \geq R$. Proposition 2.15 yields the assertion. □

3. Multiplicity of solutions

As a by-product of the previous results, we obtain the following existence result. Compared with the results of [5] here we can get infinitely many solutions, not necessarily bounded.

Theorem 3.1. *Assume that* $\varphi \in C^2(\mathbb{R})$ *satisfies the properties of Definition* 2.1, (2.25) *and let* $N \geq 3$. *Moreover, let* $j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfy* (2.1)–(2.2), $\xi \mapsto j(x, s, \xi)$ *be strictly convex, and*

$$j(x, -s - \xi) = j(x, s, \xi) \quad \textit{for a.e. } x \in \Omega \textit{ and all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.1}$$

$$j_s^\sharp(x, s, \xi)s \geq 0 \quad \textit{for all } |s| \geq R^\sharp \textit{ and some } R^\sharp \geq 0. \tag{3.2}$$

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *be continuous, satisfying* (2.27) *with* $2 < p < 2^*(1 - \beta)$,

$$g(x, -s) = -g(x, s) \quad \textit{for a.e. } x \in \Omega \textit{ and all } s \in \mathbb{R}, \tag{3.3}$$

$G(x, s) \geq 0$ *for* $|s| \geq R$ *and the joint conditions* (1.7) *and* (2.26), *for some* $R \geq 0$. *Then,*

$$\begin{cases} -\operatorname{div}(j_\xi(x, u, \nabla u)) + j_s(x, u, \nabla u) = g(x, u) & \textit{in } \Omega, \\ \quad u = 0 & \textit{on } \partial\Omega \end{cases} \tag{3.4}$$

admits a sequence (u_n) *of generalized solutions in the sense of Definition* 2.4. *Furthermore,*

$$\frac{2N}{N+2} < q < \frac{N}{2} \Longrightarrow u_n \in L^{\frac{Nq(1-\beta)}{N-2q}}(\Omega),$$
$$q > \frac{N}{2} \Longrightarrow u_n \in L^\infty(\Omega),$$

in the notations of assumptions (2.27). *In particular, if* $q > N/2$, *it follows that* $u_h \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *are solutions in distributional sense.*

Proof. Of course, $\xi \mapsto j^\sharp(x, s, \xi)$ is strictly convex. By assumptions (2.1)–(2.2), (2.27), (1.7) and (2.26), in light of Propositions 2.3, 2.9, 2.10 and 2.15 and taking into account the sign condition (3.2) for $j^\sharp$, [21, assumptions (1.1)–(1.4), (1.7), (2.2), (2.4) and the variant (1.7) for $j^\sharp$ of conditions (1.9) and (2.3) joined together which still guarantees the boundedness of Palais–Smale sequences] are satisfied for $j^\sharp$ and $g^\sharp$ for some $R^\sharp$. Also, since φ is odd, (3.1) yields

$$j^\sharp(x, -s, -\xi) = j(x, \varphi(-s), -\varphi'(-s)\xi) = j(x, -\varphi(s), -\varphi'(s)\xi) = j^\sharp(x, s, \xi)$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and, analogously, (3.3) yields

$$g^\sharp(x, -s) = g(x, \varphi(-s))\varphi'(-s) = g(x, -\varphi(s))\varphi'(s) = -g^\sharp(x, s)$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Then, we are allowed to apply [21, Theorem 2.1] and obtain a sequence $(v_h) \subset H_0^1(\Omega)$ of generalized solutions of (2.17) in the sense of [21], namely

$$j_\xi^\sharp(x, v_h, \nabla v_h) \cdot \nabla v_h \in L^1(\Omega), \qquad j_s^\sharp(x, v_h, \nabla v_h) v_h \in L^1(\Omega),$$

and

$$\int_\Omega j_\xi^\sharp(x, v_h, \nabla v_h) \cdot \nabla \psi + \int_\Omega j_s^\sharp(x, v_h, \nabla v_h) \psi = \int_\Omega g^\sharp(x, v_h) \psi \quad \text{for all } \psi \in V_{v_h}.$$

In particular, (v_n) is a sequence of $H_0^1(\Omega)$ generalized solutions of problem (2.17) in the sense of Definition 2.4. The desired existence assertion now follows from Proposition 2.6 for $u_n = \varphi(v_n)$. Concerning the summability, if $a^\sharp \in L^r(\Omega)$ and $|g^\sharp(x, s)| \le a^\sharp(x) + b|s|^{(N+2)/(N-2)}$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, then, by [21, Theorem 7.1], a generalized solution $v \in H_0^1(\Omega)$ of problem (2.17) belongs to $L^{Nr/(N-2r)}(\Omega)$ for any $2N/(N+2) < r < N/2$ and to $L^\infty(\Omega)$ for all $r > N/2$. Since g is subjected to (2.27), by Proposition 2.10, we also get the final conclusions. $\square$

Remark 3.2. We believe that Theorem 3.1 remains true if (3.2) is substituted by (1.6).

Remark 3.3. For $\beta = 0$, the summability of solutions coincides with the standard one.

By exploiting a multiplicity result of [18] which merely uses the relaxed sign condition (2.28) it is possible to provide a fully invariant version of Theorem 3.1 with respect to the sign condition (2.28), that we state in the following theorem.

Theorem 3.4. *Assume that $\varphi \in C^2(\mathbb{R})$ satisfies the properties of Definition* 2.1, (2.25) *and let $N \ge 3$. Moreover, let $j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy* (2.1)–(2.2), *let $\xi \mapsto j(x, s, \xi)$ be strictly convex satisfying* (3.1) *and* (2.28). *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous, satisfying* (2.27) *with $2 < p < 2^*(1 - \beta)$, and assume that* (3.3), (1.7) *and* (2.26) *hold, for some $R \ge 0$. Assume in addition that*

$$\lim_{s \to \infty} \frac{g(x, s)}{s} = \infty, \qquad j_\xi(x, s, \xi) \cdot \xi \le 2j(x, s, \xi) \tag{3.5}$$

and that $j^\sharp(x, s, \xi) \le C|\xi|^2$ and $j_s^\sharp(x, s, \xi) \le C|\xi|^2$ for some $C > 0$. Then, (3.4) *admits a sequence (u_n) of generalized solutions with the same regularity conclusion of Theorem* 3.1.

Proof. It is sufficient to follow step by step the argument of the proof of Theorem 3.1 by using [18, Theorem 1.2] in place of [21, Theorem 2.1], noticing that the conditions in (3.5) are invariant under the diffeomorphism φ. $\square$

The next proposition yields a class of j, which is the one studied in [5] (condition (3.6) below is precisely condition (1.3) in [5]), satisfying the assumptions of Theorem 3.1.

Proposition 3.5. *Assume that $j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is of the form*

$$j(x,s,\xi) = \frac{1}{2}a(x,s)|\xi|^2,$$

where $a(x,\cdot) \in C^1(\mathbb{R},\mathbb{R}^+)$ for a.e. $x \in \Omega$. Assume furthermore that there exists $R \geq 0$ such that

$$-2\beta a(x,s) \leq D_s a(x,s)(1+|s|)\operatorname{sign}(s) \leq 0 \tag{3.6}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ with $|s| \geq R$. Let $\varphi \in C^2(\mathbb{R})$ be a diffeomorphism according to Definition 2.1 which in addition satisfies

$$\varphi''(s) - \frac{\beta\varphi'(s)^2}{1+\varphi(s)} \geq 0 \quad \textit{for all } s \in \mathbb{R} \textit{ with } s \geq 1. \tag{3.7}$$

Then there exist $\nu^\sharp > 2$, $\delta^\sharp > 0$ and $R^\sharp > 0$ such that

$$sj_s^\sharp(x,s,\xi) \geq 0, \qquad \nu^\sharp j^\sharp(x,s,\xi) - (1+\delta^\sharp)j_\xi^\sharp(x,s,\xi)\cdot\xi - j_s^\sharp(x,s,\xi)s \geq 0$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$ and every $s \in \mathbb{R}$ with $|s| \geq R^\sharp$.

Proof. Let $R^\sharp \geq 1$ be such that $|\varphi(s)| \geq R$ for all $s \in \mathbb{R}$ with $|s| \geq R^\sharp$. Then, by (3.6), for all $s \geq R^\sharp$ we have $\varphi(s) \geq R$ and

$$\begin{aligned} j_s^\sharp(x,s,\xi) &= \Big[D_s a(x,\varphi(s))(\varphi'(s))^3 + 2\varphi'(s)\varphi''(s)a(x,\varphi(s))\Big]|\xi|^2/2 \\ &\geq a(x,\varphi(s))\varphi'(s)\left[\frac{-\beta\varphi'(s)^2}{1+\varphi(s)} + \varphi''(s)\right]|\xi|^2. \end{aligned}$$

Recalling that $a(x,\varphi(s))$ and $\varphi'(s)$ are positive and by (3.7), one gets

$$j_s^\sharp(x,s,\xi) \geq 0.$$

Similarly, if $s \leq -R^\sharp$, again by (3.6), we have $\varphi(s) \leq -R$ and

$$j_s^\sharp(x,s,\xi) \leq a(x,\varphi(s))\varphi'(s)\left[\frac{\beta\varphi'(s)^2}{1+|\varphi(s)|} + \varphi''(s)\right]|\xi|^2,$$

and so that $j_s^\sharp(x,s,\xi) \leq 0$, again due to (3.7), since being φ and φ'' odd and φ' even yields

$$\varphi''(s) + \frac{\beta\varphi'(s)^2}{1+|\varphi(s)|} \leq 0 \quad \text{for all } s \in \mathbb{R} \text{ with } s \leq -1.$$

The second inequality in the assertion follows from Corollary 2.16 (applied with $g = 0$), since $\xi \mapsto j(x,s,\xi)$ is 2-homogeneous and $j_s(x,s,\xi)s \leq 0$ for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$ and any $|s| \geq R$. □

Remark 3.6. In the statement of Proposition 3.5, in place of condition (3.6), one could consider the following slightly more general assumption: there exists $R \geq 0$ such that

$$-2\beta|s|a(x,s) \leq D_s a(x,s)(b(x)+s^2)\operatorname{sign}(s) \leq 0 \tag{3.8}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ with $|s| \geq R$, for some measurable function $b : \Omega \to \mathbb{R}$ such that $\nu^{-1} \leq b(x) \leq \nu$, for some $\nu > 0$. This condition is satisfied for instance by $a(x,s) = (b(x) + s^2)^{-\beta}$ with b measurable and bounded between positive constants.

Remark 3.7. When the maps $s \mapsto j^\sharp(x,s,\xi), j_s^\sharp(x,s,\xi), j_\xi^\sharp(x,s,\xi)$ are bounded, the variational formulation of (2.17) can be meant in the sense of distributions (see Proposition 2.8). For instance, as it can be easily verified, this occurs for the a mentioned in Remark 3.6, $a(x,s) = (b(x) + s^2)^{-\beta}$.

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